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DERIVATION OF NEWTONIAN-TYPE INTEGRATION COEFFICIENTS AND SOME APPLICATIONS TO ORBIT CALCULATIONS

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16. Abstract

Recursive relations for the computation of coefficients defining a general class of Newtonian-type integration and quadrature formulas are developed using finite difference operator techniques. These relations are expressed as functions of parameters in such a way that by specifying certain integral values, the well-known recursive relations for the coefficients in methods such as those of Adams, Cowell, and Nystrom are easily obtained.

Some applications of the formulas available in this class are also presented. The applications discussed include "starting" techniques for orbital integration and multi-revolution integration.

Finally, a description of a computer program used to generate these coefficients is presented, and tables of coefficients defining the more commonly used methods are given.

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CHAPTER I

INTRODUCTION

It is well known that finite difference operator techniques can be used to obtain many useful interpolation and integration formulas. In particular, recursive relations for the coefficients of various integration or quadrature methods can be obtained which are, in contrast to those obtainable by the method of undetermined coefficients, readily amenable to automatic computation.

In this report, these difference operator techniques are used to construct generalized operators which define the coefficients of a large class of stable integration formulas of the Newton interpolatory type. The resulting recursive relations have been programmed and used to compute the coefficients associated with the various popular integration formulas such as those of Adams, Cowell, and Nyström, as well as formulas of the Newton-Cotes type, which have applications in block, single, and multistep starting algorithms. A special application to the numerical integration of satellite orbits in multirevolution steps is also presented.

Finally, a computer program which performs the calculations with rational arithmetic is described, and the coefficients associated with some of the well-known techniques are tabulated.

It should be remarked that this report does not intend to present new formulas (although some of those derived are not easily found in the literature, in particular those pertaining to multirevolution starters) but to present a unified approach to many types of formulas which are currently being used to solve a variety of problems.

CHAPTER II

FORMULATION

A. Difference Operators and Identities

Let n be a positive integer, s and h any real numbers, and f a real-valued function defined on an interval $[a, b]$ that is n-times continuously differentiable; that is, $f \epsilon C^n[a, b]$. Consider the linear operators Δ_n, ∇_n, E_n^s, D^n, I defined by

$$\Delta_n f(x) = f(x + nh) - f(x) \quad \text{(forward difference)},$$

$$\nabla_n f(x) = f(x) - f(x - nh) \quad \text{(backward difference)},$$

$$E_n^s f(x) = f(x + snh) \quad \text{(shifting)},$$

$$D^n f(x) = f^{(n)}(x) \quad \text{(differentiation)},$$

and

$$If(x) = f(x) \quad \text{(identity)}.$$

Interpreting equality between expressions containing these operators in the usual way,* some well known relations between these operators are the following:

$$\Delta_n = \frac{\nabla_n}{I - \nabla_n}, \tag{1}$$

$$E_n = (I + \Delta_n) = (I - \nabla_n)^{-1}, \tag{2}$$

and

$$e^{hD} = (I + \Delta) = E. \tag{3}$$

From these identities, it follows that

$$\nabla_{sn} = [I - E^{-sn}] = [I - E_n^{-s}], \tag{4}$$

$$hD = -\log(I - \nabla), \tag{5}$$

and

$$\Delta = (I - \nabla_n)^{-1/n} - I. \tag{6}$$

Using (4) and (5), we can immediately form the identity

$$\nabla_K^L = h^L \left\{ \left[\frac{I - E^{-K}}{-\log(I - \nabla)} \right]^L E^{-J} \right\} E^J D^L, \tag{7}$$

*Generally with respect to some class of polynomials. See Reference 1 concerning the calculus of finite differences.

where L is any positive integer and J and K are real. In this identity, powers of operators are defined in the usual way.

Our goal is to find expressions for coefficients $\gamma_i = \gamma_i(J, K, L)$ so that (7) can be expressed in powers of ∇ as

$$\nabla_K^L = h^L \left[\sum_{i=0}^{\infty} \gamma_i(J, K, L) \nabla^i \right] E^J D^L . \tag{8}$$

Once this is accomplished, we will show how (8) can be used to yield integration formulas for initial value problems of the form

$$y^{(L)}(x) = f(x, y) , \tag{9}$$

with the initial values $y(x_0)$, $y'(x_0)$, ... $y^{(L-1)}(x_0)$.

In similar fashion, using (3), (4), and (5), we have the identity

$$\nabla_K - E^{-K} \sum_{i=1}^{L-1} \frac{(KhD)^i}{i!} = \left[\frac{E^{-(J+K)} \sum_{i=L}^{\infty} \frac{(KhD)^i}{i!}}{\nabla^L} \right] \left[\frac{\nabla}{-\log (I - \nabla)} \right]^L E^J (hD)^L , \tag{10}$$

where L is any positive integer and J and K are real, and again we are to find expressions for coefficients $\alpha_i = \alpha_i(J, K, L)$ so that (10) can be expressed as

$$\nabla_K = \sum_{i=1}^{L-1} \frac{(Kh)^i}{i!} D^i E^{-K} + h^L \sum_{i=0}^{\infty} \alpha_i(J, K, L) \nabla^i D^J E^L . \tag{11}$$

Equation (11) will be used to develop multistep starting formulas for systems of the form (9), especially for the case $L = 2$, which is of interest for orbit trajectory computations.

Finally, for applications in the theory of multirevolution integration of satellite orbits, we require the operator identity

$$\nabla_{KN} = \left[\frac{\left(I - E_N^{-K}\right) E_N^{-J}}{\left(I - \nabla_N^{-1/N}\right) - I} \right] E_N^J \Delta , \tag{12}$$

which follows directly from relations (4) and (6). As before, we will seek an expansion of (12) of the form

$$\nabla_{KN} = N \left[\sum_{i=0}^{\infty} \beta_i(J, K, N) \nabla_N^i \right] E_N^J \Delta \tag{13}$$

and will use it to develop multirevolution predictor-corrector and starting formulas.

B. Series Expansions

Before performing the expansions (8), (11), and (13), we will require the following series identities:

$$[-\log (1 - x)]^L = x^L \sum_{i=0}^{\infty} L!\, H_i^{(L)} x^i\,, \qquad \text{for } L \geq 1\,, \tag{14}$$

where

$$H_i^{(1)} = \frac{1}{i+1}\,,$$

$$H_i^{(L)} = \frac{\sum_{j=0}^{i} H_j^{(L-1)}}{i + L}\,,$$

$$(1 - x)^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} x^i\,, \tag{15}$$

and

$$1 - (1 - x)^n = x \sum_{i=0}^{\infty} (-1)^i \binom{n}{i+1} x^i\,. \tag{16}$$

Identity (14) can be proved by induction as follows: For $L = 1$,

$$[-\log (1 - x)] = x \sum_{i=0}^{\infty} \frac{x^i}{i+1}$$

is well known. Assuming the identity is true for $L = n$, we have

$$\begin{aligned} [-\log (1 - x)]^{n+1} &= D^{-1} D [-\log (1 - x)]^{n+1}\,, \\ &= D^{-1} \left\{ (n + 1)[-\log (1 - x)]^n (1 - x)^{-1} \right\}. \end{aligned}$$

By assumption,

$$\begin{aligned} [-\log (1 - x)]^{n+1} &= D^{-1} \left[(n+1)! \sum_{i=0}^{\infty} x^{i+n} \sum_{j=0}^{i} H_j^{(n)} \right], \\ &= (n+1)! \sum_{i=0}^{\infty} \frac{x^{(i+n+1)}}{i+n+1} \sum_{j=0}^{i} H_j^{(n)}\,. \end{aligned}$$

Hence, by definition of $H_j^{(n+1)}$,

$$[-\log (1 - x)]^{n+1} = (n + 1)!x^{n+1} \sum_{i=0}^{\infty} H_i^{(n+1)} x^i,$$

and hence (14) is true for all L. Identity (15) is well known and (16) follows directly from (15).

We now proceed to expand the operator expression (7) in powers of ∇. Using (16), we first have

$$I - E^{-K} = I - (I - \nabla)^K = \nabla \sum_{i=0}^{\infty} b_i(K)\nabla^i , \tag{17}$$

where

$$b_i(K) = (-1)^i \binom{K}{i+1} ,$$

so that the factor $\left[I - E^{-K}\right]^L$ can be expressed, by repeated series multiplication, as

$$\left[I - E^{-K}\right]^L = \nabla^L \sum_{i=0}^{\infty} e_i(K, L)\nabla^i , \tag{18}$$

where, if $L = 1$,

$$e_i(K, 1) = b_i(K) ,$$

and, if $L > 1$,

$$e_0 = K^L$$

and

$$e_i(K, L) = \sum_{j_{L-1}=0}^{i} \left[\cdots \left\{ \sum_{j_2=0}^{j_3} \left[\sum_{j_1=0}^{j_2} b_{j_1} b_{j_2-j_1} \right] b_{j_3-j_2} \right\} \cdots b_{i-j_{L-1}} \right], \quad i = 1, 2, 3 \ldots , \tag{19}$$

where j_n are dummy indices. Next, we have

$$E^{-J} = (I - \nabla)^J = \sum_{i=0}^{\infty} (-1)^i \binom{J}{i} \nabla^i . \tag{20}$$

So by multiplying these last two factors, we get

$$\left(I - E^{-K}\right)^L E^{-J} = \nabla^L \sum_{i=0}^{\infty} f_i(J, K, L)\nabla^i , \tag{21}$$

where

$$f_i(J, K, L) = \sum_{j=0}^{i} (-1)^{i-j}\binom{J}{i-j} e_j(K, L).$$

Finally, we have, from (14),

$$[-\log (I - \nabla)]^L = \nabla^L \sum_{i=0}^{\infty} L! H_i^{(L)} \nabla^i. \tag{22}$$

We can now form, by series division, the result

$$\left[\frac{I - E^{-K}}{-\log (I - \nabla)}\right]^L E^{-J} = \sum_{i=0}^{\infty} \gamma_i(J, K, L)\nabla^i, \tag{23}$$

where

$$\left.\begin{aligned} &\gamma_0 = K^L \\ \text{and} \\ &\gamma_i = f_i - \sum_{j=1}^{i} L! H_j^{(L)} \gamma_{i-j}, \quad i = 1, 2, 3 \ldots, \end{aligned}\right\} \tag{24}$$

and these are precisely the coefficients required in (8). In the following sections, this relation will be used to obtain various integration formulas.

Next, we wish to expand identity (10) in powers of ∇. We first have that

$$\sum_{l=L}^{\infty} \frac{(KhD)^l}{l!} = \sum_{l=L}^{\infty} K^l \frac{[-\log (I - \nabla)]^l}{l!},$$

and by (14) we have

$$\frac{[-\log (I - \nabla)]^l}{l!} = \nabla^l \sum_{i=0}^{\infty} H_i^{(l)} \nabla^i,$$

so that by collecting terms in powers of ∇, we have

$$\sum_{l=L}^{\infty} \frac{(KhD)^l}{l!} = \sum_{i=0}^{\infty} c_i(K, L)\nabla^{i+L}, \tag{25}$$

where

$$c_i(K, L) = \sum_{j=0}^{i} K^{L+j} H_{i-j}^{(L+j)} . \tag{26}$$

Next, we have, from (20),

$$E^{-(J+K)} = \sum_{i=0}^{\infty} (-1)^i \binom{J+K}{i} \nabla^i ,$$

and hence the first factor on the right-hand side of (10) can be expressed as

$$\left[\frac{E^{-(J+K)} \sum_{i=L}^{\infty} \frac{(KhD)^i}{i!}}{\nabla^L}\right] = \sum_{i=0}^{\infty} d_i(J, K, L) \nabla^i , \tag{27}$$

where

$$d_0 = c_0,$$

and

$$d_i = \sum_{j=0}^{i} (-1)^{j-i} \binom{J+K}{j-i} c_j . \tag{28}$$

Moreover, since

$$\left[\frac{\nabla}{-\log (I - \nabla)}\right]^L = \left[\frac{I - E^{-1}}{-\log (I - \nabla)}\right]^L$$

is just the expanded portion of identity (7) with $K = 1$, $J = 0$, we have

$$\left[\frac{\nabla}{-\log (I - \nabla)}\right]^L = \sum_{i=0}^{\infty} \gamma_i(0, 1, L) \nabla^i , \tag{29}$$

where γ_i are given in (24). Hence, we can see that the $\alpha_i = \alpha_i(J, K, L)$, required in (11), are given by

$$\left.\begin{aligned} \alpha_0 &= d_0 \gamma_0 \\ \text{and} \quad \alpha_i &= \sum_{j=0}^{i} d_j \gamma_{i-j} . \end{aligned}\right\} \tag{30}$$

Finally, we wish to expand identity (12) in powers of ∇_n. By (2), we can see that expanding

$$\left(I - E_N^{-K}\right)E_N^{-J}$$

in powers of ∇_n is precisely the expansion of

$$(I - E^{-K})E^{-J}$$

in powers of ∇ so that, by (21), we have

$$\left(I - E_N^{-K}\right)E_N^{-J} = \nabla_N \sum_{i=0}^{\infty} f_i(J, K, 1)\nabla_N^i .$$

Finally, expanding the factor

$$(I - \nabla_N)^{-1/N} - I = \sum_{i=1}^{\infty} (-1)^i \binom{-1/N}{i} \nabla_N^i ,$$

$$= \frac{\nabla_n}{N} \sum_{i=0}^{\infty} g_i(N)\nabla_N^i ,$$

where

$$g_i(N) = (-1)^{i+1} \binom{-1/N}{i+1},$$

we obtain the required $\beta_i \equiv \beta_i(J, K, N)$ in (13),

and

$$\left.\begin{aligned} \beta_0 &= f_0 , \\ \beta_i &= f_i(J, K, 1) - \sum_{j=1}^{i} g_j(N)\beta_{i-j} . \end{aligned}\right\} \qquad (31)$$

Before proceeding to the applications of these operator expansions, we remark that the identities (7), (10), and (12), which we have expanded, were selected because of the specific applications we had in mind, otherwise their selection was arbitrary. Also, the methods used to obtain the expansions were essentially the same in all three formulas, and if the need arose for another formula derivable from Newtonian operator methods the same techniques would be applicable. Finally, these "techniques" involve little more than some elementary series algebra and result in formulas amenable to automatic computations.

C. Integration Formulas and Measures of Accuracy

Applying the operators (8) or (11) to an arbitrary, sufficiently smooth function $y(x)$, we see that the resulting relation expresses differences of values of the function in terms of differences of its Lth derivative. For example, by (8), we have

$$\nabla_K^L y(x) = h^L \sum_{i=0}^{\infty} \gamma_i(J, K, L) \nabla^i y^{(L)}(x + Jh) . \tag{32}$$

This is precisely the type of relation one needs to numerically integrate initial value problems of the form (9). Note however that Equation (8) is valid only with respect to a class of polynomials, in which case the sums are finite, and that for an arbitrary function $y(x)$, even when sufficiently smooth, the corresponding series (32) may fail to converge. These same remarks hold for Equation (11); hence we wish to find an expression for the error resulting from the truncation of (8) or (11) after n terms when applied to such a function. To this end, we wish to estimate the difference operators (Reference 2) L_h and G_h, defined by

$$L_h[y(x)] = \nabla_K^L y(x) - h^L \sum_{i=0}^{n} \gamma_i(J, K, L) \nabla^i y^{(L)}(x + Jh) , \tag{33}$$

and

$$G_h[y(x)] = \nabla_K y(x) - \sum_{i=1}^{L-1} \frac{(Kh)^i}{i!} y^{(i)}(x - Kh) - h^L \sum_{i=0}^{n} \alpha_i(J, K, L) \nabla^i y^{(L)}(x + Jh) , \tag{34}$$

where $y(x)$ is assumed, for convenience, to be an infinitely differentiable function defined on some interval $[a, b]$ with the property that for any x and $x + rh$, contained in $[a, b]$, and $n \geq 0$, we have

$$h^n y^{(n)}(x + rh) = \sum_{m=n}^{\infty} \frac{r^{m-n} h^m}{(m - n)!} y^{(m)}(x) . \tag{35}$$

Next, by using the identity

$$\nabla_n^m y(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} y(x - inh) ,$$

where n, m are arbitrary positive integers, we see that (33) and (34) can be expressed, in ordinate form, as

$$L_h[y(x)] = \sum_{i=0}^{L} (-1)^i \binom{L}{i} y(x - iKh) - h^L \sum_{i=0}^{n} \xi(\gamma) y^{(L)}[x + (J - i)h] \tag{36}$$

and

$$G_h[y(x)] = y(x) - y(x - Kh) - \sum_{i=1}^{L-1} \frac{(Kh)^i}{i!} y^{(i)}(x - Kh) - h^L \sum_{i=0}^{n} \xi_i(\alpha) y^{(L)}[x + (J - i)h], \tag{37}$$

where

$$\xi_i(\gamma) = (-1)^i \sum_{m=i}^{n} \binom{m}{i} \gamma_m$$

and

$$\xi_i(\alpha) = (-1)^i \sum_{m=i}^{n} \binom{m}{i} \alpha_m .$$

Further, applying the Taylor expansion (35), we get

$$L_h[y(x)] = \sum_{i=0}^{L} (-1)^i \binom{L}{i} \left[\sum_{m=0}^{\infty} \frac{(-iK)^m}{m!} h^m y^{(m)}(x) \right] - \sum_{i=0}^{n} \xi_i(\gamma) \left[\sum_{m=L}^{\infty} \frac{(J - i)^{m-L}}{(m - L)!} h^m y^{(m)}(x) \right] ,$$

$$= \sum_{m=0}^{\infty} \left[\sum_{i=0}^{L} \frac{(-1)^i \binom{L}{i} (-iK)^m}{m!} \right] h^m y^{(m)}(x) - \sum_{m=L}^{\infty} \left[\sum_{i=0}^{n} \frac{\xi_i(\gamma)(J - i)^{m-L}}{(m - L)!} \right] h^m y^{(m)}(x) . \tag{38}$$

Similarly,

$$G_h[y(x)] = y(x) - \sum_{m=0}^{\infty} \frac{(-K)^m}{m!} h^m y^{(m)}(x) - \sum_{i=1}^{L-1} \frac{K^i}{i!} \sum_{m=i}^{\infty} \frac{(-K)^{m-i}}{(m - i)!} h^m y^{(m)}(x)$$

$$- \sum_{i=0}^{n} \xi_i(\alpha) \left[\sum_{m=L}^{\infty} \frac{(J - i)^{m-L}}{(m - L)!} h^m y^{(m)}(x) \right] ,$$

$$= - \sum_{m=1}^{\infty} \delta_m K^m h^m y^{(m)}(x) - \sum_{m=L}^{\infty} \left[\sum_{i=0}^{n} \frac{\xi_i(\alpha)(J - i)^{m-L}}{(m - L)!} \right] h^m y^{(m)}(x) , \tag{39}$$

where

$$\delta_m = \sum_{j=0}^{m} \frac{(-1)^{m-j}}{j!(m - j)!} , \quad \text{for } 1 \le m \le L - 1$$

and

$$\delta_m = \sum_{j=0}^{L-1} \frac{(-1)^{m-j}}{j!(m-j)!}, \quad \text{for } m > L - 1 .$$

So we see that the operators L_h and G_h can be expressed as a series in powers of $h^m y^{(m)}(x)$ as

$$\left.\begin{aligned} L_h[y(x)] &= \sum_{m=0}^{\infty} A_m h^m y^{(m)}(x) \\ \text{and} \qquad G_h[y(x)] &= \sum_{m=0}^{\infty} B_m h^m y^{(m)}(x) , \end{aligned}\right\} \tag{40}$$

where

$$\left.\begin{aligned} A_m &= \sum_{i=1}^{L} \frac{(-1)^i \binom{L}{i} (-iK)^m}{m!} \\ B_m &= \sum_{j=0}^{m} \frac{(-1)^{m-j} K^m}{j!(m-j)!} \end{aligned}\right\} \text{for } m \leq L - 1$$

and (41)

$$\left.\begin{aligned} A_m &= \sum_{i=1}^{L} \frac{(-1)^i \binom{L}{i} (-iK)^m}{m!} - \sum_{i=0}^{n} \frac{\xi_i(\gamma)(J-i)^{m-L}}{(m-L)!} \\ B_m &= -\left[\sum_{j=0}^{L-1} \frac{(-1)^{m-j} K^m}{j!(m-j)!} + \sum_{i=0}^{n} \frac{\xi_i(\alpha)(J-i)^{m-L}}{(m-L)!}\right] \end{aligned}\right\} \text{for } m > L - 1 .$$

Now the operators $L_h[y(x)]$ and $G_h[y(x)]$ are said to be of order p if

$$\left.\begin{aligned} L_h[y(x)] &= O(h^{p+L}) \\ \text{and} \qquad G_h[y(x)] &= O(h^{p+L}) \end{aligned}\right\} + \text{higher order terms} ,$$

which will be the case if and only if

$$A_m = B_m = 0 , \quad \text{for } 0 \leq m < p + L$$

in the expansions (40).

The order of an integration formula of the form (32), with a finite number of terms, can be defined as the order of its associated difference operator. In the following, we will adopt this convention, and we will assume the truncation error associated with formulas derived from (8) or (11) in the form

$$\left.\begin{array}{l} A_{p+L} h^{p+L} y^{(p+L)}(\xi) \\ \text{or} \\ B_{p+L} h^{p+L} y^{(p+L)}(\xi), \end{array}\right\} \qquad [41(a)]$$

where ξ is some point in the integration interval, so that we are essentially neglecting the higher order terms in the error expansion by assuming that some generalized mean-value theorem is applicable (see for example Reference 2, p. 247).

In a similar fashion, we would like to measure the error resulting from the truncation of operator (13) after n terms when applied to a sufficiently smooth function. We remark that although this formula does not relate functional values to differences of a derivative of this function it could be considered an integration formula if one considers the first order approximation of hD as Δ. In actual practice, this formula will be used in conjunction with an independent integrator which computes the necessary functional values for the right-hand side of (13). The details of such an application are described in Chapter III, Section C of this report. At this point we simply define a measure of accuracy of such a formula in precisely the same fashsions as was done for (8) or (11). Thus, we wish to estimate the difference operator

$$H_h[y(x)] = \nabla_{KN} y(x) - N \sum_{i=0}^{n} \beta_i(J, K, N) \nabla_N^i \Delta y(x + JNh), \tag{42}$$

where again $y(x)$ is assumed to be sufficiently smooth so that (35) holds. As before, we begin by expressing H_h in ordinate form:

$$H_h[y(x)] = y(x) - y(x - KNh) - N \sum_{i=0}^{n} \xi_i(\beta) \Delta y[x + N(J - i)h], \tag{43}$$

where

$$\xi_i(\beta) = (-1)^i \sum_{m=i}^{n} \binom{m}{i} \beta_m .$$

Expanding the terms as before, we get

$$H_h[y(x)] = -\sum_{m=1}^{\infty} \frac{(-KN)^m}{m!} h^m y^{(m)}(x) - N \sum_{i=0}^{n} \xi_i(\beta) \sum_{m=1}^{\infty} \frac{[N(J - i) + 1]^m - [N(J - i)]^m}{m!} h^m y^{(m)}(x) .$$

So we see that H_h can be expressed in powers of $h^m y^{(m)}(x)$ as

$$H_h[y(x)] = \sum_{m=1}^{\infty} C_m h^m y^{(m)}(x) , \tag{44}$$

where

$$C_m = -\frac{1}{m!}\left\{(-KN)^m + N \sum_{i=0}^{n} [N(J-i)+1]^m - [N(J-i)]^m\right\}.$$

The operator $H_h[y(x)]$ is defined to be order p if

$$H_h[y(x)] = O(h^{p+1}) + \text{higher order terms} .$$

As for the integration formulas (8) or (11), the truncation error associated with formulas derived from (13) will be in the form

$$C_{p+1} h^{p+1} y^{(p+1)}(\xi) ,$$

where C_{p+1} is the first nonzero coefficient in (44).

We remark at this point that throughout this report, the error terms associated with various formulas for quadrature or integration derived from these operators will be omitted. For the methods given in the appendix, the error terms and orders presented were obtained directly from expansions (40) and (44), which are, in general, only estimates. No attempt was made to rigorously determine their sharpest form or estimate their magnitude. It was felt that such an analysis would, in general, be lengthy, difficult, and outside the main thoughts of this report which revolve about the idea of using computer-oriented arithmetic to derive useful numerical methods. Rigorous estimates can, of course, be found readily in the literature.

CHAPTER III

APPLICATIONS

A. Multistep Quadrature and Integration Formulas for First- and Second-Order Systems

We begin this section by indicating how the operator identity (8) can be used to define some well-known quadrature methods used to obtain approximations of integrals of the form

$$\int_a^b f(x)\,dx\,.$$

Letting

$$h = \frac{(b-a)}{n}\,,$$

where n is some positive integer, and

$$x_i = a + ih\,,$$

we seek the coefficients W_i, so that

$$\int_{x_0}^{x_n} f(x)\,dx = h\sum_{i=0}^{n} W_i(n)f(x_{n-i}) + R\,. \tag{45}$$

To this end, let $L = 1$, $J = 0$ and $K = n$ in (8) and, applying the operator to the function $F(x)$, where

$$F'(x) = f(x)\,,$$

we have, retaining n terms and omitting the truncation error,

$$\nabla_n F(x) = h\sum_{i=0}^{n} \gamma_i(0, n, 1)\nabla^i F'(x)\,,$$

which can be rewritten for $x = x_n$ as

$$F(x_n) - F(x_0) = \int_{x_0}^{x_n} f(x)\,dx \approx h\sum_{i=0}^{n} \gamma_i \nabla^i f(x_n). \tag{46}$$

We see that since (45) can be considered the ordinate form of (46), it is simply required that expressions for the γ_i be obtained. From (24) we have

$$\gamma_0 = n,$$

and

$$\gamma_i = f_i(0, n, 1) - \sum_{j=1}^{i} H_j^{(1)} \gamma_{i-j}.$$

Now from (21) we have

$$\begin{aligned} f_i(0, n, 1) &= \sum_{j=0}^{i} (-1)^{i-j} \binom{0}{i-j} e_j, \\ &= e_i(n, 1), \\ &= b_i(n), \\ &= (-1)^i \binom{n}{i+1}, \end{aligned}$$

and, by (14),

$$H_j^{(1)} = \frac{1}{j+1}.$$

Therefore

$$\gamma_0 = n$$

and

$$\gamma_i = (-1)^i \binom{n}{i+1} - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1}, \tag{47}$$

and finally, converting (46) to ordinate form, we have

$$W_i(n) = (-1)^i \sum_{m=i}^{n} \binom{m}{i} \gamma_m(0, n, 1). \tag{48}$$

For example, for $n = 2$, we obtain

$$\gamma_0 = 2, \quad \gamma_1 = -2, \quad \gamma_2 = \frac{1}{3},$$

$$W_0 = \frac{1}{3}, \quad W_1 = \frac{4}{3}, \quad W_2 = \frac{1}{3},$$

and we have the well-known Simpson rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f_2 + 4f_1 + f_0].$$

Similarly, for $n = 3$, we obtain

$$\gamma_0 = 3, \quad \gamma_1 = -\frac{9}{2}, \quad \gamma_2 = \frac{9}{4}, \quad \gamma_3 = -\frac{3}{8},$$

$$W_0 = \frac{3}{8}, \quad W_1 = \frac{9}{8}, \quad W_2 = \frac{9}{8}, \quad W_3 = \frac{3}{8},$$

which yields Newton's 3/8 rule:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}h[f_3 + 3f_2 + 3f_1 + f_0],$$

and in general, the coefficients defined by (47) and (48) yield the Newton-Cotes formulas of the closed type. The Newton-Cotes formulas of the open type, i.e., those which do not involve the ordinates at the ends of the integration interval, can be obtained from (8) by letting $K = n$, $L = 1$, $J = -1$, and retaining $n - 2$ terms:

$$\nabla_n F(x_n) = \int_{x_0}^{x_n} f(x)\,dx = h \sum_{i=0}^{n-2} \gamma_i(-1, n, 1)\nabla^i f_{n-1}.$$

As before, we have from (24),

$$\gamma_0 = n$$

and

$$\gamma_i = f_i(-1, n, 1) - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1},$$

where

$$f_i(-1, n, 1) = \sum_{j=0}^{i} (-1)^{i-j}\binom{-1}{i-j} e_j,$$

$$= \sum_{j=0}^{i} b_j(n), \quad \text{since } \binom{-1}{i-j} = (-1)^{i-j}, \text{ and } L = 1,$$

$$= \sum_{j=0}^{i} (-1)^j \binom{n}{j+1},$$

and hence

$$\gamma_0 = n$$

and (49)

$$\gamma_i = \sum_{j=0}^{i} (-1)^j \binom{n}{j+1} - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1},$$

which, together with (48) with $J = -1$, can be used to define formulas of the type

$$\int_{x_0}^{x_n} f(x)\,dx = h \sum_{i=0}^{n-2} W_i(n) f(x_{n-1-i}).$$

For example, for $n = 3$, we have

$$\gamma_0 = 3, \qquad \gamma_1 = -\frac{3}{2},$$

$$W_0 = \frac{3}{2}, \qquad W_1 = \frac{3}{2},$$

and

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{2}[f_2 + f_1].$$

Likewise for $n = 4$, we get the formula

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{4h}{3}[2f_3 - f_2 + 2f_1],$$

and so forth (see Reference 1, pp. 73–74).

Next we consider some well-known integration methods derivable from (8) for initial value problems of form (9). For the case when $L = 1$, we wish to examine multistep methods of the Newton type in the form

$$\nabla_K y(x) = h \sum_{i=0}^{n} \gamma_i(J, K, 1) \nabla^i y'(x + Jh).$$

First taking the values $K = 1$, $J = -1$, we have

$$\gamma_0(-1, 1, 1) = 1$$

and

$$\gamma_i(-1,1,1) = f_i(-1,1,1) - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1}.$$

Now,

$$f_i(-1,1,1) = \sum_{j=0}^{i} (-1)^{i-j}\binom{-1}{i-j} e_j(1,1),$$

$$= \sum_{j=0}^{i} b_j(1), \quad \text{since } \binom{-1}{i-j} = (-1)^{i-j};$$

but

$$b_j(1) = (-1)^j\binom{1}{j+1} = \begin{cases} 1 \text{ if } j = 0, \\ 0 \text{ if } j > 0. \end{cases}$$

So we have

$$\gamma_0(-1,1,1) = 1$$

and (50)

$$\gamma_i(-1,1,1) = 1 - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1},$$

which defines the well-known Adams-Bashforth integration predictor formula

$$y(x) - y(x-h) = h \sum_{i=0}^{n} \gamma_i(-1,1,1)\nabla^i f(x-h) \tag{51}$$

for solving initial value problems

$$\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0. \end{aligned} \tag{52}$$

In similar fashion, the often-used associated corrector formula, the Adams-Moulton, can be obtained with $K = 1$, $J = 0$: we have

$$\gamma_0(0,1,1) = 1$$

and

$$\gamma_i(0,1,1) = f_i(0,1,1) - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1},$$

where

$$f_i(0,1,1) = \sum_{j=0}^{i} (-1)^{i-j}\binom{0}{i-j} e_j(1,1),$$

$$= e_i(1,1) = b_i(1) = 0, \text{ if } i > 0.$$

So we have

$$\left.\begin{aligned} &\gamma_0(0,1,1) = 1 \\ \text{and} \\ &\gamma_i(0,1,1) = -\sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1}, \end{aligned}\right. \qquad (53)$$

and we obtain the formula

$$y(x) - y(x-h) = h\sum_{i=0}^{n} \gamma_i(0,1,1)\nabla^i f(x),$$

which is often used to solve (52).

Another popular formula, known as Nyström's, can be obtained with $K = 2$, $J = -1$:

$$\gamma_0(-1,2,1) = 2$$

and

$$\gamma_i(-1,2,1) = f_i(-1,2,1) - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1},$$

and, as before,

$$f_i(-1,2,1) = \sum_{j=0}^{i} (-1)^{i-j}\binom{-1}{i-j} e_j(2,1),$$

$$= \sum_{j=0}^{i} b_j(2);$$

but for any $j > 1$,

$$b_j(2) = (-1)^j\binom{2}{j+1} = 0.$$

We therefore have $f_i = 1$ for all $i \geq 1$, and hence

$$\left.\begin{aligned} &\gamma_0(-1,2,1) = 2 \\ \text{and} \\ &\gamma_i(-1,2,1) = 1 - \sum_{j=1}^{i} \frac{\gamma_{i-j}}{j+1}, \end{aligned}\right\} \qquad (54)$$

which defines the formula

$$y(x) - y(x - 2h) = h \sum_{i=0}^{n} \gamma_i(-1, 2, 1)\nabla^i f(x - h),$$

defining a predictor for (52).

We now examine methods of the form ($L = 2$):

$$\nabla_K^2 y(x) = h^2 \sum_{i=0}^{n} \gamma_i(J, K, 2)\nabla^i y''(x + Jh),$$

for the solution of initial value problems of the form

$$\begin{aligned} y''(x) &= f(x, y), \\ y(x_0) &= y_0, \\ y'(x_0) &= y_0', \end{aligned} \tag{55}$$

which are of the type that frequently occurs in orbit trajectory computations. Analogous to the case when $L = 1$, we will obtain the predictor with $J = -1$ and $K = 1$. We have, from (24),

$$\gamma_0(-1, 1, 2) = 1$$

and

$$\gamma_i(-1, 1, 2) = f_i(-1, 1, 2) - \sum_{j=1}^{i} 2! H_j^{(2)} \gamma_{i-j}.$$

Now

$$f_i(-1, 1, 2) = \sum_{j=0}^{i} (-1)^{i-j} \binom{-1}{i-j} e_j(1, 2),$$

$$= \sum_{j=0}^{i} e_j(1, 2);$$

but

$$e_j(1, 2) = \sum_{l=0}^{j} b_l(1) b_{i-l}(1),$$

$$= (-1)^i \sum_{l=0}^{j} \binom{1}{l+1}\binom{1}{j-l+1},$$

$$= \begin{cases} 1 \text{ if } j = 0, \\ 0 \text{ if } j > 0. \end{cases}$$

Also

$$H_j^2 = \frac{\sum_{l=0}^{j} \frac{1}{l+1}}{j+2},$$

so we see that

$$\left.\begin{aligned} \gamma_0(-1, 1, 2) &= 1 \\ \gamma_i(-1, 1, 2) &= 1 - \sum_{j=1}^{i} \frac{2\left(\sum_{l=0}^{j} \frac{1}{l+1}\right)}{j+2} \gamma_{i-j} \end{aligned}\right\} \tag{56}$$

are the coefficients defining the Störmer predictor formula

$$\nabla^2 y(x) = h^2 \sum_{i=0}^{n} \gamma_i(-1, 1, 2) \nabla^i y''(x - h)$$

for solving (55).

Finally, in similar fashion, we can obtain the associated corrector formula, known as Cowell's method, using $J = 0$ and $K = 1$:

$$\nabla^2 y(x) = h^2 \sum_{i=0}^{n} \gamma_i(0, 1, 2) \nabla^i y''(x),$$

where

$$\gamma_0(0, 1, 2) = 1$$

and

$$\gamma_i(0, 1, 2) = - \sum_{m=1}^{i} \frac{2 \sum_{l=0}^{j} \frac{1}{l+1}}{j+2} \gamma_{i-j}. \tag{57}$$

We remark here that recursions (50), (53), (54), (56), and (57) are well known (see for example Reference 2)* and usually are obtained by other procedures. We note however, that the recursion (24) could be used to generate all these formulas in a unified fashion.

*See also Maury, J. L., and Brodsky, G. P., "Cowell Type Numerical Integration as Applied to Satellite Orbit Computation," NASA X-553-69-46, April 1969.

We next proceed to other applications of the general operators (8), (11), and (13). The various integration and quadrature formulas that can be generated by (8) through selection of values for J, K, and L should be clear at this point. It is noteworthy that by using nonintegral values of J and K, we can obtain a variety of interpolation methods which, together with the integration methods required here, could readily be used for stepsize modification or to compute output at nonstep points.

It is also noteworthy that in addition to defining multistep methods, operator (8) can be used to define block-type Runge-Kutta methods (Reference 3); for example, the five-point formulas

$$\nabla_K y_K = h \sum_{i=0}^{4} \gamma_i(-J, K, 1) \nabla^i y'(x_K + Jh), \tag{58}$$

where $K = 1, 2, 3, 4$, and $J = 4 - K$, are precisely those required by such algorithms.

Various combinations of the J, K, L parameters that were considered to be of general interest were used to compute the coefficients appearing in the appendix.

B. Multistep Starting Formulas and Algorithms

As is well known, a serious drawback of using multistep methods to integrate differential systems of the form (9) is the requirement that an independent method be employed to obtain the necessary starting values. The problem is, in general, that the starting procedure used is not so efficient as the multistep integrator, requiring either more derivative evaluations per step or a smaller stepsize to maintain accuracy or both. In the present applications, in orbit and physical parameter estimation programs, this problem becomes even more serious. This is so for the following reasons:

(1) The force models employed in such programs are generally sophisticated, so that the running time of the program is directly proportional to the total number of derivative evaluations performed.

(2) The number of equations to be integrated is frequently of the order of 100 or more.

(3) It is often the case that the integrator must restart many times during the trajectory calculation because of discontinuities introduced in the accelerations at discrete points in the orbit. These discontinuities may be due to thrusting, large solar radiation-shadow impulses, or a variety of other possibilities.

The starting methods that are currently used in such applications are frequently based on either Runge-Kutta or power-series formulas. Another method that is known, but perhaps less frequently used, is based on multistep formulas. Algorithms based on these methods may often involve a more complicated computer program, but experimentation has indicated (Reference 4)* that such methods require fewer derivative evaluations than do the other methods.

In this section, we present the formulas and describe the multistep starting algorithms for the case of second-order systems [$L = 2$ in (9)], since this case applies to orbit computations. The generalization of these ideas to more arbitrary differential systems should be clear from this example. Assume that we wish to apply the predictor-corrector formulas [see (56), (57)]

*See also Peabody, P. R., "DODS Numerical Integration," Computer Sciences Corporation internal communication, September 1963.

$$\left.\begin{aligned}\nabla^2 y_{n+1} &= h^2 \sum_{i=0}^{k} \gamma_i(-1, 1, 2)\nabla^i y''_n \qquad \text{(predictor)}\\ \nabla^2 y_{n+1} &= h^2 \sum_{i=0}^{k} \gamma_i(0, 1, 2)\nabla^i y''_{n+1} \qquad \text{(corrector)}\end{aligned}\right\} \tag{59}$$

to integrate the differential system

$$y'' = f(x, y)$$

with given initial values

$$y(x_0) = y_0$$

$$y'(x_0) = y'_0.$$

Fixing k in (59), we can convert the formulas to ordinate form,

$$\nabla^2 y_{n+1} = h^2 \sum_{i=0}^{k} \sigma_i(k) y''_{n-i} \tag{60}$$

and

$$\nabla^2 y_{n+1} = h^2 \sum_{i=0}^{k} \sigma_i^*(k) y''_{n+1-i}\,. \tag{61}$$

We readily see that to start applying these formulas, we require the starting values

$$\{y_i, y''_i\} i = (0, k) \tag{62}$$

so that we could then use (60) to obtain y_{k+1} ($n = k$), followed by successive applications of (61) to obtain corrected values of y_{k+1} and thus completing the first step. We would then repeat this predictor-corrector cycle with $N = k + 1,\ k + 2$, and so on. Hence we are required to compute the values of (62) given y_0, y'_0, and $y''_0 = f(x_0, y_0)$. To this end we consider the operator (11), truncated after k terms, with $L = 2$, applied to y_k:

$$\nabla_K y_K = Khy'_0 + h^2 \sum_{i=0}^{k} \alpha_i(J, K, 2)\nabla^i y''_{K+J}\,,$$

where the $\alpha_i(J, K, 2)$ are given in (30). Letting $K = 1, 2, 3, \ldots k$ and $J = k - K$ in this expression, and converting each formula to ordinate form, we obtain the formulas

$$y_K - y_0 = Khy'_0 + h^2 \sum_{i=0}^{k} \xi_i(K) y''_{k-i}\,, \qquad K = (1, k)\,, \tag{63}$$

where, as before,

$$\xi_i(K) = (-1)^i \sum_{m=i}^{k} \binom{m}{i} a_m(k - K, K, 2).$$

The idea behind the method is to use (63) as one would a corrector formula, solving it by successive iterations. More precisely, let $y_i^{(m)}, y_j^{(m)}$, denote the mth approximation of the required values and assume these are known; then the $(m + 1)$st approximation is given by

$$y_K^{(m+1)} = y_0 + Khy_0' + h^2 \sum_{i=0}^{k} \xi_i(K) y_{k-i}^{(m)''}$$

and (64)

$$y_K^{(m+1)''} = f\left(x_K, y_K^{(m+1)}\right), \quad K = (1, k).$$

The convergence of such a scheme can be proved in the same manner as one does for the corrector formula (61) (see, for example, Reference 2, p. 216). In fact it can be shown that if C is the Lipschitz constant associated with $f(x, y)$, or is bound on df/dy, then the method converges if h is such that

$$\max_{1 \le K \le k} \left[h^2 C \sum_{i=0}^{k} |\xi_i(K)\right] < 1.$$

All that remains to be determined is a method to obtain the first approximation. In general, if h is sufficiently small, a sufficiently accurate first approximation is given by

$$\left.\begin{aligned} y_K^{(1)} &= y_0 + Khy_0' + \frac{Kh^2}{2} y_0'' \\ y_K^{(1)''} &= f\left(x_K, y_K^{(1)}\right) \end{aligned}\right\} K = (1, k). \tag{65}$$

In orbit trajectory calculations, a more accurate first approximation can be obtained by using an analytic two-body solution. These approximations are generally not expensive to compute and can considerably reduce the number of required successive iterations of (64).

Another scheme which will generally reduce the number of required iterations of (64) is to place the given initial values in the center (assuming k is even) of the required starting values, so that they become

$$\{y_i, y_i''\} i = \left(-\frac{k}{2}, \frac{k}{2}\right). \tag{66}$$

The above algorithm can be used in this case with the changes $K = (1, k)$ to $K = (-k/2, k/2)$, $K \neq 0$, and $J = k - K$ to $J = (k/2) - K$.

We remark again that for higher values of L in (9), similar methods can be constructed from (11). Also, for $L = 1$, it is clear that (8) can be used to obtain starting formulas, in fact formulas (58) are precisely what we require for a five-point formula. Finally, we note that for equations containing derivatives, for example,

$$y^{(L)}(x) = f(x, y, y', \ldots y^{(L-r)}), \quad r \geq 1, \tag{67}$$

the methods for different values of L can be used simultaneously to obtain the necessary starting values. This same remark applies, of course, to the basic integration of (67) using methods derivable from (8).

C. Multirevolution Integration, Multirevolution Starting Formulas, and Algorithms

To improve the efficiency of lifetime study and long-range prediction calculations, a method has been proposed that integrates orbits in multirevolution steps. This method is well known (for example, Reference 5) and will be only outlined in this section. The recent interest in using orbit generation programs for lifetime studies and planning interplanetary missions prompted the inclusion of these multirevolution integration techniques in this report. Since starting procedures for such methods have not appeared in the popular literature, these are also included in the analysis.

Essentially, the method of multirevolution integration involves combining a usual short-step numerical integrator with a procedure which steps the calculations ahead in multirevolution increments. This stepping procedure is similar to the usual predictor-corrector process in that it extrapolates the orbital elements N revolutions ahead and then, starting with these extrapolated values, computes successive corrections.

We begin by outlining this basic algorithm (a more detailed description was given by Velez*). Let f_j denote the value of an orbital element at the descending node of the jth revolution, and let N be the number of revolutions to be stepped. If k is the order of the highest difference to be retained in (13), we have for $K = 1$, $J = -1$, the multirevolution predictor, applied to f_{j+N}:

$$\nabla_N f_{j+N} = N \sum_{i=0}^{k} \beta_i(-1, 1, N) \nabla_N^i (\Delta f_j), \tag{68}$$

where we are using

$$\Delta f_j = f_{j+1} - f_j,$$

that is, $h = 1$ revolution, so that

$$E_N^{-1}(\Delta f_i) = \Delta f_{i-N},$$

and where the β_i are given by (31) as

$$\beta_0(-1, -1, N) = f_0(-1, 1, 1)$$

*Velez, C. E., "Numerical Integration of Orbits in Multirevolution Steps," NASA X-542-67-341, January 1967.

and

$$\beta_i(-1,-1,N) = f_i(-1,1,1) - \sum_{j=1}^{i} g_j(N)\beta_{i-j}\,.$$

Now, as in (50), we have

$$f_i(-1,1,1) = 1 \text{ for all } i\,,$$

so we have, using the definition of $g_j(N)$,

$$\beta_0(-1,1,N) = 1$$

and

$$\beta_i(-1,1,N) = 1 - \sum_{j=1}^{i} (-1)^{j+1}\binom{1/N}{j+1}\beta_{i-j}\,.$$

The associated multirevolution corrector can be obtained from (13) with $K = 1$, $J = 0$ applied to f_{j+N}.

$$\nabla_N f_{j+N} = N\sum_{i=0}^{k} \beta_i(0,1,N)\nabla_N^i(\Delta f_{j+N})\,, \tag{69}$$

where, since

$$f_i(0,1,1) = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i > 0 \end{cases},$$

we have

$$\beta_0(0,1,N) = 1$$

and

$$\beta_i(0,1,N) = -\sum_{j=1}^{i} (-1)^{j+1}\binom{1/N}{j+1}\beta_{i-j}\,.$$

Further, we can express (68) and (69) in ordinate form:

$$f_{j+N} - f_j = N\sum_{i=0}^{k} \xi_i(k)\Delta f_{j-iN} \tag{70}$$

and

$$f_{j+N} - f_j = N\sum_{i=0}^{k} \xi_i^*(k)\Delta f_{j+N(1-i)}\,, \tag{71}$$

where ξ_i and ξ_i^* are defined in the usual way in terms of the β_i. The basic algorithm can now be simply described as follows:

(1) Compute the starting values

$$\Delta f_i, \quad i = 0, N, 2N, \ldots, kN.$$

(2) Using the value $j = kN$, use (70) to predict the values of the orbital elements at the descending node of the $[(k+1)N]$th revolution.

(3) Using the short step integrator, and starting with these extrapolated values, integrate one revolution to obtain their values at the $[(k+1)N+1]$th descending node and compute the difference $\Delta f_{(k+1)N}$.

(4) Using this difference and formula (71), correct the values of $f_{(k+1)N}$ successively to convergence, repeating step (3) for each iteration.

(5) Repeat steps (2) to (4) with $j = (k+1)N, (k+2)N, \ldots,$ and so on.

It is easily seen that efficient starting procedures are essential to the overall efficiency of the above algorithm, both for the short step integrations in step 3 and for the starting values of step 1. These integrators require a knowledge of the orbital elements at the descending nodes of the first $kN + 1$ revolutions. In lieu of performing a short step integration over all these revolutions, the following method, analogous to those discussed in Section B, could be used.

Consider the operator (13), truncated after k terms applied to f_{kN},

$$\nabla_{KN} f_{KN} = N \sum_{i=0}^{k} \beta_i(J, K, N) \nabla_N^i (\Delta f_{KN+J}),$$

where we let $K = 1, 2, 3, \ldots, k$ and $J = k - K$. Converting each such formula to the ordinate form, we have

$$f_{KN} - f_0 = N \sum_{i=0}^{k} \xi_i(K) \Delta f_{(k-i)N}, \quad K = (1, k), \tag{72}$$

where, of course,

$$\xi_i(K) = (-1)^i \sum_{m=i}^{k} \binom{m}{i} \beta_m(k - K, K, N).$$

And again, as in the case of using (63), the idea is to solve (72) by successive iterations. If we let $f_j^{(m)}$ and $\Delta f_j^{(m)}$ denote the mth approximation of these values and assume they are known, the $(m+1)$st approximation is given by

$$\left.\begin{aligned} f_{KN}^{(m+1)} &= f_0 + N \sum_{i=0}^{k} \xi_i(K) \Delta f_{(k-i)N}^{(m)} \\ &\text{and} \\ \Delta f_{KN}^{(m+1)} &= f_{KN+1}^{(m+1)} - f_{KN}^{(m+1)}, \quad K = (1, k), \end{aligned}\right\} \tag{73}$$

where for each K, $f_{KN+1}^{(m+1)}$ is obtained by the short step integration of the elements to this node, using as initial values $f_{KN}^{(m+1)}$. Note that the f_0 are just the values of the orbital elements at the node of the epoch revolution and are hence known.

The convergence of this scheme can be proved in precisely the same manner as for (64). The first approximation of the starting values can readily be obtained from a two-body solution, and the idea of placing the f_0, Δf_0 at the center of the required starting values, to improve convergence, can easily be formulated.

We see that for each iteration, the method requires at most the short step integration of k revolutions, so that the overall efficiency of the method would generally be considerably improved over the starting procedure based on a short step integration over $kN + 1$ revolutions, especially for large N.

CHAPTER IV

PROGRAM DESCRIPTION

The series expansions of these generalized operators were programmed in Fortran IV on the Univac 1108 using a rational arithmetic package to eliminate the deterioration that would have occurred using floating point. In the program, each rational number was represented by two contiguous, double-precision, floating-point words containing integral values. Input to the program consists of values for the variables J, K, and L. The output is tables of coefficients for the nonsummed* difference form, the nonsummed ordinate forms, and, where applicable, the summed ordinate forms. The nonsummed ordinate form is used by the program to determine the truncation error that defines the order of the method. The program consists of a main routine which determines the generalized operator to be used, a subroutine to create the ordinate forms, several subroutines to calculate the truncation error and order, an assembly language format routine, and the rational arithmetic package.

The rational arithmetic package consists of–

(1) GCD–a function using Euclid's algorithm to compute the Greatest Common Divisor of two numbers

$$[a_1, a_2] = \text{GCD} > 0\,,$$

where GCD = 1 if a_1 and a_2 are 0 or if a_1 or a_2 is not integral.

(2) ADD–a subroutine that performs rational addition defined by the following algorithm:

$$\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_1\left(\dfrac{d_2}{[d_1, d_2]}\right) + n_2\left(\dfrac{d_1}{[d_1, d_2]}\right)}{d_2\left(\dfrac{d_1}{[d_1, d_2]}\right)} = \frac{\dfrac{n_3}{[n_3, d_3]}}{\dfrac{d_3}{[n_3, d_3]}} = \frac{n_4}{d_4}\,.$$

(3) SUB–a subroutine that performs rational subtraction defined by

$$\frac{n_1}{d_1} - \frac{n_2}{d_2} = \frac{n_1}{d_1} + \frac{(-n_2)}{d_2} = \frac{n_3}{d_3}\,.$$

*For a discussion of the summed form of the integration formulas, see Reference 1, page 327, or Maury and Brodsky, mentioned earlier.

CHAPTER V

SUMMARY AND CONCLUSIONS

The recursive relations (24), (30), and (31) defining the coefficients of the operator identities (8), (11), and (13), respectively, were found to be an effective means of obtaining the coefficients of a broad spectrum of quadrature, interpolation, and integration formulas of the Newtonian type. The methods used to obtain these relations were elementary and allow considerable flexibility of the basic operator identities, easily yielding formulas of the Newtonian type not derivable directly from the operators considered in this report.

Although not every possible application of these operators was discussed in detail (e.g. applications to interpolation) the applications presented should give the reader a sufficiently broad exposure to the capabilities, and also the limitations, of the difference operator technique.

Finally, it is remarked that the coefficients defining the Adams-Cowell integration formulas (50), (53), (56), and (57) and the starters (63) are currently being used successfully in our orbit determination systems. Also, the multirevolution algorithm was tested and found to be an effective means of saving computer time for long arc calculations. The results of these tests were reported by Velez and mentioned earlier. It is expected that the multirevolution starters presented in this report will improve this process considerably.

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National Aeronautics and Space Administration
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APPENDIX

NUMERICAL EXAMPLES

Tables of Coefficients in Rational Form

The following tables of coefficients are grouped according to usage. (For example, a table of predictor coefficients and a table of corrector coefficients form a group, the tables of coefficients for a multistep starter form another, and so on.) Preceding each group is a brief discussion and the pertinent operator equation. Each table within a group is preceded by the value of J, the shift exponent; K, the block step variable; and L, the derivative order. Each table is composed of two sets of coefficients: a set for use in the difference formulation and a set for use in the ordinate formulation. The local error constant (see Equations [41(a)]) precedes each set of coefficients for the ordinate form formulas.

Table Group I

The two tables in this group are the Adams-Bashforth predictor coefficients and the Adams-Moulton corrector coefficients. These are used to solve the first-order differential equation

$$y' = f(x, y),$$

and, with Equation (8), can be expressed as

$$y(x) - y(x - h) = h \sum_{i=0}^{10} \gamma_i(J, 1, 1)\nabla^i y'(x + Jh),$$

where $J = -1, 0$, and by (24),

$$\gamma_0(J, 1, 1) = 1,$$

$$\gamma_i(J, 1, 1) = f_i(J, 1, 1) - \sum_{j=1}^{i} \frac{\gamma_{i-j}(J, 1, 1)}{j + 1},$$

and

$$\left.\begin{aligned} f_i(-1, 1, 1) &= 1 \\ f_i(0, 1, 1) &= 0 \end{aligned}\right\} i = 1, 2, 3 \ldots.$$

The ordinate 11-point formulas are then given by

$$y(x) - y(x - h) = h \sum_{i=0}^{10} \xi_i(\gamma) y'[x + (J - i)h],$$

where

$$\xi_i(\gamma) = (-1)^i \sum_{m=i}^{10} \binom{m}{i} \gamma_m(J, 1, 1).$$

J = −1 K = 1 L = 1

Difference form	Ordinate form, order = 11, local error constant 4 777 223/17 418 240
γ_0 = 1	ξ_0 = 2 132 509 567/479 001 600
γ_1 = 1/2	ξ_1 = −2 067 948 781/119 750 400
γ_2 = 5/12	ξ_2 = 1 572 737 587/31 933 440
γ_3 = 3/8	ξ_3 = −1 921 376 209/19 958 400
γ_4 = 251/720	ξ_4 = 3 539 798 831/26 611 200
γ_5 = 95/288	ξ_5 = −82 260 679/623 700
γ_6 = 19 087/60 480	ξ_6 = 2 492 064 913/26 611 200
γ_7 = 5 257/17 280	ξ_7 = −186 080 291/3 991 680
γ_8 = 1 070 017/3 628 800	ξ_8 = 2 472 634 817/159 667 200
γ_9 = 25 713/89 600	ξ_9 = −52 841 941/17 107 200
γ_{10} = 26 842 253/95 800 320	ξ_{10} = 26 842 253/95 800 320

J = 0 K = 1 L = 1

Difference form	Ordinate form, order = 11, local error constant −4 671/788 480
γ_0 = 1	ξ_0 = 26 842 253/95 800 320
γ_1 = −1/2	ξ_1 = 164 046 413/119 750 400
γ_2 = −1/12	ξ_2 = −296 725 183/159 667 200
γ_3 = −1/24	ξ_3 = 12 051 709/3 991 680
γ_4 = −19/720	ξ_4 = −33 765 029/8 870 400
γ_5 = −3/160	ξ_5 = 2 227 571/623 700
γ_6 = −863/60 480	ξ_6 = −21 677 723/8 870 400
γ_7 = −275/24 192	ξ_7 = 23 643 791/19 958 400
γ_8 = −33 953/3 628 800	ξ_8 = −12 318 413/31 933 440
γ_9 = −8 183/1 036 800	ξ_9 = 9 071 219/119 750 400
γ_{10} = −3 250 433/479 001 600	ξ_{10} = −3 250 433/479 001 600

Table Group II

The two tables in this group are the Störmer predictor coefficients and the Cowell corrector coefficients. The formulas are used to solve the second-order differential equation

$$y'' = f(x, y)$$

and, with Equation (8), can be expressed as follows:

$$y(x) - 2y(x - h) + y(x - 2h) = h^2 \sum_{i=0}^{10} \gamma_i(J, 1, 2)\nabla^i y''(x + Jh) ,$$

where $J = -1, 0$ and, by Equations (24),

$$\gamma_0(J, 1, 2) = 1 ,$$

$$\gamma_i(J, 1, 2) = f_i(J, 1, 2) - \sum_{j=1}^{i} 2H_j^{(2)}\gamma_{i-j}(J, 1, 2) ,$$

and

$$\left.\begin{aligned} f_i(-1, 1, 2) &= 1 \\ f_i(0, 1, 2) &= 0 \end{aligned}\right\} i = 1, 2, 3, \ldots .$$

The ordinate 11-point formulas are then given by

$$y(x) - 2y(x - h) + y(x - 2h) = h^2 \sum_{i=0}^{10} \xi_i(\gamma)y''[x + (J - i)h] ,$$

with $\xi_i(\gamma)$ as previously defined in Table Group I.

J = −1 K = 1 L = 2

Difference form		Ordinate form, order = 11, local error constant 4 671/78 848	
γ_0 =	1	ξ_0 =	263 465 639/159 667 200
γ_1 =	0	ξ_1 =	−296 725 183/79 833 600
γ_2 =	1/12	ξ_2 =	1 742 930 263/159 667 200
γ_3 =	1/12	ξ_3 =	−424 402 351/19 958 400
γ_4 =	19/240	ξ_4 =	2 337 301 223/79 833 600
γ_5 =	3/40	ξ_5 =	−1 155 556 697/39 916 800
γ_6 =	863/12 096	ξ_6 =	1 637 523 683/79 833 600
γ_7 =	275/4 032	ξ_7 =	−29 064 973/2 851 200
γ_8 =	33 953/518 400	ξ_8 =	539 999 083/159 667 200
γ_9 =	8 183/129 600	ξ_9 =	−53 797 223/79 833 600
γ_{10} =	3 250 433/53 222 400	ξ_{10} =	3 250 433/53 222 400

J = 0 K = 1 L = 2

Difference form		Ordinate form, order = 11, local error constant −24 377/13 305 600	
γ_0 =	1	ξ_0 =	3 250 433/53 222 400
γ_1 =	−1	ξ_1 =	3 124 027/3 193 344
γ_2 =	1/12	ξ_2 =	−57 128 921/159 667 200
γ_3 =	0	ξ_3 =	16 745 741/19 958 400
γ_4 =	−1/240	ξ_4 =	−88 645 069/79 833 600
γ_5 =	−1/240	ξ_5 =	42 375 577/39 916 800
γ_6 =	−221/60 480	ξ_6 =	−2 342 533/3 193 344
γ_7 =	−19/6 048	ξ_7 =	7 139 837/19 958 400
γ_8 =	−9 829/3 628 800	ξ_8 =	−18 674 153/159 667 200
γ_9 =	−407/172 800	ξ_9 =	1 838 819/79 833 600
γ_{10} =	−330 157/159 667 200	ξ_{10} =	−330 157/159 667 200

Table Group III

The two tables in this group are the predictor formula and the corrector formula coefficients. The formulas may be used to solve the third-order differential equation

$$y''' = f(x, y).$$

With Equation (8), these can be expressed as follows:

$$\nabla^3 y(x) = h^3 \sum_{i=0}^{10} \gamma_i(J, 1, 3)\nabla^i y'''(x + Jh),$$

where $J = -1, 0$ and, by Equations (24),

$$\gamma_0(J, 1, 3) = 1,$$

$$\gamma_i(J, 1, 3) = f_i(J, 1, 3) - \sum_{j=1}^{i} 6H_j^{(3)}\gamma_{i-j}(J, 1, 3),$$

and where the f_i are given in Equation (21). The ordinate 11-point formulas are as defined in Table Groups I and II.

J = −1 K = 1 L = 3

Difference form	Ordinate form, order = 11, local error constant 24 377/2 956 800
γ_0 = 1	ξ_0 = 6 275 141/11 404 800
γ_1 = −1/2	ξ_1 = 10 485 877/79 833 600
γ_2 = 0	ξ_2 = 16 745 741/13 305 600
γ_3 = 0	ξ_3 = −167 287/63 360
γ_4 = 1/240	ξ_4 = 50 087 159/13 305 600
γ_5 = 1/160	ξ_5 = −50 469 451/13 305 600
γ_6 = 221/30 240	ξ_6 = 4 522 591/1 663 200
γ_7 = 95/12 096	ξ_7 = −3 020 741/2 217 600
γ_8 = 9 829/1 209 600	ξ_8 = 2 419 061/5 322 240
γ_9 = 2 849/345 600	ξ_9 = −7 261 259/79 833 600
γ_{10} = 330 157/39 916 800	ξ_{10} = 330 157/39 916 800

J = 0 K = 1 L = 3

Difference form	Ordinate form, order = 11, local error constant −61/2 280 960
γ_0 = 1	ξ_0 = 330 157/39 916 800
γ_1 = −3/2	ξ_1 = 36 662 533/79 833 600
γ_2 = 1/2	ξ_2 = 15 601 049/26 611 200
γ_3 = 0	ξ_3 = −100 921/950 400
γ_4 = 1/240	ξ_4 = 5 935/66 528
γ_5 = 1/480	ξ_5 = −757 019/13 305 600
γ_6 = 1/945	ξ_6 = 124 909/4 435 200
γ_7 = 11/20 160	ξ_7 = −3 251/316 800
γ_8 = 47/172 800	ξ_8 = 34 189/13 305 600
γ_9 = 19/161 280	ξ_9 = −6 271/15 966 720
γ_{10} = 439/15 966 720	ξ_{10} = 439/15 966 720

Table Group IV

The tables in this group are the coefficients of the starter formulas that can be used to form the starting values

$$(y_i, y_i'), \quad i = -5, -4, \ldots -1, 1, 2, \ldots, 5$$

required by the formulas given in Table Group I to solve the first-order differential equation

$$y' = f(x, y)$$

with the initial value

$$y(x_0) = y_0 .$$

These formulas can be expressed in difference form as

$$y(x_0 + Kh) - y(x_0) = h \sum_{i=0}^{10} \gamma_i(J, K, 1) \nabla^i y'(x_K + Jh) ,$$

where J and K assume the values

J	10	9	8	7	6	4	3	2	1	0
K	−5	−4	−3	−2	−1	1	2	3	4	5

The coefficients $\gamma_i(J, K, 1)$ are given by Equation (24), and ordinate 11-point formulas are as defined in Table Groups I and II.

J = 10 K = −5 L = 1

Difference form		Ordinate form, order = 11, *local error constant* −202 025/38 320 128	
γ_0 =	1	ξ_0 =	114 985/19 160 064
γ_1 =	75/2	ξ_1 =	−320 875/4 790 016
γ_2 =	−1 525/12	ξ_2 =	311 375/912 384
γ_3 =	6 175/24	ξ_3 =	−838 375/798 336
γ_4 =	−49 775/144	ξ_4 =	2 325 625/1 064 448
γ_5 =	92 785/288	ξ_5 =	−89 035/24 948
γ_6 =	−2 543 875/12 096	ξ_6 =	2 306 375/1 064 448
γ_7 =	84 375/896	ξ_7 =	−2 793 625/798 336
γ_8 =	−3 955 625/145 152	ξ_8 =	2 996 375/6 386 688
γ_9 =	184 625/41 472	ξ_9 =	−8 183 125/4 790 016
γ_{10} =	−5 256 425/19 160 064	ξ_{10} =	−5 256 425/19 160 064

$J = 9 \qquad K = -4 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant 61/93 555
$\gamma_0 = 1$	$\xi_0 = -367/467\ 775$
$\gamma_1 = 28/1$	$\xi_1 = 4\ 099/467\ 775$
$\gamma_2 = -260/3$	$\xi_2 = -1\ 387/31\ 185$
$\gamma_3 = 156/1$	$\xi_3 = 2\ 996/22\ 275$
$\gamma_4 = -8\ 114/45$	$\xi_4 = -13\ 462/51\ 975$
$\gamma_5 = 1\ 250/9$	$\xi_5 = 424/155\ 925$
$\gamma_6 = -67\ 192/945$	$\xi_6 = -85\ 226/51\ 975$
$\gamma_7 = 21\ 808/945$	$\xi_7 = -14\ 972/31\ 185$
$\gamma_8 = -8\ 449/2\ 025$	$\xi_8 = -216\ 617/155\ 925$
$\gamma_9 = 7/25$	$\xi_9 = -158\ 327/467\ 775$
$\gamma_{10} = 547/93\ 555$	$\xi_{10} = 547/93\ 555$

$J = 8 \qquad K = -3 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant −827/3 942 400
$\gamma_0 = 1$	$\xi_0 = 127/394\ 240$
$\gamma_1 = 39/2$	$\xi_1 = -1\ 857/492\ 800$
$\gamma_2 = -219/4$	$\xi_2 = 40\ 433/1\ 971\ 200$
$\gamma_3 = 693/8$	$\xi_3 = -17\ 247/246\ 400$
$\gamma_4 = -6\ 747/80$	$\xi_4 = 34\ 983/197\ 120$
$\gamma_5 = 8\ 253/160$	$\xi_5 = -5\ 149/7\ 700$
$\gamma_6 = -43\ 021/2\ 240$	$\xi_6 = -847\ 491/985\ 600$
$\gamma_7 = 497/128$	$\xi_7 = -42\ 903/35\ 200$
$\gamma_8 = -12\ 881/44\ 800$	$\xi_8 = -773\ 809/1\ 971\ 200$
$\gamma_9 = -25/3\ 584$	$\xi_9 = 1\ 613/98\ 560$
$\gamma_{10} = -1\ 851/1\ 971\ 200$	$\xi_{10} = -1\ 851/1\ 971\ 200$

J = 7 K = −2 L = 1

Difference form		Ordinate form, order = 11, local error constant 263/7 484 400	
γ_0 =	1	ξ_0 =	−263/7 484 400
γ_1 =	12/1	ξ_1 =	263/748 440
γ_2 =	−91/3	ξ_2 =	−3 439/2 494 800
γ_3 =	125/3	ξ_3 =	793/623 700
γ_4 =	−2 999/90	ξ_4 =	7 213/415 800
γ_5 =	688/45	ξ_5 =	−62 389/155 925
γ_6 =	−13 613/3 780	ξ_6 =	−102 569/83 160
γ_7 =	41/140	ξ_7 =	−252 449/623 700
γ_8 =	119/16 200	ξ_8 =	8 249/356 400
γ_9 =	13/14 175	ξ_9 =	−9 707/3 742 200
γ_{10} =	251/1 496 880	ξ_{10} =	251/1 496 880

J = 6 K = −1 L = 1

Difference form		Ordinate form, order = 11, local error constant −14 797/191 600 64	
γ_0 =	1	ξ_0 =	14 797/95 800 320
γ_1 =	11/2	ξ_1 =	−32 309/17 107 200
γ_2 =	−149/12	ξ_2 =	1 746 433/159 667 200
γ_3 =	117/8	ξ_3 =	−163 459/3 991 680
γ_4 =	−6 731/720	ξ_4 =	3 216 337/26 611 200
γ_5 =	4 277/1 440	ξ_5 =	−379 571/623 700
γ_6 =	−19 087/60 480	ξ_6 =	−14 296 081/26 611 200
γ_7 =	−275/24 192	ξ_7 =	1 394 959/19 958 400
γ_8 =	−7 297/3 628 800	ξ_8 =	−493 837/31 933 440
γ_9 =	−7/12 800	ξ_9 =	292 531/119 750 400
γ_{10} =	−90 817/479 001 600	ξ_{10} =	−90 817/479 001 600

$J = 4 \qquad K = 1 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant −14 797/191 600 640
$\gamma_0 = 1$	$\xi_0 = 90\,817/479\,001\,600$
$\gamma_1 = -9/2$	$\xi_1 = -292\,531/119\,750\,400$
$\gamma_2 = 95/12$	$\xi_2 = 493\,837/31\,933\,440$
$\gamma_3 = -161/24$	$\xi_3 = -1\,394\,959/19\,958\,400$
$\gamma_4 = 1\,901/720$	$\xi_4 = 14\,296\,081/26\,611\,200$
$\gamma_5 = -95/288$	$\xi_5 = 379\,571/623\,700$
$\gamma_6 = -863/60\,480$	$\xi_6 = -3\,216\,337/26\,611\,200$
$\gamma_7 = -13/4\,480$	$\xi_7 = 163\,459/3\,991\,680$
$\gamma_8 = -3\,233/3\,628\,800$	$\xi_8 = -1\,746\,433/159\,667\,200$
$\gamma_9 = -2\,497/7\,257\,600$	$\xi_9 = 32\,309/17\,107\,200$
$\gamma_{10} = -14\,797/95\,800\,320$	$\xi_{10} = -14\,797/95\,800\,320$

$J = 3 \qquad K = 2 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant 263/7 484 400
$\gamma_0 = 1$	$\xi_0 = -251/1\,496\,880$
$\gamma_1 = -8/1$	$\xi_1 = 9\,707/3\,742\,200$
$\gamma_2 = 37/3$	$\xi_2 = -8\,249/356\,400$
$\gamma_3 = -9/1$	$\xi_3 = 252\,449/623\,700$
$\gamma_4 = 269/90$	$\xi_4 = 102\,569/83\,160$
$\gamma_5 = -14/45$	$\xi_5 = 62\,389/155\,925$
$\gamma_6 = -37/3\,780$	$\xi_6 = -7\,213/415\,800$
$\gamma_7 = -1/756$	$\xi_7 = -793/623\,700$
$\gamma_8 = -23/113\,400$	$\xi_8 = 3\,439/2\,494\,800$
$\gamma_9 = 0$	$\xi_9 = -263/748\,440$
$\gamma_{10} = 263/7\,484\,400$	$\xi_{10} = 263/7\,484\,400$

$J = 2 \qquad K = 3 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant −827/3 942 400
$\gamma_0 = 1$	$\xi_0 = 1\,851/1\,971\,200$
$\gamma_1 = -21/2$	$\xi_1 = -1\,613/98\,560$
$\gamma_2 = 57/4$	$\xi_2 = 773\,809/1\,971\,200$
$\gamma_3 = -75/8$	$\xi_3 = 42\,903/35\,200$
$\gamma_4 = 237/80$	$\xi_4 = 847\,491/985\,600$
$\gamma_5 = -51/160$	$\xi_5 = 5\,149/7\,700$
$\gamma_6 = -29/2\,240$	$\xi_6 = -34\,983/197\,120$
$\gamma_7 = -13/4\,480$	$\xi_7 = 17\,247/246\,400$
$\gamma_8 = -7/6\,400$	$\xi_8 = -40\,433/1\,971\,200$
$\gamma_9 = -7/12\,800$	$\xi_9 = 1\,857/492\,800$
$\gamma_{10} = -127/394\,240$	$\xi_{10} = -127/394\,240$

$J = 1 \qquad K = 4 \qquad L = 1$

Difference form	Ordinate form, order = 11, local error constant 61/93 555
$\gamma_0 = 1$	$\xi_0 = -547/93\,555$
$\gamma_1 = -12/1$	$\xi_1 = 158\,327/467\,775$
$\gamma_2 = 44/3$	$\xi_2 = 216\,617/155\,925$
$\gamma_3 = -28/3$	$\xi_3 = 14\,972/31\,185$
$\gamma_4 = 134/45$	$\xi_4 = 85\,226/51\,975$
$\gamma_5 = -14/45$	$\xi_5 = -424/155\,925$
$\gamma_6 = -8/945$	$\xi_6 = 13\,462/51\,975$
$\gamma_7 = 0$	$\xi_7 = -2\,996/22\,275$
$\gamma_8 = 13/14\,175$	$\xi_8 = 1\,387/31\,185$
$\gamma_9 = 13/14\,175$	$\xi_9 = -4\,099/467\,775$
$\gamma_{10} = 367/467\,775$	$\xi_{10} = 367/467\,775$

J = 0 K = 5 L = 1

Difference form		Ordinate form, order = 11, local error constant −202 025/38 320 128	
γ_0	= 1	ξ_0	= 5 256 425/19 160 064
γ_1	= −25/2	ξ_1	= 8 183 125/4 790 016
γ_2	= 175/12	ξ_2	= −2 996 375/6 386 688
γ_3	= −75/8	ξ_3	= 2 793 625/798 336
γ_4	= 425/144	ξ_4	= −2 306 375/1 064 448
γ_5	= −95/288	ξ_5	= 89 035/24 948
γ_6	= −275/12 096	ξ_6	= −2 325 625/1 064 448
γ_7	= −275/24 192	ξ_7	= 838 375/798 336
γ_8	= −175/20 736	ξ_8	= −311 375/912 384
γ_9	= −25/3 584	ξ_9	= 320 875/4 790 016
γ_{10}	= −114 985/19 160 064	ξ_{10}	= −114 985/19 160 064

Table Group V

The tables in this group are the coefficients of the starter formulas that can be used to form the starting values

$$(y_i, y_i''), \quad i = -5, -4, \ldots -1, 1, 2, \ldots, 5$$

required by the formulas given in Table Group II to solve the second-order differential equation

$$y'' = f(x, y)$$

with initial values

$$y(x_0) = y_0,$$

and

$$y'(x_0) = y_0'.$$

These formulas can be expressed in difference form as

$$y(x_0 + Kh) - y(x_0) = Khy' + h^2 \sum_{i=0}^{10} \gamma_i(J, K, 2)\nabla^i y''(x_K + Jh),$$

where J and K assume the values shown in Table Group IV, the $\gamma_i(J, K, 2)$ are given by Equation (24), and the ordinate 11-point formulas are as defined in Table Groups I and II.

$J = 10 \qquad K = -5 \qquad L = 2$

Difference form		Ordinate form, order = 11, local error constant 1 918 325/1 162 377 216	
$\gamma_0 =$	25/2	$\xi_0 =$	−77 425/38 320 128
$\gamma_1 =$	−250/3	$\xi_1 =$	62 875/2 737 152
$\gamma_2 =$	5 875/24	$\xi_2 =$	−1 539 875/12 773 376
$\gamma_3 =$	−30 125/72	$\xi_3 =$	208 625/532 224
$\gamma_4 =$	133 375/288	$\xi_4 =$	−5 942 875/6 386 688
$\gamma_5 =$	−43 975/126	$\xi_5 =$	10 314 625/3 193 344
$\gamma_6 =$	4 404 125/24 192	$\xi_6 =$	22 426 625/6 386 688
$\gamma_7 =$	−4 680 625/72 576	$\xi_7 =$	5 650 375/1 596 672
$\gamma_8 =$	4 191 125/290 304	$\xi_8 =$	21 348 625/12 773 376
$\gamma_9 =$	−8 183 125/4 790 016	$\xi_9 =$	21 621 125/19 160 064
$\gamma_{10} =$	202 025/3 483 648	$\xi_{10} =$	202 025/3 483 648

$J = 9 \qquad K = -4 \qquad L = 2$

Difference form		Ordinate form, order = 11, local error constant −1 556/70 945 875	
$\gamma_0 =$	8/1	$\xi_0 =$	−52/467 775
$\gamma_1 =$	−152/3	$\xi_1 =$	758/467 775
$\gamma_2 =$	416/3	$\xi_2 =$	−356/31 185
$\gamma_3 =$	−9 664/45	$\xi_3 =$	8 368/155 925
$\gamma_4 =$	1 864/9	$\xi_4 =$	−6 584/31 185
$\gamma_5 =$	−40 616/315	$\xi_5 =$	280 124/155 925
$\gamma_6 =$	9 784/189	$\xi_6 =$	532 184/155 925
$\gamma_7 =$	−181 096/14 175	$\xi_7 =$	2 704/1 485
$\gamma_8 =$	3 346/2 025	$\xi_8 =$	23 756/22 275
$\gamma_9 =$	−1 094/18 711	$\xi_9 =$	122/1 701
$\gamma_{10} =$	−124/93 555	$\xi_{10} =$	−124/93 555

$J = 8 \qquad K = -3 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant 247 319/1 793 792 000
γ_0 = 9/2	ξ_0 = −1 063/3 942 400
γ_1 = −27/1	ξ_1 = 6 511/1 971 200
γ_2 = 549/8	ξ_2 = −10 833/563 200
γ_3 = −3 831/40	ξ_3 = 1 029/14 080
γ_4 = 12 711/160	ξ_4 = −88 827/394 240
γ_5 = −4 425/112	ξ_5 = 280 821/197 120
γ_6 = 50 319/4 480	ξ_6 = 4 345 149/1 971 200
γ_7 = −35 451/22 400	ξ_7 = 464 187/492 800
γ_8 = 225/3 584	ξ_8 = 7 443/71 680
γ_9 = 17/7 040	ξ_9 = −529/78 848
γ_{10} = 1 693/3 942 400	ξ_{10} = 1 693/3 942 400

$J = 7 \qquad K = -2 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant 1 303/21 021 000
γ_0 = 2/1	ξ_0 = −263/1 871 100
γ_1 = −34/3	ξ_1 = 263/149 688
γ_2 = 80/3	ξ_2 = −131/12 474
γ_3 = −1 502/45	ξ_3 = 159/3 850
γ_4 = 2 117/90	ξ_4 = −41 543/311 850
γ_5 = −5 651/630	ξ_5 = 111 973/124 740
γ_6 = 1 466/945	ξ_6 = 35 932/31 185
γ_7 = −119/2 025	ξ_7 = 263/5 670
γ_8 = −103/113 400	ξ_8 = 3 587/623 700
γ_9 = 589/3 742 200	ξ_9 = −707/534 600
γ_{10} = 109/935 550	ξ_{10} = 109/935 550

$J = 6 \qquad K = -1 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant 5 512 813/145 297 152 000
$\gamma_0 = 1/2$	$\xi_0 = -14\,797/191\,600\,640$
$\gamma_1 = -8/3$	$\xi_1 = 90\,817/95\,800\,320$
$\gamma_2 = 139/24$	$\xi_2 = -1\,763\,939/319\,334\,400$
$\gamma_3 = -2\,333/360$	$\xi_3 = 166\,919/7\,983\,360$
$\gamma_4 = 5\,539/1\,440$	$\xi_4 = -10\,111\,819/159\,667\,200$
$\gamma_5 = -2\,713/2\,520$	$\xi_5 = 31\,494\,553/79\,833\,600$
$\gamma_6 = 275/3\,456$	$\xi_6 = 14\,797/82\,944$
$\gamma_7 = 8\,563/1\,814\,400$	$\xi_7 = -60\,917/1\,900\,800$
$\gamma_8 = 6\,533/7\,257\,600$	$\xi_8 = 466\,157/63\,866\,880$
$\gamma_9 = 30\,577/119\,750\,400$	$\xi_9 = -79\,829/68\,428\,800$
$\gamma_{10} = 87\,299/958\,003\,200$	$\xi_{10} = 87\,299/958\,003\,200$

$J = 4 \qquad K = 1 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant −5 512 813/145 297 152 000
$\gamma_0 = 1/2$	$\xi_0 = 87\,299/958\,003\,200$
$\gamma_1 = -7/3$	$\xi_1 = -79\,829/68\,428\,800$
$\gamma_2 = 103/24$	$\xi_2 = 466\,157/63\,866\,880$
$\gamma_3 = -1\,387/360$	$\xi_3 = -60\,917/1\,900\,800$
$\gamma_4 = 475/288$	$\xi_4 = 14\,797/82\,944$
$\gamma_5 = -1\,231/5\,040$	$\xi_5 = 31\,494\,553/79\,833\,600$
$\gamma_6 = -199/24\,192$	$\xi_6 = -10\,111\,819/159\,667\,200$
$\gamma_7 = -409/259\,200$	$\xi_7 = 166\,919/7\,983\,360$
$\gamma_8 = -3\,391/7\,257\,600$	$\xi_8 = -1\,763\,939/319\,334\,400$
$\gamma_9 = -263/1\,496\,880$	$\xi_9 = 90\,817/95\,800\,320$
$\gamma_{10} = -14\,797/191\,600\,640$	$\xi_{10} = -14\,797/191\,600\,640$

J = 3 K = 2 L = 2

Difference form	Ordinate form, order = 11, local error constant $-1\,303/21\,021\,000$
$\gamma_0 = 2/1$	$\xi_0 = 109/935\,550$
$\gamma_1 = -26/3$	$\xi_1 = -707/534\,600$
$\gamma_2 = 44/3$	$\xi_2 = 3\,587/623\,700$
$\gamma_3 = -538/45$	$\xi_3 = 263/5\,670$
$\gamma_4 = 409/90$	$\xi_4 = 35\,932/31\,185$
$\gamma_5 = -71/126$	$\xi_5 = 111\,973/124\,740$
$\gamma_6 = -19/945$	$\xi_6 = -41\,543/311\,850$
$\gamma_7 = -52/14\,175$	$\xi_7 = 159/3\,850$
$\gamma_8 = -23/22\,680$	$\xi_8 = -131/12\,474$
$\gamma_9 = -263/748\,440$	$\xi_9 = 263/149\,688$
$\gamma_{10} = -263/1\,871\,100$	$\xi_{10} = -263/1\,871\,100$

J = 2 K = 3 L = 2

Difference form	Ordinate form, order = 11, local error constant $-247\,319/1\,793\,792\,000$
$\gamma_0 = 9/2$	$\xi_0 \quad 1\,693/3\,942\,400$
$\gamma_1 = -18/1$	$\xi_1 = -529/78\,848$
$\gamma_2 = 225/8$	$\xi_2 = 7\,443/71\,680$
$\gamma_3 = -849/40$	$\xi_3 = 464\,187/492\,800$
$\gamma_4 = 1\,203/160$	$\xi_4 = 4\,345\,149/1\,971\,200$
$\gamma_5 = -123/140$	$\xi_5 = 280/821/197\,120$
$\gamma_6 = -141/4\,480$	$\xi_6 = -88\,827/394\,240$
$\gamma_7 = -129/22\,400$	$\xi_7 = 1\,029/14\,080$
$\gamma_8 = -21/12\,800$	$\xi_8 = -10\,833/563\,200$
$\gamma_9 = -299/492\,800$	$\xi_9 = 6\,511/1\,971\,200$
$\gamma_{10} = -1\,063/3\,942\,400$	$\xi_{10} = -1\,063/3\,942\,400$

$J = 1 \qquad K = 4 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant 1 556/70 945 875
$\gamma_0 = 8/1$	$\xi_0 = -124/93\,555$
$\gamma_1 = -88/3$	$\xi_1 = 122/1\,701$
$\gamma_2 = 128/3$	$\xi_2 = 23\,756/22\,275$
$\gamma_3 = -1\,376/45$	$\xi_3 = 2\,704/1\,485$
$\gamma_4 = 472/45$	$\xi_4 = 532\,184/155\,925$
$\gamma_5 = -376/315$	$\xi_5 = 280\,124/155\,925$
$\gamma_6 = -8/189$	$\xi_6 = -6\,584/31\,185$
$\gamma_7 = -104/14\,175$	$\xi_7 = 8\,368/155\,925$
$\gamma_8 = -26/14\,175$	$\xi_8 = -356/31\,185$
$\gamma_9 = -34/66\,825$	$\xi_9 = 758/467\,775$
$\gamma_{10} = -52/467\,775$	$\xi_{10} = -52/467\,775$

$J = 0 \qquad K = 5 \qquad L = 2$

Difference form	Ordinate form, order = 11, local error constant −1 918 325/1 162 377 216
$\gamma_0 = 25/2$	$\xi_0 = 202\,025/3\,483\,648$
$\gamma_1 = -125/3$	$\xi_1 = 21\,621\,125/19\,160\,064$
$\gamma_2 = 1\,375/24$	$\xi_2 = 21\,348\,625/12\,773\,376$
$\gamma_3 = -2\,875/72$	$\xi_3 = 5\,650\,375/1\,596\,672$
$\gamma_4 = 3\,875/288$	$\xi_4 = 22\,426\,625/6\,386\,688$
$\gamma_5 = -1\,525/1\,008$	$\xi_5 = 10\,314\,625/3\,193\,344$
$\gamma_6 = -1\,375/24\,192$	$\xi_6 = -5\,942\,875/6\,386\,688$
$\gamma_7 = -125/10\,368$	$\xi_7 = 208\,625/532\,224$
$\gamma_8 = -1\,375/290\,304$	$\xi_8 = -1\,539\,875/12\,773\,376$
$\gamma_9 = -6\,625/2\,395\,008$	$\xi_9 = 62\,875/2\,737\,152$
$\gamma_{10} = -77\,425/38\,320\,128$	$\xi_{10} = -77\,425/38\,320\,128$

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