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FINAL REPORT

ON

AN INVESTIGATION OF NONLINEAR INTERACTION PHENOMENA IN THE IONOSPHERE

This report covers the period June 1, 1964 to October 1, 1969

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3. H. C. Hsieh and J. E. Rowe, "The Necessary Conditions for Amplification of an Electromagnetic Wave Interacting with a Drifting Electron Stream," Tech. Report No. 88, November 1965.

4. H. C. Hsieh and J. E. Rowe, "Nonlinear Electromagnetic Wave Propagation in a Magnetoactive Finite Temperature Plasma," Tech. Report No. 94, September 1966.

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1. J. E. Rowe, "Collision Effects in Nonlinear Beam-Plasma Interactions," Presented at the Conference on Microwave Behavior of Ferrimagnetics and Plasmas, London, September 1965.

2. J. E. Rowe and H. C. Hsieh, "A Nonlinear Analysis of the Interaction Between a Drifting Electron Beam and an Electromagnetic Wave," Presented at the Seventh Annual Meeting of the Division of Plasma Physics of the American Physical Society, San Francisco, Calif., November 1965.

3. H. C. Hsieh and J. E. Rowe, "The Necessary Conditions for Amplification of an Electromagnetic Wave Interacting with a Drifting Electron Stream," Presented at the Seventh Annual Meeting of the Division of Plasma Physics of the American Physical Society, San Francisco, Calif., November 1965.

4. H. C. Hsieh, "Effect of Transverse Electrostatic Field on the Propagation of Circularly Polarized Electromagnetic Waves in a Finite Temperature Plasma," Presented at the Eighth Annual Meeting of the Division of Plasma Physics of the American Physical Society, Boston, Mass., November 1966.

5. H. C. Hsieh and J. E. Rowe, "Nonlinear Electromagnetic Wave Propagation in a Finite Temperature Magneto-Plasma," Presented at the Eighth Annual Meeting of the Division of Plasma Physics of the American Physical Society, Boston, Mass., November 1966.

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FINAL REPORT

ON

AN INVESTIGATION OF NONLINEAR INTERACTION PHENOMENA IN THE IONOSPHERE

1. Introduction (J. E. Rowe)

The research conducted under this grant was directed toward improving our understanding of VLF emission phenomena in the ionosphere. Several different mechanisms have been investigated with particular emphasis being placed upon investigating various nonlinear interaction phenomena. The results of these various studies have been published and a reprint of each is attached.

A second type of theoretical study carried out was on the subject of anomalous diffusion in a cylindrical plasma column in the region above the critical magnetic field. A summary of the results of this investigation is given in the next section. A complete report on the study will be published as a doctoral thesis under the program.

2. Anomalous Diffusion in a Cylindrical Plasma Column Above the Critical Field

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Staff: R. Hu

The principal purpose of this study was to investigate the anomalous diffusion due to the helical instability in a cylindrical plasma column immersed in a strong external axial magnetic field. Three effects which were hitherto considered to be negligible to the overall phenomenon are examined in detail here; namely, electron inertia, finite-amplitude perturbations and the ion-cyclotron frequency.

The effect of electron inertia on anomalous diffusion is investigated by including this particular term in the electron fluid equations. However, the combination of these fluid equations with the electron inertia term and the equations of continuity for ionized gases is too complicated to obtain any analytical insight. A transformation is made using the method of averaging to overcome this difficulty. The transformed equations reveal that the electron inertia does not play an important role in the anomalous radial diffusion so long as the electron-cyclotron frequency is much larger than the collisional damping frequency between electrons and neutrals. Nonetheless, the analysis reveals that a cylindrical plasma column subjected to an axial static magnetic field is constantly vibrating about the axis of the external magnetic field. This vibration of the plasma column should be observed in the plasma column unless some other unknown processes disrupt it. In fact, the vibration should be observed in nearly all types of low-temperature plasmas immersed in a strong axial magnetic field since they obey the same set of diffusion equations.

A finite-amplitude helical perturbation with one stable nonsteady mode of oscillation is investigated and thus the assumption that the radial diffusion is ambipolar is no longer necessary. Galerkins' method is then used to transform the nonlinear differential equations into nonlinear algebraic equations. A nonlinear dispersion relation is then derived after an appropriate change of variables. The marginal stability criterion and the perturbed wavelength can then be obtained from the imaginary part of the dispersion relation while the perturbed wave frequency is obtained from the real part of the dispersion relation.

The dimensionless numerical values for the marginal stability criterion, perturbed wavelength and frequency are then calculated and plotted using an IBM-360 computer system. The curves indicate marginal stability in the sense that it is a necessary but not sufficient condition for the instability to occur. A comparison between theory and experiments can be accomplished by discarding the unnecessary condition that the rate of growth is zero. The comparison between the theory and the experimental results of Sheffield's is as follows: The calculated axial wavelengths are about one-half of the measured values while the calculated rotational frequencies are twice that of the measured ones. The velocities of the perturbed wave, which is the product of perturbed wavelength and frequency, are approximately the same in both theory and experiments. Furthermore, the increase of the perturbed wave frequency with decreasing pressure is also in agreement with experiments. These agreements are considered satisfactory because there are many uncertainties in the experimental data.

The effect of the cyclotron frequency has been investigated by taking η, which is the ratio of ion-to-electron temperature, in the expressions for the marginal stability criterion, perturbed wavelength and frequency to have four successive values of 1, 0.1, 0.01 and 0. It is found that even in the extreme case of η equal to unity, the contribution of a finite ion-cyclotron frequency to anomalous diffusion is small and can be neglected. However, there are terms which are of the same order as those due to the electron-cyclotron frequency which are missing because of the omission of the effect of the ion-cyclotron frequency. The overall effect is that while the order of magnitude remains the same, the value of the magnetic field which gives rise to anomalous diffusion changes.

Hydrodynamic Analysis of Noise in a Finite Temperature Electron Beam

H. C. Hsieh

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Hydrodynamic Analysis of Noise in a Finite-Temperature Electron Beam*

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(Received 4 March 1965)

On the basis of a small-signal, one-dimensional analysis, a set of basic macroscopic differential equations, governing the fluctuations in quantities such as the electron-beam temperature, the mean velocity, and the current density, has been derived by taking moments of the Liouville equation with respect to the velocity variable. This set of differential equations expresses the conservations of charge, momentum, and energy, and is valid for an arbitrary amount of velocity spreading and includes the effect of heat conduction.

A system of differential equations, governing the correlation among the fluctuations in the mean velocity, current density, and beam temperature, is also derived. The relationship among the various noise parameters along the electron beam is obtained in the form of a system of differential equations whose solution gives detailed information on the variation of the noisiness parameter along the beam. The solution of the system of differential equations thus derived is also discussed.

I. INTRODUCTION

THE analysis of a multivelocity electron beam by the density-function method has been discussed by Siegman,[1] and using this method of analysis the noise propagation in a one-dimensional space-charge-limited diode has been investigated numerically by Siegman, Watkins, and Hsieh.[2] The result of their numerical analysis shows that the noise parameters defined by Haus[3] do not remain invariant as the beam passes through a multivelocity region, which suggests that both the self-power and cross-power density spectra of shot noise fluctuations can undergo considerable modification in propagation through the potential minimum region. In particular, the quantity ($S-\Pi$), which determines the theoretical minimum noise figure of a beam-type amplifier, decreases considerably below its value at the cathode.

Although the microscopic density-function method of analysis of Siegman *et al.*[2] is rigorous, it is also intricate, and depends upon solving a complicated partial differential equation for representative solutions. On the other hand, there exists a simpler macroscopic "hydrodynamical" model of an electron beam introduced by Hahn,[4] which may also describe at least the first-order effects of velocity spread. This model has been used by Parzen[5,6] and Goldstein[5] in a discussion of traveling-wave-tube gain, and later by Berghammer and Bloom[7] in their discussion of the nonconservation of the noise parameters in a multivelocity electron beam with sufficiently small but nonzero velocity spread. These latter authors have demonstrated the possibility of obtaining an equivalent transmission-line equation for a beam with a small velocity spread and have also discussed the case of a drifting beam in some detail.

In this paper an attempt is made to develop a method of analysis of noise in a multivelocity electron beam based on a hydrodynamic model, which adequately takes into account the effect of heat conduction as well as temperature fluctuations along the beam.

Based on a small-signal, one-dimensional analysis, a set of basic differential equations governing the fluctuations in the mean electron beam velocity, the current density, and the electron beam temperature is derived by taking the moments of Liouville's equation (collision-free Boltzmann equation) with respect to the velocity variable. The relationships among the various noise parameters along the electron beam are derived in the form of a system of ordinary differential equations whose solution yields the desired information on the variation of noise parameters. The solution of the system of differential equations, thus derived, is discussed briefly.

II. DERIVATION OF THE BASIC DIFFERENTIAL EQUATIONS GOVERNING A MULTI-VELOCITY ELECTRON BEAM

The Boltzmann equation for a one-dimensional, nonrelativistic, collision-free electron beam is written as

$$\frac{\partial F(x,u,t)}{\partial t}+u\frac{\partial F(x,u,t)}{\partial x}+\eta E(x,t)\frac{\partial F(x,u,t)}{\partial u}=0, \quad (1)$$

where $E(x,t)$ is the longitudinal electric field intensity and η is the charge-to-mass ratio, with m being the electronic mass and e the electronic charge which is taken as a negative value. The distribution function $F(x,u,t)dxdu$ denotes the charge density in the interval dx at the instant t due to electrons with velocities between u and $u+du$. Taking the zero-, first-, and second-order moments of Eq. (1) with respect to the velocity variable u, then integrating by parts, with the assumption that $F(x, \pm\infty, t)=0$, and in view of the fact that u and x are independent variables, yields the following three macroscopic equations. These express the idea of conservation of charge, conservation of mo-

* This work was supported by the Rome Air Development Center under Contract No. AF30(602)-3569.
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mentum, and the conservation of energy, respectively;

$$\partial\rho/\partial t+\partial J/\partial x=0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho\langle u^2\rangle)-\eta\rho E=0, \tag{3}$$

and

$$\frac{\partial}{\partial t}(\rho\langle u^2\rangle)+\frac{\partial}{\partial x}(\rho\langle u^3\rangle)-2\eta\rho vE=0, \tag{4}$$

where the macroscopic charge density, mean velocity, and convection current density of the electron beam are defined, respectively, as

$$\rho(x,t)=\int_{-\infty}^{\infty}Fdu,$$

$$v(x,t)=\frac{1}{\rho}\int_{-\infty}^{\infty}uFdu,$$

and

$$J(x,t)=\int_{-\infty}^{\infty}uFdu=\rho v; \tag{5}$$

and the mean values of u^n are defined by

$$\langle u^n\rangle=\frac{1}{\rho}\int_{-\infty}^{\infty}u^nFdu. \tag{6}$$

It is to be noted that

$$\langle u\rangle=v, \tag{7a}$$

$$\langle u^2\rangle=v^2+\langle(u-v)^2\rangle, \tag{7b}$$

and

$$\langle u^3\rangle=v^3+3v\langle(u-v)^2\rangle+\langle(u-v)^3\rangle. \tag{7c}$$

In view of the fact that the electron beam temperature $T(x,t)$ is related to the mean-square deviation of the velocity by

$$kT(x,t)/m=\langle(u-v)^2\rangle, \tag{8}$$

where k is the Boltzmann constant, Eqs. (2)–(4) can be written as follows:

$$\partial\rho/\partial t+\partial J/\partial x=0, \tag{9}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(Jv)-\eta\rho E=-\frac{\partial}{\partial x}\left(\rho\frac{kT}{m}\right), \tag{10}$$

and

$$\frac{\partial}{\partial t}(\rho v^2)+\frac{\partial}{\partial x}(Jv^2)-2\eta JE=-\frac{\partial}{\partial t}\left(\rho\frac{kT}{m}\right)$$
$$-\frac{\partial}{\partial x}\left[3J\left(\frac{kT}{m}\right)+\rho\langle(u-v)^3\rangle\right]. \tag{11}$$

It is noticed that the right-hand sides of Eqs. (10) and (11) indicate the effect of the presence of beam velocity spreading and they vanish as the velocity spread approaches zero, leading to the familiar form of the equation of motion and the kinetic power theorem of the single-velocity theory. Furthermore, the last term on the right-hand side of Eq. (11), when it is divided by a factor (-2η), represents the divergence of the energy flow density. The first member of this term represents the internal energy carried by the average velocity which is often referred to as convection and the second member corresponds to the energy carried by (heat) conduction.

For convenience, let us define the thermal current density Q (i.e., the rate of transfer of kinetic energy associated with the random motion per unit area per second) as follows:

$$Q(x,t)=\frac{1}{-2\eta}\rho\langle(u-v)^3\rangle=\frac{1}{-2\eta}\int_{-\infty}^{\infty}(u-v)^3Fdu. \tag{12}$$

Then, by multiplying Eq. (10) by $(2v)$ and subtracting from Eq. (11), with the aid of Eq. (9), Eq. (11) can be written in the following manner:

$$2v\frac{\partial}{\partial x}\left(\rho\frac{kT}{m}\right)=\frac{\partial}{\partial t}\left(\rho\frac{kT}{m}\right)+3\frac{\partial}{\partial x}\left(J\frac{kT}{m}\right)-2\eta\frac{\partial Q}{\partial x}. \tag{13}$$

Assume that all quantities of interest have the following form:

$$G(x,t)=G_0(x)+G_1(x)e^{j\omega t}, \tag{14}$$

with ω being the angular radian frequency. Equations (5), (9), (10), and (13) yield the following set of dc equations:

$$J_0=\rho_0v_0, \tag{15}$$

$$dJ_0/dx=0, \tag{16}$$

$$\frac{d}{dx}(J_0v_0)-\eta\rho_0E_0=-\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right), \tag{17}$$

and

$$2v_0\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)=3\frac{d}{dx}\left(J_0\frac{kT_0}{m}\right)-2\eta\frac{dQ_0}{dx}, \tag{18}$$

and the following set of ac equations (under the small-signal assumption):

$$J_1=\rho_0v_1+v_0\rho_1, \tag{19}$$

$$j\omega\rho_1+dJ_1/dx=0, \tag{20}$$

$$j\omega J_1+\frac{d}{dx}(J_0v_1+J_1v_0)-\eta(\rho_0E_1+\rho_1E_0)$$
$$=-\frac{d}{dx}\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right), \tag{21}$$

and

$$2v_1\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)+2v_0\frac{d}{dx}\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right)$$

$$=j\omega\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right)+3\frac{d}{dx}\left(J_0\frac{kT_1}{m}+J_1\frac{kT_0}{m}\right)$$

$$-2\eta\frac{dQ_1}{dx}. \quad (22)$$

The set of dc equations can be solved with the aid of the electrostatic scalar potential function, which satisfies Poisson's equation, and the dc density function. The ac quantities, v_1, ρ_1, J_1, T_1, Q_1, and E_1 are of interest to us in the study of noise in the electron beam.

For a one-dimensional beam (or in an open-circuited diode) the total alternating current density may be considered to be zero, so that the alternating convection current density J_1 and the ac electric field E_1 are related by

$$E_1=-J_1/j\omega\epsilon_0, \quad (23)$$

where ϵ_0 is the dielectric constant *in vacuo*.

Let us now assume that the alternating thermal current density Q_1 is invariant along the beam, i.e.,

$$dQ_1/dx=0 \quad (24)$$

so that it is only necessary to use three ac quantities to characterize the ac behavior of the beam, in view of relations (19), (23), and (24). In the present paper it has been decided to work with the quantities v_1, J_1, and T_1 and for convenience consider the ratio of the ac to dc quantities, namely, (J_1/J_0), (v_1/v_0), and (T_1/T_0).

After some algebraic manipulation, the following set of differential equations is obtained (see Appendix A for the details):

$$\frac{d\tilde{X}_l(x)}{dx}=\sum_{m=1}^{3}\tilde{a}_{lm}(x)\tilde{X}_l(x) \quad l=1, 2, 3, \quad (25)$$

in which the symbol $\sim$ denotes a complex quantity and thus $\tilde{X}_{lm}(x)$ and $\tilde{a}_{lm}(x)$ are complex quantities although the independent variable x is real. In the system of Eq. (25) the dependent variables $\tilde{X}_l(x)$ are defined as

$$\tilde{X}_1(x)=\frac{\tilde{J}_1(x)}{J_0(x)},\quad \tilde{X}_2(x)=\frac{\tilde{v}_1(x)}{v_0(x)},\quad \text{and } \tilde{X}_3(x)=\frac{\tilde{T}_1(x)}{T_0(x)} \quad (26a)$$

and the coefficients $\tilde{a}_{lm}(x)$ are given by

$$\tilde{a}_{lm}(x)=b_{lm}(x)+jC_{lm}(x),\quad l=1, 2, 3 \quad m=1, 2, 3, \quad (26b)$$

with

$$\tilde{a}_{11}=-j\beta_e,\quad \tilde{a}_{12}=j\beta_e,\quad \tilde{a}_{13}=0,$$

$$\tilde{a}_{21}=-\frac{\delta h}{\Delta}+j\frac{\beta_e}{\Delta}\left(h+\frac{\omega_p^2}{\omega^2}\right),$$

$$\tilde{a}_{22}=-\frac{2}{\Delta}\frac{d}{dx}\ln v_0-j\frac{\beta_e}{\Delta}(1+h),$$

$$\tilde{a}_{23}=\frac{3h}{\Delta}\frac{d}{dx}\ln v_0+j\frac{\beta_e}{\Delta}h,$$

$$\tilde{a}_{31}=\frac{\delta}{\Delta}(1-h)-j\frac{2\beta_e}{\Delta}\left(h+\frac{\omega_p^2}{\omega^2}\right),$$

$$\tilde{a}_{32}=\frac{4}{\Delta}\frac{d}{dx}\ln v_0+j\frac{2\beta_e}{\Delta}(1+h)$$

and

$$\tilde{a}_{33}=\frac{1}{\Delta}\left[\delta(1-3h)-6h\frac{d}{dx}\ln v_0\right]-j\frac{\beta_e}{\Delta}(1-h),$$

with

$$\Delta(x)=1-3h(x). \quad (26c)$$

The wavenumbers $\beta_e(x)$, plasma angular frequency $\omega_p(x)$, velocity spreading parameter $h(x)$, and heat conduction parameter $\delta(x)$ are defined as follows:

$$\beta_e(x)=\omega/v_0(x),\quad \omega_p^2(x)=\eta\rho_0(x)/\epsilon_0,$$
$$h(x)=kT_0(x)/mv_0^2(x),$$

and

$$\delta(x)=\frac{-2\eta}{J_0}\left(\frac{m}{kT_0}\right)\frac{dQ_0}{dx}. \quad (26d)$$

Now let the function $\tilde{\Phi}_{ln}(x)$ be defined as follows:

$$\tilde{\Phi}_{ln}(x)=\tilde{X}_l(x)\tilde{X}_n^*(x) \quad l=1, 2, 3 \quad n=1, 2, 3, \quad (27)$$

where the symbol * denotes the complex conjugate. It is to be noted that, in a language of the generalized harmonic analysis,[8] $\tilde{\Phi}_{ln}(x)$ represents the spectra of the correlation; for example, if $l=n$ it represents the spectrum of the autocorrelation of a random function, e.g., the current-, velocity-, or beam-temperature fluctuation in our case, and if $l\neq n$ it represents the spectrum of the cross-correlation of the random functions. These spectra and their respective correlation functions are related by a Fourier transform pair.

Since $\tilde{\Phi}_{ln}(x)$ is a complex quantity it can always be expressed in the following form:

$$\tilde{\Phi}_{ln}(x)=\Pi_{ln}(x)+j\Lambda_{ln}(x), \quad (28)$$

where Π_{ln} and Λ_{ln} are real quantities. Then the functions $\tilde{\Phi}_{ln}(x)$, $\Pi_{ln}(x)$, and $\Lambda_{ln}(x)$ can be shown to have

[8] Y. W. Lee, *Statistical Theory of Communication* (John Wiley & Sons, Inc., New York, 1960), Chap. 2.

the following properties:

$$\tilde{\Phi}_{ln}=(\tilde{\Phi}_{nl})^*,\quad \Pi_{ln}=\Pi_{nl},\quad \Lambda_{ln}=-\Lambda_{nl}\quad \text{for } l=1,2,3,\quad n=1,2,3,$$

$$\Phi_{ll}=\Pi_{ll} \text{ and } \Lambda_{ll}=0 \text{ for } l=1,2,3. \tag{29}$$

Upon differentiating Eq. (27) with respect to the real variable x and using Eqs. (25) and (29) one obtains

$$\frac{d\tilde{\Phi}_{ln}(x)}{dx}=\sum_{m=1}^{3}\left[\tilde{a}_{lm}(x)\tilde{\Phi}_{mn}(x)+\tilde{a}_{nm}^*(x)\tilde{\Phi}_{ml}^*(x)\right]\quad l=1,2,3 \text{ and } n=1,2,3. \tag{30}$$

The following system of first-order ordinary real differential equations is then obtained with the aid of Eq. (28):

$$\frac{d\Pi_{ln}}{dx}=\sum_{m=1}^{3}\left[(b_{lm}\Pi_{mn}+b_{nm}\Pi_{ml})-(C_{lm}\Lambda_{mn}+C_{nm}\Lambda_{ml})\right]\quad l=1,2,3,\quad n=1,2,3 \tag{30a}$$

and

$$\frac{d\Lambda_{ln}}{dx}=\sum_{m=1}^{3}\left[(C_{lm}\Pi_{mn}-C_{nm}\Pi_{ml})+(b_{lm}\Lambda_{mn}-b_{nm}\Lambda_{ml})\right]\quad l\neq n. \tag{30b}$$

It is observed that there are nine correlation functions which need to be considered, namely auto- and cross-correlation of the current fluctuations, velocity fluctuations and temperature fluctuations. Although there are 18 parameters Π_{ln} and Λ_{ln}, for $l=1,2,3$ and $n=1,2,3$ involved, since Eq. (29) represents nine conditions of constraint, it is necessary only to use nine parameters to specify the correlations. Consequently, the conditions are to be imposed on Eq. (30) in such a way that Eq. (30a) gives six equations and Eq. (30b) gives three equations.

It is interesting to note that in the case of a single-velocity beam there are only four parameters needed to specify the correlation; however, nine are needed here.

The conventionally defined noise parameters, Ψ, Φ, Π, Λ, and S, introduced by Haus[3] are related to the Π_{ln} and Λ_{ln} as follows (on the basis of per unit bandwidth and per unit beam cross-sectional area):

$$\begin{aligned}\Psi&=(4\pi)^{-1}J_0^2\Pi_{11},\\ \Phi&=(4\pi)^{-1}(v_0^4/\eta^2)\Pi_{22},\\ \Pi&=(4\pi)^{-1}(v_0^2J_0/\eta)\Pi_{21},\\ \Lambda&=(4\pi)^{-1}(v_0^2J_0/\eta)\Lambda_{21},\\ S&=(4\pi)^{-1}(v_0^2J_0/\eta)S_{21},\end{aligned} \tag{31a}$$

where

$$S_{21}=\left[\Pi_{22}\Pi_{11}-\Lambda_{21}^2\right]^{\frac{1}{2}} \tag{31b}$$

and the noisiness parameter $N(x)$ can be expressed as

$$N(x)=\frac{2\pi}{kT_0}(S-\Pi)=\frac{V_0I_0}{kT_0}n(x)=\left(\frac{J_0}{2e}\right)\frac{1}{h(x)}n(x). \tag{32a}$$

The dc kinetic voltage V_0, the direct beam current I_0, and the dimensionless parameter $n(x)$ are defined by

$$V_0=-v_0^2/2\eta,\quad I_0=-J_0$$

and

$$n(x)=S_{21}(x)-\Pi_{21}(x). \tag{32b}$$

The theoretical minimum noise figure for a beam-type amplifier may be written as

$$F_{\min}=1+(V_0I_0/kT_0)n(x), \tag{33}$$

where $T_0(x)$ is the dc electron beam temperature.

In order to know how $N(x)$ varies along the beam, it is necessary to find out the variations of Π_{11}, Π_{22}, Π_{21}, and Λ_{21} with distance by solving the system of differential equations given by Eqs. (30a) and (30b), with the coefficients $\tilde{a}_{lm}$ being given by Eq. (26c).

It is also of interest to note that, upon differentiating Eq. (31a), and with the aid of Eq. (30a) and (30b), and using the fact that $\Lambda_{ll}=0$ for $l=1, 2, 3$, $b_{1m}=0$ for $m=1, 2, 3$, and $C_{13}=0$, the following relationship is obtained governing the spatial rate of change of the conventionally defined noise parameters:

$$d\Psi/dx=-(J_0^2/2\pi)C_{12}\Lambda_{21},$$

$$\frac{d\Phi}{dx}=\left(-\frac{4}{v_0}\frac{dv_0}{dx}\right)\Phi+\frac{v_0^4}{2\pi\eta^2}\left[b_{21}\Pi_{21}+b_{22}\Pi_{22}+C_{21}\Lambda_{21}+b_{23}\Pi_{32}-C_{23}\Lambda_{32}\right],$$

$$\frac{d\Pi}{dx}=\left(-\frac{2}{v_0}\frac{dv_0}{dx}\right)\Pi+\frac{v_0^2J_0}{4\pi\eta}\left[b_{21}\Pi_{11}+b_{22}\Pi_{21}+(C_{11}-C_{22})\Lambda_{21}+b_{23}\Pi_{31}-C_{23}\Lambda_{31}\right],$$

$$\frac{d\Lambda}{dx}=\left(-\frac{2}{v_0}\frac{dv_0}{dx}\right)\Lambda+\frac{v_0^2J_0}{4\pi\eta}\left[C_{21}\Pi_{11}-C_{12}\Pi_{22}+(C_{22}-C_{11})\Pi_{21}+b_{22}\Lambda_{21}+C_{23}\Pi_{31}+b_{23}\Lambda_{31}\right]. \tag{34}$$

It is observed that the rate of change of Ψ, Φ, Π, and Λ does depend upon the functions $\tilde{\Phi}_{31}$ and $\tilde{\Phi}_{32}$, which represent the spectrum of the cross-correlation of the beam temperature fluctuations and the current fluctuations, and that of the temperature fluctuations and velocity fluctuations, respectively.

III. DISCUSSION OF THE SOLUTION OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS DERIVED

The systems of ordinary linear differential equations (25) and (30) can be solved if the coefficients $\tilde{a}_{lm}(x)$ are known, and the fluctuation in the quantities such as the current, the velocity, and the beam temperature, and their correlation along the electron beam can be determined when the input-plane boundary conditions are specified. Since $v_0(x)$, $\rho_0(x)$, and $T_0(x)$ are obtainable from $F_0(x,u)$, the coefficients $\tilde{a}_{lm}(x)$ can be determined once the dc density function $F_0(x,u)$ is known.

For an electron beam in a drift region, where $dv_0/dx=0$, the coefficients $\bar{a}_{lm}$ become independent of the position variable x, and δ also becomes zero. Consequently, the solution to the systems [Eqs. (25) and (30)] is obtained by a standard Laplace-transform technique, which is rather straightforward and simple. On the other hand, for a beam in an accelerating region, once the coefficients $\bar{a}_{lm}(x)$ as functions of the position x are known, a numerical method, such as the Runge–Kutta method,[9] can be employed for solving the systems Eqs. (25) and (30).

Case I. Drifting Beam

For a drifting beam, the system of Eq. (25) has a traveling-wave solution, which can be easily shown as follows: After taking the Laplace transformation of the system (25) with respect to the spatial variable x, the following set of algebraic equations is obtained:

$$\sum_{m=1}^{3} D_{lm}(p)y_m(p)=X_l(0) \quad l=1,\,2,\,3, \tag{35}$$

where

$$y_l(p)=\int_0^\infty X_l(x)e^{-px}dx \tag{36a}$$

and

$$D_{lm}(p)=(\delta_{lm}p-a_{lm})\,, \tag{36b}$$

in which δ_{lm} is the Kroneker delta, equal to one for $l=m$, and to zero for $l\neq m$. The term $X_l(0)$ appearing in Eq. (35) denotes the values of $X_l(x)$ at $x=0$, the input plane to the drift region.

From Cramer's rule the solution of the set of Eq. (35) can be expressed as follows:

$$y_m(p)=\sum_{l=1}^{3}\frac{N_{lm}(p)}{D(p)}X_l(0) \quad m=1,\,2,\,3, \tag{37}$$

where $D(p)$ is the determinant of the set of the transformed Eq. (35) with an order of 3, and $N_{lm}(p)$ is the cofactor of the element D_{lm} in the determinant $D(p)$, which is formed from $D(p)$ by striking out the row and column containing the element D_{lm} and prefixing the sign factor $(-1)^{l+m}$.

After taking the inverse transformation of the system (37), it is found that

$$X_m(x)=\sum_{l=1}^{3}X_l(0)\sum_{k=1}^{3}\frac{N_{lm}(p_k)}{D'(p_k)}e^{p_kx} \qquad 0\leq x \qquad m=1,\,2,\,3 \tag{38a}$$

where

$$D'(p_k)=dD/dp\,|_{p=p_k} \tag{38b}$$

provided that the rational fraction $[N_{lm}(p)/D(p)]$ has only a first-order pole, where p_k is the root of the characteristic equation $D(p)=0$.

[9] J. B. Scarborough, *Numerical Mathematical Analysis* (The John Hopkins Press, Baltimore, Maryland, 1962), 5th ed., p. 301.

In view of the fact that in a drift region, $dv_0/dx=0$, and from Eqs. (17), (18), and (26d), δ must be zero, so that the coefficients $\bar{a}_{lm}$ become purely imaginary quantities, and it can be easily shown that the characteristic equation has the following form:

$$D(p)=p^3+ap^2+bp+c=0\,, \tag{39}$$

in which

$$a=j\beta_eA,\quad b=(j\beta_e)^2B,\quad \text{and}\quad c=(j\beta_e)^3C, \tag{40a}$$

with

$$\begin{aligned}A&=1+2/\Delta,\\ B&=\Delta^{-1}(3-\omega_p{}^2/\omega^2),\\ C&=\Delta^{-1}(1-\omega_p{}^2/\omega^2).\end{aligned} \tag{40b}$$

Now by letting

$$p=j\beta_e\gamma, \tag{41}$$

Eq. (39) becomes

$$\gamma^3+A\gamma^2+B\gamma+C=0\,, \tag{42}$$

which can be arranged in the following form, when A, B, and C are given by Eq. (40b),

$$(\gamma+1)[\Delta\gamma^2+2\gamma+(1-\omega_p{}^2/\omega^2)]=0. \tag{43}$$

Note that Eq. (43) has three distinct roots and consequently Eq. (39) has the following roots:

$$\begin{aligned}p_1&=-j\beta_e\\ p_2&=-j(\beta_e/\Delta)[1-\{1-\Delta(1-\omega_p{}^2/\omega^2)\}^{\frac{1}{2}}]\\ p_3&=-j(\beta_e/\Delta)[1+\{1-\Delta(1-\omega_p{}^2/\omega^2)\}^{\frac{1}{2}}].\end{aligned} \tag{44a}$$

Furthermore note that as $h\rightarrow 0$, $\Delta\rightarrow 1$ and

$$\begin{aligned}p_2&\rightarrow -j\beta_e(1-\omega_p/\omega)=-j(\beta_e-\beta_p),\\ p_3&\rightarrow -j\beta_e(1+\omega_p/\omega)=-j(\beta_e+\beta_p),\end{aligned} \tag{44b}$$

which are the familiar expressions for the single-velocity theory, where $\beta_p=\omega_p/v_0$ is the plasma wavenumber. In view of the fact that the time harmonic variation has been assumed in the present discussion with the aid of Eq. (44a), Eq. (38a) represents the superposition of three propagating waves, all in the positive x direction, but with different phase velocities. There is one kinematic wave with phase velocity equal to the dc beam velocity, and the other two corresponding to the fast- and the slow-space-charge waves. Thus it can be concluded that a drifting beam, with an arbitrary amount of velocity spread, can support one kinematic and two space-charge waves.

Case II. Space-Charge-Limited Diode

It is obvious, from Eqs. 5 and 14, that the density function $F(x,u,t)$ is of the form

$$F(x,u,t)=F_0(x,u)+F_1(x,u)\cdot e^{j\omega t}.$$

It is well known that the dc density function $F_0(x,u)$ which satisfies the dc part of Eq. (1) and at the same time meets the proper boundary condition at the cathode has the following form:

$$F_0(x,u)=2\alpha J_0 S(u-w)e^{-\alpha(u^2-w^2)}, \tag{45a}$$

where

$$\alpha=m/2kT_c$$

and

$$J_0=J_0(x_m)=J_s\exp[-e\varphi_0(x_m)/kT_c], \tag{45b}$$

in which J_s is the saturation or total value of emission current density, and T_c is the cathode temperature (in degrees Kelvin) and $\varphi_0(x_m)$ is the dc potential at the potential minimum, $x=x_m$.

The function $S(u-w)$ is the usual unit step function:

$$\begin{aligned} S(u-w)&=0 \quad \text{for} \quad (u-w)<0\\ &=1 \quad \text{for} \quad (u-w)\geq 0. \end{aligned} \tag{45c}$$

The function $w(x)$ is defined and related to the dc electric potential function $\varphi_0(x)$ as follows:

$$w(x)=\mp w_0(x) \tag{45d}$$

with

$$w_0(x)=[-2\eta\{\varphi_0(x)-\varphi_0(x_m)\}]^{\frac{1}{2}}. \tag{45e}$$

In Eq. (45d) the upper sign is to be used for the α region, which is between the cathode and potential minimum, and the lower sign is for the β region, which is between the potential minimum and the anode.

Having assumed the form of the dc density function, the quantities ρ_0, v_0, (kT_0/m), and Q_0 can be obtained and expressed as follows:

$$\rho_0(x)=\int_{-\infty}^{\infty}F_0du=(\pi\alpha)^{\frac{1}{2}}J_0e^{\alpha w^2}[1-\mathrm{erf}(\alpha^{\frac{1}{2}}w)], \tag{46a}$$

$$v_0(x)=\frac{1}{\rho_0}\int_{-\infty}^{\infty}uF_0du=\frac{1}{(\pi\alpha)^{\frac{1}{2}}}\left[\frac{e^{-\alpha w^2}}{1-\mathrm{erf}(\alpha^{\frac{1}{2}}w)}\right], \tag{46b}$$

$$\frac{kT_0(x)}{m}=\frac{1}{\rho_0}\int_{-\infty}^{\infty}(u-v_0)^2F_0du = (kT_c/m)-v_0^2+v_0w, \tag{46c}$$

and

$$Q_0(x)=\frac{1}{-2\eta}\int_{-\infty}^{\infty}(u-v_0)^3F_0du = (J_0/-2\eta)[(1/\alpha)+w^2-v_0^2-3kT_0/m], \tag{46d}$$

where $w(x)$ is given by Eq. (45d) and the error function $\mathrm{erf}(Y)$ is defined as

$$\mathrm{erf}(Y)=\frac{2}{\sqrt{\pi}}\int_0^Y e^{-y^2}dy. \tag{46e}$$

It is to be noted that in the above equations, ρ_0, v_0, (kT_0/m), and Q_0 are expressed essentially in terms of the dc potential function $\varphi_0(x)$ through Eqs. (45d) and (45e) and these quantities are continuous at the potential minimum $x=x_m$, where $w=0$. On the other hand, $\varphi_0(x)$ must satisfy Poisson's equation:

$$d^2\varphi_0/dx^2=-(\pi\alpha)^{\frac{1}{2}}(J_0/\epsilon_0)e^{\alpha w^2}[1-\mathrm{erf}(\alpha^{\frac{1}{2}}w)] \tag{47}$$

which has been solved numerically by Langmuir.[10]

In view of the fact that the electrostatic field intensity $E_0(x)$ is derivable from the dc potential function $\varphi_0(x)$ by

$$E_0(x)=-d\varphi_0/dx \tag{48}$$

which is consistent with the requirement that $\varphi_0(x)$ must satisfy Poisson's equation, and upon substitution of Eq. (48) into Eq. (17), it is found that

$$\eta\frac{d\varphi_0}{dx}+v_0\frac{dv_0}{dx}=\frac{-1}{\rho_0}\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right), \tag{49}$$

which is equivalent to the following equation:

$$dv_0/dw=2\alpha(v_0-w)v_0, \tag{50}$$

in which v_0 is considered to be a function of w, since ρ_0, (kT_0/m), and φ_0 are expressible in terms of w. Similarly upon substitution of Eq. (46d) into Eq. (18), Eq. (50) is again obtained.

It is important to observe that the function $v_0(w)$ given by Eq. (46b) does satisfy the differential equation (50). Thus it indicates two interesting facts:

1. The form assumed for the dc density function $F_0(x,u)$ given by Eq. (45a) is consistent with the assumption that the dc potential function must satisfy Poisson's equation.
2. The quantities ρ_0, v_0, (kT_0/m), and Q_0 do satisfy the differential equations (16), (17), and (18).

These facts, in turn, ensure that once the potential function φ_0, which satisfies Poisson's equation, is specified, the quantities ρ_0, v_0, (kT_0/m), and Q_0 are properly determined and are given by Eqs. (46a–d), respectively, in such a way that the laws of conservation of charge, momentum, and energy are satisfied.

Therefore, once the dc potential distribution in the region under consideration is specified, the functions $h(x)$ and $\delta(x)$ are determined from Eq. (26d), and so are the coefficients $\tilde{a}_{lm}(x)$ in the systems (25) and (30). Having determined the coefficients $\tilde{a}_{lm}(x)$, the systems (25) and (30) can be solved by the Runge–Kutta method with the properly imposed input-boundary conditions.

IV. CONCLUDING REMARKS

The heat conduction along the electron beam has been properly taken into account in the present paper by introducing the heat conduction parameter $\delta(x)$, which is defined in Eq. (26d). The parameter $\delta(x)$ is related to the velocity spreading parameter $h(x)$ and the dc mean velocity $v_0(x)$ by the following relation,

[10] I. Langmuir, Phys. Rev. 21, 419 (1923).

from Eq. (A7):

$$\delta(x) = -[d/dx \ln h(x) + 4d/dx \ln v_0(x)], \qquad (51)$$

which is due to the law of conservation of energy, and $\delta(x)$ does depend upon the spatial rate of variation of $h(x)$ and $v_0(x)$. On the other hand, the thermal current density $Q_0(x)$, which has a dimension of joules per sec per unit area, may be put in the following form:

$$Q_0(x) = -\lambda_0(x)[dT_0(x)/dx], \qquad (52)$$

where λ_0 is the thermal conductivity of the electron beam, which in general depends upon the collision force, ρ_0 and T_0, and is governed by Eqs. (18), (26d), and (52). It is to be observed that for an adiabatic flow λ_0 can be set equal to zero so that δ will be zero also. However, for an isothermal flow, λ_0 becomes very large and δ need not be zero. It is interesting to note that for an adiabatic flow, since $\delta(x)$ can be set equal to zero, Eq. (51) implies that the quantity (hv_0^4) or $(kT_0/m)v_0^2$ is invariant along the beam, which suggests that the quantity (T_0/ρ_0^2) is also invariant.[7] On the other hand, in a drift region, since $Q_0(x)$ is independent of x, $\delta(x)$ will be zero from Eq. (26d). Thus it suggests that the thermal effect (heat conduction effect) in a drifting beam can be neglected.

It should be pointed out that no specific assumption has been made with regard to the input-plane boundary conditions in deriving the systems of equations (25) and (30). However, for a special case in which $\delta = 0$, for instance, in an adiabatic flow, and when it is further assumed that the following relation holds at the input plane, for example, at the cathode surface,

$$T_1/T_0 = 2\rho_1/\rho_0 \quad \text{at} \quad x = 0, \qquad (53)$$

the system of equations is reduced to that obtained by Berghammer and Bloom,[7] which is demonstrated in Appendix B.

While the density-function method involves solving a rather complicated partial differential equation, which must also deal with the Dirac delta function, the present method of analysis of signal and noise propagation along the electron beam involves solving a system of linear ordinary first-order differential equations, whose solution is obtainable by relatively simple and straightforward methods.

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APPENDIX A. DERIVATION OF THE SYSTEM OF EQUATION (25)

First note the following identity:

$$\frac{dC}{dx} - \frac{dD}{dx} = D\frac{d}{dx}\left(\frac{C}{D}\right) + \left(\frac{C}{D} - 1\right)\frac{dD}{dx}. \qquad (A1)$$

From Eqs. (15) and (19)

$$J_1/J_0 = v_1/v_0 + \rho_1/\rho_0. \qquad (A2)$$

Subtraction of Eq. (16) from Eq. (20) with the aid of Eqs. (A1) and (A2) gives

$$j\beta_e\left(\frac{J_1}{J_0}\right) - j\beta_e\left(\frac{v_1}{v_0}\right) + \frac{d}{dx}\left(\frac{J_1}{J_0}\right) = 0, \qquad (A3)$$

where

$$\beta_e = \omega/v_0.$$

Subtraction of Eq. (17) from Eq. (21), with the aid of Eqs. (A1) and (23) gives, after using Eqs. (16), (17), and (A2):

$$j\beta_e\left(1 - \frac{\omega_p^2}{\omega^2}\right)\left(\frac{J_1}{J_0}\right) + \frac{d}{dx}\left(\frac{v_1}{v_0} + \frac{J_1}{J_0}\right) + h\frac{d}{dx}\left(\frac{T_1}{T_0} + \frac{J_1}{J_0} - \frac{v_1}{v_0}\right)$$

$$+ \frac{2}{v_0}\frac{dv_0}{dx}\left(\frac{v_1}{v_0}\right) + \left[\frac{1}{J_0 v_0}\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)\right]\left(\frac{T_1}{T_0}\right) = 0, \qquad (A4)$$

where

$$\omega_p^2 = \eta\rho_0/\epsilon_0, \quad h = kT_0/mv_0^2.$$

Similarly, first subtracting Eq. (18) from Eq. (22), and with the aid of Eq. (A1), then dividing it through by the factor $J_0(kT_0/m)$ yields, after using Eqs. (18) and (A2),

$$\frac{d}{dx}\left(\frac{J_1}{J_0} + \frac{2v_1}{v_0} + \frac{T_1}{T_0}\right) + j\beta_e\left(\frac{T_1}{T_0} + \frac{J_1}{J_0} - \frac{v_1}{v_0}\right)$$

$$+ 2\eta\left(\frac{m}{J_0 kT_0}\right)\frac{dQ_0}{dx}\left(\frac{T_1}{T_0} + \frac{J_1}{J_0}\right) - 2\eta\left(\frac{m}{J_0 kT_0}\right)\frac{dQ_1}{dx} = 0. \qquad (A5)$$

Defining the heat conduction parameter $\delta(x)$ as

$$\delta(x) = \frac{-2\eta}{J_0}\left(\frac{m}{kT_0}\right)\frac{dQ_0}{dx}, \qquad (A6)$$

Eq. (18) can be written as follows, after it is divided through by a factor $J_0(kT_0/m)$:

$$\frac{d}{dx}\ln\left(\rho_0\frac{kT_0}{m}\right) = -\delta(x) - 3\frac{d}{dx}\ln v_0. \qquad (A7)$$

After making the following definitions

$$X_1(x) = J_1(x)/J_0(x), \quad X_2(x) = v_1(x)/v_0(x),$$
$$\text{and} \quad X_3(x) = T_1(x)/T_0(x), \qquad (A8)$$

Eqs. (A3), (A4), and (A5) can be arranged into the following system of equations with the assumption (24):

$$\frac{dX_1}{dx}=\sum_{m=1}^{3} A_{1m}X_m,$$

$$(1+h)\frac{dX_1}{dx}+(1-h)\frac{dX_2}{dx}+h\frac{dX_3}{dx}=\sum_{m=1}^{3} A_{2m}X_m,$$

$$\frac{dX_1}{dx}+2\frac{dX_2}{dx}+\frac{dX_3}{dx}=\sum_{m=1}^{3} A_{3m}X_m, \quad \text{(A9)}$$

where

$$A_{11}=-j\beta_e,\quad A_{21}=-j\beta_e\left(1-\frac{\omega_p^2}{\omega^2}\right),\quad A_{31}=\delta-j\beta_e,$$

$$A_{12}=j\beta_e,\qquad A_{22}=-2\frac{d}{dx}\ln v_0,\qquad A_{32}=j\beta_e,$$

$$A_{13}=0,\qquad A_{23}=\delta h+3h\frac{d}{dx}\ln v_0,\qquad A_{33}=\delta-j\beta_e. \quad \text{(A10)}$$

Upon solving algebraically for (dX_1/dx), (dX_2/dx), and (dX_3/dx), in terms of X_1, X_2, and X_3, from the system (A9), with the aid of Cramer's rule, one obtains

$$\frac{dX_1}{dx}=\sum_{m=1}^{3} a_{1m}X_m,$$

$$\frac{dX_2}{dx}=\sum_{m=1}^{3} a_{2m}X_m, \quad \text{(A11)}$$

$$\frac{dX_3}{dx}=\sum_{m=1}^{3} a_{3m}X_m,$$

where

$$a_{1m}=A_{1m},$$
$$a_{2m}=\Delta^{-1}[A_{2m}-hA_{3m}-A_{1m}],$$
$$a_{3m}=\Delta^{-1}[(1-h)A_{3m}-2A_{2m}+(1+3h)A_{1m}],$$
$$\Delta=1-3h \quad \text{for} \quad m=1, 2, 3, \quad \text{(A12)}$$

and upon substituting Eq. (A10) into Eq. (A12), Eq. (26c) is obtained.

APPENDIX B. DISCUSSION OF THE SYSTEM OF EQ. (34) FOR A SPECIAL CASE

For the case $\delta=0$, Eq. (A9) becomes

$$2dX_2/dx+dX_3/dx=-j\beta_eX_3 \quad \text{(B1)}$$

and from Eqs. (A2) and (A3), one has

$$\frac{dX_2}{dx}=-j\beta_e\left(\frac{\rho_1}{\rho_0}\right)-\frac{d}{dx}\left(\frac{\rho_1}{\rho_0}\right). \quad \text{(B2)}$$

After combining Eqs. (B1) and (B2), the following is obtained:

$$\frac{d}{dx}\left(\frac{T_1}{T_0}-2\frac{\rho_1}{\rho_0}\right)+j\beta_e\left(\frac{T_1}{T_0}-2\frac{\rho_1}{\rho_0}\right)=0, \quad \text{(B3)}$$

which has a solution of the form

$$\frac{T_1}{T_0}-\frac{2\rho_1}{\rho_0}=K\exp\left(-j\int_0^x \beta_e(y)dy\right), \quad \text{(B4)}$$

where K is a constant of integration, which is to be determined by the input-plane boundary conditions.

Suppose that $K=0$, as has been used by Berghammer and Bloom[7]; then

$$T_1/T_0=2\rho_1/\rho_0 \quad \text{(B5)}$$

or equivalently

$$X_3=2(X_1-X_2). \quad \text{(B6)}$$

Now from Eqs. (27) and (28), the following relations are evolved:

$$\Pi_{31}=2[\Pi_{11}-\Pi_{21}],\quad \Lambda_{31}=-2\Lambda_{21} \quad \text{(B7)}$$

and

$$\Pi_{32}=2[\Pi_{21}-\Pi_{22}],\quad \Lambda_{32}=-2\Lambda_{21}. \quad \text{(B8)}$$

Upon substituting the relations (B7) and (B8) into the system of equations (34), with the aid of Eq. (31a) and using the fact that $b_{21}=0$ for $\delta=0$,

$$d\Psi/dx=-2B_0\Lambda,$$
$$d\Phi/dx=-2M_0\Phi+2R_0\Pi+2X_0\Lambda,$$
$$d\Pi/dx=R_0\Psi-M_0\Pi+N_0\Lambda, \quad \text{(B9)}$$
$$d\Lambda/dx=X_0\Psi-B_0\Phi-N_0\Pi-M_0\Lambda,$$

where

$$B_0=(J_0\eta/v_0^2)C_{12},$$
$$M_0=-(2/v_0)(dv_0/dx)-b_{22}+2b_{23},$$
$$R_0=(2v_0^2/\eta J_0)b_{23}, \quad \text{(B10)}$$
$$X_0=(v_0^2/\eta J_0)(C_{21}+2C_{23}),$$
$$N_0=(C_{11}-C_{22}+2C_{23}).$$

When the coefficients b_{lm} and C_{lm} given by Eq. (26c) are substituted into Eq. (B10), our Eqs. (B9) and (B10) become Eqs. (22) and (19) of Berghammer and Bloom,[7] respectively.

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A theory of ionospheric thermal radiation

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Abstract—The ionosphere is considered as a dissipative medium in which the random thermal motions of the charged particles act as a source of thermal radiation. Attention has been focused on the electrons colliding with ions and neutral particles in the ionosphere. A method of analysis has been developed with the aid of the Maxwell and Langevin equations based on a linear, macroscopic, fluctuating electromagnetic field theory. The spectral density of the random-current source function is derived in terms of the conductivity tensor of the ionosphere.

The ionosphere is divided into a large number of incremental volume elements, each containing an ionized medium which represents an anisotropic elementary radiating system, characterized by the spectral density of the source function. The radiation characteristic of the radiating system observed at a point located outside of the source region is obtained with the aid of the potential functions which relate the thermal electromagnetic fields at the observation point to their source function. Based on the superposition principle, general expressions have been derived for w_0, the thermal noise power generated per unit volume, per unit bandwidth, from any given source region V_s of the ionosphere, and for $P_0(f, V_s)$, the available thermal noise per unit bandwidth at a receiving antenna. These expressions are valid for most regions of interest in the ionosphere where the electron collision process plays a major role in the thermal radiation and they are not limited in frequency range.

1. Introduction

It is well known that because the ionosphere acts as an absorber of radio waves, it can also act as an emitter of thermal radio noise. It has been conclusively demonstrated by various workers (Pawsey *et al.*, 1951; Gardner, 1954; Dowden, 1960; Little *et al.*, 1961) that the thermal emission from the D-region can, under favorable conditions, be observed with a dipole antenna. For example, Pawsey *et al.* (1951) have identified and measured the thermal radiation from the ionosphere in the vicinity of 2 Mc/s in the temperate latitude.

It appears that usually the thermal radiation has been neglected because its level is exceedingly low as illustrated by Pawsey *et al.* (1951) and it does not constitute an appreciable source of interference in radio communication. However, the noise radiated from a plasma (e.g. the ionosphere) is not necessarily a detrimental effect in all cases, as it is in communication, since if the spectral distribution of the emitted energy is characteristic of the plasma properties, a measurement of radiation provides specific information on the plasma. For example, knowledge of the radiated power gives a measure of the electron temperature in the plasma and this has been used as a powerful diagnostic technique.

It is well known that the thermal radiation from dissipative bodies is due to the random thermal motion of the charges in the body. If the body is at a uniform temperature, one approach that may be used for studying radiation may be called the integral approach. The body as a whole is considered to be nonradiating and the power that is absorbed from its surroundings, which is assumed to be at the temperature of the body, can be computed. This power is set equal to the power radiated by the body. In this approach no attempt is made to determine the noise current fluctuations that are the cause of the thermal radiation. In those cases in which the temperature of the body is nonuniform this approach fails.

Another approach, which may be called the 'Nyquist source treatment' (RYTOV, 1959; HAUS, 1961; VANWORMHOUDT and HAUS, 1962), focuses attention upon the sources of the radiation and determines their relevant statistical properties. Once these are known, the determination of the radiation is conceptually a simple problem, although mathematical difficulties usually arise.

In the present study, the 'Nyquist source treatment' is adopted and the ionosphere is considered as an anisotropic dissipative medium in which the random thermal motions of the charged particles act as a source of the thermal radiation. It is further postulated that in the ionosphere a linear constitutive local relation exists between the driven a.c. conduction current density $\mathbf{J}_d$ and an applied a.c. electric field intensity $\mathbf{E}$ of the form

$$\mathbf{J}_d(\omega, \mathbf{r}) = \boldsymbol{\sigma}(\omega, \mathbf{r}) \cdot \mathbf{E}(\omega, \mathbf{r}), \tag{1}$$

where $\boldsymbol{\sigma}$ is the conductivity tensor of the ionosphere, and a function of the angular frequency ω and position variable $\mathbf{r}$ which characterize the medium under consideration. A small-signal analysis is made throughout the present paper.

2. DERIVATION OF THE CONDUCTIVITY TENSOR

For a macroscopic analysis the Langevin equation can be used effectively to describe the motion of an electron, and it can be expressed as follows:

$$m\frac{\partial \mathbf{v}}{\partial t} + m\nu\mathbf{v} = e[\mathbf{E} + \mathbf{v} \times \mathbf{B}], \tag{2}$$

where $\mathbf{B}(\mathbf{r})$ is the static geomagnetic field, $\nu(r)$ is the average electronic collision frequency with ions and neutral particles, e, m and $\mathbf{v}$ are the electronic charge taken as a negative value, mass and velocity respectively.

On the other hand the convection density $\mathbf{J}$ is related to the velocity $\mathbf{v}$ by

$$\mathbf{J} = N_0 e\mathbf{v}, \tag{3}$$

where $N_0(\mathbf{r})$ is the electron number density.

Assuming the time harmonic variation $e^{j\omega t}$ for the quantities of interest, upon elimination of $\mathbf{v}$ from equations (2) and (3) the following relationship is established:

$$U\mathbf{J} + j(\mathbf{J} \times \mathbf{Y}) = -j\omega\varepsilon_0 X\mathbf{E}, \tag{4}$$

where

$$X = \frac{\omega_p^2}{\omega^2}, \qquad \omega_p^2 = \frac{N_0 e^2}{m\varepsilon_0}$$

$$\mathbf{Y} = \frac{e\mathbf{B}}{m\omega}, \qquad Y = \frac{\omega_b}{\omega} = \frac{-e\,|\mathbf{B}|}{m\omega},$$

$$Z = \frac{\nu}{\omega}, \qquad U = 1 - jZ, \tag{5}$$

in which ω_p and ω_b are the plasma and gyrofrequencies of the electrons, respectively, and ε_0 is the dielectric constant of vacuum.

On the other hand, the geomagnetic field $\mathbf{B}$ can be approximated by a dipole field which is induced by a uniformly magnetized spherical Earth, and may be expressed (MORGAN, 1959) as

$$\mathbf{B} = \frac{Ma^3}{3} \nabla \left(\frac{\cos \theta}{r^2}\right), \tag{6}$$

where the space variables r and θ denote, respectively, the radial and polar angular coordinates of the geomagnetic spherical coordinate system with its origin located at the center of the Earth, and the constants M and a are the magnetization and the radius of the Earth respectively. By adopting this model of the geomagnetic field, the r-component Y_1, the θ-component Y_2 and the φ-component Y_3 of the vector $\mathbf{Y}$ are given by

$$Y_1 = 2G \cos \theta, \qquad Y_2 = G \sin \theta \qquad \text{and} \qquad Y_3 = 0, \tag{7}$$

where

$$G = \left(\frac{-e}{\omega m}\right) \frac{M}{3} \left(\frac{a}{r}\right)^3. \tag{8}$$

Furthermore, by writing a vector as a column matrix the vector equation (4) may be conveniently expressed in the following matrix form:

$$\underline{\underline{\gamma}}\,\underline{\mathrm{J}} = \underline{\mathrm{E}} \tag{9a}$$

or equivalently in tensor notation as

$$\boldsymbol{\gamma} \cdot \mathbf{J} = \mathbf{E}, \tag{9b}$$

where the resistivity matrix $\underline{\underline{\gamma}}$ is defined as

$$\underline{\underline{\gamma}} = \{\gamma_{\alpha\beta}\}, \qquad \alpha,\beta = 1,2,3, \tag{10}$$

with its elements being given by

$$\begin{aligned} \gamma_{11} &= \gamma_{22} = \gamma_{33} = \frac{jU}{\omega\varepsilon_0 X}, \\ \gamma_{12} &= -\gamma_{21} = 0, \\ \gamma_{13} &= -\gamma_{31} = \frac{G \sin \theta}{\omega\varepsilon_0 X}, \\ \gamma_{23} &= -\gamma_{32} = \frac{-2G \cos \theta}{\omega\varepsilon_0 X}, \end{aligned} \tag{11}$$

and with its determinant $|\underline{\underline{\gamma}}|$ given by

$$|\underline{\underline{\gamma}}| = \frac{jU}{(\omega\varepsilon_0 X)^3} [Y^2 - U^2] \tag{12}$$

in which

$$Y^2 = G^2(1 + 3 \cos^2 \theta). \tag{13}$$

$|\underline{\underline{\gamma}}|$ can be zero only for a special situation where $\nu = 0$ and $\omega = \omega_b$ occur simultaneously. Since $\nu = 0$ is not of interest to the present study, $|\underline{\underline{\gamma}}|$ can be considered to possess an inverse, which is denoted by $\underline{\underline{\sigma}}$ and is referred to as the 'conductivity matrix', i.e.,

$$\underline{\underline{\sigma}}\underline{\underline{\gamma}} = \underline{\underline{I}}, \tag{14}$$

where $\underline{\underline{I}}$ is the unit matrix. Consequently, from equations 9a and 14, $\underline{J}$ can be expressed in terms of $\underline{E}$ explicitly as

$$\underline{J} = \underline{\underline{\sigma}}\underline{E} \tag{15a}$$

or in a tensor notation as

$$\mathbf{J} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{15b}$$

where

$$\sigma_{\alpha\beta} = DC_{\alpha\beta}, \qquad \alpha,\beta = 1,2,3, \tag{16}$$

where

$$D = j\omega\varepsilon_0 \frac{X}{U(U^2 - Y^2)}$$

with

$$\begin{aligned}
C_{11} &= (Z^2 - 1 + 4G^2 \cos^2 \theta) + j2Z \\
C_{22} &= (Z^2 - 1 + G^2 \sin^2 \theta) + j2Z \\
C_{33} &= (Z^2 - 1) + j2Z \\
C_{12} &= C_{21} = 2G^2 \sin \theta \cos \theta \\
C_{13} &= -C_{31} = -(Z + j)G \sin \theta \\
C_{23} &= -C_{32} = 2(Z + j)G \cos \theta.
\end{aligned} \tag{17}$$

3. Noise Power Radiated from the Ionosphere

A body with a non-uniform temperature distribution is not in the thermodynamic equilibrium. However, in those cases in which the distribution function of charge carriers deviates only slightly from the equilibrium distribution (so as to produce heat and current flow), and this includes all cases for which a temperature can be reasonably defined, it would be expected that the radiated noise power could still be computed as the superposition of the noise power radiated from the various volume elements of the body. In this case each element at a particular temperature radiates the same noise power it would radiate at equilibrium at the same temperature. Such an analysis calls for an approach to the fluctuation problem that considers each differential volume element separately as an absorber and emitter of noise power. It calls for the introduction of a source term into Maxwell's equations analogous to the source term of the Langevin equation in the theory of Brownian motion.

Although Maxwell's equations and the constitutive relation are sufficient to solve most electromagnetic problems, they are insufficient for noise studies. The current density derived from the constitutive relation represents only the current driven by the electromagnetic fields. Besides this driven current, the current density fluctuation caused by the random motion of the charge must be considered. This can be taken into account by introducing into Maxwell's equations a random driving current density distribution which is independent of the electromagnetic fields, i.e.

$$\nabla \times \mathbf{e} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t} \tag{18}$$

and

$$\nabla \times \mathbf{h} = \varepsilon_0 \frac{\partial \mathbf{e}}{\partial t} + \mathbf{i}, \tag{19}$$

where $\mathbf{e}$ and $\mathbf{h}$ are the time-dependent electric and magnetic fields, respectively, and $\mathbf{i}$ is the current density, μ_0 is the permeability of vacuo. The current density $\mathbf{i}$ in equation 19 consists of two parts. First of all there is the 'driven' component $\mathbf{i}_d$ that is produced by the electric field $\mathbf{e}$ and is related to $\mathbf{e}$ by equation 15. The spontaneous noise fluctuations of the field at thermal equilibrium can be taken into account by another current component of $\mathbf{i}$ in equation 19, the source current density $\mathbf{K}(t, \mathbf{r})$, a statistical quantity which is a stationary function of time.

3.1 *The dyadic spectral density of current source functions*

In the study of problems involving radiation of noise power, it is convenient to introduce Fourier transformations in time of all field quantities in equations 18 and 19. In the present case all random time functions are stationary and, strictly speaking, they do not possess Fourier transformations. However, this difficulty may be overcome by constructing a periodic substitute function (Rytov, 1959; Haus, 1961) according to the definition

$$\mathbf{F}(t, \mathbf{r}, T) = \mathbf{F}(t, \mathbf{r}), \qquad \text{for} \qquad -\frac{T}{2} < t < \frac{T}{2}$$

and

$$\mathbf{F}(t + nT, \mathbf{r}, T) = \mathbf{F}(t, \mathbf{r}, T). \tag{20}$$

These substitute functions have Fourier transformations of the form

$$\mathbf{F}(\omega, \mathbf{r}, T) = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{F}(t, \mathbf{r}, T)\, e^{-j\omega t}\, dt. \tag{21}$$

In the limit as $T \to \infty$, the substitute functions are indistinguishable from their originals. The spectral density of any noise process can be obtained directly from the ensemble average of products of these Fourier components. Thus, the dyadic spectral density of $\mathbf{F}$ is given by

$$\mathbf{S}_\mathrm{F}(\omega, \mathbf{r}, \mathbf{r}') = \lim_{T\to\infty} \frac{T}{2\pi} \langle \mathbf{F}(\omega, \mathbf{r}, T)\mathbf{F}^*(\omega, \mathbf{r}', T)\rangle_{\mathrm{avg}}, \tag{22}$$

where the symbol * denotes the complex conjugate.

It should be noted that the spectral analysis of the periodic substitute function leads to a discrete spectrum extending over negative, as well as positive, frequencies. With lines at frequency interval $\Delta f = (1/T)$ the expression

$$\langle 2\mathbf{F}(\omega, \mathbf{r}, T)\mathbf{F}^*(\omega, \mathbf{r}', T)\rangle_{\mathrm{avg}} = 4\pi\Delta f \mathbf{S}_\mathrm{F}(\omega, \mathbf{r}, \mathbf{r}'). \tag{23}$$

may be identified in the limit of large T as 'the mean-square fluctuation of $\mathbf{F}$ in the frequency interval $\Delta f'$. Furthermore, for a stationary time function $\mathbf{F}$ (BLANC-LAPIERSE and FORTET, 1953),

$$\frac{T}{2\pi} \langle \mathbf{F}(\omega, \mathbf{r}, T)\mathbf{F}^*(\omega', \mathbf{r}, T)\rangle_{\text{avg}} = 0, \qquad \omega \neq \omega'. \tag{24}$$

Precisely this kind of treatment must be kept in mind in applying the formal expansion of the Fourier integral and using the spectral amplitude densities in the study of electromagnetic fluctuation on which the present paper is based.

As a matter of convenience, for a particular physical variable, the lower case letter is used for the stationary time function and for its periodic substitute function, while the capital letter is used for its Fourier transform in the following discussion. For example, it is obvious, from equations (18) and (19), with the aid of equation (15b) that the Fourier amplitude of the periodic substitute functions is related in the following manner:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{25}$$

and

$$\nabla \times \mathbf{H} = j\omega\varepsilon_0\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E} + \mathbf{K}. \tag{26}$$

Suppose that a region of the ionosphere under study is divided into a large number of sufficiently small elementary volume elements such that within each one of these elementary volumes the medium may reasonably be assumed to be uniform at a certain temperature T_0. Strictly speaking these elementary volume elements should be made to approach zero. On the other hand, they have to be kept large enough to contain a large number of charge carriers in order that statistical arguments may be applied. A tensor-conductivity description of the medium as given by equation (15b) is possible only because the current in an elementary volume depends upon the electric field in the same volume, but not upon its derivatives, that is, upon the value of the electric field in the neighboring elementary volumes. In view of this fact, it is quite reasonable to expect that the source current caused by the random motion of the charge carriers in two neighboring elementary volumes are uncorrelated. In other words, if $\mathbf{r}$ and $\mathbf{r}'$ denote the points belonging to two different elementary volumes, then $\mathbf{K}(\omega, \mathbf{r})$ and $\mathbf{K}(\omega, \mathbf{r}')$ are not correlated and the dyadic spectral density of $\mathbf{K}$ has the form

$$\mathbf{S}_{\mathbf{K}}(\omega, \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\psi(\omega, r), \tag{27}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the usual Dirac delta function.

On the other hand, an elementary volume element may be considered as a linear network containing a noise source in thermal equilibrium and the technique developed in the theory of linear noise networks (HAUS, 1961; VANWORMHOUDT and HAUS, 1962), which makes use of the generalized Nyquist theorem, can be applied. Using the concept of a linear network, for example, HAUS (1961) has obtained a simple expression for $\psi(\omega, \mathbf{r})$ as follows:

$$\psi(\omega, \mathbf{r}) = \frac{kT_0(r)}{2\pi}[\boldsymbol{\sigma}(\omega, \mathbf{r}) + \boldsymbol{\sigma}^\dagger(\omega, \mathbf{r})], \tag{28}$$

where k is the Boltzmann constant and the symbol dagger ($\dagger$) indicates the complex-conjugate transpose of the conductivity matrix $\boldsymbol{\sigma}$. If the average volume density of thermal energy $\tau(r)$ in J/m^3 is introduced, defined as the ratio of the amount of thermal energy generated within an elementary volume ΔV to the volume ΔV, then from equations (27) and (28) one has

$$\mathbf{S}_K(\omega, \mathbf{r}) = \frac{\tau(r)}{2\pi}[\boldsymbol{\sigma}(\omega, \mathbf{r}) + \boldsymbol{\sigma}\dagger(\omega, \mathbf{r})] \tag{29}$$

and from equation (23)

$$\langle 2\mathbf{K}(\omega, \mathbf{r})\mathbf{K}^*(\omega, r)\rangle_{\text{avg}} = 2\Delta f\tau(r)[\boldsymbol{\sigma}(\omega, \mathbf{r}) + \boldsymbol{\sigma}\dagger(\omega, \mathbf{r})], \tag{30}$$

which may be given alternatively in its component form with the aid of equation (17) as follows:

$$\langle 2K_\alpha(\omega, r)K_\beta^*(\omega, r)\rangle_{\text{avg}} = \tau(r)\Delta f L_{\alpha\beta}(\omega, r), \tag{31}$$

where

$$L_{\alpha\beta}(\omega, r) = 4\omega\varepsilon_0\left(\frac{XZ}{1 + Z^2}\right)[l_{\alpha\beta} + jm_{\alpha\beta}], \qquad \alpha,\beta = 1, 2, 3, \tag{32}$$

with

$$l_{11} = \frac{1}{Q(Y, Z)}[(1 + Z^2)(1 + Z^2 + Y^2) + (Z^2 + Y^2 - 3)4G^2\cos^2\theta],$$

$$l_{22} = \frac{1}{Q(Y, Z)}[(1 + Z^2)(1 + Z^2 + Y^2) + (Z^2 + Y^2 - 3)G^2\sin^2\theta],$$

$$l_{33} = \frac{1}{Q(Y, Z)}[(1 + Z^2)(1 + Z^2 + Y^2)],$$

$$l_{12} = l_{21} = \frac{1}{Q(Y, Z)}[(Z^2 + Y^2 - 3)G^2\sin 2\theta],$$

$$l_{13} = l_{31} = l_{23} = l_{32} = 0,$$

$$m_{11} = m_{22} = m_{33} = m_{12} = m_{21} = 0,$$

$$m_{13} = -m_{31} = \frac{1}{Q(Y, Z)}[2(1 + Z^2)G\sin\theta],$$

$$m_{23} = -m_{32} = \frac{-1}{Q(Y, Z)}[4(1 + Z^2)G\cos\theta]$$

and

$$Q(Y, Z) = (Y^2 + Z^2 - 1)^2 + 4Z^2. \tag{33}$$

It is observed that $Y = 0$ when $G = 0$. In this case, $l_{\alpha\beta} = 1$ if $\alpha = \beta$ and $l_{\alpha\beta} = 0$ if $\alpha \neq \beta$, while $m_{\alpha\beta}$ becomes zero regardless of whether $\alpha = \beta$ or $\alpha \neq \beta$. This suggests that the tensor $\{L_{\alpha\beta}\}$ appearing in equation (31) becomes a scalar and the medium becomes isotropic. This is perfectly reasonable since when $G = 0$ the geomagnetic field is completely absent.

It is also interesting to note that for the case

$$Y^2 + Z^2 = 3 \tag{34}$$

$l_{\alpha\beta}$ again becomes either equal to unity or to zero according to whether $\alpha = \beta$ or $\alpha \neq \beta$ and

$$m_{13} = -m_{31} = \tfrac{1}{2}G\sin\theta$$

and

$$m_{23} = -m_{32} = -G \cos \theta. \tag{35}$$

3.2 *Time average thermal noise power radiated*

In view of the fact that for the periodic field the average time rate of change of stored energy is zero, the total average power radiated from a system of a current may be given by (STRATTON, 1941)

$$W = \oint_S \bar{\mathbf{p}} \cdot d\mathbf{S} = -\frac{1}{2} Re \int_V (\mathbf{E} \cdot \mathbf{I}^*) \, dV, \tag{36}$$

where $\mathbf{p}$ is the Poynting vector. Thus radiation can be calculated either by integrating the normal component of the Poynting vector over a closed surface S including all sources or by integrating the power expended per unit volume over the current distribution. In the present discussion the latter approach is taken.

Keeping in mind that the concern here is with the random current distribution and since the time average power radiated per unit volume, $w(\omega, \mathbf{r})$, is given by

$$w(\omega, \mathbf{r}) = \tfrac{1}{2} Re[\mathbf{K}^* \cdot \mathbf{E}], \qquad \mathrm{W/m^3}, \tag{37}$$

in which $\mathbf{K}$ is the cause and $\mathbf{E}$ is its effect, and with the aid of equation (9), $w(\omega, \mathbf{r})$ becomes

$$w(\omega, \mathbf{r}) = \tfrac{1}{2} Re[\mathbf{K}^* \cdot (\boldsymbol{\gamma} \cdot \mathbf{K})] = \tfrac{1}{2} Re[\underline{K}\dagger \underline{\underline{\gamma}} \underline{K}]. \tag{38}$$

The substitution of equation (11) into equation (38) yields

$$w(\omega, r) = \frac{Z}{2\omega\varepsilon_0 X} [K_1 K_1^* + K_2 K_2^* + K_3 K_3^*]. \tag{39}$$

On the other hand, with the aid of equations (31) and (32), the thermal noise power generated per unit volume, per unit bandwidth, $w_0(f, r)$, may be given as

$$w_0(f, \mathbf{r}) = kT_0 \left(\frac{Z^2}{1 + Z^2}\right) [l_{11} + l_{22} + l_{33}], \tag{40}$$

where l_{11}, l_{22} and l_{33} are given in equation (33).

It is interesting to observe that w_0 given in equation (40) does not depend explicitly upon the electron number density N_0 since it does not contain the parameter X.

4. OBSERVATION OF THERMAL RADIATION FROM THE IONOSPHERE

The rigorous determination of the radiation intensity within the emitting region of the ionosphere must be based on the study of the electromagnetic wave propagation in an anisotropic absorbing medium, in which each volume element can act as an emitter as well as an absorber of the thermal radiation. However, this problem is not discussed in the present paper.

Nevertheless, it is of interest and of a considerable practical importance to know about the characteristics of noise power received from the ionospheric thermal radiation at a detecting antenna located outside of the source region.

In view of the fact that the relation of the radiation fields to their sources is most readily found in terms of potential functions, and since the information with regard to some statistical properties of the random source current function $\mathbf{K}$ is available from Section 3, the retarded vector potential function is introduced here and expressed in complex form as $\mathbf{A}(\omega, x_\alpha)e^{j\omega t}$, with

$$\mathbf{A}(\omega, x_\alpha) = \frac{\mu_0}{4\pi}\int_{V_S} \frac{\mathbf{K}(\omega, x_\alpha')\exp(-jk_0R(x_\alpha, x_\alpha'))}{R(x_\alpha, x_\alpha')}\, dV', \tag{41}$$

where x_α and x_α' denote the coordinates of the observation point and the source point respectively, and $R(x_\alpha, x_\alpha')$ is the distance between them. $V_s(x_\alpha')$ is the volume of the source region under investigation and k_0 is the wave number. In the present discussion x_α is taken in the air and x_α' is taken in the ionosphere.

It should be observed that equation (41) signifies superposition of the solutions of the inhomogeneous wave equation

$$\nabla^2\mathbf{A} + k_0{}^2\mathbf{A} = -\mathbf{K}, \tag{42}$$

where $k_0{}^2 = \omega^2\mu_0\varepsilon_0$ and corresponds to a source at the point x_α' given by $\mathbf{K} = C\delta(x_\alpha - x_\alpha')$, with $\delta(x_\alpha - x_\alpha')$ being the usual Dirac delta function. On the other hand the retarded scalar potential function $\Phi(\omega, x_\alpha)$ is related to the vector potential by (Stratton, 1941)

$$\nabla \cdot A + j\omega\mu\varepsilon\Phi = 0, \tag{43}$$

which expresses the idea of conservation of charge. It should be noted that equation (43) is valid in free space (air) whereas it is only an approximation in a region of the conducting medium in which $|\sigma/j\omega\varepsilon_0| \ll 1$.

It is well known that the electromagnetic fields at an observation point x_α, taken in air, can be derived from these potential functions by

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{44}$$

and

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}, \tag{45}$$

where the spatial differential operator ∇ should be understood as ∇_{x_α}, which only operates on the function of x_α. The utilization of potential functions is particularly convenient because space differentiation ∇, under the sign of the volume integration, does not touch $\mathbf{K}(\omega, x_\alpha')$ and thereby the field intensities $\mathbf{E}$ and $\mathbf{H}$ in the same manner do not contain derivatives of $\mathbf{K}$.

Upon substitution of equations (41) and (43) into equations (44) and (45) and if only a $1/R$ dependent radiation field is taken into account, the electric and magnetic fields may be written as

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\frac{1}{4\pi}\int_{V_s} [\mathbf{k} \times (\mathbf{K} \times \mathbf{k})]\frac{e^{-j\mathbf{k}\cdot\mathbf{R}}}{R}\, dV' \tag{46}$$

and

$$\mathbf{H} = \frac{1}{4\pi}\int_{V_s} (\mathbf{K} \times \mathbf{k})\frac{e^{-j\mathbf{k}\cdot\mathbf{R}}}{R}\, dV' \tag{47}$$

in which the propagation vector $\mathbf{k} = \mathbf{n}k_0$ is introduced and the unit vector $\mathbf{n}$ is defined as $\mathbf{R}/R$ so that $\mathbf{k}$ and $\mathbf{R}$ are in the same direction.

The electromagnetic fields given by equations (46) and (47) can be considered as the random thermal electromagnetic fields since their source function $\mathbf{K}$ is a random, statistical quantity. The time average power flow density at the observation point x_α may be considered now with the aid of the Poynting vector defined as

$$\bar{\mathrm{p}} = \tfrac{1}{2}Re[\mathrm{E} \times \mathrm{H}^*]. \tag{48}$$

It is not difficult to show that the substitution of equations (46) and (47) into equation (48) yields

$$\bar{\mathrm{p}}(\omega, x_\alpha) = \frac{k_0 \Delta f}{2\lambda^2} \int_{V_s} \mathbf{n}(x_\alpha, x_\alpha') \left[\frac{kT_0 XZ}{1 + Z^2}\right] \frac{\Gamma(x_\alpha, x_\alpha')}{R^2(x_\alpha, x_\alpha')} \, dV', \tag{49}$$

where λ is the free-space wavelength, Z, X and T_0 are functions of the source point coordinate x_α' and $\Gamma(x_\alpha, x_\alpha')$ is defined by

$$\Gamma(x_\alpha, x_\alpha') = (1 - n_1^2)l_{11} + (1 - n_2^2)l_{22} + (1 - n_3^2)l_{33} - 2n_1 n_2 l_{12}, \tag{50}$$

in which n_1, n_2 and n_3 are the components of the unit vector $\mathbf{n}$ along r-, θ- and φ-coordinate axes, and l_{11}, l_{22}, l_{33} and l_{12} are given in equation (33).

It should be observed that equation (49) is based on the concept that the radiation intensity in any solid angle can be treated as energy, transferable in a bundle of plane, nonextinguishable waves whose normals are included in the solid angle. In a homogeneous isotropic medium the direction of the vector of energy flux coincides with the wave normal (RYTOV, 1959). The unit vector $\mathbf{n}(x_\alpha, x_\alpha')$ indicates the direction of propagation of the wave orginating at the source point x_α'.

Since the time average Poynting vector $\bar{\mathrm{p}}(\omega, x_\alpha)$ is determined, the noise power received from the ionospheric thermal radiation at the receiving antenna can be obtained by taking a proper surface integral of $\bar{\mathrm{p}}(\omega, x_\alpha)$ over the aperture of the antenna A_0,

$$P(\omega) = \int_{A_0} \bar{\mathrm{p}}(\omega, x_\alpha) \cdot \mathbf{ds}, \tag{51}$$

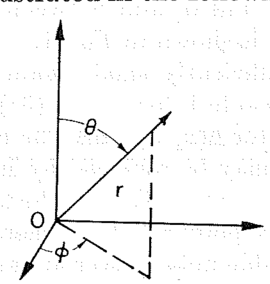
where $\mathbf{ds} = \mathbf{n}_0 \, ds$, with $\mathbf{n}_0$ being a unit vector normal to the differential surface area ds.

It should be noted that $P(\omega)$, given by equation (51) can be regarded as the available noise power at the receiving antenna in the frequency interval between f and $f + \Delta f$. On the other hand, from an elementary antenna theory (KRAUS, 1950), if the receiving antenna is properly oriented for maximum response, the available noise power $P(\omega)$ can be given by

$$P(\omega) = A_e p_0(\omega), \tag{52}$$

where $p_0(\omega)$ is the time average Poynting vector at the position of the receiving antenna and $A_e = (\lambda^2)/\Omega_a$ is the effective area of the antenna, with Ω_a being the solid angle through which all of the power radiated would stream if the power per unit solid angle equaled the maximum value of radiation intensity over the beam area.

In order to determine $\mathrm{p}(\omega, x_\alpha)$ from equation (49) the source region V_s, which is determined by the beam area of the receiving antenna, must be specified and the integrand must be expressed as a function of conveniently chosen coordinate variables. Although the parameter $l_{\alpha\beta}$ was expressed in spherical coordinate variables (r, θ, φ) in the previous section it is not difficult to see that the integration can conveniently be introduced with respect to the solid angle, subtended at the observation point, instead of carrying out the volume integration in a spherical coordinate system as in equation (49); this is illustrated in the following discussion.



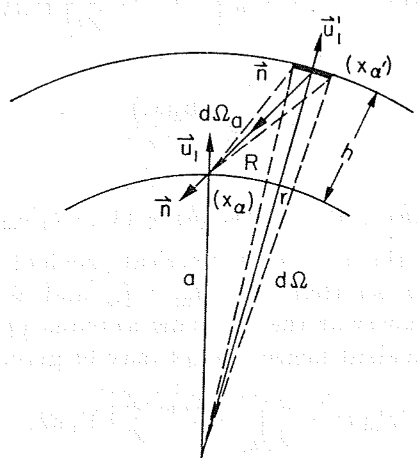


Fig. 1. Coordinate system and definition of variables. (a) Geomagnetic spherical coordinate system. (b) Geometrical relation between the source points x_α' and the observation points x_α.

If $d\Omega$ and $d\Omega_a$ denote, respectively, the differential solid angle subtended at the origin (the center of the Earth) and at the observation point (on the surface of the Earth) by a source located at x_α', then it is not difficult to see that with the aid of of Fig. 1

$$\frac{R^2 d\Omega_a}{|(\mathbf{n} \cdot \mathbf{u}_1')|} = r^2\, d\Omega, \tag{53}$$

where $\mathbf{u}_1'(x_\alpha')$ is the radial unit vector at the source point. The radial component of the noise power flow density received by the antenna located on the surface of the Earth may be given as follows, with the aid of equation (49):

$$\bar{p}_r(\omega, x_\alpha) = \mathbf{p}(\omega, x_\alpha) \cdot \mathbf{u}_1(x_\alpha) = \frac{k_0 \Delta f}{2\lambda^2} \int_{h_0}^{h_1} \int_{\Omega_a} y(x_\alpha, x_\alpha') \cdot \left[\frac{k T_0 X Z}{1 + Z^2}\right] \Gamma d\Omega_a dh, \tag{54}$$

where

$$y(x_\alpha, x_\alpha') = \frac{\mathbf{n}(x_\alpha, x_\alpha') \cdot \mathbf{u}_1(x_\alpha)}{|\mathbf{n}(x_\alpha, x_\alpha') \cdot \mathbf{u}_1'(x_\alpha')|} = \frac{\cos \psi_0}{\cos \psi_0'} \tag{55}$$

and Ω_a is the solid angle representing the beam area of the receiving antenna and $r = a + h$ is used in the derivation. The angles ψ_0 and ψ_0' appearing in equation 55 are those between $\mathbf{n}$ and $\mathbf{u}_1$ and between $\mathbf{n}$ and $\mathbf{u}_1'$ respectively, and they are related geometrically as is shown in Fig. 1.

If an antenna of sufficiently small beam area is used for measurement, some approximation can be made in equation (54). That is to say, if Ω_a is sufficiently small, then the unit vector $\mathbf{n}(x_\alpha, x_\alpha')$ may be considered as a constant vector within the solid angle Ω_a, and may be replaced by $\bar{\mathbf{n}}(x_\alpha, q_\alpha)$, where q_α is the representative source point lying on the axis of Ω_a and the factor y given in equation (55) becomes independent of the source point x_α' also. Therefore from equations (52) and (54), the expression for the available noise power at the receiving antenna is

$$P_r(\omega) = \frac{k_0 \Delta f}{2} \bar{y} \int_{h_0}^{h_1} \left[\frac{kT_0XZ}{1 + Z^2}\right] \bar{\Gamma}\, dh, \tag{56}$$

where

$$\bar{y} = \frac{\bar{\mathbf{n}} \cdot \mathbf{u}_1(x_\alpha)}{\bar{\mathbf{n}} \cdot \mathbf{u}_1'(q_\alpha)} \tag{57}$$

and

$$\bar{\Gamma} = (1 - \bar{n}_1^2)l_{11}(h) + (1 - \bar{n}_2^2)l_{22}(h) + (1 - \bar{n}_3^2)l_{33}(h) - 2\bar{n}_1\bar{n}_2 l_{12}(h). \tag{58}$$

It is observed that for the case of a vertical incident measurement, $\mathbf{u}_1 = \mathbf{u}_1'$, $\bar{n}_1 = 1$ and $\bar{n}_2 = \bar{n}_3 = 0$, so that $\bar{\Gamma}_0 = l_{22} + l_{33}$ and $\bar{y} = 1$. Consequently the available thermal noise power at the receiving antenna per unit bandwidth, $P_0(f)$, for the case of vertical incident measurement may be given by

$$P_0(f) = \frac{\pi}{\lambda} \int_{h_0}^{h_1} \left[\frac{kT_0XZ}{1 + Z^2}\right] \bar{\Gamma}_0\, dh. \tag{59}$$

It is interesting to note that for a special case $Y = 0$ (corresponding to the absence of a geomagnetic field), $l_{22} = l_{33} = 1$ and $\bar{\Gamma}_0 = 2$. Furthermore, if $Z^2 \ll 1$, then equation (59) is reduced essentially to the same form as that used by many workers (Pawsey *et al.*, 1951; Gardner, 1954; Dowden, 1960; Little *et al.*, 1961; Davis, 1960; Whitehead, 1959).

5. Concluding Remarks

The attention has been focused in the present study on the effect of colliding electrons under the assumption that the effect of the motion of ions in the region of the ionosphere of interest is negligible.

The general expressions derived for w_0, the thermal noise power generated per unit volume, per unit bandwidth, from any given source region V_s of the ionosphere, and for $P_0(f)$, the available thermal power per unit bandwidth received at the detecting antenna due to the radiation from V_s, are valid for all frequency ranges and for most regions of interest in the ionosphere, i.e. where the electron collision process plays a major role. Once V_s is specified, the profiles of $T_0(h)$, $N_0(h)$ and

$\nu(h)$ obtained from the experimental observations (NICOLET, 1959; THOMAS, 1959; KANE, 1960*a*, 1960*b*; ATAEV, 1959; SCHLAPP, 1959) can be used for the evaluation of w_0 and $P_0(f)$. Thus the detailed information with regard to the spectral distribution of the thermal energy radiated from the ionosphere can be obtained with the aid of a numerical integration of the expressions derived in the present paper.

It is indeed desirable that the present theory be tested and verified with some sort of experimental observation, e.g. a laboratory experiment. In other words, if the ionospheric plasma condition can be realistically represented with a laboratory experiment, then it will permit a study of the characteristics of thermal radiation in great detail and a test of the soundness of the present theory.

It should be pointed out that the present analysis may not be as rigorous as a microscopic treatment using the Boltzmann transport equation with the proper collision integral. However, this method of analysis does offer a simple and direct way of analyzing the thermal radiation from an anisotropic ionized medium and its radiation characteristics.

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SHORT PAPER

In future, Short Papers will replace the Research Notes which appeared in previous Numbers. For details about Short Papers see "Message From the Editor-in-Chief" (Vol. 27. No. 10, p. 1033, 1965) and "Information for Contributors" (Inside back cover)

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Characteristics of ionospheric thermal radiation

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Abstract—A theoretical observation of the characteristics of ionospheric thermal radiation is made using a linear theory, based on a macroscopic concept with the aid of fluctuating electromagnetic field theory. The thermal noise power generated in the ionosphere per unit volume, per unit frequency bandwidth, and the available thermal noise power at a receiving antenna per unit frequency bandwidth are calculated. The spectral distributions of ionospheric thermal radiation are obtained and discussed in detail.

A study of the calculated power level of the thermal noise generated in the ionosphere shows that it is exceedingly low and decreases rather rapidly with an increase of frequency f. A large portion of the noise signal generated appears to be in the frequency range of $f < 10^7$ c/s, with the microwave noise signal being negligible, and appears to come mainly from the region between 60 km and 100 km of the ionosphere. Furthermore, the study reveals that the available noise power at a receiving antenna depends upon the geographical location of the antenna in general and that the power level is higher in the equatorial zone than in the polar cap zone.

For a noise signal frequency of less than 10^8 c/s, the power level increases monotonically with an increase of polar angle θ from $\theta = 0$ and reaches its maximum value at $\theta = 90°$, where it is at least a hundred times greater than at $\theta = 0$. For a noise signal with a frequency of 10^9 c/s, the available noise power has its maximum at $\theta \simeq 50°$, where the power level is comparable to that of a noise signal with a frequency of 10^6 c/s. For a noise signal of a frequency above 5×10^9 c/s, the angular dependence disappears.

1. INTRODUCTION

IN A recent theoretical study made by the author on ionospheric thermal radiation, the expressions for w_0, the thermal noise power generated per unit volume, per unit frequency bandwidth, and for P_0, the available noise power at the receiving antenna per unit frequency bandwidth for the case of vertical incident measurement, have been derived, based on a macroscopic concept with the aid of the fluctuating electromagnetic field theory using the Maxwell-Langevin equations. It is the purpose of the present paper to estimate theoretically the radiation characteristics of ionospheric thermal noise by investigating the expressions for w_0 and P_0 with the aid of experimentally observed available data (e.g. LEGALLEY and ROSEN, 1964; JACKSON and KANE, 1959; JACKSON and SEDDON, 1958) for the profile of the electron number density.

The expressions for w_0 and P_0 are given as follows (the derivation of these expressions is presented in Hsieh (1966)):

$$w_0(f, r, \theta) = kT_0 \left(\frac{Z^2}{1 + Z^2}\right)(1 + 2l_{33}) \tag{1}$$

and

$$P_0(f, \theta) = \int_{h_0}^{h_0+\Delta h} \frac{\pi f}{c} \left(\frac{kT_0XZ}{1 + Z^2}\right) [l_{22} + l_{33}]\, dh, \tag{2}$$

where

$$l_{22}(f, r, \theta) = \frac{(1 + Z^2)(1 + Z^2 + Y^2) + (Y^2 + Z^2 - 3)\, G^2 \sin^2 \theta}{(Y^2 + Z^2 - 1)^2 + 4Z^2},$$

$$l_{33}(f, r, \theta) = \frac{(1 + Z^2)(1 + Z^2 + Y^2)}{(Y^2 + Z^2 - 1)^2 + 4Z^2},$$

$$X = \left(\frac{\omega_p}{\omega}\right)^2,$$

$$\omega_p^2 = \frac{N_0 e^2}{m\varepsilon_0},$$

$$Y^2 = \left(\frac{\omega_b}{\omega}\right)^2 = G^2[1 + 3 \cos^2 \theta],$$

$$Z = \frac{\nu}{\omega},$$

$$G = -\frac{M}{3}\frac{e}{m}\left(\frac{a}{r}\right)^3 \frac{1}{\omega} \text{ and}$$

$$r = a + h. \tag{3}$$

In the above expressions, r and θ are the radial and polar angular variables, respectively, in the geomagnetic polar coordinate system whose origin is located at the center of the Earth. The electronic charge e is taken as a negative value and m is the electronic mass. ω_p and ω_b are the electron plasma and gyrofrequencies, respectively, and ω is the angular frequency of the noise signal under consideration. c is the speed of light in vacua and ε_0 is the dielectric constant of vacuum. h is the height above sea level; f is the radiation frequency; a is the radius of the Earth; M is the magnetization of the Earth; and k is the Boltzmann constant. Finally N_0, T_0 and ν are electronic number density, temperature and collision frequency, respectively.

2. Asymptotic Expressions

When the radius of the Earth a is taken as 6370 km and the magnetization of the Earth M is taken as $(0{\cdot}935)/(4\pi)$ A/m² (Morgan, 1959), the factor $G(\omega, r)$ given in equation (3) can be approximated by

$$G(\omega, r) \simeq \frac{6{\cdot}95 \times 10^8}{f}\left\{1 - 3\left(\frac{h}{a}\right)\right\}, \qquad \text{for } \left(\frac{h}{a}\right) \ll 1. \tag{4}$$

Consequently G is practically invariant with respect to h for the range of height up to 200 km. On the other hand, Z, being equal to $\nu(h)/\omega$, varies considerably with h in the same region of the ionosphere. It is of interest to note that when the parameters Z and G satisfy the following conditions:

$$4Z^2 \ll 1, \qquad 3 \ll G^2 \qquad \text{(Case A)} \tag{5a}$$

or

$$4Z^2 \ll 1, \quad 4G^2 \ll 1 \qquad \text{(Case B)}, \tag{5b}$$

Equation (1) takes the following simple form:

$$w_{03}(f, h) = kT_0 Z_3{}^2 \tag{6a}$$

or

$$w_{04}(f, h) = 3kT_0 Z_4{}^2. \tag{6b}$$

On the other hand, for $|\Delta h| \ll h_0$, equation (2) takes the following form:

$$P_{03}(f, \theta) = \xi_0(\theta, h_0) kT_0 X_3 Z_3 \left(\frac{\Delta h}{\lambda_3}\right) \pi \tag{7a}$$

or

$$P_{04}(f, \theta) = 2kT_0 X_4 Z_4 \left(\frac{\Delta h}{\lambda_4}\right) \pi \tag{7b}$$

with

$$\xi_0(\theta, h_0) = \frac{2 + G^2 \sin^2 \theta}{G^2(1 + 3\cos^2 \theta)}, \tag{8}$$

where the subscripts 3 and 4 are introduced in w_0 and P_0 to indicate the fact that Case A and Case B are being considered respectively. Furthermore, for the region of the ionosphere between 85 km and 200 km, the range of frequency which satisfies the conditions (5a) and (5b) can be given as follows:

$$1{\cdot}6 \text{ mc} \leq f \leq 42 \text{ mc}, \qquad \text{for Case A} \tag{9a}$$

or

$$14 \text{ kmc} \leq f, \qquad \text{for Case B}. \tag{9b}$$

Thus equations (6a) and (7a) can be regarded as the asymptotic expressions for w_0 and P_0 for the radio-frequency range and equations (6b) and (7b) are the asymptotic expressions for the microwave-frequency range respectively.

3. Calculation

It is easily seen that once the electron temperature T_0, number density N_0 and collision frequency ν are known, calculation of the quantities w_0 and P_0 is straightforward. Unfortunately the exact knowledge of T_0, N_0 and ν as functions of position is not available at the present time. In view of the fact that many of the results of radio observation can be explained by assuming that the ionosphere is horizontally stratified, that is, that the electron density and collision frequency are functions only of the height h, this assumption is adopted for the present investigation. Furthermore, it is assumed that under the thermal equilibrium conditions the electron temperature T_0 is equal to the background gas temperature T_g. The midday mean electron number density profile $N(h)$, based on seven experimental observations—one by partial reflection technique, two by cross-modulation technique and four by rocket experiment—is used for the D-region (60 km–90 km) and the other profiles, based on two rocket observations, are used for the height range between 90 km and 180 km.

The value of the average electron collision frequency ν with the neutral particles and ions can be given approximately as follows:

$$\nu = \langle \nu_m \rangle + \langle \nu_{ei} \rangle, \tag{10}$$

where $\langle \nu_m \rangle$ and $\langle \nu_{ei} \rangle$ denote the average electron collision frequency with neutral particles and with ions respectively (e.g. Shkarofsky, 1961). They are given in terms of atmospheric parameters as follows:

$$\langle \nu_m(h) \rangle = 1{\cdot}9944 n_M T_0 \times 10^{-7}, \qquad \text{for } h \leq 180 \text{ km}, \tag{11}$$

where n_M is the number density of air, and

$$\langle \nu_{ei} \rangle = 8{\cdot}375 N \log \Lambda \times 10^{-6} / T_0^{3/2} \tag{12}$$

with

$$\Lambda = 1{\cdot}24 \times 10^7 \, (T_0^3/N)^{1/2}, \tag{13}$$

where N is the ion density and T_0 is the electron temperature. In the present calculation it is assumed that $N_0 = N$ and $T_0 = T_g$. ν, given in equation (10), is calculated with the aid of tabulated data for n_M and T_g (e.g. Kallmann-Bijl, 1961). The mks system of units is employed in the present study.

4. Discussion of Results

The results of calculation of the noise power spectrum show that for a given value of θ and h, w_0 decreases with an increase in f as shown in Fig. 1 except in the vicinity of $f = 10^9$ c/s. The rate of decrease is higher in the h.f. range than in the l.f. range. There is a hump, i.e. a relative minimum and a relative maximum, in the range

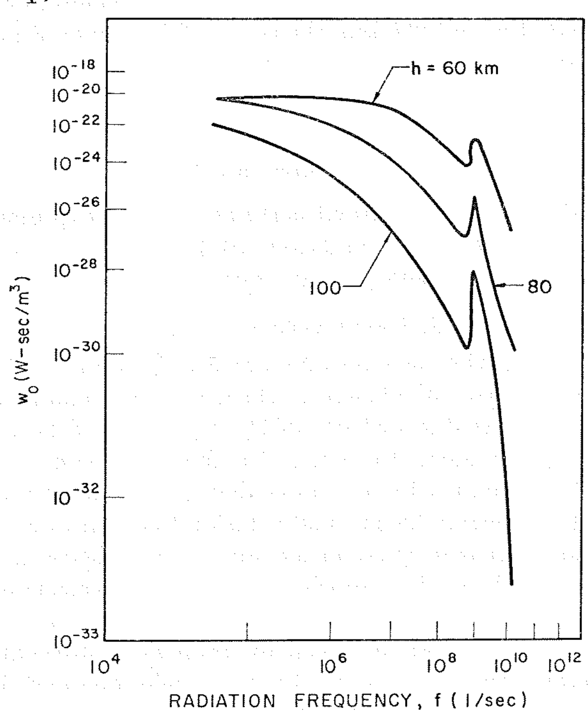

Fig. 1. The frequency spectrum of the thermal energy volume density with height as parameter for $\theta = 45°$.

$10^8 < f < 5 \times 10^9$ c/s and this hump appears to be largest for $\theta = 45°$ and it tends to disappear for $\theta = 90°$.

It is of interest to note that for the frequency range specified by the inequalities (9a) and (9b), equations (6a) and (6b) both suggest that w_0 is inversely proportional to the square of the frequency f and directly proportional to the square of the collision frequency ν for the region between 85 km and 200 km of the ionosphere. In addition, a comparison between equations (6a) and (6b) shows that at a given height h, since $dw_{04}/df = 3dw_{03}/df$, the rate of decrease of w_0 in the h.f. range is three times greater than in the l.f. range (e.g. see the case of $h = 100$ km in Fig. 1). Thus the observation made in Fig. 1 on the behaviour of w_0 can be predicted quite well by the asymptotic expressions for w_0 given by equations (6a) and (6b). It should be pointed out, however, that equations (6a) and (6b) are not applicable for a frequency range in the vicinity of $f = 10^9$ c/s and therefore a prediction of the behaviour of w_0 in this region has to be made by studying equation (1).

On the other hand, as shown in Fig. 2, for a given $f \leq 10^8$ c/s, P_0 increases rapidly with an increase of θ in the region $0 \leq \theta \leq 20°$ and increases gradually in the region

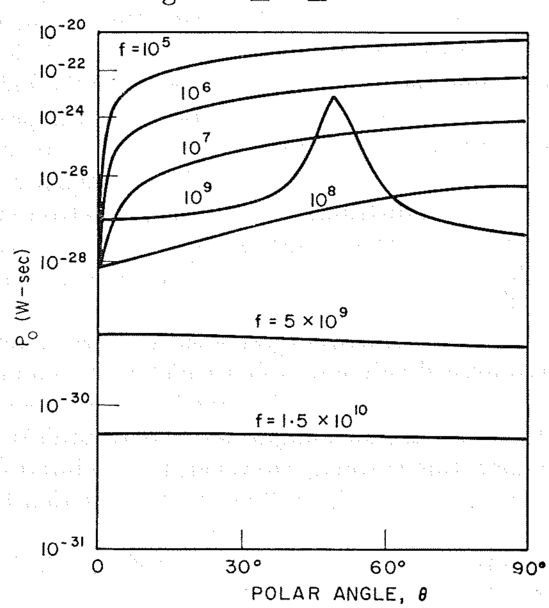

Fig. 2. The polar angular spectrum of available noise power at the receiving antenna per unit frequency bandwidth from the source lying between 60 km and 180 km.

$\theta > 20°$, then reaches its maximum value at $\theta = 90°$. For a given $f \geq 7{\cdot}5 \times 10^9$ c/s, P_0 becomes independent of θ. However, for the noise signal with $f = 10^9$ c/s, P_0 increases gradually from $\theta = 0$ to about 30°, then increases rather rapidly to reach its maximum value at about $\theta = 50°$, and finally decreases as θ increases further. The values of P_0 in the region of $\theta = 0$ and in the vicinity of $\theta = 90°$ are about the same. The maximum value of P_0 at $\theta = 50°$ for $f = 10^9$ c/s is comparable to that for $f = 10^6$ c/s. It should be noted that the asymptotic expression (7b) which predicts P_0 for the microwave range is independent of θ, while equation (7a) indicates that P_0, for the radio-frequency range, does depend upon θ in the manner specified by equation (8), which is consistent with the above observation made on the behaviour of P_0 in Fig. 2.

5. Conclusions

The spectral distribution of ionospheric thermal noise power has been obtained. At a given position (h, θ) in the ionosphere there is at least ten thousand times more thermal noise power per unit volume, per unit frequency bandwidth, generated in the frequency range $5 \times 10^4 \leq f \leq 10^6$ c/s than in the range $f \geq 10^9$ c/s. w_0, the thermal noise power generated per unit volume, per unit frequency bandwidth, decreases monotonically with an increase in the height h, which is reasonable in view of the fact that the electronic collision process plays a major role in the type of radiation under consideration, and since the collision frequency $\nu(h)$ decreases with an increase in height. w_0 appears to be independent of the polar angle θ except for the case of $f = 10^9$ c/s in which it has its maximum value at $\theta = 45°$.

The observation of the spectrum of the available noise power at the receiving antenna $P_0(f, \theta)$, per unit frequency bandwidth, with the source regions between 60 km and 180 km of the ionosphere, shows that in general it decreases with an increase of the frequency f and increases with an increase of the polar angle θ.

It should be pointed out that most of the uncertainty in the present calculation arises from a lack of complete knowledge concerning the electron temperature and the electron distribution function. Furthermore, the results shown above are based on the ideal assumption of a Maxwellian distribution, and equal electron and gas temperatures. Consequently the results of the present study should be regarded as a good qualitative analysis of the thermal radiation from the ionosphere rather than a rigorous quantitative analysis.

The calculations indicate that the power level of the thermal radiation originating in the ionosphere is indeed exceedingly low, a fact which is commonly believed. For example, at $\theta = 90°$ for $f = 10^5$ c/s, P_0 is of the order of 10^{-20} W c/s and for $f = 10^6$ c/s, it is of the order of 10^{-22} W c/s, which might not be detectable without the aid of an exceedingly sensitive detecting system. However, it is of interest to observe the dependence of P_0 on the polar angle θ, as well as on the radiation frequency f.

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Cutoff conditions for transverse circularly polarized electromagnetic waves in a finite temperature electro-magneto-ionic medium

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Abstract—The dispersion relation for wave propagation in a homogeneous, electrically neutral electron gas subject to crossed static electric and magnetic fields is derived using the coupled Maxwell–Boltzmann–Vlasov equations. The cutoff condition for transverse circularly polarized electromagnetic waves is obtained from the derived dispersion relation. The variation of the cutoff frequency ω_0 with the static electric field E_0, magnetostatic field B_0, the electron number density N and the electron gas temperature T is discussed. For example it is shown that for a given value of B_0 and N, either a decrease of the electron gas temperature T or an increase of static electric field E_0 will cause the cutoff frequency of the left-hand circularly polarized wave to increase while the cutoff frequency of the right-hand circularly polarized wave decreases. A possible application of the theory to the study of electromagnetic wave propagation in the region of ionospheric plasma, where both static electric and static magnetic fields are present, is indicated.

1. Introduction

It is well known that the propagation of electromagnetic waves in a magnetoionic medium can be characterized by the Appleton equation. A detailed discussion on this subject has been given by Ratcliffe (1959). The magnetoionic medium is defined as one in which free electrons and heavy ions are situated in a uniform magnetic field and are distributed with statistical uniformity, so that there is no resultant space charge.

The cutoff conditions for the circularly polarized electromagnetic wave in a cold plasma have been discussed in detail by Heald and Wharton (1965). Recently, in the course of investigating the effect of a transverse static electric field on the propagation of the circularly polarized electromagnetic wave in a finite temperature plasma, this author (Hsieh, 1966) has shown that the presence of a static electric field may cause the cutoff frequency of the electromagnetic wave to shift. The purpose of this paper, therefore, is to discuss the cutoff conditons for the circularly polarized waves in a homogeneous, electrically neutral electron gas, subjected to crossed electrostatic and magnetostatic fields. This system is referred to as the electro-magneto-ionic medium. The thermal motion of an electron is considered although ion motion and collision effects are assumed to be negligible. The present discussion is based on a small-amplitude, one-dimensional analysis in which all time-varying quantities are assumed to have harmonic dependence of the form: $\exp[j(\omega t - kz)]$, where ω and k are the wave angular frequency and propagation constant respectively. z and t denote, respectively, the spatial and time variables.

2. Dispersion Relations

Consider all quantities of interest to be composed of two parts; the time-independent part denoted by the subscript "0" and the time-dependent part denoted by the subscript "1", e.g. the magnetic flux density $\mathbf{B}$ and electric field intensity $\mathbf{E}$ are written as

$$\mathbf{B} = \mathbf{B}_0(z) + \mathbf{B}_1(z, t) \quad \text{and} \quad \mathbf{E} = \mathbf{E}_0(z) + \mathbf{E}_1(z, t)$$

and the electron distribution function f is written as

$$f = f_0(z, \mathbf{v}) + f_1(z, \mathbf{v}, t).$$

Suppose that the applied static electric field $\mathbf{E}_a$ and magnetostatic field $\mathbf{B}_0$ are directed along the positive y- and positive z-directions respectively.

The dynamic electromagnetic fields in the electron gas are governed by Maxwell's field equation, which is expressed in the following manner:

$$\mathbf{E}_\pm = \frac{-j\left(\dfrac{\omega e}{\varepsilon_0}\right)}{(\omega^2 - c^2k^2)} \int v_r \, \mathrm{e}^{\pm j\varphi} f_1 \, d^3v,$$

$$E_{1z} = \frac{-je}{\omega \varepsilon_0} \int v_z f_1 \, d^3v, \tag{1}$$

where $c = 1/\sqrt{(\mu_0\varepsilon_0)}$ is the speed of light in free space, and μ_0 and ε_0 denote the permeability and dielectric constant of vacuum; $d^3v = (v_r \, dv_r \, d\varphi \, dv_z)$ is a volume element in velocity space. On the other hand, the electron distribution is described by the Boltzmann–Vlasov equation, which is written as

$$\left(j(\omega - kv_z) + \omega_z \frac{\partial}{\partial\varphi}\right) f_1 - a_-\left(\frac{\partial f_1}{\partial v_r} + \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right) \mathrm{e}^{j\varphi} - a_+\left(\frac{\partial f_1}{\partial v_z} - \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right)\mathrm{e}^{-j\varphi}$$

$$= \frac{e}{m} M_-(f_0)E_-\mathrm{e}^{j\varphi} + \frac{e}{m} M_+(f_0)E_+\mathrm{e}^{-j\varphi} + \frac{e}{m} E_{1z}\frac{\partial f_0}{\partial v_z}, \tag{2}$$

where

$$E_\mp = \tfrac{1}{2}(E_{1x} \mp jE_{1y}), \; B_\mp = \tfrac{1}{2}(B_{1x} \mp jB_{1y}),$$

$$a_\mp = \mp j\frac{eE_a}{2m}, \qquad \omega_z \equiv \left(\frac{\mathrm{e}B_0}{m}\right),$$

$$M_\mp(f_0) = \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_0}{\partial v_r} \mp \frac{j}{v_r}\frac{\partial f_0}{\partial\varphi}\right) + \frac{kv_r}{\omega}\frac{\partial f_0}{\partial v_z}\right],$$

$$v_x = v_r \cos\varphi, \qquad v_y = v_r \sin\varphi. \tag{3}$$

Here E_- and E_+ denote the left- and right-hand circularly polarized components of electric field respectively. E_{1z} is the longitudinal component of electric field. ω_z is the electron cyclotron frequency.

Suppose that the time-varying distribution function f_1 is written in the form.

$$f_1(z, t, v_r, v_z, \varphi) = f_-(z, t, v_r, v_z)\mathrm{e}^{j\varphi} + f_+(z, t, v_r, v_z)\mathrm{e}^{-j\varphi} + g(z, t, v_r, v_z), \tag{4}$$

in which the first, second and third terms of the right-hand side can be regarded as the left-hand circularly polarized, the right-hand circularly polarized wave, and the longitudinal components of the distribution function respectively. Then in view of the fact that equation (2) must be valid for an arbitrary value of φ, the substitution of equation (4) into equation (2) yields a system of equations, expressing the functions f_-, f_+ and g in terms of the electric field components E_-, E_+ and E_{1z} as follows:

$$\begin{aligned} f_- &= k_{11}E_- + k_{12}E_+ + k_{13}E_{1z}, \\ f_+ &= k_{21}E_- + k_{22}E_+ + k_{23}E_{1z}, \\ g &= k_{31}E_- + k_{32}E_+ + k_{33}E_{1z}, \end{aligned} \tag{5}$$

where

$$k_{11} = \frac{-j\dfrac{e}{m}M_-(f_0)}{(b+\omega_z)}, \qquad k_{12} = 0, \qquad k_{13} = \frac{-\dfrac{e}{m}a_-\dfrac{\partial}{\partial v_r}\left(\dfrac{\partial f_0}{\partial v_z}\right)}{b(b+\omega_z)},$$

$$k_{21} = 0, \qquad k_{22} = \frac{-j\dfrac{e}{m}M_+(f_0)}{(b-\omega_z)}, \qquad k_{23} = \frac{-\dfrac{e}{m}a_+\dfrac{\partial}{\partial v_r}\left(\dfrac{\partial f_0}{\partial v_z}\right)}{b(b-\omega_z)},$$

$$k_{31} = \frac{-2\dfrac{e}{m}\dfrac{a_+}{v_r}M_-(f_0)}{b(b+\omega_z)}, \qquad k_{32} = \frac{-2\dfrac{e}{m}\dfrac{a_-}{v_r}M_+(f_0)}{b(b-\omega_z)},$$

$$k_{33} = \frac{-j\dfrac{e}{m}\dfrac{\partial f_0}{\partial v_z}}{b} + j\frac{4a_-a_+}{v_r}\frac{\dfrac{e}{m}\dfrac{\partial}{\partial v_r}\dfrac{\partial f_0}{\partial v_z}}{b(b^2-\omega_z^2)}, \qquad b \equiv (\omega - kv_z). \tag{6}$$

Upon combining equations (1) and (5), the following set of algebraic equations governing the electric field components is obtained:

$$\begin{aligned} E_- &= R_{11}E_- + R_{12}E_+ + R_{13}E_{1z}, \\ E_+ &= R_{21}E_- + R_{22}E_+ + R_{23}E_{1z}, \\ E_{1z} &= R_{31}E_- + R_{32}E_+ + R_{33}E_{1z}, \end{aligned} \tag{7}$$

where

$$R_{1l} = \frac{-j\left(\dfrac{\omega e}{\varepsilon_0}\right)}{2(\omega^2 - c^2k^2)}\int_{-\infty}^{\infty}\int_0^{\infty} v_r^2 \int_0^{2\pi} (k_{1l} + k_{2l}e^{-j2\varphi} + k_{3l}e^{-j\varphi})\, d\varphi\, dv_r\, dv_z,$$

$$R_{2p} = \frac{-j\left(\dfrac{\omega e}{\varepsilon_0}\right)}{2(\omega^2 - c^2k^2)}\int_{-\infty}^{\infty}\int_0^{\infty} v_r^2 \int_0^{2\pi} (k_{1p}e^{j2\varphi} + k_{2p} + k_{3p}e^{j\varphi})\, d\varphi\, dv_r\, dv_z,$$

$$R_{3q} = \frac{-je}{\omega\varepsilon_0}\int_{-\infty}^{\infty} v_z \int_0^{\infty} v_r \int_0^{2\pi} (k_{1q}e^{j\varphi} + k_{2q}e^{-j\varphi} + k_{3q})\, d\varphi\, dv_r\, dv_z,$$

in which l, p, and q take the values of 1, 2, and 3. The coefficients k_{pq} are given in equations (6).

The dispersion relation for the system is readily given from equation (7) and is expressed as

$$D(\omega, k) = \begin{vmatrix} (R_{11} - 1) & R_{12} & R_{13} \\ R_{21} & (R_{22} - 1) & R_{23} \\ R_{31} & R_{32} & (R_{33} - 1) \end{vmatrix} = 0. \tag{8}$$

It should be noted that equations (7) indicate the possibility of coupling between the characteristic modes. Once the time-independent distribution function f_0 is known, then the coefficients k_{pq} are determined so that R_{pq} can be evaluated. f_0, which must satisfy the time-independent part of the Boltzmann–Vlasov equation for the case of a homogeneous, electrically neutral electron gas subjected to crossed static electric and magnetic fields, can be written in a Maxwellian form as follows:

$$f_0 = n_0 \exp\{-\alpha[(v_x - u)^2 + v_y^2 + v_z^2]\}, \tag{9}$$

where $\alpha \equiv (m/2KT)$ and $u \equiv (E_0/B_0)$. K is the Boltzmann constant, T and u denote the temperature and drift velocity of an electron respectively, and n_0 is the normalization constant.

Based on equation (9) the elements of the determinant in dispersion relation (8) can be given as follows:

$$R_{11} = \frac{X}{(1-\eta)}\left[\frac{1}{1+Y} + \frac{\gamma}{2}\left(\delta + \frac{1}{(1+Y)^3}\right)\right],$$

$$R_{12} = \frac{\delta X}{(1-\eta)}\left[\left(1 + \frac{\delta}{2}\frac{Y}{(1-Y)}\right) + \frac{\gamma}{2}\left(1 + \frac{\delta}{2}\frac{Y}{(1-Y)^3}\right)\right],$$

$$R_{13} = \frac{\sqrt{(\delta\gamma)}X}{2(1-\eta)}\left[\left(2 - \frac{\delta}{2}\right) + \frac{\left(\frac{\delta}{2}\right) - 1}{(1+Y)^2}\right],$$

$$R_{21} = \frac{\delta X}{(1-\eta)}\left[\left(1 - \frac{\delta}{2}\frac{Y}{1+Y}\right) + \frac{\gamma}{2}\left(1 - \frac{\delta}{2}\frac{Y}{(1+Y)^3}\right)\right],$$

$$R_{22} = \frac{X}{(1-\eta)}\left[\frac{1}{1-Y} + \frac{\gamma}{2}\left(\delta + \frac{1}{(1-Y)^3}\right)\right],$$

$$R_{23} = \frac{\sqrt{(\delta\gamma)}X}{2(1-\eta)}\left[\left(2 - \frac{\delta}{2}\right) + \frac{(\delta/2 - 1)}{(1-Y)^2}\right],$$

$$R_{31} = \sqrt{(\delta\gamma)}X\left(1 - \frac{\delta}{2}\frac{(1-\delta)Y}{(1+Y)^2}\right),$$

$$R_{32} = \sqrt{(\delta\gamma)}X\left(1 + \frac{\delta}{2}\frac{(1-\delta)Y}{(1-Y)^2}\right),$$

$$R_{33} = X\left[1 + \frac{\delta(1 - \frac{3}{2}\delta)Y^2}{(1-Y^2)}\right], \tag{10}$$

where

$$X \equiv \left(\frac{\omega_p}{\omega}\right)^2, \qquad Y \equiv \left(\frac{\omega_z}{\omega}\right), \qquad \eta \equiv \left(\frac{c^2k^2}{\omega^2}\right),$$

$$\gamma \equiv \frac{1}{\alpha v_0^{\,2}}, \qquad \delta \equiv \alpha u^2, \qquad v_0 \equiv \left(\frac{\omega}{k}\right), \qquad \alpha \equiv \left(\frac{m}{2KT}\right),$$

and provided that

$$\delta^2 \ll 1, \qquad \gamma^2 \ll 1 \quad \text{and} \quad |\sqrt{(\alpha v_0)}(1 \pm Y)|^4 \gg 1. \tag{11}$$

The assumption that conditions (11) are satisfied is equivalent to assuming that the drift velocity u is much smaller than the thermal velocity, while the thermal velocity is much smaller than the phase velocity of the transverse electromagnetic wave.

It is of interest to note that as $\delta \to 0$, $R_{pq} \to 0$, when $p \neq q$, so that the off-diagonal terms of the determinant in equation (8) vanish, which suggests that the coupling between the modes disappears. In this case equation (8) becomes

$$(R_{11} - 1)(R_{22} - 1)(R_{33} - 1) = 0, \tag{12}$$

which in turn yields three dispersion equations for the uncoupled characteristic modes, as shown in equation (13):

$$1 = \frac{X}{(1-\eta)}\left(\frac{1}{1+Y} + \frac{\gamma}{2}\frac{1}{(1+Y)^3}\right), \tag{13a}$$

$$1 = \frac{X}{(1-\eta)}\left(\frac{1}{1-Y} + \frac{\gamma}{2}\frac{1}{(1-Y)^3}\right), \tag{13b}$$

$$1 = X. \tag{13c}$$

Equations (13) are, rspectively, the dispersion equations for the left-hand circularly polarized wave, the right-hand circularly polarized wave, and the longitudinal plasma oscillation. It should be noted that as $\gamma \to 0$, which is the case when $T \to 0$, equations (13a) and (13b) are reduced to the familar expression in the cold-plasma magnetoionic theory.

3. The Cutoff Conditions for the Transverse Electromagnetic Waves

The cutoff of electromagnetic wave propagation occurs when its propagation constant k becomes zero, in which case $\eta = 0$ and $\gamma = 0$ so that, from equation (10), $R_{13} = R_{23} = R_{31} = R_{32} = 0$. Equation (8) becomes

$$[(r_{11} - 1)(r_{22} - 1) - r_{12}r_{21}](R_{33} - 1) = 0, \tag{14}$$

where r_{pq} is the value of R_{pq} for $\eta = 0$ and $\gamma = 0$. Therefore the cutoff conditions for the transverse electromagnetic wave are given by

$$(r_{11} - 1)(r_{22} - 1) - r_{12}r_{21} = 0, \tag{15}$$

which can be conveniently written as

$$\frac{\delta^2 y_0^{\,2}(1-\delta)}{x^4} = y_0^{\,2} - \left(x - \frac{1}{x}\right)^2, \tag{16}$$

where $y_0 \equiv (\omega_z/\omega_p)$ is the ratio of the electron cyclotron frequency ω_z to the electron plasma frequency ω_p and $x \equiv (\omega_0/\omega_p)$ with ω_0 being the cutoff frequency of the transverse electromagnetic wave. Once the values of y_0 and δ are specified, equation (16) can be solved for x so that (ω_0/ω_p) is determined. However, the variation of ω_0 with respect to δ can easily be observed with the aid of a graphical method illustrated below:

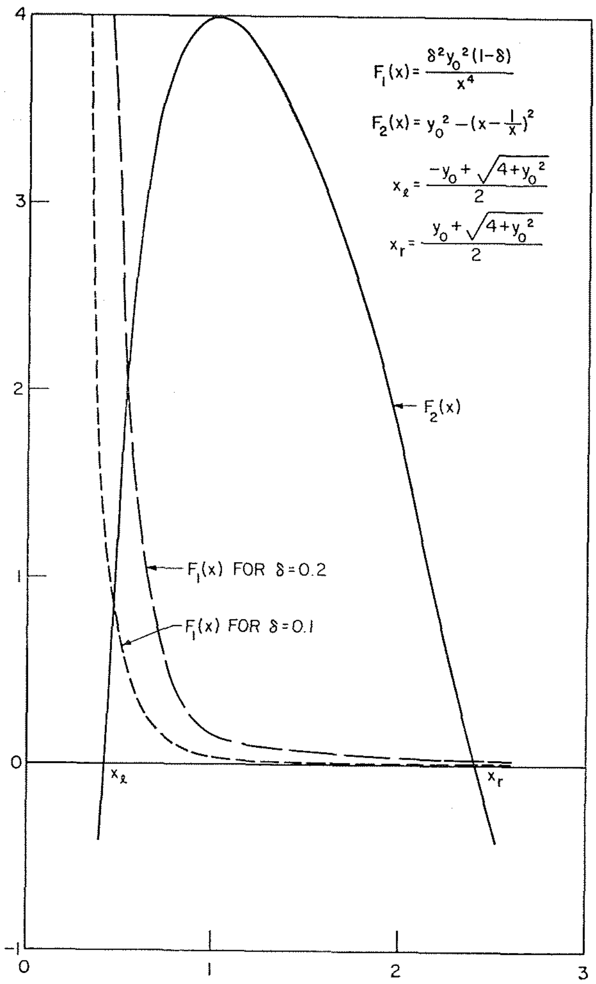

Fig. 1. Illustration of graphical solution of equation (16).

Let $F_1(x)$ be the left-hand side and $F_2(x)$ be the right-hand side of equation (16). If F_1 vs. x and F_2 vs. x are plotted in the same plane, as illustrated in Fig. 1, then the intersections of the two plots provides the real roots of equation (16). Once y_0 is given, the curve of $F_2(x)$ is determined, and if δ is also specified, then $F_1(x)$ is completely determined. Thus the intersection point of the two plots is readily determined. It should be noted that when $\delta = 0$ the F_1 curve coincides with the x-axis, and if its intersections with the F_2 curve are denoted by x_l and x_r, then they are given by

$$x_l = \frac{-y_0 + \sqrt{(y_0^2 + 4)}}{2} \quad \text{and} \quad x_r = \frac{y_0 + \sqrt{(y_0^2 + 4)}}{2}. \tag{17}$$

x_l determines ω_{0l}, the cutoff frequency of the left-hand circularly polarized wave, and

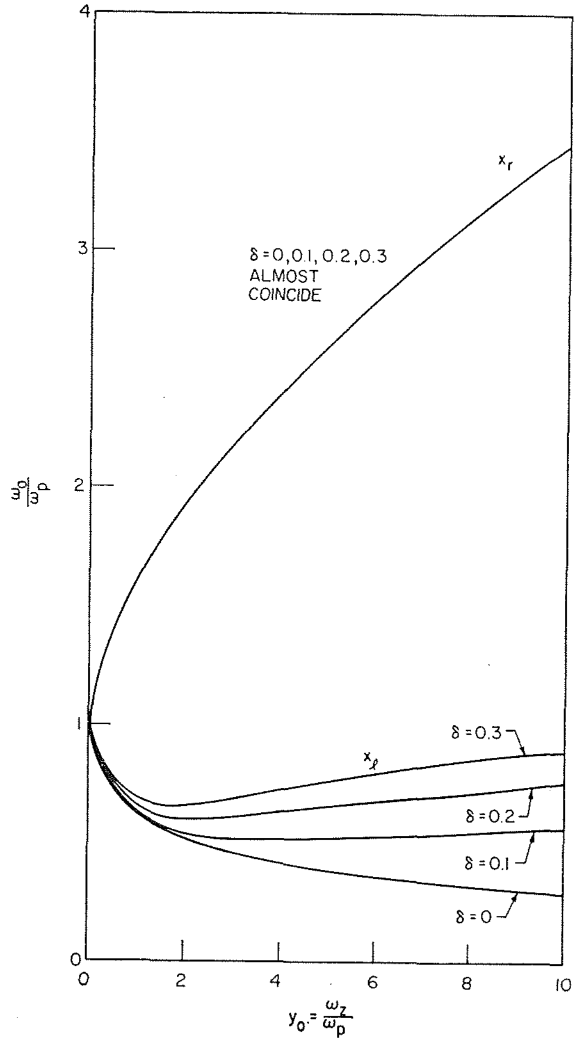

Fig. 2. The plot of (ω_0/ω_p) vs. (ω_z/ω_p) with δ as parameter.

x_r determines ω_{0r}, the cutoff frequency of the right-hand circularly polarized wave. It is easily seen from Fig. 1 that an increase in the parameter δ leads to an increase in ω_{0l}, but to a slight decrease in ω_{0r}. Numerical illustration of equation (16) is given in Figs. 2–4. The normalized cutoff frequency $x = (\omega_0/\omega_p)$ as a function of the normalized cyclotron frequency $y_0 = (\omega_z/\omega_p)$ for various values of the parameter $\delta = (m/2KT)(E_0^2/B_0^2)$ is shown in Fig. 2. The plot of x vs. δ for various values of y_0 is

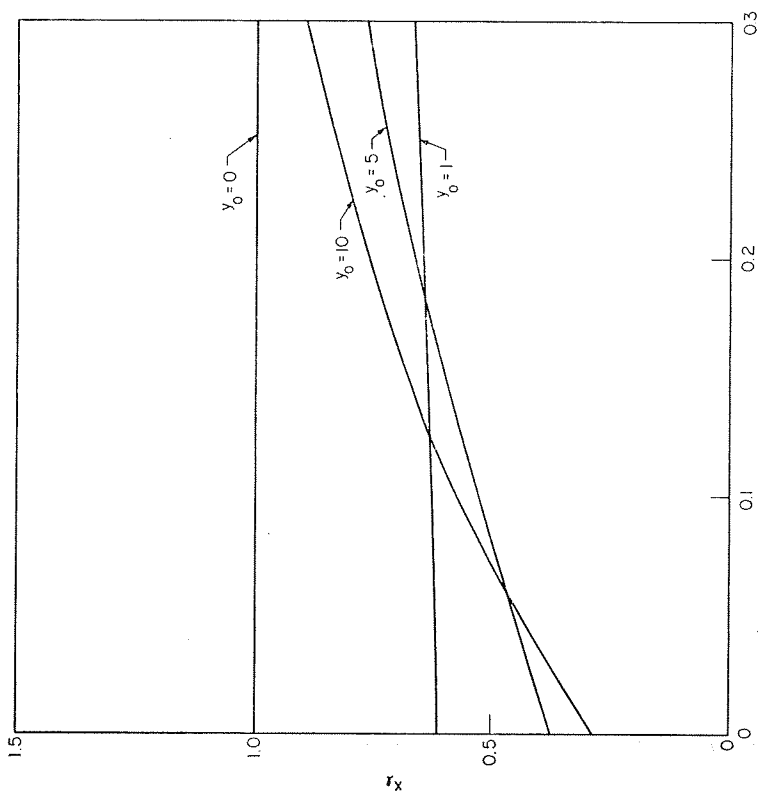

Fig. 3. The plot of (ω_0/ω_p) vs. δ with $y_0 = (\omega_z/\omega_p)$ as parameter.

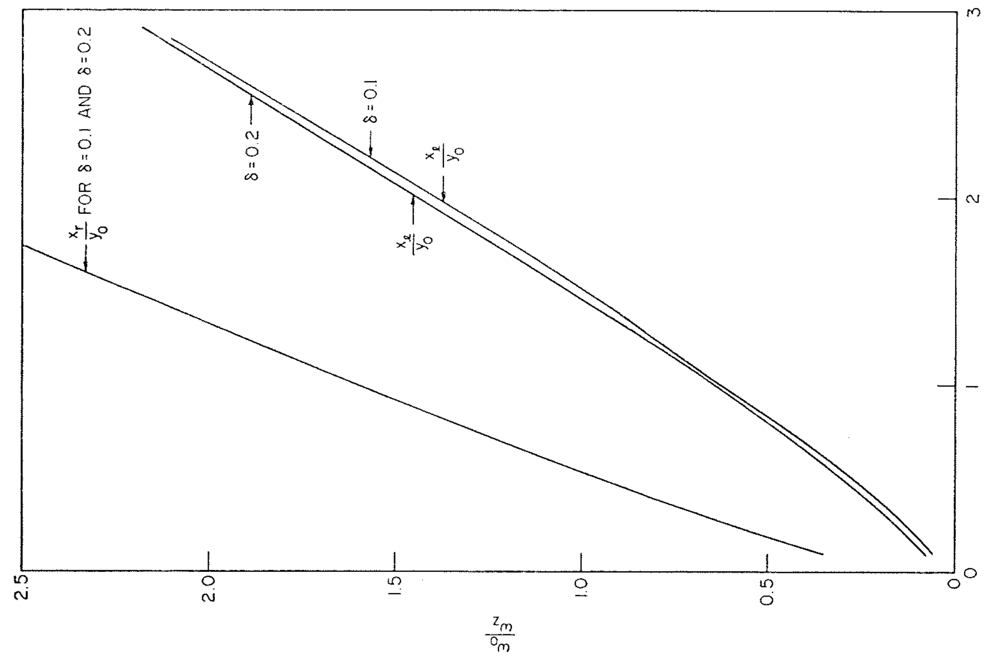

Fig. 4. The plot of (ω_0/ω_z) vs. (ω_p/ω_z) with δ as parameter.

shown in Fig. 3, and the plot of (ω_0/ω_z) vs. (ω_p/ω_z) for various values of δ is shown in Fig. 4.

It is of interest to observe that in Fig. 2 x_r increases with y_0 while x_l decreases with y_0 in the range of small y_0, for the case $\delta > 0{\cdot}1$, and increases with y_0 in the range of large y_0. A change in the value of δ appears to affect the value of x_l significantly, but has only a slight effect on x_r. The variation of x_l with δ for a given value of y_0 is shown in Fig. 3. x_l appears to increase with δ faster for the larger value of y_0 than for the small value of y_0. It is observed in Fig. 4 that for a given B_0 and δ, the cutoff frequency of transverse electromagnetic waves increases with the electron density monotonically. In view of the fact that δ is proportional to the square of the static electric field E_0 and inversely proportional to the temperature T, the above observations suggest that for a given B_0 either a decrease in T or an increase in E_0 will cause ω_{0l} to increase while ω_{0r} decreases slightly in the range of the value of y_0 chosen.

4. Concluding Remarks

It should be noted that the cutoff frequency ω_0 in the cutoff condition, equation (16), is expressed in terms of the electron cyclotron frequency ω_z, the electron plasma frequency ω_p, and δ which is the square of the ratio of the electron drift velocity (E_0/B_0) to the thermal velocity $\sqrt{(2KT/m)}$. Since ω_z is proportional to the static magnetic field B_0 and ω_p is proportional to $\sqrt{N}$, with N being the number density, equation (16) gives the relation between ω_0, E_0, B_0, N and T. Suppose that E_0, B_0, and N are known in the region of an electron gas under consideration; then by observing the cutoff frequency of a circularly polarized electromagnetic wave, the temperature T of the electron gas can be, in principle, determined with the aid of equation (16). This suggests a possible diagnostic technique for the measurement of electron gas temperature.

On the other hand, most analyses of electromagnetic wave propagation in the ionospheric plasma in the past appear to have been concerned primarily with the effect of the Earth's magnetic field, but little or no attention has been given to the effect of static electric field which may be present in the ionosphere. However, the existence of electrostatic fields in the ionosphere and magnetosphere has been postulated by various workers in the studies of various ionospheric phenomena, for example, in the formation of F-region irregularities (e.g. Dagg, 1957; Farley, 1959, 1960; Spreiter and Briggs, 1961).

The method of analysis developed in this paper can be profitably applied to the investigation of transverse circularly polarized electromagnetic wave propagation along the geomagnetic field line in the ionospheric plasma in which the presence of a transverse static electric field may be important. For example, a whistler propagating along the geomagnetic field line between two hemispheres might encounter a region of the ionosphere or magnetosphere in which a static electric field is perpendicular to the geomagnetic field. Then a natural question arises as to what is the effect of this static electric field on the propagation characteristics of electromagnetic waves, if there is any? To answer this type of question e.g. dispersion equation (8) together with equations (10) can be used.

Furthermore it should be pointed out that it is not difficult to extend the present method of analysis to include the effect of ion motion, as well as collisions. In this case the element R_{pq} in equations 10 and the cutoff condition will be modified. The application of the theory developed in this paper to the ionospheric plasma is to be considered in detail in a future paper.

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Effect of longitudinal electrostatic field on the whistler mode propagation in a warm magneto-ionic medium

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Abstract—The dispersion equation for a whistler mode propagation in a warm plasma, subjected to parallel static electric and magnetic fields, is derived by using a linearized coupled Vlasov–Maxwell equation. From the derived dispersion equation, the amplitude constant α and phase constant β of the whistler mode are expressed in terms of static electric field E_0, static magnetic field B_0, the electron number density N_0, the electron temperature and the wave angular frequency ω. The effect of a weak static electric field on the propagation of a whistler mode is investigated in detail; the whistler mode may be amplified or attenuated according to whether $\mathbf{E}_0$ and $\mathbf{B}_0$ are in the same direction or in the opposite direction. The spatial rate of change of the wave amplitude and phase velocity of the whistler mode increase with $|E_0|$ in general. For a low-frequency wave propagation, α is directly proportional to E_0, ω, and N_0, and inversely proportional to $B_0{}^3$.

A whistler mode propagation in the magnetosphere is also considered. The results of study show that the effects of a static electric field on the propagation characteristic of a whistler mode are likely to be more important in the region of low geomagnetic latitude and high altitude rather than in the high latitude region.

1. Introduction

Most analyses of electromagnetic wave propagation in the ionospheric plasma in the past appear to have been concerned primarily with the effect of the earth's magnetic field (e.g. Ratcliffe, 1959, Ginzburg and Budden, 1961), but little or no attention has been given to the effect of a static electric field which may be present in the ionosphere. This is perhaps because, in general, the static electric field effects were presumed to be small. However the existence of a static electric field in the ionosphere or magnetosphere has been postulated by various workers in the studies of various ionospheric phenomena, for example, in the formation of F-region irregularities (Dagg, 1957; Alfvén, 1964; Willmore, 1966). It is also believed that the existence of turbulence in the magnetosphere must necessarily lead to the existence of a weak static electric field along the direction of the magnetostatic field in the steady state (Reid, 1965). Recently the experimental evidence of electrostatic field in auroral ionosphere has been reported by Mozer and Bruston (1967); e.g., the component of the d.c. electric field along the magnetic field direction had a magnitude as large as 20 mV/m. The importance of electric fields to magnetospheric and ionospheric phenomena appears to be well established by experiment and theory.

In view of the above, a natural question arises as to what, if any, is the effect of an electrostatic field on the propagating electromagnetic wave along the earth's magnetic field line in the ionospheric plasma. The answer to this question will be of interest since the electromagnetic wave under consideration might be of a man-made radio signal used in a communication or used as a diagnostic tool for probing the plasma condition. It might also be of a radio noise of natural origin, such as whistler or v.l.f. emission, propagating in the ionosphere.

The whistler mode propagation through the ionospheric plasma has been discussed in detail by a number of workers (Storey, 1953; Ratcliffe, 1959; Helliwell and Morgan, 1959; Smith, 1961; Scarf, 1962; Gallet, 1964; Helliwell, 1965). On the other hand, the effect of a static electric field on the longitudinal propagation of circularly polarized waves in a finite temperature magneto-plasma has been investigated by this author. It is found that the presence of a transverse static electric field causes the cutoff frequency of the electromagnetic wave to shift (Hsieh, 1967*a*); in addition, it leads to a coupling of the longitudinal mode to a transverse circularly polarized mode (Hsieh, 1967*b*), whereas the presence of a longitudinal static electric field may significantly affect the amplitude and phase of the electromagnetic waves.

The purpose of this paper, therefore, is to discuss in detail the effect of a longitudinal static electric field upon the whistler mode propagation in the magneto-ionic medium, as defined by Ratcliffe (1959). The thermal motion of an electron is considered although ion motion and collision effects are assumed to be negligible. The present discussion is based on a small-amplitude, one-dimensional analysis in which all time-varying quantities are assumed to have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the wave angular frequency and propagation constant respectively. z and t denote, respectively, the spatial and time variables.

2. Dispersion Relation

Consider all quantities of interest to be composed of two parts; the time-independent part denoted by the subscripts '0', and the time-dependent part denoted by the subscript '1', e.g., the magnetic flux density $\mathbf{B}$ and electric field intensity $\mathbf{E}$ are written as

$$\mathbf{B} = \mathbf{B}_0(z) + \mathbf{B}_1(z, t) \text{ and } \mathbf{E} = \mathbf{E}_0(z) + \mathbf{E}_1(z, t)$$

and the electron distribution function f is written as

$$f = f_0(z, \mathbf{v}) + f_1(z, \mathbf{v}, t).$$

Suppose that the static electric field $\mathbf{E}_0$ and the magnetostatic field $\mathbf{B}_0$ are both directed along the positive z-direction. The dynamic electromagnetic fields in the electron gas are governed by Maxwell's field equation, which is expressed in the following manner:

$$E_{\pm} = \frac{-j\left(\dfrac{\omega e}{\varepsilon_0}\right)}{(\omega^2 - c^2k^2)} \int v_r e^{\pm j\varphi} f_1 \, d^3v,$$

$$E_{1z} = \frac{-je}{\omega\varepsilon_0} \int v_z f_1 \, d^3v, \tag{1}$$

where $c = 1/\sqrt{(\mu_0\varepsilon_0)}$ is the speed of light in free space, and μ_0 and ε_0 denote the permeability and dielectric constant of vacuum; $d^3v = (v_r \, dv_r \, d\varphi \, dv_z)$ is a volume element in velocity space. On the other hand, the electron distribution is described by the Vlasov equation, which is written as

$$\left[j(\omega - kv_z) + \omega_z \frac{\partial}{\partial \varphi}\right] f_1 - a_z \frac{\partial f_1}{\partial v_z} = \frac{e}{m} M_-(f_0)E_- e^{j\varphi} + \frac{e}{m} M_+(f_0)E_+ e^{-j\varphi} + \frac{e}{m} E_{1z} \frac{\partial f_0}{\partial v_z}, \tag{2}$$

where

$$\begin{aligned} E_\mp &= \tfrac{1}{2}(E_{1x} \mp jE_{1y}), & B_\mp &= \tfrac{1}{2}(B_{1x} \mp jB_{1y}), \\ a_z &\equiv \left(\frac{eE_0}{m}\right), & \omega_z &= \left(\frac{eB_0}{m}\right), \\ M_\mp(f_0) &= \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_0}{\partial v_r} \pm \frac{j}{v_r}\frac{\partial f_0}{\partial \varphi}\right) + \frac{kv_r}{\omega}\frac{\partial f_0}{\partial v_z}\right], \\ v_x &= v_r \cos\varphi, & v_y &= v_r \sin\varphi. \end{aligned} \tag{3}$$

Here E_- and E_+ denote the left and right-hand circularly polarized components of electric field respectively. E_{1z} is the longitudinal component of electric field. ω_z is the electron cyclotron frequency.

Suppose that the time-varying distribution function f_1 is written in the form

$$f_1(z, t, v_r, v_z, \varphi) = f_-(z, t, v_r, v_z)e^{j\varphi} + f_+(z, t, v_r, v_z)e^{-j\varphi} + g(z, t, v_r, v_z) \tag{4}$$

in which the first, second and third terms of the right-hand side can be regarded as the left-hand circularly polarized, the right-hand circularly polarized, and the longitudinal components of the distribution function respectively. Then in view of the fact that equation (2) must be valid for an arbitrary value of φ, the substitution of equation (4) into equation (2) yields a system of equations, expressing the function f_-, f_+ and g in terms of the electric field components E_-, E_+ and E_{1z} as follows:

$$\begin{aligned} j(\omega - kv_z + \omega_z)f_- - a_z \frac{\partial f_-}{\partial v_z} &= \frac{e}{m} M_-(f_0)E_-, \\ j(\omega - kv_z - \omega_z)f_+ - a_z \frac{\partial f_+}{\partial v_z} &= \frac{e}{m} M_+(f_0)E_+, \\ j(\omega - kv_z)g - a_z \frac{\partial g}{\partial v_z} &= \frac{e}{m}\frac{\partial f_0}{\partial v_z} E_{1z}. \end{aligned} \tag{5}$$

For the case where f_-, f_+ and g have v_z dependence of the form $e^{-\alpha_e v_z^2}$, in which $\alpha_e \equiv (m/2KT)$, they can be expressed explicitly in terms of E_-, E_+ and E_{1z} as follows:

$$f_\mp = \frac{\frac{e}{m} M_\mp(f_0)E_\mp}{j(\tilde{b} \pm \omega_z)} \quad \text{and} \quad g = \frac{\frac{e}{m}\frac{\partial f_0}{\partial v_z} E_{1z}}{j\tilde{b}}, \tag{6}$$

where

$$\tilde{b} = (\omega - \tilde{k}v_z), \qquad \tilde{k} = k + jK_1 \qquad \text{and} \qquad K_1 \equiv \left(\frac{eE_0}{KT}\right).$$

K and T denote the Boltzmann constant and the electronic temperature respectively.

Upon substituting equation (6) into equation (4) then combining with equation (1) the following set of equations is obtained:

$$1+\frac{\pi\left(\frac{\omega e}{\varepsilon_0}\right)}{(\omega^2-c^2k^2)}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\frac{e}{m}M_{\mp}(f_0)}{(\tilde{b}\pm\omega_z)}v_r^2\,dv_r\,dv_z=0 \tag{7a}$$

and

$$1+\frac{2\pi e}{\omega\varepsilon_0}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e}{m}\frac{1}{\tilde{b}}\frac{\partial f_0}{\partial v_z}v_r v_z\,dv_r\,dv_z=0. \tag{7b}$$

Equation (7b) is the dispersion relation for the longitudinal mode. Equation (7a) represents the dispersion relation for the transverse circularly polarized modes. The upper sign in this equation is taken for the left-hand circularly polarized mode and the lower sign is for the right-hand circularly polarized mode, which is of interest to the present investigation. Once the time-independent distribution function f_0 is known, the indicated integration in equation (7a) can be carried out. f_0, which must satisfy the time-independent part of the Vlasov equation for the case of a sufficiently weak static electric field in an electrically neutral electron gas, can be written in a Maxwellian form as follows:

$$f_0=N_0\left(\frac{\alpha_e}{\pi}\right)^{3/2}\exp\left[-\alpha_e(v_r^2+v_z^2)\right], \tag{8}$$

where $\alpha_e=(m/2KT)$ and N_0 is the electron number density. Substitution of equation (8) into equation (7a) yields, for the right-hand circularly polarized mode, the following dispersion relation:

$$1-\frac{c^2k^2}{\omega^2}=\frac{X}{(1-Y)}\left[1+\frac{V_1^2\tilde{k}^2}{2\omega^2}\frac{1}{(1-Y)^2}\right], \tag{9}$$

where

$$X\equiv\left(\frac{\omega_p^2}{\omega^2}\right),\qquad Y\equiv\left(\frac{\omega_z}{\omega}\right),\qquad V_1\equiv\sqrt{\left(\frac{2KT}{m}\right)},$$

$$\tilde{k}\equiv(k+jK_1),\qquad K_1\equiv\left(\frac{eE_0}{KT}\right),\qquad \omega_z\equiv\left(\frac{eB_0}{m}\right),$$

$$\omega_p\equiv\sqrt{\left(\frac{e^2N_0}{m\varepsilon_0}\right)}.$$

It should be pointed out that the derivation of equation (9) involves an evaluation of an integral of the form

$$G_0(x)\equiv\frac{j}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\alpha_e v_z^2}}{(v_z-\chi)}\,dv_z, \tag{10}$$

where $\chi=(1-Y)\omega/\tilde{k}$.

This integral has been discussed in detail by Stix (1962) and his result is used here. When the term representing the Landau or cyclotron damping is neglected and by taking only the first two terms of its asymptotic expansion, $G_0(\chi)$ can be given as

$$G_0(\chi) = \frac{-j}{\sqrt{(\alpha_e\chi)}}\left(1 + \frac{1}{2\alpha_e\chi^2}\right) \tag{11}$$

provided that $|\sqrt{\alpha_e\chi}|^4 \gg 1$ is satisfied. The case where $|\sqrt{\alpha_e\chi}| < 1$ has been discussed by SCARF (1962) in connection with the study of LANDAU damping and attenuation of whistlers.

It is of interest to note that when $V_1 = 0$, equation (9) is reduced to the familiar dispersion equation in the cold-plasma magneto-ionic theory.

3. PROPAGATION CONSTANTS

Equation (9) is a quadratic in k and can be conveniently rearranged as

$$Pk^2 + j2Sk - (Q + W) = 0, \tag{12}$$

where

$$P = \frac{c^2}{\omega^2}\left[1 - \frac{\omega^2}{c^2}\frac{2S}{K_1}\right],$$

$$W = 2K_1S, \qquad S = \frac{K_1V_1^2}{4\omega^2}\frac{X}{(1 - Y)^3}$$

and

$$Q = 1 - \frac{X}{(1 - Y)}. \tag{13}$$

For a real wave angular frequency ω, the factor P, Q, S, and W are real and equation (12) yields two complex roots which can be given as follows:

$$k = \pm\beta - j\alpha, \tag{14a}$$

where

$$\alpha = \frac{S}{P},$$

and

$$\beta = \sqrt{\left(\frac{(Q + W)}{P} - \alpha^2\right)}. \tag{14b}$$

In view of the fact that the time and spatial dependence is assumed to be in the form $\exp\,[j(\omega t - kz)]$, k given by equation (14a) with the upper sign corresponds to the propagation constant of the forward wave, and that with the lower sign corresponds to the propagation constant of the backward wave. Moreover a positive α represents the attenuation, while a negative α represents the amplification of the wave. Having determined the complex propagation constant k, the variation of the amplitude and phase of the wave with respect to various system parameters can be examined in detail. For convenience of discussion the amplitude coefficient (α/β_0) and phase coefficient (β/β_0), where $\beta_0 \equiv (\omega/c)$, are expressed as

$$\left(\frac{\alpha}{\beta_0}\right) = -\tfrac{1}{2}\delta Z \qquad \text{and} \qquad \left(\frac{\beta}{\beta_0}\right) = n_0C, \tag{15a}$$

where

$$Z = \frac{r}{(1 - \frac{1}{2}\tau r)}, \qquad n_0 = \sqrt{\left(\frac{1 + q}{1 - \frac{1}{2}\tau r}\right)}$$

and

$$C = \left[1 - 2\frac{\delta^2}{\tau}\frac{r}{(1 + q)}\left(1 + \frac{\tau}{8}Z\right)\right]^{1/2} \tag{15b}$$

in which

$$q \equiv \frac{X}{(Y - 1)}, \qquad r \equiv \frac{X}{(Y - 1)^3}, \qquad \tau \equiv \left(\frac{2KT}{mc^2}\right)$$

and

$$\delta \equiv \left(\frac{eE_0}{mc\omega}\right). \tag{15c}$$

It should be noted that the amplitude coefficient (α/β_0) is proportional to δ, which in turn is proportional to E_0. For a whistler mode of interest, $(\tau r)/2 < 1$ so that $Z > 0$. Consequently the amplitude constant α can be either positive or negative according to whether $E_0 < 0$ or $E_0 > 0$. In view of the fact that the direction of $\mathbf{B}_0$ is taken in the positive z-direction, which is also that of the wave propagation, $E_0 < 0$ means that the electrostatic field is directed in the direction opposite to the wave vector, while $E_0 > 0$ signifies that $\mathbf{E}_0$ is in the direction of the wave vector. For the case $\alpha > 0$ (i.e., $E_0 < 0$) the whistler mode under consideration suffers an attenuation, whereas for the case $\alpha < 0$ (i.e., $E_0 > 0$) it experiences an amplification. The attenuation corresponds to the absorption of whistler mode electromagnetic wave energy by the plasma electrons, while the amplification corresponds to the situation of extraction of energy from the plasma.

On the other hand it should be noted that the phase coefficient (β/β_0) is nothing but the refractive index (c/v_0), with v_0 being the phase velocity of the wave, and n_0 representing the refractive index for the case of zero-static electric field, while the factor C represents the correction factor due to the presence of the static electric field. Thus the effects of E_0 upon the propagation of the whistler mode under consideration are characterized by equation (15a). The numerical illustration of equation (15a) is shown in Figs. 1 and 2 for a conveniently chosen set of parameters. The variations of the amplitude coefficient (α/β_0), and the phase coefficients (β/β_0) with the parameters X and Y for a fixed value of δ and τ, are illustrated in Figs. 1(a) and 1(b) respectively. It is observed from Fig. 1(a) that (α/β_0) increases monotonically with X when Y is fixed, whereas it decreases as Y increases when X is fixed. Moreover the rate of increase of (α/β_0) with respect to X decreases as Y increases. Thus, Fig. 1(a) suggests that the spatial rate of change of the wave amplitude increases with the electron number density N_0 when the wave angular frequency ω and magnetostatic field B_0 are fixed, whereas it decreases with an increase in the strength of the magnetostatic field when N_0 is fixed.

Figure 1(b) indicates that the phase coefficient (β/β_0) increases with X when Y is fixed, whereas it decreases as Y increases when X is fixed. Thus, Fig. 1(b) suggests that the phase velocity of the whistler mode under consideration decreases as the

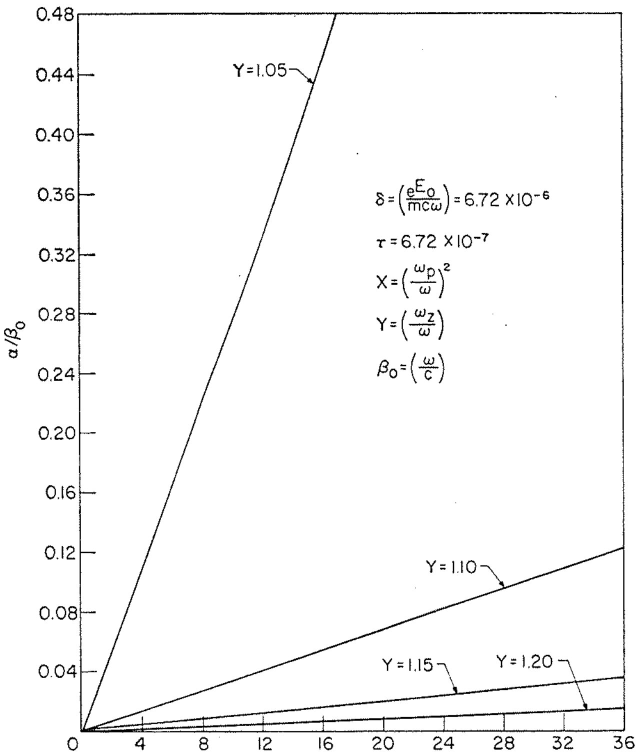

Fig. 1(a). The plot of (α/β_0) vs. X with Y as parameter for $\delta = 6{\cdot}72 \times 10^{-6}$ and $\tau = 0{\cdot}672 \times 10^{-6}$ (or T = 2000°K)

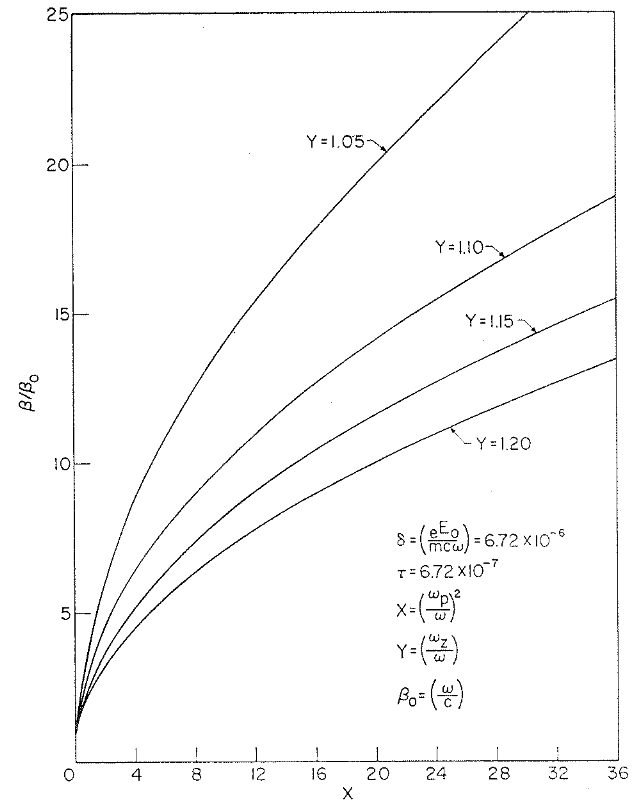

Fig. 1(b). The plot of (β/β_0) vs. X with Y as parameter for $\delta = 6{\cdot}72 \times 10^{-6}$ and $\tau = 0{\cdot}672 \times 10^{-6}$ (or T = 2000°K).

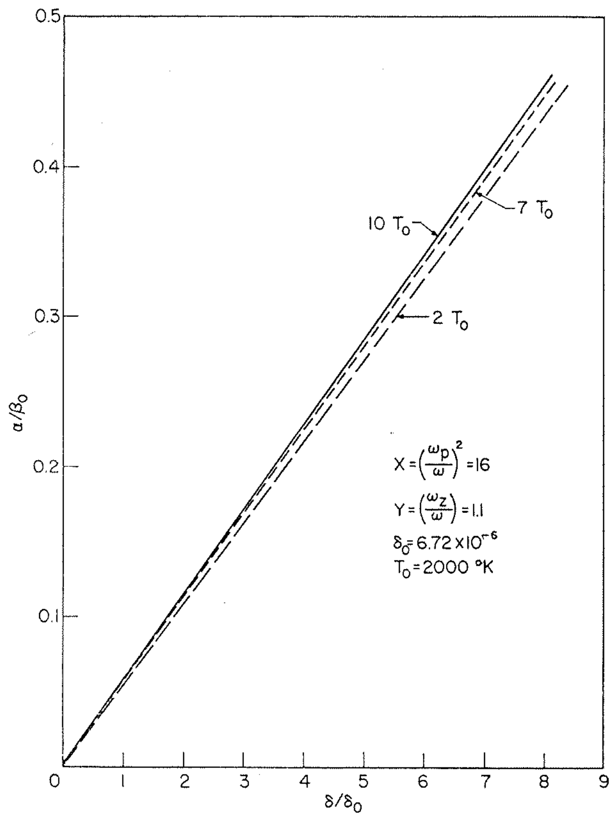

Fig. 2(a). The plot of (α/β_0) vs. δ with τ as parameter for $X = 16{\cdot}0$ and $Y = 1{\cdot}1$.

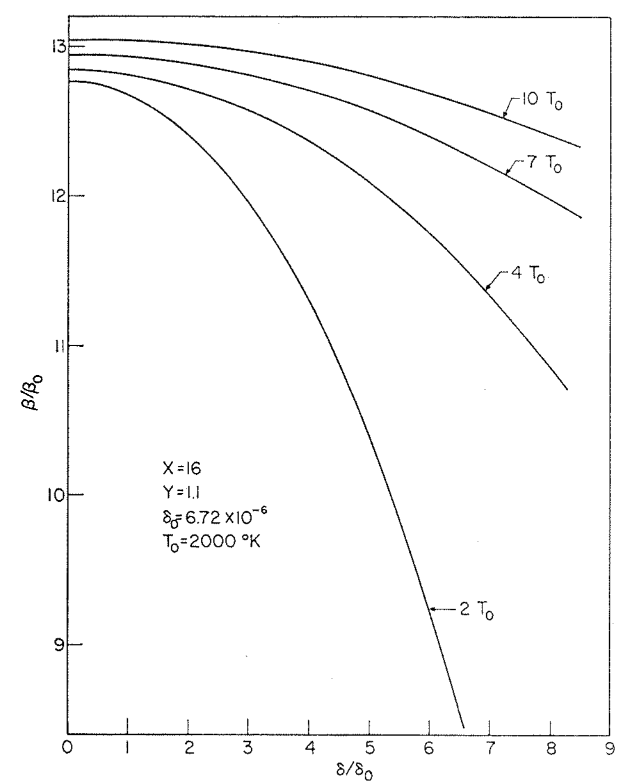

Fig. 2(b). The plot of (β/β_0) vs. δ with τ as parameter for $X = 16{\cdot}0$ and $Y = 1{\cdot}1$.

electron number density N_0 increases when ω and B_0 are fixed, whereas it increases as B_0 increases when N_0 is fixed.

On the other hand, the variation of (α/β_0) and (β/β_0) with the parameters δ and τ for a fixed value of X and Y is shown in Figs. 2(a) and 2(b) respectively. It is observed from Fig. 2(a) that (α/β_0) is directly proportional to the parameter δ and it increases with τ when δ is fixed. Thus, Fig. 2(a) suggests that the spatial rate of change of the wave amplitude increases proportionally with E_0 when N_0, B_0, ω and τ are fixed. Figure 2(b) indicates that (β/β_0) decreases with δ when τ is fixed, whereas it increases with τ when δ is fixed. Figure 2(b) thus suggests that for a given value of N_0, B_0, and ω, the phase velocity of the whistler mode increases with $|E_0|$ when T is fixed, whereas it decreases as T increases when E_0 is fixed.

Figures 2(a) and 2(b) together tend to suggest that the presence of an electrostatic field in the system may modify both the amplitude and phase of the whistler mode under consideration significantly. On the other hand, a change in the electron temperature T appears to have only little effect upon the amplitude, but significant effect on the phase of the whistler mode.

It is of interest to note that when the parameters X, Y, and τ are such that

$$1 \ll Y \ll X \qquad \text{and} \qquad |X/Y^3| \ll 1/\tau, \tag{16}$$

equations (15) yield

$$(\alpha/\beta_0) = -\tfrac{1}{2}\delta\sigma^2/Y \qquad \text{and} \qquad (\beta/\beta_0) = \sigma\sqrt{\left(Y^2 - \frac{2\delta^2}{\tau}\right)}, \tag{17}$$

where $\sigma \equiv (\omega_p/\omega_z)$. Equation (17) can also be written as

$$\alpha = -\frac{1}{2}\left(\frac{E_0}{cB_0}\right)\left(\frac{\omega_p{}^2}{\omega_z{}^2}\right)\left(\frac{\omega}{c}\right) \tag{18a}$$

and

$$\left(\frac{v_0}{c}\right) = \sqrt{\frac{\omega_z\omega}{\omega_p{}^2}}\frac{1}{\sqrt{1-\left(\frac{m}{KT}\right)\left(\frac{E_0}{B_0}\right)^2}} \tag{18b}$$

which suggests that, for the low frequency whistler mode propagation, the spatial rate of change of wave amplitude α is directly proportional to E_0, N_0, and ω, and inversely proportional to $(B_0{}^3)$. In other words, for larger N_0, there will be more electrons available for participating in the exchange of energy with the electromagnetic wave. On the other hand, the phase velocity v_0 of the whistler mode is proportional to $\sqrt{(\omega)}$ and it increases as E_0 increases regardless of the algebraic sign of E_0, whereas it decreases as the electron temperature T increases.

4. 'Whistler Mode' Propagation in Magnetosphere

A class of the electromagnetic wave of natural origin known as 'whistler' and 'v.l.f. emissions' can propagate through the ionosphere and magnetosphere along the geomagnetic field line in the whistler mode (Helliwell, 1965) when the wave frequency ω is in the proper range such that $\omega < \omega_z < \omega_p$.

The magnetosphere can be regarded as consisting of a neutral (hydrogen) plasma imbedded in the geomagnetic field. The background plasma density varies in space and time and the field is approximately a dipole subject to small perturbation and the distortion of the solar wind. Above the ionosphere (e.g., 1000 km) the medium is slowly varying electromagnetically in the sense that the change in number density N_0 is small in the space of a wavelength (Ratcliffe, 1959). Therefore, the propagation characteristics of the uniform medium which are given in the preceding section are applicable locally in the magnetosphere under a quiescent condition. If the path integral of the amplitude constant α is assumed to describe the net growth or decay of wave amplitude, then the amplification or absorption of power in a wave which propagates through a slowly varying medium is given by

$$A(d\mathrm{B}) = -10 \log_{10} \left[\exp \left\{ \int_{s_1,\ \mathrm{path}}^{s_2} 2\alpha(\omega, s)\, ds \right\}\right], \tag{19}$$

where α is the local amplitude exponent defined by equations (15a) or (18a), and $s_1 \leqq s \leqq s_2$ specifies the region where $\mathbf{E}_0$ and $\mathbf{B}_0$ are parallel and directed along the direction of wave propagation. ds is a differential path length along the geomagnetic field line. It should be noted that α depends on the quantities ω_z, ω_p, T and E_0, which are in turn dependent on position along the path in general. To evaluate the integral in equation (19), therefore, a knowledge of the variation of system parameters ω_z, ω_p, T, and E_0 is required.

If the geomagnetic field is approximated by a pure dipole field, then the whistler path is defined by the equation for a field line

$$\frac{R}{R_0} = \frac{\cos \theta}{\cos \theta_0}, \tag{20}$$

where (R/R_0) is the radial distance in earth radii, θ is the geomagnetic latitude, θ_0 is the latitude of the path at the earth's surface and R_0 is the radius of the earth. The electron cyclotron frequency along this path has the form

$$\omega_z = \omega_{z0} \frac{(1 + 3 \sin^2 \theta)^{1/2}}{\cos^6 \theta}, \tag{21}$$

where ω_{z0} is its equatorial value ($= 5{\cdot}50 \times 10^6 \cos^6 \theta_0$ rad./sec).

As for the electron plasma frequency variation along this path the so-called "gyro-frequency model" has been used by several authors (Storey, 1953; Gallet, 1959; Smith, 1961; Dowden, 1961) in the study of whistlers. In this model it is assumed that the electron density tends to be proportional to the strength of the earth's magnetic field, i.e.,

$$\omega_p{}^2 = \kappa \omega_z, \tag{22}$$

where κ is a constant which, in general, may depend on θ_0. Equation (22) represents a good empirical model for $45° \leqq \theta_0 \leqq 65°$.

On the other hand the electron temperature T within the magnetosphere is not uniform in general (Chapman, 1960) so that T varies along the whistler path. However for most whistler analyses T has been regarded as constant along the path.

With regard to the strength and variation of the electrostatic field E_0 along the geomagnetic field line, it appears that no experimentally observed data has ever been reported in the literature. In his theoretical estimate Reid (1965) showed that if the source level at which the electrostatic potential originates were taken at the equatorial plane, which lies 42560 km above the surface of the earth, measured along the field line originating at 65° geomagnetic latitude, then a longitudinal electrostatic field will be about 5×10^{-7} V/m throughout the magnetosphere. However this estimate does involve a certain amount of guesswork as to the magnitude of source potential. Thus using equations (15) and (20), and assuming a uniform temperature and electrostatic field along the path in the magnetosphere, the absorption or amplification A, given by equation (19), can in principle be evaluated. However, for an illustration, it is of interest to consider the case in which the wave angular frequency ω is much smaller than the minimum gyrofrequency along the path, ω_{z0}, i.e., $\omega \ll \omega_{z0}$. If the electron temperature T in the magnetosphere is assumed to be in the range between 2000°K and 20,000°K, then the value of (V_1/c) will be in the range between $0{\cdot}82 \times 10^{-3}$ and $2{\cdot}57 \times 10^{-3}$. Since along the whistler path $\omega_z < \omega_p$, it is not difficult to see that the inequalities (16) are satisfied, so that equations (18a) and (18b) are applicable to the analysis of the low-frequency components of a whistler or v.l.f. emission in the magnetosphere. Consequently, with the aid of equations (21) and (22), equations (18a) and (18b) become respectively

$$\alpha = -0{\cdot}975 \times 10^{-6}\left(\frac{E_0\omega\kappa}{\omega_{z0}^2}\right)\psi(\theta), \text{ nepers/m} \tag{23a}$$

$$\frac{v_0}{c} = \sqrt{\frac{\omega}{\kappa}}\left[1 - \left(4{\cdot}52 \times 10^7 \frac{E_0}{T\omega_{z0}}\right)^2 \psi(\theta)\right]^{-1/2}, \tag{23b}$$

where

$$\psi(\theta) = \frac{\cos^{12}\theta}{(1 + 3\sin^2\theta)}. \tag{23c}$$

The plot of $\psi(\theta)$ vs. θ (in Fig. 3a) shows that the function $\psi(\theta)$ decreases monotonically with the geomagnetic latitude θ from $\psi(0) = 1$ to $\psi(45°) = 0{\cdot}01$. The value of $\psi(\theta)$ for $\theta > 45°$ is negligibly small compared to $\psi(0)$. Consequently equation (23a) suggests that the amplitude constant α along the whistler path under consideration has a larger value in the lower latitude region, near the equatorial zone, than in the high latitude region.

On the other hand the absorption or amplification along the path, from the equatorial plane to a height of h km above the earth's surface, can be given by equation 19 as

$$A(d\text{B}) = 54\left(\frac{E_0\omega\kappa}{\omega_{z0}^2}\right)\Psi(\theta_1) \tag{24a}$$

where

$$\Psi(\theta_1) = \int_0^{\theta_1} \frac{\cos^{13}\theta}{(1 + 3\sin^2\theta)^{1/2}}\, d\theta \tag{24b}$$

in which θ_1 is given by

$$\frac{\cos\theta_0}{R_0} = \frac{\cos\theta_1}{(R_0 + h)}. \tag{24c}$$

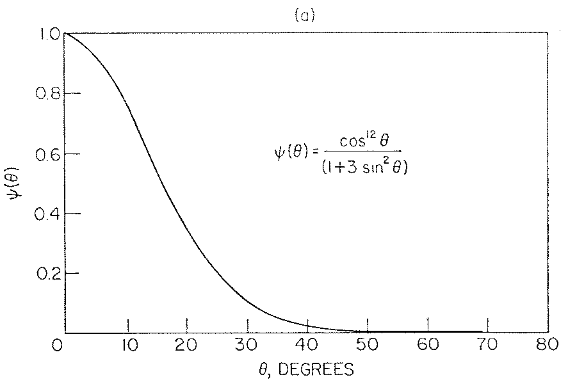

Fig. 3(a). The plot of $\psi(\theta)$ vs. θ.

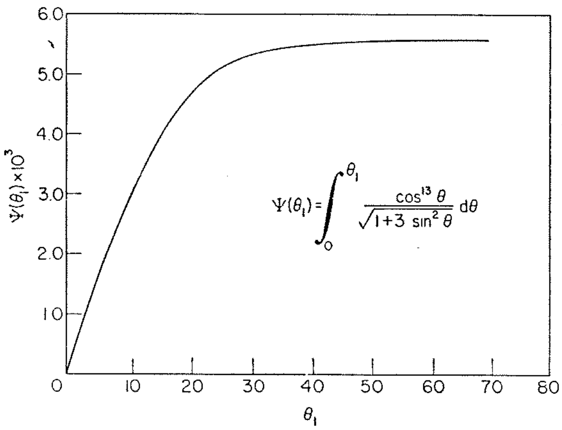

Fig. 3(b). The plot of $\Psi(\theta_1)$ vs. θ_1.

The plot of $\Psi(\theta_1)$ vs. θ_1 (in Fig. 3(b)) shows that the function $\Psi(\theta_1)$ increases monotonically from 0 at $\theta_1 = 0$ to $5{\cdot}5 \times 10^{-3}$ at $\theta_1 = 45°$, then $\Psi(\theta_1)$ remains constant for $\theta_1 > 45°$.

It should be noted that there is a path which has frequently been considered by various workers (e.g., SMITH, 1961; BRICE, 1964; LIEMOHN, 1967) in the study of whistlers. It is the path specified by $\theta_0 = 60°$ or $R(\theta = 0) = 4R_0$, which gives $\omega_{z0} = 8{\cdot}6 \times 10^4$ rad./sec or $f_z = 13{\cdot}7$ kc/sec. Along this path the value of κ, which appeared in equation 22, can be taken as $\kappa = 2\pi \times 10^6$ rad./sec. Then for the frequency range $f \leqq 1$ kc/sec, equations (23a) and (24) can be used. For an illustration, suppose that the electrostatic field along the path is taken as $E_0 = -5 \times 10^{-7}$ V/m; then a whistler mode electromagnetic wave with a frequency of one kc/sec will suffer an attenuation of $2{\cdot}64 \times 10^{-12}$ nepers/m at most. In this case the absorption for $\theta_1 = 45°$ will be $8{\cdot}03 \times 10^{-7}$ dB which is negligibly small. However, when a path with larger values of θ_0 and E_0 is considered, both α and A can be increased considerably.

5. CONCLUDING REMARKS

In considering a whistler mode propagation an attempt has been made to take into account the effect of a weak static electric field which might be present in a warm magneto-ionic medium. The equilibrium distribution function of the electron, f_0, is assumed to be Maxwellian. Furthermore, it is also assumed that the static electric field E_0 in the direction of the magnetostatic field $\mathbf{B}_0$ is sufficiently weak so that the drift motion in the direction of wave propagation is small and does not modify the distribution function f_0 significantly. The drift velocity of electrons due to $\mathbf{E}_0$ is assumed to be negligibly small in comparison to the wave phase velocity. Thus the medium under consideration is regarded as essentially stationary under a weak static electric field.

The effect of the presence of electrostatic field on a whistler mode electromagnetic wave propagation is characterized by equation (15). The method of analysis discussed in Section 3 is most likely to be adequate for consideration of a whistler mode propagation in the magnetosphere where the collision effect is usually regarded as negligible. The application of the theory is illustrated in Section 4 for a low frequency wave. The effects of electrostatic field E_0 on the propagation of these low frequency waves is characterized by equation (23), which suggests that the electrostatic field effect on the amplitude and phase velocity of the whistler mode will likely be more significant in the region of low geomagnetic latitude and high altitude rather than in the high latitude region. In view of the lack of knowledge regarding the strength and variation of the electrostatic field in the magnetosphere, a reasonably accurate numerical estimation of the value of α or A is not possible at this time. However, the method illustrated in Section 4 for estimating the value of α or A can be used profitably when more data regarding E_0 become available in the future. Finally, it should be pointed out that the method of analysis developed in this paper can be extended to include such effects as ion-motion, collisions, and the drift motion of a plasma particle under the influence of a strong electrostatic field. If this is done then it can also be used for the consideration of electrostatic field effects on a whistler mode propagation in the ionosphere, particularly in the F-region where collision effects may be significant and the electrostatic fields are believed to be present.

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Whistler mode propagation in the ionosphere in the presence of a longitudinal static electric field*

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Abstract—The effect of a longitudinal electrostatic field on whistler mode propagation has been studied in a region of the ionosphere where the influence of particle collisions and ion motion may be significant. As an illustration, the propagation in two cases of interest has been considered; one deals with the E-region in the neighborhood of 100 km height, and the other with the F-region in the neighborhood of 250 km.

It is shown that the presence of a weak static electric field $\mathbf{E}_0$ tends to reduce the attenuation of the forward whistler mode when the magnetostatic field $\mathbf{B}_0$ and the electrostatic field $\mathbf{E}_0$ are in the same direction. Moreover for a sufficiently large value of $\mathbf{E}_0$ the whistler mode may experience an amplification. On the other hand, when $\mathbf{E}_0$ and $\mathbf{B}_0$ are in opposite directions an increase in $|\mathbf{E}_0|$ tends to increase the attenuation of the wave. The variation of the attenuation constant, phase constant, and the phase velocity of the whistler mode with the signal frequency and with the electrostatic field is also discussed in detail.

1. Introduction

The whistler mode propagation in a warm collisionless electro-magneto-ionic medium has been studied recently by Hsieh (1967) using the coupled Maxwell–Vlasov equations. The method of analysis given by Hsieh (1967) is suitable for the analysis of whistler mode propagation in the magnetosphere but it is not adequate for consideration of the propagation in the lower ionosphere where the particle collision effects are known to play an important role. However this method of analysis can be easily extended so that the effect of collisions can be taken into account. This is done by using the Boltzmann equation with a properly assumed collision term instead of the Vlasov equation.

The purpose of the present paper is to discuss the effect of a longitudinal electrostatic field on whistler mode propagation in a region of the ionosphere where the effects of collisons and ion motion may be of significance.

In the present paper the following assumptions are made:

1. All quantities of interest are to be composed of two parts, a time-independent part and a time-varying part.
2. Small amplitude conditions are satisfied.
3. A one-dimensional analysis is applicable.
4. All time-dependent quantities have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the angular frequency and the propagation constant respectively; t and z are the time and spatial variables respectively.

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5. The effective collision frequency of electrons ν_e and that of positive ions ν_i are independent of the particle velocity.

6. Electrical neutrality is satisfied.

7. The equilibrium distribution function of electrons and ions has the form of a Maxwellian.

8. The electrostatic field $\mathbf{E}_0$, which is directed parallel to the magnetostatic field $\mathbf{B}_0$, is sufficiently weak so that the drift velocities of charged particles are much smaller than the phase velocity of the whistler mode.

2. Dispersion Equation

Using the procedure of Hsieh (1967) and by combining the time-varying parts of the coupled Maxwell–Boltzmann equations the dispersion equation of the right-hand circularly polarized (whistler) mode for the system under consideration can be obtained and expressed as follows:

$$1 - \frac{c^2k^2}{\omega^2} = \frac{X_1}{W_1}\left(1 + \frac{V_1{}^2k_1{}^2}{2\omega^2}\frac{1}{W_1{}^2}\right) + \frac{X_2}{W_2}\left(1 + \frac{V_2{}^2k_2{}^2}{2\omega^2}\frac{1}{W_2{}^2}\right), \tag{1}$$

where

$$W_1 = [(1 - Y_1) - jZ_1], \qquad k_1 = (k + jK_1),$$

$$W_2 = [(1 - Y_2) - jZ_2], \qquad k_2 = (k + jK_2),$$

$$X_1 = \left(\frac{\omega_p}{\omega}\right)^2, \qquad Y_1 = \left(\frac{\omega_z}{\omega}\right), \qquad Z_1 = \left(\frac{\nu_e}{\omega}\right),$$

$$X_2 = \left(\frac{\Omega_p}{\omega}\right)^2, \qquad Y_2 = \left(\frac{\Omega_z}{\omega}\right), \qquad Z_2 = \left(\frac{\nu_i}{\omega}\right),$$

$$V_1 = \left(\frac{2KT_e}{m}\right)^{1/2}, \qquad K_1 = \left(\frac{eE_0}{KT_e}\right), \qquad \omega_z = \left(\frac{eB_0}{m}\right),$$

$$\omega_p = \left(\frac{e^2n}{m\varepsilon_0}\right)^{1/2}, \qquad V_2 = \left(\frac{2KT_i}{M}\right)^{1/2}, \qquad K_2 = \left(\frac{-eE_0}{KT_i}\right),$$

and

$$\Omega_z = \left(\frac{-eB_0}{M}\right), \qquad \Omega_p = \left(\frac{e^2N}{M\varepsilon_0}\right)^{1/2},$$

in which c = the speed of light in free space,
ε_0 = the dielectric constant of vacuum,
e = the electron charge,
m = the mass of electron,
M = the mass of positive ion,
T_e = the electron temperature,
T_i = the ion temperature,
K = the Boltzmann constant,
ν_e = the effective collision frequency of the electron and
ν_i = the effective collision frequency of the positive ion.

It should be noted that equation (1) is a quadratic in k and can conveniently be written as follows:

$$\tilde{A}\left(\frac{\tilde{k}}{\beta_0}\right)^2 + \tilde{B}\left(\frac{\tilde{k}}{\beta_0}\right) - \tilde{C} = 0, \tag{2a}$$

where

$$\begin{aligned}\tilde{A} &= \left[1 + \tau X\left(\frac{1}{\tilde{W}_1^{\,3}} + \frac{\xi^2}{\theta}\frac{1}{\tilde{W}_2^{\,3}}\right)\right],\\ \tilde{B} &= j2\delta X\left(\frac{1}{\tilde{W}_1^{\,3}} - \frac{1}{\tilde{W}_2^{\,3}}\right),\\ \tilde{C} &= \left[1 - X\left(\frac{1}{\tilde{W}_1} + \frac{\xi}{\tilde{W}_2}\right)\right],\end{aligned} \tag{2b}$$

in which

$$\tilde{W}_1 \equiv W_1, \qquad \tilde{W}_2 \equiv W_2, \qquad X \equiv X_1, \qquad Y \equiv Y_1,$$

$$\tau \equiv \left(\frac{KT_e}{mc^2}\right), \qquad \delta \equiv \left(\frac{eE_0}{mc\omega}\right), \qquad \xi \equiv \left(\frac{m}{M}\right), \qquad \theta \equiv \left(\frac{T_e}{T_i}\right),$$

$$\beta_0 = \left(\frac{\omega}{c}\right). \tag{2c}$$

The symbol $\sim$ appearing in equation (2) is introduced to emphasize the fact that the quantity under consideration is a complex quantity. In the present discussion the wave angular frequency ω is regarded as a real quantity and the propagation constant $\tilde{k}$ is regarded as a complex quantity, which can be written as

$$\tilde{k} = (\beta - j\alpha), \tag{3}$$

where α and β are the amplitude and phase constants respectively. In view of the fact that the time and spatial dependence is assumed to be of the form

$$\exp\left[j(\omega t - kz)\right] = \exp\left[-\alpha z + j(\omega t - \beta z)\right],$$

a positive β represents the forward wave, while a negative β denotes the backward wave. It is apparent that positive α represents attenuation and negative α represents amplification of the wave. The two roots of equation (2a) give the propagation constants of the forward and backward waves. Once the system parameters are specified, equation (2a) can be solved for $(\tilde{k}/\beta_0) = [(\beta/\beta_0) - j(\alpha/\beta_0)]$. Thus the investigation of the behavior of roots of equation (2a) provides information with regard to the variation of the amplitude and phase of the whistler mode with the various system parameters.

3. Whistler Mode Propagation in the E- and F-Regions of the Ionosphere

The E- and F-regions of the ionosphere are regarded as consisting of a partially ionized gas in which electrons, positive ions and neutral particles may interact with each other. To study the propagation of a whistler mode in such regions a knowledge of the plasma parameter is required. For example, in order to calculate the net growth or decay in amplitude of a whistler wave which propagates through a slowly varying medium, knowledge of variation of such quantities as n, N, β_0, E_0, T_e, T_i, ν_e and ν_i along the whistler path is required.

A theoretical analysis made by Nicolet (1953) shows that the electron collision frequency in the ionosphere depends on the neutral particle concentration in D- and E-regions and on the electron concentration in F-region. The collision frequency also depends on the temperature. For example, above the $F2$-layer, ν_{ei}, the collision frequency of electrons with ions, can be approximately given by [Nicolet, 1953]

$$\nu_{ei} = [34 + 8{\cdot}36 \log_{10} (T^{3/2}/n^{1/2})]nT^{-3/2}, \tag{4a}$$

where n is the electron density and T is the temperature of the plasma.

In the case of a plasma consisting of electrons and protons, the effective collision frequency of protons with electrons should be (see e.g., Ferraro and Plumpton, 1961)

$$\nu_p = (m/m_p)^{1/2}\nu_{ei}, \tag{4b}$$

where m_p is the mass of the proton.

Some experimental data with regard to the profiles of number density and temperature of electron and positive ions in the ionosphere can be found in Valley (1965).

There is little information available in the literature as to the strength and spatial variation of the electrostatic field E_0 in the ionosphere. However, Mozer and Bruston (1967) recently reported sounding rockets measurements of $E_0 = 20$ mV/m in the auroral ionosphere.

As an illustration of the method of analysis, the propagation characteristics of a whistler mode with frequency in the range between 1 and 20 kc/sec, and with $|E_0| < 10$ mV/m, are examined for two special cases of interest; Case I deals with the E-region in the neighborhood of height $h = 100$ km, and Case II deals with the F-region in the neighborhood of $h = 250$ km. The parameters chosen for this example are given below.

Cases	Region	ω_p rad./sec	ω_z rad./sec	T_e°K	T_i	ν_e rad./sec	ν_i rad./sec	$\xi = \frac{m}{M}$.
I	$h = 100$ km	$2{\cdot}3 \times 10^7$	10^7	270°	270°	$3{\cdot}14 \times 10^5$	$6{\cdot}28 \times 10^3$	$4{\cdot}0 \times 10^{-4}$
II	$h = 250$ km	4×10^7	$8{\cdot}8 \times 10^6$	2600°	1300°	$3{\cdot}14 \times 10^4$	$1{\cdot}83 \times 10^2$	$0{\cdot}34 \times 10^{-4}$

The values of parameters ω_p, ω_z, T_e, and T_i are taken from the data given by Valley (1965). The value of ν_e for Case I is taken from Shkarofsky (1961), and the value of ν_e for Case II is derived with the aid of equation (4a). Moreover for both cases it is assumed that $(\nu_i/\nu_e) = (m/M)^{1/2}$. For Case II the positive ion under consideration is taken to be an oxygen ion.

Having specified the physical parameters of the region under consideration, for example, the variation of the amplitude and phase of a whistler mode with the wave angular frequency for different values of E_0 can be examined. The plots of α vs. ω, β vs. ω and (v_{ph}/c) vs. ω, with v_{ph} denoting the phase velocity of the whistler mode are

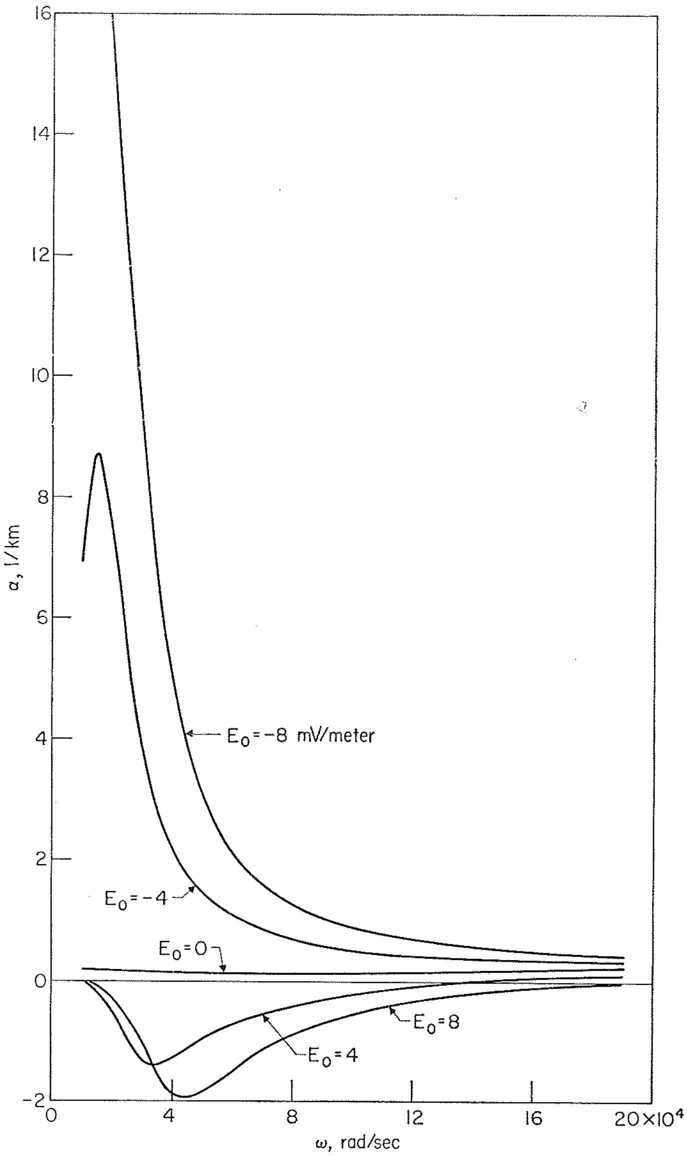

Fig. 1a. α vs. ω with E_0 as parameter for Case I.

shown in Figs. 1a, 1b, and 1c respectively for Case I, and the corresponding plots are shown in Figs. 2a, 2b, and 2c respectively for Case II.

4. Discussion of results

It is well known that particle collisions lead to the attenuation of the whistler mode in the ionosphere. It is noted from Figs. 1a and 2a that when $E_0 = 0$ the attenuation constant α is essentially invariant with respect to the signal frequency. However, with the presence of an electrostatic field α does vary with the signal

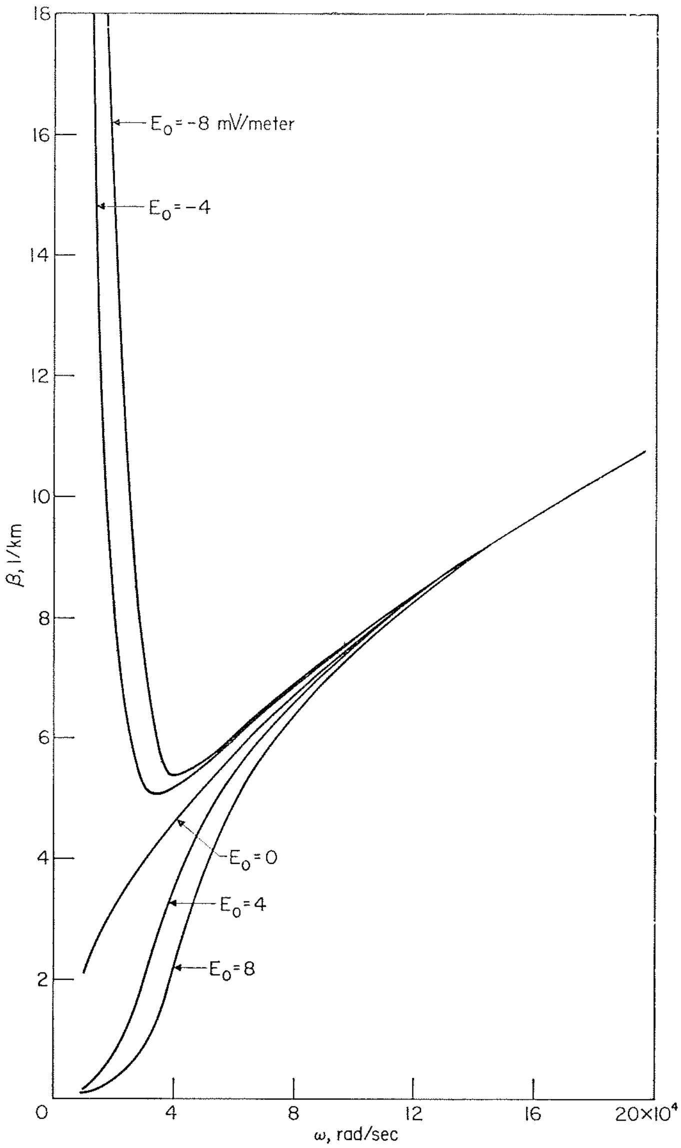

Fig. 1b. β vs. ω with E_0 as parameter for Case 1.

frequency. For $E_0 > 0$, the presence of E_0 leads to a reduction of the attenuation and if E_0 is sufficiently large then α may become negative; thus the wave is amplified. It should be noted that when $E_0 > 0$, for the parameters chosen, α is negative except in the range $\omega > 15 \times 10^4$ rad/sec with $E_0 = 4$ mV/m in Case I. This suggests that when $\mathbf{E}_0$ and $\mathbf{B}_0$ are both in the direction of the wave vector, the whistler mode experiences an amplification. Moreover when $E_0 < 0$, α is positive so that the wave experiences an attenuation when $\mathbf{E}_0$ and $\mathbf{B}_0$ are oppositely directed. $|\alpha|$ increases with $|E_0|$ in the range $\omega \geqq 4 \times 10^4$ rad/sec for Case I and in the range $\omega \geqq 3 \times 10^4$

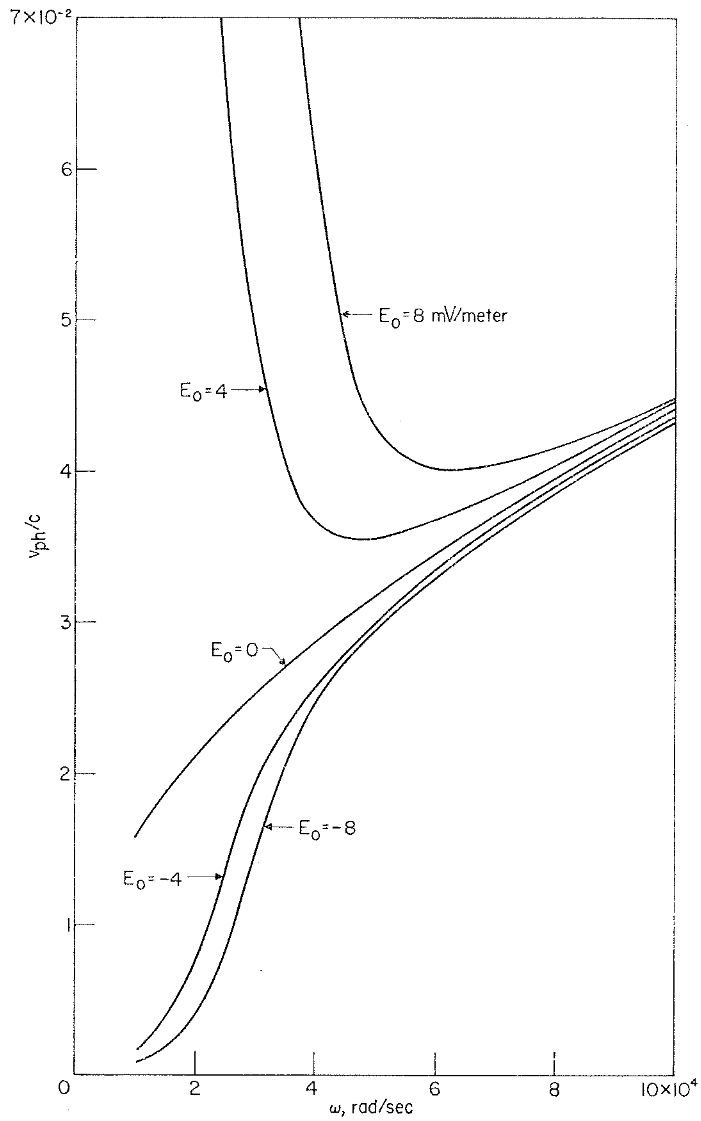

Fig. 1c. (v_{ph}/c) vs. ω with E_0 as parameter for Case I.

rad/sec for Case II. The maximum of $|\alpha|$ for both Cases I and II appears to lie somewhere in the low frequency range of the spectrum; $\omega \leqq 5 \times 10^4$ rad/sec for $|E_0| < 8$ mV/m. The comparison of Fig. 1a with Fig. 2a shows that the extent of the effect of E_0 on α is greater in Case II than in Case I. However it should be noted that in Case I it is assumed that $T_e = T_i$, whereas in Case II $T_e = 2T_i$.

On the other hand, from Figs. 1b and 2b, it is observed that when $E_0 = 0$ the phase constant β increases monotonically with the signal frequency. When $E_0 > 0$, the group velocity $v_g \equiv d\omega/d\beta$ is positive and varies with the signal frequency. However, when $E_0 < 0$, $d\omega/d\beta < 0$ in the low frequency range and $d\omega/d\beta > 0$ in the

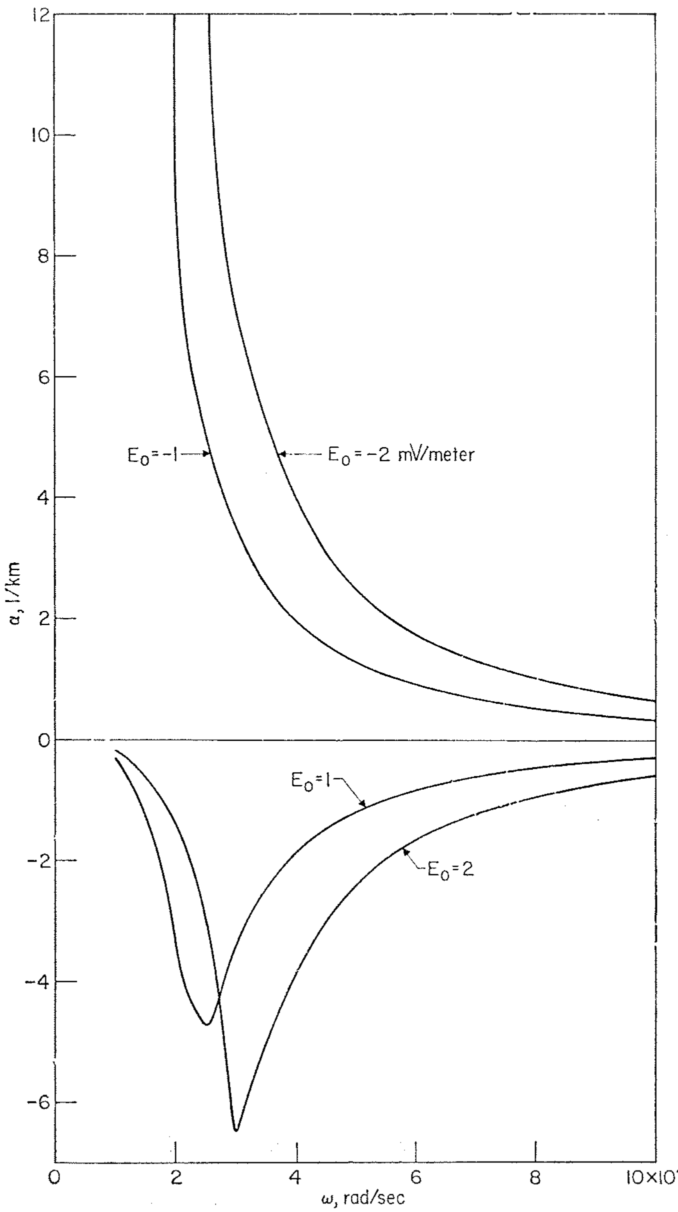

Fig. 2a. α vs. ω with E_0 as parameter for Case II.

high frequency range. Furthermore it is observed that in Case I, when $E_0 > 0$, the wavelength of the whistler mode increases with $|E_0|$ and when $E_0 < 0$, it decreases as $|E_0|$ increases. For Case II the wavelength increases with $|E_0|$ regardless of the algebraic signs of E_0. Finally it is observed that in Case I (see Fig. 1c) the phase velocity v_{ph} of the whistler mode increases with E_0 when $E_0 > 0$ and decreases with $|E_0|$ when $E_0 < 0$. On the other hand, in Case II (see Fig. 2c), v_{ph} increases with $|E_0|$ regardless of the algebraic sign of E_0.

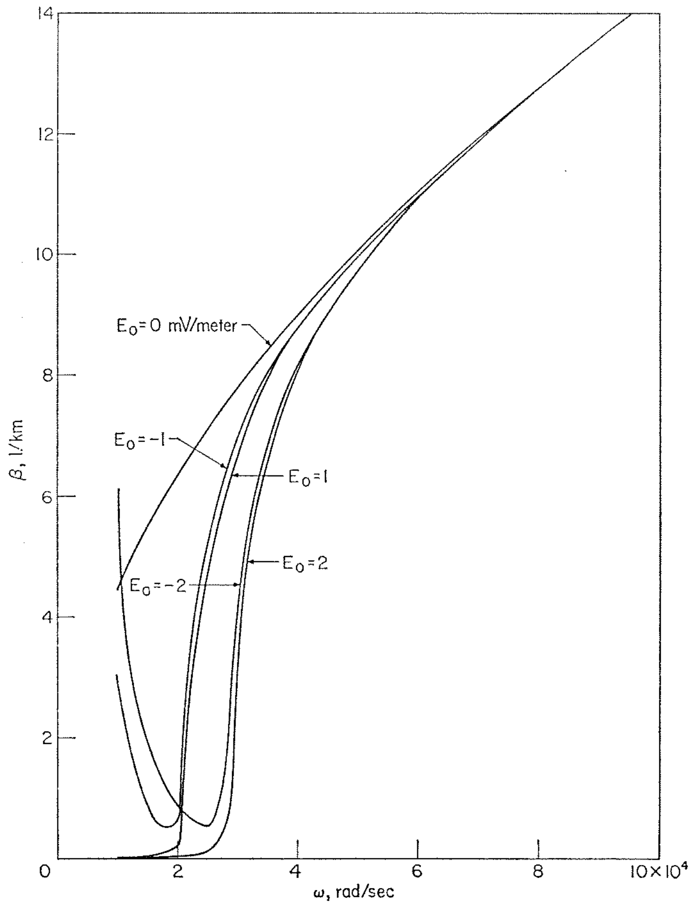

Fig. 2b. β vs. ω with E_0 as parameter for Case II.

It is also of interest to note that Figs. 1 and 2 both suggest that the extent of influence of E_0 on the change in the amplitude, wavelength and phase velocity of the whistler mode under consideration in both Cases I and II is much greater in the low frequency range of the spectrum, e.g., $\omega < 6 \times 10^4$ rad/sec, than in the high frequency range, e.g., $\omega > 6 \times 10^4$ rad/sec.

5. Concluding Remarks

In the present paper it is assumed that the collision frequencies of the electron and positive ions are independent of the particle velocity. The classical Appleton–Hartree equation used for propagation in the ionosphere has this inherent assumption. Because the electron elastic collision frequency with nitrogen molecules in the atmosphere varies as the square of the electron speed (Phelps and Pack, 1959) a large discrepancy between classical theory and ionospheric experiments can be expected.

However, Shkarofsky (1961) has derived a generalized Appleton–Hartree equation applicable to any variation of electron collision frequency with electron

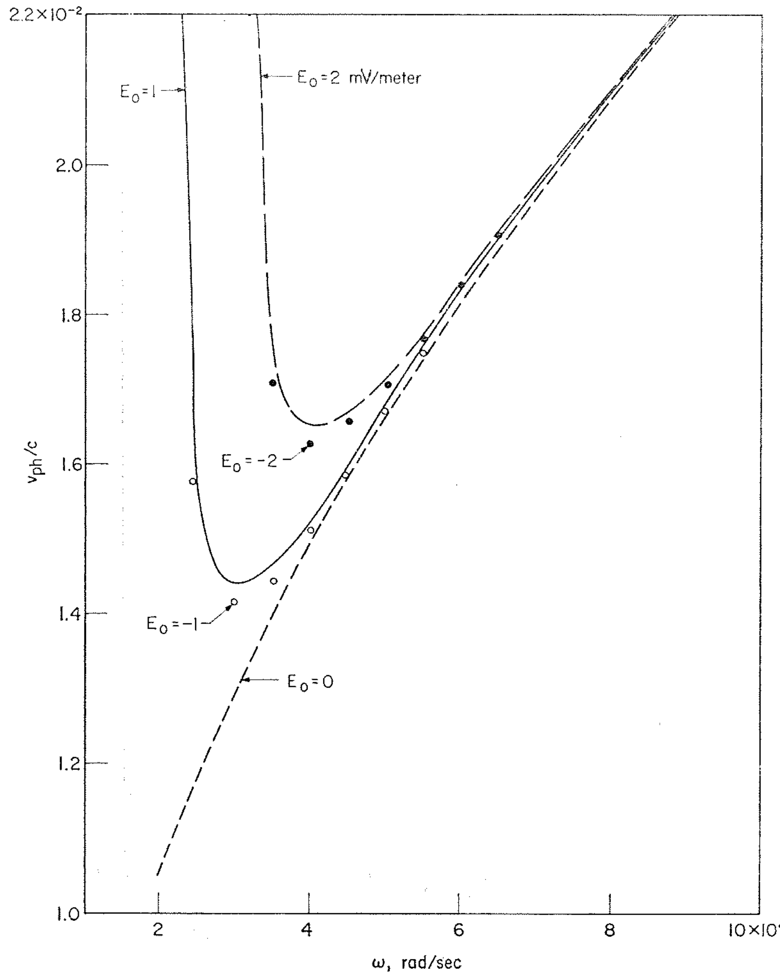

Fig. 2c. (v_{ph}/c) vs. ω with E_0 as parameter for Case II.

speed and any degree of ionization. His analysis suggests that the classical Appleton–Hartree equation should be applicable with no correction necessary for the region of the ionosphere and the range of parameters under consideration in the present paper. Thus the velocity-independent collision frequency model considered here is reasonable.

Most of the uncertainty in the calculation arises from a lack of knowledge of the electrostatic field strength and the collision frequency of positive ions in the region investigated. Consequently the numerical illustration shown in Figs. 1 and 2 only provide information with regard to the order of magnitude and a general behavior of the propagation constant. Although only the forward whistler mode has been considered in these figures, it is not difficult to analyze the propagation characteristic of the backward whistler mode, which is also characterized by equation (2a).

The frequency range used for the illustration lies within the range over which a class of naturally occurring radio noise in the ionosphere and magnetosphere such as a whistler and VLF emission is observed (Helliwell, 1965). The result presented in this paper should be useful for the study of the propagation of a whistler or VLF emission in the region of ionosphere where the presence of electrostatic field may be of significance.

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Propagation of Circularly Polarized Electromagnetic Waves in a Finite Temperature Electromagnetoplasma

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The dispersion relation for circularly polarized electromagnetic waves in a warm two-component plasma subject to parallel static electric and magnetic fields has been derived from the linearized coupled Boltzmann–Maxwell equations with the collision frequency assumed to be independent of the particle velocity. The effect of a weak longitudinal electrostatic field, $\mathbf{E}_0$, on the propagation characteristic of the right- and left-hand circularly polarized waves in an isothermal electron–proton plasma is examined in detail and illustrated numerically for a conveniently chosen set of the system parameters. For the right-hand polarized wave the electrostatic field effect is found to be significant for a wave with frequency ω in the vicinity of the electron cyclotron frequency $\omega_z \equiv (eB_0/m)$. For example, for a given ω and $\delta \equiv (eE_0/mc\omega) > 0$ an increase in δ, or in E_0, leads to the increase or decrease of the attenuation constant α, of the wave according to whether $Y \equiv (\omega_z/\omega) < 1$ or $Y > 1$. Moreover, for $Y = 1.10$, when $\delta < 0$ (i.e., when $\mathbf{E}_0$ and the wave vector $\mathbf{k}$ are oppositely directed) α increases with $|\delta|$. On the other hand, when $\delta > 0$ an increase in $|\delta|$ causes α to decrease and for a sufficiently large value of δ, α may become negative so that the wave may experience a spatial growth.

I. INTRODUCTION

A theory of growing electromagnetic waves was advanced some years ago by Bailey[1–3] in his electromagnetoionic theory, which is an extension of the well-known magnetoionic theory of Appleton and others. The basis of Bailey's theory consists of the following physical laws:

(1) Maxwell's law governing the behavior of electromagnetic fields.

(2) The conservation of electron and positive ions.

(3) Maxwell's law for the transfer of momentum in a mixture of different kinds of particles.

The analysis of the dispersion relation for the system, derived from the above macroscopic laws under the small-amplitude condition, led Bailey[1] to predict the amplification of a plane wave, within a certain frequency band, in an ionized medium pervaded by static electric and magnetic fields which are both parallel to the direction of wave propagation. Bailey then applied his theory to explain the excess noise radiation observed in sunspot.[4] However, Bailey's[3] theory of amplified circularly polarized waves in an ionized medium was first criticized by Twiss[5] who argued that the growing wave, which Bailey interprets as an amplified wave, can only be excited by reflection at the boundary. A critical analysis of Bailey's theory was also given later by Piddington,[6] who shows that the electromagnetoionic theory predicts spurious growing waves which do not correspond to any interchange of energy between the gas and the field, but are attributed to the movement of the observer and emitter relative to the gas particles. Even with the additional ion motion the mass drift of the electrons and ions together introduce no new wave form in the electromagnetoionic theory although drift does modify the existing waves. While these authors are concerned primarily with the amplification aspects, no attention has been given to the other aspects of the propagation characteristics.

On the other hand, studies of electromagnetic wave propagation based on a macroscopic small-signal theory have been made by several authors for a drifting cold magnetoplasma[7–10] and a stationary two-component warm plasma.[11]

Recently, in the course of examining the dispersion relations for a finite temperature electromagnetoplasma some interesting effects on the propagation of circularly polarized electromagnetic waves due to static electric fields have been observed.[12] For example, the presence of an applied transverse static electric field causes the cutoff frequency to shift[13] and in addition results in the coupling of the longitudinal mode to a transverse circularly polar-

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zed mode; these are discussed elsewhere. On the other hand, when the applied static electric field is directed parallel to the static magnetic field, the presence of a static electric field may significantly affect the amplitude and phase of the electromagnetic wave.

The purpose of this paper, therefore, is to discuss in detail the effect of an applied longitudinal static electric field upon the propagation characteristics of a transverse circularly polarized wave which travels along the static magnetic field in a finite temperature unbounded two-component plasma. In the present discussion the coupled Boltzmann–Maxwell equation is used.

II. BASIC EQUATION

The electron distribution function $f(\mathbf{r}, \mathbf{v}, t)$ and the ion distribution function $F(\mathbf{r}, \mathbf{v}, t)$ for this plasma are governed by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f = \nu_e(f_0 - f) \quad (1a)$$

and

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F + \frac{e}{M}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v F = \nu_i(F_0 - F), \quad (1b)$$

where m and M denote the electron and ion masses, respectively, and e is the electronic charge which is taken as a positive quantity. ν_e and ν_i are the frequencies of collision of electrons with positive ions and of ions with electrons, respectively. These collision frequencies are assumed to be independent of the particle velocity. f_0 and F_0 are the equilibrium distribution functions of the electron and ion, respectively.

The electromagnetic fields in the plasma are governed by the Maxwell equations

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla\cdot\mathbf{D} = \rho, \quad \text{and} \quad \nabla\cdot\mathbf{B} = 0. \quad (2)$$

The electric displacement vector $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$ are, respectively, related to the electric field intensity $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ in the following manner:

$$\mathbf{D} = \epsilon_0\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_0\mathbf{H}, \quad (3)$$

where ϵ_0 and μ_0 denote the dielectric constant and the permeability of vacua.

The convection current density $\mathbf{J}$ and the charge density ρ may be written in terms of the distribution function as

$$\mathbf{J} = e\int \mathbf{v}(F - f)\, d^3v \quad \text{and} \quad \rho = e\int (F - f)\, d^3v. \quad (4)$$

Consider all quantities of interest to be composed of two parts, a time-independent part and a time-varying part, which are denoted by the subscripts 0 and 1, respectively,

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), & \mathbf{E} &= \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}, t); \\ \mathbf{J} &= \mathbf{J}_0(\mathbf{r}) + \mathbf{J}_1(\mathbf{r}, t), & \rho &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t); \\ f &= f_0(\mathbf{r}, \mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t), & F &= F_0(\mathbf{r}, \mathbf{v}) + F_1(\mathbf{r}, \mathbf{v}, t). \end{aligned} \quad (5)$$

In the present paper the following assumptions are made:

(1) Small-amplitude conditions are satisfied so that the terms involving the product of time-dependent quantities are negligible.

(2) A one-dimensional analysis is applicable, i.e., all quantities vary only with one spatial variable, and $\partial/\partial x = \partial/\partial y = 0$ in a rectangular Cartesian coordinates system.

(3) All time-varying quantities have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the angular frequency and the propagation constant, respectively.

Based on the above assumptions, the substitution of Eq. (5) into Eqs. (1)–(4) results in two sets of differential equations, one of which governs the time-independent quantities and the other governs the time-varying quantities. The former is given by

$$v_z \frac{\partial f_0}{\partial v_z} - \frac{e}{m}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v f_0 = 0, \quad (6a)$$

$$v_z \frac{\partial F_0}{\partial v_z} + \frac{e}{M}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v F_0 = 0, \quad (6b)$$

$$\frac{\partial E_{0x}}{\partial x} = 0, \quad \frac{\partial E_{0y}}{\partial y} = 0, \quad \frac{\partial E_{0z}}{\partial z} = \frac{\rho_0(z)}{\epsilon_0}, \quad (6c)$$

$$\frac{\partial B_{0y}}{\partial z} = -\mu_0 J_{0x}, \quad \frac{\partial B_{0x}}{\partial z} = \mu_0 J_{0y}, \quad \frac{\partial B_{0z}}{\partial z} = 0, \quad (6d)$$

$$\mathbf{J}_0 = e\int \mathbf{v}(F_0 - f_0)\, d^3v, \quad (6e)$$

and

$$\rho = e\int (F_0 - f_0)\, d^3v, \quad (6f)$$

where $d^3v = dv_x\, dv_y\, dv_z$ denotes the volume element in velocity space. On the other hand, in discussing the set of equations relating the time-varying quantities it is convenient to consider the following transformation of variables:

$$v_x = v_r \cos\varphi \quad \text{and} \quad v_y = v_r \sin\varphi, \quad (7a)$$

zed mode; these are discussed elsewhere. On the other hand, when the applied static electric field is directed parallel to the static magnetic field, the presence of a static electric field may significantly affect the amplitude and phase of the electromagnetic wave.

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and

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The convection current density $\mathbf{J}$ and the charge density ρ may be written in terms of the distribution function as

$$\mathbf{J} = e\int \mathbf{v}(F - f)\, d^3v \quad \text{and} \quad \rho = e\int (F - f)\, d^3v. \quad \text{(4)}$$

Consider all quantities of interest to be composed of two parts, a time-independent part and a time-varying part, which are denoted by the subscripts 0 and 1, respectively,

$$\begin{aligned} &\mathbf{B} = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), && \mathbf{E} = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}, t); \\ &\mathbf{J} = \mathbf{J}_0(\mathbf{r}) + \mathbf{J}_1(\mathbf{r}, t), && \rho = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t); \quad \text{(5)} \\ &f = f_0(\mathbf{r}, \mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t), && F = F_0(\mathbf{r}, \mathbf{v}) + F_1(\mathbf{r}, \mathbf{v}, t). \end{aligned}$$

In the present paper the following assumptions are made:

(1) Small-amplitude conditions are satisfied so that the terms involving the product of time-dependent quantities are negligible.

(2) A one-dimensional analysis is applicable, i.e., all quantities vary only with one spatial variable, and $\partial/\partial x = \partial/\partial y = 0$ in a rectangular Cartesian coordinates system.

(3) All time-varying quantities have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the angular frequency and the propagation constant, respectively.

Based on the above assumptions, the substitution of Eq. (5) into Eqs. (1)–(4) results in two sets of differential equations, one of which governs the time-independent quantities and the other governs the time-varying quantities. The former is given by

$$v_z\frac{\partial f_0}{\partial v_z} - \frac{e}{m}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v f_0 = 0, \quad \text{(6a)}$$

$$v_z\frac{\partial F_0}{\partial v_z} + \frac{e}{M}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v F_0 = 0, \quad \text{(6b)}$$

$$\frac{\partial E_{0x}}{\partial x} = 0, \quad \frac{\partial E_{0y}}{\partial y} = 0, \quad \frac{\partial E_{0z}}{\partial z} = \frac{\rho_0(z)}{\epsilon_0}, \quad \text{(6c)}$$

$$\frac{\partial B_{0y}}{\partial z} = -\mu_0 J_{0x}, \quad \frac{\partial B_{0x}}{\partial z} = \mu_0 J_{0y}, \quad \frac{\partial B_{0z}}{\partial z} = 0, \quad \text{(6d)}$$

$$\mathbf{J}_0 = e\int \mathbf{v}(F_0 - f_0)\, d^3v, \quad \text{(6e)}$$

and

$$\rho = e\int (F_0 - f_0)\, d^3v, \quad \text{(6f)}$$

where $d^3v = dv_x\, dv_y\, dv_z$ denotes the volume element in velocity space. On the other hand, in discussing the set of equations relating the time-varying quantities it is convenient to consider the following transformation of variables:

$$v_x = v_r\cos\varphi \quad \text{and} \quad v_y = v_r\sin\varphi, \quad \text{(7a)}$$

and

$$E_\pm = \tfrac{1}{2}(E_{1x} \pm jE_{1y}) \text{ and } B_\pm = \tfrac{1}{2}(B_{1x} \pm jB_{1y}). \quad \text{(7b)}$$

Then, by introducing the following parameters:

$$\omega_c = \left(\frac{e\mathbf{B}_0}{m}\right), \quad \mathbf{a} = \left(\frac{e\mathbf{E}_0}{m}\right), \quad \mathbf{\Omega}_c = \left(\frac{-e\mathbf{B}_0}{M}\right),$$

and

$$\mathbf{A} = \left(\frac{-e\mathbf{E}_0}{M}\right),$$

the time-varying parts of Eqs. (2)–(4) can be combined to give

$$2\left(\frac{\omega^2}{c^2} - k^2\right)E_\pm = j\omega\mu_0 e \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} e^{\pm j\varphi}(F_1 - f_1)v_r^2\, d\varphi\, dv_r\, dv_z \quad \text{(8a)}$$

and

$$E_{1z} = \frac{je}{\omega\epsilon_0}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}(F_1 - f_1)v_r v_z\, d\varphi\, dv_r\, dv_z. \quad \text{(8b)}$$

Moreover, the time-varying parts of Eq. (1a), with the aid of Eqs. (7), can be written as follows:

$$\left[j(\omega - kv_z) + \nu_e + \omega_z\frac{\partial}{\partial\varphi}\right]f_1 - a_z\frac{\partial f_1}{\partial v_z} - \left[a_-\left(\frac{\partial f_1}{\partial v_r} + \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right) + \omega_-\frac{v_z}{v_r}\frac{\partial f_1}{\partial\varphi} + j\omega_-\, D(f_1)\right]e^{j\varphi} - \left[a_+\left(\frac{\partial f_1}{\partial v_r} - \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right) + \omega_+\frac{v_z}{v_r}\frac{\partial f_1}{\partial\varphi} - j\omega_+\, D(f_1)\right]e^{-j\varphi}$$
$$= \frac{e}{m}M_-(f_0)E_-e^{j\varphi} + \frac{e}{m}M_+(f_0)E_+e^{-j\varphi} + \frac{e}{m}E_{1z}\frac{\partial f_0}{\partial v_z} - \frac{e}{m}B_{1z}\frac{\partial f_0}{\partial\varphi}, \quad \text{(9a)}$$

where the differential operator D is defined by

$$D(\ ) \equiv \left(v_r\frac{\partial}{\partial v_z} - v_z\frac{\partial}{\partial v_r}\right)$$

and

$$\omega_\pm = \tfrac{1}{2}(\omega_{cx} \pm j\omega_{cy}), \qquad a_\pm = \tfrac{1}{2}(a_x \pm ja_y),$$

$$M_\pm(f_0) \equiv \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_0}{\partial v_r} \mp j\frac{1}{v_r}\frac{\partial f_0}{\partial\varphi}\right) + \frac{kv_r}{\omega}\frac{\partial f_0}{\partial v_z}\right]. \quad \text{(9b)}$$

On the other hand, the time-varying part of Eq. (1b) which governs the F_1 function, is easily obtained by replacing $a_\pm$, a_z, $\omega_\pm$, ω_z, ν_e, f_1, and f_0 with $A_\pm$, A_z, $\Omega_\pm$, Ω_z, ν_i, F_1, and F_0, respectively, in Eqs. (9).

It should be noted that E_- and E_+ in Eq. (9a) are the left- and right-hand circularly polarized components of electric field, respectively.

III. DISPERSION RELATIONS

Suppose that the positive z direction is taken in the direction of the magnetostatic field $\mathbf{B}_0$, i.e., $B_{0x} = B_{0y} = 0$ so that $\omega_\pm = 0$. Since $\nabla\cdot\mathbf{B} = 0$, B_{1z} must be independent of z, and it is taken to be zero in the present discussion (which is reasonable for a longitudinal propagation). Moreover, consider that the time-varying electron distribution function f_1 is composed of three parts and may be written as

$$f_1 = f_-(z, t, v_r, v_z)e^{j\varphi} + f_+(z, t, v_r, v_z)e^{-j\varphi} + g(z, t, v_r, v_z). \quad \text{(10)}$$

Since Eq. (9a) must be valid for an arbitrary value of φ, the substitution of Eq. (10) into Eq. (9a) yields the following system of equations:

$$j(\omega - j\nu_e - kv_z + \omega_z)f_- - a_z\frac{\partial f_-}{\partial v_z} - a_-\frac{\partial g}{\partial v_r} = \frac{e}{m}M_-(f_0)E_-, \quad \text{(11a)}$$

$$j(\omega - j\nu_e - kv_z - \omega_z)f_+ - a_z\frac{\partial f_+}{\partial v_z} - a_+\frac{\partial g}{\partial v_r} = \frac{e}{m}M_+(f_0)E_+, \quad \text{(11b)}$$

and

$$j(\omega - j\nu_e - kv_z)g - a_z\frac{\partial g}{\partial v_z} - \frac{2a_-}{v_r}f_+ - \frac{2a_+}{v_r}f_- = \frac{e}{m}\frac{\partial f_0}{\partial v_z}E_{1z} \quad \text{(11c)}$$

which clearly suggests the possibility of coupling between the transverse mode and the longitudinal mode when a_+ and a_- are nonzero, which is the case when the transverse electrostatic field is present. This case has been examined and discussed elsewhere.[13]

In the present investigation it is assumed that $a_\pm = 0$, i.e., $E_{0x} = E_{0y} = 0$, since the effect of the longitudinal electrostatic field is of primary concern. Suppose that f_-, f_+, and g have the v_z dependence of the form $\exp(-\alpha_1 v_z^2)$, in which $\alpha_1 = m/(2KT_e)$; then f_-, f_+, and g can be explicitly expressed in terms of E_-, E_+, and E_{1z}, respectively, from Eqs. (11) as

$$f_\mp = \frac{(e/m)M_\mp(f_0)E_\mp}{j(\tilde{b}_1 \pm \omega_z)} \quad \text{and} \quad g = \frac{(e/m)(\partial f_0/\partial v_z)E_{1z}}{j\tilde{b}_1}, \quad \text{(12)}$$

where

$\tilde{b}_1 = (\tilde{\omega}_1 - \tilde{k}_1 v_z)$, $\tilde{\omega}_1 = (\omega - j\nu_e)$, $\tilde{k}_1 = (k + jK_1)$,

and

$$K_1 = \left(\frac{eE_0}{KT_e}\right).$$

K and T_e denote the Boltzmann constant and the electron temperature, respectively. Thus combining Eqs. (10) and (12) the distribution function f_1 is expressed in terms of $E_\pm$ and E_{1z}. Similarly the time-varying ion distribution function F_1 can be written as

$$F_1 = F_-(z, t, v_r, v_z)e^{i\varphi} + F_+(z, t, v_r, v_z)e^{-i\varphi} + G(z, t, v_r, v_z), \qquad (13)$$

where

$$F_\pm = \frac{-(e/M)M_\mp(F_0)E_\mp}{j(\bar{b}_2 \pm \Omega_z)}$$

and

$$G = \frac{-(e/M)(\partial F_0/\partial v_z)E_{1z}}{j\bar{b}_2}, \qquad (14)$$

in which

$$\bar{b}_2 = (\bar{\omega}_2 - \bar{k}_2 v_z),\ \bar{\omega}_2 = (\omega - j\nu_i),\ \bar{k}_2 = (k + jK_2),$$

and

$$K_2 = \left(\frac{-eE_0}{KT_i}\right).$$

M and T_i denote the mass and temperature of the ion, respectively,

Upon substituting F_1 and f_1, given by Eqs. (13) and (10), respectively, into Eqs. (8) the following set of equations is obtained:

$$1 + \frac{\pi(\omega e/\epsilon_0)}{(\omega^2 - c^2k^2)} \cdot \int_{-\infty}^{\infty}\int_0^{\infty}\left[\frac{(e/m)M_\mp(f_0)}{(\bar{b}_1 \pm \omega_z)} + \frac{(e/M)M_\mp(F_0)}{(\bar{b}_2 \pm \Omega_z)}\right]v_r^2\, dv_r\, dv_z = 0 \qquad (15a)$$

and

$$1 + \frac{2\pi e}{\omega\epsilon_0} \cdot \int_{-\infty}^{\infty}\int_0^{\infty}\left(\frac{e}{m}\frac{1}{\bar{b}_1}\frac{\partial f_0}{\partial v_z} + \frac{e}{M}\frac{1}{\bar{b}_2}\frac{\partial F_0}{\partial v_z}\right)v_r v_z\, dv_r\, dv_z = 0. \qquad (15b)$$

Equation (15a) represents the dispersion relation for the transverse circularly polarized modes; the upper sign is to be taken for the left-hand circularly polarized mode and the lower sign is for the right-hand circularly polarized mode. Equation (15b) is the dispersion relation for the longitudinal mode. In the following discussion, Eq. (15a) is examined in detail.

It should be noted that when $E_0 = 0$, $K_1 = K_2 = 0$, and when $\nu_e = \nu_i = 0$, $\bar{\omega}_1 = \bar{\omega}_2 = \omega$, so that $\bar{b}_1 = \bar{b}_2 = (\omega - kv_z)$, then Eq. (15a) is reduced to those given by Montgomery and Tidman.[14]

For a one-dimensional analysis in a Maxwellian plasma, f_0 and F_0 can be written as

$$f_0 = n\left(\frac{\alpha_1}{\pi}\right)^{\frac{3}{2}} \exp\left[-\alpha_1(v_r^2 + v_z^2) + \frac{e\Phi(z)}{KT_e}\right] \qquad (16a)$$

and

$$F_0 = N\left(\frac{\alpha_2}{\pi}\right)^{\frac{3}{2}} \exp\left[-\alpha_2(v_r^2 + v_z^2) - \frac{e\Phi(z)}{KT_i}\right], \qquad (16b)$$

in which the electric scalar potential $\Phi(z)$ is related to static electric field by $E_{0z} = -d\Phi/dz$. n and N are the electron and positive ion concentrations, respectively. It is easily verified that f_0 and F_0 given by Eqs. (16) satisfy Eqs. (6a) and (b), respectively. Furthermore, they yield that $J_{0x} = J_{0y} = J_{0z} = 0$ which implies that the static magnetic field is independent of z. The space-charge density ρ_0 is given by

$$\rho_0(z) = eN \exp\left(\frac{-e\Phi}{KT_i}\right) - en \exp\left(\frac{e\Phi}{KT_e}\right). \qquad (17)$$

Suppose that Φ is sufficiently small so that

$$\left|\frac{e\Phi}{KT_e}\right| \ll 1 \quad \text{and} \quad \left|\frac{e\Phi}{KT_i}\right| \ll 1 \qquad (18)$$

and the condition of electrical neutrality is also satisfied, i.e., $n = N$. Then ρ_0 vanishes and from Eq. (6c), E_{0z} will be independent of z.

In the present discussion E_{0z} is assumed to be constant and denoted by E_0.

Having assumed the form of the functions f_0 and F_0, the indicated integration in Eq. (15a) can be carried out to give

$$1 - \frac{c^2k^2}{\omega^2} = \frac{X_1}{W_1}\left(1 + \frac{V_1^2\bar{k}_1^2}{2\omega^2}\frac{1}{W_1^2}\right) + \frac{X_2}{W_2}\left(1 + \frac{V_2^2\bar{k}_2^2}{2\omega^2}\frac{1}{W_2^2}\right), \qquad (19)$$

where

$$W_1 = [(1 \pm Y_1) - jZ_1], \qquad W_2 = [(1 \pm Y_2) - jZ_2],$$

$$\bar{k}_1 = (k + jK_1), \qquad \bar{k}_2 = (k + jK_2),$$

$$X_1 = \left(\frac{\omega_p}{\omega}\right)^2, \qquad Y_1 = \left(\frac{\omega_z}{\omega}\right),$$

$$Z_1 = \left(\frac{\nu_e}{\omega}\right), \qquad K_1 = \left(\frac{eE_0}{KT_e}\right),$$

[14] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill Book Company, Inc., New York, 1964), Chap. 10.

$$X_2 = \left(\frac{\Omega_p}{\omega}\right)^2, \quad Y_2 = \left(\frac{\Omega_z}{\omega}\right), \quad Z_2 = \left(\frac{\nu_i}{\omega}\right),$$

$$K_2 = \left(\frac{-eE_0}{KT_i}\right), \quad \omega_z = \left(\frac{eB_{0z}}{m}\right), \quad \omega_p = \left(\frac{e^2 n}{m\epsilon_0}\right)^{\frac{1}{2}},$$

$$\Omega_z = \left(\frac{-eB_{0z}}{M}\right), \quad \Omega_p = \left(\frac{e^2 N}{M\epsilon_0}\right)^{\frac{1}{2}}.$$

It should be pointed out that the derivation of Eq. (19) involves an evaluation of an integral of the form

$$G_0(\chi) \equiv \frac{j}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\exp(-\alpha v_z^2)}{(v_z - \chi)}\, dv_z, \tag{20a}$$

where χ may take on the value of $(\omega/\tilde{k}_{1,2})(1 - jZ_{1,2})$ or $(\omega/\tilde{k}_{1,2})W_{1,2}$. This integral has been discussed in detail by Stix[15] and his results are used here. When the term representing the Landau or cyclotron damping is neglected and taking only the first two terms of its asymptotic expansion, $G_0(\chi)$ can be given by

$$G_0(\chi) = \frac{-j}{\alpha^{\frac{1}{2}}\chi}\left(1 + \frac{1}{2\alpha\chi^2}\right), \tag{20b}$$

provided that $|\alpha^{\frac{1}{2}}\chi|^4 \gg 1$ is satisfied. It is of interest to note that when $V_1 = V_2 = 0$, Eq. (19) is reduced to the familiar dispersion equation in the cold-plasma magnetoionic theory. On the other hand, when $E_0 = 0$, $K_1 = K_2 = 0$, so that $\tilde{k}_1 = \tilde{k}_2 = k$, and when $\nu_1 = \nu_2 = 0$, $\tilde{\omega}_1 = \tilde{\omega}_2 = \omega$ so that Eq. (19) is reduced to those given by Heald and Wharton.[16]

IV. PROPAGATION CONSTANT

Equation (19) is a quadratic in k and can conveniently be written as

$$\tilde{A}\left(\frac{\tilde{k}}{\beta_0}\right)^2 + \tilde{B}\left(\frac{\tilde{k}}{\beta_0}\right) - \tilde{C} = 0, \tag{21a}$$

where

$$\tilde{A} = \left(1 + \frac{\tau_1 X_1}{\tilde{W}_1^3} + \frac{\tau_2 X_2}{\tilde{W}_2^3}\right),$$

$$\tilde{B} = j2\left(\frac{\delta_1 X_1}{\tilde{W}_1^3} + \frac{\delta_2 X_2}{\tilde{W}_2^3}\right), \tag{21b}$$

$$\tilde{C} = \left(1 - \frac{X_1}{\tilde{W}_1} - \frac{X_2}{\tilde{W}_2}\right),$$

in which

$$\tau_1 = \left(\frac{KT_e}{mc^2}\right), \quad \tau_2 = \left(\frac{KT_i}{Mc^2}\right), \quad \delta_1 = \left(\frac{eE_0}{mc\omega}\right),$$

$$\delta_2 = \left(\frac{-eE_0}{Mc\omega}\right) \quad \text{and} \quad \beta_0 = \left(\frac{\omega}{c}\right).$$

The symbol $\sim$ appearing in Eqs. (21) is introduced to emphasize the fact that the quantity under consideration is a complex quantity. In the present discussion the wave angular frequency ω is regarded as real and the propagation constant $\tilde{k}$ is regarded as a complex quantity which can be written as

$$\tilde{k} = (\beta - j\alpha), \tag{22}$$

where α and β are the amplitude and phase constants, respectively. Since the time and spatial dependence is assumed to be in the form $\exp[j(\omega t - \tilde{k}z)] = \exp[-\alpha z + j(\omega t - \beta z)]$, the forward and backward waves are represented by positive and negative values of β, respectively. On the other hand, the attenuation and amplification of the wave are represented by a positive and negative value of α, respectively. Once the system parameters are specified, Eq. (21a) can be solved for $(\tilde{k}/\beta_0) = (\beta/\beta_0 - j\alpha/\beta_0)$ and the propagation characteristics can be examined.

It should be noted that for an electrically neutral plasma the electronic and ionic parameters are related by $X_2 = \xi X_1$, $Y_2 = -\xi Y_1$, $\tau_2 = (\xi/\theta)\tau_1$ and $\delta_2 = -\delta_1/\xi$, where $\xi = (m/M)$ and $\theta = (T_e/T_i)$. Furthermore, the collision frequencies ν_e and ν_i are, in general, dependent on the density and temperature. For a fully ionized gas ν_e is determined by the electron–ion encounter, while for a partially ionized gas ν_e is determined by electron–neutral encounter and electron–ion encounter.[17] For the purpose of illustration, a fully ionized gas, consisting of electrons and protons, is considered here. For a Maxwellian isothermal plasma (i.e., $T_e = T_i$) ν_{ei} can be given by[18]

$$\nu_{ei} = 3.63 \times 10^{-6}\left(\frac{n}{T_e^{\frac{3}{2}}}\right) \ln \Lambda, \tag{23}$$

where

$$\Lambda = 1.24 \times 10^7 \left(\frac{T_e^3}{n}\right)^{\frac{1}{2}},$$

in which n is the electron concentration in mks units. On the other hand, the effective collision frequency for proton–electron encounters can be given by[19]

$$\nu_p = \left(\frac{m}{m}\right)^{\frac{1}{2}} \nu_e \simeq \frac{1}{43}\nu_e, \tag{24}$$

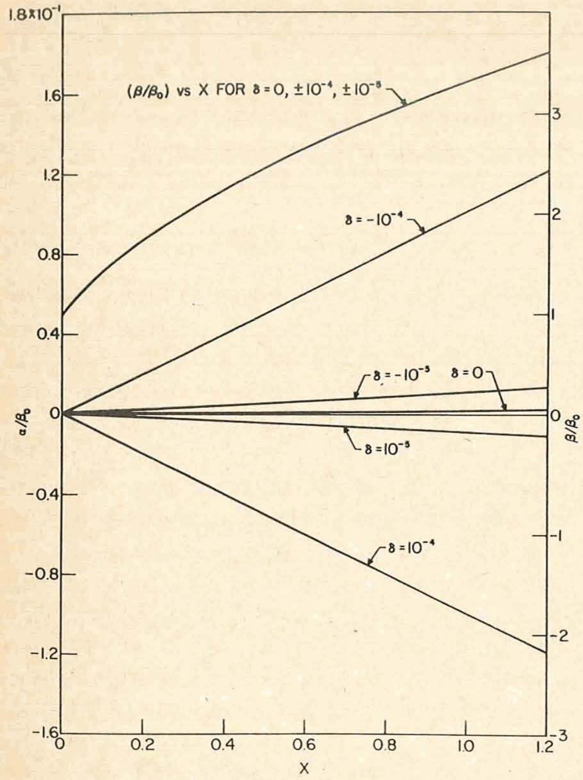
[15] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), Chap. 8.

[16] M. A. Heald and C. B. Wharton, *Plasma Diagnostics with Microwaves* (John Wiley & Sons, Inc., New York, 1965), Chap. 3.

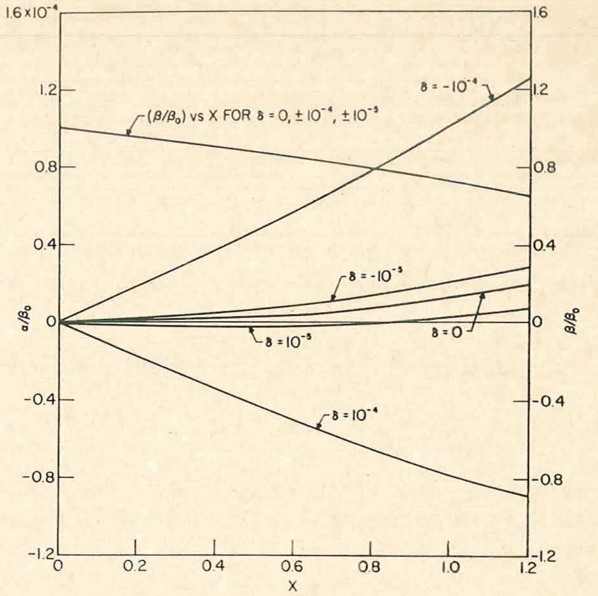
[17] B. S. Tanenbaum and D. Mintzer, Phys. Fluids 5, 1226 (1962).

[18] M. P. Bachynski and B. W. Gibbs, Phys. Fluids 9, 520 (1966).

[19] V. C. Ferraro and C. Plumpton, *An Introduction to Magneto-Fluid Mechanics* (Clarendon Press, Oxford, England, 1961), Chap. 8.



(a)



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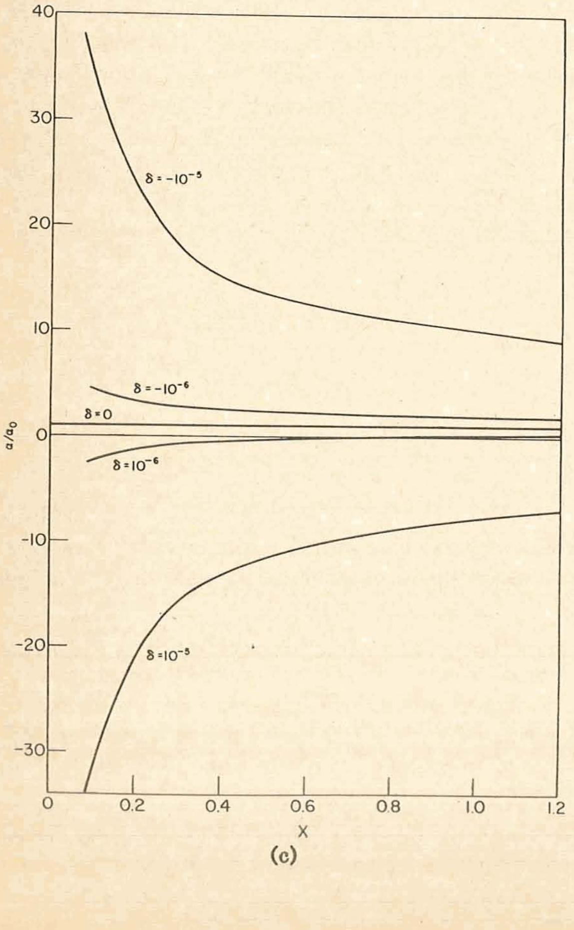

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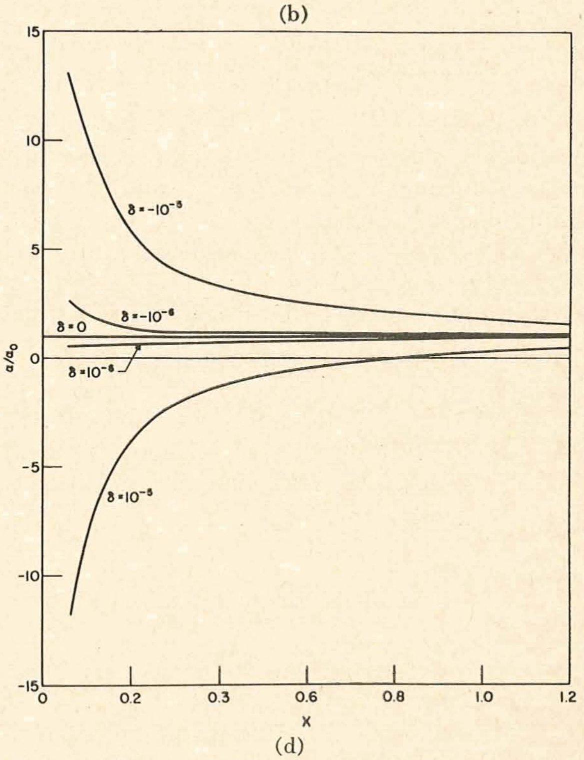


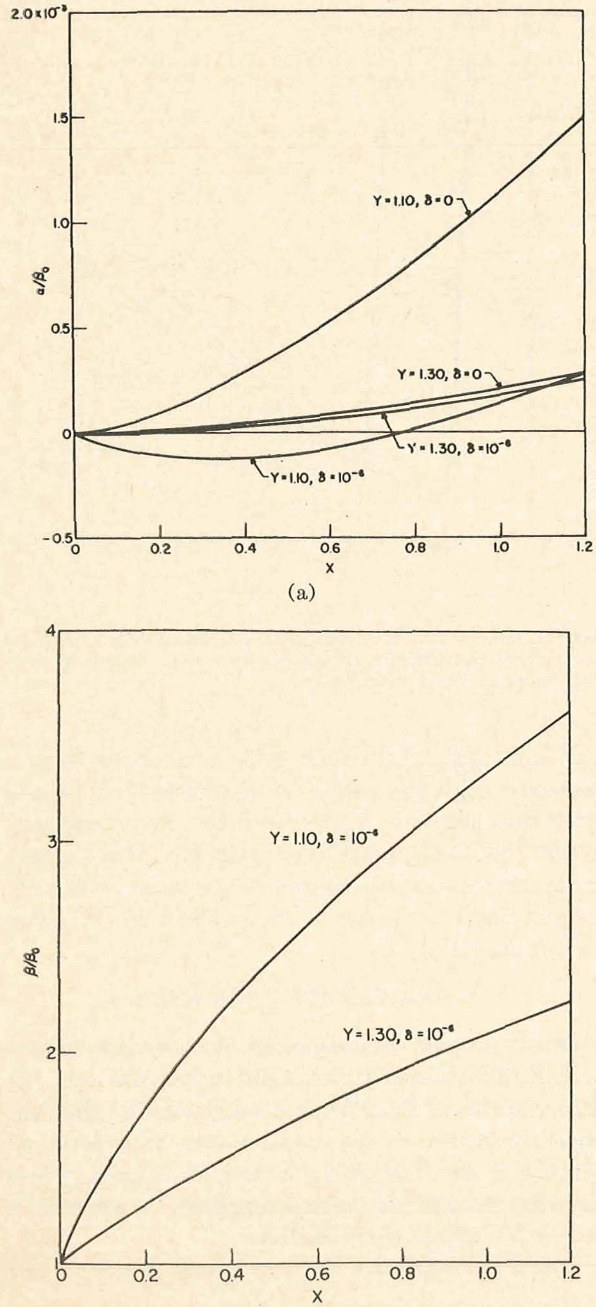
(d)

FIG. 1. (a) Variation of the amplitude coefficient (α/β_0) and the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave in an isothermal electron–proton plasma, with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (b) Variation of the amplitude coefficient (α/β_0) and the phase coefficient (β/β_0) with electron density (X) for left-hand circularly polarized wave in an isothermal electron–proton plasma, with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (c) Variation of the normalized amplitude constant (α/α_0) with the normalized electron density (X) for right-hand circularly polarized wave in an isothermal electron-proton plasma with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (d) Variation of the normalized amplitude constant (α/α_0) with the normalized electron density (X) for left-hand circularly polarized wave in an isothermal electron–proton plasma with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec.

where m_p is the proton mass, therefore, $Z_2 = \xi^{\frac{1}{2}} Z_1$. The variations of (α/β_0) and (β/β_0) are illustrated numerically in Figs. 1–3 for an isothermal electron–proton plasma (i.e., $\xi = 1/1836$ and $\theta = 1$) with the aid of Eqs. (23) and (24).

In these illustrations the frequency of the wave under consideration is taken as 1 GHz. The plots of amplitude coefficient (α/β_0) and phase coefficient (β/β_0) versus the electron density (X) for different values of the parameter $\delta \equiv (eE_0/mc\omega)$, in the case $Y = 1.10$, are shown in Figs. 1. Figure 1(a) deals with the right-hand circularly polarized wave, while Fig. 1(b) is for the left-hand circularly polarized wave. From both Figs. 1(a) and 1(b), it is observed, that for a given set of system parameters a change in δ has a profound effect on the amplitude, but it has practically no effect on the phase of the circularly polarized wave. Furthermore, when $\delta < 0$ (i.e., when the longitudinal electrostatic field $\mathbf{E}_0$ is directed opposite to the wave vector, $\mathbf{k}$) an increase in $|\delta|$ leads to an increase in the attenuation of the wave. On the other hand, when $\delta > 0$ (i.e., when $\mathbf{E}_0$ and $\mathbf{k}$ are in the same direction) an increase in $|\delta|$ leads to the reduction of the wave attenuation. In this case if δ is sufficiently large, α may become negative so that the wave may experience an amplification rather than an attenuation. However, it is difficult to give a comprehensive physical interpretation for gain in wave amplitude when $\delta > 0$, without a detailed analysis of the dynamic behavior of the charged particles or the studies of the energy conversion process between the particles and the electromagnetic wave. This is not done in the present paper; however, the question of energy conversion process will be considered in a future paper. The result of the present simple-minded theory appears to suggest that the effect of electrostatic field on the amplitude of the electromagnetic wave is evident. While the collision process in the plasma tends to randomize the order motion, the introduction of electrostatic field in the wave direction tends to reorganize the motion of the particle in such a way as to make the exchange of energy between the particles and the wave easier. In other words, it makes the extraction of particle energy in the plasma easier. Comparison of Figs. 1(a) and (b) shows that the amplitude coefficient for the right-hand circularly polarized wave is three orders of magnitude greater than that of the left-hand circularly polarized wave.

In the interest of emphasizing the change in the amplitude constant caused by the electrostatic field, the plots of α/α_0 vs X for the right-hand circularly polarized and the left-hand circularly polarized waves are shown in Fig. 1(c) and Fig. 1(d), respectively, where α_0 denotes the amplitude constant for the case of zero static electric field and also represents the rate of collision damping. In these figures, it is observed that for a given X, $|\alpha/\alpha_0|$ increases as $|\delta|$, which suggests that the effect of E_0 on the



(b)

FIG. 2. (a) Variation of the amplitude coefficient (α/β_0) with electron density (X) for right-hand circularly polarized wave with $Y>1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec. (b) Variation of the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave with $Y > 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec.

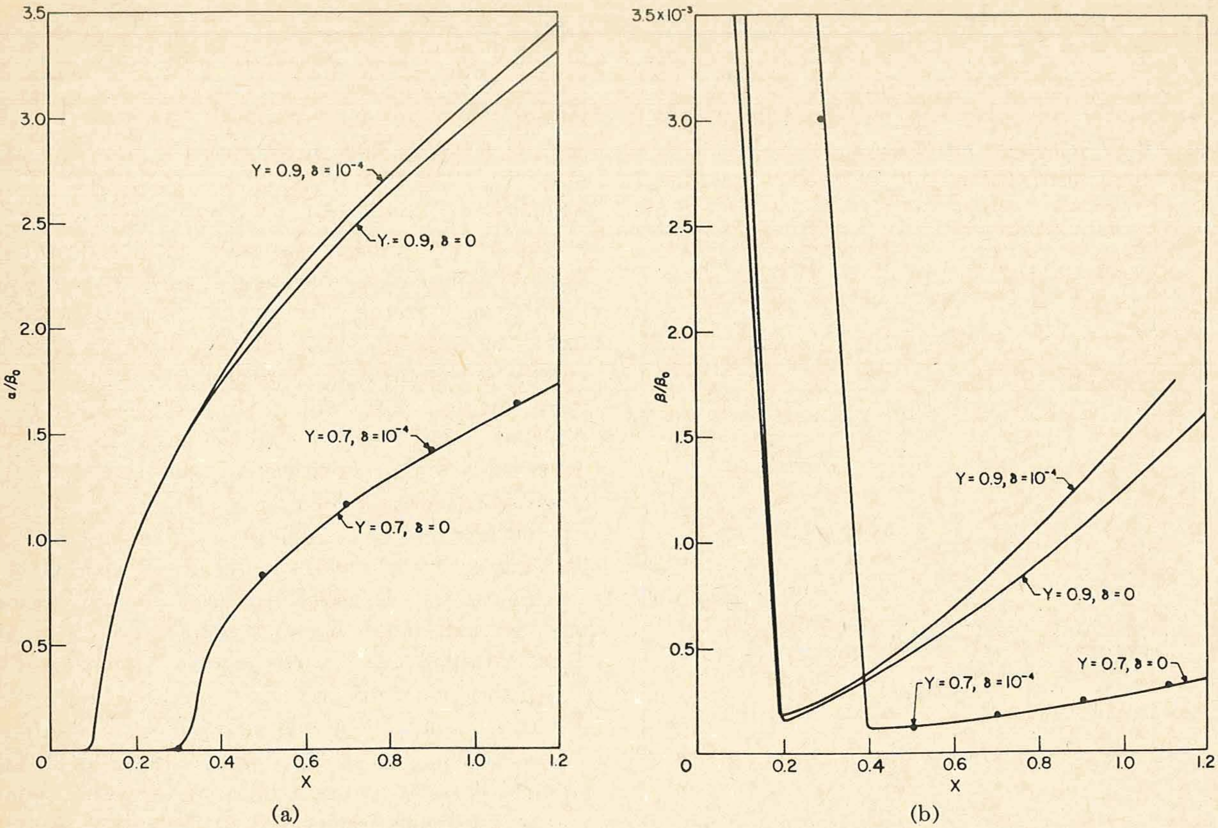


FIG. 3. (a) Variation of the amplitude coefficient (α/β_0) with electron density (X) for right-hand circularly polarized wave with $Y < 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec. (b) Variation of the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave with $Y < 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec.

amplitude constant α increases as $|E_0|$. On the other hand, the change in the amplitude constant α is more drastic in the range of small X than in the range of large X; i.e., the effect of E_0 on the change in amplitude of the electromagnetic wave is greater in the region of low electron number density than in the region of high number density.

The effects of change in the strength of magnetostatic field $\mathbf{B}_0$ upon the plots of (α/β_0) and (β/β_0) vs X, for the right-hand circularly polarized wave, are illustrated in Figs. 2 for the case $Y > 1$ and in Figs. 3 for the case $Y < 1$. It is observed that in the case $Y > 1$ [see Figs. 2(a) and (b)], for a given value of X, and $\delta = 0$ an increase in $Y \equiv (eB_0/m\omega)$ causes both α and β to decrease, which suggests that an increase in $|\mathbf{B}_0|$ reduces the attenuation and increases the phase velocity of the wave. On the other hand, in the case $Y < 1$ [see Figs. 3(a) and (b)] for $X > 0.4$ an increase in Y causes both α and β to increase so that an increase in $|\mathbf{B}_0|$ leads to an increase of attenuation and reduction of phase velocity of the wave.

It is also of interest to note, by comparing Figs. 2(a) and 3(a), that when $\delta > 0$ an increase in δ decreases α for the case $Y > 1$, whereas it increases α for the case $Y < 1$. On the other hand, comparison of Figs. 2(b) and 3(b) suggests that the presence of E_0 does modify the phase of the wave somewhat for the case $Y < 1$, but it has no effect on the phase for the case $Y > 1$.

V. CONCLUDING REMARKS

In the present discussion the electrostatic electric field E_0 is assumed to be sufficiently weak and the drift velocity of the plasma is much smaller than the phase velocity of the wave under consideration. Thus, it is assumed that the medium through which the electromagnetic wave propagates is essentially stationary rather than drifting.

It is shown that the effect of a weak static electric field, directed along the direction of wave propagation, on the amplitude and phase of the right-hand circularly polarized wave is most significant when the wave frequency is in the vicinity of the electron cyclotron frequency, e.g., for cases $Y = 1.1$ or 0.9.

A constant collision frequency model has been used in the present discussion. For the system parameters chosen for illustration this assumption is reasonable. The effect of a velocity-dependent collision frequency on the Appleton–Hartree equation of magnetoionic theory has been discussed by Shkarofsky.[20]

It should be pointed out that in the present discussion a Maxwellian plasma has been considered, i.e., the time-independent distribution functions of electron and positive ions are assumed to be a Maxwellian. Furthermore, it is also assumed that the time-varying distribution function of electron and positive ions has a Maxwellian distribution in the direction of wave propagation. It should be noted that the latter assumption may not in general be valid. The reasonableness of this assumption might be tested by an experimental investigation which is to be considered in the future. However, if this assumption is not too unreasonable, then the result of the present theory suggests that the introduction of an electrostatic field in the direction of wave propagation may reduce the attenuation of the wave. It is of interest to note that with a proper strength of E_0 it may also lead to an amplification of the circularly polarized electromagnetic wave in a warm collisional two-component magnetoplasma.

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Focusing of a Relativistic Electron Flow*

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The trajectory differential equation governing the motion of relativistic electrons is derived in terms of the scalar and vector potentials of the system using the principle of least action. The conservation of energy and momentum are used to develop the paraxial-ray differential equation describing the beam radius of a laminar-flow relativistic electron beam. The focusing of the electron beam in drift and accelerating regions has been examined and the conditions for perfect balancing and nonspreading of a laminar-flow drifting beam are also derived. It is shown that the equilibrium condition for Brillouin flow is that $2\omega_L^2-\omega_p^2[1-(u_z/c)^2]=0$, where ω_L is the Larmor precession frequency and ω_p the electron–plasma frequency. u_z and c denote, respectively, the axial beam velocity and the speed of light in vacuum. The variation of the normalized ripple amplitude and the scallop wavelength of a drifting beam is discussed. The profile of a beam accelerated in a uniform longitudinal electrostatic field is also illustrated.

I. INTRODUCTION

The subject of electron-beam focusing has been investigated extensively by various workers[1–11] interested in their use, e.g., in microwave tubes and linear electron accelerators. In the dynamic analysis of electron beams encountered in most microwave beam-type devices[1–6] relativistic effects are usually negligible, whereas, in the focusing of charged particles in a linear electron accelerator, relativistic effects (self-focusing and pinching) play a primary role. Recently the focusing of a high-intensity electron beam in an accelerating tube has been discussed theoretically by Meshkov and Chirikov,[7] and previously the magnetic self-focusing of partially neutralized relativistic electron beams drifting in the absence of any longitudinal field has been analyzed theoretically for a variety of idealized stream conditions[8–10] and observed experimentally.[11]

It appears that little attention has been given to the

* This work was supported by the National Aeronautics and Space Administration.

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study of the dynamics of a relativistic electron flow under the influence of its own fields in the presence of externally applied static electric and magnetic fields. The purpose of the present study is to investigate the focusing of a relativistic cylindrical electron beam in the presence of applied static axial electric and magnetic fields, assuming that the effects of radiation and collisions are negligible, and further that the transverse velocity is small in comparison with the axial velocity. The principle of "least action" is used in deriving the trajectory differential equation of the charged particle.[12]

II. TRAJECTORY DIFFERENTIAL EQUATIONS

The canonical momentum $\mathbf{p}$ of an electron moving at a relativistic velocity $\mathbf{u}$ may be written as

$$\mathbf{p}=m\mathbf{u}+e\mathbf{A}, \tag{1}$$

where $m=m_0/[1-(u/c)^2]^{1/2}$ and m_0 and e are, respectively, the rest mass and charge (negative quantity) of an electron, c denotes the speed of light in vacuum, $\mathbf{A}$ is the magnetic vector potential, and u denotes the magnitude of the velocity vector $\mathbf{u}$.

The principle of least action is conveniently written as

$$\delta\int_{x_{\alpha1}}^{x_{\alpha2}}\mathbf{p}\cdot d\mathbf{l}=\int_{x_{\alpha1}}^{x_{\alpha2}}(m\mathbf{u}+e\mathbf{A})\cdot d\mathbf{l}=0, \tag{2}$$

where $\mathbf{l}$ is the position vector, $x_{\alpha1}$ and $x_{\alpha2}$ denote the coordinates of the end points of the line integral, and δ is the usual variational operator. By taking one of the three spatial coordinates, e.g., z, as the independent variable, and defining $r'\equiv dr/dz$ and $\theta'\equiv d\theta/dz$ in a cylindrical coordinate system, Eq. (2) can be written as follows:

$$\delta\int_{z_1}^{z_2}\{mu[r'^2+(r\theta')^2+1]^{1/2}+e(A_rr'+A_\theta r\theta'+A_z)\}dz=0. \tag{3}$$

The Eulerian equations for this system will yield the differential equations for the electron orbit provided that the magnitude of the velocity u is known as a function of z. Since, according to the principle of least action, the varied path satisfies the law of conservation of energy, u can be expressed in terms of the kinetic energy T and the rest energy ϵ_0 of the particle, i.e.,

$$(u/c)/[1-(u/c)^2]^{1/2}=(T^2+2T\epsilon_0)^{1/2}/\epsilon_0, \tag{4}$$

since $mc^2=\epsilon_0+T$. Thus, after introducing Eqs. (1) and (4), the variational equation, Eq. (3), can be written as follows:

$$\delta\int_{z_1}^{z_1}Pdz=0, \tag{5}$$

[12] W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Co., Inc., Cambridge, Mass, 1955), Chap. 23.

where $P\equiv(\tau^2+2\tau)^{1/2}(r'^2+r^2\theta'^2+1)^{1/2}+(ec/\epsilon_0)(A_rr'+A_\theta r\theta'+A_z)$ with $\tau\equiv(T/\epsilon_0)$ and $\epsilon_0\equiv m_0c^2$. Then the Eulerian equations for $r(z)$ and $\theta(z)$ can be obtained from

$$(d/dz)(\partial P/\partial r')-(\partial P/\partial r)=0 \tag{6a}$$

and

$$(d/dz)(\partial P/\partial\theta')-(\partial P/\partial\theta)=0. \tag{6b}$$

For an axially symmetric system, $\partial P/\partial\theta=0$ and Eqs. (5) and (6) are combined to give

$$\{(\tau^2+2\tau)^{1/2}(r^2\theta')/[r'^2+(r\theta')^2+1]^{1/2}\}+(ec/\epsilon_0)A_\theta r=M_0, \tag{7}$$

where M_0 is a constant of integration, independent of z. It is observed that Eq. (7) expresses the conservation of the θ component of the canonical momentum since

$$(r\theta')/(r'^2+r^2\theta'^2+1)^{1/2}=(r\dot\theta)/u, \tag{8}$$

where the dot denotes the time derivative. It is easily shown that Eq. (7) is equivalent to the following familiar relationship:

$$rp_\theta+eA_\theta r=K_0, \tag{9}$$

in which $p_\theta=m_0(r\dot\theta)/[1-(u/c)^2]^{1/2}$ and $K_0\equiv m_0cM_0$. A differential equation for $r(z)$ can be obtained by combining Eqs. (5), (6a), and (7):

$$r''-(R/2w)[(\partial w/\partial r)-r'(\partial w/\partial z)]$$
$$=(-R^{3/2}/w^{1/2})(ec/\epsilon_0)[(\partial A_r/\partial z)-(\partial A_z/\partial r)], \tag{10}$$

where $R\equiv(1+r'^2)$, $w\equiv\xi(1-\eta^2)$, $\xi\equiv(\tau^2+2\tau)$, and

$$\eta\equiv\xi^{-1/2}[(M_0/r)-(ecA_\theta/\epsilon_0)]. \tag{11}$$

In view of the fact that w is a function of ξ and η, and ξ depends in turn on τ, which is the ratio of the particle kinetic energy T to the rest energy ϵ_0, once $T(r,z)$ and $\mathbf{A}(r,z)$ are known, then Eq. (10) can be solved for $r(z)$ for a properly specified set of input conditions; $\theta(z)$ then can be determined directly from Eq. (7).

It should be noted that η can be written as $\eta=(u_\theta/u)\equiv r\dot\theta/u$, and r' as $r'=u_r/u_z\equiv\dot r/\dot z$, where $u^2=(u_r^2+u_\theta^2+u_z^2)$. If the radial velocity u_r is much smaller than the axial velocity u_z, i.e., $r'^2\ll1$, then $R\approx1$.

III. PARAXIAL RAY EQUATIONS FOR AN AXIALLY SYMMETRIC FLOW

Under the condition $r'^2\ll1$, Eq. (10) takes the following form:

$$r''=w^{-1}\{(\tau+1)[(\partial\tau/\partial r)-r'(\partial\tau/\partial z)]+\eta(\xi)^{1/2}(M_0/r^2)$$
$$+\eta(\xi)^{1/2}(ec/\epsilon_0)\cdot[(\partial A_\theta/\partial r)-r'(\partial A_\theta/\partial z)]\}$$
$$-w^{-1/2}(ec/\epsilon_0)[(\partial A_r/\partial z)-(\partial A_z/\partial r)]. \tag{12}$$

The law of conservation of the particle energy can be expressed as

$$T+e\Psi=\text{constant}, \tag{13a}$$

so that

$$\partial\tau/\partial r = (-e/\epsilon_0)(\partial\Psi/\partial r)$$

and

$$\partial\tau/\partial z = (-e/\epsilon_0)(\partial\Psi/\partial z). \tag{13b}$$

In addition, if γ and β are defined as

$$\gamma \equiv (1-\beta^2)^{-1/2} \quad \text{and} \quad \beta \equiv (u/c), \tag{14a}$$

then from Eq. (4), $\xi = (\gamma^2\beta^2)$ so that $(\tau+1) = (1+\xi)^{1/2} = \gamma$ and from Eqs. (9) and (11)

$$\eta(\xi)^{1/2} = [(M_0/r) - (ecA_\theta/\epsilon_0)] = \gamma(u_\theta/c) \tag{14b}$$

and

$$w = \xi(u_z/u)^2.$$

As a result Eq. (12) can be written as

$$r'' = (1/\gamma u_z^2)\{(-e/m_0)[(\partial\Psi/\partial r) - r'(\partial\Psi/\partial z)] - u_\theta(\partial/\partial r)[(M_0c/r) - (e/m_0)A_\theta] - (e/m_0)u_\theta r' \times (\partial A_\theta/\partial z)\} - (\gamma u_z)^{-1}(e/m_0)[(\partial A_r/\partial z) - (\partial A_z/\partial r)], \tag{15}$$

where $\Psi(r, z)$ and $\mathbf{A}(r, z)$ are, respectively, the scalar and vector potentials of the system under consideration. The potentials, in general, consist of two parts (one part due to an external source and the other due to the electron stream space charge and current). Furthermore, these potentials must satisfy the following partial differential equations:

$$\nabla^2\Psi = -\rho/\epsilon \quad \text{and} \quad \nabla\times(\nabla\times\mathbf{A}) = \mu_0\rho\mathbf{u}, \tag{16}$$

where ρ is the space-charge density and μ_0 and ϵ denote the permeability and permittivity of vacuum, respectively.

For an axially symmetric system Poisson's equation becomes

$$r^{-1}(\partial/\partial r)[r(\partial\Psi/\partial r)] + (\partial^2\Psi/\partial z^2) = -\rho_0\gamma/\epsilon \tag{17a}$$

and for a steady-state static condition $\nabla\cdot\mathbf{A}$ can be taken as zero so that Eqs. (16) give, in component form,

$$(\partial/\partial r)[r^{-1}(\partial/\partial r)(rA_r)] + (\partial^2 A_r/\partial z^2) = -\mu_0\rho_0\gamma u_r, \tag{17b}$$

$$(\partial/\partial r)[r^{-1}(\partial/\partial r)(rA_\theta)] + (\partial^2 A_\theta/\partial z^2) = -\mu_0\rho_0\gamma u_\theta \tag{17c}$$

and

$$r^{-1}(\partial/\partial r)[r(\partial A_z/\partial z)] + (\partial^2 A_z/\partial z^2) = -\mu_0\rho_0\gamma u_z, \tag{17d}$$

where ρ_0 is the rest charge density which is a negative quantity for electrons and is assumed to be constant in the present investigation. It should be noted that, relativistically, charge density and current density are simply different aspects of the same thing. If ρ_0 is a "proper" charge density in a frame where charges are at rest, then $\rho = \gamma\rho_0$ gives the transformation from a charge density at rest to a charge density in a nonproper frame, ensuring the invariance of the total charge. This can be seen as follows: A spatial volume element dv is related to a proper spatial volume dv_0 by $dv = dv_0/\gamma$, since only one dimension (e.g., the z direction if u_z is comparable to c) suffers a Lorentz contraction and hence $\rho dv = \rho_0 dv_0$ and the charge within a given boundary remains invariant. The law of conservation of energy [Eq. (13)] can be written as

$$m_0\gamma c^2 + e\Psi = m_0c^2 \tag{18a}$$

and the concept of the conservation of canonical momentum is expressed as

$$m_0\gamma u_r + eA_r = 0, \tag{18b}$$

$$m_0\gamma u_\theta + eA_\theta = K_0/r \tag{18c}$$

and

$$m_0\gamma u_z + eA_z = 0. \tag{18d}$$

Combining Eqs. (17) and (18) yields

$$r^{-1}(\partial/\partial r)[r(\partial\gamma/\partial r)] + (\partial^2\gamma/\partial z^2) = \beta_0^2\gamma, \tag{19a}$$

$$(\partial/\partial r)[r^{-1}(\partial/\partial r)(rU)] + (\partial^2 U/\partial z^2) = \beta_0^2 U, \tag{19b}$$

$$(\partial/\partial r)[r^{-1}(\partial/\partial r)(rV)] + (\partial^2 V/\partial z^2) = \beta_0^2 V \tag{19c}$$

and

$$r^{-1}(\partial/\partial r)[r(\partial W/\partial r)] + (\partial^2 W/\partial z^2) = \beta_0^2 W, \tag{19d}$$

where

$$U \equiv (\gamma u_r), \qquad V \equiv (\gamma u_\theta), \qquad W \equiv (\gamma u_z),$$

$$\beta_0^2 \equiv (\omega_p^2/c^2) \quad \text{and} \quad \omega_p^2 \equiv (e\rho_0/m_0\epsilon).$$

Equations (19) are linear partial differential equations and can be solved by the standard technique of separation of variables. For example, it is easily observed that the general solution of Eqs. (19) can be written in product form $[R(r)Z(z)]$, where $R(r)$ denotes a linear combination of the nth-order modified Bessel functions of the first kind, $I_n(\alpha r)$, the second kind, $K_n(\alpha r)$, where $\alpha \equiv (k^2+\beta_0^2)^{1/2}$, with k being the separation constant. For $k \neq 0$, $Z(z)$ is a periodic sinusoidal function of z, whereas for $k=0$ it is a linear function of z. The solutions of Eqs. (19a) and (19d) involve the zero-order modified Bessel function, while that of Eqs. (19b) and (19c) involve the first-order modified Bessel function.

When a solid electron beam is considered, the quantities γ, U, V, and W all must remain finite along the axis $r=0$, and consequently the modified Bessel function of the second kind is not permissible in the solution of Eqs. (19). Thus,

$$\gamma(r, \zeta) = M_1 I_0(\alpha r)\cos k\zeta, \tag{20a}$$

$$U(r, \zeta) = M_3(k/\alpha)I_1(\alpha r)\sin k\zeta, \tag{20b}$$

$$V(r, \zeta) = M_2 I_1(\alpha r)\cos k\zeta \tag{20c}$$

and

$$W(r, \zeta) = M_3 I_0(\alpha r)\cos k\zeta, \tag{20d}$$

in which $\nabla\cdot\mathbf{A} = 0$ has been used, and $\zeta \equiv (z - z_0)$. M_1,

M_2, M_3, and z_0 are the constants of integration which are yet to be determined.

From Eqs. (18) and (20), the potential in the beam region, allowing for the presence of applied static fields, can be expressed as follows:

$$\Psi(r,\zeta)=-E_0\zeta+(m_0c^2/-e)M_1I_0(\alpha r)\cos k\zeta+(m_0c^2/e),$$

$$A_r(r,\zeta)=(m_0/-e)(k/\alpha)M_3I_1(\alpha r)\sin k\zeta,$$

$$A_\theta(r,\zeta)=(m_0/-e)M_2I_1(\alpha r)\cos k\zeta+(K_0/er)$$

and

$$A_z(r,\zeta)=(m_0/-e)M_3I_0(\alpha r)\cos k\zeta, \qquad (21)$$

in which $\zeta=0$, i.e., $z=z_0$ denotes the entry into the interaction region under consideration.

It should be noted that any longitudinal inhomogeneity, which may exist in the system, is represented by the constant k. For example, in an infinitely long homogeneous beam k can be taken as zero. For a constant velocity drifting beam, E_0 vanishes. In this case, $u_z=(M_3/M_1)$, which is independent of z. For an accelerating beam, $E_0\neq 0$ and it can easily be observed from Eqs. (18), (20), and (21) that u_z does depend upon z. For a laminar flow, i.e., nonintersecting electron trajectories that are sufficiently well confined in the transverse direction, only the motion of the boundary electron need be considered. Thus, Eq. (15) can be used for the investigation of the variation of the radius of the beam boundary as a function of axial distance.

Let y be the beam radius under consideration, and suppose that the quantity (αy) satisfies the following inequality:

$$(\alpha y)^4 \ll 1.$$

Then the Bessel functions may be expanded as follows:

$$I_0(\alpha y)\approx 1+\tfrac{1}{4}(\alpha y)^2$$

and

$$I_1(\alpha y)\approx \tfrac{1}{2}(\alpha y)+\tfrac{1}{16}(\alpha y)^3. \qquad (22)$$

Thus upon substituting Eqs. (20) and (21) into Eq. (15), with the approximation (22), the following differential equation is obtained:

$$y''+y'\big((G_2/\cos k\zeta)[1-\tfrac{1}{4}(\alpha^2y^2)]+\{G_1^2y^2[1-\tfrac{1}{4}(\alpha^2y^2)]-\delta_0^2\}k\tan k\zeta\big)+y[G_1^2+\tfrac{1}{2}(\beta_0^2-\alpha^2\delta_0^2)]+\tfrac{1}{16}(\alpha^2y^3)(\alpha^2\delta_0^2-\beta_0^2)=0, \qquad (23)$$

where

$$G_2\equiv[(eE_0/m_0)(M_1/M_3^2)], \qquad G_1\equiv(\alpha M_2/2M_3),$$
$$\delta_0\equiv(cM_1/M_3).$$

ω_p is the electron-beam plasma angular frequency for a beam with infinite lateral extent. When the beam cross section is finite, such as is frequently the case, the plasma angular frequency is much smaller than that for an infinite beam due to the effect of the conducting boundary surrounding the beam. The effect of the finiteness of the beam size on the plasma frequency can be taken into account by replacing ω_p by $\omega_q\equiv R_p\omega_p$, with R_p denoting the plasma reduction factor. Formulas are available for the determination of R_p in the case of a solid cylindrical beam passing through a metal tube.[13] The constants M_1, M_2, and M_3 can be determined in terms of the physical and geometrical parameters specified at the input plane $\zeta=0$. It is obvious from Eqs. (20) and (21) that $M_1=\gamma_{0,0}=[1+(-e\Psi_{0,0}/m_0c^2)]$, where $\Psi_{0,0}$ denotes the axial beam voltage at the input plane $\zeta=0$. From Eq. (20d), $M_3=\gamma_{0,0}u_z(0,0)=M_1u_{z,0}$. If the axial beam current at the entrance plane is denoted by $\bar{I}_0$, then

$$\bar{I}_0=\rho_0\int_0^{y_0}W2\pi r dr$$
$$=(\pi y_0^2)\rho_0M_3[2I_1(\alpha y_0)/\alpha y_0], \qquad (24)$$

where y_0 denotes the injection beam radius. In view of the fact that $\gamma_{0,0}=1/[1-(u_{z,0}/c)^2]^{1/2}$, specification of $\Psi_{0,0}$ determines $u_{z,0}$ and $\gamma_{0,0}$ as well as M_1 and M_3. $\bar{I}_0$ and y_0 must also be specified in order to determine ω_p. On the other hand, M_2 can be expressed in terms of the input θ-component velocity, from Eq. (20c) as follows:

$$M_2I_1(\alpha y_0)=\gamma(y_0,0)u_\theta(y_0,0)=M_1I_0(\alpha y_0)u_\theta(y_0,0).$$

It is convenient to introduce a pitch angle parameter for the boundary electron at the input plane which is defined as

$$\tan\psi_0\equiv u_\theta(y_0,0)/u_z(y_0,0) \qquad (25)$$

so that G_1 can be expressed as

$$G_1=y_0^{-1}\tan\psi_0. \qquad (26)$$

Thus specification of the velocity ratio (u_θ/u_z) at $r=y_0$ and $\zeta=0$ determines the parameter G_1; ψ_0 is small since u_θ is assumed to be much smaller than u_z.

It should be noted that the solution of Eq. (23) provides information on the profile of the electron beam in the accelerating region. Suppose that the normalized perturbation in beam radius, Y, and the normalized axial distance x are respectively defined as

$$Y\equiv(y-y_0)/y_0 \quad\text{and}\quad x\equiv(\beta_0\zeta), \qquad (27)$$

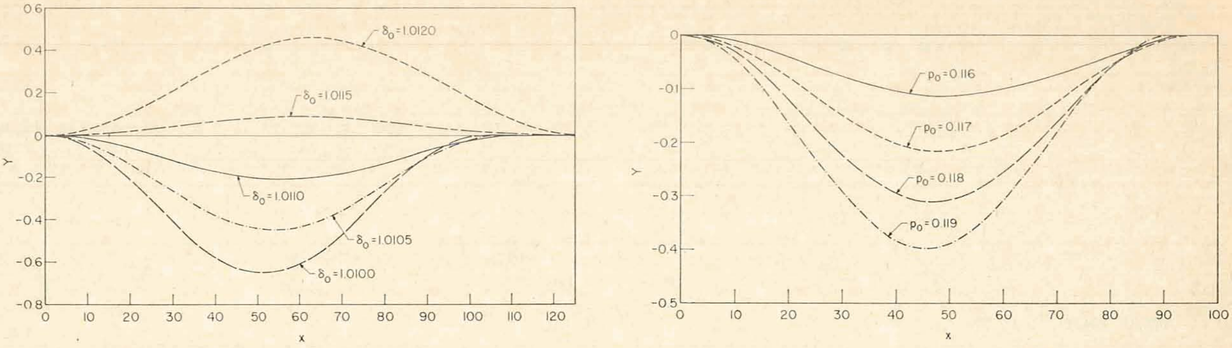
where y_0 denotes the injection beam radius. Then Eq. (23) becomes

$$(d^2Y/dx^2)+(dY/dx)\{(g_2/\cos\nu x)[1-\tfrac{1}{4}\sigma^2(1+Y)^2]+\nu\tan\nu x[p_0^2(1+Y)^2-\tfrac{1}{4}\sigma^2p_0^2(1+Y)^4-\delta_0^2]\}+(1+Y)[(p_0^2/\mu^2)+g_3]-(1+Y)^3\tfrac{1}{8}\sigma^2g_3=0, \qquad (28)$$

where

$$g_2=a_0(\delta_0/\mu)(\delta_0^2-1)^{1/2}, \qquad \sigma^2=\mu^2(1+\nu^2),$$
$$g_3=\tfrac{1}{2}[1-(1+\nu^2)\delta_0^2]$$

[13] G. M. Branch and T. G. Mihran, IRE Trans. Electron Devices ED-2, 3 (1955).



(a) (b)

FIG. 1.(a) Velocity dependence of electron-beam profile when $Y^2 \ll 1$. ($\mu=1.0$, $p_0=0.1$, $\nu=a_0=0$). (b) Effect of transverse velocities on electron-beam profile when $Y^2 \ll 1$. ($\delta_0=1.015$, $\mu=1.0$, $\nu=a_0=0$).

and

$$\nu \equiv (k/\beta_0), \qquad \mu \equiv (\beta_0 y_0), \qquad p_0 \equiv \tan\psi_0,$$

$$a_0 \equiv (eE_0 y_0/mc^2).$$

Before discussing the numerical solution of Eq. (28), it is instructive to consider an interesting special case in which the electron beam is longitudinally uniform, so that $\nu \equiv 0$, and the perturbation in the beam radius is small, i.e., $Y^2 \ll 1$.

Case I. Constant Velocity

In the drift region, where there is no longitudinal electrostatic field $g_2=0$, under the conditions $\nu=0$ and $Y^2 \ll 1$, Eq. (28) reduces to

$$(d^2Y/dx^2)+h_1 Y+h_0=0, \tag{29}$$

where

$$h_1=(p_0^2/\mu^2)-\tfrac{1}{2}(\delta_0^2-1)(1-\tfrac{3}{8}\mu^2)$$

and

$$h_0=(p_0^2/\mu^2)-\tfrac{1}{2}(\delta_0^2-1)(1-\tfrac{1}{8}\mu^2).$$

The complete solution of Eq. (29) consists of two parts—a complementary solution and a constant term representing the particular integral. The form of the complementary solution depends upon the algebraic sign of h_1. When $h_1<0$, the complementary solution of Eq. (29) takes the form of a nonperiodic exponential function of x, which implies that the beam radius y grows exponentially with the axial distance ζ, so that the beam is continually spreading. However, when $h_1>0$, the complementary solution of Eq. (29) takes the form of a periodic (sinusoidal) function of x, so that the beam is rippling. For $h_1=0$, the solution of Eq. (29) has a quadratic dependence on x. Consequently the condition $h_1>0$ can be regarded as the condition for nonspreading of the drifting beam. The general solution of Eq. (29) for the case $h_1>0$ is given by

$$Y(x)=[Y_0'/(h_1)^{1/2}]\sin(h_1)^{1/2}x + [Y_0+(h_0/h_1)]\cos(h_1)^{1/2}x-(h_0/h_1), \tag{30}$$

where Y_0 and Y_0' are, respectively, the normalized deviation in beam radius and the slope of the beam boundary at the entry to the drift region. When a parallel-flow beam is launched into the drift region, i.e., $Y_0=0$ and $Y_0'=0$, Eq. (30) yields

$$Y(x)=(h_0/h_1)[\cos(h_1)^{1/2}x-1]. \tag{31}$$

In this case, the beam profile is characterized by two factors: (1) the amplitude of the beam ripple, and (2) the ripple wavelength (scalloping wavelength). Equation (31) indicates that $Y(x)$ varies between 0 and $-2(h_0/h_1)$. Since a laminar flow is being considered and $y \geqq 0$, $(2h_0/h_1)$ must be less than unity. The conditions $h_1>0$ and $2h_0<h_1$ can be combined to give

$$\tfrac{1}{2}\mu^2(\delta_0^2-1)(1-\tfrac{3}{8}\mu^2)<p_0^2<\tfrac{1}{2}\mu^2(\delta_0^2-1)(1+\tfrac{1}{8}\mu^2). \tag{32}$$

Thus, when the beam parameters μ, δ_0, and p_0 are so chosen that condition (32) is satisfied, a rippling beam results. The profile given by Eq. (31) is illustrated in Figs. 1 for a conveniently chosen set of parameters. The normalized amplitude of the beam ripple denoted by Y_m and the characteristic wavelength λ are respectively given by

$$Y_m=(h_0/h_1) = [2p_0^2-\mu^2(\delta_0^2-1)(1-\tfrac{1}{8}\mu^2)]/[2p_0^2-\mu^2(\delta_0^2-1)(1-\tfrac{3}{8}\mu^2)]$$

and

$$\lambda=2/\beta_0(h_1)^{1/2} = 2\pi y_0/[p_0^2-\tfrac{1}{2}\mu^2(\delta_0^2-1)(1-\tfrac{3}{8}\mu^2)]^{1/2}. \tag{33}$$

It should be observed that when $h_0=0$, $Y(x)=0$, so that the beam radius remains constant in the drift region (balanced flow). The condition for balanced flow, therefore, is written as

$$p_0^2=\tfrac{1}{2}\mu^2(\delta_0^2-1)(1-\tfrac{1}{8}\mu^2). \tag{34}$$

Note that $\delta_0=(c/u_{z,0})$ and $\mu=(\beta_0 y_0)$ can be expressed in terms of the input axial beam voltage $\Psi_{0,0}$ and the

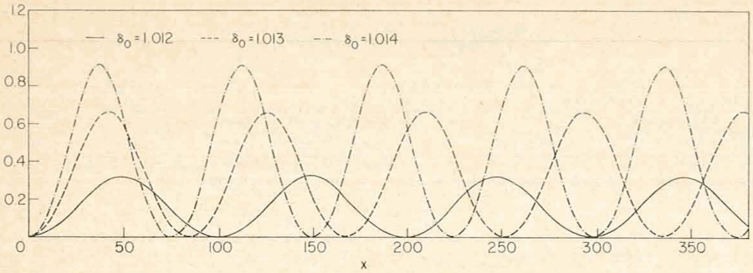


FIG. 2. General dependence of beam profile on velocity. ($\mu=1.0$, $p_0=0.1$, $\nu=a_0=0$).

axial beam current $\bar{I}_0$ as

$$\delta_0=\{1-[1/(1+Q)^2]\}^{-1/2}$$

and

$$\mu^2=4(\{1+(\bar{I}_0/2D)[(\delta_0^2-1)]^{1/2}\}^{1/2}-1), \quad (35)$$

where

$$Q\equiv(-e\Psi_{0,0}/m_0c^2)$$

and

$$D\equiv(\pi c^3 m_0\epsilon/e)=-4.26\times10^3 \text{ A}.$$

When $\bar{I}_0$ and δ_0 are given in such a way that $\mu^2\ll1$, then Eqs. (34) and (35) become, respectively,

$$p_0^2=\tfrac{1}{2}\mu^2(\delta_0^2-1) \quad (36a)$$

and

$$\mu^2=(\bar{I}_0/D)(\delta_0^2-1)^{1/2}. \quad (36b)$$

It is of interest to note that for most electron-beam devices the condition $\mu^2\ll1$ is satisfied so that Eq. (36a) is applicable. In the case of Brillouin flow, Busch's theorem gives the angular velocity $\dot{\theta}$ at the entry to the drift region as the Larmor precession frequency, $\omega_L=(|e|B_0/2m_0)$, with B_0 denoting the uniform applied static axial magnetic flux density in the region under consideration. Consequently $u_\theta(y_0,0)=y_0\omega_L$ and $(p_0/y_0)=G_1=\omega_L/u_{z,0}$. Since $\delta_0=(c/u_{z,0})$ and $\mu=(\beta_0y_0)$, Eq. (36a) can be written as

$$\omega_L^2-\tfrac{1}{2}\omega_p^2[1-(u_{z,0}/c)^2]=0. \quad (37)$$

It is observed that the last term of the left-hand side of Eq. (37) represents the relativistic focusing effect due to the θ component of magnetic field. When $(u_{z,0}/c)^2\ll1$, Eq. (37) reduces to $2\omega_L^2=\omega_p^2$, which is the familiar equilibrium expression for nonrelativistic Brillouin flow.

Case II. Accelerating Beam

For the case in which a longitudinally uniform beam is accelerated by a uniform longitudinal static field, ($E_0\neq0$ and $\nu=0$), Eq. (28) reduces under the condition $Y^2\ll1$ to

$$(d^2Y/dx^2)+g_2(dY/dx)+h_2Y+h_2=0, \quad (38a)$$

where

$$h_2=(p_0^2/\mu^2)-\tfrac{1}{2}(\delta_0^2-1).$$

The complementary solution of Eq. (38a) has the form e^{sx}, where s satisfies the following algebraic equation:

$$s^2+g_2s+h_2=0. \quad (38b)$$

In the accelerating region, E_0 must be negative and since e is negative, $g_2>0$. Thus, Eq. (38b) has a pair of complex conjugate roots with a negative real part when $4h_2>g_2^2$. In this case, the complementary solution of Eq. (38a) is in the form of a damped oscillation such that the fluctuation in the beam radius is stable. On the other hand, when a longitudinal nonuniformity is permitted, ν must be different from zero. As an illustration, Eq. (28) is solved numerically for the input conditions $Y_0=Y_0'=0$ and the results are shown in Figs. 2–4.

IV. DISCUSSION OF RESULTS

The profiles of a uniform drifting electron beam, under the restriction $Y^2\ll1$, are illustrated in Fig. 1(a) for different values of axial beam velocity, $\delta_0=(c/u_z)$, and in Fig. 1(b) for various values of the input azimuthal to axial velocity ratio $p_0=(u_0/u_z)$. It is observed in Fig. 1(a) that if $\Psi_{0,0}$ and $\bar{I}_0$ are adjusted so that μ is kept constant then the characteristic wavelength of the beam λ increases as δ_0 increases. This fact is also evident from Eq. (33). The normalized amplitude of the beam ripple $|Y_m|$ has its minimum value of zero when $h_0=0$, and for $h_0<0$, $|Y_m|$ increases as δ_0 increases. The plots of Fig. 1(b) indicate that for given values of $\Psi_{0,0}$ and $\bar{I}_0$, λ decreases as p_0 increases, and thus the pinch effect increases as p_0 increases. The variation of Y_m and λ with the system parameters for a drifting beam can be easily studied by inspecting Eqs. (33).

The profile of a uniform beam in a drift region without the restriction of $Y^2\ll1$ is shown for various

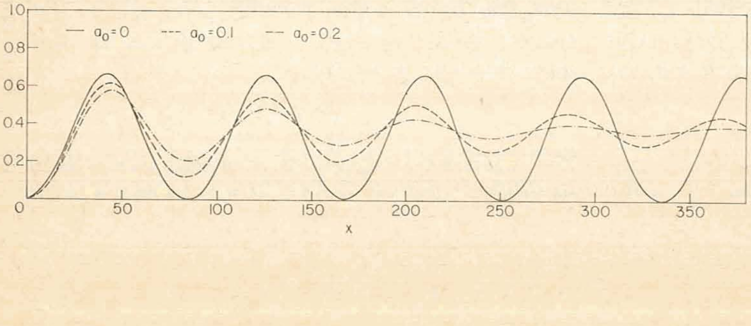


FIG. 3. Effect of a longitudinal accelerating static electric field. ($\delta_0=1.013$, $\mu=1.0$, $p_0=0.1$, $\nu=0$).

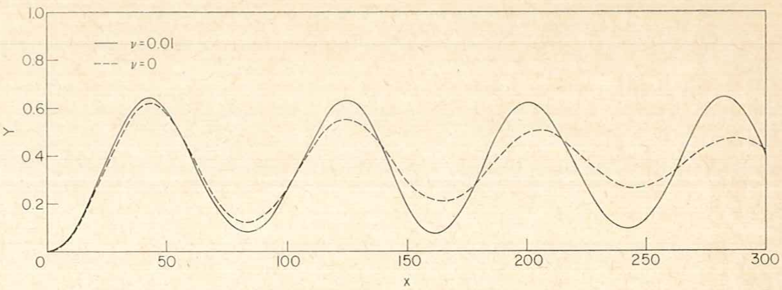


FIG. 4. Effect of a longitudinal inhomogeneity. ($\delta_0=1.013$, $\mu=1.0$, $p_0=0.1$, $a_0=0.1$).

values of velocity δ_0 in Fig. 2. It is observed that the maximum beam radius is an almost periodic function of the axial distance and the amplitude of oscillation in the beam boundary decreases as δ_0 decreases, i.e., as the axial beam velocity approaches the velocity of light. The remarks made above in regard to Fig. 1(a) are also applicable to Fig. 2. The effect of a longitudinally uniform static accelerating electric field E_0 on the profile of a uniform beam is shown in Fig. 3. It is noteworthy that the fluctuation in beam radius is damped as E_0 is introduced since $a_0=(eE_0y_0/mc^2)\neq 0$. The maximum normalized deviation in beam radius decreases as E_0 increases and the amplitude of oscillation in beam radius decreases as x increases. This is consistent with the observation made in Fig. 2 since, as the beam is accelerated and at large value of x, the axial beam velocity is increased so that δ_0 is decreased, thus reducing the amplitude of oscillation of the beam boundary. Finally, the profile of a beam with a small nonuniformity in the accelerating region is illustrated in Fig. 4. The nonuniformity has the effect of increasing the amplitude of oscillation of the beam boundary.

V. CONCLUSIONS

In the present paper the analysis of relativistic electron flows has been generalized to account for radial variations in the electron velocity. The condition for nonspreading of a laminar-flow drifting electron beam is given by inequality (32), which is expressed in terms of the beam parameters p_0, μ, and δ_0, which are related to the ratio of transverse to longitudinal velocity at the input, the beam current, and the axial beam voltage. It is shown that for the case $\mu^2\ll 1$, which is rather common in many experimental systems, the equilibrium condition for Brillouin flow with the relativistic correction is given by Eq. (37). The derived condition (37) implies that for a fixed ω_p the higher the axial velocity of the beam the less applied axial magnetic field that is needed to obtain a perfectly balanced flow. In the present investigation, since it is assumed that the transverse velocity is much smaller than the axial velocity, the axial component of self-induced magnetic field would be much smaller than the θ component of self-induced magnetic field and consequently the dominant self-focusing effect is due to the Lorentz force from the θ component of self-magnetic field and the axial beam velocity.

If a linear beam is launched at the entry of the drift region, in which case p_0 is zero, then from Eq. (34) it is necessary that either $\delta_0=1$ or $\mu^2=8$ for the beam to be perfectly balanced. However, for normal laboratory operating conditions, the latter condition, i.e., $\mu^2=8$, is rarely satisfied. Consequently, a perfectly balanced flow would not likely be obtained unless the axial beam velocity is nearly equal to the speed of light in vacuum.

Nonzero values of $\nu=(k/\beta_0)$ indicate the existence of a nonuniformity along the beam. The cause of the nonuniformity may be due to various factors, e.g., for short beams the beam termination will effect the overall beam configuration due to reflections. Also when a static spatially periodic electric field is used in the focusing of the beam, k can be determined from the spatial periodicity of the applied static electric field.

The system parameters used for Figs. 1–4 are conveniently chosen to illustrate the method of analysis and generally correspond to the physical conditions existing in accelerator devices rather than those of microwave beam devices. However, it is not difficult to make similar calculations for the parameters which represent closely the physical condition encountered in any experimental system.

The method of analysis developed here can be extended to include the effects of positive ions which may be present when a partially neutralized beam is considered, provided that the potentials Ψ and $\mathbf{A}$ in the system are properly modified.

LIST OF SYMBOLS

T = the particle kinetic energy,
ϵ_0 = the particle rest energy,
M_0 = the integration constant [see text following Eq. (9)],
M_1 = the integration constant [see text following Eq. (23)],
M_2 = the integration constant [see text following Eq. (24)],
M_3 = the integration constant [see text following Eq. (23)],
K_0 = the equivalent angular momentum of a particle,
$\mathbf{A}$ = the vector potential,
ω_L = the Larmor angular precession frequency,
ω_p = the electron radian plasma frequency,
$\mathbf{p}$ = the particle linear momentum,
ψ = the scalar potential.