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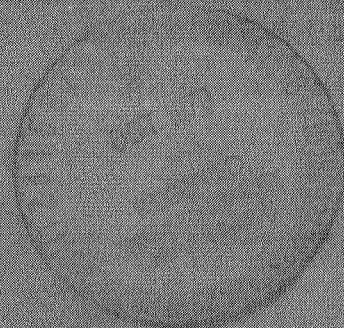
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GREEN'S FUNCTIONS FOR STURM-LIOUVILLE PROBLEMS

Jack Simons



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GREEN'S FUNCTIONS FOR STURM-LIOUVILLE PROBLEMS

by

Jack Simons

Theoretical Chemistry Institute

University of Wisconsin

Madison, Wisconsin 53706

ABSTRACT

It is shown how the Green's function for any inhomogeneous Sturm-Liouville (S-L) problem can be constructed once two independent solutions of the homogeneous problem are known. The general technique is illustrated by considering two specific S-L problems. When the parameter appearing in the S-L equation is equal to an eigenvalue of the homogeneous S-L problem an added difficulty arises. The method of constructing a generalized Green's function which deals with this difficult case is also treated by example.

I. General Sturm-Liouville Theory

The inhomogeneous Sturm-Liouville (S-L) problem can be written in the following form¹

$$\frac{d}{dx} \left(p \frac{d\psi}{dx} \right) - q \psi + \lambda \rho \psi = -f, \quad a \leq x \leq b \quad (1)$$

with the boundary conditions

$$\begin{aligned} (\alpha \frac{d\psi}{dx} + \beta \psi)_{x=a} &= \phi_a \\ (\alpha \frac{d\psi}{dx} + \beta \psi)_{x=b} &= \phi_b \end{aligned} \quad (2)$$

The known functions $p(x)$, $q(x)$, $\rho(x)$, and $f(x)$ are restricted only in that

- (i) $p(x) \geq 0$, $\rho(x) \geq 0$ for $a \leq x \leq b$
- (ii) $p(x)$ is differentiable for $a \leq x \leq b$.

The constant λ is a parameter of the problem; α and β are constants which serve to make the boundary conditions as general as possible for the S-L problem.

The Green's function $G(x,y)$ is introduced by seeking a solution of Eq. (1) having the specific form given below:

$$\psi(x) = \lim_{\eta \rightarrow 0+} \left\{ \int_a^{x-\eta} G(x,y) f(y) dy + \int_{x+\eta}^b G(x,y) f(y) dy \right\}. \quad (3)$$

For convenience Eq. (3) is written in shorthand as follows

$$\psi(x) = \int_a^b G(x,y) f(y) dy. \quad (4)$$

Substituting the expression for $f(y)$ given in Eq. (1) into Eq. (4) results in

$$\psi(x) = - \int_a^b G(x,y) \left[\frac{d}{dy} \left(p \frac{d\psi}{dy} \right) - q \frac{d\psi}{dy} + \lambda p \psi \right] dy \quad (5)$$

After integration by parts Eq. (5) becomes

$$\begin{aligned} \psi(x) = & - \int_a^b \left[\frac{d}{dy} \left(p \frac{dG}{dy} \right) - q G + \lambda p G \right] \psi(y) dy \\ & - \lim_{\eta \rightarrow 0^+} \left[p \left(\frac{d\psi}{dy} G - \frac{dG}{dy} \psi \right) \right]_a^{x-\eta} \\ & - \lim_{\eta \rightarrow 0^+} \left[p \left(\frac{d\psi}{dy} G - \frac{dG}{dy} \psi \right) \right]_{x+\eta}^b \end{aligned} \quad (6)$$

Based on Eq. (6) we now present four conditions which the correct Green's function must satisfy for the right hand side of Eq. (3) to correctly yield the function $\psi(x)$. We take as the first property of the Green's function that it satisfies the homogeneous S-L equation

$$\frac{d}{dy} \left(p \frac{dG}{dy} \right) - q G + \lambda p G = 0 \quad (7)$$

in the intervals $a \leq y < x \leq b$, $a \leq x < y \leq b$, i.e. everywhere except at the point $x = y$. With this the integral term in Eq. (6) vanishes identically. The second property is that $G(x,y)$ satisfies the same boundary conditions (Eq. (2)) as ψ at $y = a$ and $y = b$. This implies that two of the boundary terms appearing in Eq. (6) are equal to zero:

$$\begin{aligned}
& \left[p \left(\frac{d\psi}{dy} G - \frac{dG}{dy} \psi \right) \right]_a^b \\
&= \frac{1}{\alpha} \left[p \left(\alpha \frac{d\psi}{dy} G - \alpha \frac{dG}{dy} \psi \right) \right]_a^b \\
&= -\frac{1}{\alpha} \left[p (\beta \psi G - \beta G \psi) \right]_a^b = 0 \quad (8)
\end{aligned}$$

Equation (6) is therefore reduced to the following simple form:

$$\begin{aligned}
\psi(x) &= -\lim_{\eta \rightarrow 0+} \left[p \left(\frac{d\psi}{dy} G - \frac{dG}{dy} \psi \right) \right]_{x+\eta}^{x-\eta} \\
&= -\lim_{\eta \rightarrow 0+} \left\{ p(x) \left[\frac{dG}{dy}(x, x+\eta) - \frac{dG}{dy}(x, x-\eta) \right] \psi(x) \right. \\
&\quad \left. + p(x) [G(x, x+\eta) - G(x, x-\eta)] \frac{d\psi}{dy}(x) \right\} \quad (9)
\end{aligned}$$

Use has been made of the facts that $p(x)$ and $\psi(x)$ are continuous functions in the range $a \leq x \leq b$.

The remaining two restrictions on $G(x, y)$ are easily inferred from Eq. (9). Clearly if $G(x, y)$ is a continuous function of y for all x and y in the interval $[a, b]$, and if $\frac{dG(x, y)}{dy}$ is a continuous function of y except for a jump discontinuity of magnitude $-\frac{1}{p(x)}$ at the point $y = x$:

$$\lim_{\eta \rightarrow 0+} \left(\frac{dG}{dy}(x, x+\eta) - \frac{dG}{dy}(x, x-\eta) \right) = -1/p(x) \quad (10)$$

then the right hand side of Eq. (9) will yield the desired function $\psi(x)$.

In review then, we see that the correct Green's function $G(x,y)$, which when substituted into Eq. (3) gives the solution $\Psi(x)$ to the S-L problem, exhibits the following four properties:

- (i) $G(x,y)$ obeys the homogeneous S-L equation, Eq. (7)
- (ii) $G(x,y)$ obeys the same boundary condition as $\Psi(y)$ (i.e. Eq. (2))
- (iii) $G(x,y)$ is continuous for all x and y in $[a,b]$
- (iv) $\frac{dG}{dy}(x,y)$ is continuous for all x and y in $[a,b]$ except for a jump discontinuity of magnitude $-1/p(x)$ at the point $y = x$, (Eq. (10)).

As we will soon demonstrate by example, these four properties are sufficient to determine $G(x,y)$ unless the parameter λ is an eigenvalue of the homogeneous S-L equation. This more difficult case will be treated after we discuss two illustrative examples.

II. Two Simple Examples

To understand how the four properties (i) - (iv) can be used to determine the Green's function $G(x,y)$, let us consider a specific S-L problem

$$\frac{d}{dx} \left(x \frac{d\psi}{dx} \right) - \frac{n^2}{x} \psi + k^2 x \psi = -f(x), \quad 0 \leq x \leq b$$

with boundary conditions

$$\psi(b) = 0$$

$$\psi(0) \text{ finite.}$$

(11)

In the interval $0 \leq y < x \leq b$ $G(x,y)$ should satisfy

$$\frac{d}{dy} \left(y \frac{dG(x,y)}{dy} \right) - \frac{n^2}{y} G(x,y) + k^2 y G(x,y) = 0.$$

Therefore $G(x,y)$ can be expanded in terms of the two independent solutions of the homogeneous differential equation:

$$G(x,y) = A(x) J_n(ky) + D(x) Y_n(ky), \quad 0 \leq y < x \leq b. \quad (12)$$

The functions $A(x)$ and $D(x)$ are to be determined by using the four properties of $G(x,y)$. $J_n(ky)$ and $Y_n(ky)$ are the two independent solutions of the Bessel equation

$$\frac{d^2}{dz^2} W(z) + \frac{1}{z} \frac{dW}{dz} - \frac{n^2}{z^2} W + W = 0, \quad (13)$$

for $z = ky$. Likewise for the interval $0 \leq x < y \leq b$, $G(x,y)$ can be written as

$$G(x,y) = B(x) J_n(ky) + C(x) Y_n(ky), \quad 0 \leq x < y \leq b. \quad (14)$$

The boundary condition that $G(x,0)$ be finite implies that $D(x) = 0$ because $Y_n(ky)$ is not finite at $y = 0$. The second boundary condition $G(x,b) = 0$ takes the form

$$B(x) J_n(kb) + C(x) Y_n(kb) = 0. \quad (15)$$

Because $G(x,y)$ is continuous everywhere in $[a,b]$ it must be continuous at the point $y = x$ for all x in $[a,b]$. This property is easily written as

$$A(x) J_n(kx) = B(x) J_n(kx) + C(x) Y_n(kx). \quad (16)$$

The condition that $\frac{dG}{dy}$ have a jump discontinuity at $y = x$ leads to the equation

$$B(x) J_n'(kx) + C(x) Y_n'(kx) - A(x) J_n'(kx) = -1/x, \quad (17)$$

where

$$J_n'(kx) \equiv \frac{d J_n(kx)}{dx}$$

and

$$Y_n'(kx) \equiv \frac{d Y_n(kx)}{dx}. \quad (18)$$

Equations (15) - (17) can be solved simultaneously for the three functions $A(x)$, $B(x)$, and $C(x)$. The resulting solutions are given by:

$$A(x) = x^{-1} [J_n(kx) Y_n(kb) - J_n(kb) Y_n(kx)] [J_n(kb) Y_n(kx) J_n'(kx) - J_n(kb) J_n(kx) Y_n'(kx)]^{-1} \quad (19)$$

$$B(x) = x^{-1} J_n(kx) Y_n(kb) [J_n(kb) Y_n(kx) J_n'(kx) - J_n(kb) J_n(kx) Y_n'(kx)]^{-1} \quad (20)$$

$$C(x) = -x^{-1} J_n(kx) [Y_n(kx) J_n'(kx) - J_n(kx) Y_n'(kx)]^{-1} \quad (21)$$

By using various properties of the $J_n(kx)$ and $Y_n(kx)$ these expressions can be simplified considerably¹. Once this is done the Green's function $G(x,y)$ given in Eqs. (12) and (14) can be written as follows:

$$G(x,y) = -\frac{\pi J_n(ky)}{2 J_n(kb)} [J_n(kx) Y_n(kb) - Y_n(kx) J_n(kb)]$$

for $0 \leq Y \leq X \leq b$ and (22)

$$G(x, y) = -\frac{\pi J_n(kx)}{2 J_n(kb)} [J_n(ky) Y_n(kb) - Y_n(ky) J_n(kb)] \quad (23)$$

for $0 \leq x \leq y \leq b$

provided that $J_n(kb) \neq 0$. That is, provided that k^2 is not an eigenvalue of the homogeneous Bessel equation.

Thus we see that the four properties of $G(x, y)$ discussed above can be used to calculate the Green's function once we know two independent solutions of the homogeneous S-L differential equation. Notice that nothing is said about boundary conditions, until $G(x, y)$ is considered, that is the two independent solutions of the homogeneous equation will not, in general, satisfy all the boundary conditions of the problem.

As a second example of the general method let us consider the following S-L problem²

$$\frac{d^2 \psi}{dx^2} - \beta^2 x^2 \psi + \epsilon \psi = -f(x) \quad (24)$$

with boundary conditions

$$\psi(\pm \infty) = 0.$$

If we define the real parameter n by

$$n = \epsilon/2\beta - \frac{1}{2}$$

then two independent solutions of the homogeneous equation are given by³ $D_n(x\sqrt{2\beta})$ and $D_{-n-1}(ix\sqrt{2\beta})$. These Weber functions can be expressed in terms of the confluent hypergeometric functions of the

third kind (V_1 and V_2) as³:

$$D_m(z) = z^{m/2} e^{-\frac{1}{4}z^2} e^{\frac{i\pi m}{2}} V_2\left(-\frac{m}{2}/\frac{1}{2}/\frac{z^2}{2}\right)$$

and

$$D_{m-1}(iz) = z^{-\frac{1}{2}(m+1)} e^{\frac{i\pi(m+1)}{2}} e^{\frac{1}{4}z^2} V_1\left(-\frac{m}{2}/\frac{1}{2}/-\frac{z^2}{2}\right) \quad (25)$$

Because $G(x,y)$ must also be a solution of the homogeneous equation for $y < x$ and for $x < y$, we can immediately write

$$G(x,y) = A(x) D_n(y\sqrt{2\beta}) + B(x) D_{n-1}(iy\sqrt{2\beta}), \quad y < x \quad (26)$$

and

$$G(x,y) = C(x) D_n(y\sqrt{2\beta}) + E(x) D_{n-1}(ix\sqrt{2\beta}), \quad x < y \quad (27)$$

The conditions that $G(x, +\infty)$ be zero imply that $E(x) = 0$ and $A(x) = 0$ because $D_{-n-1}(i\infty\sqrt{2\beta})$ and $D_n(-\infty\sqrt{2\beta})$ are unbounded. Continuity of $G(x,y)$ at the point $y = x$ (for all x in $(-\infty, \infty)$) requires

$$B(x) D_{n-1}(ix\sqrt{2\beta}) = C(x) D_n(x\sqrt{2\beta}), \quad -\infty < x < \infty \quad (28)$$

The equation for the jump discontinuity of $\frac{dG}{dy}$ at $y = x$ reads as follows:

$$C(x) D_n'(x\sqrt{2\beta}) - B(x) D_{n-1}'(ix\sqrt{2\beta}) = -1, \quad -\infty < x < \infty \quad (29)$$

By solving Eqs. (28) and (29) we arrive at the following expression for $B(x)$ and $C(x)$:

$$B(x) = D_n(x\sqrt{2\beta}) \left[D_n(x\sqrt{2\beta}) D'_{-n-1}(x\sqrt{2\beta}) - D_{-n-1}(x\sqrt{2\beta}) D'_n(x\sqrt{2\beta}) \right]^{-1} \quad (30)$$

$$C(x) = D_{-n-1}(x\sqrt{2\beta}) \left[D_n(x\sqrt{2\beta}) D'_{-n-1}(x\sqrt{2\beta}) - D_{-n-1}(x\sqrt{2\beta}) D'_n(x\sqrt{2\beta}) \right]^{-1} \quad (31)$$

Defining $\Delta(x)$ as the denominator appearing in Eqs. (30) and (31), the final expression for $G(x,y)$ can be written as

$$G(x,y) = D_n(x\sqrt{2\beta}) D_{-n-1}(y\sqrt{2\beta}) \Delta^{-1}(x), \quad y < x \quad (32)$$

$$G(x,y) = D_{-n-1}(x\sqrt{2\beta}) D_n(y\sqrt{2\beta}) \Delta^{-1}(x), \quad x < y. \quad (33)$$

III. The Case When λ is an Eigenvalue

The treatment given above is only valid if the parameter ϵ is not an eigenvalue of the homogeneous differential equation with the same boundary conditions as the original S-L problem. If ϵ is equal to some eigenvalue, say ϵ_j , where j is a non-negative integer then a generalized Green's function⁴, which can still be used in Eq. (2) to give a solution $\Psi(x)$ of the inhomogeneous equation, can be constructed if the inhomogeneity $f(x)$ is orthogonal to the normalized solution $U_j(x)$ of the homogeneous S-L problem belonging to the eigenvalue ϵ_j , i.e.

$$\int_a^b U_j(x) f(x) dx = 0. \quad (34)$$

Assuming that Eq. (34) is true for the specific case of interest, we then proceed to construct $G(x,y)$ by using the same four properties discussed above with one modification. The requirement that $G(x,y)$ satisfy the homogeneous S-L differential equation for $x < y$ and for $y < x$ is replaced by the following property:

$$\frac{d^2}{dy^2} G(x,y) - \beta^2 y^2 G + \epsilon_j G = C_j e^{-\frac{\beta x^2}{2} - \frac{\beta y^2}{2}} H_j(x\sqrt{\beta}) C_j H_j(y\sqrt{\beta}) \quad (35)$$

where

$$C_j = \frac{1}{\sqrt{2^j j!}} \left(\frac{\beta}{\pi}\right)^{1/4} \quad (36)$$

is the normalization constant for the harmonic oscillator eigenfunction, and

$$H_j(z) = (-1)^j e^{z^2} \frac{d^j}{dz^j} e^{-z^2} \quad (37)$$

is the j^{th} Hermite polynomial. The function

$$C_j e^{-\frac{\beta x^2}{2}} H_j(x\sqrt{\beta})$$

is just the Weber function $D_j(x\sqrt{2\beta})$ normalized to unity.

To see that the quantity

$$\int_{-\infty}^{\infty} G(x,y) f(y) dy$$

is an acceptable solution $\Psi(x)$ of the S-L problem, we use the following identity for $f(y)$:

$$\frac{d^2 \Psi}{dy^2} - \beta^2 y^2 \Psi + \epsilon_j \Psi = -f(y), \quad (38)$$

to write

$$\int_{-\infty}^{\infty} G(x, y) f(y) dy = - \int_{-\infty}^{\infty} G(x, y) \left[\frac{d^2 \psi}{dy^2} - \beta^2 y^2 \psi + \epsilon_j \psi \right] dy, \quad (39)$$

Integrating by parts and using both Eq. (35) and the boundary conditions on $G(x, y)$ and $\Psi(y)$, we obtain

$$\int_{-\infty}^{\infty} G(x, y) f(y) dy = C_j e^{-\frac{\beta x^2}{2}} H_j(x/\sqrt{\beta}) \int_{-\infty}^{\infty} C_j e^{-\frac{\beta y^2}{2}} H_j(y/\sqrt{\beta}) \psi(y) dy + \psi(x). \quad (40)$$

That $\Psi(x)$ can, without loss of generality, be chosen to be orthogonal to $U_j(x)$ is easily shown:

Replace $\Psi(x)$ by $\Psi(x) - cU_j(x)$; then

$$\begin{aligned} & \frac{d^2}{dy^2} (\psi - c v_j) - \beta^2 y^2 (\psi - c v_j) + \epsilon_j (\psi - c v_j) \\ &= -f(y) - c \left[\frac{d^2 v_j}{dy^2} - \beta^2 y^2 v_j + \epsilon_j v_j \right] \\ &= -f(y) \end{aligned} \quad (41)$$

Thus $\Psi - cU_j$ is an equally valid solution of the inhomogeneous equation. Hence we can choose the constant c such that

$$\int_{-\infty}^{\infty} (\psi - c v_j)(x) v_j(x) dx = 0, \quad (42)$$

i.e. we can choose to find a solution of the inhomogeneous S-L equation which is orthogonal to U_j .

With $\Psi(x)$ orthogonal to $U_j(x)$, Eq. (40) reads

$$\int_{-\infty}^{\infty} G(x,y) f(y) dy = \Psi(x), \quad (43)$$

which is the desired identity. To assure that the function $\Psi(x)$ is orthogonal to $U_j(x)$ it is sufficient to constrain the Green's function $G(x,y)$ to be orthogonal to $U_j(x)$:

$$\int_{-\infty}^{\infty} G(x,y) U_j(y) dy = 0. \quad (44)$$

This constraint will shortly be used along with the other four properties of $G(x,y)$ to construct the generalized Green's function $G(x,y)$.

Before we can write the form of $G(x,y)$ we must know the general solution of the differential equation given below:

$$\frac{d^2}{dy^2} F - \beta^2 y^2 F + \epsilon_j F = C_j e^{-\frac{\beta y^2}{2}} H_j(y\sqrt{\beta}) = U_j(y). \quad (45)$$

The solution of Eq. (45) requires the knowledge of two independent solutions of the corresponding homogeneous equation

$$\frac{d^2}{dy^2} \phi - \beta^2 y^2 \phi + \epsilon_j \phi = 0. \quad (46)$$

Clearly one solution is given by

$$U_j(y) = C_j e^{-\frac{\beta y^2}{2}} H_j(y\sqrt{\beta}) \quad (47)$$

which is proportional to the Weber function $D_j(y\sqrt{2\beta})$.

A second solution $T_j(y\sqrt{2\beta})$ of Eq. (46) can be constructed as follows:

$$T_j(y\sqrt{2\beta}) = \lim_{n \rightarrow j} \left(\frac{D_n(y\sqrt{2\beta}) - (-1)^j D_n(-y\sqrt{2\beta})}{n-j} \right). \quad (48)$$

Because both $D_n(y\sqrt{2\beta})$ and $D_n(-y\sqrt{2\beta})$ are solutions of the equation

$$\frac{d^2 \phi}{dy^2} - \beta^2 y^2 \phi + \epsilon \phi = 0, \quad n = \frac{\epsilon}{2\beta} - \frac{1}{2} \quad (49)$$

the particular combination given by

$$\frac{D_n(y\sqrt{2\beta}) - (-1)^j D_n(-y\sqrt{2\beta})}{n-j}$$

must also be a solution of Eq. (49). Therefore as we allow n to approach the integer j , the solution $T_j(y\sqrt{2\beta})$ obeys the limiting value of Eq. (49), i.e. Eq. (46).

To simplify the expression for $T_j(y\sqrt{2\beta})$ given in Eq. (48) let us expand the "n-dependence" of the two Weber functions $D_n(y\sqrt{2\beta})$ and $D_n(-y\sqrt{2\beta})$ in a Taylor series about the point $n = j$:

$$D_n(z) = D_j(z) + (n-j) \left. \frac{dD_n(z)}{dn} \right|_{n=j} + \dots \quad (50)$$

$$D_n(-z) = D_j(-z) + (n-j) \left. \frac{dD_n(-z)}{dn} \right|_{n=j} + \dots \quad (51)$$

Using the fact that for n equal to a non-negative integer $D_n(-z) = (-1)^n D_n(z)$, we easily obtain from Eqs. (50) and (51) that

$$T_j(y\sqrt{2\beta}) = \left. \frac{dD_n(y\sqrt{2\beta})}{dn} \right|_{n=j} - (-1)^j \left. \frac{dD_n(-y\sqrt{2\beta})}{dn} \right|_{n=j}. \quad (52)$$

These derivatives are assumed to be known functions.

Given the two solutions $D_j(y\sqrt{2\beta})$ and $T_j(y\sqrt{2\beta})$ of Eq. (46) we can immediately⁵ write down the general solution of Eq. (45)

$$F(y) = \alpha_1 D_j(y\sqrt{2\beta}) + \alpha_2 T_j(y\sqrt{2\beta}) + \int dw \Delta'(w) [D_j(w\sqrt{2\beta}) T_j(y\sqrt{2\beta}) - T_j(w\sqrt{2\beta}) D_j(y\sqrt{2\beta})] C_j e^{-\frac{\beta w^2}{2}} H_j(w\sqrt{\beta}) \quad (53)$$

where α_1 and α_2 are arbitrary constants and

$$\Delta(w) = D_j(w\sqrt{2\beta}) \frac{d}{dw} T_j(w\sqrt{2\beta}) - T_j(w\sqrt{2\beta}) \frac{d}{dw} D_j(w\sqrt{2\beta}) \quad (54)$$

The integral appearing in Eq. (53) is an indefinite integral. For brevity we will refer to this integral as $F_j(y)$.

Knowing the general solution of Eq. (45), we can now write the form which $G(x,y)$ must have if it is to be a solution of Eq. (35):

$$G(x,y) = A(x) D_j(y\sqrt{2\beta}) + B(x) T_j(y\sqrt{2\beta}) + V_j(x) F_j(y), \quad \text{for } y < x \quad (55)$$

and

$$G(x,y) = C(x) D_j(y\sqrt{2\beta}) + E(x) T_j(y\sqrt{2\beta}) + V_j(x) F_j(y), \quad \text{for } x < y. \quad (56)$$

The conditions that $G(x, \pm\infty)$ be zero imply that

$$B(x) = -V_j(x) \lim_{y \rightarrow -\infty} (F_j(y) / T_j(y\sqrt{2\beta})) \\ = -b V_j(x) \quad (57)$$

and

$$\begin{aligned} E(x) &= -V_j(x) \lim_{y \rightarrow \infty} (F_j(y) / T_j(y\sqrt{2\beta})) \\ &\equiv -e V_j(x). \end{aligned} \quad (58)$$

Continuity of $G(x,y)$ at the point $y = x$ for all x in $(-\infty, \infty)$ requires

$$A(x) D_j'(x\sqrt{2\beta}) - b V_j(x) T_j'(x\sqrt{2\beta}) = C(x) D_j'(x\sqrt{2\beta}) - e V_j(x) T_j'(x\sqrt{2\beta}),$$

for $-\infty < x < \infty$ (59)

The equation for the jump discontinuity of $\frac{dG}{dy}$ at $y = x$ reads as follows:

$$\begin{aligned} C(x) D_j'(x\sqrt{2\beta}) - e V_j(x) T_j'(x\sqrt{2\beta}) - A(x) D_j'(x\sqrt{2\beta}) \\ + b V_j(x) T_j'(x\sqrt{2\beta}) = -1, \quad -\infty < x < \infty. \end{aligned} \quad (60)$$

Notice that Eqs. (59) and (60) can not be solved for both unknown functions $A(x)$ and $C(x)$; the difference $A(x) - C(x)$ arises in both equations.

By using Eqs. (57) - (59) we can write the previous expression for $G(x,y)$ as:

$$\begin{aligned} G(x,y) &= C(x) D_j(y\sqrt{2\beta}) + (b-e) (V_j(x) T_j(x\sqrt{2\beta}) / D_j(x\sqrt{2\beta})) D_j(y\sqrt{2\beta}) \\ &\quad - b V_j(x) T_j(y\sqrt{2\beta}) + V_j(x) F_j(y), \quad \text{for } y < x \end{aligned} \quad (61)$$

$$G(x,y) = C(x) D_j(y\sqrt{2\beta}) - e U_j(x) T_j(y\sqrt{2\beta}) \\ + U_j(x) F_j(y), \text{ for } x < y. \quad (62)$$

The remaining unknown $C(x)$ is to be determined from Eq. (44):

$$\int_{-\infty}^x G(x,y) U_j(y) dy + \int_x^{\infty} G(x,y) U_j(y) dy = 0, \quad (44)$$

and the requirement⁴ that $G(x,y)$ be symmetric

$$G(x,y) = G(y,x). \quad (63)$$

From Eqs. (61) and (62) it is clear that the proper choice of $C(x)$

is

$$C(x) = \alpha_j (F_j(x) - b T_j(x\sqrt{2\beta}) + \delta_j D_j(x\sqrt{2\beta})),$$

where the constant α_j arises from the fact that $U_j(x)$ is proportional to $D_j(x\sqrt{2\beta})$

$$U_j(x) = \left(\frac{\beta}{\pi}\right)^{1/4} \frac{1}{j!} D_j(x\sqrt{2\beta}) \equiv \alpha_j D_j(x\sqrt{2\beta}), \quad (64)$$

and δ_j can be determined from Eq. (44).

The final expression for $G(x,y)$ is then given by

$$G(x,y) = \alpha_j [F_j(x) D_j(y\sqrt{2\beta}) + D_j(x\sqrt{2\beta}) F_j(y) + \delta_j D_j(x\sqrt{2\beta}) D_j(y\sqrt{2\beta}) \\ - e T_j(x\sqrt{2\beta}) D_j(y\sqrt{2\beta}) - b D_j(x\sqrt{2\beta}) T_j(y\sqrt{2\beta})], \text{ for } y < x \quad (65)$$

$$G(x, y) = \alpha_j \left[F_j(x) D_j(y/\sqrt{2\beta}) + D_j(x/\sqrt{2\beta}) F_j(y/\sqrt{2\beta}) + \delta_j D_j(x/\sqrt{2\beta}) D_j(y/\sqrt{2\beta}) \right. \\ \left. - b T_j(x/\sqrt{2\beta}) D_j(y/\sqrt{2\beta}) - c D_j(x/\sqrt{2\beta}) T_j(y/\sqrt{2\beta}) \right], \text{ for } x < y. \quad (66)$$

Hence the most general solution of Eq. (24) which vanishes at $x = \pm \infty$ ($\varepsilon = \varepsilon_j$) can be written as follows:

$$\psi(x) = a_1 D_j(x/\sqrt{2\beta}) + \int_{-\infty}^{\infty} G(x, y) f(y) dy. \quad (67)$$

The constant a_1 is arbitrary.

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REFERENCES AND NOTES

1. The treatment presented here is taken largely from Dettman's book Mathematical Methods in Physics and Engineering, McGraw-Hill, 1962.
This is not original research.
2. This equation arises in a quantum-mechanical treatment of the one-dimensional harmonic oscillator.
3. For a discussion of Weber functions and confluent hypergeometric functions of the third kind see Morse and Feshbach's book Methods of Theoretical Physics, McGraw-Hill, 1953.
4. See page 198 of Reference 1 for the logical development of the theory of generalized Green's functions.
5. See page 529 of Vol. I of Reference 3.