ALMOST PERIODIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR VOLTERRA SYSTEM. II

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ABSTRACT

This paper studies forced almost periodic oscillations in a nonlinear system of two Volterra integral equations. It is a sequel paper to an earlier paper on the same topic. Earlier results are improved in two ways. First it is shown that the oscillatory solution is Lyapunov stable under small perturbations in the coefficients of the equation. Secondly it is shown that whenever the coefficients are quasiperiodic and analytic, the almost periodic oscillation is in the same class.

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I. Introduction.

In this paper we continue the study of forced oscillations in a nonlinear system of Volterra integral equations of the form

$$(1.1)\qquad x_1(t) = f_1(t) - \int_0^t a_1(t-s)g_1(s,x_1(s),x_2(s))ds - \int_0^t a_2(t-s)g_2(s,x_1(s),x_2(s))ds,$$

$$x_2(t) = f_2(t) - \int_0^t a_2(t-s)g_1(s,x_1(s),x_2(s))ds - \int_0^t a_1(t-s)g_2(s,x_1(s),x_2(s))ds.$$

In an earlier paper [1] sufficient conditions were given so that the solutions $x_1(t)$ and $x_2(t)$ tend to certain almost periodic limiting functions $P_1(t)$ and $P_2(t)$ as $t \to \infty$. In this paper we shall improve the previous results in two ways. Firstly, it will be shown that this oscillatory behavior is stable under small perturbations in the functions f_i and g_i. That is the solution of the perturbed problem is oscillatory and is near the solution of the unperturbed problem. Secondly, we give rather weak sufficient conditions in order that the limiting functions $P_1(t)$ and $P_2(t)$ are analytic in t.

We shall follow the notation in [1] and shall freely use results from that paper. In particular we rewrite (1.1) in the vector form

$$x(t) = f(t) - \int_0^t A(t-s)G(s,x(s))ds \tag{E}$$

where $A(t)$ is the appropriate matrix and x, f and G are two dimensional column vectors. The vector norm will be

$$|x| = \max \{|x_1|, |x_2|\}, \text{ if } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The symbol Q will always denote the special matrix

$$Q = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1.2}$$

Note that $Q = Q^{-1}$ and that $|Q| = \sqrt{2}$. For any $N > 0$ the resolvent of $NQA(t)Q$ will be denoted by $R_N(t)$. Using $R_N(t)$, the variation of constants formula and the change of variables $x = Qy + f(t)$ one car rewrite system (E) in the equivalent form

$$(E_N) \qquad y(t) = \int_0^t R_N(t-s)\{y(s) - G_N(s,y(s))\}ds$$

where $G_N(t,y) = QG(t,Qy+f(t))N^{-1}$.

System (1.1) arises in a natural way from the initial - boundary value problem

$$(1.3) \qquad \begin{aligned} u_t &= u_{xx} && (t > 0,\ 0 < x < \pi), \\ u(0,x) &= F(x) && (0 < x < \pi), \end{aligned}$$

and

$$u_x(t,0) = g_1(t,u(t,0),u(t,\pi)),\ u_x(t,\pi) = = -g_2(t,u(t,0),u(t,\pi)),$$

for all $t > 0$. In particular, we have in mind boundary conditions motivated by C. C. Lin's theory of superfluidity (see [2] or [3]):

$$(1.4) \qquad g_1(t,x_1) = B_1(x_1-c_1 \sin k_1t)^3,\ g_2(t,x_2) = = B_2(x_2-c_2 \sin k_2t)^3.$$

As an application of the results proved here and in [1] we shall prove the following result.

Theorem 1. *Consider the problem* (1.3-4) *with* $F_0 \in C^2[0,\pi]$. *Then given any* $B > 0$ *there exists* $\varepsilon > 0$ *such that if* $|B_i - B| < \varepsilon$ *for* $i = 1,2$ *and if* $F \in C^2[0,\pi]$ *with*

$$\Sigma_{j=0}^{2} \max_x |F_0^{(j)}(x) - F^{(j)}(x)| < \varepsilon$$

then the boundary functions $u(t,0)$ *and* $u(t,\pi)$ *tend asymptotically as* $t \to \infty$ *to almost periodic limiting functions* $P_1(t)$ *and* $P_2(t)$. *The functions* $u(t,0)$, $u(t,\pi)$, $P_1(t)$ *and* $P_2(t)$ *all vary continuously with* F, B_1, B_2, C_1 *and* C_2. *Moreover, each* $P_i(t)$ *has the form* $P_i(t) = P_i(k_1 t, k_2 t)$ *where* $P_i(\theta_1,\theta_2)$ *is real analytic in* (θ_1,θ_2) *and is* 2π*-periodic in each of its two variables.*

II. Perturbation Results.

Assume that the coefficient functions f, G and A in (E) satisfy the following hypotheses:

(A1) $a_1(t) = 1 + 2\sum_{n=1}^{\infty} \exp(-n^2 t)$, $a_2(t) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp(-n^2 t)$

and

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_2(t) & a_1(t) \end{pmatrix}.$$

(A2) $f(t)$ is continuous and bounded on $0 \leq t < \infty$.

(A3) $G(t,x)$ is continuous in (t,x) for all $t \geq 0$, $|x| < \infty$, G is locally Lipschitz continuous in x and $G(t,0) \equiv 0$.

(A4) The function $G(t,x_1,x_2)$ has the special form

$$G(t,x_1,x_2) = \begin{pmatrix} g(t,x_1) \\ g(t,x_2) \end{pmatrix}$$

where $g(t,y)$ is an odd nondecreasing function of y which is bounded in $t \in (-\infty,\infty)$ uniformly for y on any compact subset of $(-\infty,\infty)$.

The following hypotheses are related to (A4):

(A5) There exist positive numbers N and K such that if $|y| \leq K$ then $|y-G_N(t,y)| < K$ uniformly for all $t \in (-\infty,\infty)$. Here $y \in R^2$ is a two dimensional column vector and G_N is the function defined below (E_N).

(A6) There exist positive numbers N, K_0 and K_1 such that if $|y| \leq K$, then $|y-G_N(t,y)| \leq K_0 < K_1$ uniformly for all $t \in (-\infty,\infty)$.

The proof of Lemma 3 in [1] is actually a proof of the following stronger result.

Lemma 1. Suppose G satisfies (A3) and (A4) and $b \geq \sup\{|f(t)|: t \geq 0\}$. Then for any $M > \sqrt{2}\, b$ and for any ε in the interval $0 < \varepsilon < b$ there exists a number $N > 0$ such that (A5) is true with $K = M + \varepsilon$. Moreover, if L is a constant such that

$$L \geq \sup\{|G(t,y)|: -\infty < t < \infty,\ |x| \leq 5M\},$$

then N depends only on the numbers M, ε and L.

Using Lemma 1 we now prove:

Lemma 2. Suppose G satisfies (A3) and (A4). Then G satisfies

(A6) <u>where</u> N <u>and</u> $K = K_1$ <u>are the numbers obtained in Lemma 1 above.</u>

Proof. Let $K = K_1$ and N be given by Lemma 1.
We must show that there exists a number $K_0 < K_1$ such that (A6) is true. For a contradiction we suppose there is no such K_0. Then for each positive integer n there exists numbers y_n and t_n such that

$$|y_n - G_N(t_n, y_n)| \geq K - 1/n.$$

By possibly taking a subsequence we may assume that $y_n \to y_0$, $G_N(t_n, y_n) \to g_0$ and $f(t_n) \to f_0$ as $n \to \infty$. Note that $|y_0| \leq K$. Define $x_n = Qy_n + f(t_n)$ so that $x_n \to x_0 = Qy_0 + f_0$.

Write x_0 and $\gamma = NQg_0$ in terms of their components

$$x_0 = \mathrm{col}(x_{01}, x_{02}), \quad \gamma = NQg_0 = \mathrm{col}(\gamma_1, \gamma_2).$$

Define

$$g^*(z) = \begin{cases} 0 & \text{if } z = 0 \\ |\gamma_1| & \text{if } z = |x_{01}| \\ |\gamma_2| & \text{if } z = |x_{02}| \end{cases} .$$

Extend $g^*(z)$ linearly between the points 0, $|x_{01}|$ and $|x_{02}|$, extend g^* as a constant on the remaining part of the half line $z \geq 0$ and let $g^*(-z) = -g^*(z)$ when $z < 0$. Define $G^*(x_1,x_2) = \mathrm{col}\,(g^*(x_1), g^*(x_2))$ for all x_1 and x_2. Note that $G^*(x_0) = NQg_0$.

The function G^* defined in this way satisfies (A3) and (A4) and has the same upper bound L as the function G in Lemmas 1 and 2. Since $|y_0| \leq K$, then Lemma 1 implies that

$$|y_0 - QG^*(Qy_0+f_0)N^{-1}| < K.$$

On the other hand it follows by the construction of G^* and the choice of y_0 and g_0 that

$$|y_0 - QG^*(Qy_0+f_0)N^{-1}| = |y_0 - QG^*(x_0)N^{-1}|$$
$$= |y_0 - g_0| \geq K.$$

This contradiction completes the proof. Q.E.D.

Theorem 2. Suppose (A1-4) are true and $b = \sup\{|F(t)|: t \geq 0\}$. Consider the perturbation problem

$$\text{(PE)}\qquad X(t) = F(t) - \int_0^t A(t-s)\{G(s,X(s)) + P(s,X(s))\}ds,$$

where P is continuous in (t,x). Then for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that if

$$\sup\{|P(t,x)| : t \geq 0, |Q(x-F(t))| < \sqrt{2}\, b + \varepsilon\} < \delta$$

then the solution $X(t)$ of (PE) exists for all $t \geq 0$ and satisfies the inequality

$$|Q(X(t) - F(t))| \leq \sqrt{2}\, b + \varepsilon.$$

Proof. Define $H(t,x) = G(t,x) + P(t,x)$ and let $H_N(t,y) = QH(t,Qy + F(t))N^{-1}$. Given $K_1 = \sqrt{2}\, b + \varepsilon$ choose N and K_0 using Lemma 2. Choose $\delta < N(K_1-K_0)|Q|^{-1}$ so that

$$|QP(t,Qy - F(t))N^{-1}| \leq |Q|\delta N^{-1} < K_1 - K_0.$$

If $|y| \leq K_1$ then by the choice of δ one has

$$\begin{aligned} |H_N(t,y)| &\leq |G_N(t,y)| + |QP(t,Qy - F(t))N^{-1}| \\ &< K_0 + (K_1-K_0) = K_1. \end{aligned}$$

Thus $H(t,x)$ satisfies (A3) and (A5). Now apply Theorem 4 of [1]. Q.E.D.

Theorem 3. *Suppose the coefficients* f, A *and* G *of* (E) *satisfy* (A1-4). *Define* $\|f\| = \sup\{|f(t)|: t \geq 0\}$. *Suppose that given any* $A > 0$ *there exists a positive, continuous, increasing function* $\alpha(u)$ *such that*

$$\{g(t,u+x) - g(t,x)\}/u \geq \alpha(|u|) \qquad (|u| \geq A)$$

uniformly for all $t \geq 0$ *and all* x *such that* $|Q(x-f(t))| \leq \sqrt{2}\,\|f\| + 4$. *Then given any* $\varepsilon > 0$ *there exists a positive number* δ *such that whenever*

i) $F(t)$ *is any continuous function satisfying* $\|f-F\| = \sup\{|f(t) - F(t): t \geq 0\} < \delta$,

ii) $P(t,x)$ *is any continuous function satisfying* $\sup\{|P(t,x)|: t \geq 0,\ |Q(x - f(t))| \leq \sqrt{2}\|f\| + 4\} < \delta$,

and

iii) $X(t)$ *is the unique solution of* (PE),

then $|x(t) - X(t)| \leq \varepsilon$ *for all* $t \geq 0$.

Proof. Define $y(t) = x(t) - X(t)$, $\varphi(t) = f(t) - F(t)$ and $H(t,y) = G(t,y+X(t)) - G(t,X(t))$. Then one has

$$y(t) = \varphi(t) - \int_0^t A(t-s)\{H(s,y(s)) - P(s,X(s))\}ds$$

or symbolically

$$y = \varphi - A*\{H(y) - P(X)\}.$$

Let $Y = Qy$, $A_N = NQAQ$ and $H_N(t,Y) = QH(t,QY)N^{-1}$ so that

$$(2.1) \quad Y = (Q\varphi + A_N*\{Y - H_N(Y) + QP(X)\}N^{-1}) - A_N*Y.$$

If R_N is the resolvent of A_N, that is

$$(2.2) \qquad R_N = A_N - A_N*R_N,$$

then any equation of the form $Y = S - A_N*Y$ may be written in the equivalent form $Y = S - R_N*S$. Applying this to (2.1) and using the relation (2.2) one can calculate

$$Y = Q\varphi + A_N*\{Y - H_N(Y) + QP(X)N^{-1}\} - R_N*Q\varphi$$
$$- R_N*A_N*\{Y - H_N(Y) + QP(X)N^{-1}\},$$

$$(2.3) \qquad Y = Q\varphi - R_N*Q\varphi + R_N*\{Y - H_N(Y) + QP(X)N^{-1}\}$$

or

$$(2.3') \quad Y(t) = Q\varphi(t) - Q\int_0^t R_N(t-s)\varphi(s)ds +$$
$$\int_0^t R_N(t-s)\{Y(s) - H_N(s,Y(s)) + QP(s,X(s)N^{-1}\}ds.$$

Define $S_0 = \{(t,x)\colon t \geq 0, |Q(x - F(t))| \leq \sqrt{2}\|F\| + 1\}$ and let $S_1 = \{(t,x)\colon t \geq 0, |Q(x - f(t))| \leq \sqrt{2}\|f\| + 4\}$. If $\|\varphi\| = \|f-F\| < 1$, and if $(t,x) \in S_0$ then

$$
\begin{aligned}
|Q(x - f(t))| &\leq |Q(x - F(t))| + |Q|\,\|f - F\| \\
&\leq \sqrt{2}\|F\| + 1 + \sqrt{2}\cdot 1 \leq \sqrt{2}(\|f\| + 1) + 1 + \sqrt{2} \\
&\leq \sqrt{2}\|f\| + 4.
\end{aligned}
$$

Therefore, $S_0 \subset S$, if $\|\varphi\| < 1$. By Theorem 2 there exists a number $\delta_0 > 0$ such that if $|P(t,x)| < \delta_0$ on S_0 then $X(t)$ exists for all $t \geq 0$ and $(t,X(t)) \in S_0$.

Write $H(t,x)$ in the form $H(t,x) = \text{col}\,(M_1x_1, M_2x_2)$ where

$$M_j(t,x) = \{g(t,x_j+X_j(t)) - g(t,X_j(t))\}/x_j \qquad (j = 1,2).$$

Then $Y - H_N(t,Y)$ can be written in the form

$$Y - H_N(t,Y) = A(t,Y)Y, \; A = \begin{pmatrix} 1 - \dfrac{M_1+M_2}{2N} & \dfrac{M_1-M_2}{2N} \\ \dfrac{M_1-M_2}{2N} & 1 - \dfrac{M_1+M_2}{2N} \end{pmatrix}$$

If $M(t,x) = \text{col}\,(M_1(t,x_1), M_2(t,x_2))$ and if $|M(t,QY| < N$ then the norm of the matrix A is

$|A| = 1 - |M(t,QY)|/N = \max \{1-M_1/N,\ 1-M_2/N\}$. For any number $K > 0$ if $|Y| \leq K/2$, then since $|A| < 1$ one has $|AY| \leq K/2$. If $K/2 \leq |Y| \leq K$, then either $|Y_1+Y_2|$ or $|Y_1-Y_2| \geq K/2$. Therefore, the hypotheses of the theorem imply that $|M(t,QY)| \geq \alpha(K/2) > 0$ for some function $\alpha(u)$. This means that

$$|Y-H_N(t,Y)| \leq 1 - \alpha(K/2)/N \qquad (K/2 \leq |Y| \leq K).$$

Consequently, for any given $\varepsilon > 0$, if $K = \varepsilon/\sqrt{2}$ then there exist positive numbers N and K_0 such that if $|Y| \leq \varepsilon/\sqrt{2}$ then $|Y - H_N(t,Y)| \leq K_0 < \varepsilon/\sqrt{2}$. The number δ in the conclusion of the present theorem will be chosen so that $\delta \leq \min \{\delta_0, 1\}$ and such that $4\|f - F\| + 2|P(t,x)|N^{-1} \leq 6\delta \leq \varepsilon - \sqrt{2}\, K_0$ for all $(t,x) \in S_1$. For this choice of δ we shall show that $|y(t)| = |x(t) - X(t)| \leq \varepsilon$ for all $t \geq 0$. Equivalently, since $Q = Q^{-1}$ and $|Q| = \sqrt{2}$, then we may show that $|Y(t)| = |Q(x(t) - X(t))| \leq \varepsilon/\sqrt{2}$.

Let $W = \{z \in C[0,\infty)\colon |z(t)| \leq \varepsilon/\sqrt{2}$ for all $t \geq 0\}$ and let TZ be the map defined by the right hand side of (2.3), that is

$$TZ(t) = Q\varphi(t) - Q\int_0^t R_N(t-s)\varphi(s)ds + \int_0^t R_N(t-s)\{Z(s) - H_N(s,Z(s)) - QP(s,X(s))N^{-1}\}ds.$$

By Lemma 1 of [1] the matrix $R_N \in L^1(0,\infty)$ and $\int_0^t |R_N(t-s)|ds \leq 1$ for all $t \geq 0$. Therefore, if $z \in W$,

$$\begin{aligned}|TZ(t)| &\leq \sqrt{2}\,\|\varphi\| + \sqrt{2}\,\|\varphi\| \int_0^t |R_N(t-s)|ds + \\ &+ N^{-1}\sqrt{2} \max_{S_1} |P(t,x)| \int_0^t |R_N(t-s)|ds + \int_0^t |R_N(t-s)|K_0 ds \\ &\leq 2\sqrt{2}\,\|\varphi\| + \sqrt{2} \max_{S_1} |P(t,x)|N^{-1} + K_0 \\ &\leq 6\delta + K_0 \leq \varepsilon/\sqrt{2}\ .\end{aligned}$$

This shows that $Tz \in W$ if $z \in W$. If the space $C[0,\infty)$ is given the topology of uniform convergence on bounded subsets of $[0,\infty)$, then it becomes a locally convex linear topological space with the additional property that $T: C[0,\infty) \to C[0,\infty)$ is a completely continuous map. Since W is a closed bounded convex subset of $C[0,\infty)$ and T maps W into itself, then the Schauder fixed point theorem applies. This means that (2.3') has at least one solution $Y(t)$ such that $|Y(t)| \leq \varepsilon/\sqrt{2}$ for all $t \geq 0$. But the function $H(t,Y)$ is locally Lipschitz continuous in Y so that the solution of (2.3') is unique, that is $Y(t) = Q(x(t) - X(t))$. Q.E.D.

III. Quasi-Periodic Functions.

Let $k_1, k_2, \ldots, k_m$ be positive constants which are linearly independent over the integers. Let k denote the vector $k = (k_1, k_2, \ldots, k_m)$ with $m \geqq 1$.

Definition. A function $\varphi(t)$ will be called quasiperiodic with fundamental frequencies k if and only if there exists a function $\Phi(\theta) = \Phi(\theta_1, \theta_2, \ldots, \theta_m)$ continuous in θ and periodic in each variable θ_j of period 2π such that

$$\varphi(t) = \Phi(kt) = \Phi(k_1 t, k_2 t, \ldots, k_m t), \quad -\infty < t < \infty.$$

Each quasiperiodic is easily seen to be almost periodic. If $m = 1$ so that $k = k_1$ then the quasiperiodic function is actually periodic.

According to the results in [1] if $x = Qy + f(t)$ then for any $N > 0$ the function $y(t)$ solves (E_N). Conditions are given which guarantee that $y(t)$ tends asymptotically to an almost periodic function $p(t)$ where

$$(3.1) \qquad p(t) = \int_{-\infty}^{t} R_N(t-s)\{p(s) - G_N(s, p(s))\}ds, \quad -\infty < t < \infty.$$

The function p is the unique solution of (3.1) if N is sufficiently large.

The aim in this section is to give sufficient conditions in order that $p(t) = P(kt)$ is quasiperiodic and $P(\theta)$ is analytic in θ. Assume:

(A7) $G(t,x) = \gamma(kt,x)$ and $f(t) = \varphi(kt)$ are quasiperiodic in t with fundamental frequencies k. Moreover, $\gamma(\theta,x)$ and $\varphi(\theta)$ are real analytic functions of (θ,x) and θ respectively in regions

$$(3.2) \qquad U(\delta_0) = \{(\theta,x): |\mathrm{Im}\,\theta_j|, |\mathrm{Im}\,x_i| < \delta_0, -\infty < \mathrm{Re}\,x_i, \mathrm{Re}\,\theta_j < \infty \text{ for } 1 \leq j \leq m \text{ and } i = 1,2\}$$

and

$$(3.3) \qquad D(\delta_0) = \{\theta: |\mathrm{Im}\,\theta_j| < \delta_0, -\infty < \mathrm{Re}\,\theta_j < \infty \text{ for } 1 \leq j \leq m\}.$$

Under this assumption it follows that the function

$$\gamma_N(kt,y) = Q\gamma(kt,Qy + \varphi(kt))N^{-1}$$

is also quasiperodic and analytic in $U(\delta_0)$. If the solution of (3.1) was quasiperodic, say $p(t) = P(kt)$, then (3.1) could be rewritten as

$$P(kt) = \int_{-\infty}^{t} R_N(t-s)\{P(ks) - \gamma_N(ks,P(ks))\}ds$$

$$= \int_{0}^{\infty} R_N(s)\{P(kt-ks) - \gamma_N(kt-ks,P(kt-ks))\}ds.$$

Since $k = (k_1, k_2, \ldots, k_n)$ is a vector of linearly independent frequencies and $P(\theta)$ is continuous in θ, then this is equivalent to the equation

$$(3.4) \qquad P(\theta) = \int_0^\infty R_N(s)\{P(\theta-ks) - \gamma_N(\theta-ks, P(\theta-ks))\}ds.$$

Conversely, if $P(\theta)$ is any continuous solution of (3.4) such that $P(\theta)$ is 2π-periodic in each variable θ_j, then $p(t) = P(kt)$ will solve (3.1). Therefore, our problem is reduced to finding an analytic and periodic solution of (3.4).

For any $\delta > 0$ the symbols $D(\delta)$ or $U(\delta)$ will denote regions defined in the manner of (3.2) and (3.3). Using this notation we now prove the following:

Theorem 4. Suppose (A1-3) and (A6-7) are true. Then there exists a $\delta > 0$ such that (3.4) has a solution $P(\theta)$ which is real analytic in $\theta \in D(\delta)$ and 2π-periodic in each variable θ_j.

Proof. Let N, K_0 and K_1 be the numbers given by (A6). For any δ in the interval $0 < \delta < \delta_0$ let $\mathscr{F}(\delta)$ denote the set of a functions $Z(\theta)$ real analytic in $\theta \in D(\delta)$ and 2π-periodic in each variable θ_j. If $\mathscr{F}(\delta)$ is given the topology of uniform convergence on compact subsets of $D(\delta)$, then this family becomes a locally convex linear topological space over the real numbers. Define

$$S = \{Z \in \mathscr{F}(\delta): |Z(\theta)| \leq K_1 \text{ for all } \theta \in D(\delta)\}$$

where K_1 is the constant in (A6). Then S is a closed, convex, nonempty and compact subset of $\mathscr{F}(\delta)$. Since (A6) is true for $G(t,x) = \gamma(kt,x)$ and since $k = (k_1, k_2, \ldots, k_m)$ is a vector with linearly independent components, then

$$|y - \gamma(\theta,y)| \leq K_0 < K_1 \quad \text{if} \quad |y| \leq K_1, \quad (\theta,y) \in U(\delta)$$

and (θ,y) is real. By continuity there exists a number δ with $0 < \delta < \delta_0$ such that

$$|y - \gamma(\theta,y)| \leq K_1 \quad \text{if} \quad |y| \leq K_1 \quad \text{and} \quad (\theta,y) \in U(\delta)$$

where (θ,y) is now allowed to be complex. This is the appropriate δ.

For any $Z \in S$ define

$$TZ(\theta) = \int_0^\infty R_N(s)\{Z(\theta-ks) - \gamma_N(\theta-ks, Z(\theta-ks))\}ds, \quad \theta \in D(\delta).$$

By Lemma 1 of [1] the matrix $R_N(t) \in L^1(0,\infty)$ with $\int_0^\infty |R_N(t)|dt \leq 1$. This means that $TZ(\theta)$ is well defined, $TZ \in \mathscr{F}(\delta)$ and

$$|TZ(\theta)| \leq \int_0^\infty |R_N(s)|K_1 ds \leq K_1.$$

In particular T maps S into S continuously.. By the Schauder fixed point theorem T has a fixed point. Q.E.D.

IV. Outline of the Proof of Theorem 1.

The results in section 2 of [1] show that (1.3) is equivalent to (E) with $x_1(t) = u(t,0)$ and $x_2(t) = u(t,\pi)$, with

$$f_1(t) = F_0/2 + \sum_{n=1}^{\infty} F_n \exp(-n^2 t), \quad f_2(t) = F_0/2 + \sum_{k=1}^{\infty} F_n (-1)^n \exp(-n^2 t)$$

and with

$$F_n = (2/\pi)\int_0^{\pi} F(x) \cos nx dx.$$

It is easy to prove that $f_1(t)$ and $f_2(t)$ vary continuously in the uniform norm over $0 \leq t < \infty$ as F varies in the norm of $C^2[0,\pi]$. The results in section 2 above show that $x_1(t)$ and $x_2(t)$ vary continuously (again in the uniform norm over $0 \leq t < \infty$) as f and g vary.

The results in section 6 of [1] are sufficient to see that $x_1(t)$ and $x_2(t)$ are asymptotic to almost periodic functions $p_1(t)$ and $p_2(t)$ such that $p(t) = \mathrm{col}\,(p_1(t), p_2(t))$ solves (3.1). If k_1 and k_2 are linearly independent, then Theorem 4 above implies that $p(t)$ is analytic and quasiperiodic with fundamental frequencies k_1 and k_2. Finally, note that since $|p(t) - x(t)| \to 0$ as $t \to \infty$ where $p(t)$ is almost periodic and $x(t)$ varies continuously with f and g, then $p(t)$ varies continuously (in the uniform norm over $-\infty < t < \infty$) with f and g.

If k_1 and k_2 are linearly dependent over the integers, then the same conclusion follows but with $p(t)$ a periodic function.

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