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TECHNICAL MEMORANDUM

TRANSFORMATION OF A POTENTIAL FUNCTION UNDER COORDINATE TRANSLATIONS

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DATE-August 13, 1970

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AUTHOR(S)- S. L. Levie, Jr.

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ABSTRACT

Starting with a potential function written as an infinite series of spherical harmonics, the coordinate system to which the potential is referred is given a pure translation. Known transformation properties of spherical harmonics then are used to produce an equivalent series valid in the new system. The procedure yields expressions for the series coefficients in the new system, showing them to be linear combinations of the old coefficients. Two applications are discussed. One involves modeling the contribution of mascons to the lunar potential, and the other involves the study of complete earth perturbations on a high-altitude lunar satellite.

BA-145A (8-68)

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SUBJECT: Transformation of a Potential Function Under Coordinate Translations
Case 310

DATE: August 13, 1970

FROM: S. L. Levie, Jr.

TM-70-2014-7

TECHNICAL MEMORANDUM

1. INTRODUCTION

The gravitational potential of a body is commonly written as the infinite series

$$U(r,\theta,\phi) = \frac{\mu}{r}\left\{1 + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left(\frac{a}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta) \cdot \left[C_{\ell m}\cos m\phi + S_{\ell m}\sin m\phi\right]\right\}. \quad (1)$$

In this expression, (r,θ,ϕ) are the spherical polar coordinates of a point in space relative to a body-fixed rectangular coordinate frame which will be denoted by 0, μ is the body's constant of attraction, and a is a length usually taken as the body's mean radius. $P_{\ell}^{m}(\cos\theta)$ denotes the associated Legendre polynomial defined in Section 3.

Suppose the expansion coefficients $C_{\ell m}$ and $S_{\ell m}$ are known such that the series gives a precise representation of the potential at all points outside the body for which $r>a$. Then the question arises: How is $U(r,\theta,\phi)$ transformed under a pure translation of 0?

The purpose of this paper is to answer the question in full detail, to provide several special cases of the solution, and to indicate two applications of it. One application involves modeling the contribution of mascons to the lunar potential, and the other involves the study of complete earth perturbations on a high altitude lunar satellite.

The solution begins with some general results about spherical harmonics, given by James [1]. These results are developed for the problem at hand, utilizing associated Legendre polynomials normalized consistent with a recommendation of the International Astronomical Union [2, p. 173]. This normalization casts the solution in the idiom of astrodynamics.

2. SUMMARY

The question posed in the introduction was: How is $U(r,\theta,\phi)$ transformed under a pure translation of O? In particular, let $T(R,\theta_0,\phi_0)$ be a translation operator which takes the original coordinate frame O into the new frame O' leaving the physical body fixed. (R,θ_0,ϕ_0) are the spherical polar coordinates of the origin of O expressed in the frame O'. Then the physical point (r,θ,ϕ) assumes the new representation (r',θ',ϕ') after the translation, and the potential expansion $U(r,\theta,\phi)$ is taken to $U'(r',\theta',\phi')$ by the translation -- that is

$$U(r,\theta,\phi) \xrightarrow[T(R,\theta_0,\phi_0)]{} U'(r',\theta',\phi'). \qquad (2)$$

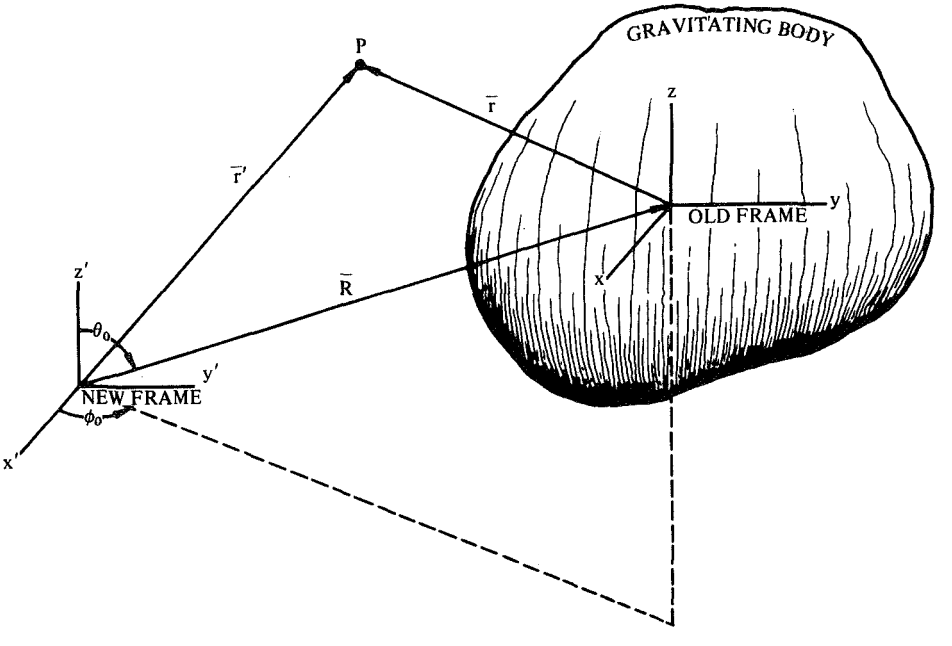


FIGURE 1 - GEOMETRY OF A COORDINATE SYSTEM TRANSLATION.

$\bar{R}$ DENOTES THE TRANSLATION VECTOR (R, θ_0, ϕ_0) [IN SPHERICAL POLAR COORDINATES] WHICH LOCATES THE OLD FRAME RELATIVE TO THE NEW. $\bar{r}$ IS THE VECTOR (r, θ, ϕ) TO AN ARBITRARY POINT P IN THE OLD FRAME, AND $\bar{r}'$ IS THE VECTOR (r', θ', ϕ') WHICH LOCATES P IN THE NEW FRAME. THE POTENTIAL AT P IS INDEPENDENT OF THE COORDINATE SYSTEM WITH RESPECT TO WHICH IT IS COMPUTED.

The numbers U and U' evaluated at the same physical point will be equal. The geometry of the problem is illustrated in Figure 1.

Based on the work of James [1], two cases will be developed. For the exterior case, for which r'>R, it will be shown that

$$U'(r',\theta',\phi') = \frac{\mu}{r'} \left\{ 1 + \sum_{k=1}^{\infty} \sum_{m=0}^{k} \left(\frac{a}{r'}\right)^k P_k^m(\cos\theta') \cdot \left[E_{km} \cos m\phi' + W_{km} \sin m\phi' \right] \right\} \quad (r'>a). \tag{3}$$

For the interior case, for which r'<R, it will be shown that

$$U'(r',\theta',\phi') = \frac{\mu}{R} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(\frac{r'}{R}\right)^k P_k^m(\cos\theta') \cdot \left[E_{km} \cos m\phi' + W_{km} \sin m\phi' \right] \quad (R>a). \tag{4}$$

In these equations, (r',θ',ϕ') must be outside the physical body if the new potential is to be physically meaningful. It may be observed that (3) and (4) both solve Laplace's equation, as a potential function must, so the essential results of this paper are the formulas for the new constants, E_{km} and W_{km}, which are the expansion coefficients appropriate for the new coordinate frame. They are linear combinations of the old coefficients, and the parameters of the translation appear in the coefficients of these combinations. The formula for the coefficients for the exterior case is given in (17), and the formula for the interior case is given in (34).

The development of the exterior case is given in Section 4, and special forms of it are given in Section 5. The interior case is developed in Section 6, and special forms of it are given in Section 7. A brief summary of the properties of the associated Legendre polynomials used in this paper is given in the next section.

3. ASSOCIATED LEGENDRE POLYNOMIALS

When dealing with associated Legendre polynomials, it is important to understand clearly which of several common normalization conventions is being used. The associated Legendre polynomials used in this paper are defined, for integer values of ℓ and m, by

$$P_\ell^m(x) = \begin{cases} \dfrac{(1-x^2)^{m/2}}{2^\ell \ell!} \dfrac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell & \text{if } |m| \leq \ell \\ 0 & \text{if } |m| > \ell, \end{cases} \tag{5}$$

which is valid for $x^2 \leq 1$. The first several of these polynomials are listed in [3]. These polynomials have the symmetry property

$$P_\ell^{-m}(x) = (-)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \tag{6}$$

and the normalization property

$$\int_{-1}^{1} P_\ell^m(x) \, P_n^m(x) \, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell n}. \tag{7}$$

When $m=0$, these functions are the customary polynomials of Legendre, denoted by $P_\ell(x)$.

The definition (5) is consistent with one recommended by the International Astronomical Union [2,p.173]. It differs from the one used by James in [1]. His polynomials may be obtained by multiplying the ones defined in (5) by the Schmidt normalization factor

$$[(2-\delta_{m0})(\ell-m)!/(\ell+m)!]^{1/2} .$$

In the equations in the following sections, an associated Legendre polynomial without an argument will appear occasionally. The argument will be understood to be cos θ_0, i.e.,

$$P_\ell^m \equiv P_\ell^m(\cos\ \theta_0). \tag{8}$$

4. THE EXTERIOR CASE

In order to develop the results of James, (1) must be rewritten as the sum of solid harmonics U_ℓ. Thus

$$U(r,\theta,\phi) = \sum_{\ell=0}^{\infty} U_\ell(r,\theta,\phi), \tag{9}$$

where

$$U_\ell(r,\theta,\phi) = \frac{\mu}{r}\left(\frac{a}{r}\right)^\ell \sum_{m=0}^{\ell} P_\ell^m(\cos\ \theta)\ [C_{\ell m}\cos\ m\phi + S_{\ell m}\sin\ m\phi]. \tag{10}$$

Equations (9) and (1) are equivalent if one takes $C_{00}=1$. To facilitate a later discussion, we shall adopt the convention

$$\begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} = 0 \qquad \text{if } m>\ell \text{ or } m<0.$$

$S_{\ell 0}$ clearly may be set to zero, for all ℓ.

In [1,(31)], James shows that under the condition r'>R, which defines the exterior case, solid harmonics transform as

$$U_\ell(r,\theta,\phi) \xrightarrow[T(R,\theta_0,\phi_0)]{} U'_\ell(r',\theta',\phi') \tag{11}$$

with

$$U_\ell'(r',\theta',\phi') = \frac{\mu}{r'} \sum_{k=\ell}^{\infty} \sum_{m=0}^{k} \frac{R^{k-\ell}}{r'^{k}} \frac{a^{\ell}}{(k-\ell)!} P_k^m(\cos\theta') \cdot \left[C_{km}^{(\ell)} \cos m\phi' + S_{km}^{(\ell)} \sin m\phi'\right] . \quad (12)$$

The coefficients in this series are a consequence of [1,(32)]. They are

$$\begin{pmatrix} C_{km}^{(\ell)} \\ S_{km}^{(\ell)} \end{pmatrix} = (k-\ell)! \sum_{j=0}^{\ell} \left\{ C_{\ell j} \left[\alpha_{km}^{\ell j} \begin{pmatrix} \cos(m-j)\phi_0 \\ \sin(m-j)\phi_0 \end{pmatrix} P_{k-\ell}^{m-j} + \beta_{km}^{\ell j} \begin{pmatrix} \cos(m+j)\phi_0 \\ \sin(m+j)\phi_0 \end{pmatrix} P_{k-\ell}^{m+j} \right] + S_{\ell j} \left[\alpha_{km}^{\ell j} \begin{pmatrix} -\sin(m-j)\phi_0 \\ \cos(m-j)\phi_0 \end{pmatrix} P_{k-\ell}^{m-j} + \beta_{km}^{\ell j} \begin{pmatrix} \sin(m+j)\phi_0 \\ -\cos(m+j)\phi_0 \end{pmatrix} P_{k-\ell}^{m+j} \right] \right\} . \quad (13)$$

The constants in (13) are

$$\alpha_{km}^{\ell j} = \frac{(k-m)!}{(\ell-j)!\,(k+m-\ell-j)!} \tag{14}$$

$$\beta_{km}^{\ell j} = \frac{(-)^j (k-m)!\,(1-\delta_{m0})}{(\ell-j)!\,(k+m-\ell+j)!} \cdot \tag{15}$$

It follows from (2), (9), (11) and (12) that the full potential expressed in the new frame O', obtained by translating O, is

$$U'(r',\theta',\phi') = \frac{\mu}{r'} \sum_{\ell=0}^{\infty} \sum_{k=\ell}^{\infty} \sum_{m=0}^{k} \frac{R^{k-\ell}}{r'^{k}} \frac{a^{\ell}}{(k-\ell)!} P_k^m(\cos\theta')$$

$$\cdot \left[C_{km}^{(\ell)} \cos m\phi' + S_{km}^{(\ell)} \sin m\phi' \right] , \tag{16}$$

where the condition $r'>a$ arises to guarantee convergence. This can be recast in the form of (3) by moving the ℓ-sum as far to the right as possible. Doing so leads directly to the potential coefficients E_{km} and W_{km} which are valid in the new frame. They are

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = \sum_{\ell=0}^{k} \sum_{j=0}^{\ell} \left(\frac{R}{a}\right)^{k-\ell}$$

$$\cdot \left\{ C_{\ell j} \left[\alpha_{km}^{\ell j} \begin{pmatrix} \cos(m-j)\phi_0 \\ \sin(m-j)\phi_0 \end{pmatrix} P_{k-\ell}^{m-j} \right. \right.$$

$$+ \beta_{km}^{\ell j} \begin{pmatrix} \cos(m+j)\phi_0 \\ \sin(m+j)\phi_0 \end{pmatrix} P_{k-\ell}^{m+j} \Big]$$

$$+ S_{\ell j} \Big[\alpha_{km}^{\ell j} \begin{pmatrix} -\sin(m-j)\phi_0 \\ \cos(m-j)\phi_0 \end{pmatrix} P_{k-\ell}^{m-j}$$

$$+ \beta_{km}^{\ell j} \begin{pmatrix} \sin(m+j)\phi_0 \\ -\cos(m+j)\phi_0 \end{pmatrix} P_{k-\ell}^{m+j} \Big] \Big\} . \qquad (17)$$

The new (k,m) coefficient thus displays in linear combination all the old coefficients whose first index is less than or equal to k.

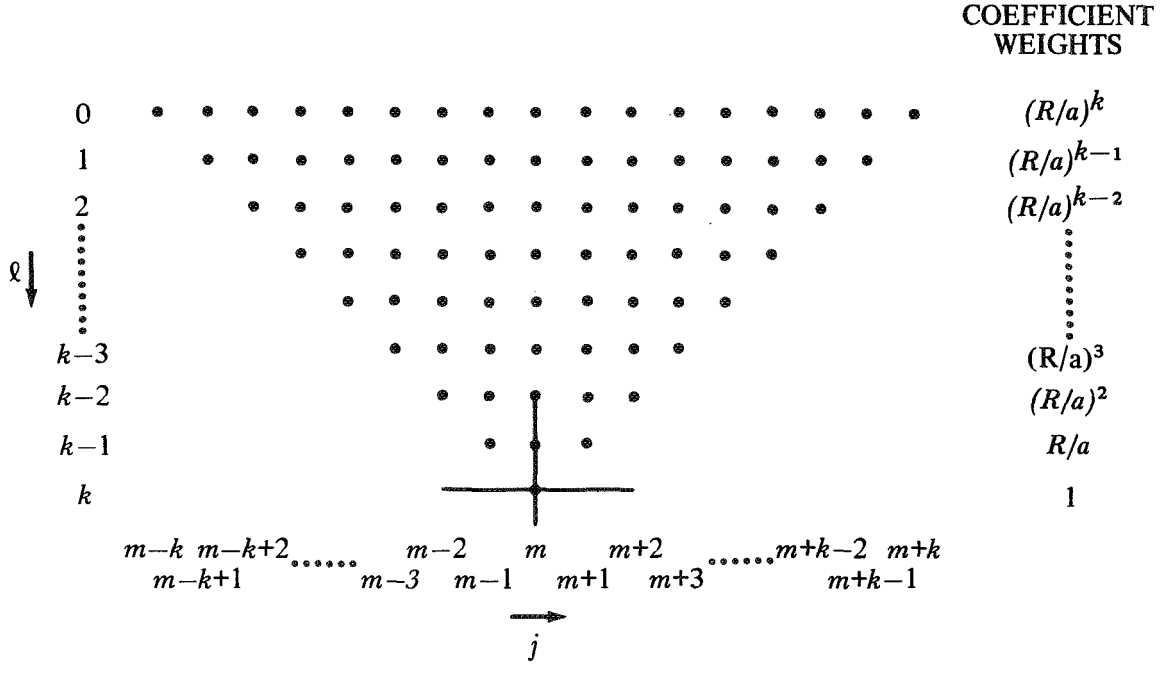


FIGURE 2 - LOCUS OF (ℓ, j) COMBINATIONS MAKING NON-ZERO CONTRIBUTIONS TO (17), INCLUDING COEFFICIENT WEIGHTS

Not all of the terms indicated in (17) are actually present, due to the indicial requirement on the associated Legendre polynomial, which is given in (5). This requirement may be used to show that the only index combinations allowable in (17) are those in the triangle shown in Figure 2. (Note that the triangle's vertex is at $\ell=k$, $j=m$.) Moreover, for many of these combinations -- the ones for which $j>\ell$ or $j<0$ -- the coefficients $C_{\ell j}$ and $S_{\ell j}$ are non-existent. The index combinations for which $C_{\ell j}$ and $S_{\ell j}$ are not zero a fortiori are shown in Figure 3. It follows that the only non-zero contributions to the double sum in (1) have index combinations which lie in the intersection of the sets illustrated in Figures 2 and 3.

By direct calculation of the $\ell=k$, $j=m$ terms in (17), it can be shown that they contribute just

$$\begin{pmatrix} C_{km} \\ S_{km} \end{pmatrix}$$

to the sum. Hence the new (k,m) coefficient equals the old (k,m) coefficient plus a perturbation induced by the translation.

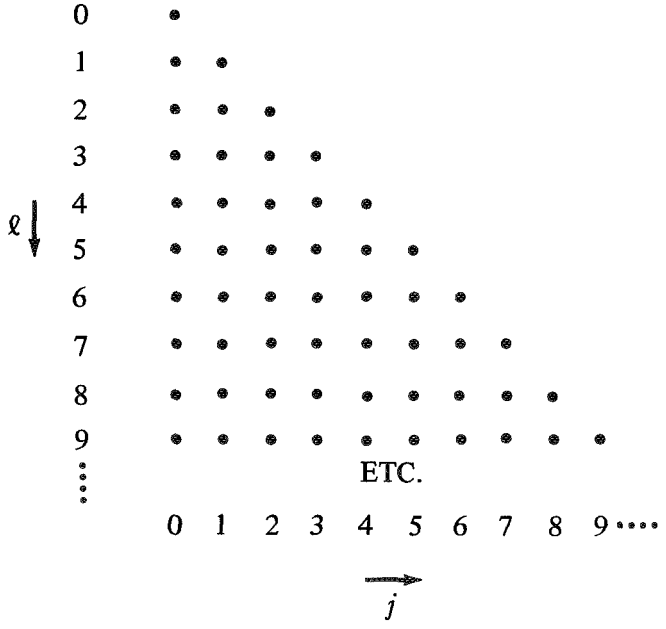


FIGURE 3 - LOCUS OF (ℓ, j) COMBINATIONS FOR WHICH THE OLD EXPANSION COEFFICIENTS NEED NOT BE ZERO.

The presence of the factor $(R/a)^{k-\ell}$ in (17) imposes a weighting hierarchy on the contributions of the old coefficients $C_{\ell j}$ and $S_{\ell j}$. The weights for each ℓ-value are in Figure 2 at the right. For displacements of the coordinate system for which R<<a, the weighting hierarchy restricts the old coefficients' influence on the new (k,m) coefficient: only near neighbors of (k,m) are important.

From this and the customary smallness of all gravitational coefficients relative to C_{00}, one can conclude that the gravitational potential expansion coefficients of the earth and moon are affected only insignificantly by the small uncertainties in their center of mass locations, except for the (1,0) and (1,1) coefficients. The transformation formulas for these coefficients are

$$\left.\begin{aligned} E_{10} &= C_{10} + z/a \\ E_{11} &= C_{11} + x/a \\ W_{11} &= S_{11} + y/a \end{aligned}\right\} , \qquad (18)$$

where (x,y,z) are the cartesian coordinates of the true center of mass in the new frame. The new frame is located at the assumed center of mass and the old frame at the true center of mass. It can be shown [4] that $C_{10} = C_{11} = S_{11} = 0$, so the corresponding coefficients in the assumed center of mass frame just equal the fractional center of mass errors.

It can be observed that the terms in (17) which contain β contribute infrequently to the transformation, due to the large superscript, $m+j$, on the accompanying associated Legendre polynomial. The locus of index combinations for which these terms are not zero may be determined in the following way. Begin with the triangular locus of Figure 2, overlaying it on Figure 3 by locating the vertex at (k,m). Now translate the triangle along its left side until the vertex rests at $(k-m,0)$. The contributing terms lie in the intersection of this new triangle and the underlying array of Figure 3, except for the case $m=0$, for which there are no contributing terms due to the definition of β. It is found that although the β terms contribute nothing to zonals, they contribute very frequently to the

transformation of near zonal harmonics. This frequency decreases until the sectorials are reached, to which only the C_{00} term makes a non-zero contribution.

5. SPECIAL FORMS OF THE EXTERIOR CASE

As an important special case of equations (3) and (17) consider the gravitating body to be a point mass at the origin. Then all the old coefficients disappear except $C_{00} = 1$, the parameter a automatically cancels out, and the potential in the new frame becomes simply

$$U'(r',\theta',\phi') = \frac{\mu}{r'} \sum_{\ell=0}^{\infty} \left(\frac{R}{r'}\right)^{\ell} P_{\ell}(\cos \gamma'), \tag{19}$$

where γ' is the angle between the vectors to the field point (r',θ',ϕ') and to the old coordinate origin (R,θ_0,ϕ_0). The addition theorem for Legendre polynomials was used in obtaining this. Equation (19) may be recognized as the classical formula for the exterior potential due to a point mass located at (R,θ_0,ϕ_0).

Now consider the case of a translation of O along its negative z-axis. Relative to O' this appears as a motion of O along the positive z'-axis. Hence the parameters of the translation are $(Z',0,0)$. Using these values in (17), the potential coefficients for the new frame may be shown to be

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = \sum_{\ell=0}^{k} \left(\frac{Z'}{a}\right)^{k-\ell} \alpha_{km}^{\ell m} \begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} . \tag{20}$$

For this set of indices, $\alpha_{km}^{\ell m}$ is a binomial coefficient. Referring to Figure 3, it can be seen that the z-translation induces a linear combination of only old coeffieicents which lie on the line drawn upward from $\ell=k$, $j=m$. Notice that there is no mixing of the C's and S's.

The potential of an axially symmetric lunar mascon may be determined with the help of (20). This development begins with the fact that the potential coefficients in the old frame may be written [4] as the integrals

$$\begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} = \frac{(2-\delta_{m0})}{a^{\ell} M} \frac{(\ell-m)!}{(\ell+m)!} \int_{\text{mascon}} \rho^{\ell} P_{\ell}^{m}(\cos \alpha) \begin{pmatrix} \cos m\beta \\ \sin m\beta \end{pmatrix} dM, \qquad (21)$$

in which (ρ,α,β) are the spherical polar coordinates of an element of mass in the mascon, whose total mass is M. If the z-axis of the old frame is selected to coincide with the mascon's symmetry axis, then (21) becomes

$$\begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} = \begin{pmatrix} \delta_{m0} \\ 0 \end{pmatrix} C_{\ell 0} \qquad (22)$$

where*

$$C_{\ell 0} = \frac{1}{a^{\ell} M} \int_{\text{mascon}} \rho^{\ell} P_{\ell}(\cos \alpha) dM. \qquad (23)$$

*If the mascon is assumed to be a homogeneous oblate spheroid of semimajor axis a and eccentricity e, these coefficients in a frame with origin at the mascon's center and z-axis along the symmetry axis are

$$C_{\ell 0} = \begin{cases} 3(-)^{\ell/2} e^{\ell} \Big/ ((\ell+1)(\ell+3)) & \ell=\text{even} \\ 0 & \ell=\text{odd}. \end{cases}$$

The semimajor axis is the dimensional parameter to be used in the series (1), which converges only for points for which r>ae. An equivalent expansion is given in [5] along with a full functional form valid at all points outside the mascon.

It follows from (20), (22), and (23) that the potential coefficients of the mascon after translation are

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = \begin{pmatrix} \delta_{m0} \\ 0 \end{pmatrix} E_{k0} \ , \tag{24}$$

where

$$E_{k0} = \sum_{\ell=0}^{k} \left(\frac{Z'}{a}\right)^{k-\ell} \alpha_{k0}^{\ell 0} C_{\ell 0} . \tag{25}$$

Putting (24) into (3) gives the potential due to an axially symmetric mascon at Z' on the z'-axis. It is

$$U'(r',\theta',\phi') = \frac{\mu}{r'} \left\{ 1 + \sum_{k=1}^{\infty} \left(\frac{a}{r'}\right)^{k} P_k(\cos\theta') E_{k0} \right\} . \tag{26}$$

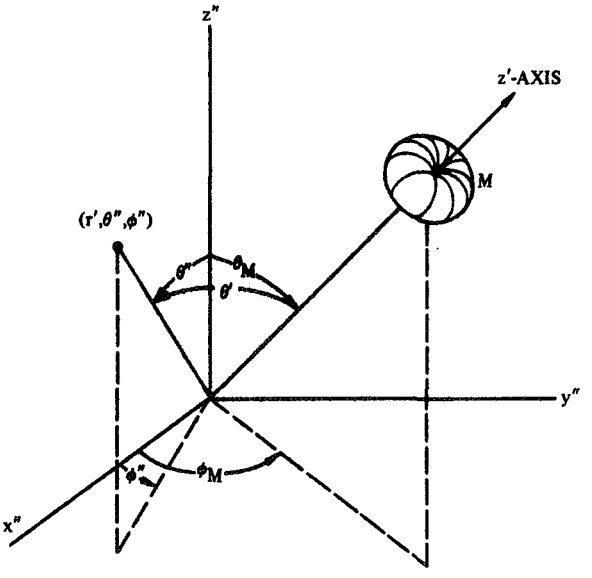


FIGURE 4 - GEOMETRY USED IN THE ADDITION THEOREM TO ROTATE AN AXIALLY SYMMETRIC MASCON TO POLAR COORDINATES (θ_M, ϕ_M).

It remains to generalize (26) for the case of the same mascon located at the arbitary point (Z', θ_M, ϕ_M). To do so, consider rotating to a new coordinate system in which the mascon is at (Z', θ_M, ϕ_M) and the invariant field point is now at (r', θ'', ϕ''), as illustrated in Figure 4. Then the addition theorem for Legendre polynomials, in the form

$$P_k(\cos\theta') = \sum_{m=0}^{k} (2-\delta_{m0}) \frac{(k-m)!}{(k+m)!} P_k^m(\cos\theta'') P_k^m(\cos\theta_M) \cos m(\theta''-\phi_M), \quad (27)$$

may be used in (26) to obtain

$$U''(r',\theta''\phi'') = \frac{\mu}{r'} \left\{ 1 + \sum_{k=1}^{\infty} \sum_{m=0}^{k} \left(\frac{a}{r'}\right)^k P_k^m(\cos\theta'') \right.$$

$$\left. \cdot \left[E_{km}'' \cos m\phi'' + W_{km}'' \sin m\phi'' \right] \right\}, \quad (28)$$

where

$$\begin{pmatrix} E_{km}'' \\ W_{km}'' \end{pmatrix} = E_{k0}(2-\delta_{m0}) \frac{(k-m)!}{(k+m)!} P_k^m(\cos\theta_M) \begin{pmatrix} \cos m\phi_M \\ \sin m\phi_M \end{pmatrix} . \quad (29)$$

Notice that the final potential (28) has the same form as (3) but that the appropriate coefficients are now given by (29).

For the case of a completely unsymmetric mascon, a general procedure is to use (21) to obtain the coefficients in a body-centered frame and then to use (17), the formula for a general translation. In the event there is a preferred frame in which the integration is simplified, an alternative procedure

is to integrate (21) in the preferred frame, to transform the resulting coefficients to new ones valid in a frame of the proper orientation, and then to translate to the final frame, using (17). The rotational transformation needed in the second step is given in [6], which is consistent with the conventions of this paper.

6. THE INTERIOR CASE

In the previous two sections, the transformation formulas valid for the case $r'>R$ were developed and discussed. A separate development is required when $r'<R$, in order to have a convergent series. This dichotomy is familiar from the study of the attraction of point masses [7]. The development for $r'<R$, the interior case, will be presented in this section.

In [1,(34)] James shows that for the interior case solid harmonics transform to

$$U_\ell'(r',\theta',\phi') = \frac{\mu}{R} \sum_{k=0}^{\infty} \left(\frac{r'}{R}\right)^k \left(\frac{a}{R}\right)^\ell \sum_{m=0}^{k} P_k^m(\cos\theta') \cdot \left[C_{km}^{(\ell)} \cos m\phi' + S_{km}^{(\ell)} \sin m\phi'\right] , \tag{30}$$

in which the coefficients may be shown to be

$$\begin{pmatrix} C_{km}^{(\ell)} \\ S_{km}^{(\ell)} \end{pmatrix} = \sum_{j=0}^{\ell} \left\{ C_{\ell j} \left[\gamma_{km}^{\ell j} \begin{pmatrix} \cos(m-j)\phi_0 \\ \sin(m-j)\phi_0 \end{pmatrix} P_{k+\ell}^{m-j} + \eta_{km}^{\ell j} \begin{pmatrix} \cos(m+j)\phi_0 \\ \sin(m+j)\phi_0 \end{pmatrix} P_{k+\ell}^{m+j} \right] \right.$$

$$+ S_{\ell j} \left[\gamma_{km}^{\ell j} \begin{pmatrix} -\sin(m-j)\phi_0 \\ \cos(m-j)\phi_0 \end{pmatrix} P_{k+\ell}^{m-j} + \eta_{km}^{\ell j} \begin{pmatrix} \sin(m+j)\phi_0 \\ -\cos(m+j)\phi_0 \end{pmatrix} P_{k+\ell}^{m+j} \right] \Bigg\} \quad . \tag{31}$$

The constants appearing in (31) are

$$\gamma_{km}^{\ell j} = \frac{(-)^{\ell-j}}{2} \frac{(k-m+\ell+j)!}{(k+m)!(\ell-j)!} (2-\delta_{m0}) \tag{32}$$

$$\eta_{km}^{\ell j} = \frac{(-)^{\ell}}{2} \frac{(k-m+\ell-j)!}{(k+m)!(\ell-j)!} (2-\delta_{m0}) . \tag{33}$$

The full potential expressed in the new frame O' is the sum over ℓ of the solid harmonics given in (30). Performing this summation and rearranging the sums yields (4), for which the condition $a<R$ arises to ensure convergence. The potential coefficients in (4) may be shown to be

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \left(\frac{a}{R}\right)^{\ell} \left\{ C_{\ell j} \left[\gamma_{km}^{\ell j} \begin{pmatrix} \cos(m-j)\phi_0 \\ \sin(m-j)\phi_0 \end{pmatrix} P_{k+\ell}^{m-j} \right. \right.$$

$$+ \eta_{km}^{\ell j} \begin{pmatrix} \cos(m+j)\phi_0 \\ \sin(m+j)\phi_0 \end{pmatrix} P_{k+\ell}^{m+j} \Big]$$

$$+ S_{\ell j} \Big[\gamma_{km}^{\ell j} \begin{pmatrix} -\sin(m-j)\phi_0 \\ \cos(m-j)\phi_0 \end{pmatrix} P_{k+\ell}^{m-j}$$

$$+ \eta_{km}^{\ell j} \begin{pmatrix} \sin(m+j)\phi_0 \\ -\cos(m+j)\phi_0 \end{pmatrix} P_{k+\ell}^{m+j} \Big] \Big\} \quad . \tag{34}$$

Unlike the previous case, the new (k,m) coefficient is a linear combination of all the old coefficients. Furthermore, for this case no restriction on the premissible (ℓ,j) combinations arises from the indicial requirement on the associated Legendre polynomials. There is a weighting hierarchy, however, due to the presence of $(a/R)^{\ell}$ in (34). Since $(a/R)<1$, the old coefficients with large ℓ-values contribute only insignificantly to the new coefficients, the old low-order field being dominant.

Equations (4) and (34) provide a suitable starting point for computing general earth perturbations on a high altitude lunar satellite, since they allow the earth's potential to be expressed in selenocentric coordinates.* The complete procedure for computing these perturbations is non-trivial and will be deferred to a subsequent paper. In outline, however, the procedure involves converting (4) to a function of the

*In this application, (4) and (34) should be regarded as the result of translating a frame which has been obtained by rotating the geographic frame into proper orientation. The transformation of a potential function under coordinate rotations is given in [6], which is consistent with the conventions of this paper.

satellite's orbital elements and using the resulting function as part of the disturbing function in Lagrange's planetary equations. The rest of the disturbing function consists of the non-spherical portion of the lunar potential plus a dynamical consistency term which accounts for the complete earth-moon attraction.

7. SPECIAL FORMS OF THE INTERIOR CASE

Consider the case in which a body-centered coordinate system is translated by a distance which is very large with respect to the body's size parameter a. Then one expects that in the new frame, the body's effect will be that of a point mass. That (34) yields this result may be seen by setting R>>a, which allows all terms but the $\ell=0$ term to be dropped, giving

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = (2-\delta_{m0}) \frac{(k-m)!}{(k+m)!} \begin{pmatrix} \cos m\phi_0 \\ \sin m\phi_0 \end{pmatrix} P_k^m . \qquad (35)$$

Putting (35) into (4) yields for the new potential

$$U'(r',\theta',\phi') = \frac{\mu}{R} \sum_{k=0}^{\infty} \left(\frac{r'}{R}\right)^k P_k(\cos \gamma'), \qquad (36)$$

where γ' is the angle between the vectors to the field point (r',θ',ϕ') and to the old coordinate origin (R,θ_0,ϕ_0).

The addition theorem for Legendre polynomials was used in obtaining (36). This formula may be recognized as the classical formula for the interior potential due to a point mass located at (R,θ_0,ϕ_0).

To parallel the discussion in Section (5), consider also the case of a translation of O along its negative z'-axis, which appears as a motion of O along the positive z'-axis of O'. Taking the translation parameters to be (Z',0,0), it follows from (34) that the potential coefficients in the new frame are

$$\begin{pmatrix} E_{km} \\ W_{km} \end{pmatrix} = \sum_{\ell=0}^{\infty} \left(\frac{a}{\bar{z}'}\right)^{\ell} \gamma_{km}^{\ell m} (1+\delta_{m0}) \begin{pmatrix} C_{\ell m} \\ S_{\ell m} \end{pmatrix} . \qquad (37)$$

The new (k,m) coefficient is thus a linear combination of all the coefficients of the same m-value, but with no mixing of C's and S's.

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REFERENCES

1. James, R. W., "Transformation of Spherical Harmonics Under Change of Reference Frame," Geophys. J. R. Astr. Soc., 17, pp. 305-16 (1969).

2. Transactions of the International Astronomical Union, Vol. XIB - Proceedings of the Eleventh General Assembly Berkley 1961, D. H. Sadler, Ed., Academic, New York (1962).

3. Emde, P., and Jahnke, E., Tables of Functions with Formulae and Curves, Dover, New York, Fourth Edition.

4. Battin, R. H., Astrounautical Guidance, McGraw-Hill, New York (1964).

5. MacMillan, W. D., The Theory of the Potential, Dover, New York (1958).

6. Levie, Jr., S. L., "Transformation of a Potential Function Under Coordinate Rotations," JAS, to be published.

7. Jackson, J. D., Classical Electrodynamics, Wiley, New York (1962).

