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FINAL REPORT

RESEARCH GRANT NGR 10-019-01

"Propagation & Confinement of Light Beams"

NASA RESEARCH GRANT NGR-10-019-001

FINAL REPORT

CONTENTS



TITLE: "Propagation and Confinement of Light Beams"
SPONSORING INSTITUTION: Florida Technological University
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This report was prepared by Dr. Harvey N. Rexroad, Principal Investigator, and Communicated to Headquarters Contracts Division, Code DHC-4, National Aeronautics and Space Administration, Washington, D.C. 20546 by Leslie L. Ellis, Director of University Research, Florida Technological University.

# I. SUMMARY REPORT

The time schedule for performing the work sponsored by this grant has a natural division into three distinct parts.

Part 1. May 15, 1968 to early Spring 1969

Part 2. Extenuating circumstances. Principal investigator had open-heart surgery in June 1969 and a pacemaker installed in September 1969.

Part 3. Basically from January 1970 to August 1, 1970.

Part 1 was a productive period. It started with a series of seminars at West Virginia University, covering a variety of topics, which contributed to some of the material in the notes that were completed in Part 3. A paper was delivered at the Spring (1969) meeting of the American Physical Society in Washington on one aspect of the work, "Maximum Power Transfer Coefficient Between Two Confocal Apertures". A paper on this subject was well in progress when the principal investigator moved to Florida Technological University in August 1968. Professor Henderson then collaborated with this part of the work and contributed ideas that changed the basic format of some of the proofs. It appeared in the November, 1969 issue of J. Opt. Soc. of America as a joint publication. At this stage it was thought that progress was being made on the computer solution of the resonator problem with an output coupling aperture. This work led to a revision of Chapters IV, Sections D and E, of the notes, and the addition of Appendices G and H.

Some revision of the objectives of the grant also occurred during this period. Because there was no way for the principal investigator to pay himself as a consultant or even to really buy released time, as originally planned, the work time for the grant was contributed during evenings, holidays, and weekends, and the corresponding money (with approval from NASA) saving was used to establish a $CO_2$ laser laboratory at Florida Technological University and to begin some thin film studies.

Part 2. The only accomplishments during this period was to edit the galley proof for the November, 1969, article in JOSA.

Part 3 was another productive period. Chapters V, VI, VII, and Appendices I, J, K, L, M, N of the notes on Laser Optics" were completed. A simplified version of a graphical representation of the law of propagation of a gaussian beam was discovered, and will likely lead to a publication. Preparation of the manuscript is about half complete. The spatial coherence of optical resonators was investigated and meaningful, but not earth shaking, conclusions were reached. It is possible that my derivation of conjugate relations for mutual coherence for the case of a virtual entrance pupil is a new result. The treatment of coherence in the notes raises some new questions concerning microscope illumination and the action of iterated phase transformers. It was discovered early in this period that the computer method for solving the resonator problem with output coupling aperture, although technically correct, was not very practical. Progress in building a $CO_2$ laser was slower than I had hoped, but, nonetheless, satisfactory. One laser is now complete except for one small machine shop job on one of the mirror mounts. All these things are described more completely in the notes or in the Technical Report that follows.

## II. FINANCIAL REPORT

All funds were committed prior to the August 1, 1970, terminating date of this grant. There are a few small accounts that are outstanding that will be cleaned up in the near future. The FTU office of Finance and Accounting will send their official report in the near future.

My records are as follows:

| | | |
|---|---|---|
| Fees and Wages | | |
| Prof. A. D. Levine, Consultant | $1,300.00 | |
| Prof. W. E. Vehse, Consultant | 435.00 | |
| Prof. W. M. Squire, Consultant | 357.50 | |
| Principal Investigator, Consultant | 1,300.00 | |
| Principal Investigator, released time | 654.60 | |
| Rodney Hamilton, Student Assistant | 195.00 | |
| Linda Stover, Typist | 47.03 | |
| Margaret Cooper, Drafting | 25.00 | |
| | | 4,334.13 |
| Capital Equipment | | |
| IR Detector | 1,945.00 | |
| Mirrors, lenses | 1,927.00 | |
| Mounts and Translational Stages | 1,130.00 | |
| Pressure Gauges | 459.00 | |
| | | 5,461.00 |
| Expenses | | |
| Glassware, Dewars | 115.37 | |
| Chemicals and Salt Flats | 263.69 | |
| Duplicating, Library Service, Computer | 223.06 | |
| Travel | 89.90 | |
| Publication Fee | 191.50 | |
| Electronic Parts, Tools, Apparatus | 421.35 | |
| | | 1,304.87 |
| Overhead and Indirect Cost @20% Provisional | | 1,400.00 |
| | Total | 12,500.00 |
| University Contribution (through faculty salary) | | 657.90 |
| | TOTAL | $13,157.90 |

# III. TECHNICAL REPORT

## (a) Maximum Power Transfer Between Confocal Apertures

This work was culminated with a publication. Reprints were sent to the Optical Systems Branch and the Grants and Contracts Office at a prior date. For completeness, this reprint constitutes the last pages of the first two of these final reports.

## (b) Technical Notes on "Laser Optics": Distribution List

Revisions to Chapter IV, Section D and E, along with Appendices G and H, were distributed at a prior date. This report contains the recent additions to this work that begins with Chapter V and Appendix I. The next two pages are reproductions of the Table of Contents. For completeness, the first two copies of this report contain the old notes as well as recent additions. Copies of these notes have been sent to:

| | |
|---|---|
| 2 copies: complete set: | NASA Optical System Branch<br>Atten: Nelson McAvoy, Code 524 |
| 1 copy: complete set: | Dr. John R. Bolte, FTU |
| 1 copy: complete set: | Dr. William C. Oelfke, FTU |
| 1 copy: complete set: | Rodney Hamilton, FTU |
| 1 copy: complete set: | Dean Robert Kersten, FTU |
| 2 copies: complete set (with this final report) | NASA Grants and Contracts Office |
| 8 copies: recent additions (with this final report) | NASA Grants and Contracts Office |
| 1 copy: recent additions (with this final report) | Office of University Research<br>Florida Technological University |
| 1 copy: recent additions (with this final report) | NASA Optical Systems Branch<br>c/o Nelson McAvoy, Code 524 |

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(c) Radiation Pattern from a Laser with an Output Coupling Iris

The problem to be solved is clearly specified by Equation (46) and Appendices G and H (for combining solutions) of the notes. The integral Equation (46) has the form

$$\psi_\ell(u_2)=\epsilon_\ell\int_{\frac{b_1}{a_1}c}^{c}\int_{\frac{b_2}{a_2}c}e^{i\left(\frac{1}{2}u_0^2G_2+u_1^2G_1\right)}J_\ell(u_1u_0)J_\ell(u_1u_2)\psi_\ell(u_0)u_0\,du_0\,u_1\,du_1 \quad (46)$$

In the following, the subscript "$\ell$" is dropped, but we keep in mind the fact that two solutions, $\psi_{\ell+1}$ and $\psi_{\ell-1}$, must be combined to form one Cartesian component $E_{x\ell}$ for each mode.

The plan originally proposed was to represent $\psi$ by several points (say 25) across the mirror aperture (radially outward from center to edge). The value of $\psi$ at points between the chosen 25 was then to be obtained using the Lagrange interpolation formula

$$\psi(u)=\sum_{j=0}^{25}\ell_j(u)\,\psi_j \quad .$$

Integrations would then be performed using something like a Gauss integration formula, where it appeared that 48 or more points would be required to achieve the required accuracy,

$$\int_{\frac{b_2}{a_2}c}^{c}f(u)\,du \simeq C_2\sum_{k=1}^{48}w_k\,f(u_k)$$

Combining the last two equations with (46) then gives

$$\psi_m=\epsilon C_1C_2\sum_{n=0}^{25}\sum_{j=1}^{48}\sum_{k=1}^{48}w_j w_k u_j u_k e^{i\left[\frac{1}{2}u_j^2G_2+u_k^2G_1\right]}\times$$

$$J(u_ku_j)\,J(u_ku_m)\,\ell_n(u_m)\,\psi_n \quad .$$

The result is thus a 25x25 non-Hermitian complex matrix to solve for the 25 complex values that define $\psi$. Since there are 625 matrix elements, each with 48x48 = 2304 terms in the summation, a total of 48x48x625 = 1,440,000 terms must be computed for each matrix. Each term involves the calculation of two Bessel functions of large argument, one Lagrange coefficient, one sine function, one cosince function, etc. If 50 basic number manipulations are needed for each Bessel function, and 40 for the rest of an element computation, there are 1,440,000,x50x50x40 = $1.44\times10^{11}$ basic number manipulations. At 100 microsecods per manipulation, this "brute force method" still represents 0.456 years to compute the elements of one matrix for one "$\ell$" value.

Instead of consuming one year of computer time to solve one problem, the principal investigator decided it was a more prudent investment to look for better methods, which has been the case. There has, however, been no major break through for a practical solution of this problem. The group at Bell Laboratories has apparently encounted similar difficulities in achieving accurate answers to a general problem of this type. McCumber (recent publication), however, has continued to obtain results beyond those for special cases by T. Li and H. Zucker (BSTJ, 57 (1967), pp 984-986 and his earlier work (BSTJ, 44 (1965), pp 333-363.

It should be pointed out that the single pass symmetric situation is beginning to look feasible using this method. The time required to obtain the elements of one matrix would be in the order of a few 3 to 4 hours.

(d) Laser Laboratory: Thin Films

With the help of Dr. John R. Bolte and an assistant, Bruce Stockton, who both contributed time to this project, we now have one $CO_2$ laser nearly complete, and have begun construction of another. All that remains on the first is a small mechanical alteration in one of the mirror mounts. Because we are still in the learning stage on this project, it is reasonable to expect at least another year will be needed to bring anything new to fruition. At the moment, we are strongly attracted to experiments that involve the interaction of light with the depletion layer in semiconductor junctions.

(e) Graphical Solution of Propagation Laws for a Gaussian Beam

A graphic representation of Gaussian Beam Optics is described in Section C of Chapter VII (pp 81-90) that is simpler than analagous methods (all related through conformal mapping) described by Collins (App. Opt. 3, 1964) pp 1263-1274), Chu (BSTJ, Feb. (1966), pp 287-299), Gordon (BSTJ, July (1964), pp 1826 27), and Kogelnik and Li (App. Opt., 5, (1966), pp 1550-1567. This method has a direct correspondence to the laboratory situation in that a traveling light beam is represented by a horizontal straight line with a linear scale, and is a correspondingly convenient teaching aid for the principles of Gaussian beam propagation. After checking one additional reference, the principal investigator plans to send a short article to Applied Optics.

(f) Illumination of a Microscope

An equation is derived to relate the mutual coherence function between conjugate points in the entrance and exit pupils. This derivation on pages 102-105 of the notes for the case of a virtual entrance pupil may be something new. This result has been used by the principal investigator to examine ways to improve the degree of coherence when illuminating a microscope slide with an incoherent extended source. The purpose in mind is, of course, to increase the resolving power of the instrument. All attempts thus far have, in the last analysis, amounted physically to focusing the source to as small as an area as possible at some plane ahead of the microscope slide.

The derivation given by Zernike (Physica, V, (1938), pp 785-795) is a little different than that leading to Equation (146) of the notes. If diffraction of the condensing lens is taken into account (instead of extending limits of integration to infinity, as we did for a real entrance pupil), the conclusion is: *The degree of coherence in a plane illuminated through a lens is the same, whether a source of uniform brightness be imaged on the plane or placed directly behind the lens.* With this scheme of illumination, it follows that the degree of coherence is also independent of lens aberrations and that a cheap condensing lens is just as good as a well corrected achromat.

If, instead of focusing the source on the slide, the source is first focused to a point well in front of the slide, the degree of coherence is greatly enhanced. (Also, more expensive lenses are required.) The principle investigator is not aware of the extent to which these ideas are used in actual practice, but is in the process of collecting several references on the subject to see if there is anything new here.

(g) Resonator Mutual Coherence

The treatment of spatial mutual coherence in resonators in Section K of Chapter VI (pp 68 to 75 of the notes) is different from anything this outhor has seen, even though the end results fall pretty much into the *"already well known"* domain. The idea that mutual coherence can be improved by passage through an iterated system of phase transformers is something that should be tested experimentally. An experiment is being designed to do this very thing.

(h) Beam Spot Size

Although too late to include in the notes, the following derivation for beam spot size on the two mirrors of a resonator verifies Gordon and Kogelnik's result stated on page 99 of the notes.

The law of propagation can be written (Equation 112):

$$\left.\begin{aligned} w_0^2 &= \frac{w^2}{1+\left(\frac{\pi w^2}{\lambda R}\right)^2} \\ z &= \frac{R}{1+\left(\frac{\lambda R}{\pi w^2}\right)^2} \end{aligned}\right\} \quad (1)$$

With mirrors placed at $z_2$ and $z_1$, these give:

$$\left.\begin{aligned} \frac{a_1}{1+a_1^2\left(\frac{1}{R_1}\right)^2} &= \frac{a_2}{1+a_2^2\left(\frac{1}{R_2}\right)^2} \\ d = z_2 - z_1 &= \frac{R_2}{1+\left(\frac{R_2}{a_2}\right)^2} - \frac{R_1}{1+\left(\frac{R_1}{a_1}\right)^2} \end{aligned}\right\} \quad (2)$$

Now let $M_2$ be a mirror located at a positive distance $z_2$ and with positive $R$, according to sign conventions in the propagation laws. The two cases to consider for $M_1$ are show in Figure 2. In both cases, the rules for computing the parameter g for the cavity give

(3) $$\begin{cases} g_2 = 1 - \dfrac{d}{R_2} \\[2ex] g_1 = 1 + \dfrac{d}{R_1} \end{cases}$$

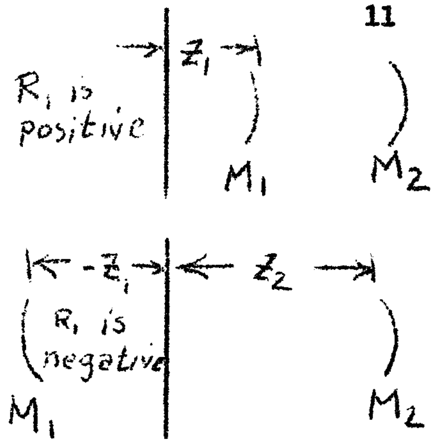


Figure 2.

The result of substituting into (2) is

$$\frac{a_1}{d^2 + a_1^2(1-g_1)^2} = \frac{a_2}{d^2 + a_2(1-g_2)^2}$$

$$1 = \frac{(1-g_2)a_2^2}{(1-g_2)^2 a_2^2 + d^2} + \frac{(1-g_1)a_1^2}{d^2 + (1-g_1)^2 a_1^2}$$

Substitution of the first for one of the denominators of the second gives

$$d^2 + (1-g_1)^2 a_1^2 = a_1^2(1-g_1) + a_1 a_2 (1-g_2)$$

and alternately

$$d^2 + (1-g_2)^2 a_2^2 = a_2^2(1-g_2) + a_1 a_2 (1-g_1),$$

which can also be written:

$$\left.\begin{aligned} d^2 &= a_1^2 g_1 (1-g_1) + a_1 a_2 (1-g_2) \\ d^2 &= a_2^2 g_2 (1-g_2) + a_1 a_2 (1-g_1), \end{aligned}\right\} (3)$$

and result in the following quadratic expression for $a_1/a_2$

$$\left(\frac{a_1}{a_2}\right)^2 g_1(1-g_1) - \left(\frac{a_1}{a_2}\right)(g_2 - g_1) - g_2(1-g_2) = 0$$

The solution is

$$\frac{a_1}{a_2} = \frac{g_2}{g_1}$$

and when put into (3) yields

$$a_1 a_2 = \frac{d^2}{1 - g_1 g_2} .$$

Thus,

$$\frac{w_1}{w_2} = \sqrt{\frac{g_2}{g_1}} \quad , \text{ and } \quad w_1 w_2 = \frac{\left(\frac{\lambda d}{\pi}\right)}{\sqrt{1 - g_1 g_2}} .$$

# PUBLICATIONS

A. Rexroad, Harvey N., "Maximum Power Transfer Coefficient Between Two Confocal Apertures", Bull. Am. Phys. Soc. 14, No. 4, (1969), p. 619, paper FC9.

B. Rexroad, Harvey N., and B. J. Henderson, "Maximum Power-Transfer Coefficient Between Two Confocal Apertures", J. Opt. Soc. Am. 59, (1969), pp. 1415-1421.

C. Rexroad, Harvey N., "Laser Optics", A set of notes prepared for the Optical Systems Branch of NASA as described in III-b above.

D. "Graphical Representation of Gaussian Beam Optical Systems". Preparation of a manuscript is about half complete. One more reference article must be obtained before it is sent to Applied Optics. Basis for the article is the material of Chapter VII, Section C of notes (see III-b above).

E. The material described in Sections f, g, and h of the Technical Report are the possible basis for additional publications. The thin film work of Section (d) is in early stages of development, but could result in publications.

## V. PLANS AND RECOMMENDATIONS

### (a) Technical Work and Experiments

One result of this grant has been the training of the principal investigator in the field of Laser Optics. He is very knowledgeable of the classical aspects of this subject, and, with a little effort, can interpret literature on the quantum field theory and statistical aspects. I hope you will feel free to use him as a consultant, if the need arises. By continuing work along these same lines, the knowledge and capabilities will be expanded. For the immediate future, theoretical projects outlined under III-e, III-f, and III-g will be completed. Plans are then to examine the details of basic interference experiments from the point of view of Quantum Electrodynamics.

### (b) Future Grants

I do not wish to propose another grant of this type at this time. After we get things going smoothly at this new University, and it becomes possible to buy released time for the principal investigator, I'll reconsider.

The one thing that is desperately needed is money to support students. In addition to the educational benefits to the students involved, they will provide some relief to my work load (and to that of other faculty members) so more of the future work described can be accomplished. I will call soon to explore the possibility of supporting one or two part-time student assistants over the next two years at a cost of about $3,000 per year for the two of them.

# LASER OPTICS

By

Harvey N. Rexroad

Compiled for NASA Optical Systems Division, Goddard Space Flight Center, Greenbelt Maryland as a part of the work sponsored by NASA Research Grant NGR 10 019 001 and through ASEE-NASA Faculty Fellowships in 1967 and 1968. The author is presently Professor and Chairman of the Department of Physics at Florida Technological University, Orlando, Florida 32816

Orlando (1970)

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# CHAPTER V

# WAVEGUIDE APPROACH

## A. BASIC EQUATIONS

In Appendix A it was shown that the vector Helmholtz equation $\nabla^2\vec{E} + K^2\vec{E} = 0$, can be written in a form that is more suitable for a beam generally directed along the $\pm z$ axis, and with a $z$ dependence of $e^{\pm ihz}$. These results beginning with Equation A38 are

$$\boxed{\begin{aligned} &\nabla_t^2\vec{E} + \sigma^2\vec{E} = 0 \\ \text{and}\quad &\nabla_t^2\vec{B} + \sigma^2\vec{B} = 0\,, \\ \text{where}\quad &\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2}\,, \\ &\sigma = \sqrt{k^2-h^2} \\ \text{and}\quad &k = \frac{2\pi}{\lambda}\sqrt{\epsilon\mu} \end{aligned}} \tag{5.1}$$

The components transverse to the $z$ axis were given by:

$$\boxed{\begin{aligned} B_t &= \frac{1}{k^2-h^2}\left[\nabla_t\left(\frac{\partial B_z}{\partial z} + i\frac{\epsilon\mu\omega}{c}\hat{z}\times\nabla_t E_z\right)\right] \\ E_t &= \frac{1}{k^2-h^2}\left[\nabla_t\left(\frac{\partial E_z}{\partial z} - \frac{i\omega}{c}\hat{z}\times\nabla_t B_z\right)\right] \end{aligned}} \tag{5.2}$$

The boundary conditions at a perfectly conducting surface with normal $\hat{n}$ are:

$$\boxed{\hat{n} \times \vec{E} = 0 \quad \text{and} \quad \hat{n} \cdot \vec{B} = 0} \; . \tag{5.3}$$

B. SOLUTION IN CARTESIAN COORDINATES

In Carlesian coordinates a transverse component of electric field is given by (From 5.1):

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \sigma^2 E_x = 0 \; , \tag{5.4}$$

and has a solution

$$\boxed{\begin{aligned} E_x &= A \sin(k_x x + b_x) \sin(k_y y + b_y) e^{\pm i h z} \\ &\text{where } k_x^2 + k_y^2 = \sigma^2 \end{aligned}} \tag{5.5}$$

The general solution may then be obtained by a superposition of expressions of this type, where all values of $k_x$, $k_y$ and $k_z$ consistent with

$$\boxed{k_x^2 + k_y^2 + h^2 = k^2} \tag{5.6}$$

are possible. In waveguides the boundary conditions restrict $k_x$ and $k_y$ to one or another of certain eigen values.

From Equation (5.2) it can be seen that TEM modes are only possible if

$$\boxed{\begin{array}{l} k^2 - h^2 = \sigma^2 = 0 \\ \qquad \text{TEM modes} \end{array}} \quad . \tag{5.7}$$

If waves are very nearly TEM waves, as is the case for the optical resonator equations derived using Fresnel zone diffraction formula and the small angle approximation, then $\sigma$ is almost zero. For such waves, we put

$$\boxed{h = \sqrt{k^2 - \sigma^2} \approx k - \frac{\sigma^2}{2k}} \tag{5.8}$$

The general solution using (5.5) then becomes:

$$\boxed{\begin{array}{l} E_x = e^{\pm i k z} \displaystyle\int_0^\infty A(\sigma) \sin(k_x x + b_x) \sin(k_y y + b_y) e^{\mp i \frac{\sigma^2}{2k} z} \sigma \, d\sigma \\ \sigma^2 = k_x^2 + k_y^2 \approx 0 \end{array}} \tag{5.9}$$

The factor $\sigma$ has been inserted to account for the degeneracy. To see how this comes about, notice that in a rectangle of dimensions $L \times L$, the solutions must have the form $\sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right)$, so that they vanish at the boundaries, $x = 0$ & $L$, and $y = 0$ & $L$. Thus, $k_x = \frac{n_x \pi}{L}$, and $k_y = \frac{n_y \pi}{L}$, where $n_x$ and $n_y$ are integers. Since

$$\sigma^2 = k_x^2 + k_y^2 = \frac{\pi}{L}(n_x^2 + n_y^2);$$

$$\sigma^2 = \frac{\pi}{L} n^2 \; ; \text{ where } n^2 = n_x^2 + n_y^2 ,$$

the number of possible values of $n$ determines the number of solutions having the same $\sigma$. If we plot $n_y$ against $n_x$, each point corresponds to a possible $(n_x, n_y)$. These points

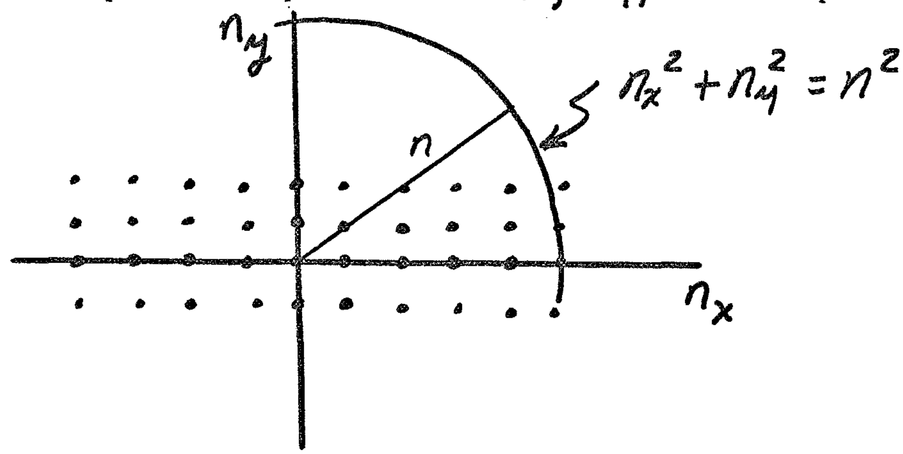


Figure 5.1. Illustrating the number of states with the same $\sigma^2$ = the number of points along a circle.

are uniformly distributed over the plane, and each occupies a unit area. The points on the circle $n_x^2 + n_y^2 = n^2$ are those corresponding to the same $\sigma^2 = \frac{\pi}{L} n^2$. Thus, the number, $N$, of solutions with $\sigma^2 \leq \frac{\pi}{L} n^2$ is the area of the circle $\pi n^2 = N$. Then, $dN = 2\pi n\, dn = L\, \sigma\, d\sigma$ is the number with $\sigma$ between $\sigma$ and $\sigma + d\sigma$, and is proportional to $\sigma\, d\sigma$.

## C. COMPARISON WITH QUANTUM MECHANICS

Substitution of

$$E_x = \psi(x,y,z)\, e^{ihz} \tag{5.10}$$

into the Helmholtz equation, $(\nabla^2 + k^2)E_x = 0$, gives:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + 2ih\frac{\partial \psi}{\partial z} + (k^2 - h^2)\psi = 0 .$$

If $k^2 \approx h^2$, we can again put $h = \sqrt{k^2 - \sigma^2} \approx k - \frac{\sigma^2}{2k}$ to obtain:

$$\nabla_t^2 \psi + \frac{\partial^2 \psi}{\partial z^2} + 2ik\frac{\partial \psi}{\partial z} - 2i\sigma^2\frac{\partial \psi}{\partial z} + \left(\sigma^2 - \frac{\sigma^4}{4k^2}\right)\psi = 0. \quad (5.11)$$

Now suppose that $\psi$ is a very slowly varying function of $z$. For such a wave, the $\frac{\partial^2 \psi}{\partial z^2}$ term is negligible compared to the $2ik\frac{\partial \psi}{\partial z}$ term. Also, since $\sigma$ is small, the last two terms are of no importance. Equation (5.10) then reduces to

$$\boxed{\nabla_t^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0} . \quad (5.12)$$

Now consider the quantum mechanical description of a particle that has been directed along the $z$ axis with speed $v$. In the Schrödinger equation, $\nabla^2 \psi + \frac{2mi}{\hbar}\frac{\partial \psi}{\partial t} = 0$, we can put $\frac{\partial \psi}{\partial t} = v\frac{\partial \psi}{\partial z}$, $\lambda = \frac{h}{mv}$ and $k = \frac{2\pi}{\lambda}$. Again, if $\frac{\partial^2 \psi}{\partial z^2}$ is neglected compared to $2ik\frac{\partial \psi}{\partial z}$, we obtain

$$\boxed{\nabla_t^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0} , \quad (5.13)$$

which is identical to Equation (5.12) above. Thus, the analogy between the spreading of the transverse cross section of a light beam with that of a wave packet representing a particle appears rather strong. Although approximations were used to show this analogy, it is not obvious that they are essential. It should be remarked that Maxwell's equations have the same form in Classical Electrodynamics as in Quantum Electrodynamics. The intensity patterns for such problems turn out to be identical. This connection between classical and quantum electrodynamics is elucidated in "Quantum Mechanics" by Leonard Schiff (McGraw-Hill) pp. 390-95.

# CHAPTER VI

# PARTIALLY COHERENT POLYCHROMATIC LIGHT

## A. INTRODUCTION*

If properly applied, the principles of physics and mathematics that were known as early as 1963, and possibly as soon as 1955, appear to agree with experiments thus far performed in this field. It is, however, difficult to glean from present treatments the pertinent points needed to answer many basic questions concerning the coherence of light beams, interference patterns, and the precise conditions required for observing beats between two signals.

The purpose of this chapter is first to collect the fundamental ideas into a reasonably complete, but nonetheless, terse, package; and secondly to treat some of the problems of interest in laser work. The new things that have evolved or things that have been clarified are:

(1) Monochromatic light has been assumed for much of the work in the earlier chapters. It is known, but not often emphasized, that it is really the Fourier transforms of the field components that obey the basic diffraction integral equation. Technically, it follows that the foregoing integral equations for laser modes really apply to the Fourier transforms.

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*A large part of the introductory material of this section follows the treatment in Born and Wolf[4]. The best and most complete treatment to date is that of Mandel and Wolf[13]. A fairly recent book by Klauder and Sudarshan[14] is also highly recommended but is more concerned with quantum aspects that are not treated in the present work.

(2) It is also known that the Fourier transform of the mutual coherence function $\tilde{\Gamma}$ (a tilda denotes Fourier transform) obeys a wave equation. Propagation laws and a diffraction integral can, consequently, be obtained for $\tilde{\Gamma}$. An integral equation for laser modes can be obtained for $\tilde{\Gamma}$ that is analagous to Equation 45. The upshot of this revelation is that idealized laser light has an almost complete mutual space coherence as well as the better recognized temporal, and corresponding long coherence length. The latter, of course, arises from the very narrow frequency spread that is achieved with the combination of a high resonator Q along with the superregenerative effect of the lasing medium. (Coherence lengths of a few hundred kilometers have been achieved).

and

(3) The important factor for obtaining beats between two signals is temporal coherence and not merely the narrowness of the frequency spread, as is sometimes wrongly assumed. Beats are possible when $\Delta f \gg f_2 - f_1$ provided the frequency components have the required temporal coherence. On the other hand, the condition $f_2 - f_1 \gg \Delta f$ assures the required temporal coherence.

The discussion of this chapter makes use of Classical Electrodynamics. Although a different viewpoint can be gained using Quantum Electrodynamics and a number representation of a boson field, so far as the author knows, the experimental results predicted by the two theories for the experiments considered are in agreement. To answer questions concerning the actual thermodynamic equilibrium density of photons with certain energy and phase, for example, it is necessary to resort to the quantum theory.

## B. REPRESENTATION OF ACTUAL POLYCHROMATIC FIELDS

At a given point $\vec{r}$ in space and instant of time, $t$, the light coming from a source is generally made up of light from a very large number of different sources, each having atomic dimensions. By the *principle of superposition,* a Cartesian component (such as $E_x$ or $E_y$) of field, denoted by $\psi^{(r)}$, is obtained by adding together the contributions from each oscillator.

$$\psi^{(r)}(t) = \sum_{\nu} a_{\nu} \cos(\varphi_{\nu} - 2\pi\nu t), \qquad (50)$$

where each frequency, $\nu$, has a phase $\varphi_{\nu}$ and amplitude, $a_{\nu}$.
If $a(\nu)d\nu$ represents the amplitude for waves with frequency between $\nu$ and $\nu + d\nu$, this equation can be written:

$$\left.\begin{aligned} \psi^{(r)}(t) &= \int_0^{\infty} a(\nu)\cos(\varphi(\nu) - 2\pi\nu t)\, d\nu, \text{ or} \\ \psi^{(r)}(t) &= \mathcal{R}e \int_0^{\infty} a(\nu) e^{i(\varphi(\nu) - 2\pi\nu t)} d\nu . \end{aligned}\right\} \qquad (51)$$

It is convenient to define $\Psi(t)$ as:

$$\boxed{\begin{aligned} &\Psi(t) \equiv \int_0^{\infty} \left(a(\nu)e^{i\phi(\nu)}\right) e^{-i2\pi\nu t} = \int_0^{\infty} \tilde{\psi}(\nu) e^{-i\omega t} d\nu \\ &\omega = 2\pi\nu \quad ; \quad \tilde{\psi} = a e^{i\varphi} \end{aligned}} \qquad (52)$$

Clearly, $\Psi(t)$ is uniquely specified by the actual signal $\psi^{(r)}(t)$, as is $\psi^{(i)}(t)$, where

$$\boxed{\Psi(t) = \psi^{(r)}(t) + i\psi^{(i)}(t)} \qquad (53)$$

It is a mathematical convenience that leads us to include both amplitude and phase in one symbol

$$\boxed{\tilde{\psi}(\nu) \equiv a(\nu)\, e^{i\Phi(\nu)}} \quad . \qquad (54)$$

From a strict mathematical point of view, we could begin with the fact that any real sectionally continuous function, such as $2\,\psi^{(r)}(t)$ , can be represented by the FOURIER INTEGRAL THEOREM*

$$\boxed{\begin{array}{l} 2\psi^{(r)}(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \tilde{\psi}(\omega)\, e^{-i\omega t}\, d\omega \\ \omega = 2\pi\nu \end{array}} \quad . \qquad (55)$$

Writing this as the sum of two integrals,

$$2\psi^{(r)}(t) = \int_0^{\infty} \tilde{\psi}(\nu)\, e^{-i\omega t}\, d\nu + \int_0^{\infty} \tilde{\psi}(-\nu)\, e^{+i\omega t}\, d\nu \, ,$$

and recalling that $\psi^{(r)}$ is real, it is apparent that the second of the integrals must be the complex conjugate of the first.

Therefore,

$$\boxed{\tilde{\psi}(-\nu) = \tilde{\psi}^{*}(+\nu)} \qquad (56)$$

and,

$$\boxed{\psi^{(r)}(t) = \mathrm{Re} \int_0^{\infty} \tilde{\psi}(\nu)\, e^{-i\omega t}\, d\nu} \, , \qquad (57)$$

where Re means "real part of". Again we define $\psi(t)$ and $\psi^{(i)}(t)$ so that

$$\psi(t) = \psi^{(r)}(t) + i\,\psi^{(i)}(t) \quad ,$$

and

$$\psi(t) = \int_0^{\infty} \tilde{\psi}(\nu)\, e^{-i\omega t}\, d\nu \quad . \qquad (58)$$

*Throughout this treatise a *tilda* will be used to denote the FOURIER TRANSFORM, and an asterisk for the complex conjugate.

Because of the reality requirement for $\psi^{(r)}(t)$, not only $\psi^{(r)}(t)$, but also $\psi^{(i)}(t)$ and $\psi(t)$ <u>are completely specified by the positive frequencies</u>.

The frequency spectrum of a real signal is found with well established methods. Equation (55) is multiplied by $e^{+i\omega' t}dt$ and integrated from $-\infty$ to $+\infty$:

$$2\int_{-\infty}^{\infty}\psi^{(r)}(t)\,e^{i\omega' t}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\psi}(\nu)\,e^{-i2\pi(\nu-\nu')t}d\nu\,dt,$$

which because of (55) is:

$$= \int_{-\infty}^{\infty}\tilde{\psi}(\nu)\,\delta(\nu-\nu')\,d\nu = \tilde{\psi}(\nu').$$

Thus,

$$\boxed{\tilde{\psi}(\nu) = 2\int_{-\infty}^{\infty}\psi^{(r)}(t)\,e^{i2\pi\nu t}dt} \qquad (59)$$

, or

from (58)

$$\boxed{\tilde{\psi}(\nu)_{\substack{\text{POSITIVE}\\ \text{FREQUENCIES}}} = \int_{-\infty}^{\infty}\psi(t)\,e^{i2\pi\nu t}dt \quad ; \quad \tilde{\psi}(-\nu) = \tilde{\psi}^{*}(\nu)}.$$

<u>*Amplitude Modulation.*</u> As an example, a 100% amplitude modulated carrier has two positive frequency components. That is:

$$\psi^{(r)}(t) = 2\cos\delta t\,\cos\Omega t = \cos(\Omega+\delta)t + \cos(\Omega-\delta)t,$$

so that the two positive frequency components are $\Omega+\delta$ and $\Omega-\delta$.

. Using (59), we obtain,

$$\tilde{\psi}(\nu) = 2\int_{-\infty}^{\infty}\left[\frac{e^{i(\Omega+\delta)t}+e^{-i(\Omega+\delta)t}}{2}+\frac{e^{i(\Omega-\delta)t}+e^{-i(\Omega-\delta)t}}{2}\right]e^{i2\pi\nu t}\,d$$

$$\tilde{\psi}(\nu) = \delta(\omega-(\Omega+\delta))+\delta(\omega+(\Omega+\delta)) + \delta(\omega-(\Omega-\delta)) + \delta(\omega+(\Omega-\delta)) .$$

Examples of *frequency* and *phase modulation* are given in Appendix I. A useful relation for combining any two sinusoidal signals is given in Appendix J.

Summary

The *superposition* of signals of different frequencies and amplitudes

$$\boxed{\begin{aligned}&\psi^{(r)}(t) = \int_0^{\infty} a(\nu)\cos(\varphi(\nu)-2\pi\nu t)\,d\nu = \mathrm{Re}\int_0^{\infty}\tilde{\psi}(\nu)e^{-i\omega t}\,d\nu,\\ &\tilde{\psi}(\nu) = a(\nu)e^{i\varphi(\nu)}, \quad \omega = 2\pi\nu\end{aligned}} \quad (6($$

has lead to:

$$\boxed{\begin{aligned}&\psi = \psi^{(r)} + i\,\psi^{(i)}, \qquad \tilde{\psi}(-\nu) = \tilde{\psi}^{*}(+\nu)\\ &\tilde{\psi}(\nu) = 2\int_{-\infty}^{\infty}\psi^{(r)}(t)e^{i\omega t}\,dt \;; \quad \tilde{\psi}(\nu)_{\substack{\text{POSITIVE}\\ \text{FREQUENCIES}}} = \int_{-\infty}^{\infty}\psi(t)e^{i\omega t}\,dt\\ &\psi(t) = \int_0^{\infty}\tilde{\psi}(\nu)e^{-i\omega t}\,d\nu\end{aligned}} \quad (61)$$

Using

$$\psi^{(r)}(t) = \int_0^\infty a(\nu) \cos\left[\varphi(\nu) - \omega t\right] d\nu$$

and

$$\psi^{(i)}(t) = \int_0^\infty a(\nu) \sin\left[\varphi(\nu) - \omega t\right] d\nu \qquad (61a)$$

along with

$$a(-\nu) = a(+\nu) \quad \text{and} \quad \varphi(-\nu) = -\varphi(+\nu),$$

the useful relations:

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\psi^{(r)}(t)\right]^2 dt &= \int_{-\infty}^{\infty}\left[\psi^{(i)}(t)\right]^2 dt = \frac{1}{2}\int_{-\infty}^{\infty}\psi(t)\,\psi^*(t)\,dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} a(\nu)^2 d\nu = \frac{1}{2}\int_{-\infty}^{\infty}\tilde{\psi}(\nu)\,\tilde{\psi}^*(\nu)\,d\nu = \int_0^{\infty}\tilde{\psi}(\nu)\,\tilde{\psi}^*(\nu)\,d\nu\end{aligned} \qquad (62)$$

can be derived.

## C. HILBERT TRANSFORM RELATIONS

Starting with the last of equations (61),

$$\psi(t) = \int_0^\infty a(\nu)\, e^{i(\varphi(\nu)-\omega t)}\, d\nu \;,$$

we define a function $\psi(z)$ of the complex variable

$$z = t + i\tau$$

as

$$\psi(z) \equiv \int_0^\infty a(\nu)\, e^{i(\varphi-\omega z)}\, d\nu = \int_0^\infty a(\nu)\, e^{+\omega\tau}\, e^{i(\varphi-\omega t)}\, d\nu \, .$$

Clearly, this function, and also $\frac{\psi(z)}{z-t}$, satisfied the Cauchy-Riemann equations, $\frac{\partial\psi^{(r)}}{\partial t} = \frac{\partial\psi^{(i)}}{\partial\tau}$ and $\frac{\partial\psi^{(r)}}{\partial\tau} = -\frac{\partial\psi^{(i)}}{\partial t}$, and is therefore analytic. This permits the use of the Cauchy integral theorem

$$\oint \frac{\psi(z')dz'}{z'-t} = 2\pi i \sum \text{Residues} \;,$$

where the path of integration has been selected to be the lower half plane of Figure 7. Thus,

$$\int_R^{t+\epsilon} \frac{\psi(t')dt'}{t'-t} + \int_{2\pi}^{\pi} \frac{\psi(t+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta} d\theta + \int_{t-\epsilon}^{-R} \frac{\psi(t')dt'}{t'-t} + \int_C \frac{\psi(z')dz'}{z'-t} = 2\pi i \sum_k \mathcal{R}_k$$

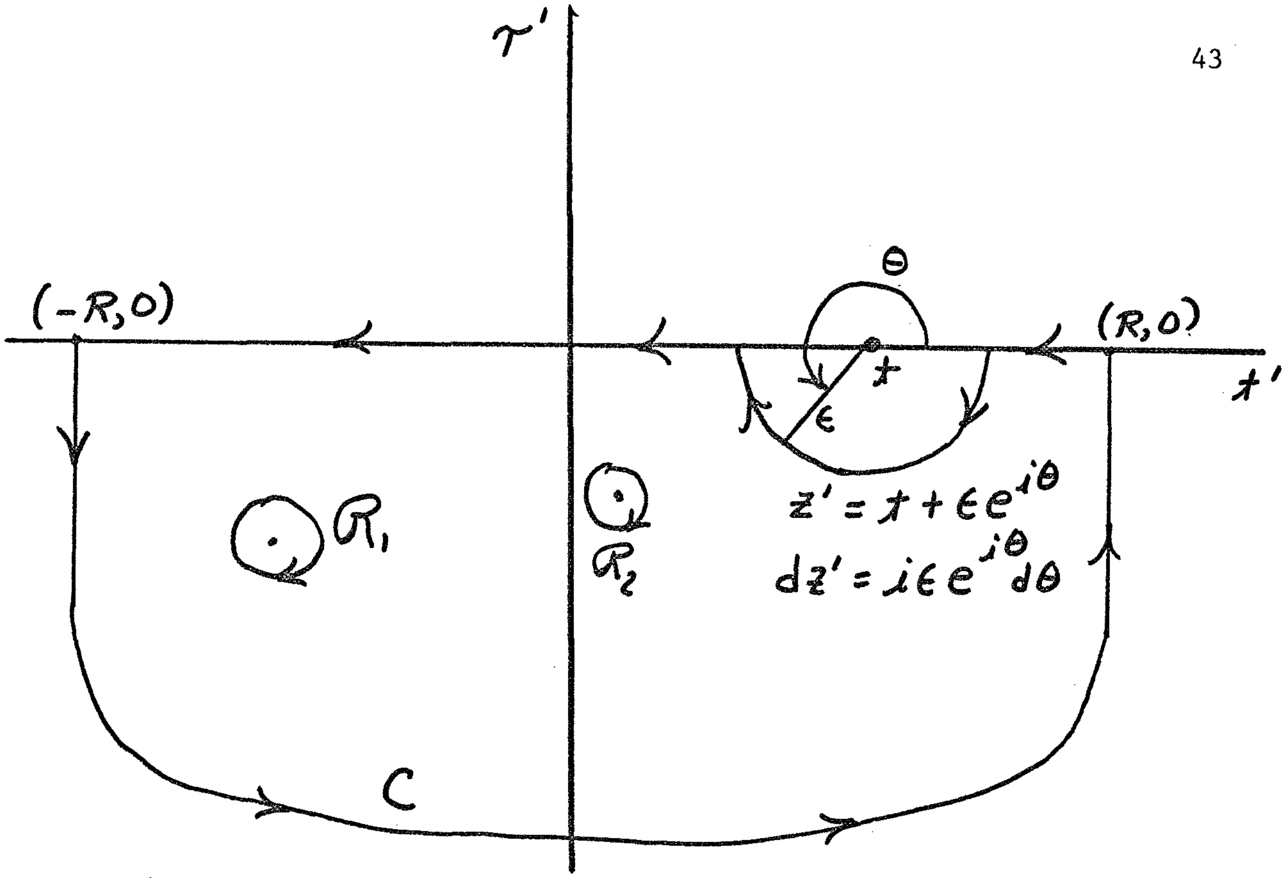



Figure 7. Path of integration used to derive the HILBERT TRANSFORM (or KRAMERS KRONIG) relations for $\psi^{(r)}$ and $\psi^{(i)}$.

is to be evaluated as $R \to \infty$, $\epsilon \to 0$, and $C$ enlarges to cover the entire half plane. Because of the large magnitude of $z$ in the denominator near the real axis at $R = \pm\infty$, and of the factor $e^{-|\tau|\omega}$, that vanishes when $|\tau| \to \infty$, the fourth of these integrals vanishes. The first and third give the Cauchy principle value (which we denote by P). Thus,

$$-P\int_{-\infty}^{\infty} \frac{\psi(t')\,dt'}{t' - t} - i\pi\,\psi(t) = 2\pi i \sum_k \mathcal{R}_k \quad .$$

If there are no poles in the lower half plane, separation of the real and imaginary parts gives:

$$\boxed{\begin{aligned} \psi^{(r)}(t) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\psi^{(i)}(t')\,dt'}{t'-t} \\ \text{and} \qquad \psi^{(i)}(t) &= +\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\psi^{(r)}(t')\,dt'}{t'-t} \end{aligned}} \qquad (63)$$

and $\psi^{(r)}$ and $\psi^{(i)}$ are said to satisfy the HILBERT transform relations. These relations hold true for any analytic function having no singularities in the lower half complex plane, and whose integral over the curve $C$ vanishes as $R \to \infty$. When applied to magnetic susceptibility, $\chi = \chi' - i\chi''$, these are known as the *KRAMERS-KRONIG relations*.

## D. COHERENCE (INTRODUCTION AND DEFINITIONS)

The result of the *superposition* of amplitures according to Equation (50) or (51) is, at least for a sufficiently short time, a sinusoidal function having some average frequency and complex amplitude. In actual cases this average frequency and complex amplitude changes with time not only in a regular way which one obtains if the components superposed remain fixed, but also because the set of oscillators contributing to the field will vary in a random way. If the frequency and complex amplitude do not change appreciably in a time $\Delta t$, we refer to the signal as having a *coherence time* of $\Delta t$. The *coherence length* is the distance the wave travels in the coherence time: $\Delta l = c\,\Delta t$. When a screen is illuminated by an extended incoherent source, the fluctuations at two points on the screen will be correlated provided that for all source points the path difference does not exceed the coherence length $c\,\Delta t$. We are thus lead to the concept of a *region of coherence* around any point in a wave field.

The physical quantities that can be observed when random fluctuations occur usually involve averages over periods of time. The *mutual coherence function,* $\Gamma$, and the *complex degree of coherence,* $\gamma$, which we now define, are the important parameters in many experiments that result from averaging over time. For the fields $\psi(P_1, t_1)$ and $\psi(P_2, t_2)$ at points $P_1$ and $P_2$ in space (or at the same point but with different optical paths or past histories denoted by $P_1$ and $P_2$), we define:

$$\Gamma(P_1 P_2 t_1, t_2) = \Gamma(P_1 P_2 \tau) = \Gamma_{12}(\tau) \equiv \langle \psi(P_1 t_1)\psi^*(P_2 t_2) \rangle$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \psi(P_1, t_2+\tau)\psi^*(P_2, t_2)dt_2, \quad (64)$$
$$\text{where} \quad \tau = t_1 - t_2$$

In practical calculations, the field quantities can be defined as zero at times outside the range $2T$. In this way, the limits of integration can be extended from $-\infty$ to $\infty$ so that the FOURIER integral theorem can be used. In most cases, it can be assumed that the ensemble is *stationary*, which means that the average does not depend on the origin of time. That is

$$\Gamma_{12}(\tau) = \langle \psi(P_1, t+\tau)\psi^*(P_2, t) \rangle$$
$$= \langle \psi(P_1, \tau)\psi^*(P_2, 0) \rangle \; ; \quad \text{stationary ensemble} \quad (65)$$

The *mutual coherence function* is called the *autocorrelation function* when $P_1 = P_2$, and the *cross correlation function* when $P_1$ and $P_2$ are different. The mutual coherence function at a point $P$ is proportional to the average intensity. Some authors, in fact, write

$$I(P) \equiv |\Gamma_{11}(0)| \quad (66)$$

Here we sometimes put

$$\Gamma(P_1 P_1\, 0) = \Gamma(\vec{r}_1\, \vec{r}_1\, 0) = \Gamma_{11}(0) = \Gamma_1(0)\,. \tag{67}$$

The *complex degree of coherence* is then defined:

$$\boxed{\gamma_{12}(\tau) \equiv \frac{\Gamma_{12}(\tau)}{\sqrt{|\Gamma_{11}(0)|\,|\Gamma_{22}(0)|}}}\,. \tag{68}$$

Schwarz's inequality, $\int |f|^2 dt \int |g|^2 dt \geq \left| \int f^* g\, dt \right|$,

then guarantees that

$$\boxed{0 \leq \left| \gamma_{12}(\tau) \right| \leq 1}\,. \tag{69}$$

## E. INTERFERENCE; YOUNG'S EXPERIMENT

Consider two signals $\psi(P_1 t_1)$ and $\psi(P_2 t_2)$ with mutual coherence $\Gamma_{12}(\tau)$. If these signals are projected over equal optical paths to arrive at a common point Q, the total signal at $Q$, by the principle of superposition, is:

$$\psi(Q) = \psi(P_1 t_1) + \psi(P_2 t_2),$$

so that

$$\begin{aligned} I(Q) &= \langle \psi(Q,t)\, \psi^*(Q\, t) \rangle \\ &= \langle \psi(P_1 t_1)\psi^*(P_1\, t_1)\rangle + \langle \psi(P_2 t_2)\, \psi^*(P_2\, t_2)\rangle \\ &\quad + \langle \psi(P_1 t_1)\psi^*(P_2 t_2)\rangle + \langle \psi(P_2 t_2)\psi^*(P_1 t_1)\rangle . \end{aligned}$$

The imaginary parts of the last two terms cancel one another, leaving

$$I(Q) = \Gamma_{11}(0) + \Gamma_{22}(0) + 2\, Re\, \Gamma_{12}(\tau)$$

or

$$\boxed{I(Q) = I_1 + I_2 + 2\sqrt{I_1 I_2}\;\; Re\, \gamma_{12}(\tau)} \quad (70)$$

This result is quite general and points out the importance of the *degree of coherence* for determining the intensities in interference patterns.

Young's experiment illustrated in Figure 8 is a good example of the use of these ideas. In this experiment $|\gamma_{12}|$ may be less than unity because light arriving at Q over the different optical paths $P_1$ and $P_2$ was emitted from S at different times. In fact,

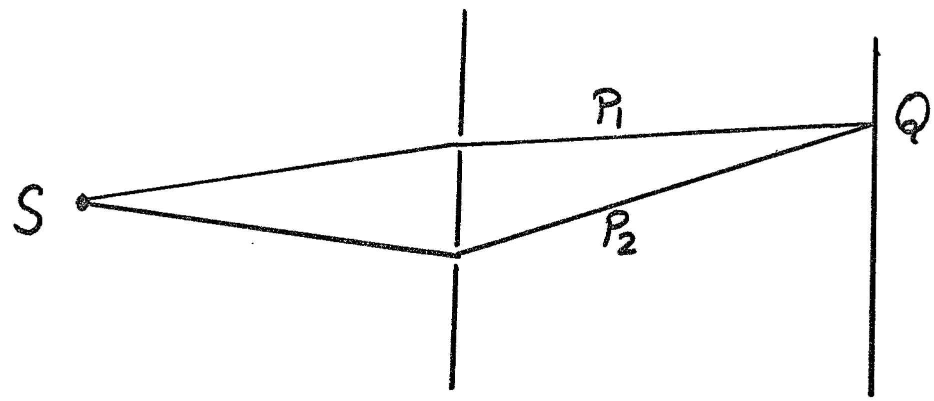



Figure 8. Experimental arrangement for Young's experiment.

we expect that if the difference in path length $c\,(t_2 - t_1)$ is greater than $c$ times the coherence time of the source, visible interference fringes will not be possible. The *visibility of fringes* is defined as

$$\boxed{\mathcal{V} \equiv \frac{I_{max} - I_{min}}{I_{max} + I_{min}}} \qquad (71)$$

The general appearance of $\Gamma$ can be obtained by combining Equations (52) and (64), which gives

$$\Gamma_{12}(t_1,t_2)=\frac{1}{2T}\int_{-T}^{T}dt\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_1(\nu)a_2(\nu')e^{i[\varphi_1(\nu)-\varphi_2(\nu')]}e^{-i2\pi(\nu t_{20}-\nu' t_{10})}e^{-i2\pi(\nu-\nu')t}\,d\nu\,d\nu'$$

$$\text{where}\quad t_1=t_{10}+t\quad\text{and}\quad t_2=t_{20}+t\,.$$

The last factor, $e^{-i2\pi(\nu-\nu')t}$, is a rapidly oscillating function of time when $\nu$ is appreciably different from $\nu'$; and the time integration is zero for these conditions. If $\nu=\nu'$, the integral over time gives $2T$. Thus, very nearly

$$\Gamma_{12}(\tau)\cong\int_{-\infty}^{\infty}a_1(\nu)\,a_2(\nu)\,e^{i(\varphi_1(\nu)-\varphi_2(\nu))}e^{-i2\pi\nu\tau}\,d\nu\,;$$

where $t_2-t_1=t_{20}-t_{10}=\tau$.

If the amplitudes $a_1(\nu)$ and $a_2(\nu)$ are appreciable only in a narrow range between $\bar\nu-\frac{\Delta\nu}{2}$ and $\bar\nu+\frac{\Delta\nu}{2}$, it is convenient to put $\nu=\bar\nu+\nu'$, to get

$$\Gamma_{12}(\tau)\approx e^{-i2\pi\bar\nu\tau}\left[\int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}}a_1(\bar\nu+\nu')a_2(\bar\nu+\nu')e^{i[\varphi_1(\bar\nu+\nu')-\varphi_2(\bar\nu+\nu')]}e^{-i2\pi\nu'\tau}\,d\nu'\right].$$

In Young's experiment, the quantity $\tau$ is the time required to travel a few wavelengths, and has a typical size of $\frac{\bar{\lambda}}{c} = \frac{1}{\bar{\nu}}$. A typical magnitude of $2\pi\nu'\tau$ is then $2\pi\nu'/\bar{\nu}$ with an extreme value of $\pi\left(\frac{\Delta\nu}{\bar{\nu}}\right)$. If $\pi\left(\frac{\Delta\nu}{\bar{\nu}}\right) <<< 1$, so that $2\pi\nu'\tau$ can be replaced by zero over the range of integration, the quantity in brackets will be independent of $\tau$. If $\varphi_1 = \varphi_2$ it is also a real number. For $\varphi_1 = \varphi_2$ and a very sharp spectral line; and a long time coherence of the source:

$$\Gamma_{12} = \text{constant} \times e^{-i\bar{\omega}\tau}$$

$$\gamma_{12} = e^{-i\bar{\omega}\tau}$$

$$I(Q) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\bar{\omega}\tau).$$

If $I_1 = I_2$, the visibility becomes $\mathcal{V} = \cos(\bar{\omega}\tau)$.

If $\tau$ arises purely from a difference in path lengths, $\Delta l$,

$$\mathcal{V} = \cos\left(2\pi\frac{\Delta l}{\bar{\lambda}}\right).$$

More generally,

$$\Gamma_{12} = A(\tau)\, e^{i\Phi(\tau)}\, e^{-i\bar{\omega}\tau},$$

so that

$$I(Q) = I_1 + I_2 + 2\sqrt{I_1 I_2}\, A(\tau)\cos(\bar{\omega}\tau - \Phi(\tau)).$$

Other examples of interference with partially coherent light will be considered after we first examine the field equations and propagation properties of the mutual coherence function $\Gamma$.

## F. FIELD EQUATIONS FOR $\Gamma$

If the complete frequency spectrum at a point $\vec{r}$ in space of a component of field amplitude is known, that component can be found from Equation (52):

$$\psi(\vec{r},t) = \int_0^\infty \tilde{\psi}(\vec{r},\nu) e^{-i2\pi\nu t} d\nu. \tag{72}$$

In Equations (61a) $a(\vec{r},\omega)$ and $\varphi(\vec{r},\omega)$ must now be regarded as functions of position. It follows immediately from these equations that if $\psi^{(r)}$ satisfies the wave equation, so does $\psi^{(i)}$. Consequently, $\psi(\vec{r},t)$ satisfies the wave equation. Operation on Equation (72) with $\nabla^2$ gives

$$\nabla^2 \psi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{r},t) = 0$$

$$\int_0^\infty \left(\nabla^2 \tilde{\psi} + \left(\frac{2\pi\nu}{c}\right)^2 \tilde{\psi}\right) e^{-i2\pi\nu t} d\nu = 0,$$

or

$$\int_0^\infty (\nabla^2 + k^2)\tilde{\psi}(\vec{r},\nu) e^{-i2\pi\nu t} d\nu = 0.$$

If this is now multiplied by $e^{i2\pi\nu' t}dt$ and integrated between the limits $-\infty$ to $+\infty$, the Fourier integral theorem gives

$$\int_0^\infty (\nabla^2 + k^2)\tilde{\psi}(\vec{r},\nu)\,\delta(\nu'-\nu)\,d\nu = 0$$

Consequently,

$$\boxed{(\nabla^2 + k^2)\tilde{\psi}(\vec{r},\nu) = 0}. \tag{73}$$

For polychromatic light, it is strictly the Fourier transform, $\tilde{\psi}$, of transverse field components that satisfy the Helmholtz equation.

The mutual coherence function,

$$\Gamma(\vec{r}_1, t_1; \vec{r}_2, t_2) = \Gamma(\vec{r}_1, t+\tau; \vec{r}_2, t) = \Gamma(\vec{r}_1, \vec{r}_2, \tau),$$

satisfies a wave equation. If $\vec{r}_2$ is held fixed, and we operate with $\nabla_1^2$ the result is

$$\nabla_1^2 \Gamma = \frac{1}{2T}\int_{-\infty}^{\infty}\left[\nabla_1^2 \psi(\vec{r}_1, t_1)\right]\psi^*(\vec{r}_2, t_2)\, dt .$$

Since

$$\nabla_1^2 \psi(\vec{r}_1, t_1) = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t_1^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial \tau^2}, \text{ we obtain}$$

$$\nabla_1^2 \Gamma = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\left\{\frac{1}{2T}\int_{-T}^{T}\psi(\vec{r}_1, t+\tau)\,\psi^*(\vec{r}_2, t)\,dt\right\}$$

$$= \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial \tau^2} .$$

Thus,

$$\boxed{\left(\nabla_1^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right)\Gamma_{F2}(\vec{r}_1, \vec{r}_2, \tau) = 0} \tag{74}$$

The subscript $F2$ is used to emphasize the fact that one point is held fixed. In this equation $\Gamma_{F2}$ propagates according to a wave equation and takes different values at points $\vec{r}_1$ and time differences $\tau$. Because rapid time variations of light waves can not be observed, this description in terms of $\tau = t_1 - t_2$ is especially useful. If point $\vec{r}_1$, is held fixed while $\vec{r}_2$ varies, a similar result is obtained:

$$\boxed{\left(\nabla_2^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right)\Gamma_{F1}(\vec{r}_1 \vec{r}_2 \tau) = 0} \tag{75}$$

The frequency spectrum of $\Gamma$ can be treated in the same way as that of $\psi$.. $\tilde{\Gamma}$ is defined through

$$\boxed{\Gamma(\vec{r}_1 \vec{r}_2 \tau) = \int_0^{\infty}\tilde{\Gamma}(\vec{r}_1 \vec{r}_2 \nu)\, e^{-i2\pi\nu\tau}\, d\nu} \tag{76}$$

By repeating the procedure used to obtain (73) from (72) and the wave equation, it can be shown that both $\tilde{\Gamma}_{F1}$ and $\tilde{\Gamma}_{F2}$ satisfy the Helmholtz equation:

$$\boxed{\begin{aligned} \left(\nabla_1^2 + k^2\right)\tilde{\Gamma}_{F2}(\vec{r}_1, \vec{r}_2, \nu) = 0 \\ \left(\nabla_2^2 + k^2\right)\tilde{\Gamma}_{F1}(\vec{r}_1, \vec{r}_2, \nu) = 0 \end{aligned}} \qquad (77)$$

## G. PROPAGATION OF MUTUAL COHERENCE

If use is made of the fact that $\Gamma$ satisfies the wave equations (74) and (75), the mutual coherence between two field points $\vec{r}_1$ and $\vec{r}_2$ can be computed in terms of the values of $\Gamma$

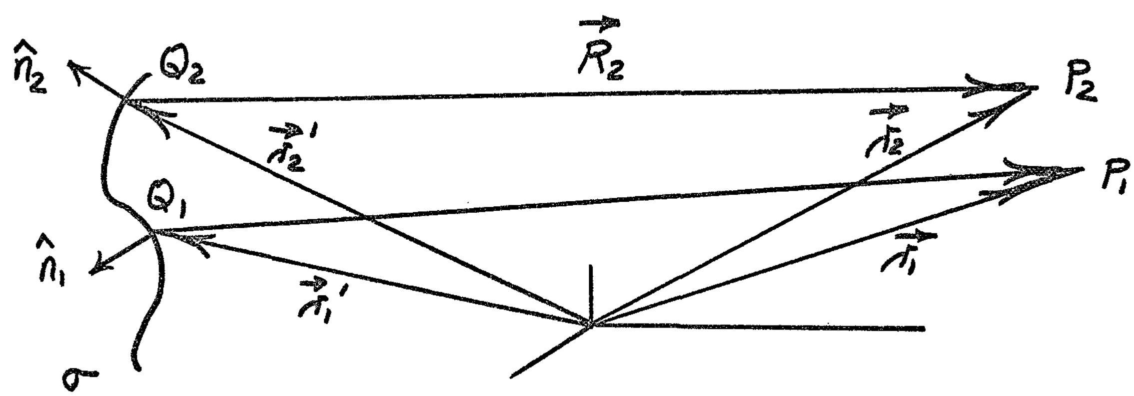



Figure 9. Definitions of the various vectors used in finding $\Gamma(P_1 P_2 \tau)$ in terms of its values $\Gamma(Q_1 Q_2 \tau)$ on source $\sigma$.

on some surface $\sigma$. The solutions of Equations (74) and (75) can be expressed in terms of the sources which leads to Equation (4D). The Kirchhoff surface integral representation of (4D) is:

$$\psi^{(r)}(\vec{r},t) = \frac{1}{4\pi}\oint_S \hat{n}\cdot\left[\frac{\nabla'\psi^{(r)}(\vec{r}\,',t')}{R} - \frac{\hat{R}}{R^2}\psi^{(r)} - \frac{\hat{R}}{cR}\frac{\partial\psi^{(r)}}{\partial t'}\right]_{RET} dS' \tag{76}$$

Using this equation, $\Gamma^{(r)}(\vec{r}_1\,\vec{r}_2\,t_1\,t_2)$ can be expressed in terms of $\Gamma^{(r)}(\vec{r}_1\,'\,\vec{r}_2\,t_1,t_2)$. In the same way $\Gamma^{(r)}(\vec{r}_1\,'\,\vec{r}_2\,t_1\,t_2)$ can be expressed in terms of $\Gamma^{(r)}(\vec{r}_1\,'\,\vec{r}_2\,'\,t_1\,t_2)$. These equations are:

$$\Gamma^{(A)}(\vec{r}_1\vec{r}_2\,t_1 t_2) = \frac{1}{(4\pi)}\int_\sigma \hat{n}_1\cdot\left[\frac{\nabla_1'\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2\,t_1 t_2)}{R_1} - \frac{\hat{R}_1}{R_1^2}\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2\,t_1 t_2) - \frac{\hat{R}_1}{cR_1}\frac{\partial\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2\,t_1 t_2)}{\partial t_1}\right]_{t_1\ \mathrm{RET}} dS_1 \tag{77}$$

and

$$\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2\,t_1 t_2) = \frac{1}{4\pi}\int_\sigma \hat{n}_2\cdot\left[\frac{\nabla_2'\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2{}'\,t_1 t_2)}{R_2} - \frac{\hat{R}_2}{R_2^2}\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2{}'\,t_1\,t_2) - \frac{\hat{R}_2}{cR_2}\frac{\partial\Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2{}'\,t_1 t_2)}{\partial t_2}\right]_{t_2\ \mathrm{RET}} dS_2 \tag{78}$$

Substitution of (78) into (77) then gives the desired result:

$$\begin{aligned}\Gamma^{(A)}(\vec{r}_1\vec{r}_2\,t_1\,t_2) = \frac{1}{4\pi^2}\int_\sigma\int_\sigma dS_1 dS_2 \Bigg\{ &\frac{\hat{n}_1\cdot\nabla_1'(\hat{n}_2\cdot\nabla_2'\Gamma)}{R_1R_2} \\ &+ p_1p_2\left(\frac{\Gamma}{R_1R_2} + \frac{1}{c^2}\frac{\partial^2\Gamma}{\partial t_1 t_2}\right) - \frac{1}{R_1R_2}\left[p_1\hat{n}_2\cdot\nabla_2'\Gamma + p_2\hat{n}_1\cdot\nabla_1'\Gamma\right] \\ &- \frac{1}{cR_2}p_1\,\hat{n}_2\cdot\nabla_2'\frac{\partial\Gamma}{\partial t_1} - \frac{1}{cR_1}p_2\,\hat{n}_1\cdot\nabla_1'\frac{\partial\Gamma}{\partial t_2} \\ &+ \frac{p_1p_2}{c}\left(\frac{1}{R_2}\frac{\partial\Gamma}{\partial t_1} + \frac{1}{R_1}\frac{\partial\Gamma}{\partial t_2}\right)\Bigg\}_{\substack{\text{BOTH } t_1\,\&\,t_2\\ \text{RETARDED}}},\end{aligned} \tag{79}$$

where $\Gamma$ on the right side is

$$\left.\begin{aligned}\Gamma &= \Gamma^{(A)}(\vec{r}_1{}'\vec{r}_2{}'\,t_1\,t_2), \quad \text{and} \\ p_1 &\equiv \frac{\hat{n}_1\cdot\hat{R}_1}{R_1}\ ; \quad p_2 \equiv \frac{\hat{n}_2\cdot\hat{R}_2}{R_2},\end{aligned}\right\} \tag{80}$$

and the gradients are all source gradients.

The subscript "Ret" means that the entire expression must be evaluated at times $t_1 - \frac{R_1}{c}$ and $t_2 - \frac{R_2}{c}$. Thus,

$$\left[\Gamma^{(A)}(\vec{x}_1', \vec{x}_2', t_1, t_2)\right]_{\substack{t_1 \text{ RET} \\ t_2 \text{ RET}}} = \Gamma^{(A)}\left(\vec{x}_1', \vec{x}_2', t_1 - \frac{R_1}{c}, t_2 - \frac{R_2}{c}\right). \tag{81}$$

Again it is convenient to define

$$\tau \equiv t_1 - t_2 \quad , \quad \text{whereby}$$

$$\frac{\partial}{\partial t_1} = \frac{\partial}{\partial \tau} \qquad \text{and} \qquad \frac{\partial}{\partial t_2} = -\frac{\partial}{\partial \tau} \quad , \tag{82}$$

and to make the *stationary assumption* that $\Gamma^{(A)}$ is independent of the time selected for beginning the averaging process:

$$\left[\Gamma^{(A)}(\vec{x}_1', \vec{x}_2', t_1, t_2)\right]_{\substack{t_1 \& t_2 \\ \text{RET}}} = \Gamma^{(A)}\left(\vec{x}_1', \vec{x}_2', \tau - \frac{R_1 - R_2}{c}\right). \tag{83}$$

With these substitutions, Equation (79) becomes:

$$\begin{aligned}\Gamma^{(A)}(P_1 P_2 \tau) = \frac{1}{(4\pi)^2} \int_\sigma \int_\sigma dS_1\, dS_2 \Bigg\{ & \frac{\hat{n}_1 \cdot \nabla_1'(\hat{n}_2 \cdot \nabla_2' \Gamma)}{R_1 R_2} \\ & + p_1 p_2 \left(\frac{\Gamma}{R_1 R_2} - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial \tau^2}\right) - \frac{1}{R_1 R_2}\left(p_1 \hat{n}_2 \cdot \nabla_2' \Gamma + p_2 \hat{n}_1 \cdot \nabla_1' \Gamma\right) \\ & + \frac{p_2}{c R_1} \hat{n}_1 \cdot \nabla_1' \frac{\partial \Gamma}{\partial \tau} - \frac{p_1}{c R_2} \hat{n}_2 \cdot \nabla_2' \frac{\partial \Gamma}{\partial \tau} + \frac{p_1 p_2}{c}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\frac{\partial \Gamma}{\partial \tau} \Bigg\}_{\text{RET}}. \end{aligned} \tag{84}$$

where

$$p_i = \frac{\hat{n}_i \cdot \hat{R}_i}{R_i} \quad \text{and} \quad \Gamma = \Gamma^{(A)}\left(\vec{x}_1', \vec{x}_2', \tau - \frac{R_1 - R_2}{c}\right). \tag{85}$$

The methods of treating real signals that are summarized in Equations (60) and (61) can also be used for $\Gamma$. For this purpose, we put

$$\boxed{\begin{aligned} &\tilde{\Gamma}(\vec{r}_1\,\vec{r}_2\,\Omega) \equiv 2\int_{-\infty}^{\infty} \Gamma^{(r)}(\vec{r}_1\,\vec{r}_2\,\tau)\, e^{i\Omega\tau}\, d\tau \\ &\text{and} \\ &\Gamma(\vec{r}_1\,\vec{r}_2\,\tau) = \frac{1}{2\pi}\int_0^{\infty} \tilde{\Gamma}(\vec{r}_1\,\vec{r}_2\,\Omega)\, e^{-i\Omega\tau}\, d\Omega \end{aligned}} \tag{86}$$

Equation (84) can then be multiplied by $2e^{i\Omega\tau}d\tau$ and integrated from $-\infty$ to $\infty$. A parts integration can be used where derivatives with respect to $\tau$ occur. If we assume that $\Gamma$ & $\frac{\partial\Gamma}{\partial\tau}$ have zero values when $\tau = +\infty$ or $-\infty$, a condition we might call "the impossibility of perfect coherence", the result is:

$$\boxed{\begin{aligned} \tilde{\Gamma}(\vec{r}_1\,\vec{r}_2\,\Omega) = \frac{1}{(4\pi)^2}\int_\sigma\int_\sigma dS_1\,dS_2\, e^{iK(R_1-R_2)}\Bigg\{ &\frac{\hat{n}_1\cdot\nabla_1'(\hat{n}_2\cdot\nabla_2'\tilde{\Gamma})}{R_1R_2} \\ &+ p_1p_2\left(\frac{1}{R_1R_2}+K^2\right)\tilde{\Gamma} - \frac{p_1\,\hat{n}_2\cdot\nabla_2'\tilde{\Gamma}}{R_1\,R_2}(1-iKR_1) \\ &- \frac{p_2\,\hat{n}_1\cdot\nabla_1'\tilde{\Gamma}}{R_1\,R_2}(1+iKR_2) - \frac{iKp_1p_2}{c}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\tilde{\Gamma}\Bigg\}, \\ \text{where}\quad p_i = \frac{\hat{n}_i\cdot\hat{R}_i}{R_i}\,,\quad K = \frac{\Omega}{c}\,, \\ \text{and}\quad \tilde{\Gamma} = \tilde{\Gamma}(\vec{r}_1{}'\,\vec{r}_2{}'\,\Omega) \end{aligned}} \tag{87}$$

## H. FAR FIELD SOLUTION

If $R_1$ and $R_2$ are very large compared to the dimensions of the source, it is convenient to put

$$\left.\begin{aligned} \vec{R}_1 &= \vec{R}_1^{\,o} + \vec{\rho}_{\Delta 1} \\ \text{and} \qquad \vec{R}_2 &= \vec{R}_2^{\,o} + \vec{\rho}_{\Delta 2} \end{aligned}\right\} \tag{88}$$

and to let $(\theta_1, \varphi_1)$ and $(\theta_2, \varphi_2)$ be the spherical coordinates that specify the directions of the vectors $\vec{R}_1$ and $\vec{R}_2$ respectively. After integration over the small dimensions of $\vec{\rho}_{\Delta 1}$ and $\vec{\rho}_{\Delta 2}$, the various terms in (87) then have the form

$$T = \frac{f(\theta_1 \varphi_1 \theta_2 \varphi_2)\, e^{iK(R_1^o - R_2^o)}}{(R_1^o)^n (R_2^o)^n}$$

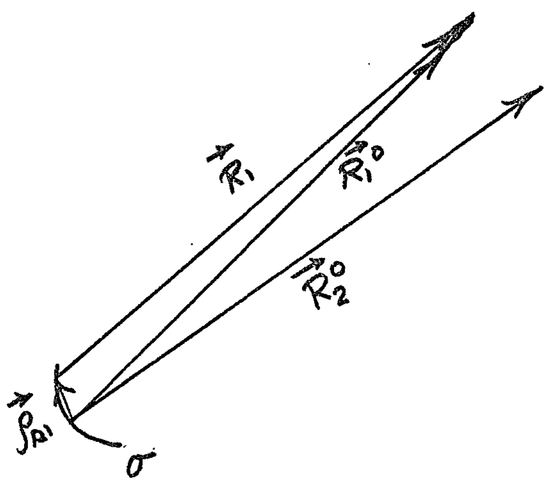



Figure 10. In the far field it is convenient to put $\vec{R}_1 = \vec{R}_1^{\,o} + \vec{\rho}_{\Delta 1}$, where $\vec{R}_1^{\,o}$ is a constant.

The gradient of this function then becomes:

$$-\nabla_1' T = +\nabla_1 T = \left(\hat{R}_1^o \frac{\partial}{\partial R_1^o} + \frac{\hat{\theta}_1}{R_1^o} \frac{\partial}{\partial \theta_1} + \frac{\hat{\Phi}_1}{R_1^o \sin\theta_1} \frac{\partial}{\partial \varphi_1}\right) T$$

$$= \left[ iKf \hat{R}_1^o \left(1 + \frac{ni}{KR_1^o}\right) + \frac{\hat{\theta}_1}{R_1^o} \frac{\partial f}{\partial \theta_1} + \frac{\hat{\Phi}_1}{R_1^o \sin\theta_1} \frac{\partial f}{\partial \varphi_1} \right] \frac{e^{iK(R_1^o - R_2^o)}}{(R_1)^n (R_2)^n}.$$

As $\mathcal{R}_1^o$ and $\mathcal{R}_2^o$ become large, the first term dominates the function, so that

$$\boxed{\begin{array}{l}\underline{\text{Far Field}}\\ \quad -\nabla_1'\tilde{\Gamma} = +\nabla_1\tilde{\Gamma} = iK\hat{R}_1\tilde{\Gamma}\\ \quad -\nabla_2'\tilde{\Gamma} = +\nabla_2\tilde{\Gamma} = -iK\hat{R}_2\tilde{\Gamma}\end{array}} \tag{89}$$

where we have used $\hat{R}_1 \approx \hat{R}_1^o$ and $\hat{R}_2 \approx \hat{R}_2^o$ . In the far field it is not surprising that the space gradients, or changes with distance, turn out to be along the direction of propagation.

## I. $\tilde{\Gamma}_{12}$ PROPAGATION IN AN OPTICAL RESONATOR

In an optical resonator, or an iterated system of phase transformers, the light is generally directed along the axis of the system and has the character of radiation in the far field of another source as illustrated in Figure 11. The far field Equations (89) can then be used to compute the source gradients in Equation (87). With $\nabla_1' \tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega) = i K \hat{\Delta}_1 \tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega)$

and $\nabla_2' \tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega) = -i K \hat{\Delta}_2 \tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega)$,

we obtain

$$\boxed{\begin{aligned} \tilde{\Gamma}(\vec{r}_1 \vec{r}_2 \Omega) = \frac{1}{(4\pi)^2} \int_\sigma \int_\sigma dS_1 dS_2 \left( \frac{\tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega)}{R_1 R_2} \right) e^{iK(R_1 - R_2)} \times \\ \Big\{ K^2 R_1 R_2 (q_1 q_2 + p_1 p_2 + q_1 p_2 + q_2 p_1) + p_1 p_2 \\ + iK \Big[ R_2 p_1 q_2 - R_1 p_2 q_1 - \frac{p_1 p_2}{c} \Big( \frac{1}{R_2} - \frac{1}{R_1} \Big) \Big] \Big\}, \\ \text{where } p_i = \frac{\hat{n}_i \cdot \hat{R}_i}{R_i} \text{ and } q_i = \frac{\hat{n}_i \cdot \hat{\Delta}_i}{R_i} \end{aligned}} \quad (90)$$

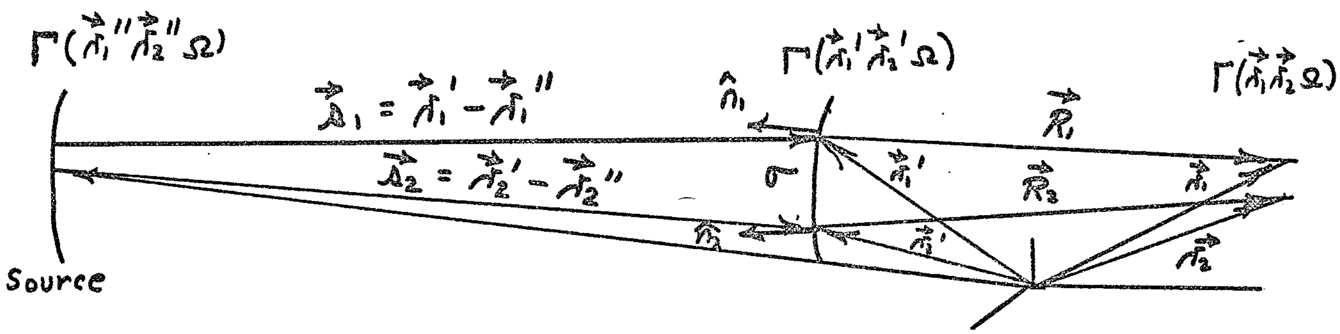



Figure 11. Illustrating the idea that light leaving the mirror of an optical resonator (or a lens of iterated phase transformers) has the character of radiation in the far field of another source.

The same approximations that lead to the BASIC DIFFRACTION INTEGRALS FOR OPTICAL RESONATORS (Eqns. 20, 21, 22), will again be invoked. Namely,

$$\hat{n}_1 \cdot \hat{R}_1 = \hat{n}_2 \cdot \hat{R}_2 = \hat{n}_1 \cdot \hat{\Delta}_1 = \hat{n}_2 \cdot \hat{\Delta}_2 = -1$$

$$p_1 = q_1 = \frac{-1}{R_1} \quad , \quad p_2 = q_2 = \frac{-1}{R_2}$$

$R_1 = R_2 = d$ except in the phase factor where

*Rectangular Coordinates*

$$R_1 = d + \frac{g}{2d}(x_1^2 + y_1^2) + \frac{g'}{2d}(x_1'^2 + y_1'^2) - \frac{1}{d}(x_1 x_1' + y_1 y_1')$$

$$R_2 = d + \frac{g}{2d}(x_2^2 + y_2^2) + \frac{g'}{2d}(x_2'^2 + y_2'^2) - \frac{1}{d}(x_2 x_2' + y_2 y_2') \qquad (91)$$

*Cylindrical Coordinates*

$$R_1 = d + \frac{g}{2d}\rho_1^2 + \frac{g'}{2d}\rho_1'^2 - \frac{\rho_1 \rho_1'}{d}\cos(\theta_1' - \theta_1)$$

$$R_2 = d + \frac{g}{2d}\rho_2^2 + \frac{g'}{2d}\rho_2'^2 - \frac{\rho_2 \rho_2'}{d}\cos(\theta_2' - \theta_2)$$

where

$g = 1 - \frac{d}{R}$ and $g' = 1 - \frac{d}{R'}$ , and $R$ & $R'$ are radii of the mirrors.

,

and leads to:

$$\boxed{\begin{aligned}&\tilde{\Gamma}(\vec{r}_1,\vec{r}_2,\Omega) = \frac{\Phi(\vec{r}_1,\vec{r}_2)}{(4\pi)^2 d^2}\int_\sigma\int_\sigma dS_1' dS_2'\, e^{i\frac{K}{d}[\quad]}\,(1+4K^2d^2)\,\tilde{\Gamma}(\vec{r}_1{}',\vec{r}_2{}',\Omega)\\ &\text{where}\\ &[\quad] = \tfrac{1}{2}g(\rho_1^2-\rho_2^2)+\tfrac{1}{2}g'(\rho_1'^2-\rho_2'^2)+\rho_2\rho_2'\cos(\theta_2'-\theta_2)-\rho_1\rho_1'\cos(\theta_1'-\theta_1)\end{aligned}} \quad (92)$$,

where a reflection coefficient $\Phi(\vec{r}_1,\vec{r}_2)$ has been included so that $\tilde{\Gamma}(\vec{r}_1,\vec{r}_2,\Omega)$ is the mutual coherence for the field leaving the mirror.

## J. THE VAN CITTERT - ZERNIKE THEOREM

If Equation (92) above is multiplied by $\frac{1}{2\pi} e^{-i\Omega\tau} d\Omega$ and the integration is performed from $0$ to $\infty$, the dependence on $\tau$ can be recovered since

$$\Gamma(\vec{r}_1 \vec{r}_2 \tau) = \frac{1}{2\pi} \int_0^\infty e^{-i\Omega\tau} \tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega)\, d\Omega .$$

This calculation is performed in Appendix L, where the result for a Gaussian amplitude distribution of frequencies is given by Equation L8. According to this appendix, if we put

$$\tilde{\Gamma}(\vec{r}_1{}' \vec{r}_2{}' \Omega) = \frac{F(\vec{r}_1{}')\, \delta(\vec{r}_2{}' - \vec{r}_1{}')}{\sqrt{\Delta\Omega}}\, e^{-\left[\frac{\Omega - \bar{\Omega}}{\sqrt{2}\,\Delta\Omega}\right]^2} , \qquad (93)$$

The result will be,

$$\Gamma(\vec{r}_1 \vec{r}_2 \tau) = \frac{\phi\, e^{-i\bar{\Omega}\tau}}{(4\pi)^2 d^4\, 2\sqrt{2\pi\Delta\tau}} \int_\sigma\!\!\int_\sigma dS_1 dS_2\, e^{i\bar{\Omega}\alpha}\, e^{-\left(\frac{\tau-\alpha}{\sqrt{2}\,\Delta\tau}\right)} \{\text{factor}\} F(\vec{r}_1{}')\, \delta(\vec{r}_2{}' - \vec{r}_1{}')$$

where

$$\alpha = \frac{1}{cd}\left[\frac{1}{2} g(\rho_1^2 - \rho_2^2) + \frac{1}{2} g'(\rho_1'^2 - \rho_2'^2) + \rho_2\rho_2' \cos(\theta_2' - \theta_2) - \rho_1\rho_1' \cos(\theta_1' - \theta_1)\right]$$

$$\{\text{factor}\} = 1 + 2\bar{K}^2 d^2\left(1 + 2\sqrt{\tfrac{2}{\pi}}\,\delta + \delta^2\right) - 2\left(1 + \sqrt{\tfrac{2}{\pi}}\right) i(\tau-\alpha)\bar{\Omega}\,\delta^2 - \left[(\tau-\alpha)\bar{\Omega}\,\delta^2\right]^2 ,$$

$$\delta = \frac{\Delta\Omega}{\bar{\Omega}} , \quad K = \frac{\Omega}{c} , \quad \bar{K} = \frac{\bar{\Omega}}{c} , \quad \Delta\Omega\,\Delta\tau = 1 .$$

Here it has been assumed that the cross correlations are all zero. The delta function then puts $\vec{r}_2{}' = \vec{r}_1{}' \equiv \vec{r}\,'$ when one integration is performed, giving

$$\Gamma(\vec{r}_1,\vec{r}_2,\tau) = \frac{\phi e^{-i\bar{\Omega}\tau} e^{i\frac{Ka}{2d}(\rho_1^2-\rho_2^2)}}{(4\pi)^2 d^4 2\sqrt{2\pi\Delta\tau}} \int_\sigma dS' e^{-\left[\frac{\tau-\alpha}{\sqrt{2}\Delta\tau}\right]^2} e^{i\frac{\bar{K}}{d}\rho'[\rho_2\cos(\theta'-\theta_2)-\rho_1\cos(\theta'-\theta_1)]} \{factor\} F(\vec{r}\,') \quad . \qquad (94)$$

The integral in this equation is

$$I_\sigma = \int_0^a \rho' d\rho' \int_0^{2\pi} d\theta' e^{-\left[\frac{\tau-\alpha}{\sqrt{2}\Delta\tau}\right]^2} e^{i\frac{\bar{K}}{d}\rho'\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\cos(\theta'+\beta)} \{factor\} F(\vec{r}\,') \qquad (95)$$

where $\beta = \tan^{-1}\frac{y_2-y_1}{x_2-x_1}$ .

At this point we make the usual approximations

$$\boxed{\begin{array}{l} \delta \ll 1,\ (\bar{K}d)^2 \gg 1,\ \text{so that } \{factor\} = 4\bar{K}^2 d^2; \\ \text{and } \Delta\tau \text{ is sufficiently long that} \\ \quad e^{-\left[\frac{\tau-\alpha}{\sqrt{2}\Delta\tau}\right]^2} \simeq 1 \end{array}}$$

This last approximation simply limits the range of $\tau$ over which the final equations are valid. Typically, $\alpha_{max} \approx 10^{-13}$ ; so we are requiring $\frac{\tau}{\Delta\tau} < (\sim \frac{1}{100})$ . It is also convenient to put

$$d_{12} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

= distance between points $\vec{r}_1$ & $\vec{r}_2$ .

Thus, $\rho_1^2 - \rho_2^2 = 2d\, d_{12}$

Also, assuming that $F(\vec{r}\,') = F(r')$ , (94) reduces to

$$\Gamma(\vec{r}_1,\vec{r}_2,\tau) = \frac{4\phi e^{-i\bar{\Omega}\tau} e^{i\bar{K}g\,d_{12}}}{(4\pi)^2 d^2\, 2\sqrt{2\pi\Delta\tau}} \bar{K}^2 \int_0^a \rho' d\rho' \int_0^{2\pi} d\theta' e^{i\frac{\bar{K}d_{12}}{d}\rho'\cos(\theta'-\beta)} F(r') \, .$$

If $F(r')$ is constant, the definite integrals

$$\boxed{\begin{aligned}&\int_0^{2\pi} e^{i\mathfrak{z}\rho'\cos(\theta'+\beta)}\,d\theta' = 2\pi J_0(\mathfrak{z}\rho') \\ &\text{and} \\ &\int_0^{a} J_0(\mathfrak{z}\rho')\rho'\,d\rho' = \frac{a}{\mathfrak{z}} J_1(a\mathfrak{z}) = a^2\,\frac{J_1(a\mathfrak{z})}{(a\mathfrak{z})}\end{aligned}} \qquad (96)$$

can be used to integrate this expression; to obtain

$$\Gamma_{12}(\tau) = \frac{\phi F\sqrt{2\pi}\,\bar{K}^2}{8\pi^2\sqrt{\Delta\tau}}\, e^{-i\bar{\Omega}\tau} e^{ig\bar{K}d_{12}} \left(\frac{a}{d}\right)^2 \left[\frac{J_1\left(\frac{\bar{K}a\,d_{12}}{d}\right)}{\left(\frac{\bar{K}a\,d_{12}}{d}\right)}\right]$$

where by,

$$\boxed{\gamma_{12}(\tau) = \left[\frac{J_1\left(\frac{\bar{K}a\,d_{12}}{d}\right)}{\left(\frac{\bar{K}a\,d_{12}}{d}\right)}\right] e^{-i\bar{\Omega}\tau} e^{ig\bar{K}d_{12}}} \qquad (97)$$

The function $J_1(\mathfrak{z})/\mathfrak{z}$ is small except near $\mathfrak{z}=0$. It first vanishes when $\mathfrak{z} = 3.83$. Thus, the distance between two points required for an appreciable degree of coherence is $d_{12}^{(c)}$, given by $\bar{K}\,d_{12}^{(c)}\left(\frac{a}{d}\right) < 3.83$.

The frequencies present in can be examined by combining

$$\tilde{\Gamma}(\vec{r}_1,\vec{r}_2\,\Omega) = \int_{-\infty}^{\infty} \Gamma(\vec{r}_1\,\vec{r}_2\,\tau)\, e^{i\Omega\tau}\,d\tau$$

$$\Gamma(\vec{r}_1\,\vec{r}_2\,\tau) = \frac{1}{2T}\int_{-T}^{T} \psi(r_1, t+\tau)\,\psi^*(\vec{r}_2, t)\,dt$$

$$\psi(\vec{r}_1\,t) = \frac{1}{2\pi}\int_0^{\infty} \tilde{\psi}(\vec{r}_1\,\omega_1)\,e^{-i\omega_1 t}\,d\omega_1$$

and $$\psi(\vec{r}_2\,t) = \frac{1}{2\pi}\int_0^{\infty} \tilde{\psi}(\vec{r}_2\,\omega_2)\,e^{-i\omega_2 t}\,d\omega_2\,,$$

to obtain:

$$\tilde{\Gamma}(\vec{r}_1,\vec{r}_2,\Omega) = \frac{1}{2T}\int_{-T}^{T} dt\, e^{i(\omega_2-\omega_1)t}\int_{-\infty}^{\infty} d\tau\, e^{i(\Omega-\omega_1)\tau}\int_0^{\infty}\frac{d\omega_1}{2\pi}\int_0^{\infty}\frac{d\omega_2}{2\pi}\,\tilde{\psi}(\vec{r}_1,\omega_1)\,\tilde{\psi}^*(\vec{r}_2,\omega_2).$$

Since the averaging over $t$ can be extended to very long limits, the first integration gives a delta function, $\delta(\omega_2-\omega_1)$. The second term also gives a delta function, $\delta(\Omega-\omega_1)$.

Thus, there is no contribution to $\tilde{\Gamma}$ except where $\Omega$ is a frequency that is common to both $\tilde{\psi}(\vec{r}_1,\omega_1)$ and $\tilde{\psi}(\vec{r}_2,\omega_2)$. *The frequencies, $\Omega$, present in $\tilde{\Gamma}$ are just those that are mutually common to both $\tilde{\psi}_1$ and $\tilde{\psi}_2$.*

For the sun, the central common frequency is about $\frac{1}{2}\times 10^{15}$ cycles per second, corresponding to $\bar{k} = 10^5\ cm^{-1}$. With $\frac{a}{d} = .009$ radians, the distance over which an appreciable mutual coherence occurs is $\bar{k}d_{12}^{(c)}\left(\frac{a}{d}\right) < 3.83$ ; $10^5 d_{12}^{(c)} \times (.009) < 3.83$ ; $d_{12}^{(c)} < (\sim .004)\, cm$.

## K. RESONATOR SELF CONSISTENT COHERENCE

The remarkable thing about the Van Cittert-Zernike result is that a non zero cross correlation is obtained starting with a zero cross correlation over the entire original source. How this comes about is illustrated in Figure 12. The field at point $P_1^{(1)}$ can be obtained by integration over the source. Since the field at $P_2^{(1)}$ depends on an integration over the same source, it is not surprising that they are now

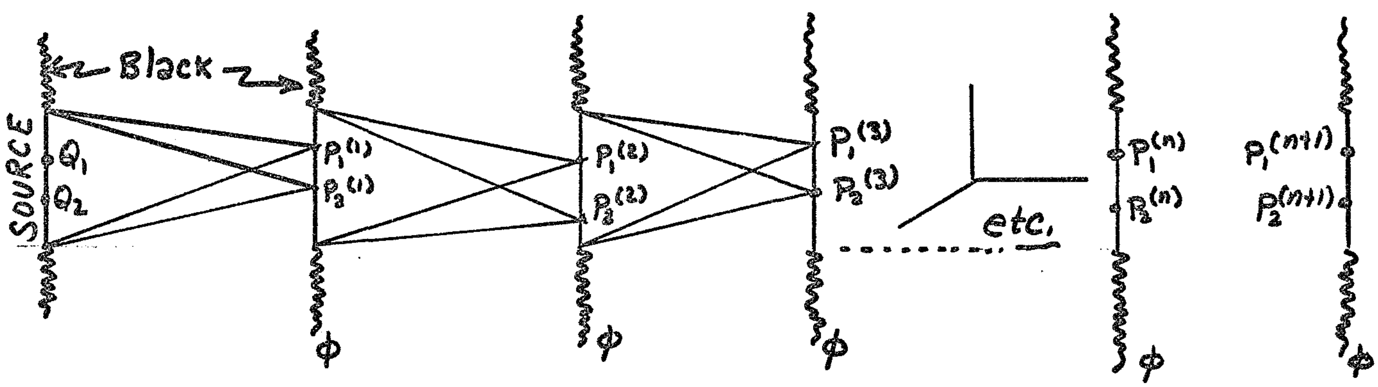



Figure 12. Illustrating the improvement of mutual coherence as light propagates through an iterated system of phase transformers. At the start $\Gamma(Q_1, Q_2\ \tau) = 0$ when $Q_1 \neq Q_2$, but is different from zero over a short range of $\tau$ when $Q_1 = Q_2$. According to the Van Cittert-Zernike theorem $\Gamma(P_1^{(1)} P_2^{(1)} \tau)$ is different from zero for short separations of the points $P_1^{(1)}$ and $P_2^{(1)}$.

correlated statistically. An increase in temporal coherence would also be expected because the correlated fields would give the greater contributions to fields at $P_1^{(1)}$ and $P_2^{(1)}$ when the integrations are performed. Thus, the statistical correlation should improve with each iteration. If we assume that this process converges to some limiting value, eventually

$$\boxed{\tilde{\Gamma}_{n+1}(\rho_1\theta_1\rho_2\theta_2\Omega) = \epsilon\,\tilde{\Gamma}_n(\rho_1\theta_1\rho_2\theta_2\Omega)\,,} \tag{98}$$

where $\epsilon$ is a complex constant.

Using Equation (92) to express $\Gamma_{n+1}$ in terms of $\Gamma_n$ leads to the integral equation;

$$\epsilon\tilde{\Gamma}(\rho_1\theta_1\rho_2\theta_2\Omega) = \frac{\Phi(\rho_1\theta_1\rho_2\theta_2)}{(4\pi)^2 d^4}\int_\sigma\int_\sigma dS_1' dS_2' e^{i\frac{K}{d}[\;]}(1+4K^2d^2)\,\tilde{\Gamma}(\rho_1'\theta_1'\rho_2'\theta_2'\Omega) \tag{99}$$

$$[\;] = \tfrac{1}{2}g(\rho_1^2-\rho_2^2)+\tfrac{1}{2}g'(\rho_1'^2-\rho_2'^2)+\rho_2\rho_2'\cos(\theta_2'-\theta_2)-\rho_1\rho_1'\cos(\theta_1'-\theta_1)$$

This expression should be compared with the single pass mode equation developed earlier. The requirement for the field to repeat itself was

$$\tilde{\psi}_{n+1}(\rho\theta\omega) = e\,\tilde{\psi}_n(\rho\theta\omega)\,. \tag{100}$$

Using the basic diffraction integral of Equation (24) to compute $\tilde{\psi}_{n+1}$ in terms of $\tilde{\psi}_n$ and dropping the subscript $n$ gives:

<u>Mode equation</u>

$$e\,\tilde{\psi}(\rho\theta\omega) = \frac{-ik\,e^{ikd}\,\Gamma(\rho\theta)}{2\pi d}\int_\sigma dS'\,e^{i\frac{k}{d}[\frac{1}{2}\rho^2 g+\frac{1}{2}\rho'^2 g'-\rho\rho'\cos(\theta'-\theta)]}\,\tilde{\psi}(\rho'\theta'\omega) \tag{101}$$

For the case of rectangular symmetry with $g' = 0$, the focus condition, both Equations (99) and Equation (101) can be solved.

For mirrors, or phase transformers, of dimensions $2a \times 2b$, we put

$$\tilde{\Gamma} = X_1(x_1)\,Y_1(y_1)\,X_2(x_2)\,Y_2(y_2)$$

$$\epsilon = \epsilon_{x1}\,\epsilon_{x2}\,\epsilon_{x3}\,\epsilon_{x4}$$

$$\tilde{\psi} = \mathcal{X}(x)\,\mathcal{Y}(y)$$

and $e = e_x e_y$. Equations (99) and (101) then reduce to

$$\left.\begin{aligned}
X_1(x_1) &= \frac{1}{\epsilon_{x1}} \sqrt[4]{\frac{\phi\,(1+4K^2 d^2)\, e^{i\frac{Kg}{2d}(\beta^2-\beta_2^2)}}{(4\pi)^2 d^4}} \int_{-a}^{a} e^{-i\frac{K}{d}x_1 x_1'}\, X_1(x_1')\,dx_1' \\
X_2(x_2) &= \frac{1}{\epsilon_{x2}} \sqrt[4]{\frac{\phi\,(1+4K^2 d^2)\, e^{i\frac{Kg}{2d}(\beta_1^2-\beta_2^2)}}{(4\pi)^2 d^4}} \int_{-a}^{a} e^{i\frac{K}{d}x_2 x_2'}\, X_2(x_2')\,dx_2' \\
\mathcal{X}(x) &= \frac{1}{e_x} \sqrt{\frac{-ik\, e^{ikd}\, r\, e^{i\frac{kg}{2d}\rho^2}}{2\pi d}} \int_{-a}^{a} e^{-i\frac{k}{d}xx'}\, \mathcal{X}(x')\,dx'
\end{aligned}\right\} \quad (102)$$

and similar expressions for the $y$'s with integration limits changed to $-b$ to $+b$.

The solutions of these integral equations are the prolate spheroidal angle function which satisfy:

$$\boxed{\begin{aligned} &2i^{n} R'_{on}(\alpha,1)\, S_{on}(\alpha,t) = \int_{-1}^{1} e^{i\alpha t s}\, S_{on}(\alpha,s)\, ds \\ &\text{and} \\ &2i^{-n} R'_{on}(\alpha,1)\, S_{on}(\alpha,t) = \int_{-1}^{1} e^{-i\alpha t s}\, S_{on}(\alpha,s)\, ds \end{aligned}} \qquad (103)$$

The change of variable, $x_1' = as$ and $x_1 = at$, reduces the first of Equations (102) to the form

$$X_1(at) = \frac{1}{\epsilon_{x1}} \sqrt[4]{\phantom{xx}} \int_{-1}^{1} e^{-i\left(\frac{Ka^2}{d}\right)} X_1(as)\, ds .$$

Comparison with the second of Equations (103) then gives:

$$2i^{-1} R'_{on}\left(\frac{Ka^2}{d}, 1\right) = \frac{\epsilon_{x1}}{\sqrt[4]{\phantom{xx}}}$$ and

$$X_1(at) = S_{on}\left(\frac{Ka^2}{d}, t\right)$$ or,

$$X_1(x_1) = S_{on}\left(\frac{Ka^2}{d}, \frac{x_1}{a}\right).$$

Combining these results, we have:

<u>Rectangular symmetry</u>. <u>Resonator self consistent solution</u>

$$\tilde{\Gamma}_{nmpq}(x_1 x_2 y_1 y_2 \Omega) = (\text{Const})\, S_{on}\left(\frac{Ka^2}{d}, \frac{x_1}{a}\right) S_{om}\left(\frac{Ka^2}{d}, \frac{x_2}{a}\right) S_{op}\left(\frac{Kb^2}{d}, \frac{y_1}{b}\right) S_{oq}\left(\frac{Kb^2}{d}, \frac{y_2}{b}\right)$$

$$\epsilon_{nmpq} = 16\, i^{-n+m-p+q} R'_{on}\left(\frac{Ka^2}{d}, 1\right) R'_{om}\left(\frac{Ka^2}{d}, 1\right) R'_{op}\left(\frac{Kb^2}{d}, 1\right) R_{oq}\left(\frac{Kb^2}{d}, 1\right) \frac{\phi\, e^{i\frac{Kg}{2d}\frac{(\rho_1^2-\rho_2^2)}{(1+4K^2d^2)}}}{(4\pi)^2 d^4} \qquad (104)$$

*next page*

$$\boxed{\begin{aligned}\tilde{\Psi}_{mn}(x\,y\,w) &= (const)\, S_{on}\left(\frac{ka^2}{d}, \frac{x}{a}\right) S_{om}\left(\frac{kb^2}{d}, \frac{y}{b}\right) \\ e_{mn} &= 4i^{-n-m-1} R'_{om}\left(\frac{ka^2}{d}, 1\right) R'_{on}\left(\frac{kb^2}{d}, 1\right) \frac{\sigma k e^{ikd} e^{i\frac{kq}{2d}\rho^2}}{2\pi d}\end{aligned}} \qquad \begin{matrix}(104)\\ \text{Cont.}\end{matrix}$$

If $\frac{Ka^2}{d}$ is sufficiently large, the angle functions can be replaced by

$$S_{on}\left(\frac{Ka^2}{d}, \frac{x\sqrt{\frac{K}{d}}}{\sqrt{\frac{Ka^2}{d}}}\right) = \frac{1}{\sqrt{\sqrt{\pi}\,2^n n!}}\, e^{-\frac{1}{2}\left(x\sqrt{\frac{K}{d}}\right)^2} H_n\left(x\sqrt{\frac{K}{d}}\right) \tag{46F}$$

as explained in Appendix F. Actually, this approximation is rather good even for Fresnel numbers, $\frac{Ka^2}{2\pi d}$ , as small as 3 or 4. Thus, for each frequency $\Omega$ and for the lowest mode, we have;

$$\tilde{\Gamma}_{0000}(\vec{\rho}_1 \vec{\rho}_2\, \Omega) = (const)\; e^{-\frac{K}{2d}(\rho_1^2 + \rho_2^2)} . \tag{105}$$

The total $\tilde{\Gamma}$ is then obtained by adding the ones together for each frequency. If a Gaussian distribution is assumed, and this is multiplied by $\frac{1}{2\pi}\, e^{-i\Omega\tau} d\Omega$ and intergrated, the final total mutual coherence function is obtained:

$$\Gamma_{0000}(\vec{\rho}_1 \vec{\rho}_2\, \tau) = \frac{A}{2\pi\sqrt{\Delta\Omega}} \int_0^{\infty} e^{-\left[\frac{\Omega - \bar{\Omega}}{\sqrt{2}\,\Delta\Omega}\right]^2} e^{-\frac{K}{2d}(\rho_1^2 + \rho_2^2)} e^{-i\Omega\tau}\, d\Omega .$$

This integral has been evaluated in Appendix L where it can be seen that this gives:

$$\Gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau) = \frac{A}{2\sqrt{2\pi\Delta\tau}}\, e^{-i\bar{\Omega}\tau}\, e^{-i\frac{\bar{\Omega}}{2cd}(\rho_1^2+\rho_2^2)}\, e^{-\left[\frac{\tau-\frac{i}{2cd}(\rho_1^2+\rho_2^2)}{\sqrt{2}\,\Delta\tau}\right]^2}$$

$$= \frac{A}{2\sqrt{2\pi\Delta\tau}}\, e^{-i\bar{\Omega}\tau}\, e^{i\frac{\tau}{cd}\frac{(\rho_1^2+\rho_2^2)}{2(\Delta\tau)^2}}\, e^{-i\frac{\bar{K}}{2d}(\rho_1^2+\rho_2^2)} \times e^{-\left(\frac{\tau}{\sqrt{2}\Delta\tau}\right)^2}\, e^{+\frac{(\rho_1^2+\rho_2^2)^2}{8c^2d^2(\Delta\tau)^2}}.$$

Putting $\Delta l = c\,\Delta\tau.$

$$\Gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau) = \frac{A}{2\sqrt{2\pi\Delta\tau}}\, e^{-i\tau\left[\bar{\Omega}-\frac{(\rho_1^2+\rho_2^2)}{2d(\Delta l)}\right]}\, e^{-i\frac{\bar{K}}{2d}(\rho_1^2+\rho_2^2)}\, e^{-\left(\frac{\tau}{\sqrt{2}\Delta\tau}\right)^2}\, e^{+\frac{(\rho_1^2+\rho_2^2)^2}{8d^2c^2(\Delta\tau)^2}}.$$

$$\Gamma_{0000}(\vec{\rho}_1\vec{\rho}_1\,0) = \frac{A}{2\sqrt{2\pi\Delta\tau}}\, e^{-i\frac{\bar{K}}{2d}(\rho_1^2+\rho_1^2)}\, e^{\frac{(\rho_1^2+\rho_1^2)^2}{8d^2(\Delta l)^2}}.$$

So that the complex degree of coherence is:

$$\gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau) \equiv \frac{\Gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau)}{\sqrt{\left|\Gamma_{0000}(\vec{\rho}_1\vec{\rho}_1\,0)\right|\left|\Gamma_{0000}(\vec{\rho}_2\vec{\rho}_2\,0)\right|}}$$

$$\gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau) = \frac{e^{-\left[\frac{\tau}{\sqrt{2}\Delta\tau}\right]^2}\, e^{-i\tau\left[\bar{\Omega}-\frac{\rho_1^2+\rho_2^2}{2d\,\Delta l}\right]}\, e^{-i\frac{\bar{K}}{2d}(\rho_1^2+\rho_2^2)}\, e^{\frac{\rho_1^4+2\rho_1^2\rho_2^2+\rho_2^4}{8d^2(\Delta l)^2}}}{e^{\frac{4\rho_1^4+4\rho_2^4}{8d^2(\Delta l)^2}}}$$

$$\boxed{\gamma_{0000}(\vec{\rho}_1\vec{\rho}_2\,\tau) = e^{-\left(\frac{\tau}{\sqrt{2}\Delta\tau}\right)^2}\, e^{-\frac{(3\rho_1^4-2\rho_1^2\rho_2^2+3\rho_2^4)}{8d^2(\Delta l)^2}}\, e^{-i\left\{\tau\left(\bar{\Omega}-\frac{\rho_1^2+\rho_2^2}{2d\,\Delta l}\right)+\frac{\bar{K}}{2d}(\rho_1^2+\rho_2^2)\right\}}} \qquad (105\ \lambda)$$

If light of equal intensity is used from points $\vec{\rho}_1$ & $\vec{\rho}_2$ to form an interference pattern, the intensity according to Equation (70) will be:

$$I \propto 1 + Re\,\gamma_{12}(\tau),$$

and the visibility will be

$$\mathcal{V} = Re\,\gamma_{12}(\tau)$$

Thus,

$$\boxed{\begin{aligned} \mathcal{V} &= e^{-\left(\frac{\tau}{\sqrt{2}\,\Delta T}\right)^2} e^{-\frac{(3\rho_1^4 - 2\rho_1^2\rho_2^2 + 3\rho_2^4)}{8d^2\,(\Delta \ell)^2}} \cos\left[\tau\bar{\Omega} - \frac{\tau(\rho_1^2+\rho_2^2)}{2d\,\Delta\ell} + \frac{\bar{K}}{2d}(\rho_1^2+\rho_2^2)\right] \\ I &\propto 1 + \mathcal{V} \end{aligned}} \quad (106)$$

The second factor does not fall off appreciably unless it is larger than say $1/2$ . With $\rho_1 = \rho_2$ , this requires:

$$\rho < \sqrt{d\,(\Delta\ell)}\,, \qquad (107)$$

a condition that is easily met in lasers. Typically, $\rho < 1\ cm$, whereas $d \sim 100\ cm$, and $\Delta\ell \sim$ 10,000,000 cm. In fact, it is probably not possible to observe this term using laser radiation. It might, however, be possible to see such an effect in an iterated system of phase transformers that is used to filter white light.

The last two phase factors in (106) are small for typical lasers and are of little importance. The result can be summarized by stating that *the degree of mutual coherence is essentially constant over the cross section of the laser beam.*

Finally, it should be remarked that this analysis did not include an active lasing medium. The effects of such a medium are well known. Basically, the frequency width per mode in such a regenerative oscillator is many times smaller than the frequency width of the passive resonator. Most of these effects can be put into the results of this paper simply by using the much smaller value of $\Delta\omega$ that one obtains with such a system.

# CHAPTER VII

# PRACTICAL LASER OPTICS

## A. INTRODUCTION

This chapter is primarily concerned with the basic laws required for the practical design of the laser resonators and for determining the propagation properties of laser beams through optical systems. The fundamental law of propagation of Gaussian beams was obtained by Boyd and Gordon[9] in 1961. An improved version of a graphical solution of this law suggested by Gordon[15] is given for the first time in the present work. The rules for image formation follow those given in a review article by Kogelnik and Li[15], and an earlier one by Kogelnik[16]. These rules include image formation, curvature, and ray tracing using the ABCD law and ray transfer matrices. These ideas naturally lead to the topic of stability and resonator frequencies in optical resonators, where additional information is given by Gordon and Kogelnik[18] and by Boyd and Kogelnik[19].

The image rules according to scalar diffraction follows the treatment of Collins[20] and of Born and Wolf[4]. The conjugate relations for the mutual coherence function between points in the entrance and in the exit pupils for a real entrance pupil are given in this reference (4). So far as this author knows, this same result for virtual entrance pupils is developed here for the first time, where there is a slightly new twist to the development in that integration over an infinite principal plane is not necessary. The conjugate relations for mutual coherence contradict the commonly believed idea that the degree of coherence in a microscope slide is identical to that obtained with an incoherent source filling the condense lens, and suggest the old idea of focusing to a point is the way to obtain spatial coherence from an incoherent source.

## B. LAW OF PROPAGATION OF A GAUSSIAN WAVE

Consider the confocal resonator shown in Figure 13 with square mirrors.

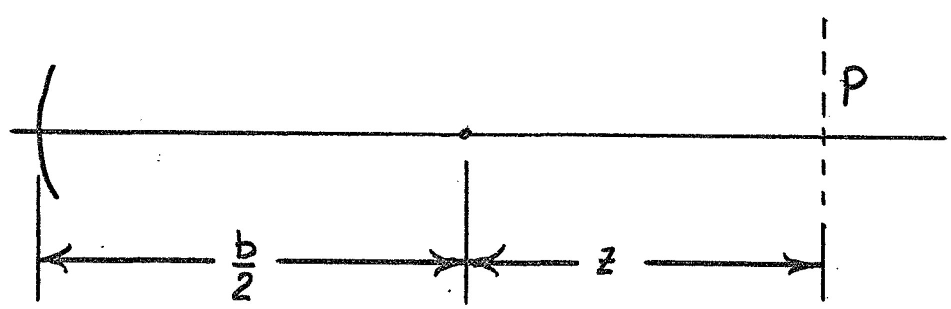



Figure 13. A part of the confocal resonator geometry used to derive Equation (107)

The modes for such a resonator are given through the integral Equations 33 and 34. As shown in Appendix F (Eqn. 38F), the Spheroidal angle functions are the solution of this integral equation, and for even a moderately high Fresnel number, these solutions reduce to the Hermite-Gauss functions of Equation 46F.

Starting with the Hermite-Gauss function fields on the left mirror of Figure 13, the basic diffraction integral can be used to obtain the fields at a plane P. In so doing, Boyd and Gordon obtained for one transverse component

$$\boxed{\begin{aligned}\psi_{nm}(x,y,z) &= (\text{const})\sqrt{\frac{2}{1+\xi^2}\,\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(m+1)\,\Gamma(n+1)}}\; H_n\!\left(\frac{x\sqrt{c}}{a}\sqrt{\frac{2}{1+\xi^2}}\right) H_m\!\left(\frac{y\sqrt{c}}{a}\sqrt{\frac{2}{1+\xi^2}}\right) \times \\ &\quad e^{-\frac{k\rho^2}{b(1+\xi^2)}}\; e^{i\left\{k\left[\frac{b}{2}(1+\xi)+\frac{\xi}{1+\xi^2}\frac{\rho^2}{b}\right]-(1+m+n)\left(\frac{\pi}{2}-\varphi\right)\right\}}, \\ \text{where}\quad & c = 2\pi\left(\frac{a^2}{\lambda d}\right) \\ & \text{mirror dimensions} = a \times a \\ & \rho^2 = x^2+y^2 \\ & \xi = \frac{2z}{b} \\ & \tan\varphi = \frac{1-\xi}{1+\xi}\end{aligned}} \tag{107}$$

This equation applies to the wave propagating to the right in Figure 13 and is valid for the wave that passes through the right mirror provided it is multiplied by the appropriate transmitting factor of the mirror. This equation is also valid for the standing waves inside the cavity when $e^{i\{\ \}}$ is replaced by $\sin\{\quad\}$, and, as a consequence, provides the basis for determining resonant frequencies. The law of propagation for such beams follows from this equation. For the $m=0, n=0$ mode (the most important) the beam width is dictated by the factor $\exp\left\{-\frac{k\rho^2}{b(1+\xi^2)}\right\}$.

If the beam width, $w$, is defined by the $1/e$ value, this factor becomes $\exp\left\{-\left(\frac{\rho}{w}\right)^2\right\}$, so that

$$\boxed{w^2 = \frac{b}{k}(1+\xi^2)} \tag{108}$$

The surfaces of constant phase are given by

$$k\left[\frac{b}{2}(1+\xi) + \frac{\xi}{1+\xi^2}\left(\frac{\rho^2}{b}\right)\right] + \varphi = constant = k\left(\frac{b}{2} + z_0\right)$$

The dependence on $z$ through $\varphi$ is very small and can be neglected, giving

$$z - z_0 \approx -\left(\frac{\xi}{1+\xi^2}\right)\frac{\rho^2}{b} \tag{109}$$

which has the appearance of $-\rho^2/2R$.

Thus, the beam radius is

$$\boxed{R = \frac{(1+\xi^2)b}{2\xi}} \tag{110}$$

The cross section of any Gaussian beam is therefore specified by $e^{-\frac{\rho^2}{w^2}} e^{-i\frac{k\rho^2}{2R}}$ and is characterized by the spot size or width, $w$, and radius, $R$, at any point $z$. Equations 108 and 109 specify these parameters as a function of $z$, and are the

$$\boxed{\begin{array}{l} \text{LAW OF PROPAGATION; GAUSSIAN BEAM} \\ w^2 = \frac{k}{b}\left[1 + \frac{4z^2}{b^2}\right] \\ 4zR = (1 + 4z^2/b^2)\,b^2 \end{array}} \tag{111}$$

These equations have a simpler form when expressed in terms of values at the beam "waist" where $z = z_0$, $R = \infty$, and $w_0 = \sqrt{b/k}$. This last result can be obtained by eliminating $\xi$ from (108) and (109) which gives $w^2 = \frac{b}{k}\left[1 + \left(\frac{kw^2}{2R}\right)^2\right]$. As $R \to \infty$ with $w$ finite, $w_0 \to \sqrt{\frac{b}{k}}$.

Elimination of "$b$" in (111) gives

Law of Propagation: Gaussian Beam

$$w^2 = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]$$

$$R = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$$

or

$$w_0^2 = \frac{w^2}{1 + \left(\frac{\pi w^2}{\lambda R}\right)^2}$$

$$z = \frac{R}{1 + \left(\frac{\lambda R}{\pi w^2}\right)^2} \qquad (112)$$

with the additional useful relation

$$w^2 w_0^2 = \left(\frac{\lambda}{\pi}\right)^2 R z$$

By defining, $\frac{1}{q} \equiv \frac{1}{R} - i\frac{\lambda}{\pi w^2}$, the law of propagation can be expressed in a single equation

Law of Propagation; Gaussian Beam

$$q = q_0 + z$$

where $\qquad (113)$

$$\frac{1}{q} \equiv \frac{1}{R} - i\frac{\lambda}{\pi w^2}$$

which is easy to verify by separation of the real and imaginary parts.

## C. GRAPHICAL SOLUTION

For a graphical solution of the propagation law, it is convenient to measure distance in units of $\lambda$ and to introduce a scale factor "$a$" through the definitions:

$$C \equiv \frac{aR}{2\lambda} \qquad F = \frac{af}{2\lambda}$$

$$W \equiv a\left(\frac{\pi w^2}{4\lambda^2}\right) \qquad a = \text{scale factor} \tag{114}$$

$$\Delta \equiv a\left(\frac{z}{\lambda}\right) \qquad P \equiv \frac{\lambda}{aq}$$

With these substitutions Equations (112) become

Law of Propagation

$$\boxed{\begin{aligned} W &= W_0\left[1 + \left(\frac{\Delta/2}{W_0}\right)^2\right] \\ C &= \frac{\Delta}{2}\left[1 + \left(\frac{W_0}{\Delta/2}\right)^2\right] \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} W_0 &= \frac{WC^2}{C^2+W^2} \\ \frac{\Delta}{2} &= \frac{CW^2}{C^2+W^2} \end{aligned}} \tag{115}$$

and the law for change of beam radius for a lens, $\frac{1}{R_{new}} = \frac{1}{R_{old}} - \frac{1}{f}$ with $w_{new} = w_{old}$ becomes

$$\boxed{\begin{aligned} \frac{1}{C_{new}} &= \frac{1}{C_{old}} - \frac{1}{F} \\ W_{new} &= W_{old} \end{aligned}} \tag{116}$$

Equations (113) becomes

$$\boxed{\begin{array}{l} P \equiv \dfrac{i}{2C} - \dfrac{i}{2W} \quad , \quad \text{and} \\ \underline{\text{Law of Propagation}} \\ \dfrac{1}{P} = \dfrac{1}{P_0} + s \\ \text{at the beam waist;} \\ s = 0 \, , \quad C = \infty \, , \quad W = W_0 \end{array}} \tag{117}$$

Combining these last equations gives:

$$\frac{1}{\dfrac{1}{2C} - \dfrac{i}{2W}} = 2iW_0 + s$$

and has solutions where the complex number, $u + iv$ , on the left is equal to that on the right;

$$u + iv = x + iy \tag{118}$$

where $u + iv = \dfrac{1}{\dfrac{1}{2C} - \dfrac{i}{2W}}$ and $x + iy = s + 2iW_0$

Separation of real and imaginary gives:

$$\left.\begin{aligned} u &= \frac{2CW^2}{C^2 + W^2} \\ v &= \frac{2WC^2}{C^2 + W^2} \end{aligned}\right\} \text{and} \qquad \begin{aligned} x &= s \\ y &= 2W_0 \end{aligned} \tag{119}$$

The first two of these equations immediately give the useful relation

$$\boxed{\frac{v}{u} = \frac{C}{W} = \text{slope}} \quad . \tag{120}$$

Elimination of $C$ and $W$ respectively gives: $u^2 + (v-W)^2 = W^2$ and $(u-C)^2 + v^2 = C^2$ . Thus, on the $(u,v)$ plane, curves of constant $W$ are circles of radius $W$ with center at $(0,W)$ , and curves of constant $C$ are circles of radius $|C|$ and center at $(C,0)$. The solution of the propagation law is given where a point $(x,y) = (\Delta, 2W_0)$ coincides with a point on the $(u,v)$ plane, each of which now defines a value of $C$ and a value of $W$.

## Summary

$$C \equiv \frac{aR}{2\lambda} \quad , \quad F \equiv \frac{af}{2\lambda} ,$$

$$W \equiv a\left(\frac{\sqrt{\pi}\, w}{2\lambda}\right)^2 , \quad \Delta \equiv \frac{az}{\lambda} \quad . \tag{121}$$

**$u$-$v$ PLANE**

$u^2+(v-W)^2 = W^2$ ; constant $W$ are circles:

Radius $= W$

Center at $(0,W)$

$$(u-c)^2 + v^2 = C^2 \; ; \quad \text{constant } c \text{ are circles: Radius} = |c| \text{ center at } (c,0)$$

$$\text{slope} = \frac{v}{u} = \frac{c}{W} \qquad (121)$$

<u>$x$-$y$ PLANE</u>

$$x = \Delta \quad ; \quad y = 2W_0$$

<u>Action of a lens</u>

$$\frac{1}{C_{new}} = \frac{1}{C_{old}} - \frac{1}{F} \quad ; \quad W_{old} = W_{new}$$

The simplifying feature of this solution is that, as the wave propagates, the point $(x,y) = (\Delta, 2W_0)$ simply moves to the right at constant distance $2W_0$ from the horizontal axis. An auxiliary scale for the plane is not necessary, changes in $\Delta$ are in the same units as the radii of circles and can be found from either the horizontal or vertical scale.

The chart thus proposed for the solution of actual problems is that of Figure 14. Numbers have not been attached to the scales because their choice in actual problems is dictated by the range of beam widths or range of operating distances "$\Delta$" one wishes to consider.

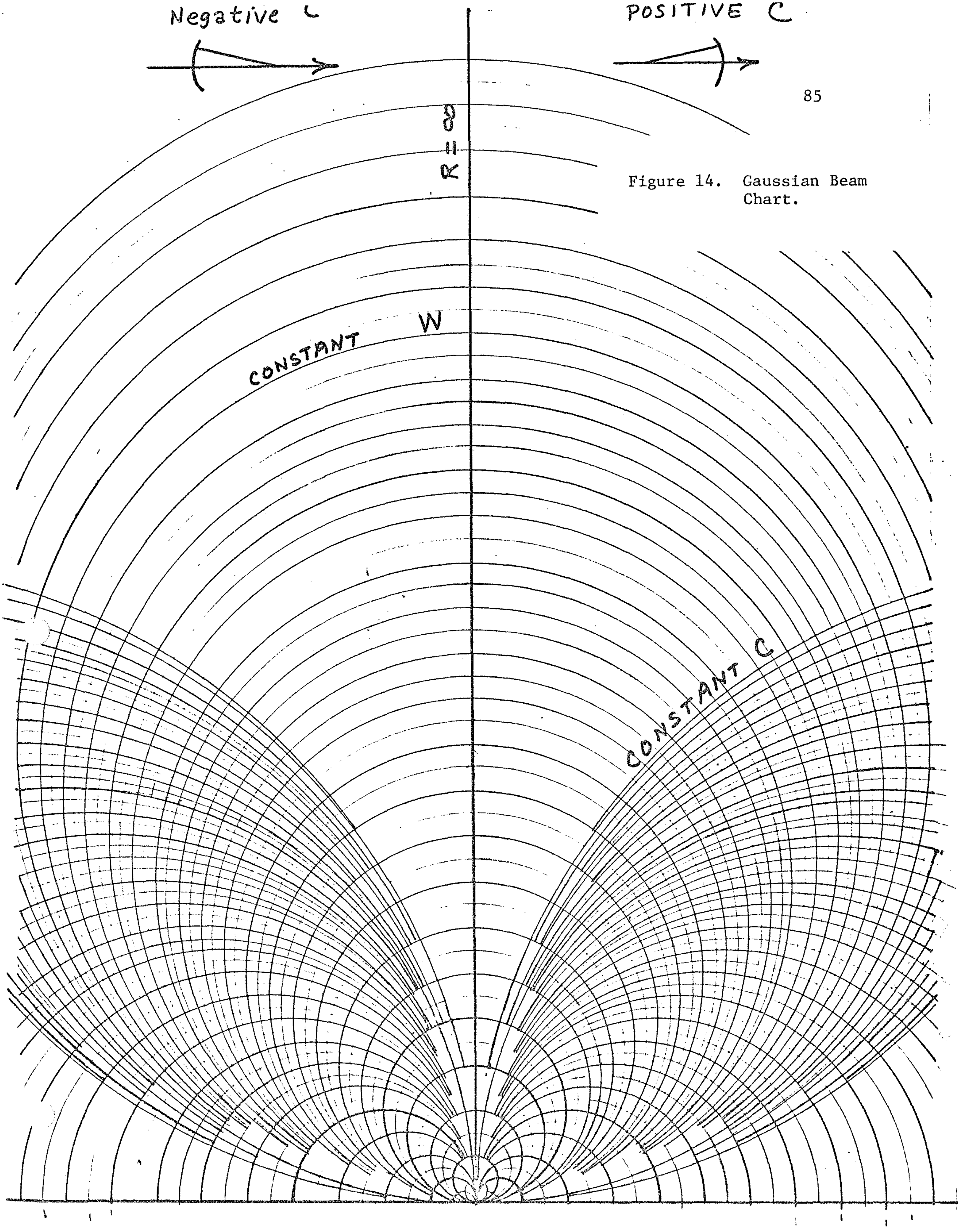



Figure 14. Gaussian Beam Chart.

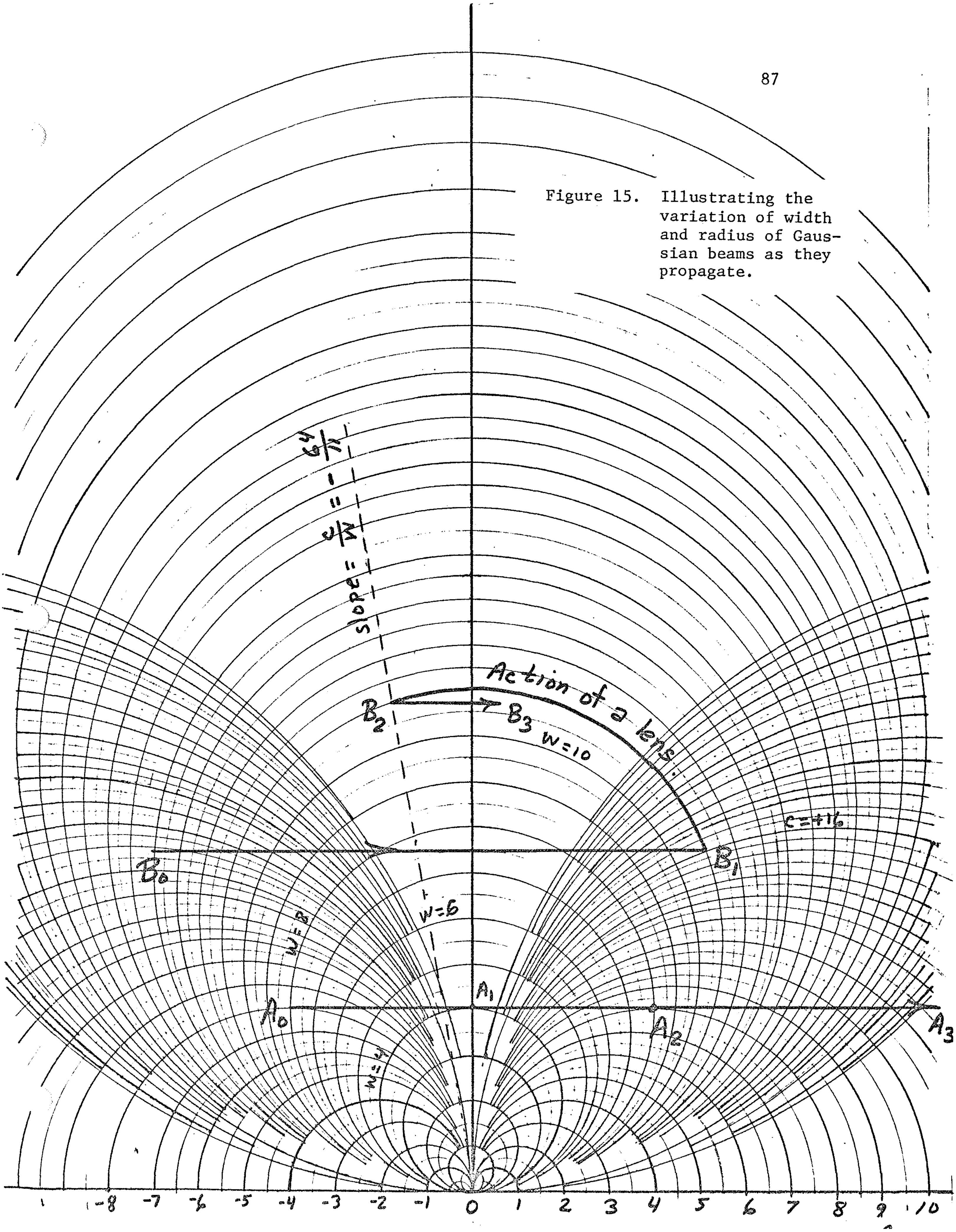
In Figure 15 line $A_0 A_3$ describes a beam of width $W = 8$ and radius $C = -8$ at $A_0$. As it moves to the right the radius approaches $-\infty$ at $A_1$ where it has a minimum width of $W_0 = 4$. The radius then positive and becomes smaller until it reaches $A_2$. Beyond $A_2$ the beam approaches a plane wave as it progresses toward the far field. The width of the beam gets continually larger throughout the journey beyond $A_1$. The trace $B_0 \ldots B_3$ illustrates the action of a positive lens of focal length $F = +12.8$ inserted at $B_1$ to change the radius from $C = +16$ to $C_{new} = -64$ with no change in $W$. The closed loops, $QRQ'$ in Figure 16 illustrates a non-confocal laser cavity. As an example of the use of this chart, suppose we wish to design a laser with the following specifications:

> one flat mirror; one curved. $\lambda = 10^{-3}$ cm.
>
> fields same as confocal
>
> Beam out of flat mirror
>
> $d = 100$ cm between mirrors
>
> beam to be expanded by factor of 2 with lenses and made parallel all in a short distance.

First we find the appropriate scale factor "a":

Distance are in the order of 100 to 200 cm, corresponding to $\Delta \sim \frac{a \times (100 \text{ to } 200)}{\lambda} = (.1 \text{ to } .2) \times 10^6 a$.

Figure 15. Illustrating the variation of width and radius of Gaussian beams as they propagate.



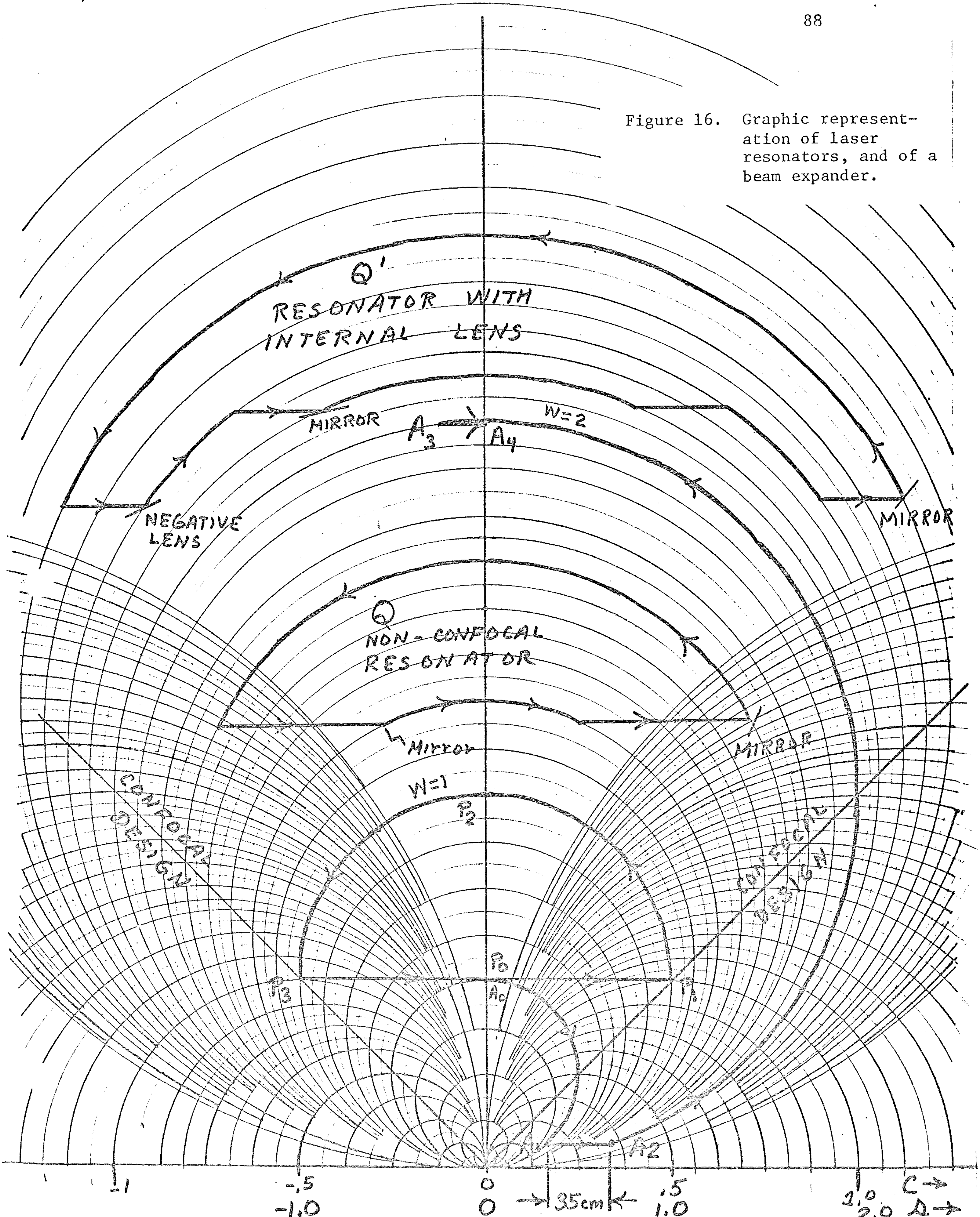



Figure 16. Graphic representation of laser resonators, and of a beam expander.

A convenient choice for "a" is $10^{-5}$, which gives (from Eqns 121) the correspondence:

$$\boxed{\begin{array}{ll} w = \dfrac{\sqrt{W}}{3.95}\ \text{cm.} & a = 10^{-5} \\ R = 200\,C\ \text{cm.} & f = 200\,F \\ z = 100\,\Delta\ \text{cm.} & \end{array}} \tag{122}$$

The curve $P_0\,P_1\,P_2\,P_3 \ldots$ etc. of Figure 16 represents such a laser.* At $P_1$, a distance of $\Delta = 1$ $(d = 100\ \text{cm})$ from the flat mirror at $z = 0$, the radius is $C = 1$ $(R = 200\ \text{cm})$ and requires a mirror with focal length $F = 1/4$ $(f = 50\ \text{cm})$ to change the curvature from + to − with the same magnitude. This mirror is placed at $P_1$ and moves the beam character around the constant $W = 1$ curve to $C = -1$ at $P_3$. The line $P_3 P_0$ is the return trip to the flat mirror.

Actually, the curved mirror could just as well be at $P_3$ in which case $P_0$ to $P_1$ also represents the beam transmitted through the flat mirror. A negative lens has been placed at the output $P_0$ which moves us to point $A_1$. The reason is so the passage from one $C$ curve to the next can be made in a shorter distance, $A_1$ to $A_2$. At $A_2$ a converging lens changes the curvature to $C =$ . At point $A_4$ we then have the desired output beam with $W = 2$ $(w = .358)$ and $R = \infty$. The focal lengths of the lenses at $P_0 = A_0$ and $A_2$ are $F_0 = -\frac{3}{16}$ $(f_0 = -37.5\ \text{cm.})$ and $F_2 = +.35$ $(f_2 = 70\ \text{cm})$; and they are 35 cm apart.

Both this and the graphic solution due to Gordon have the limitation of not having all possible points contained within a closed area as is the case with the analagous Smith chart for transmission line problems. In both, the fact that $W$ is proportional to $w^2$ makes it impossible to obtain accurate solutions to problems with large changes in $w$. In Gordon's

*Notice that the confocal requirement along with d=100 cm. from flat to curved mirror completely determines $W_o = W_{flat} = 0.5$ ($w_{flat} = .179$ cm.); $W_{curved} = 1$ ($w_{curved} = .253$ cm); and $C = 1$ ($R = 200$ cm.)

representation the $\frac{1}{W}$ and $\frac{1}{C}$ curves make up the rectangular grid, while constant $z$ curves and $\frac{1}{W}$ curves were represented by circles. The method here has the advantage of a linear scale for distance, and a rectangular grid (xy-plane) so simple that scales are not necessary for it.

## D. IMAGE LAWS; ABCD LAW; RAY TRANSFER MATRIX

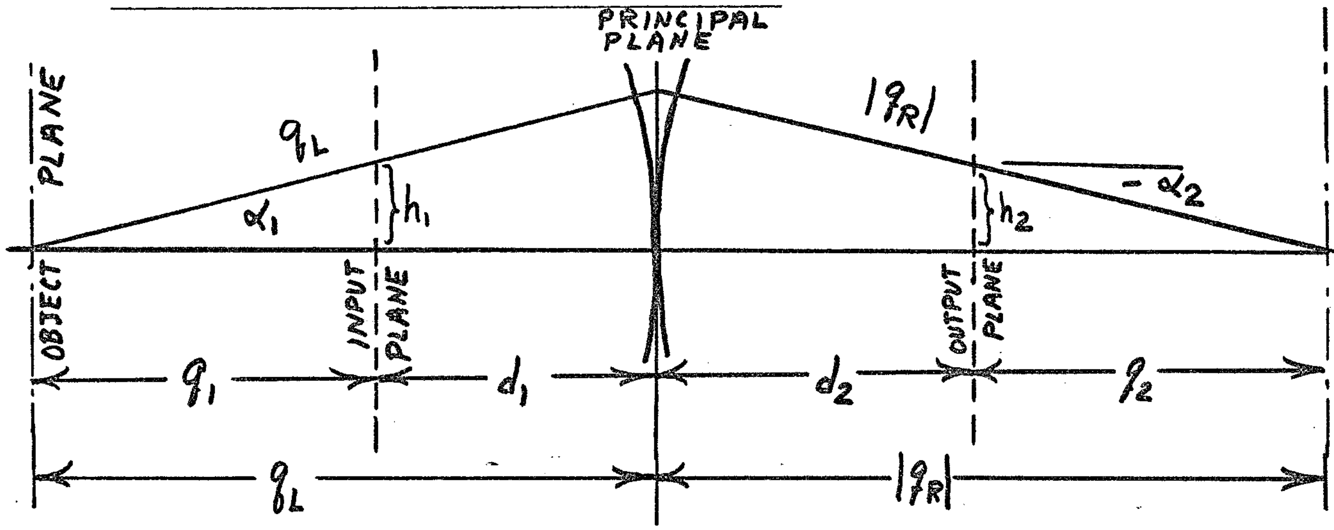



Figure 17. Symbols and sign conventions used with image laws.

The usual rules of ray optics apply, but the sign conventions used here are those shown in Figure 17, so that

$$\boxed{\begin{gathered} \frac{1}{q_L} - \frac{1}{q_R} = \frac{1}{f} \\ w_L^2 = w_R^2 \end{gathered}} \quad . \tag{123}$$

Since $q_L = q_1 + d_1$ and $q_R = q_2 - d_2$, in terms of the input and output planes shown, the first of these equations is

$$\boxed{\frac{1}{q_1 + d_1} - \frac{1}{q_2 - d_2} = \frac{1}{f}}, \tag{124}$$

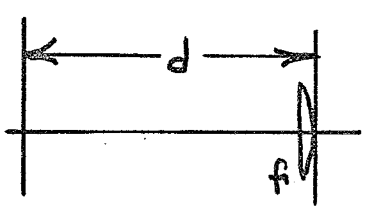
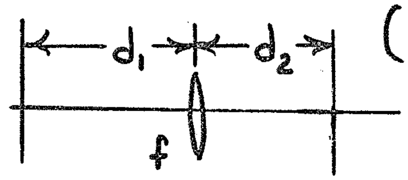
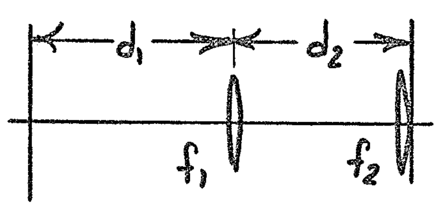
which can be solved for $q_2$ to obtain

$$q_2 = \frac{q_1\left(1 - \frac{d_2}{f}\right) + \left(d_1 + d_2 - \frac{d_1 d_2}{f}\right)}{\left(-\frac{1}{f}\right) q_1 + \left(1 - d_1/f\right)} .$$

This equation has the form

$$\boxed{q_2 = \frac{q_1 A + B}{q_1 C + D}} \tag{125}$$

and is called the ABCD law. The values of A, B, C, D for this and two other cases are given in the following

| SYSTEM | | $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ |
|---|---|---|
|  | (a) | $\begin{pmatrix} 1 & d \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{pmatrix}$ |
|  | (b) | $\begin{pmatrix} 1 - \frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -1/f & 1 - d_1/f \end{pmatrix}$ |
|  | (c) | $\begin{pmatrix} 1 - \frac{d_2}{f_1} & d_1 + d_2 - \frac{d_1 d_2}{f_1} \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{d_2}{f_1 f_2} & 1 - \frac{d_1}{f_1} - \frac{d_2}{f_2} - \frac{d_1}{f_2} + \frac{d_1 d_2}{f_1 f_2} \end{pmatrix}$ |

(126)

Equation (125) has another interpretation. The action of an optical system can be described by specifying for the output ray both the distance, $h_2$, from the optic axis and the slope, $\alpha_2$, in terms of their values at the input plane. From Figure 17 it can be seen that

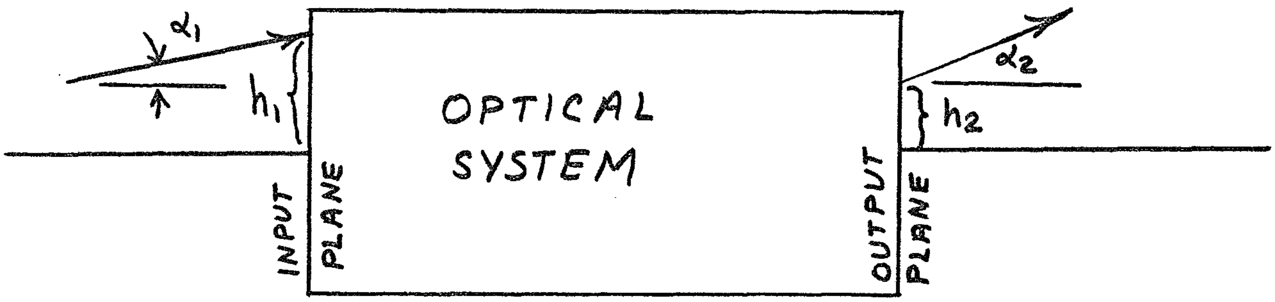



Figure 18. The effect of an optical system can be described by specifying the distance $h_2$ and slope $\alpha_2$ of a ray emerging from the output plane in terms of its values at the input plane.

$$\alpha_1 = \frac{h_1}{q_1} \quad \text{and} \quad \alpha_2 = \frac{h_2}{q_2} \tag{127}$$

where both $q_2$ and $\alpha_2$ are negative in the figure. Substitution of these values of $q_1$ and $q_2$ into Equation (125) gives

$$\frac{h_2}{\alpha_2} = \frac{h_1 A + \alpha_1 B}{h_1 C + \alpha_1 D} \tag{128}$$

Apparently, $h_2 = (const)(h_1 A + \alpha_1 B)$ and $\alpha_2 = (const)(h_1 C + \alpha_1 D)$. Since, $h_1 = h_2$ when $d_1 = d_2 = 0$ for the examples, the constant is unity. Thus,

$$\boxed{\begin{pmatrix} h_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h_1 \\ \alpha_1 \end{pmatrix}} \tag{129}$$

and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is also the *ray transfer matrix*.

## E. RESONATOR FREQUENCY

For resonators, the last factor in Equation (107) due to Boyd and Gordon is

$$\boxed{\sin\left\{k\left[\frac{b}{2}(1+\xi)+\frac{\xi}{1+\xi^2}\frac{\rho^2}{b}\right]-(1+m+n)\left(\frac{\pi}{2}-\varphi\right),\quad \text{where}\quad \xi=\frac{2z}{b},\ \text{and}\ \tan\varphi=\frac{1-\xi}{1+\xi}\right.} \tag{130}$$

A resonant cavity is formed with two mirrors such that (1) each coincides with a surface of constant phase, and (2) the phase difference between the two surfaces is $\pi$ times an integer, $q$. Thus,

$$(\text{phase})_2 = k\left[\frac{b}{2}+z_2+\frac{\xi_2\rho^2}{(1+\xi_2^2)b}\right]-(1+m+n)\left(\frac{\pi}{2}-\varphi_2\right)$$

$$(\text{phase})_1 = k\left[\frac{b}{2}+z_1+\frac{\xi_1\rho^2}{(1+\xi_1^2)b}\right]-(1+m+n)\left(\frac{\pi}{2}-\varphi_1\right)$$

Then,

$(\text{phase})_2 - (\text{phase})_1 = q\pi$ gives

Resonant Condition

$$\frac{2}{\lambda}\left\{\left[z_2+\frac{\rho^2}{b}\left(\frac{\xi_2}{1+\xi_2^2}\right)\right]-\left[z_1+\frac{\rho^2}{b}\left(\frac{\xi_1}{1+\xi_1^2}\right)\right]\right\}=q+\frac{1}{\pi}(1+m+n)(\varphi_1-\varphi_2) \tag{131}$$

where $\xi_2=\frac{2z_2}{b}$, $\xi_1=\frac{2z_1}{b}$, $\tan\varphi_1=\frac{1-\xi_1}{1+\xi_1}$, and $\tan\varphi_2=\frac{1-\xi_2}{1+\xi_2}$

A confocal resonator is formed by putting

$$z_2 = +b/2-\rho^2/2b \quad ; \quad \xi_2\approx 1$$
$$z_1 = -b/2+\rho^2/2b \quad ; \quad \xi_1\approx -1$$
$$z_2-z_1 = b = d\,.$$

and leads to

<u>Confocal Resonant Condition</u>

$$\frac{2b}{\lambda} = q + \frac{1}{2}(1+m+n) \tag{132}$$

Although small in comparison to $q$, the last term is important because modes with $m$ and $n$ different from zero are not degenerate with the $TEM_{00q}$ mode.

In practice, spherical mirrors are used, and it is sufficiently accurate to put,

$$z_2 = z_{20} - \frac{\rho^2}{b}\left(\frac{\xi_2}{1+\xi_2^2}\right) \quad ; \quad \xi_2 = \frac{2z_{20}}{b}$$

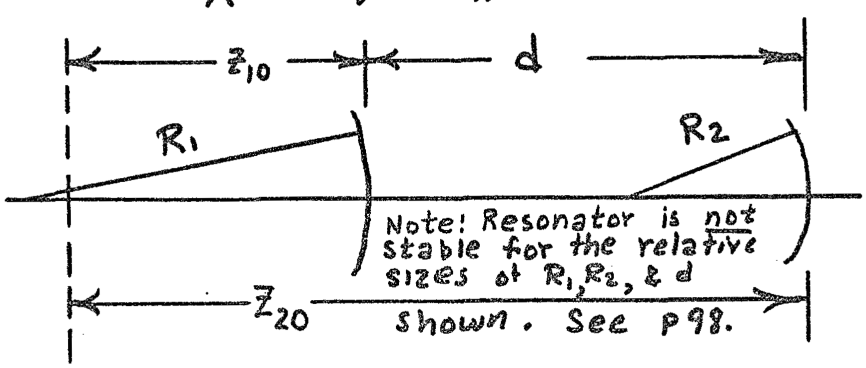
$$z_1 = z_{10} - \frac{\rho^2}{b}\left(\frac{\xi_1}{1+\xi_1^2}\right) \quad ; \quad \xi_1 = \frac{2z_{10}}{b}$$

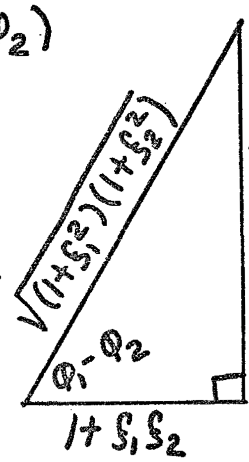
$$\text{and} \quad z_{20} - z_{10} = d$$

In Appendix M it is shown that $(\varphi_1 - \varphi_2) = \sin^{-1}\frac{\xi_2 - \xi_1}{\sqrt{(1+\xi_2^2)(1+\xi_1^2)}}$ so that the resonance condition (130) becomes:

<u>Resonance Condition: $TEM_{nmq}$ Modes</u>

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(1+m+n)(\varphi_1 - \varphi_2) \tag{133}$$



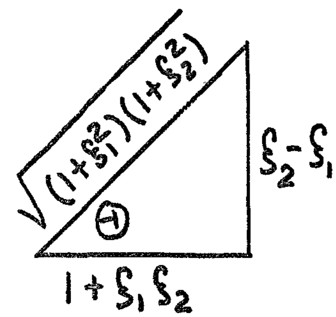

$$\frac{b}{R_1} = \frac{2\xi_1}{1+\xi_1^2} \quad ; \quad \frac{b}{R_2} = \frac{2\xi_2}{1+\xi_2^2}$$

$$\xi_1 = \frac{2z_{10}}{b}, \quad \xi_2 = \frac{2z_{20}}{b}, \quad \text{and} \quad z_{20} - z_{10} = d.$$

This result should be compared to that given by Boyd and Gordon, and also that of Boyd and Kogelnik. All three derive their results from our Equation (130), which is identical to Boyd's and Gordon's Equation (20), and all make the same approximations. The three equations have the form

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(1+m+n)\,\Theta,$$

where $\Theta$ is given by:

| Bell Sys. Tech J. <br> 40 (1961) pp. 489-508 <br> Boyd and Gordon's Eqn. (31) | Bell Sys. Tech J. <br> 41 (1962) pp. 1347-1369 <br> Boyd and Kogelnik's-Eqn.(46) | Eqn. (133) this paper |
|---|---|---|
| *Identical Mirrors* <br> $\frac{\pi}{2} - 2\tan^{-1}\left(\frac{b-d}{b+d}\right)$ <br> from which we obtain <br> $\cos^{-1}\left(\frac{b^2-d^2}{b^2+d^2}\right)$ | $\cos^{-1}\sqrt{\left(1-\frac{d}{R_1}\right)\left(1-\frac{d}{R_2}\right)}$ |   <br> $\xi_1 = \frac{2z_{10}}{b}$, $\xi_2 = \frac{2z_{20}}{b}$ <br> $z_{20}-z_{10} = d$ <br> $\frac{b}{R_1} = \frac{2\xi_1}{1+\xi_1^2}$, $\frac{b}{R_2} = \frac{2\xi_2}{1+\xi_2^2}$. |

For a resonator made with identical mirrors, $z_{20} = -z_{10} = \frac{2}{b}\frac{d}{2} = d/b$. The third column then gives: $\cos\Theta = \frac{1-(d/b)^2}{1+(d/b)^2}$, which agrees with Boyd and Gordon's equation in the first column. It does not, however, agree with the result in column two. For identical mirrors, $R_2 = -R_1 = \frac{1+(d/b)^2}{2(d/b)}$. When substituted into column two, this gives $\cos^{-1}\sqrt{\left(\frac{b^2-d^2}{b^2+d^2}\right)\left(\frac{b^2+3d^2}{b^2+d^2}\right)}$.

This factor involving $\Theta$ makes a negligible contribution to the overall distance calculation for resonance, but does show the conditions for lifting the $m,n$ degeneracy. The reason for starting with Equation (130) instead of a pure Gaussian beam, which approximates the lowest laser mode, was purely to insure that a term of this type would not be overlooked.

## F. RESONATOR STABILITY

Stability considerations can be approached through the ray transfer matrix methods. Most resonators are an iterated system of two phase transformers like that shown in Equation (126c), but with $d_1 = d_2 = d;\ f_1 = \frac{R_1}{2}$, and $f_2 = \frac{R_2}{2}$. The ray transfer matrix of such a system for one pass is:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - \frac{2d}{R_1} & 2d - \frac{2d^2}{R_1} \\ -\frac{2}{R_1} - \frac{2}{R_2} + \frac{4d}{R_1 R_2} & 1 - \frac{2d}{R_1} - \frac{4d}{R_2} + \frac{4d^2}{R_1 R_2} \end{pmatrix}.$$

Putting,

$$\boxed{g_1 = 1 - \frac{d}{R_1} \quad \text{and} \quad g_2 = 1 - d/R_2}, \qquad (134)$$

reduces this to:

$$\boxed{\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 2g_1 - 1 & 2dg_1 \\ \frac{2}{d}(2g_1 g_2 - g_1 - g_2) & 4g_1 g_2 - 2g_1 - 1 \end{pmatrix}} \qquad (135)$$

The ray transfer matrix, $T$, for "$n$" passes through such a system is:

$$T_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n .$$

The matrix of Equation (135) has the property, $AD - BC = 1$, so that Sylvester's theorem can be used to find the product of this matrix with itself

$n$ times. This theorem is derived in Appendix N, and states that:

$$\left.\begin{aligned}&\text{If } AD-BC=1, \text{ then}\\ &\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \frac{1}{\sin\theta}\begin{pmatrix} A\sin n\theta - \sin(n-1)\theta & B\sin n\theta \\ C\sin n\theta & D\sin n\theta - \sin(n-1)\theta \end{pmatrix},\\ &\text{where} \quad \cos\theta = \tfrac{1}{2}(A+D) = \tfrac{1}{2}\,\text{Trace}.\end{aligned}\right\} \tag{136}$$

From (13),

$$\cos\theta = \tfrac{1}{2}\,Trace = 2g_1 g_2 - 1.$$

A real transfer matrix for "$n$" passes can only result if $\cos\theta$ is between $-1$ and $+1$, which requires

$$-1 \leq (2g_1 g_2 - 1) \leq +1$$

$$0 \leq 2g_1 g_2 \leq 2$$

$$0 \leq g_1 g_2 \leq 1 .$$

This expression can also be stated in terms of the more general $G$'s, $\left(G_1 = \frac{a_1}{a_2} g_1 ,\; G_2 = \frac{a_2}{a_1} g_2\right)$, so that:

Stability Condition

$$\boxed{0 \leq g_1 g_2 \leq 1 \quad \text{or} \quad 0 \leq G_1 G_2 \leq 1} \tag{137}$$

A convenient chart showing the regions of possible stable resonators is shown in Figure 19. In this derivation, a converging mirror has a positive radius. It should be remarked that a similar approach can be used to obtain the stability condition for particle accelerators that use *alternating gradient focusing*.

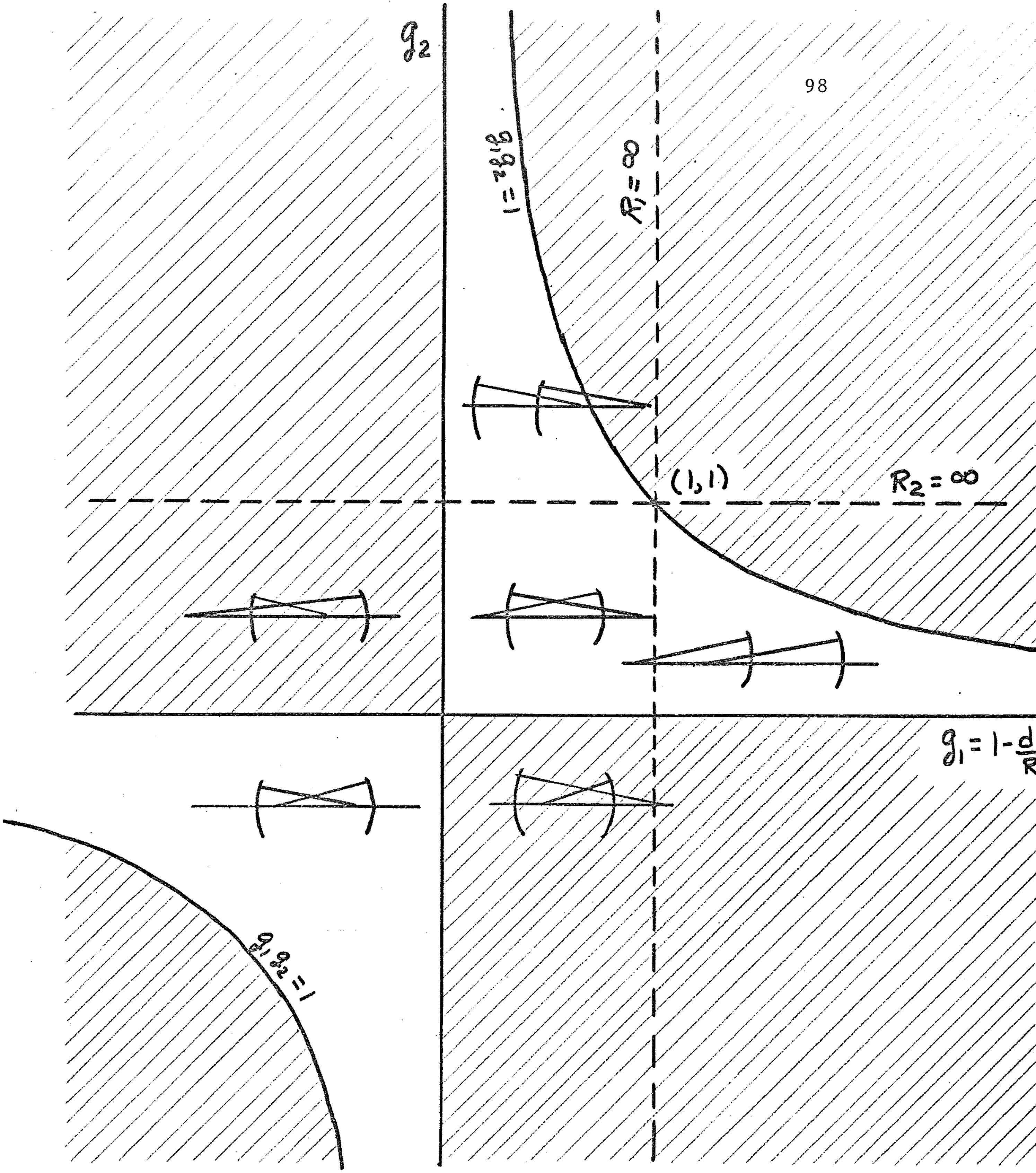



Figure 19. Illustrating the permissible values of $g_1$ and $g_2$ for optical resonators. The unstable regions are the shaded areas. A positive radius corresponds, in this scheme to a converging mirror. A confocal resonator is represented by a point at the origin.

## G. RESONATOR SPOT SIZE

The graphical method of Section C (this chapter) is a powerful tool for the design of resonator cavities. The relative spot sizes on the mirrors, radii of curvature, and length of cavity can be seen at a glance and adjusted to fit the desired design parameters. For more quantitative results, it is still the law of propagation of Equations (112) that must be satisfied. Gordon and Kogelnik [Bell Sys. Tech. J. 43, (1964), Eqn. (19)] give the equations:*

$$\left.\begin{aligned} \frac{w_1}{w_2} &= \sqrt{\frac{g_2}{g_1}} \\ w_1 w_2 &= \frac{\lambda d/\pi}{\sqrt{1-g_1 g_2}} \end{aligned}\right\} \qquad (138)$$

for determining the radii of the spots on the two mirrors.

~~*Attempts by the present author to verify these two equations have not yet been successful.~~

these eqns. are correct. See pp 10,12 of REPORT.

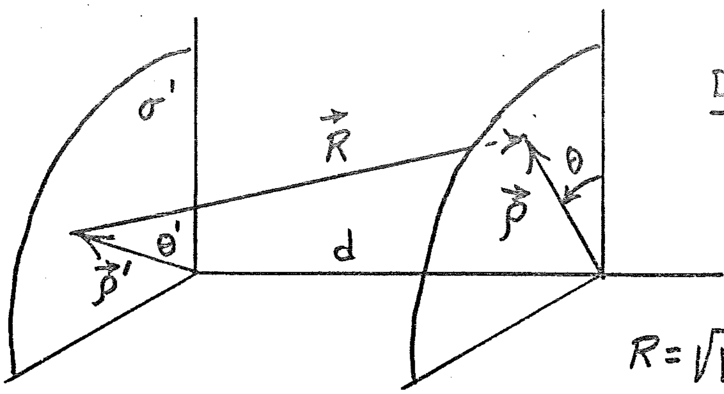
## H. IMAGE LAW FOR SCALAR FRESNEL DIFFRACTION

The basic diffraction integral from Kirchhaff's surface integral representation will now be applied to the lens situation of Figure 20. The fields to the left of the lens can be expressed in terms of those on the source, $\sigma_P$ ,

$$\tilde{\psi}_L(\vec{\rho}) = \frac{-ik}{2\pi d}\int_{\sigma_P} dS_P\, \tilde{\psi}(\vec{\rho}_P)\, e^{ik\left[P + \frac{1}{2P}\left(\rho_P^2 + \rho^2 - 2\vec{\rho}_P\cdot\vec{\rho}\right)\right]} \qquad (139)$$

The effect of the lens is to advance the phase an amount proportional to $\rho^2 = x^2 + y^2$ . For a focal length $f$ , a term $-k\left(\frac{\rho^2}{2f}\right)$ must be added to the phase. The field at points to the right of the lens then becomes

$$\tilde{\psi}_R(\vec{\rho}) = \frac{-ik}{2\pi d}\int_{\sigma_P} \tilde{\psi}(\vec{\rho}_P)\, e^{ik\left[P + \frac{1}{2}\rho^2\left(\frac{1}{P} - \frac{1}{f}\right) + \frac{1}{2P}\rho_P^2 - \frac{1}{P}\vec{\rho}_P\cdot\vec{\rho}\right]} \qquad (140)$$




Diffraction Integral

$$\tilde{\psi}(\vec{\rho}\,\Omega) = \frac{-ik}{2\pi}\int_{\sigma'} \frac{e^{ikR}}{R}\, dS'$$

$$\vec{R} = \vec{d} + \vec{\rho} - \vec{\rho}\,'$$

$$R = \sqrt{[\vec{d} + (\vec{\rho}-\vec{\rho}\,')]\cdot[\vec{d} + (\vec{\rho}-\vec{\rho}\,')]} \quad ; \quad \begin{array}{l} \vec{d}\cdot\vec{\rho} = 0 \\ \vec{d}\cdot\vec{\rho}\,' = 0 \end{array}$$

$$R \simeq d + \frac{1}{2d}(\vec{\rho}-\vec{\rho}\,')^2.$$

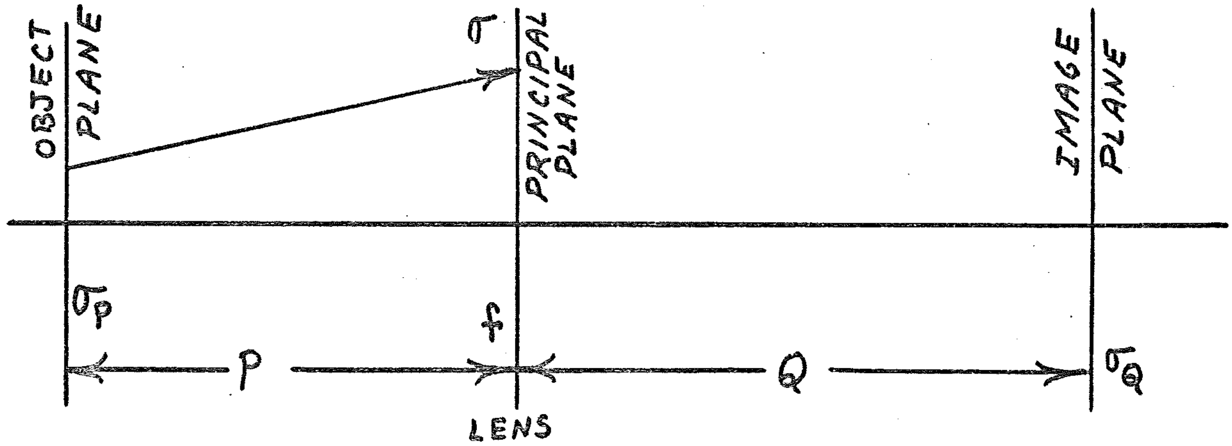



Figure 20. Application of the basic diffraction integral to the problem of image formation using a lens.

The diffraction integral can again be used to find the field at a point on the image plane in terms of its values on the principal plane:

$$\tilde{\psi}(\vec{\rho}_Q) = \frac{-ik}{2\pi Q}\int_{\sigma} dS_\sigma\, \tilde{\psi}_R(\vec{\rho})\, e^{ik[Q + \frac{1}{2Q}(\rho^2 + \rho_Q^2 - 2\vec{\rho}\cdot\vec{\rho}_Q)]} \quad . \tag{141}$$

Substituting $\tilde{\psi}_R$ from (140) into the expression gives:

$$\tilde{\psi}(\vec{\rho}_Q) = \frac{-k^2}{4\pi^2 PQ}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy \int_{\sigma_P} dS_P\, \tilde{\psi}(\vec{\rho}_P)\, e^{ik[P+Q+\frac{1}{2f}(\frac{1}{P}+\frac{1}{Q}-\frac{1}{f})+\frac{1}{2}(\frac{\rho_Q^2}{Q}+\frac{\rho_P^2}{P})]} \times e^{-ik[\frac{xx_Q+yy_Q}{Q} + \frac{xx_P+yy_P}{P}]} \quad , \tag{142}$$

where the extension of the limits of integration to include the entire principal plane of the lens is justified if the aperture is large compared to the beam diameter.

Since

$$\frac{P}{2\pi}\int_{-\infty}^{\infty} d\left(\frac{kx}{P}\right) e^{-i\left(\frac{kx}{P}\right)\left(x_P+\frac{P}{Q}x_Q\right)} = P\,\delta\left(x_P+\frac{P}{Q}x_Q\right),$$

and likewise for the $y$ integration, Equation (142) reduces to

$$\tilde{\psi}(\vec{\rho}_Q) = \frac{-P^2}{PQ}\int_{\sigma} dS_P\, \tilde{\psi}(\vec{\rho}_P)\, e^{ik\left[P+Q+\frac{1}{2}\left(\frac{\rho_Q^2}{Q}+\frac{\rho_P^2}{P}\right)\right]}\, \delta\left(\frac{x_P}{P}+\frac{x_Q}{Q}\right)\delta\left(\frac{y_P}{P}+\frac{y_Q}{Q}\right), \tag{143}$$

where use was also made of

$$\frac{1}{P}+\frac{1}{Q}=\frac{1}{f}\quad. \tag{144}$$

The remaining integral over the source, $\sigma_P$ , is zero unless

$$\vec{\rho}_P = -\frac{P}{Q}\vec{\rho}_Q \quad. \tag{145}$$

With the help of (144) and this last equation, the factor $\frac{1}{2}\left(\frac{\rho_Q^2}{Q}+\frac{\rho_P^2}{P}\right)$ reduces to $\frac{\rho_Q^2}{2f}\frac{P}{Q}$ . The image law then is:

<u>Image Law: Scalar Fresnel Diffraction</u>

$$\boxed{\begin{gathered}\tilde{\psi}_{\substack{ON\\ \sigma_Q}}(\vec{\rho}_Q,\Omega) = -\frac{P}{Q}\,\tilde{\psi}_{\substack{ON\\ \sigma_P}}\left(-\frac{P}{Q}\vec{\rho}_Q\right) e^{ik\left[P+Q+\frac{P}{Q}\frac{\rho_Q^2}{2f}\right]}\\ \vec{\rho}_P = -\frac{P}{Q}\vec{\rho}_Q\end{gathered}} \tag{146}$$

.

## I. CONJUGATE RELATIONS IN ENTRANCE AND EXIT PUPILS

Equation (146) just derived assumes a real propagation of light from the object plane $\sigma_P$ to the image plane $\sigma_Q$, so that light propagates according to the diffraction formula. This result clearly applies to conjugate points in the entrance and exit pupils of Figure 21a, where there is clearly a source plane $\sigma_P$, image plane $\sigma_Q$, and where Equation (144) is satisfied between these planes. We now show that Equation (146) is also valid for conjugate points in the entrance and exit pupils of Figures (21b) and (21c), even though the entrance pupils are virtual. In both cases, however, the relation $\frac{1}{P} + \frac{1}{Q} = \frac{1}{f}$ is still satisfied, but with $P_v = -P$ = positive number.

Let $\tilde{\Psi}(\vec{\rho}_\sigma, \Omega)$ represent the field at points just to the left of the principal plane. In finding the field at points in the entrance pupil from the basic diffraction integral, the light must be treated as though it has not passed through the lens. That is;

$$\tilde{\Psi}(\vec{\rho}_P) = \frac{-ik}{2\pi P_v} \int_\sigma dS_\sigma \tilde{\Psi}(\vec{\rho}_\sigma)\, e^{ik\left[P_v + \frac{1}{2P_v}(\rho_P^2 - \rho_\sigma^2)\right]} . \qquad (147)$$

To obtain the fields at the exit pupil, the phase must first be advanced to account for passage through the lens. Thus,

$$\tilde{\Psi}(\vec{\rho}_Q) = \frac{-ik}{2\pi Q} \int_\sigma dS_\sigma \tilde{\Psi}(\vec{\rho}_\sigma)\, e^{ik\left[Q - \frac{\rho_\sigma^2}{2f} + \frac{1}{2Q}(\vec{\rho}_Q - \vec{\rho}_\sigma)^2\right]} . \qquad (148)$$

The first of these integrals (146) can also be written

$$\underset{\substack{\text{ON} \\ \sigma_P}}{\tilde{\Psi}}\left(-\frac{P}{Q}\vec{\rho}_Q\right) = \frac{-ik}{2\pi Q}\left(\frac{Q}{P_v}\right) \int_\sigma dS_\sigma \tilde{\Psi}(\rho_\sigma)\, e^{ik\left[-P - \frac{1}{2P}\left(-\frac{P}{Q}\rho_Q\right)^2 - \frac{\rho_\sigma^2}{2P} + \frac{1}{2P}\left(-\frac{P}{Q}\right)2\vec{\rho}_Q\cdot\vec{\rho}_\sigma\right]} ,$$

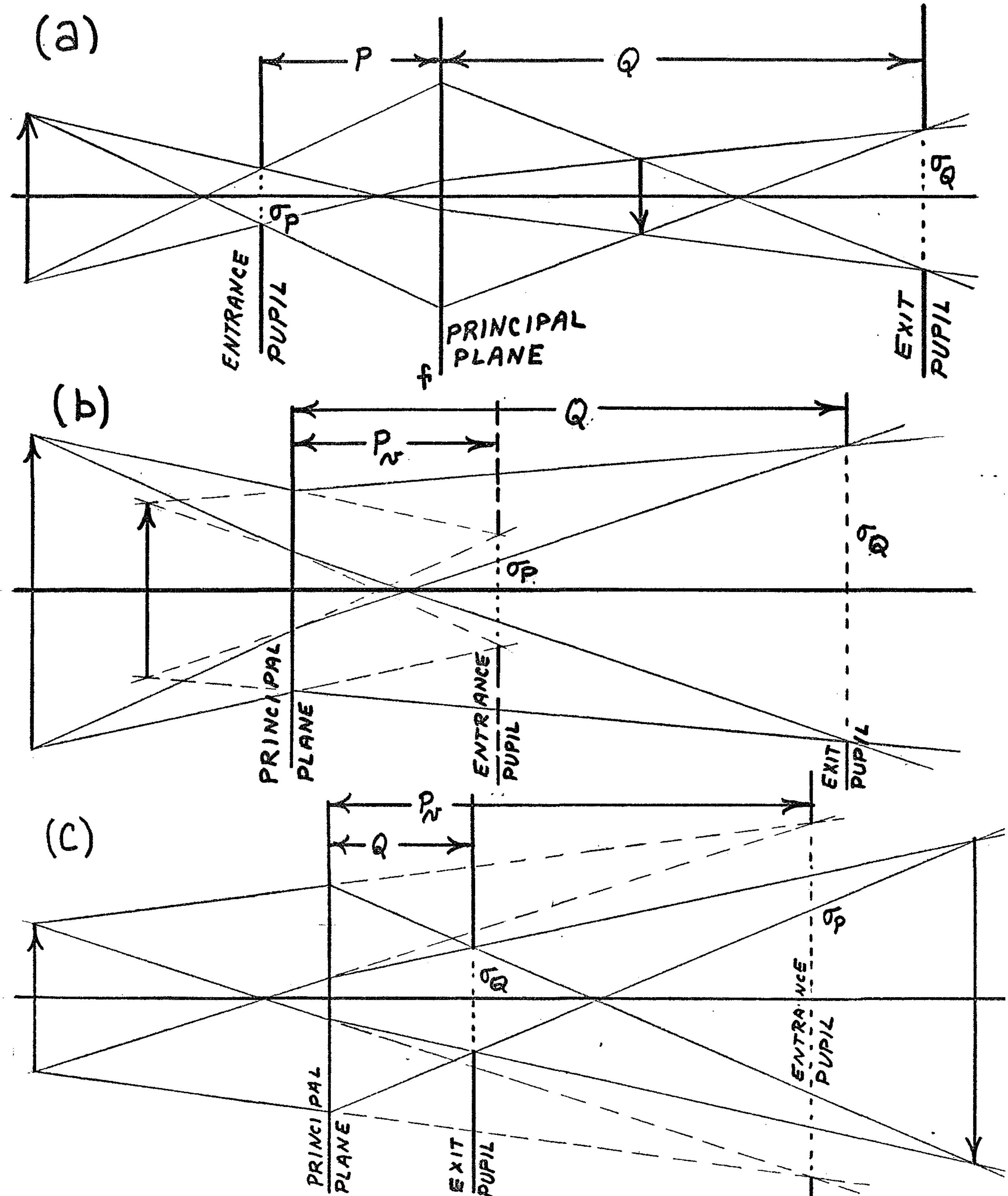



Figure 21. Optical systems for which Equation (146) is valid for conjugate points in the entrance and exit pupils. In the case of virtual entrance pupils of (b) and (c) the approximation of infinite limits on the integration over the principal plane is not necessary.

and with the help of (144) becomes

$$\tilde{\psi}_{\substack{ON\\ \sigma_P}}\left(-\frac{P}{Q}\vec{\rho}_Q\right)=\left(\frac{Q}{P_N}\right)\left\{\frac{-ik}{2\pi Q}\int_\sigma dS_\sigma\,\tilde{\psi}(\vec{\rho}_\sigma)\,e^{ik\left[\left(\frac{1}{Q}-\frac{1}{f}\right)\frac{\rho_\sigma^2}{2}-\frac{1}{Q}\vec{\rho}_Q\cdot\vec{\rho}_\sigma\right]}\right\}e^{ik\left[-Q-\frac{P}{2Q^2}\rho_Q^2\right]} \quad . \qquad (149)$$

The integration in brackets is identical to that in Equation (148). Substitution of the value from (148) gives

$$\tilde{\psi}_{\substack{ON\\ \sigma_P}}\left(-\frac{P}{Q}\vec{\rho}_Q\right)=\frac{+Q}{P_N}\left\{\tilde{\psi}_{\substack{ON\\ \sigma_Q}}(\vec{\rho}_Q)\,e^{-ik\left[P+\frac{1}{2Q}\rho_Q^2\right]}\right\}e^{-ik\left[Q+\frac{P}{2Q^2}\rho_Q^2\right]}$$

whereby

$$\boxed{\tilde{\psi}_{\substack{ON\\ \sigma_Q}}(\vec{\rho}_Q,\Omega)=\frac{-P}{Q}\,\tilde{\psi}_{\substack{ON\\ \sigma_P}}\left(-\frac{P}{Q}\vec{\rho}_Q,\Omega\right)e^{ik\left[P+Q+\frac{P}{Q}\frac{\rho_Q^2}{2f}\right]}} \qquad (150)$$

which is identical to equation (146) for the case of a real entrance pupil. In this derivation, however, it is not necessary to extend the limits of integration over the principal plane to cover the entire plane.

## J. CONJUGATE RELATIONS FOR MUTUAL COHERENCE

In each of the examples of Figure 21, the field at a point on a plane $\sigma_Q$ can be determined in terms of the field at a conjugate point on plane $\sigma_P$ by using Equation (150). It follows that $\tilde{\Gamma}$ between points on $\sigma_Q$ can be found in terms of its value at conjugate points on $\sigma_P$. One application of Equation (150) gives:

$$\underset{\sigma_Q}{\tilde{\Gamma}_{ON}}(\vec{\rho}_{Q1},\vec{\rho}_{Q2},\Omega) = -\frac{P}{Q}\tilde{\Gamma}\left(-\frac{P}{Q}\vec{\rho}_{Q1}=\vec{\rho}_{P1},\vec{\rho}_{P2},\Omega\right) e^{iK\left(P+Q+\frac{P}{Q}\frac{\rho_{Q1}^2}{2f}\right)} .$$

In the second application of (150), it is the complex conjugate, $\tilde{\psi}^*$, that is being transposed, so that

$$\tilde{\Gamma}\left(\vec{\rho}_{P1}=-\frac{P}{Q}\vec{\rho}_{Q1},\vec{\rho}_{P2}\right) = -\frac{P}{Q}\tilde{\Gamma}\left(\vec{\rho}_{P1}=\frac{-P}{Q}\vec{\rho}_{Q1},\vec{\rho}_{P2}=\frac{-P}{Q}\vec{\rho}_{Q2}\right) e^{-iK\left[P+Q+\frac{P}{Q}\frac{\rho_{Q2}^2}{2f}\right]} .$$

The result of combining these last two equations is:

$$\boxed{\begin{aligned} &\underset{\sigma_Q}{\tilde{\Gamma}_{ON}}\left(d_{12}^{(Q)},\Omega\right) = \frac{P^2}{Q^2}\underset{\sigma_P}{\tilde{\Gamma}_{ON}}\left(d_{12}^{(P)}=\frac{-P}{Q}d_{12}^{(Q)},\Omega\right) e^{i\frac{K}{2f}\frac{P}{Q}\left(\rho_{Q1}^2-\rho_{Q2}^2\right)} \\ &\text{where we also have :} \quad \frac{1}{P}+\frac{1}{Q}=\frac{1}{f} \\ &\vec{\rho}_{P1} = \vec{\rho}_1^{(P)} = -\frac{P}{Q}\vec{\rho}_1^{(Q)} \\ &\vec{\rho}_{P2} = \vec{\rho}_2^{(P)} = -\frac{P}{Q}\vec{\rho}_2^{(Q)} \quad ; \quad d_{12}^{(P)} = \frac{-P}{Q}d_{12}^{(Q)} \end{aligned}} \tag{151}$$

# APPENDIX I

# FREQUENCY AND PHASE MODULATION

## Frequency Modulation

The identities

$$\left.\begin{aligned}
\cos(z\sin\theta) &= J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\theta) \\
\sin(z\sin\theta) &= 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin[(2k+1)\theta] \\
\cos(z\cos\theta) &= J_0(z) + 2\sum_{k=1}^{\infty} (-)^k J_{2k}(z)\cos(2k\theta) \\
\text{and}\qquad \sin(z\cos\theta) &= 2\sum_{k=0}^{\infty} (-)^k J_{2k+1}(z)\cos[(2k+1)\theta]
\end{aligned}\right\}\quad (\text{I}1)$$

are useful for analyzing frequency and phase modulated signals, along with

$$\left.\begin{aligned}
\int_0^{\infty} J_n(\alpha x) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\beta x)\, dx &= \frac{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(n\sin^{-1}\beta/\alpha\right)}{\sqrt{\alpha^2-\beta^2}}, \quad [\beta<\alpha] \\
&= \infty \text{ or } 0, \quad [\beta=\alpha] \\
&= \frac{\alpha^n \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\frac{n\pi}{2}\right)}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^n}, \quad [\beta>\alpha]
\end{aligned}\right\}\quad (\text{I}2).$$

The 100% frequency modulated signal

$$\cos(\Omega t \cos\delta t), \quad \Omega>\delta$$

has the frequency spectrum (eqn. 61)

$$\tilde{\psi}(\omega) = 2\int_{-\infty}^{\infty} \cos[\Omega t\cos\delta t]\, e^{+i\omega t}\, dt,$$

$$= 2\int_{-\infty}^{\infty} \{\cos(\Omega t\cos\delta t)\cos\omega t + i\cos(\Omega t\cos\delta t)\sin\omega t\}\, dt$$

The second integral vanishes because the integrand is an odd function, leaving:

$$\tilde{\psi}(\omega) = 4\int_0^\infty \cos(\Omega t \cos\delta t)\cos\omega t\,dt$$

$$= 4\int_0^\infty \Big\{J_0(\Omega t) + 2\sum_{k=1}^{\infty}(-)^k J_{2k}(\Omega t)\cos(2k\delta t)\Big\}\cos\omega t\,dt$$

$$= 4\int_0^\infty \Big\{J_0(\Omega t)\cos\omega t + \sum_{k=1}^{\infty}(-)^k J_{2k}(\Omega t)\big[\cos(2k\delta+\omega)t + \cos(2k\delta-\omega)t\big]dt .$$

$$\tilde{\psi}(\omega) = 4\sum_{k=1}^{\infty}(-)^k\Bigg\{ \frac{\sin\left(2k\sin^{-1}\frac{2k\delta+\omega}{\Omega}\right)}{\sqrt{\Omega^2-(2k\delta+\omega)^2}}\Bigg|_{\omega<(\Omega-2k\delta)}$$

$$+ \frac{\sin\left(2k\sin^{-1}\frac{2k\delta-\omega}{\Omega}\right)}{\sqrt{\Omega^2-(2k\delta-\omega)^2}}\Bigg|_{\omega>-(\Omega-2k\delta)}$$

$$+ \frac{\Omega^{2k}\cos(k\pi)}{\sqrt{\Omega^2-(2k\delta+\omega)^2}\left[\Omega+\sqrt{\Omega^2-(2k\delta+\omega)^2}\right]^{2k}}\Bigg|_{\omega>(\Omega-2k\delta)}$$

$$+ \frac{\Omega^{2k}\cos(k\pi)}{\sqrt{\Omega^2-(2k\delta-\omega)^2}\left[\Omega+\sqrt{\Omega^2-(2k\delta-\omega)^2}\right]^{2k}}\Bigg|_{\omega<-(\Omega-2k\delta)}\Bigg\}$$

$$+ \frac{1}{\sqrt{\Omega^2-\omega^2}}\Bigg|_{\omega<\Omega},$$

where the first and third terms are the main contributors to positive frequencies when $\Omega \gg \delta$.

Phase Modulation:

For a phase modulated signal such as

$$\psi^{(m)}(t) = \cos\left[\Omega t + 2\pi\cos\delta t\right]$$

$$= \cos\Omega t\cos(2\pi\cos\delta t) - \sin\Omega t\sin(2\pi\cos\delta t),$$

the identies in I1 lead to:

$$\psi^{(m)}(t) = \cos\Omega t\Big\{J_0(2\pi) + 2\sum_{k=1}^{\infty}(-)^k J_{2k}(2\pi)\cos(2k\delta t)\Big\}$$

$$-\sin\Omega t\Big\{2\sum_{k=0}^{\infty}(-)^k J_{2k+1}(2\pi)\cos(2k\delta t)\Big\}.$$

$$\psi^{(+)}(t) = J_0(2\pi)\cos\Omega t + \sum_{k=1}^{\infty}(-)^k J_{2k}(2\pi)\left[\cos(\Omega+2k\delta)t + \cos(\Omega-2k\delta)t\right] - \sum_{k=0}^{\infty}(-)^k J_{2k+1}(2\pi)\left[\sin(\Omega+(2k+1)\delta)t + \sin(\Omega-(2k+1)\delta)t\right].$$

From equation (61), $\tilde{\Psi}(\omega) = 2\int_0^{\infty}\psi^{(+)}(t)\,e^{i\omega t}\,dt$, we then find:

$$\tilde{\Psi}(\omega) = J_0(2\pi)\,\delta(\omega-\Omega) + i\,J_1(2\pi)\left[\delta(\omega-(\Omega+\delta)) + \delta(\omega-(\Omega-\delta))\right]$$
$$+ \sum_{k=1}^{\infty}(-)^k\Big\{J_{2k}(2\pi)\left[\delta(\omega-[\Omega+2k\delta]) + \delta(\omega-[\Omega-2k\delta])\right.$$
$$\left. + i J_{2k+1}(2\pi)\left[\delta(\omega-[\Omega+(2k+1)\delta]) + \delta(\omega-[\Omega-(2k+1)\delta])\right]\right]$$

$$+ \left(\text{same negative frequencies};\ \tilde{\Psi}(-\omega) = \tilde{\Psi}^*(\omega)\right).$$

# APPENDIX J

## SUPERPOSITION OF TWO SINUSOIDAL SIGNALS

Consider

$$\psi^{(r)}(t) = A\cos(\omega_a t + a) + B\cos(\omega_b t + b).$$

Define:

$$\boxed{\begin{aligned}\Omega &= \tfrac{1}{2}(\omega_a + \omega_b)\\ \delta &= \tfrac{1}{2}(\omega_a - \omega_b)\end{aligned}} \quad \therefore \quad \begin{aligned}\omega_a &= \Omega + \delta\\ \omega_b &= \Omega - \delta.\end{aligned}$$

$$\psi^{(r)}(t) = \tfrac{1}{2}(A+B)\left[\cos(\omega_a t + a) + \cos(\omega_b t + b)\right] + \tfrac{1}{2}(A-B)\left[\cos(\omega_a t + a) - \cos(\omega_b t + b)\right].$$

To evaluate

$$\cos(\omega_a t + a) \pm \cos(\omega_b t + b) = \cos\omega_a t\cos a - \sin\omega_a t\sin a \pm \cos\omega_b t\cos b \mp \sin\omega_b t\sin b$$

$$= \tfrac{1}{2}(\cos a \pm \cos b)(\cos\omega_a t + \cos\omega_b t) + \tfrac{1}{2}(\cos a \mp \cos b)(\cos\omega_a t - \cos\omega_b t)$$

$$- \tfrac{1}{2}(\sin a \pm \sin b)(\sin\omega_a t + \sin\omega_b t) - \tfrac{1}{2}(\sin a \mp \sin b)(\sin\omega_a t - \sin\omega_b t)$$

But,

$$\cos\omega_a t \pm \cos\omega_b t = \cos\Omega t\cos\delta t - \sin\Omega t\sin\delta t \pm \cos\Omega t\cos\delta t \pm \sin\Omega t\sin\delta t$$

$$\sin\omega_a t \pm \sin\omega_b t = \sin\Omega t\cos\delta t + \cos\Omega t\sin\delta t \pm \sin\Omega t\cos\delta t \mp \cos\Omega t\sin\delta t$$

$$\therefore \cos(\omega_a t + a) \pm \cos(\omega_b t + b) =$$

$$(\cos a \pm \cos b)\cos\Omega t\cos\delta t - (\sin a \mp \sin b)\cos\Omega t\sin\delta t$$

$$- (\cos a \mp \cos b)\sin\Omega t\sin\delta t - (\sin a \pm \sin b)\sin\Omega t\cos\delta t.$$

$$= \left[(\cos a \pm \cos b)\cos\delta t - (\sin a \mp \sin b)\sin\delta t\right]\cos\Omega t$$

$$- \left[(\cos a \mp \cos b)\sin\delta t + (\sin a \pm \sin b)\cos\delta t\right]\sin\Omega t.$$

$$\psi^{(r)}(t) = \tfrac{1}{2}(A+B)\Big\{\left[(\cos a + \cos b)\cos\delta t - (\sin a - \sin b)\sin\delta t\right]\cos\Omega t$$

$$- \left[(\cos a - \cos b)\sin\delta t + (\sin a + \sin b)\cos\delta t\right]\sin\Omega t\Big\}$$

$$+ \tfrac{1}{2}(A-B)\Big\{\left[(\cos a - \cos b)\cos\delta t - (\sin a + \sin b)\sin\delta t\right]\cos\Omega t$$

$$- \left[(\cos a + \cos b)\sin\delta t + (\sin a - \sin b)\cos\delta t\right]\sin\Omega t\Big\}.$$

$$\psi^{(r)}(t) = (A+B)\Big\{\left[1 + \cos(a+b)\right]\cos\left(\delta t + \tan^{-1}\frac{\sin a - \sin b}{\cos a + \cos b}\right)\cos\Omega t$$

$$- \left[1 - \cos(a+b)\right]\sin\left(\delta t + \tan^{-1}\frac{\sin a + \sin b}{\cos a - \cos b}\right)\sin\Omega t\Big\}$$

$$+ (A-B)\Big\{\left[1 - \cos(a+b)\right]\cos\left(\delta t + \tan^{-1}\frac{\sin a + \sin b}{\cos a - \cos b}\right)\cos\Omega t$$

$$- \left[1 + \cos(a+b)\right]\sin\left(\delta t + \tan^{-1}\frac{\sin a - \sin b}{\cos a + \cos b}\right)\sin\Omega t\Big\}$$

$$[\text{illegible}] = \Big[2(A+B)\cos\left(\tfrac{a+b}{2}\right)\cos\left(\delta t + \tan^{-1}\tfrac{\sin a - \sin b}{\cos a + \cos b}\right)$$
$$+ 2(A-B)\sin\left(\tfrac{a+b}{2}\right)\cos\left(\delta t + \tan^{-1}\tfrac{\sin a + \sin b}{\cos a - \cos b}\right)\Big]\cos\Omega t$$
$$-\Big[2(A-B)\cos\left(\tfrac{a+b}{2}\right)\sin\left(\delta t + \tan^{-1}\tfrac{\sin a - \sin b}{\cos a + \cos b}\right)$$
$$+2(A+B)\sin\left(\tfrac{a+b}{2}\right)\sin\left(\delta t + \tan^{-1}\tfrac{\sin a + \sin b}{\cos a - \cos b}\right)\Big]\sin\Omega t.$$

Thus,

$$\psi^{(+)}(t) = (U_+ + U_-)\cos\Omega t - (V_+ + V_-)\sin\Omega t$$
$$= \sqrt{(U_+ + U_-)^2 + (V_+ + V_-)^2}\;\cos\left(\Omega t + \tan^{-1}\frac{V_+ + V_-}{U_+ + U_-}\right),$$

where

$$U_+ = 2(A+B)\cos\left(\frac{a+b}{2}\right)\cos\left(\delta t + \tan^{-1}\frac{\sin a - \sin b}{\cos a + \cos b}\right)$$
$$U_- = 2(A-B)\sin\left(\frac{a+b}{2}\right)\cos\left(\delta t + \tan^{-1}\frac{\sin a + \sin b}{\cos a - \cos b}\right)$$
$$V_+ = 2(A+B)\sin\left(\frac{a+b}{2}\right)\sin\left(\delta t + \tan^{-1}\frac{\sin a + \sin b}{\cos a - \cos b}\right)$$
$$V_- = 2(A-B)\cos\left(\frac{a+b}{2}\right)\sin\left(\delta t + \tan^{-1}\frac{\sin a - \sin b}{\cos a + \cos b}\right)$$

# APPENDIX K

## SUPERPOSITION OF TWO GAUSSIAN SIGNALS

Here the problem is to find

$$\boxed{\psi(t) = \int_0^\infty \tilde{\psi}(\omega)\, e^{-i\omega t}\, d\nu} \qquad (K1)$$

when

$$\boxed{\tilde{\psi}(\omega) = \frac{A}{\sqrt{\Delta\omega_a}}\, e^{-\left(\frac{\omega-\omega_a}{\sqrt{2}\,\Delta\omega_a}\right)^2} + \frac{B}{\sqrt{\Delta\omega_b}}\, e^{-\left(\frac{\omega-\omega_b}{\sqrt{2}\,\Delta\omega_b}\right)^2}} \qquad (K2)$$

These integrals have been evaluated in Appendix L. Using the first of equations (L1) gives:

$$\boxed{\begin{gathered}\psi(t) = \frac{A}{2\sqrt{2\pi\Delta t_a}}\, e^{-i\omega_a t} e^{-\left(\frac{t}{\sqrt{2}\,\Delta t_a}\right)^2} + \frac{B}{2\sqrt{2\pi\Delta t_b}}\, e^{-i\omega_b t} e^{-\left(\frac{t}{\sqrt{2}\,\Delta t_b}\right)^2} \\ \text{where} \quad \Delta\omega_a \Delta t_a = 1 \quad , \text{and} \quad \Delta\omega_b \Delta t_b = 1\end{gathered}} \qquad (K3)$$

Putting,

$$\left.\begin{aligned} a &\equiv \frac{A}{2\sqrt{2\pi\Delta t_a}}\, e^{-\left(\frac{t}{\sqrt{2}\,\Delta t_a}\right)^2} \\ \mathcal{B} &= \frac{B}{2\sqrt{2\pi\Delta t_b}}\, e^{-\left(\frac{t}{\sqrt{2}\,\Delta t_b}\right)^2} \end{aligned}\right\} \qquad (K4)$$

the real part of $\psi(t)$ can be written

$$\psi^{(R)}(t) = a(t)\cos\omega_a t + \mathcal{B}(t)\cos\omega_b t \,.$$

Using the rules of Appendix J, this becomes:

$$\psi^{(r)}(t) = \sqrt{4(a+\mathcal{B})^2\cos^2\delta t + 4(a-\mathcal{B})^2\sin^2\delta t}\;\cos\left(\Omega t + \tan^{-1}\left(\frac{a-\mathcal{B}}{a+\mathcal{B}}\tan\delta t\right)\right)$$

$$\boxed{\psi^{(r)}(t) = 2\sqrt{2}\sqrt{a^2+\mathcal{B}^2+a\mathcal{B}\cos(2\delta t)}\;\cos\left(\Omega t + \tan^{-1}\left(\frac{a-\mathcal{B}}{a+\mathcal{B}}\tan\delta t\right)\right)} \qquad (K5)$$

where $\Omega = \frac{1}{2}(\omega_a+\omega_b)$, and $\delta = \frac{1}{2}(\omega_a-\omega_b)$.

If $a(t) = \mathcal{B}(t)$, this becomes

$$\psi^{(r)}_{a=\mathcal{B}}(t) = \frac{A}{\sqrt{\pi\Delta t}}\, e^{-\left(\frac{t}{\sqrt{2}\,\Delta t}\right)^2}\sqrt{1+\cos(2\delta t)}\;\cos(\Omega t),$$

$$\psi^{(r)}_{a=\mathcal{B}}(t) = A\sqrt{\frac{2}{\pi\Delta t}}\; e^{-\left(\frac{t}{\sqrt{2}\Delta t}\right)^2} \cos(\delta t)\,\cos(\Omega t),$$

so that a beat frequency is easily detected even for very broad spectral lines and small frequency differences. For square law detection at frequency $\Omega$, the output signal is then proportional to:

$$V_{a=\mathcal{B}}(t) \propto \frac{1}{\Delta t}\, e^{-\left(\frac{t}{\Delta t}\right)^2} (1+\cos 2\delta t), \qquad (K6)$$

and has a d.c. component as well as one at frequency $2\delta = \omega_a - \omega_b$.

If $\mathcal{B}(t)$ is much weaker than a reference signal $a(t)$, it is convenient to write equation (K5) in the form:

$$\psi^{(r)}_{\mathcal{B}\ll a}(t) = 2\sqrt{2}\, a\sqrt{1+\frac{\mathcal{B}}{a}\cos(2\delta t)}\;\; \cos(\omega_a t).$$

Square law detection at frequency $\omega_a$ then produces a signal proportional to:

$$V_{B\ll A}(t) \propto \frac{1}{\sqrt{\Delta t_a}}\, e^{-\left(\frac{t}{\Delta t_a}\right)^2} \left\{A^2 + AB\sqrt{\frac{\Delta t_a}{\Delta t_b}}\, e^{-\frac{t^2}{2}\left(\frac{1}{(\Delta t_b)^2}-\frac{1}{(\Delta t_a)^2}\right)} \cos(2\delta t)\right\}. \quad (K7)$$

Since the beat signal is proportional to the product $AB$, a strong reference signal "A" is desirable.

It should be pointed out that this mixing to produce beats is a <u>large effect</u> and may be accomplished with <u>zero mutual coherence</u> between the two signals. The two signals superposed in equation (K3) have the mutual coherence function:

$$\Gamma_{12}(\tau) = \langle \psi_1(t+\tau)\psi^*(t)\rangle$$

$$= \frac{1}{2T}\int_{-T}^{T} \frac{AB}{4\pi\sqrt{\Delta t_a \Delta t_b}}\, e^{-\frac{t^2}{2}\left(\frac{1}{(\Delta t_a)^2}+\frac{1}{(\Delta t_b)^2}\right)} e^{i(\omega_2-\omega_1)t} e^{-i\omega_1\tau}\, dt. \quad \text{(K8)}$$

If $e^{-\frac{t^2}{2}\left[\frac{1}{(\Delta t_a)^2}+\frac{1}{(\Delta t_b)^2}\right]}$ is essentially constant over several complete oscillations of the term $e^{i(\omega_2-\omega_1)t}$, the value of this integral will be nearly zero. thus, the time for a complete oscillation at frequency $\omega_2-\omega_1$ must be appreciably greater than (or at least comparable to) one of the $\Delta t$'s before the integral will have a significant value. that is,

$$\frac{2\pi}{\omega_a-\omega_b} > (\sim\Delta t) \quad ; \text{otherwise } \Gamma_{12}(\tau)\approx 0.$$

$$\therefore \quad \boxed{(\omega_a-\omega_b)\Delta t < (\sim 2\pi) \quad ; \text{otherwise } \Gamma_{12}(\tau)\approx 0.} \quad \text{(K9)}.$$

It is <u>not</u>, however, necessary to satisfy this condition to obtain a strong beat frequency signal. In fact, it is just the other way. We expect that the time for either of the signals to appreciably change character $\Delta\tau$ (coherence time) must be appreciably larger than the time for an oscillation at the beat frequency before the beat note can be well defined. thus,

$$\Delta\tau > \left(\frac{\sim 2\pi}{\omega_a-\omega_b}\right)$$

$$\boxed{(\omega_a-\omega_b)\Delta\tau > (\sim 2\pi) \quad ; \text{otherwise no beat signal}} \quad \text{(K10)}.$$

# APPENDIX L

## GAUSSIAN FREQUENCY SPECTRUM

### EVALUATION OF INTEGRALS

In work with Gaussian beams and with the fundamental laser mode, it is frequently necessary to know the value of integrals of the type

$$I_n = \int_0^\infty e^{-\left(\frac{u-\Delta}{w}\right)^2} e^{-iut} u^n \, du, \qquad n = 0, 1, 2 .$$

It can be evaluated by first completing the square of the exponent

$$\left(\frac{u-\Delta}{w}\right)^2 + iut = \left(\frac{u-\Delta}{w}\right)^2 + i(ut - \Delta t) + \left(\frac{iwt}{2}\right)^2 + i\Delta t - \left(\frac{iwt}{2}\right)^2,$$

so that

$$I_n = e^{-i\Delta t} e^{-\left(\frac{wt}{2}\right)^2} w^{n+1} \int_0^\infty e^{-\left(\frac{u}{w} - \frac{\Delta}{w} + i\frac{wt}{2}\right)^2} \left(\frac{u}{w}\right)^n d\left(\frac{u}{w}\right).$$

Making the change of variable; $x = \frac{u}{w} - \frac{\Delta}{w} + \frac{iwt}{2}$, and using

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad \int_0^\infty e^{-x^2} x \, dx = \frac{1}{2}, \quad \text{and} \int_0^\infty e^{-x^2} x^2 dx = \frac{\sqrt{\pi}}{4},$$

completes the evaluation, and gives:

$$\boxed{\begin{aligned}
I_0 &= \frac{\sqrt{\pi}}{2} w e^{-i\Delta t} e^{-\left(\frac{wt}{2}\right)^2} \\
I_1 &= \frac{\sqrt{\pi}}{2} w^2 e^{-i\Delta t} e^{-\left(\frac{wt}{2}\right)^2} \left[\frac{1}{\sqrt{\pi}} + \frac{\Delta}{w} - i\frac{wt}{2}\right] \\
I_2 &= \frac{\sqrt{\pi}}{2} w^3 e^{-i\Delta t} e^{-\left(\frac{wt}{2}\right)^2} \left[\frac{1}{2} + \frac{2}{\sqrt{\pi}}\left(\frac{\Delta}{w} - \frac{iwt}{2}\right) + \left(\frac{\Delta}{w} - \frac{iwt}{2}\right)^2\right], \\
&\text{where} \\
I_n &\equiv \int_0^\infty e^{-\left(\frac{u-\Delta}{w}\right)^2} e^{-iut} u^n \, du
\end{aligned}} \tag{L1}$$

### GAUSSIAN FREQUENCY SPECTRUM

These integrals (L1) can also be used

to examine the time dependence of a signal having a frequency spectrum whose amplitude is Gaussian. In particular, Equation (92) has the form

$$\tilde{\Gamma}(\vec{r}_1, \vec{r}_2, \Omega) = \tilde{\Gamma}_{12}(\Omega) = A \int_\sigma \int_\sigma dS_1' dS_2' \, e^{i\alpha\Omega} (1 + 4d^2\Omega^2/c^2) \tilde{\Gamma}_{12}'(\Omega), \quad (L2)$$

where the definitions of $\alpha$ and $A$ are readily obtained by comparing with (92), and where we have used $K = \Omega/c$. The dependence on $\tau$ is obtained through

$$\Gamma_{12}(\tau) = \frac{1}{2\pi} \int_0^\infty e^{-i\Omega\tau} \tilde{\Gamma}_{12}(\Omega) \, d\Omega . \quad (L3)$$

After multiplying (L2) by $\frac{1}{2\pi} e^{-i\Omega\tau} d\Omega$ and integrating, we find:

$$\boxed{\Gamma_{12}(\tau) = A \int_\sigma \int_\sigma dS_1' dS_2' \left\{ \frac{1}{2\pi} \int_0^\infty e^{-i\Omega(\tau-\alpha)} (1 + 4d^2\Omega^2/c^2) \tilde{\Gamma}_{12}'(\Omega) \, d\Omega \right\}} . \quad (L4)$$

the integration

$$\boxed{J = \frac{1}{2\pi} \int_0^\infty e^{-i\Omega(\tau-\alpha)} (1 + 4d^2\Omega^2/c^2) \tilde{\Gamma}_{12}'(\Omega) \, d\Omega} \quad (L5)$$

can not be performed unless the functional dependence of $\tilde{\Gamma}'$ on $\Omega$ is known. It is instructive to learn what happens for a Gaussian amplitude distribution*. For this purpose, we put

$$\boxed{\begin{aligned} \tilde{\Gamma}_{12}'(\Omega) &= \frac{1}{\sqrt{\Delta\Omega}} F(\vec{r}_1{}', \vec{r}_2{}') e^{-\left(\frac{\Omega - \bar{\Omega}}{\sqrt{2}\,\Delta\Omega}\right)^2} \\ &= \frac{F}{\sqrt{\Delta\Omega}} e^{-\left(\frac{\Omega - \bar{\Omega}}{\sqrt{2}\,\Delta\Omega}\right)^2} \end{aligned}} \quad (L6)$$

---

* In this example the argument of $\tilde{\Gamma}'$ has no dependence on $\Omega$. It is constant.

Substitution into (L5) and making use of the integrals (L1) is a straightforward calculation. It is also convenient to define:

$$\boxed{\begin{aligned} \Delta\tau &\equiv \frac{1}{\Delta\Omega} \\ \text{and} \quad \delta &\equiv \frac{\Delta\Omega}{\bar{\Omega}} \end{aligned}} \qquad (L7)$$

the result of this calculation can then be written

$$\boxed{\begin{aligned} J = \frac{F}{2\sqrt{2\pi\Delta\tau}}\, e^{-i\bar{\Omega}\tau}\, e^{i\bar{\Omega}\alpha}\, e^{-\left(\frac{\tau-\alpha}{\sqrt{2}\,\Delta\tau}\right)^2} \Big\{ 1 + 4\bar{K}^2 d^2 \Big[ 1 + 2\sqrt{\tfrac{2}{\pi}}\,\delta \\ + \delta^2 - 2i\left(1+\delta\sqrt{\tfrac{2}{\pi}}\right)\bar{\Omega}(\tau-\alpha)\delta^2 - \left(\bar{\Omega}(\tau-\alpha)\delta^2\right)^2 \Big] \Big\} \end{aligned}} \quad L8$$

thus, there is an oscillation in $\tau$ at angular frequency $\bar{\Omega}$ (the term $e^{-i\bar{\Omega}\tau}$). The amplitude is centered about $\tau = \alpha$, and drops to $e^{-.5}$ of its central value when $\tau = \alpha \pm \Delta\tau$, just as the amplitude of L6 became $e^{-.5}$ of its' maximum at $\Omega = \bar{\Omega} \pm \Delta\Omega$. The width of the frequency spectrum of L6 (between $1/e$ points) is $2\sqrt{2}\,\Delta\Omega$, and the corresponding width in time is $2\sqrt{2}\,\Delta\tau$. Because of (L7) there are restrictions on the relative sizes of $\Delta\tau$ and $\Delta\Omega$. If $\Delta\tau$ is made smaller, the spread in frequency $\Delta\Omega$ must become larger and vice versa. For signals other than Gaussian it can be shown that the equals sign in L7 must be replaced with $\geq$. It is the same as the UNCERTAINTY PRINCIPLE $\Delta\Omega\,\Delta\tau \geq 1$, becomes $\Delta E\,\Delta t \geq \hbar$ when we put $E = \hbar\Omega$, $t = \tau$.

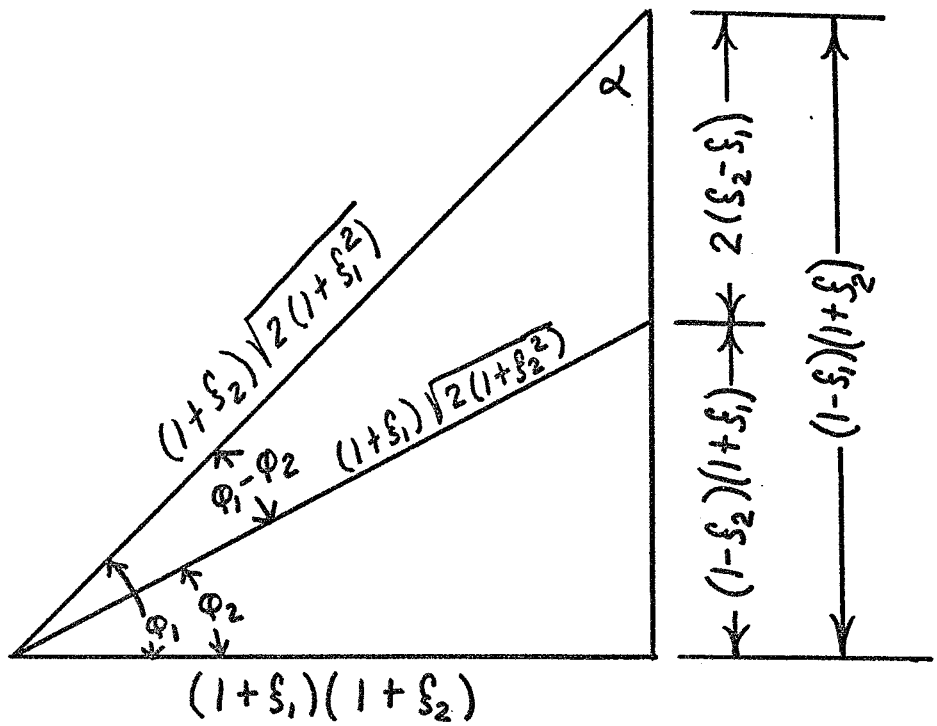
# APPENDIX M

## EVALUATION OF ($Q_1 - Q_2$)

To evaluate $\varphi_1 - \varphi_2 = \tan^{-1}\frac{1-\xi_1}{1+\xi_1} - \tan^{-1}\frac{1-\xi_2}{1+\xi_2}$, we note that it can also be written:

$$\varphi_1 - \varphi_2 = \tan^{-1}\frac{(1-\xi_1)(1+\xi_2)}{(1+\xi_1)(1+\xi_2)} - \tan^{-1}\frac{(1-\xi_2)(1+\xi_1)}{(1+\xi_2)(1+\xi_1)},$$

which can be represented by the following geometric construction:




By the law of sines: $\frac{\sin(\varphi_1-\varphi_2)}{2(\xi_2-\xi_1)} = \frac{\sin\alpha}{(1+\xi_1)\sqrt{2(1+\xi_2^2)}}$,

$$\sin(\varphi_1-\varphi_2) = \frac{2(\xi_2-\xi_1)}{(1+\xi_1)\sqrt{2(1+\xi_2^2)}}\left[\frac{(1+\xi_1)(1+\xi_2)}{(1+\xi_2)\sqrt{2(1+\xi_1^2)}}\right]$$

$$\boxed{\sin(\varphi_1-\varphi_2) = \frac{\xi_2-\xi_1}{\sqrt{(1+\xi_1^2)(1+\xi_2^2)}}}.$$

$$\cos^2(\varphi_1-\varphi_2) = 1-\sin^2(\varphi_1-\varphi_2) = 1 - \frac{(\xi_2-\xi_1)^2}{(1+\xi_1^2)(1+\xi_2^2)}$$

$$= \frac{1+\xi_1^2+\xi_2^2+\xi_1^2\xi_2^2-\xi_2^2-\xi_1^2+2\xi_1\xi_2}{(1+\xi_1^2)(1+\xi_2^2)}$$

$$\boxed{\cos^2(\varphi_1-\varphi_2) = \frac{(1+\xi_1\xi_2)^2}{(1+\xi_1^2)(1+\xi_2^2)}}.$$

# APPENDIX N

## SYLVESTER'S THEOREM

**Statement of the Theorem**

If $\quad AD-BC=1, \quad$ then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \frac{1}{\sin\theta}\begin{pmatrix} A\sin n\theta - \sin(n-1)\theta & B\sin n\theta \\ C\sin n\theta & D\sin n\theta - \sin(n-1)\theta \end{pmatrix} \qquad (N1)$$

where

$$\cos\theta = \tfrac{1}{2}(A+D).$$

**Proof by induction** (from Prof. W.E. Vehse)

The theorem is true for $n=1$ by inspection. We now show that if the theorem is true for an arbitrary $n$, it is also true for $n+1$.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{n+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{n} = \frac{1}{\sin\theta}\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}.$$

$$t_1 = A\left[A\sin n\theta - \sin(n-1)\theta\right] + BC\sin n\theta$$
$$= \sin n\theta\left[A^2 + BC + AD - AD\right] - A\sin(n-1)\theta$$
$$A+D = \cos\theta \quad , \quad AD-BC=1$$
$$= A\left[2\cos\theta\sin n\theta - \sin(n-1)\theta\right] - \sin n\theta$$
$$= A\left[\cos\theta\sin n\theta + \cos\theta\sin n\theta - \sin n\theta\cos\theta + \cos n\theta\sin\theta\right] - \sin n\theta$$
$$= A\sin(n+1)\theta - \sin n\theta$$

$$t_2 = AB\sin n\theta + BD\sin n\theta - B\sin(n-1)\theta$$
$$= B\left[(A+D)\sin n\theta - \sin(n-1)\theta\right] = B\left[2\cos\theta\sin n\theta - \sin(n-1)\theta\right]$$
$$= B\sin(n+1)\theta.$$

The calculation for $t_3$ & $t_4$ is very similar, so that:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{n+1} = \frac{1}{\sin\theta}\begin{pmatrix} A\sin(n+1)\theta - \sin n\theta & B\sin(n+1)\theta \\ C\sin(n+1)\theta & D\sin(n+1)\theta - \sin n\theta \end{pmatrix}$$

Q.E.D.

Proof using spin matrices & generating function (contributed by Prof. A. D. Levine).

Using $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, & $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

the matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ can be written

$$T = \tfrac{1}{2}(A+D)1 + \tfrac{1}{2}(A-D)\sigma_z + \tfrac{1}{2}(B+C)\sigma_x + \tfrac{i}{2}(B-C)\sigma_y .$$

Define

$$\alpha = \tfrac{1}{2}(A+D), \quad \beta_x = \frac{B+C}{2}, \quad \beta_y = \tfrac{i}{2}(B-C), \quad \beta_z = \frac{A-D}{2}$$

$$\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = \tfrac{1}{2}\sqrt{A^2 - 2AD + D^2 + 4BC} = \tfrac{1}{2}\sqrt{(A+D)^2 + 4BC - 4AD}.$$

since unitary $AD - BC = 1$

$$\boxed{\cos\theta = \tfrac{1}{2}(A+D)}$$

$$\beta = \sqrt{\cos^2\theta - 1} \quad \boxed{\beta = i\sin\theta} \quad \boxed{\eta = \sin\theta} \quad \boxed{\vec{\beta} = i\vec{\eta}}$$

then

$$\boxed{T = \alpha 1 + \vec{\beta}\cdot\vec{\sigma}}$$

Now use the generating function:

$$\boxed{\begin{aligned} \Gamma &\equiv e^{\lambda T} = \sum_n \frac{\lambda^n}{n!} T^n \\ \left.\frac{d^n\Gamma}{d\lambda^n}\right|_{\lambda=0} &= T^n \end{aligned}}$$

These spin operators have the property that

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

$$\sigma_x\sigma_y + \sigma_y\sigma_x = \sigma_y\sigma_z + \sigma_z\sigma_y + \sigma_z\sigma_x + \sigma_x\sigma_z = 0,$$

whereby,

$$\vec{\eta}\cdot\vec{\sigma} = \vec{\eta}\cdot\vec{\sigma}$$

$$(\vec{\eta}\cdot\vec{\sigma})^2 = \eta^2 1 = \eta^2$$

$$(\vec{\eta}\cdot\vec{\sigma})^3 = \eta^2(\vec{\eta}\cdot\vec{\sigma})$$

$$(\vec{\eta}\cdot\vec{\sigma})^4 = \eta^4$$

$\vdots$ etc.

as customary we omit writing the identity matrix, 1.

thus,

$$e^{i\lambda\vec{\eta}\cdot\vec{\sigma}} = 1 + \frac{i\eta\lambda\left(\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right)}{1!} - \frac{(\lambda\eta)^2}{2!} - i\frac{(\lambda\eta)^3\left(\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right)}{3!} + \cdots$$

$$= \cos(\lambda\eta) + i\,\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\sin(\lambda\eta)$$

$$= \frac{1}{2}\left[e^{i\eta\lambda}\left(1+\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right) + e^{-i\eta\lambda}\left(1-\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right)\right]$$

$$= \frac{1}{2}\left[e^{\lambda\beta}\left(1+\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right) + e^{-\beta\lambda}\left(1-\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right)\right]$$

$$\Gamma = \frac{1}{2}e^{\lambda\alpha+\lambda\beta}\left(1+\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right) + \frac{1}{2}e^{\lambda(\alpha-\beta)}\left(1-\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right)$$

$$T^n = \left.\frac{d^n\Gamma}{d\lambda^n}\right|_{\lambda=0} = \frac{1}{2}(\alpha+\beta)^n\left(1+\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right) + \frac{1}{2}(\alpha-\beta)^n\left(1-\frac{\vec{\beta}\cdot\vec{\sigma}}{i\sin\theta}\right)$$

$$\alpha+\beta = \alpha+i\eta = e^{i\theta}$$
$$\alpha-\beta = \alpha-i\eta = e^{-i\theta}$$

$$T^n = \frac{1}{2}\left(e^{in\theta}+e^{-in\theta}\right) + \frac{\vec{\beta}\cdot\vec{\sigma}}{\sin\theta}\left(\frac{e^{in\theta}-e^{-in\theta}}{2i}\right)$$

$$T^n = \cos(n\theta) + \frac{\vec{\beta}\cdot\vec{\sigma}}{\sin\theta}\sin(n\theta)$$

$$T^n = \frac{1}{\sin\theta}\left(\sin\theta\cos(n\theta) + \vec{\beta}\cdot\vec{\sigma}\sin(n\theta)\right)$$

But, $\vec{\beta}\cdot\vec{\sigma} = \begin{pmatrix} \frac{1}{2}(A-D) & B \\ C & \frac{1}{2}(D-A) \end{pmatrix}$

$$T^n = \frac{1}{\sin\theta}\begin{pmatrix} \frac{1}{2}(A-D)\sin n\theta + \sin\theta\cos(n\theta) & B\sin n\theta \\ C\sin n\theta & \frac{1}{2}(D-A)\sin n\theta + \sin\theta\cos n\theta \end{pmatrix}$$

$$\frac{1}{2}(A-D) = A-\cos\theta \quad ; \quad \frac{1}{2}(D-A) = D-\cos\theta$$

$$T^n = \frac{1}{\sin\theta}\begin{pmatrix} A\sin n\theta - \sin n\theta\cos\theta + \sin\theta\cos n\theta & B\sin n\theta \\ C\sin n\theta & D\sin n\theta - \sin n\theta\cos\theta + \sin\theta\cos n\theta \end{pmatrix}$$

$$\boxed{T^n = \frac{1}{\sin\theta}\begin{pmatrix} A\sin n\theta - \sin(n-1)\theta & B\sin n\theta \\ C\sin n\theta & D\sin n\theta - \sin(n-1)\theta \end{pmatrix}}$$

Q.E.D.

13. L. Mandel and E. Wolf, "Coherence Properties of Optical Fields", Rev. Mod. Phys. 37, (1965) pp. 231-287.

14. Klauder, John R. and E. C. G. Sudarshan, "Fundamentals of Quantum Optics", W.A. Benjamin, Inc., New York (1968).

15. Gordon, J. P., "A Circle Diagram for Optical Resonators", Bell System Tech. J. 43 , (1964) pp 1826-1827.

16. Kogelnik, H. and T. Li, "Laser Beams and Resonators", Applied Optics 5 (1966), pp 1550 - 1567.

17. Kogelnik, Herwig, "Imaging of Optical Modes -- Resonators with Internal Lenses" Bell System Tech. J., 44 (1965) pp 455-494.

18. Gordon, J. P., "Equivalence Relations Among Spherical Mirror Optical Resonators", Bell Sys. Tech. J. 43 (1964) pp 2873 - 2886.

19. Boyd, G. D., and H. Kogelnik, "Generalized Confocal Resonator Theory, Bell Sys. Tech. J. 41 (1962) pp 1347 - 1369.

20. Collins, S. A. "Analysis of Optical Resonators Involving Focusing Elements", Appl Apt. 3 (1964) pp 1263 - 1275.

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# Maximum Power-Transfer Coefficient between Two Confocal Apertures*

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This paper discusses a modification of a maximum power-transfer theorem whose essential features were developed by Alan F. Kay and by Giorgio V. Borgiotti. It is shown that a $C_2$ rotation symmetry with respect to the optic axis is a sufficient symmetry restriction for the shape of each aperture, and that the phase as well as the amplitude of the illuminating field can be included in a statement of the theorem if it is applied to confocal instead of flat surfaces. The modified statement of the theorem is that a maximum power-transfer coefficient between apertures in two confocal surfaces, whose shapes have $C_2$ symmetry, is obtained when the illumination of one surface is identical to that of a confocal resonator having the same geometry and operating in the lowest diffraction-loss eigenmode, and that the power-transfer coefficient, $T$, is then related to the full-pass diffraction loss, $D$, by $T=(1-D)^{\frac{1}{2}}$. It is further shown that the eigenfunctions of the confocal-resonator equation are identical to those of the extremum power-transfer condition, and that these functions form a complete orthogonal set. The actual solution for surfaces of rectangular shape is compared with results obtained for illumination with a gaussian-amplitude distribution. It turns out that attempts to minimize power radiated into sidelobes by using gaussian-amplitude distributions have been very close to the optimum solution of this problem.

INDEX HEADINGS: Resonant modes; Microwaves; Diffraction.

The conditions for a maximum power-transfer coefficient between two parallel planar apertures were developed in an earlier paper by Kay.[1] The similarity of these conditions to the confocal optical-resonator mode equations was first pointed out by Borgiotti,[2] who gave a statement of the theorem considered here for the amplitude distribution over flat apertures. By launching the electromagnetic wave from a confocal instead of a flat surface, his results can be stated in a more general way because the theorem then applies to the phase as well as the amplitude of the illuminating field. The double symmetry with respect to $x$ and $y$ of aperture shapes used in this earlier work is more stringent than is necessary. The theorem is derived here for apertures having $C_2$ symmetry with respect to rotations about the optic axis. The theorem then states that a maximum power-transfer coefficient between apertures in two confocal surfaces, whose shapes have $C_2$ symmetry, is obtained when the illumination of one surface is identical to that of a confocal resonator having the same geometry and operating in the lowest diffraction-loss eigenmode, and that the power-transfer coefficient, $T$, is then related to the full-pass diffraction loss, $D$, by $T=(1-D)^{\frac{1}{2}}$. It is also shown that the eigenfunctions of the confocal resonator equation are identical to those of the extremum condition for maximum power transfer. These functions also form a complete orthogonal set and can be selected such that they represent definite parity states.

The physical basis of both the self-consistent-field condition for an optical resonator and the expression for the power-transfer coefficient are briefly reviewed. Throughout, the situation considered is such that the fields at points on one confocal surface, located in the Fresnel zone of the other, can be derived from those on the other, using Kirchhoff's diffraction integral and the small-angle approximation. The derivation of the theorem given here emphasizes the reason for the symmetry restrictions on aperture shapes. Basically, the $C_2$ symmetry is required to insure that the eigenfunctions of integral equations that occur form a complete orthogonal set and that they represent definite parity states.

It has been recognized for some time that the power radiated into the side-lobes can be substantially reduced[3] by making the amplitude of the illuminating field a truncated gaussian distribution. When the power transfer of optimum gaussian distributions is compared to that for prolate-spheroidal angle-function distributions, the solution of the maximum power-transfer problem for rectangular apertures, the difference is found to be of minor importance.

## RESONATOR AND POWER-TRANSFER EQUATIONS

The physical situation considered and the symbols used are both defined in Fig. 1. The field at a point such

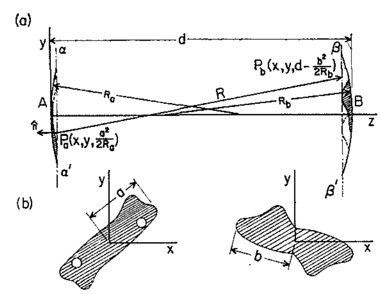


FIG. 1. (a) The dimensions and relative orientation of the spherical (or parabolic) surfaces $A$ and $B$. (b) Projections of $A$ and $B$ on the planes $\alpha\alpha'$ and $\beta\beta'$ illustrating the form of surfaces that satisfy the $C_2$ symmetry requirement.

* Supported by NASA research grant NGR-10-019-001.
[1] Alan F. Kay, Trans. IRE AP-8, 586 (1960).
[2] Giorgio V. Borgiotti, Trans. IEEE AP 14, 158 (1966).
[3] L. J. Lader and J. B. Winderman, Can. J. Phys. 44, 2765 (1966).

as $P_b$ is specified in terms of the fields on the plane screen $\alpha\alpha'$ by a form of the Kirchhoff diffraction integral given by Jackson[4]

$$\mathbf{E}_\beta(x,y)=-\frac{1}{2\pi}\int\int_A d\xi d\eta\{\mathbf{n}\times\mathbf{E}_\alpha(\xi,\eta)\}\times\nabla\frac{e^{ikR}}{R}. \quad (1)$$

The advantage of starting with this form of the diffraction integral is that the precise conditions for which a transverse component of $\mathbf{E}$ on $\alpha\alpha'$ results only in the same component on $\beta\beta'$ are immediately revealed when the vector operations are carried out. Because this is a valid approximation for the present physical situation, it is only necessary to consider one transverse component of the electric field. For the $x$ component, the fields on $\beta\beta'$ are thus related to those on $\alpha\alpha'$ by

$$E_{\beta x}(x,y)=-\frac{ik}{2\pi}\int\int_A d\xi d\eta\frac{E_{\alpha x(\xi,\eta)}}{R}\exp(ikR). \quad (2)$$

In the small-angle Fresnel zone, $R$ in the denominator is replaced by $d$, but in the phase factor includes the quadratic terms,

$$R=\left[\left(d-\frac{a^2}{2R_a}-\frac{b^2}{2R_b}\right)^2+(x-\xi)^2+(y-\eta)^2\right]^{\frac{1}{2}}$$

$$\simeq d-\frac{a^2}{2R_a}-\frac{b^2}{2R_b}+\frac{x^2+y^2+\xi^2+\eta^2}{2d}-\frac{x\xi+\eta y}{d}. \quad (3)$$

These substitutions into Eq. (2) give

$$E_{\beta x}(x,y)=-\frac{ike^{ikd}}{2\pi d}\int\int_A d\xi d\eta E_{\alpha x}(\xi,\eta)$$

$$\times\exp\left[ik\left(\frac{x^2+y^2}{2d}-\frac{b^2}{2R_b}+\frac{\xi^2+\eta^2}{2d}-\frac{a^2}{2R_a}-\frac{x\xi+y\eta}{d}\right)\right]. \quad (4)$$

If the wave were launched from the spherical surface $A$ instead of the plane $\alpha\alpha'$ and generally directed along $z$, then the field on $\alpha\alpha'$ would lag the field $F(\xi,\eta)$ on $A$ according to

$$E_{\alpha x}(\xi,\eta)=F(\xi,\eta)\exp\left[ik\left(\frac{a^2}{2R_a}-\frac{\xi^2+\eta^2}{2R_a}\right)\right]. \quad (5)$$

Actually, because there is some divergence of the wave as it traverses the very small distance $[a^2-(\xi^2+\eta^2)]/2R_a$, this equation can not be exactly true. The degree of approximation that is involved is equivalent to beginning with fields on the curved surface and then regarding the element of surface area on $A$ to be $d\xi d\eta$. This approximation is permitted here because the treatment is restricted to optical small-angle situations. Although the fields on $\alpha\alpha'$ and $A$ are different because of the phase factor, an integration over the spherical surface $A$ may be replaced by an integration over the flat projection. These same statements also apply to $\beta\beta'$ and $B$, where

$$E_{\beta x}(x,y)=G(x,y)\exp\left\{-ik\left(\frac{b^2}{2R_b}-\frac{x^2+y^2}{2R_b}\right)\right\}. \quad (6)$$

Substitution of Eq. (5) into Eq. (4) gives

$$E_{\beta x}(x,y)=-\frac{ike^{ikd}}{2\pi d}\int\int_A d\xi d\eta F(\xi,\eta)$$

$$\times\exp\left\{ik\left[-\frac{b^2}{2R_b}+\frac{x^2+y^2}{2R_b}\right]+i\frac{k}{d}\left[\frac{g_a}{2}(\xi^2+\eta^2)+\frac{g_b}{2}(x^2+y^2)-x\xi-y\eta\right]\right\}, \quad (7)$$

where the customary notation, $g_a=1-d/R_a$ and $g_b=1-d/R_b$, has been adopted.

### Power-Transfer Coefficient

If Eq. (6) is multiplied by its complex conjugate and integrated over $B$, the fractional time-average power passing through $B$ (equal to that passing through the projectionon $\beta\beta'$) becomes

$$T_{g_a}\int\int_A dudvF(u,v)\bar{F}(u,v)$$

$$=\left(\frac{k}{2\pi d}\right)^2\int\int_B dxdy\int\int_A d\xi d\eta\int\int_A dudvF(\xi,\eta)\bar{F}(u,v)$$

$$\times\exp\left\{i\frac{k}{d}\left[\frac{g_a}{2}(\xi^2-u^2+\eta^2-v^2)+x(u-\xi)+y(v-\eta)\right]\right\}. \quad (8)$$

This expression becomes simpler if the launch aperture, $A$, is confocal ($g_a=0$) with respect to the target, $B$. Physically, this restricts the launch aperture to one that is focused on the target. With $g_a=0$, the power-transfer coefficient, $T$, is

$$T=\frac{\left(\frac{k}{2\pi d}\right)^2\int\int_A dudv\int\int_A d\xi d\eta F(\xi,\eta)\bar{F}(u,v)\mathcal{K}(\xi,\eta;u,v)}{\int\int_A dudvF(u,v)\bar{F}(u,v)}, \quad (9)$$

where

$$\mathcal{K}(\xi,\eta;u,v)=\int\int_B dxdy\,\exp\left\{i\frac{k}{d}[-x(\xi-u)-y(\eta-v)]\right\}. \quad (10)$$

[4] John David Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1966), p. 287.

### Resonator Self-Consistent-Field Condition

Substitution of Eq. (6) into Eq. (7) gives the field $G$ at points $B$ in terms of those at points on $A$;

$$G(x,y)=-\frac{ike^{ikd}}{2\pi d}\int\int_A d\xi d\eta F(\xi,\eta)\times\exp i\frac{k}{d}\left\{\frac{g_a}{2}(\xi^2+\eta^2)+\frac{g_b}{2}(x^2+y^2)-x\xi-y\eta\right\}. \quad (11)$$

The field $F'$ at points on $A$ in terms of $G(x,y)$ is given by a completely analogous expression. Consequently, when these expressions are combined, $F'$ is specified in terms of $F$. The self-consistent-field condition is obtained by putting the field $F'$ after a complete pass (over and back) equal to a constant times $F$ at all points on $A$. Thus, if both mirrors have reflection coefficients of $\pm 1$, the self-consistent-field condition becomes

$$F'(u,v)=\gamma F(u,v)=-\left(\frac{k}{2\pi d}\right)^2 e^{2ikd}\int\int_A d\xi d\eta F(\xi,\eta)\int\int_B dxdy\times\exp\left\{i\frac{k}{d}\left[\frac{g_a}{2}(\xi^2+\eta^2+u^2+v^2)+g_b(x^2+y^2)-x(\xi+u)-y(\eta+v)\right]\right\}. \quad (12)$$

Putting $\Gamma=-\gamma e^{-2ikd}$, and imposing the confocal conditions, $g_a=0$ and $g_b=0$, then leads to

$$\left(\frac{2\pi d}{k}\right)^2\Gamma F(u,v)=\int\int_A d\xi d\eta F(\xi,\eta)K(\xi,\eta;u,v), \quad (13)$$

where

$$K(\xi,\eta;u,v)=\int\int_B dxdy\exp\left\{i\frac{k}{d}[-x(\xi+u)-y(\eta+v)]\right\}. \quad (14)$$

The diffraction loss for a complete pass can be found from either $\gamma$ or $\Gamma$, and is

$$D=1-|\gamma|^2=1-|\Gamma|^2. \quad (15)$$

A similar integral equation involving only one integration was used by Boyd and Gordon[5] to solve the confocal-resonator problem for identical rectangular mirrors. The full-pass equation, (13), is identical to that obtained earlier by Boyd and Kogelnic.[6] The integral equation for this problem was also found by Goubau and Schwering[7] for the condition imposed on modes that could be propagated by an iterated system of phase transformers.

Equation (13) is a homogeneous Fredholm equation of the second kind, and can be derived from a variational principle in the following way. Consider

$$\Gamma=\frac{\left(\frac{k}{2\pi d}\right)^2\int\int_A dudv\int\int_A d\xi d\eta F(\xi,\eta)\bar{F}(u,v)K(\xi,\eta;u,v)}{\int\int_A dudvF(u,v)\bar{F}(u,v)}. \quad (16)$$

If the function $F$ is varied to make $\Gamma$ an extremum ($\delta\Gamma=0$), it follows that

$$\Gamma\left(\frac{2\pi d}{k}\right)^2\int\int_A dudv(F\delta\bar{F}+\bar{F}\delta F)=\int\int_A dudv\left\{\int\int_A d\xi d\eta[K(\xi,\eta;u,v)F(\xi,\eta)\delta\bar{F}(u,v)+K(u,v;\xi,\eta)\bar{F}(\xi,\eta)\delta F(u,v)]\right\}. \quad (17)$$

Since $\delta F$ and $\delta\bar{F}$ are arbitrary and independent variations, this variational requirement is equivalent to Eq. (13), the confocal optical-resonator-mode equation, and the same condition for the complex conjugate of $F$, $\bar{F}$.

### MAXIMUM POWER-TRANSFER COEFFICIENT

Without imposing symmetry restrictions of any kind, a remarkable similarity can be noticed between the problem of finding a function $F$ in Eq. (16) to make $\Gamma$ an extremum [which led to Eq. (13)] and the problem of finding the function $F$ in Eq. (9) to make the power-transfer coefficient a maximum. The only difference is a change of the sign of $u$ and $v$ in going from $\mathcal{K}(\xi,\eta;u,v)$ to $K(\xi,\eta;u,v)$. The same variational methods used above then lead to the condition for an extremum power-transfer coefficient

$$\left(\frac{2\pi d}{k}\right)^2 TF(u,v)=\int\int_A d\xi d\eta F(\xi,\eta)\mathcal{K}(\xi,\eta;u,v),$$
$$\mathcal{K}=\int\int_B dxdy\exp\left\{i\frac{k}{d}[x(u-\xi)+y(v-\eta)]\right\}. \quad (18)$$

With the proper phase adjustment, this equation applies to non-confocal situations as well. If

$$F(\xi,\eta)\exp\left\{i\frac{k}{d}\frac{g_a}{2}(\xi^2+\eta^2)\right\}$$

in Eq. (8) is replaced by $F''(\xi,\eta)$, this same expression for the maximum power transfer can be derived for $F''$.

[5] G. D. Boyd and J. P. Gordon, Bell System Tech. J. **40**, 489 (1961).
[6] G. D. Boyd and H. Kogelnik, Bell System Tech. J. **41**, 1347 (1962).
[7] G. Goubau and F. Schwering, Trans. IRE **AP 9**, 248 (1961).

All that is needed is an adjustment of phase to keep $A$ focused on the center of $B$.

A substitution to change the sign of each of the variables $\xi$, $\eta$, $u$, $v$, $x$, and $y$ gives

$$\left(\frac{2\pi d}{k}\right)^2 TF(-u,-v)=\int\int_{A_-}d\xi d\eta F(-\xi,-\eta)\int\int_{B_-}dxdy\times\exp\left\{i\frac{k}{d}[x(u-\xi)+y(v-\eta)]\right\}. \quad (19)$$

If the surfaces $A$ and $B$ are both such that each has $C_2$ rotation symmetry with respect to the $z$ axis [i.e., for each element $dxdy$ at $(x,y)$, there is an element of area $dxdy$ at $(-x,-y)$], then Eq. (19) for $F(-u,-v)$ is identical to Eq. (18) for $F(u,v)$. The nonzero solutions of these two equations must then be the same set of independent functions. Thus, each of the one or more functions $F_j$ that correspond to the maximum value of $T$ must have the property $F_j(-u,-v)=\epsilon F_j(u,v)$, where $\epsilon$ is a constant. Hence, $F_j(u,v)=\epsilon F_j(-u,-v)=\epsilon^2F_j(u,v)$; whereby $\epsilon=\pm1$, and $F_j(u,v)=\pm F_j(-u,-v)$. When use is made of this result in Eq. (18), a substitution to change the sign of $u$ and $v$ gives

$$\pm\left(\frac{2\pi d}{k}\right)^2 T_{\max}F_j(u,v)=\int\int_A d\xi d\eta F_j(\xi,\eta)K(\xi,\eta;u,v), \quad (20)$$

which is identical to the confocal-resonator equation (13) with $T_{\max}=|\Gamma|_{\max}$; or, with the help of Eq. (15), $T_{\max}=(1-D_{\min})^{\frac{1}{2}}$. This completes the derivation of the theorem in question. It is now instructive to examine the way in which the required symmetry restrictions influence the set of eigenfunctions for the confocal-resonator equation (13), and to compare the eigenfunctions of Eqs. (13) and (18).

## COMPLETENESS AND PARITY

It is not necessary to invoke symmetry restrictions for the surface $A$ in order to show that the eigenfunctions of the extremum power-transfer condition [Eq. (18)] constitute a complete orthogonal set of functions on $A$. The $C_2$ symmetry condition for $A$, however, is used in order to obtain a correlation between these functions and those for the confocal resonator and to show that they represent definite parity states. To see how all this comes about, we recall that the sufficient conditions for the eigenfunctions of Eq. (18) to be complete[8] are

(1) The equation for the eigenvectors corresponds to some variational principle,

$$(2)\quad J=\int\int_A dudv\int\int_A d\xi d\eta F(\xi,\eta)\bar{F}(u,v)\mathcal{K}(\xi,\eta;u,v)$$

is real (the self-adjoin or hermitian condition for the corresponding operator), and

(3) $J$ is greater than zero (the corresponding operator is positive definite).

The first of these conditions is already satisfied because Eq. (18) was obtained as the result of applying a variational principle to Eq. (9). The second condition requires that the kernel be real. The imaginary part of the kernel of Eq. (18) will vanish if for every element of area $dxdy$ at $(x,y)$, there is an element $dxdy$ at $(-x,-y)$. Thus, if $B$ has $C_2$ symmetry, the kernel is real and symmetric and, as is well known, the eigenvalues are real and eigenfunctions belonging to different eigenvalues are orthogonal.

To investigate the third condition, we write the function $F$ in Eq. (18) in terms of its real and imaginary parts, $F=F_r+iF_i$. If use is then made of a trigonometric identity in the integrand of the kernel,

$$\mathcal{K}=\int\int_B dxdy\cos\frac{k}{d}[(x\xi+y\eta)-(xu+yv)],$$

the expression for $J$ can be written

$$J=\int\int_B dxdy\left\{\left[\int\int_A d\xi d\eta F_r(\xi,\eta)\cos\frac{k}{d}(x\xi+y\eta)\right]^2+\left[\int\int_A d\xi d\eta F_r(\xi,\eta)\sin\frac{k}{d}(x\xi+y\eta)\right]^2+\text{similar terms for }F_i\right\}. \quad (21)$$

In this expression, it is obvious that $J>0$ for any nonzero function, $F$, and finite areas $A$ and $B$. Thus, the only symmetry requirement needed to show that the eigenfunctions of the extremum power-transfer condition (18) form a complete orthogonal set on $A$ is the $C_2$ rotation-type symmetry for surface $B$. The fact that $T$ can never be negative is in agreement with the physical meaning of the power-transfer coefficient, which restricts $T$ to the range of values: $0<T<1$.

It has already been pointed out following Eq. (19) that if surface $A$ also has $C_2$ symmetry, then the independent functions that satisfy Eq. (18) have definite parity; i.e., $g_j(u,v)=\pm g_j(-u,-v)$. With this symmetry restriction for $A$, it is also clear that any odd function $g_k$ is orthogonal to any even function $g_j$; i.e.,

$$\int\int_A \bar{g}_k(u,v)g_j(u,v)dudv=0$$

[8] Philip M. Morse and Herman Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), p. 774.

because the integrand is an odd function and for every element $dudv$ at $(u,v)$ there is an element of area $dudv$ at $(-u, -v)$. Thus, when degeneracies occur, it is only necessary to consider linear combinations of the functions $g$ that have the same parity to form the orthogonal set. In conclusion, if $B$ has $C_2$ symmetry, the eigenfunctions of Eq. (18) form a complete orthogonal set on $A$ which we choose to be normalized so that

$$\int\int_A f_k(u,v) f_j(u,v) dudv = \delta_{kj}. \tag{22}$$

If $A$ also has $C_2$ symmetry, these functions can always be selected to have definite parity

$$f_k(u,v) = \pm f_k(-u, -v). \tag{23}$$

If both $A$ and $B$ have $C_2$ symmetry so that Eq. (23) holds, then a change of variables in Eq. (18) for one of the eigenfunctions, $f_k$, converts it into Eq. (13) for the same function $f_k$. Thus, the eigenfunctions of the confocal-resonator equation (13) are identical to those for the extremum power-transfer condition (18) when the entire problem has $C_2$ symmetry. The eigenvalues have the correspondence

$$\Gamma_k = T_k \quad \text{if } f_k \text{ is even}$$
$$\Gamma_k = -T_k \quad \text{if } f_k \text{ is odd.}$$

If instead of first examining the eigenfunctions of Eq. (18), we had started with the confocal-resonator equation (13), all of the same arguments would have been valid up to the proof of the positive definite property. The expression for Eq. (13) that is analogous to Eq. (21) has negative signs in front of terms involving the sine function. If $A$ also has $C_2$ symmetry, the outcome is now clear. The positive eigenvalues correspond to a complete set of even functions that give zero for the sine terms in Eq. (21), and the negative eigenvalues correspond to a complete set of odd functions.

An alternate derivation of the theorem in question can be given once it is known that the eigenfunctions of Eq. (13) form a complete orthogonal set with definite parity. Equations (23), (22), and (13) can then be used to integrate Eq. (9) when $F = f_0$, where $|\Gamma_0|$ is the maximum $|\Gamma_k|$. The result is

$$T_0 = |\Gamma_0|, \tag{24}$$

where we have used the fact that $T$ must be positive. When $F$ in Eq. (9) is any arbitrary sectionally continuous function, it can be expanded in the complete set

$$F(\xi,\eta) = \Sigma_k a_k f_k(\xi,\eta). \tag{25}$$

Equations (23), (22), and (13) can again be used to integrate Eq. (9), giving

$$T_F \Sigma_k |a_k|^2 = \Sigma_k |a_k|^2 (\pm\Gamma_k). \tag{26}$$

When Eq. (24) is multiplied by $\Sigma_k |a_k|^2$, and subtracted

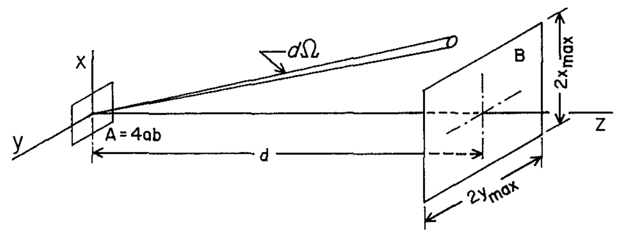


FIG. 2. Dimensions of the rectangular transmitting and receiving apertures.

from Eq. (26), the result is

$$T_F - T_0 = \frac{\Sigma_k (\pm\Gamma_k - |\Gamma_0|) |a_k|^2}{\Sigma_k |a_k|^2}. \tag{27}$$

Because $|\Gamma_0|$ is the largest magnitude of the $\Gamma$'s, the quantity on the right is obviously negative or zero. Thus, $T_0 \geq T_F$, which completes the derivation.

## RECTANGULAR APERTURES

Several previous calculations have been concerned with the radiation patterns produced by truncated gaussian illumination distributions.[3,9] Takeshita,[10] for example, compares his results for such illumination of circular apertures with optical-resonator modes for larger Fresnel numbers. The purpose of this section is to compare both gaussian and uniform-illumination patterns with those produced by the optimum power-transfer illumination for a wide range of Fresnel numbers. Rectangular geometry is selected because it is known that the prolate-spheroidal angle functions are solutions of Eq. (13) for this case.[5]

The dimensions of the rectangular apertures considered are shown in Fig. 2. Because the variables of the Helmholtz equation separate in rectangular coordinates, it is convenient to consider partial solutions of this type here. In Eq. (9) we put

$$F(\xi,\eta) = X(s) Y(p), \tag{28}$$

where

$$\xi = as, \quad u = at$$
$$\eta = bp, \quad v = bq. \tag{29}$$

In what follows, the notation

$$\alpha = (ka/d) x_{\max}, \quad \beta = (kb/d) y_{\max}$$
$$c = (ka/d) x, \qquad C = (kb/d) y \tag{30}$$

has also been used.

The differential gain is defined as the time-average power per unit solid angle $(\overline{dP}/d\Omega)$ radiated in the direction $\Omega$ divided by the time-average power through $A$ per unit solid angle, assuming uniform radiation in all directions. With the above substitutions into Eq. (9),

[9] Arden L. Buck, Proc. IEEE **AP 15**, 448 (1967).
[10] Shinya Takeshita, Trans. IEEE **AP 16**, 305 (1968).

Table I. Expressions used to compute the differential, $G(c,C) = (4\pi A/\lambda^2)g(c)g(C)$, and finite, $F(\alpha,\beta) = (4\pi A/\lambda^2)f(\alpha)f(\beta)$, gains for the various illumination distributions.

| DISTRIBUTION | $f(\alpha)$ | $g(c)$ |
|---|---|---|
| Gaussian $X(s) = e^{-(\mu s)^2}$ | $\dfrac{\int_{-1}^{1}\int_{-1}^{1}\frac{\sin\alpha(t-s)}{\alpha(t-s)}X(s)X(t)dsdt}{2\int_{-1}^{1}(X(s))^2 ds}$ | $\dfrac{\int_{-1}^{1}\int_{-1}^{1}e^{ic(t-s)}X(s)X(t)dsdt}{2\int_{-1}^{1}(X(s))^2 ds}$ |
| Uniform $X(s) = \text{constant}$ | $\sum_{n=0}^{\infty}\dfrac{(-1)^n(2\alpha)^{2n}}{(n+1)(2n+1)(2n+1)!}$ | $\left(\dfrac{\sin c}{c}\right)^2$ |
| Prolate Spheroidal Angle Function $X(s) = S_{on}(\alpha,s)$ | $\left[R^1_{on}(\alpha,1)\right]^2$ | $\dfrac{2}{N_{on}(\alpha)}\left[S_{on}(\alpha,1)R^1_{on}(\alpha,\frac{c}{\alpha})\right]^2 = \dfrac{2}{N_{on}(\alpha)}\left[R^1_{on}(\alpha,1)S_{on}(\alpha,\frac{c}{\alpha})\right]^2$ |

the $x$ and $y$ parts separate to give the differential gain as

$$G(c,C)\frac{\overline{dP}/d\Omega}{\overline{P_A}/4\pi}=G_0g(c)g(C), \tag{31}$$

where

$$G_0=4\pi A/\lambda^2,$$

and

$$g(c)=\int_{-1}^{1}\int_{-1}^{1}e^{ic(t-s)}X(s)\bar{X}(t)dsdt \Big/ 2\int_{-1}^{1}|X(s)|^2ds. \tag{32}$$

In place of the power-transfer coefficient, we define finite gain in analogy to the definition for differential gain. It is proportional to $T$, but has the advantage of becoming equal to the differential gain as area $B$ approaches zero. With the above substitutions into Eq. (9), the finite gain becomes

$$F(\alpha,\beta)\equiv\overline{P_B}/\Omega_B/\overline{P_A}/4\pi=G_0f(\alpha)f(\beta), \tag{33}$$

where

$$f(\alpha)=\int_{-1}^{1}\int_{-1}^{1}\frac{\sin\alpha(t-s)}{\alpha(t-s)}X(s)\bar{X}(t)dsdt \Big/ 2\int_{-1}^{1}|X(s)|^2ds. \tag{34}$$

The prolate-spheroidal angle functions, $S_{0n}(\alpha,s)$, obey the integral equation

$$2i^nR_{0n}^1(\alpha,1)S_{0n}(\alpha,t)=\int_{-1}^{1}e^{i\alpha ts}S_{0n}(\alpha,s)ds, \tag{35}$$

where $R_{0n}^1$ is the prolate-spheroidal radial function of the first kind. It is the property expressed by Eq. (35) that gives these angle functions much of their utility—they are, so to speak, their own finite Fourier transforms. It is straightforward to verify that $S_{0n}$ are the functions that solve Eq. (13) for the rectangular case. Equation (35) can be used to evaluate Eq. (32) to obtain the differential gain when $X(s)$ is a prolate-spheroidal angle function. An integral theorem due to Slepian and Pollak,[11]

$$\frac{2\alpha}{\pi}[R_{0n}^1(\alpha,1)]^2S_{0n}(\alpha,t)=\int_{-1}^{1}\frac{\sin\alpha(t-s)}{\alpha(t-s)}S_{0n}(\alpha,s)ds, \tag{36}$$

is useful for integrating an expression like Eq. (34) when $X(s)$ is a prolate-spheroidal angle function.

Expressions for $f(\alpha)$ and $g(c)$ that are needed for the finite and differential gains, respectively, are shown in Table I. Actual results for gaussian illumination were obtained by a numerical integration of Eqs. (32) and (34). The expression $(\sin c/c)^2$ for differential gain in the case of uniform illumination is a familar result. The expression for $f(\alpha)$ in the uniform-illumination case was obtained from the series expansion for $[\sin\alpha(t-s)/\alpha(t-s)]$. Results for the prolate angle-function distribution were computed using Eqs. (35) and (36) in (32) and (34), respectively. In these expressions, Flammer's normalization scheme was used throughout, whereby $S_{mn}(\alpha,s)$ becomes equal to the associated Legendre polynomial when $\alpha=0$ $[S_{mn}(0,s)=P_n^m(s)]$, the radial function $R_{mn}^1[\alpha,(c/\alpha)]$ approaches the spherical Bessel function $j_n(c)$ as $c/\alpha$ approaches $\infty$, and $N_{mn}(\alpha)$ is the normalizing factor

$$\int_{-1}^{1}S_{mp}(\alpha,s)S_{mn}(\alpha,s)ds=\delta_{pn}N_{mn}(\alpha).$$

The last two expressions for $g(c)$ in Table I are equivalent. One can be obtained from the other by two applications of the connecting formula, $S_{mn}(c,s) = K_{mn}^{(1)}(c)R_{mn}^1(c,s)$. Normally, it is easier to use the expression for which the arguments of $S_{0n}$ and $R_{0n}^1$

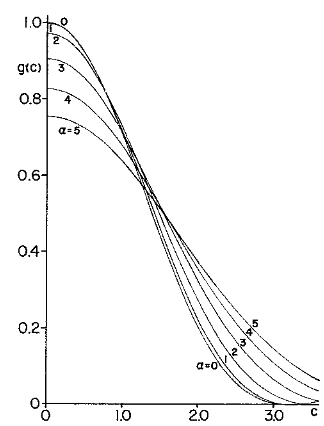


Fig. 3. Differential gain in the main lobe for an angle-function illumination of a rectangular aperture. $X(s)=S_{00}(\alpha,s)$. Gain $=4\pi Ag(c)g(C)/\lambda^2$. $c=kax/d$.

[11] D. Slepian and H. O. Pollak, Bell System Tech. J. 40, 43 (1961).

fall in their normal ranges. The $d$ coefficients given by Flammer[12] were used to obtain numerical results for the angle-function distribution.

The differential gain for the angle-function distribution is shown in Figs. 3 and 4. As $\alpha$ is increased the main lobe becomes less sharply peaked, but contains more of the total power, which is appropriate for a maximum power transfer through a receiving aperture of finite size, $x_{\max}=(d/ka)\alpha$. The improvement of finite gain (proportional to the power-transfer coefficient) that is achieved by using the angle-function distribution in place of a uniform one is shown in Fig. 5. The finite gain for the best truncated gaussian-distribution illumination is so close to the angle-function results that it would only slightly broaden the solid line if plotted on this same graph. How this comes about is indicated in part in Fig. 6. In this figure, a value of $\mu$ was selected for each value of $\alpha$ (or target size) to produce a maximum finite gain. When $\alpha=0$, the angle-function, gaussian, and uniform distributions are all in exact agreement as required by the well-known fact that a uniform distribution produces an optimum differential gain. As $\alpha$ is increased, the optimum gaussian function is seen in each case to agree remarkably well with the angle-function distribution. When $\alpha$ becomes very large, the agreement becomes even better. In fact, as $\alpha\rightarrow\infty$, the prolate-spheroidal angle functions become the Hermite–Gauss functions

$$S_{0n}(\alpha,s)\underset{\alpha\rightarrow\infty}{\longrightarrow}N_nH_n(s\sqrt{\alpha})\exp[-(\alpha/2)s^2]. \qquad (37)$$

At $\alpha=5$, the value of $\mu$ for the optimum gaussian function, 1.5280, is already close to $(\alpha/2)^{1/2}=1.579$.

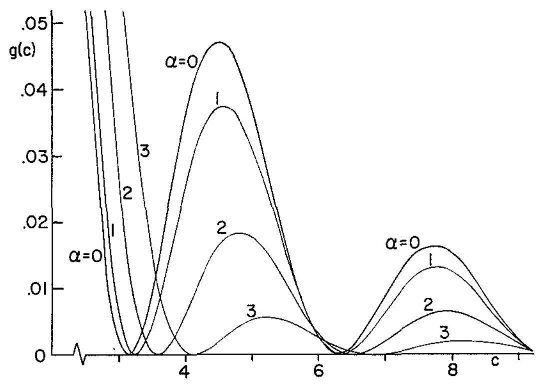

FIG. 4. Side lobes of Fig. 3.

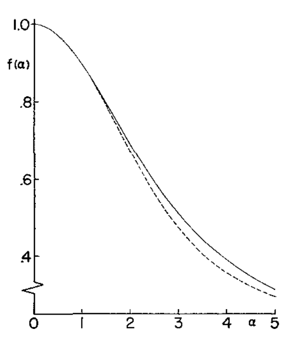

FIG. 5. Comparison of finite gain for angle-function and uniform illuminations. The optimum truncated-gaussian illumination produces a finite gain so close to the solid curve that the distinction can not be seen on this scale. $F=4\pi Af(\alpha)f(\beta)/\lambda^2$.—$X(s)=S_{00}(\alpha,s)$. ----$X(s)=$const.

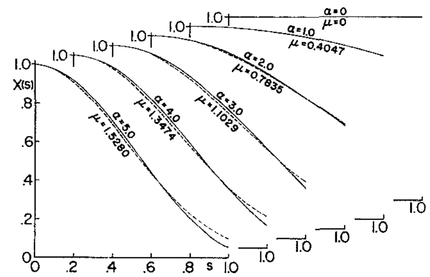

FIG. 6. Comparison of amplitude distribution functions for angle function and for the optimum gaussian-function illuminations. The origin has been translated to a new position for each value of $\alpha$. —$X(s)=S_{00}(\alpha,s)$. ----$X(s)=\exp[-(\mu s)^2]$.

## CONCLUDING REMARKS

The close relation between the lowest diffraction-loss eigenmode of a confocal resonator and the maximum power-transfer coefficient is not surprising. Both are dictated by the same physical restriction, a minimum diffraction loss. The fact that irradiance distributions can be obtained from lasers that are ideally suited for communication with distant targets may be a fortunate circumstance. The close agreement of the optimum truncated gaussian distribution with the prolate angle-function distribution for rectangular apertures is consistent with the fact that the agreement must become exact in both limiting cases of large and of very small receiving-antenna sizes.

## ACKNOWLEDGMENTS

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[12] Carson Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, Calif., 1957).