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DISSIPATIVE PERIODIC PROCESSES

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Dissipative Periodic Processes

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1. Introduction.

It seems worthwhile to consider a theory of periodic processes of sufficient generality that it can be applied to dynamical systems defined by partial differential equations (distributed parameter systems), functional differential equations (hereditary systems), systems arising in the theory of elasticity, etc. A large number of examples of such dynamical systems and more complete references can be found in the paper [1] by Hale. There the principal objective was to obtain a generalized invariance principle and to exploit this invariance to obtain a general stability theory. Hale's work was extended in [2] to periodic dynamical systems by Slemrod, and Dafermos in [3] gave an invariance principle for compact processes which include periodic processes. Some recent applications of this general stability theory can be found in [4]-[7].

The objective of this paper is to develop in the spirit of the work above a general and meaningful theory of dissipative periodic systems. Nonlinear ordinary differential equations which are periodic and dissipative were studied by Levinson [8] in 1944. More general results for ordinary differential equations can be found in [9] and [10]. For ordinary periodic differential equations one studies the iterates of a map T of a state space into itself where the map T is topological and the space is locally compact (n-dimensional Euclidean space). However, for the applications we have in mind the solutions will be unique only in the forward direction of time and the state spaces are not locally compact. Because

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13. ABSTRACT

The objective of this paper is to develop a general and meaningful theory of dissipative periodic systems. For ordinary periodic differential equations one studies the iterates of a map T of a state space into itself where the map T is topological and the space is locally compact (n-dimensional Euclidean space). However for the applications the authors have in mind, the solutions will be unique only in the forward direction of time and the state spaces are not locally compact. Because of this generalization of the results for ordinary differential equations is by no means trivial.

The basic theory of dissipative periodic processes on Banach spaces are developed in Sections 2 and 3 of the paper. How this applies to retarded functional differential equations of retarded type is discussed in the fourth section. Two sufficient conditions for dissipativeness are given in terms of Liapunov functions. They formalize the intuitive notion that many systems for large displacements dissipate energy. Application of these results is illustrated at the end of Section 4.

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2. Dissipative mappings.

Let R denote the real numbers, R^+ the nonnegative reals, and let X be a Banach space with norm $\|\cdot\|$. Consider a mapping $u: R \times X \times R^+ \to X$ and define $(t,\tau): X \to X$ for each $t \in R$ and each $\tau \in R^+$ by

$$(t,\tau)x = u(t,x,\tau). \tag{2.1}$$

Interpret $(t,\tau)x$ as the state of the system at time $t+\tau$ if initially the state of the system at time t was x. A process on a Banach space X is a mapping $u: R \times X \times R^+ \to X$ with the following properties:

2.1. u is continuous

2.2. $(t,0)x = x$

2.3. $(t+\sigma,\tau)(t,\sigma) = (t,\sigma+\tau)$.

Thus a process here is essentially what was called in [2] a "generalized nonautonomous dynamical system" and differs by the continuity condition on u from what was called a process in [3].

A process is said to be periodic of period $\omega > 0$ if $(t+\omega,\tau) = (t,\tau)$ for all $t \in R$ and all $\tau \in R^+$. For any fixed $t \in R$ there is then associated with a periodic process a continuous mapping $T: X \to X$ defined by

$$T(x) = (t,\omega)x.$$

With T^n the n^{th} iterate of T it follows from (2.3) that $T^n(x) = (t,n\omega)x$ and the sequence $T^n(x)$, $n = 0,1,2\ldots$ is called the (positive) motion or orbit through x. Since for a periodic process $(t,\tau+k\omega) = (t,\tau)(t,k\omega)$, we see that the fixed points of T^k correspond to periodic motions of the periodic process.

Thus motivated we will now spend the rest of this section studying the discrete dynamical system defined by an arbitrary continuous mapping $T: X \to X$ where X is a Banach space. We select a Banach space because we use the Schauder fixed point theorem and the property that the closed convex hull of a compact set is compact for Corollary 3.1, which is the principal result. A point y is said to be a limit point of the motion $T^n(x)$ if there exists a subsequence n_k of integers such that $n_k \to \infty$ and $T^{n_k}(x) \to y$ as $k \to \infty$. The limit set $L(x)$ is the set of all limit points of $T^n(x)$. Note that

$$L(x) = \bigcap_{j=0}^{\infty} \mathrm{Cl} \bigcup_{n=j}^{\infty} T^n(x), \tag{2.2}$$

where Cl is closure.

A set $M \subset X$ is said to be positively invariant if $T(M) \subset M$ and negatively invariant if $M \subset T(M)$. It is said to be invariant if $T(M) = M$; i.e., if it is both positively and negatively invariant. Negative invariance and the axiom of choice implies the existence on M of a right inverse T^{-1} to T. Hence we have T^n defined for all integers n (when n is negative $T^n = (T^{-1})^{-n}$) with the property that $T^{k-j} = T \cdot T^{-j}$ for all nonnegative integers k and j. Thus negative invariances implies the existence of an extension over all integers of each positive motion through a point of M and the negative extension is contained in M. Although the following lemma is essentially contained in [1], [2], and [3], the proof for the case of discrete motions is especially simple and is included. Essentially the same proof also yields the generalization given in Lemma 3.1.

Lemma 2.1. If the motion $T^n(x)$, $n = 0,1,2,\ldots$ is precompact, then the limit set $L(x)$ is nonempty, compact and invariant.

Proof: $L(x)$ is the intersection (2.2) of a descending sequence of nonempty compact sets and is therefore nonempty and compact. The continuity of T implies $L(x)$ is positively invariant. Let $y \in L(x)$. Then there exists a sequence of integers n_j such that $n_j \to \infty$ and $T^{n_j}(x) \to y$ as $j \to \infty$. We can by the precompactness assumption select a subsequence, which we still label n_j, such that $T^{n_j-1}(x) \to z$ as $k \to \infty$. Now $z \in L(x)$, and $T(z) = \lim_{j \to \infty} T^{n_j}(x) = y$ by the continuity of T. Hence $L(x) \subset T(L(x))$ which completes the proof. Taking $T^{-1}(y) = z$ gives a right inverse of T on $L(x)$.

For most applications other than ordinary differential equations the state space X is not locally compact and there is the practical difficulty of determining compactness. For many processes T smooths the initial data and with suitable topologies for the state spaces boundedness of the motion implies that the motion is precompact (see, for example, [1]). Intuitively one expects from energy considerations that real processes will be dissipative for large displacements and the notion of dissipativeness is naturally associated with boundedness. With applications in mind we develop a theory of dissipative processes based on boundedness and require a smoothing property stronger than that mentioned above.

Definition 2.1. T is said to smooth if there is a nonnegative integer

n_0 such that for each bounded set B in X there is a compact set B^* in X such that $T^n(x) \in B$ for $n = 0,1,\dots,N$ implies $T^n(x) \in B^*$ for $n = n_0, n_0+1,\dots,N$.

For ordinary differential equations every continuous T smooths with $n_0 = 0$ and for retarded functional differential equations T smooths with $n_0\omega > r$, the retardation. If T carried bounded sets into bounded sets, then Definition 2.1 would imply that T^n is compact for $n \geqq n_0$. However, even for retarded functional differential equations T may not have this property (for an example see [17]).

Definition 2.2. T is dissipative if (1) it smooths and (2) there is a bounded set B in X with the property that given $x \in X$ there is a positive integer $N(x)$ such that $T^n(x) \in B$ for $N(x) \leqq n \leqq N(x)+n_0$.

It is convenient to note first the following simple result.

Lemma 2.2. If T is dissipative, then there is a compact set K_0 in X such that given $x \in X$ there is a positive integer $n(x)$ and an open neighborhood O_x of x with the property that $T^{n(x)}(O_x) \subset K_0$.

Proof: We may always assume that the set B in Definition 2.2. is open. Taking B to be open, we define $K_0 = B^*$ (Definition 2.1). Then by continuity of T there is an open neighborhood O_x of x such that $T^n(O_x) \subset B$ for $N(x) \leqq n \leqq N(x)+n_0$. Let $n(x) = N(x)+n_0$. Then $T^{n(x)}(O_x) \subset B^* = K_0$, which completes the proof.

We now show that if T is dissipative, then the dissipativeness is uniform on compact sets in that there is a compact set K with the property that eventually the motion of each compact set is in K. This result generalizes Theorems 2.1 and 2.2 of [10] and here the proofs are both simpler and direct. If the space is locally compact (ordinary differential equations), then every continuous T smooths and T is dissipative if there is a bounded set B such that for each $x \in X$ there is an $N(x)$ such that $T^{N(x)}(x) \in B$. If the space is not locally compact, the assumption that T smooths is needed and for each $x \in X$ the motion $T^n(x)$ must remain in B long enough to smooth.

Theorem 2.1. If T is dissipative, then there is a compact set K in X with the property that given a compact set H in X there is a positive integer $N(H)$ and an open neighborhood O_H of H such that $T^n(O_H) \subset K$ for all $n \geqq N(H)$.

Proof: Let K_0 be the compact set of Lemma 2.2 and let H be any compact set in X. Then for each $x \in H$ there is an $n(x)$ and an O_x such that $T^{n(x)}(O_x) \subset K_0$. Selecting from this open covering of H a finite covering, we see that there is an $n(H)$ such that for each O_x of the finite covering there is an $i = i(x)$ such that $1 \leqq i \leqq n(H)$ and $T^i(O_x) \subset K_0$. Hence all that we need do to prove the theorem is to show that there is a positive integer $N(K_0)$ and a compact set K such that $T^n(K_0) \subset K$ for all $n \geqq N(K_0)$. Let $x \in K_0$ and let n be any positive integer greater than or equal to $n(K_0)$. There is then a least integer j, $0 \leqq j \leqq n$ such that $T^{n-j}(x) \in K_0$ and $T^{n-k} \notin K_0$ for $0 \leqq k < j$. It follows by what was shown at the beginning of the proof that $0 \leqq j \leqq n(K_0)$. Hence $T^n(x)$ is contained in the union K of $K_0, T(K_0),\dots,T^{n(K_0)}(K_0)$, which is compact and

The principal result is now an immediate consequence of Theorem 2.1 which implies that every smooth, dissipative periodic process has a periodic solution.

Corollary 2.1. If T is dissipative, then there is an integer k such that T^n has a fixed point for each $n \geq k$.

Proof: Since in a Banach space the closed convex hull of a compact set is compact, we may assume that the K of Theorem 2.1 is compact and convex. Then $T^n(K) \subset K$ for each $n \geq N(K)$ and by the Schauder fixed point theorem each such T^n has a fixed point.

There is a very special class of dissipative systems where T has a unique fixed point. For a topological map T and hence for periodic ordinary differential equations a result of this type was given in [11] (Corollary 2). (For ordinary differential equations see also [10] and for retarded functional differential equations see [12].)

Definition 2.3. T is said to be extremely stable if (1) there is a bounded motion $x, T(x), \ldots, T^n(x), \ldots$ and (2) $\|T^n(x) - T^n(y)\| \to 0$ as $n \to \infty$ for each $x, y \in X$.

Corollary 2.2. If T is smooth and extremely stable, then T has a (unique) fixed point and all motions approach this fixed point as $n \to \infty$.

Proof: Since T smooths, the bounded motion is precompact. Let L be its limit set, which by Lemma 2.1 is nonempty and compact. Since

T is extremely stable, every solution approaches L as $t \to \infty$ (all solutions have L as their positive limit set), and therefore T is certainly dissipative. By Corollary 1 we have $T^k x = x$ for some positive k and some x. Since $T(x)-x = T^{nk}(T(x)) - T^{nk}(x)$ for $n = 1, 2, \ldots$, it follows from (2) of Definition 2.3 that $T(x) = x$. Hence $L = \{x\}$ and the proof is complete.

In Section 3 we show that the fixed point is globally asymptotically stable (Theorem 3.2).

3. The limit set I.

We wish now to show that if T is dissipative then there is a compact invariant set I that is globally asymptotically stable. Just as in [8] for second order ordinary differential equations I will be the maximum compact set invariant under T.

Let K be the compact set of Theorem 2.1. Define

$$I = \bigcap_{n=0}^{\infty} T^n(K) \tag{3.1}$$

Of course, K is not unique, but it is not difficult to see that I does not depend upon K. Let K_1 be any other compact set with the same property (Theorem 2.1) that K has and let $I(K)$ be the set defined in (3.1). Then for all n sufficiently large $T^n(K) \subset K_1$ and $T^n(K_1) \subset K$. Hence $I(K_1) = I(K_2)$. Similarly it is easy to see that if n_j is any sequence of integers such that $n_j \to \infty$ as $j \to \infty$ then

$$I = \bigcap_{j=0}^{\infty} T^{n_j}(K) \tag{3.2}$$

and, in particular, for $n_1 = n(K)$

$$I = \bigcap_{j=0}^{\infty} T^{jn_1}(K). \tag{3.3}$$

It is interesting to relate I to the motion $K, T(K), \ldots, T^n(K), \ldots$. Given a set H in X we define $L(H)$, called the <u>limit set</u> of the motion through H, by

$$L(H) = \bigcap_{j=0}^{\infty} \mathrm{Cl} \bigcup_{n=j}^{\infty} T^n(H),$$

where Cl denotes closure. Then $y \in L(H)$ means there exist sequences n_i and $y_i \in H$ such that $n_i \to \infty$ and $T^{n_i}(y_i) \to y$ as $i \to \infty$. Thus when H is a single point x this is the usual limit set $L(x)$. Now just as for Lemma 2.1 we obtain

<u>Lemma 3.1</u>. If for some j sufficiently large $\bigcup_{n=j}^{\infty} T^n(H)$ is precompact, then the limit set $L(H)$ is nonempty, compact, and invariant.

<u>Thoerem 3.1</u>. Assume that T is dissipative. Then $I = L(K)$ and hence I is nonempty, compact, invariant and is the maximum invariant set in X.

<u>Proof:</u> Clearly $I \subset L(K)$. To prove the converse consider any $y \in L(K)$. Then there is a sequence of integers n_i and a sequence $x_i \in K$ such that $T^{n_i}(x_i) \to y$ as $i \to \infty$. For any positive integer j we know that $T^{n-j}(x_n) \in K$ for all n sufficiently large (Theorem 2.1). Therefore for the sequence x_i we may assume that $y_i^j = T^{n_i - j}(x_i) \to y^j \in K$ as $i \to \infty$ for any positive integer j. But clearly then $T^j(y^j) = y$ and this implies $y \in I$. Hence $I = L(K)$ and by Lemma 3.1 I is nonempty, compact and invariant. If M is any compact invariant set, then it follows immediately from Theorem 2.1 that M is in I. This completes the proof.

We recall that a set M is a <u>global attractor</u> if $T^n(x) \to M$ as $n \to \infty$ for each $x \in X$. Since each motion $T^n(x)$ is precompact (Theorem 2.1) and its limit set $L(x)$ is nonempty, compact and invariant (Lemma 2.1), it follows from the above theorem that $L(x)$ is in I for each $x \in X$. Hence I is a global attractor. For $\delta > 0$ let M^δ denote the δ-neighborhood of M ($M^\delta = \{y;\ \|y-x\| < \delta$ for some $x \in M\}$). A set M is said to be <u>stable</u> if given $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in M^\delta$ implies that $T^n(x) \in M^\varepsilon$ for all $n \geq 0$. If the set M is both stable and a global attractor it is said to be <u>globally asymptotically stable</u>. By an argument similar to that used by LaSalle to prove Theorem 3 in [13] we obtain

<u>Theorem 3.2.</u> If T is dissipative, then the set I is globally asymptotically stable.

<u>Proof:</u> We pointed out above that I is a global attractor and we need only show that I is stable.

Let us observe first how T being dissipative enters the proof. Suppose that $y_j \to y \in I$ and relative to this sequence define

$$\gamma = \{z;\ T^{n_j}(y_j) \to z \text{ as } j \to \infty \text{ for some sequence } n_j,\ z \text{ not in } I\}.$$

In the remainder of the proof $T^{n_j}(y_j) \to z$ as $j \to \infty$, $z \notin I$. Since I is compact, given $z \in \gamma$, there is an $\varepsilon > 0$ and an ε-neighborhood I^ε of I such that (1) z is not in the ε-neighborhood I^ε of I and (2) I^ε is in the open neighborhood O_I of Theorem 2.1.

Since T is continuous and I is invariant, then given any integer k we know that $T^n(y_j) \in I^\varepsilon$ for $0 \leq n \leq k$ and all j sufficiently large. This implies that $n_j \to \infty$ as $j \to \infty$. Note by Theorem 2.1 that this also implies that each such sequence $T^{n_j}(y_j)$ is in K for j sufficiently large and therefore $\gamma \subset K$. Using the standard diagonalization procedure, it is easy to see that γ is closed and hence γ is compact. We will now show that γ is invariant. Clearly $T(\gamma) \subset \gamma$. Given $z \in \gamma$ the sequence $T^{n_j - 1}(y_j)$ is precompact. If ω is a limit point of this sequence, then $T(\omega) = z$ and $\omega \in \gamma$. Hence $\gamma \subset T(\gamma)$, and γ is compact and invariant. Therefore by Theorem 3.1 the set γ must be empty since γ is not in I.

We now use this to show that I is stable. Assume that I is not stable. Then there exists for some $\varepsilon > 0$, which we can make as small as we please, sequences n_j and y_j such that $y_j \to I$ as $j \to \infty$, $T^n(y_j) \in I^\varepsilon$, $0 \leq n \leq n_j$, and $T^{n_j+1}(y_j)$ is not in I^ε. Since I is compact, we may assume that $y_j \to y \in I$ as $j \to \infty$. Otherwise, we know that there are $z_j \in I$ such that $\|y_j - z_j\| \to 0$ as $j \to \infty$. Then we could select a subsequence z_{m_j} with limit y, and clearly $y_{m_j} \to y$ as $j \to \infty$. As before we know that $T^{n_j}(y_j)$ is precompact and we may assume by again selecting a subsequence if necessary that $T^{n_j}(y_j) \to z$ as $j \to \infty$. Since $T(z)$ is not in I^ε, it is not in I, and therefore z is not in I. But then $z \in \gamma$, and this contradiction completes the proof.

4. Retarded functional differential equations.

We examine briefly how Sections 2 and 3 can be applied to retarded functional differential equations. Let R^n be a real n-dimensional vector space with norm $|\cdot|$. Given $r > 0$, $C = C([-r,0],R^n)$ will denote the space of continuous functions φ mapping $[-r,0]$ into R^n with $\|\varphi\| = \sup\{\varphi(\theta); -r \leq \theta \leq 0\}$. Let f be a continuous function taking $R \times C$ into R^n. A retarded functional differential equation is a system of the form

$$\dot{x}(t) = f(t,x_t), \tag{4.1}$$

where $\dot{x}$ is the derivative of x and $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. A function x mapping $[t_0-r,t_0+a)$ into R^n is said to be a solution of (4.1) on $[t_0,t_0+a)$ with initial value $\varphi \in C$ at t_0 if x has a continuous derivative on $[t_0,t_0+a)$ satisfying (4.1) and $x_{t_0} = \varphi$.

A brief survey of the history of functional differential equations is given in [14]. For general theorems on existence, uniqueness, continuation and continuity see [1], [4], [12] or [15]. These theorems are quite similar to those for ordinary differential equations, and we make the general assumption that f satisfies, in addition to the continuity condition above, conditions sufficient to insure uniqueness of solutions to the right. We shall also assume that the solution $x(t,t_0,\varphi)$ of (4.1) satisfying $x_{t_0}(t_0,\varphi) = \varphi$ is defined for all $t \geq t_0$. This will be implied by dissipativeness. Then $u(t_0,\varphi,\tau) = x_{t_0+\tau}(t_0,\varphi)$ is, as described in Section 2, a process on the

Banach space C. We shall assume also (1) $f(t,\varphi)$ is periodic in t with period $\omega > 0$ and (2) f maps bounded sets of $R \times C$ into R^n. If $x(t)$ is any solution of (4.1), we see that $|x(t)| < b$ for $t \in [t_0,t_0+T)$ implies $\|\dot{x}_t\| < d$ for $t \in [t_0+r,t_0+T)$. Thus corresponding to each bounded set B in C there is a compact set B^* in C such that $x_t \in B$ for $t \in [t_0,t_0+T)$ implies $x_t \in B^*$ for $t \in [t_0+r,t_0+T)$. This smoothing of the initial data was exploited by Hale in [1] although he did not use and did not need a smoothing property as strong as this one. Defining $T(\varphi) = x_{t_0+\omega}(t_0,\varphi)$ for any fixed t_0, we see that T smooths in the sense of Definition 2.1 with n_0 the least integer such that $n_0\omega > r$. We see also that T will be dissipative if there is a number b such that given $\varphi \in C$ there is a $t_1 = t_1(\varphi,t_0)$ with the property that $|x(t,t_0,\varphi)| < b$ for all $t_1 \leq t \leq t_1+n_0\omega$.

Just as with ordinary differential equations one can give in terms of a Liapunov function necessary and sufficient conditions that there exist a b such that given $\varphi \in C$ there is a $t_1 = t_1(\varphi,t_0)$ with the property that $|x(t,t_0,\varphi)| < b$ for all $t \geq t_1$ (see [10]). Here we confine ourselves to stating two sufficient conditions. These conditions and the proofs that they are sufficient are similar to the conditions and proofs for uniform asymptotic stability given in [16] (Theorems 11.1 and 11.2).

Let V be a continuous mapping of $R \times C$ into R^n. Define relative to solutions $x(t,t_0,\varphi)$ of (4.1)

$$\dot{V}(t,\varphi) = \overline{\lim_{x \to 0^+}} \frac{1}{h} [V(t+h, x_{t+h}(t,\varphi)) - V(t,\varphi)]$$

The function t satisfies the same conditions as before and the mapping T is defined above.

<u>If</u>

(1) $u(\varphi(0)) \leq V(t,\varphi) \leq v(\|\varphi\|)$ <u>where</u> u <u>and</u> v <u>are continuous and</u> $u(s) \to \infty$ <u>as</u> $s \to \infty$.

(2) $\dot{V}(t,\varphi) < -\varepsilon$ <u>for all</u> t <u>and</u> $|\varphi(0)| > b$, <u>then</u> T <u>is dissipative</u>.

If $V(t,x)$ is a continuous mapping of $R \times R^n$ into R^n, then $\dot{V}(t,x(t))$ ($x(t)$ continuous on $[t-r,t]$) is defined by $\dot{V}(t,x(t)) = \dot{W}(t,x_t)$, where $W(t,\varphi) = V(t,\varphi(0))$. Another useful sufficient condition in terms of $V(t,x)$ is

<u>If</u>

(a) $u(|x|) \leq V(t,x) \leq v(|x|)$, <u>where</u> u <u>and</u> v <u>are continuous and</u> $u(s) \to \infty$ <u>as</u> $s \to \infty$.

(b) $\dot{V}(t,x(t)) \leq -\varepsilon < 0$ <u>for all</u> $t > 0$ <u>and all</u> $x(t)$ <u>satisfying</u> $|x(t)| > b$ <u>and</u>

$$V(\xi, x(\xi)) < g(V(t,x(t))), \quad \text{on } [t-r,t]$$

<u>where</u> $g(s)$ <u>is continuous, nondecreasing and</u> $g(s) > s$ <u>for</u> $s > a > 0$, <u>then</u> T <u>is dissipative</u>.

We illustrate these results by simple one-dimensional examples (see [16] where all the examples on uniform asymptotic stable varied slightly give examples of dissipative functional differential equations).

<u>Example 1.</u> Consider the one-dimensional linear system

$$\dot{x}(t) = -ax(t) - b(t)x(t-r) + e(t)$$

where $r > 0$, $a > 0$, $b(t)$ and $e(t)$ are continuous and periodic of period ω. Let B be the amplitude of $b(t)$ (the maximum value of $|b(t)|$) and let E be the amplitude of $e(t)$. With $V(\varphi) = \frac{1}{2a}\varphi^2(0) + \mu\int_{-r}^{0}\varphi^2(\theta)d\theta$ then for any solution x_t

$$\dot{V}(x_t) = -(1-\mu)x^2(t) - \frac{b(t)}{a}x(t)x(t-r) - \mu x^2(t-r) + \frac{1}{a}e(t)x(t).$$

The first three terms are a quadratic form and we see that

$$\dot{V}(x_t) \leq -\alpha x^2(t) + \frac{1}{a} e(t)x(t)$$

if $4(1-\alpha-\mu)\mu > \frac{b^2(t)}{a^2}$, $0 \leq \mu < 1$, or, taking $2\mu = 1-\alpha$, if $b^2(t) < (1-\alpha)^2 a^2$, $0 < \alpha \leq 1$. Thus with $B < \lambda a$

$$\dot{V}(x_t) \leq -x^2(t)(1-\lambda - \frac{E}{a|x(t)|}), \quad 0 \leq \lambda < 1.$$

Hence, if $|x(t)| > \beta > \frac{E}{(1-\lambda)a}$, $\dot{V}(x_t) \leq -\varepsilon < 0$. Clearly (1) and (2) above are satisfied and the system is dissipative. In fact, since the system is linear, it is extremely stable (Section 3) and has a unique periodic solution of period ω that is globally asymptotically stable (a steady state solution).

A more detailed analysis shows, if $B < \lambda a$, $0 \leq \lambda < 1$, that the amplitude of the periodic solution is not greater than $\frac{E}{(1-\lambda)a}\sqrt{1+ar}$. This suggests it can be expected that the amplitude may increase as a decreases, as B increases, or as the delay increases. As $r \to 0$ the result is the best possible.

Example 2. Consider the one-dimensional system

$$\dot{x}(t) = f(t,x(t-\gamma(t))) + e(t), \quad 0 \leq \gamma(t) \leq r \qquad (4.2)$$

where f, γ, and e are continuous and $f(t+\omega,x) = f(t,x)$, $\gamma(t+\omega) = \gamma(t)$ and $e(t+\omega) = e(t)$ for all t and x. We assume moreover that f has a continuous partial derivative $f_2 = \frac{\partial f}{\partial x}$ such that $|f_2(t,x)| < L$ for all t and all x. Equation (4.2) is then equivalent to

$$\dot{x}(t) = f(t,x(t)) + [x(t-\gamma(t)) - x(t)]f_2(t,x(\theta(t)) + e(t),$$

where $t-r < \theta(t) < t$. Define $2V(x) = x^2$. Then for any solution $x(t)$

$$\dot{V}(x(t)) = x(t)f(t,x(t)) + x(t)[x(t-\gamma(t)) - x(t)]f_2(t,x(\theta(t)) + e(t)x(t).$$

For those solutions satisfying

$$x^2(t-\xi) \leq qx^2(t), \quad 0 \leq \xi \leq r, \quad q > 1$$

$$\dot{V}(x(t)) \leq x^2(t)\left[\frac{f(t,x(t))}{x(t)} + (1+q)L + \frac{|e(t)|}{|x(t)|}\right]$$

Hence, if for all t and all $|x| > \rho$

$$\frac{f(t,x)}{x} + 2L \leq -\alpha < 0,$$

then (a) and (b) above are satisfied with $g(s) = qs$, $q > 1$, and the system (4.2) is dissipative. If $f(t,0) < 0$, then a similar argument shows that it is dissipative if

$$\frac{f(t,x)}{x} + L\gamma(t) \leq -\alpha < 0$$

for all t and all $|x| > \rho$. For the linear system $(a(t+\omega) = a(t))$

$$\dot{x}(t) = -a(t)x(t-\gamma(t)) + e(t),$$

$$\dot{V}(x(t)) = -a(t)x(t-\gamma(t))x(t) + e(t)x(t).$$

If $0 < \alpha < a(t)$, then the same argument shows that the system is dissipative and therefore extremely stable. Here one can easily see that the amplitude of the periodic solution is not greater than E/α. In this generality the bound on the amplitude is the best possible (take $a(t) = A$, $e(t) = E$ and $\gamma(t) = 0$ for all t).

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