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JET PROPULSION LABORATORY

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

UCLA-ENG-7070
July 1970

A STUDY FOR THE COMPARATIVE EVALUATION OF ATTITUDE CONTROL SYSTEMS

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Final Report
Jet Propulsion Laboratory Contract No. 952584

School of Engineering and Applied Science
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Los Angeles, California

PREFACE

This report provides the results of the Phase 2 portion of the study, "Comparative Evaluation of Attitude Control Systems." This study was funded by the California Institute of Technology Jet Propulsion Laboratory (JPL) in accordance with Contract No. 952584.

ABSTRACT

The study described in this report had the twofold objective of i) selecting a competitive alternative to the reaction wheel attitude control system originally selected by JPL for tentative incorporation into a spacecraft for a multi-planet mission, and ii) establishing an improved basis for evaluation of the merits of the chosen alternative by increasing design efficiency beyond that incorporated in JPL preliminary studies of design alternatives.

Attempts to improve the efficiency of the dual-spin attitude control system beyond the level assumed in an earlier JPL study proceeded in two directions: i) Unsuccessful efforts were made to justify reduction in attitude control requirements involving reorientations for midcourse motor firings. ii) Methods were successfully developed to improve the efficiency of propellant utilization in accomplishing prescribed reorientations. Specifically, the problem of fuel-optimal small angle reorientation of a dual-spin vehicle is solved, with dramatic improvements over previously published responses to this problem. Results are applied (suboptimally) to the large-angle turn problem, and propellant requirements are estimated for the dual-spin vehicle on a multi-planet mission.

TABLES

FIGURES

1. INTRODUCTION

Selection of an attitude control system for a flight spacecraft must in a practical situation be based on many subjectively defined criteria. Although serious attempts are always made to assign numerical values to such comparison criteria as weight and power requirements, these numbers are properly recognized to stem ultimately from educated guesses based on experience with previous components, and in many cases the evidence of these numbers is outweighed by even more clearly subjective judgments of reliability or ease of development on schedule. These are the facts of engineering design of complex systems, and they will not be changed in the immediate future.

Even within a practical contemporary framework, however, it is recognized that quantification of design criteria is a laudable objective, and that numerical assessments of the ingredients of a design decision should be as sound as foreknowledge permits. Yet the pressures of time rarely permit the detailed development of numerical inputs, and the choice of an attitude control system is consequently always in a broad sense suboptimal.

It seems a healthy exercise for an organization engaged continuously in the selection among design alternatives to pause on occasion for an introspective period of technical assessment, in order to bring under careful study an engineering decision made previously under the pressures of project development. Such a decision is under examination in this report.

The National Aeronautics and Space Administration (NASA) has funded several studies of spacecraft designed to explore the outer planets. In such an investigation begun several years ago at the Caltech Jet Propulsion Laboratory (JPL) it was decided on the basis of a brief but intensive trade-off study that a dual-spin attitude control system did not compare favorably with a reaction wheel attitude control system. At that time a reaction wheel system was made the preliminary choice for a particular multi-planet mission vehicle, although other options remained under study. Primary remaining contenders were a reaction jet control system employing newly developed micro-thrusters, and a dual-spin attitude control system.

JPL Contract 952584 was negotiated with UCLA for comparative evaluation of attitude control systems; it was initiated in July of 1969, with the objective of providing first a brief review of the full spectrum of alternatives and then an intensive investigation of an alternative to the configuration selected by JPL. Phase I of the study resulted in a report, dated 27 September 1969, recommending deeper quantitative investigation of the dual-spin system. It was noted in that report that in the original trade-off study the dual-spin vehicle sustained a severe penalty in meeting the mission demands for commanded turns. By sharpening the turn requirement specifications and improving the efficiency of the reorientation maneuvers it seemed that one might eliminate the weight advantage originally held by the reaction wheel system over the dual-spin system. The present report is devoted largely to the investigation of this possibility.

It may be noted that the rejection of total reliance on reaction jet control in the noted Phase I report was the result of subjective evaluation of flight-readiness of the required micro-thrusters, based on examination of available literature. Since there appears to be a substantial weight advantage with reaction jet systems when recently developed micro-thrusters are used, the design decision must rest on difficult questions of reliability and engineering feasibility issues lying beyond the scope of this report or its authors' expertise.

The critical question which determines the validity of the weight estimate of the dual-spin system hinges upon interpretation of the requirements for large angle turns. Both the reaction wheel attitude control system and the dual-spin attitude control system were provided in preliminary studies with the capability of making nine large angle turns, each with the capacity for changing the vehicle to any desired orientation. Large angle turns are required for midcourse motor firings before each planetary encounter, and after all but the last encounter, so for a five planet "grand tour" mission there are nine major reorientations.

The decision to require 4π steradian pointing capability for each turn is a conservative choice for any attitude control system, but not uniformly so. The fuel cost associated with rotating a reaction wheel vehicle about any axis is zero, as long as a reversal of the turn is contemplated (assuming no violation of momentum storage requirements). The cost for a dual-spin vehicle is zero if the turn happens to be about the bearing axis (rotor spin axis), but if the axis of rotation is transverse to the bearing axis it becomes necessary to exhaust fuel as required to rotate the angular momentum vector. Thus for a dual-spin vehicle it is critical that the turn magnitudes be estimated without undue conservatism, and it is extremely important to utilize any foreknowledge of the rotation axis for required turns. Since the decision to require 4π steradian pointing capability for each of nine midcourse motor firings imposed a more severe penalty on the dual-spin system than was imposed on the reaction wheel system, the first objective of this study was to examine the midcourse trajectory correction requirements in order to determine whether or not this design constraint is truly necessary. This question is explored in Section 2.

The second objective of this study was to develop the analytical and computational tools necessary to accomplish reorientations of a dual-spin vehicle in an optimal or near-optimal manner, and then to use these tools to estimate fuel costs for orientation control during a multi-planet mission. This objective received major emphasis, and success in its achievement is the major accomplishment of this study.

Section 3 is devoted to the selection of a base-line vehicle suitable for studies of fuel-optimal methods of reorientation.

Section 4 summarizes the results of an extensive investigation of the problem of fuel-optimal small-angle reorientations of dual-spin vehicles, as appropriate for the cruise mode of a multi-planet mission. Contributions to this topic constitute the Ph.D. dissertation of V. Larson, attached to this report as an appendix.

In Section 5 the dissertation results are used to estimate fuel consumption requirements for those large-angle turns required for a multi-planet mission, as dictated by the results of Section 2.

In Section 6 the control system weight and power estimates for dual-spin and reaction wheel systems are presented as originally developed by JPL, together with modifications of fuel requirements for the dual-spin vehicle resulting from this study. In addition, the weight and power requirements of improved baseline systems studied by JPL are presented.

Recommendations for further study appear in the final section.

2. ATTITUDE CONTROL REQUIREMENTS

A grand tour mission trajectory involves encounter with as many as five planets, beginning with Jupiter. As the spacecraft approaches Jupiter (perhaps eighteen to twenty days prior to closest approach), a midcourse motor is fired to provide an incremental correction to the vehicle velocity in order to refine the trajectory towards its nominal state. Another velocity correction is necessary shortly after Jupiter encounter. Similar pairs of corrections are required in the vicinity of Uranus and subsequent planets, until the mission is completed with the final planetary encounter.

Throughout the days and years of flight through interplanetary space, an antenna of the spacecraft must maintain an earth-pointing orientation within a specified tolerance, requiring an extensive series of small-angle turns. In the interplanetary space beyond Jupiter, a tolerance of one milliradian is imposed, and this number increases linearly with distance to five milliradians near the earth. For this interplanetary or <u>cruise mode</u> of the mission, dynamic analysis based on linearized equations of motion is appropriate.

When in the neighborhood of Jupiter and subsequent planets it becomes necessary to fire a midcourse motor in the direction required for velocity correction, the antenna lock on the earth is temporarily relinquished while the vehicle orientation is changed as necessary to properly point the motor. Subsequent to motor burn the antenna is returned to an earth-pointing orientation, which is maintained until the next midcourse correction is required. This process repeats nine times for a five-planet grand tour mission.

In the absence of specific information to the contrary, it must be assumed that the velocity increment demanded for trajectory correction is of random direction, requiring the capability of reorienting the vehicle to fire the midcourse motor in any direction. This was the assumption adopted in JPL's preliminary selection of an attitude control system for a multi-planet spacecraft.

If on the other hand it could be established that velocity increments would be required only in the ecliptic plane, they by placing the midcourse motor (with autopilot) on the despun platform of a dual-spin vehicle with rotor axis normal to the orbital plane, one could accomplish the necessary reorientations with the electric motor of the despin control system, expending no propellant for angular momentum vector reorientation.

It is the objective of this chapter to determine by statistical estimation whether or not a statement between the extreme alternatives of the two preceding paragraphs would diminish the weight penalty sustained by the dual-spin attitude control system in comparison with the reaction wheel control system.*

The procedure adopted here involves the mapping of the error ellipsoid associated with the covariance matrix of position and velocity errors at orbit injection into errors at Jupiter encounter, and determining by a similar mapping the velocity correction ellipsoid (covariance matrix) required at a given point of the trajectory to cancel the indicated target error.

If Λ_I is the six by six covariance matrix of position and velocity errors at injection, and $\Lambda_{\Delta v}$ is the three by three covariance matrix of the velocity increments required to correct the trajectory at a given point of the orbit, then these matrices are related by

$$\Lambda_{\Delta v} = P \; \Lambda_I \; P^T \tag{1}$$

where P is a three by six matrix available as the product of [C] and a direction cosine matrix establishing the reference axes of matrices generated by the JPL computer program ANAPAR. Specifically, the matrix P may be written

$$[P] = [C][K^{-1}][A \quad K][U] \tag{2}$$

where [C] is the direction cosine matrix required for transformation to geocentric ecliptic reference axes from geocentric equatorial axes and

*This objective could not have been met without the analytical and computational support of A. Khatib of JPL.

the matrices A, K, and U represent matrices of partial derivatives established numerically in the ANAPAR program for any point in time.

It was the expectation of the principal investigator after discussions with JPL trajectory analysts that the velocity correction ellipsoids for the several midcourse corrections would be extremely flat, with little extension in the direction normal to the ecliptic. If the three-sigma component of the velocity correction in this direction proved to be sufficiently small, it would be possible to provide this trajectory correction capability independently of the midcourse motor, thereby precluding the necessity of large-angle turns entirely.

A numerical covariance matrix of injection errors typical of boosters in the appropriate class was provided by JPL, and the corresponding velocity correction covariance matrix was calculated for a midcourse motor burn at 540 days from launch, approximately 20 days before closest approach to Jupiter. The ellipsoid for the velocity correction did <u>not</u> have the desired flatness property, thereby frustrating the objective of eliminating the large-angle turns for the dual-spin vehicle.

Numbers used in the indicated calculation are documented in what follows:

$$
A_I = \begin{matrix}
1.46720253E+00 & -7.20787890E-02 & 8.56707000E-02 & 5.40521340E-02 & 1.87504938E-02 & -4.32557340E-02 \\
-7.20787830E-02 & 1.87511316E-01 & -3.87526230E-01 & 1.18442664E-02 & 5.23837920E-03 & -1.20019449E-02 \\
8.56707000E-02 & -3.87526260E-01 & 1.32149472E+00 & -4.29633330E-02 & -1.79779818E-02 & 4.26056820E-02 \\
5.40521400E-02 & 1.18442682E-02 & -4.29633300E-02 & 3.72084810E-03 & 1.40456463E-03 & -3.27112470E-03 \\
1.87504959E-02 & 5.23838010E-03 & -1.79779818E-02 & 1.40456466E-03 & 5.37383910E-04 & -1.25093625E-03 \\
-4.32557370E-02 & -1.20019437E-02 & 4.26056790E-02 & -3.27112500E-03 & -1.25093625E-03 & 2.91872616E-03
\end{matrix} \qquad (3)
$$

$$
U = \begin{matrix}
-6.09011960E+04 & 4.04708980E+04 & 1.81426250E+04 & -2.79750190E-03 & 6.96135670E-04 & 3.98763880E-04 \\
-7.39662340E+04 & 7.22347720E+04 & 3.13350990E+04 & -3.58427040E-03 & 1.01205770E-03 & 5.76319760E-04 \\
3.55558600E+04 & -3.29282650E+04 & -2.02794310E+04 & 1.73234430E-03 & -5.06967030E-04 & -2.45696340E-04 \\
6.70069470E+07 & -2.91947810E+07 & -1.59280240E+07 & 2.96256560E+00 & -6.67380390E-01 & -3.68901220E-01 \\
-1.39577520E+08 & 1.20043740E+08 & 5.54763180E+07 & -6.64378990E+00 & 1.80879020E+00 & 1.01082270E+00 \\
-4.21427630E+07 & 3.14532320E+07 & 2.33431470E+07 & -2.00151390E+00 & 5.59557670E-01 & 2.51241920E-01
\end{matrix} \qquad (4)
$$

$$
A = \begin{matrix}
1.25082840E+00 & 7.90575000E-02 & -1.13493310E+00 \\
-1.08379100E+01 & 3.69264300E+00 & 2.01739710E+00 \\
3.18475870E-05 & -8.20689920E-06 & -4.71707610E-06
\end{matrix} \qquad (5)
$$

$$
K = \begin{matrix}
-2.65005440E+07 & 2.27537020E+07 & 6.17851350E+06 \\
2.42958160E+08 & -1.64722720E+08 & -6.96132810E+07 \\
-6.61353600E+02 & 4.95103720E+02 & 2.09161250E+02
\end{matrix} \qquad (6)
$$

$$
C = \begin{matrix}
1.00000000E+00 & 0. & 0. \\
0. & 8\ 87413447E-01 & 4.60974374E-01 \\
0. & -4.60974374E-01 & 8.87413447E-01
\end{matrix} \qquad (7)
$$

The covariance matrix $\Lambda_{\Delta V}$ from Equations (1)-(7) is

$$\Lambda_{\Delta V} = \begin{bmatrix} 9.76255387E+00 & 2.95048724E+00 & -5.88235777E+00 \\ 2.95048721E+00 & 8.92794381E-01 & -1.78220161E+00 \\ -5.88235778E+00 & -1.78220163E+00 & 3.56300031E+00 \end{bmatrix} \quad (8)$$

providing one sigma values for velocity increments in directions x,y, and z of

$$\begin{aligned} \sigma_x &\cong 3.124 \text{ km/sec} \\ \sigma_y &\cong 0.945 \text{ km/sec} \\ \sigma_z &\cong 1.888 \text{ km/sec} \end{aligned} \quad (9)$$

Since the z axis is normal to the ecliptic plane, the corresponding ellipsoid certainly does not have a dimension in this direction of the anticipated small relative size.

The conclusion of this chapter is thus disappointingly negative. With the acknowledgment that velocity corrections in all directions have comparable statistical likelihood (as assumed in the original JPL attitude control system trade-off study), the prospect of overcoming the weight advantage held by the reaction wheel system over the dual-spin system is much reduced.

There remains the possibility of reducing propellant weight estimates by more efficient use of fuel in accomplishing the required turns. Determination of a fuel-optimal control law is a task of substantial analytical and computational complexity, as may be judged by the dissertation here attached as an appendix. Success in this endeavor cannot be expected however to accomplish the reduction in propellant weight demanded to overcome the weight difference between reaction wheel and dual-spin control systems as originally estimated by JPL.

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3. DUAL-SPIN VEHICLE PARAMETER SELECTION

In this section, the values of the dual-spin vehicle parameters used in computing the fuel-optimal controller are given. In this work, no attempt is made to find the optimal parameters; instead, the fuel-optimal controller is determined for a configuration which is believed to be appropriate for the Multi-Planet Mission. The parameters that enter the analysis are

(1) h/I_1, the ratio of the stored angular momentum h and the transverse inertia of the vehicle I_1
(2) σ, the rotor speed relative to the despun portion
(3) K, the ratio of applied moment M and transverse inertia I_1
(4) $\bar{r}$, the ratio of the rotor inertia about the spin axis J_3^R and the transverse inertia of the system I_1.

Estimates of these parameters depend on

(1) The inertia characteristics of the system
(2) The external torque environment
(3) The geometry of the vehicle
(4) The assumptions made concerning the orbit (the out-of-plane drift, etc.)
(5) The accuracy requirement.

3.1 Determination of the Estimates of the System Parameters

In this section, estimates of the system parameters are determined. The inertia characteristics of the vehicle are determined primarily by the need for a configuration which allows the mission objectives to be satisfactorily achieved. The need for such components as the

(1) despun platform
(2) antenna
(3) rotor
(4) planetary encounter instrumentation
(5) jets

coupled with a consideration of the mission requirements determines the weight and geometry of the system. In this analysis, it is assumed (somewhat arbitrarily) that the transverse inertia the symmetric vehicle is 200 slug-ft^2, and that the ratio of the inertia of the rotor about the spin axis (J_3^R) and the transverse inertia of the system (I_1) is 0.15, i.e.,

$$\tilde{r} = \frac{J_3^R}{I_1} = 0.15$$

The external torque depends on the geometry of the vehicle and its position (distance from the sun). The maximum torque due to solar radiation for the TOPS vehicle was previously estimated as 50 dyne-cm.* If the spacecraft (s/c) were configured as shown in Figure 3.1, the torque due to solar radiation would be relatively large. An indication of the magnitude of the solar radiation torque for such a vehicle at the earth's surface is provided below.

Item	Value of Equation
• Solar radiation pressure of 1 A.U.	
Absorption	$p_o \doteq 0.456 \frac{\text{dyne}}{\text{m}^2} \cong 9.7\text{x}10^{-8} \frac{\text{lb}}{\text{ft}^2}$
Perfect Reflection	$p_o = 0.91 \frac{\text{dyne}}{\text{m}^2}$
• Parameters	
Area	$A = \pi(7.5)^2 \text{ ft}^2$
Lever Arm	$\ell = 10$ ft
• Solar Radiation Torque	$T_D = p_o A\ell$
	$= 2300$ dyne-cm$=1.7\text{x}10^{-4}$ ft #

*This estimate was obtained during an informal conversation with Ed Dorroh of JPL.

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This torque is over 40 times larger than that for the TOPS vehicle.*

3.1.1 Stored Angular Momentum, h

In this section, an estimate of the angular momentum that is stored in the rotor is obtained. First, it should be noted that the optimal value depends on the nature of the mission. The mission consists primarily of two portions, viz., the

(1) cruise portion in which a large angular momentum is desirable

(2) large angle turn mode in which a small stored angular momentum is desirable.

The angular momentum h that should be stored in the rotor, for the cruise mode, is dependent upon the

(1) disturbance torque environment

(2) out-of-plane angle of the orbit (assumed small in the analysis).

By selecting a range of values for h and by using the maximum value of the external torques, the drift rate of the angular momentum vector can be determined. The drift rate is important since it determines how frequently the jets have to be fired in order to keep the angular momentum vector properly oriented. Estimates of the drift rate β (in the vicinity of Jupiter) based on

$$T_D = \frac{\Delta h}{\Delta t} = h \frac{\Delta \beta}{\Delta t} = h\dot{\beta},$$

are provided below (see Table 3.1) for various values of T_o and h.**

*This value could be decreased by mounting the rotor closer to the antenna, i.e., by decreasing the lever arm associated with the solar radiation force.

**The solar radiation torque is approximated as $T_o = T_o \cdot 1/R^2$ where T_o = solar radiation torque near the earth
R = distance from the sun in a.u. (R=5 for Jupiter).

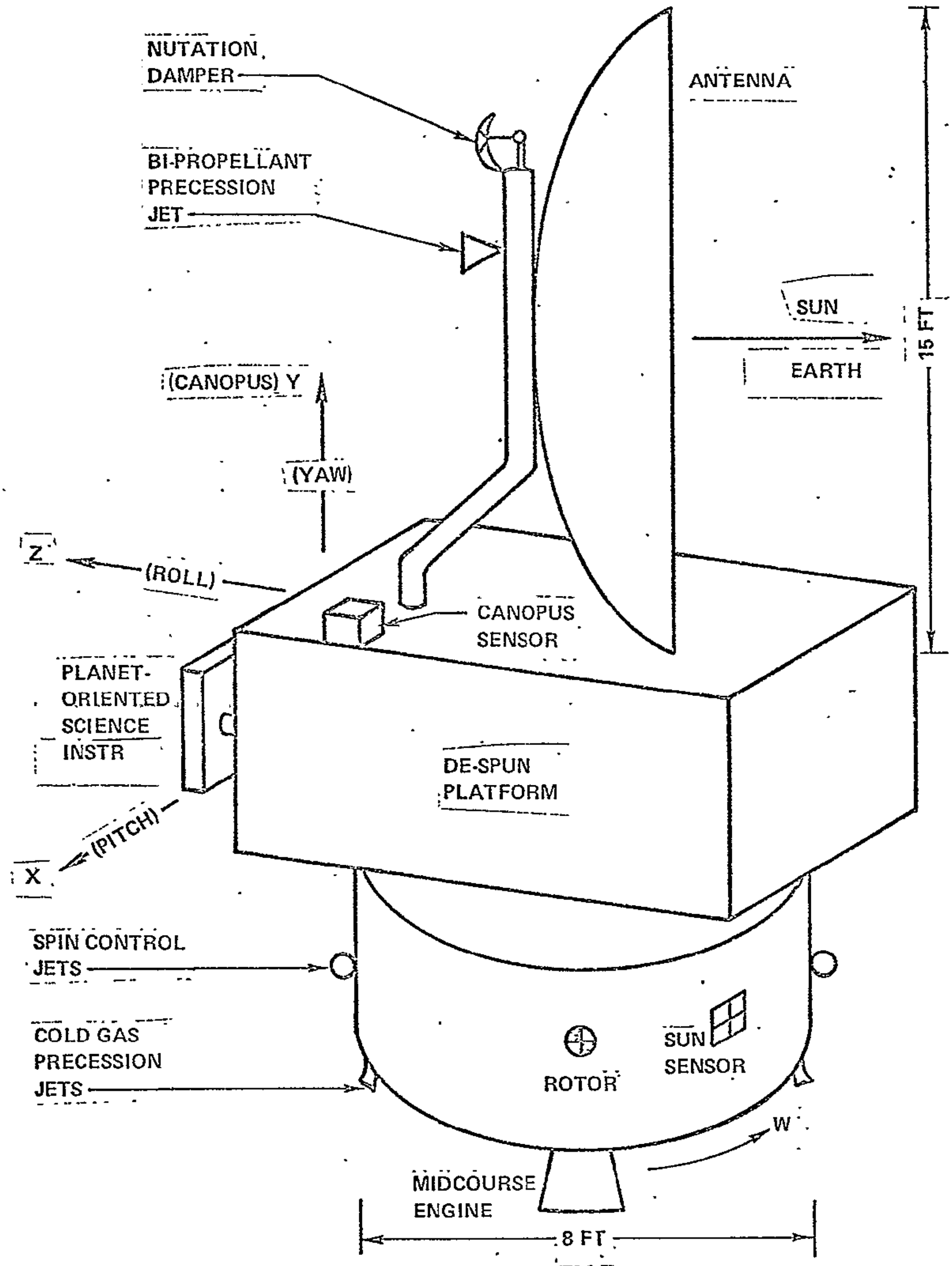


Figure 3.1 Dual-Spin Attitude Control Configuration

Table 3.1

DRIFT RATE OF ANGULAR MOMENTUM VECTOR DUE TO DISTURBANCE TORQUES (SOLAR RADIATION) NEAR JUPITER

Angular Momentum, h ft # sec	Drift Rate $\dot{\beta}^*$ $\frac{deg}{day}$ for Various T_D^{**} (Dyne-Cm)			
	$T_D = 50$	$T_D = 100$	$T_D = 500$	$T_D = 2000$
100	.0072	.0148	.0732	.292
200	.0036	.0072	.0364	.148
300	.0024	.0048	.0244	.096
500	.0016	.0028	.0148	.06

Based on the results of Table 3.1, a range of values considered appropriate for h is

$$200 \leq h \leq 300 \text{ ft-lb-sec}$$

3.1.2 Determination of the Rotor Speed

The rotor speed can be determined from the assumed values of the parameter $\bar{r} = J_3^R/I_1 = 0.15$ and the stored angular momentum h. For values of h between 100 and 500 ft-lb-sec, the corresponding rotor speed σ is computed from

$$\sigma = \frac{h}{\bar{r}\, I_1} = \frac{h}{J_3^R}$$

and is tabulated below (see Table 2.2).

Table 3.2

ROTOR SPEED VERSUS STORED ANGULAR MOMENTUM

Angular Momentum h $h = J_3^R\sigma$ (ft-lb-sec)	Rotor Speed, σ (RPM)
100	30
200	60
300	100
500	150

---

* The drift rate $\dot{\beta}$ is

$$\dot{\beta} = 0.014604 \frac{T_D}{h} \text{ deg/day}$$

**One ft-lb = 1.35582×10^7 dyne-cm.

3.1.3 Determination of the Ratio of Applied Moment and Transverse Inertia, K

The system parameter K defined by

$$K = \frac{M_A}{I_1}$$

depends on the

(1) stored angular momentum, h

(2) lever arm of jet

(3) allowed reorientation time

(4) reorientation accuracy.

From the expressions

$$M_A = h \frac{\Delta\theta}{\Delta t}$$

$$M_A = F_A r$$

the required thruster capability is

$$F_A = \frac{h}{r} \frac{\Delta\theta}{\Delta t} \quad \#$$

The term $\Delta\theta/\Delta t$, the reorientation rate, provides a measure of the desired response time. The orientation tolerance is approximately 0.06 degrees.* An estimate of the reorientation rate can be obtained by requiring that an angle corresponding to 100 times the tolerance (consistent with small angle approximation) be nulled within 30 seconds. Using this criterion, the response rate for the cruise mode becomes

$$\frac{\Delta\theta}{\Delta t} = \frac{6 \text{ deg}}{30 \text{ sec}} = 0.2 \frac{\text{deg}}{\text{sec}} = 3 \frac{\text{mr}}{\text{sec}} .$$

---

*This tolerance applies for the Jupiter to Neptune portion of the mission.

The values of K are provided below (Table 3.3) for reasonable ranges of the rotor radius, the stored angular momentum, and the reorientation rates. As seen in Table 3.3 a reasonable constraint on K is

$$0.001 \leq K \leq 0.005 \quad \frac{1}{\sec^2}$$

3.2 Values of Parameters Used in the Numerical Work

Based on the discussion provided above, the values of the dual-spin vehicle parameters to be used in the numerical determination of the fuel optimal controller are

$$\bar{r} = 0.15 \text{ (dimensionless)}$$

$$\frac{h}{I_1} = \frac{300}{200} \quad \frac{\text{ft-lb-sec}}{\text{slug-ft}^2} = 1.5 \frac{\text{rad}}{\text{sec}}$$

$$K = \frac{M_A}{I_1} = 0.001 \quad \frac{1}{\text{sec}}$$

$$\sigma = 10 \frac{\text{rad}}{\text{sec}}^{*}$$

---

*One rad/sec = 9.549 rpm.

Table 3.3

RATIO OF APPLIED MOMENT AND TRANSVERSE INERTIA*

Rotor Radius, r (ft)	Angular Momentum, h (ft # sec)	Thrust, F_A (mlb)		Ratio of Applied Moment and Transverse Inertia, $K\left(\frac{1}{sec^2}\right)$	
		$\frac{\Delta\theta}{\Delta t} = 1\ \frac{mr}{sec}$	$\frac{\Delta\theta}{\Delta t} = 3\ \frac{mr}{sec}$	$\frac{\Delta\theta}{\Delta t} = 1\ \frac{mr}{sec}$	$\frac{\Delta\theta}{\Delta t} = 3\ \frac{mr}{sec}$
2	200	100	300	0.0010	0.0030
	300	150	450	0.0015	0.0045
4	200	50	150	0.0010	0.0030
	300	75	225	0.0015	0.0045

*Computed for the case in which

$$I_1 = 200 \text{ slug-ft}^2$$

$$F = \frac{J_3^R}{I_1} = 0.15$$

4. FUEL-OPTIMAL CONTROL IN THE CRUISE MODE

In this section, the fuel-optimal control of the symmetric dual-spin vehicle described in Section 3 is briefly discussed (see Appendix 1, for a detailed treatment of this topic). The control concept is a hybrid attitude control scheme consisting of (see Figure 1.1 of Appendix 1)

(1) an active phase in which the angular momentum vector $\underline{H}$ is aligned to the desired direction $\underline{H}_D$ (angular momentum control, AMCO),

(2) a passive phase in which the nutation damper is used to complete the control objective of aligning the rotor spin axis and the desired angular momentum vector $\underline{H}_D$.

In the cruise mode, the objective of the attitude control system is to maintain the desired orientation of the rotor axis (and hence the antenna). Since the tolerance on the antenna pointing accuracy is stringent (0.06 deg at Jupiter and beyond), the deviations from the desired orientation are perforce, small. This implies that the attitude angles of the rotor axis can validly be assumed to be small during the cruise mode. A 3-axis control scheme is, of course, needed to accomplish the control task. The control of the despun portion about the spin axis is considerably simpler than the control of the vehicle about the other two axes. Essentially a motor-controlled closed-loop is used to ensure that the antenna tracks the earth. The emphasis of this study is on the control of the pitch-roll motion of the spacecraft.

The fuel-optimal control problem $\{L, \Delta, \Omega, X_0, X_1, J\}$ for the symmetric dual-spin vehicle in the cruise mode simply stated is as follows. Given

(1) the linear plant*

$$(L)\ \dot{\underline{x}} = A\,\underline{x}(t) + B(t)\,\underline{u}(t) \text{ in } C^1 \text{ in } R^{n+m+1}$$

*See the List of Symbols of Appendix 1 for a definition of symbols used.

(2) the class of admissible controllers Δ

(3) the control restraint set Ω

(4) the initial set X_o

$$X_o = \{\underline{x},t) : \underline{x}(t_o) = \underline{x}_o \quad \text{(fixed)}\}$$

(5) the target set X_1

$$X_1 = \{\underline{x},t) : g_j(\underline{x},t_1) = 0 \quad \forall j, \; t_1 \text{ free}\}$$

(6) the cost functional J (that appropriate for fuel-optimization),

the problem is to find the controller $u(t) \subset \Omega$ which

(a) takes $\underline{x}_o$ to X_1 such that the pair

$$(\underline{x}(t_1), t_1) \in X_1$$

(b) minimizes the cost functional $J(u)$.

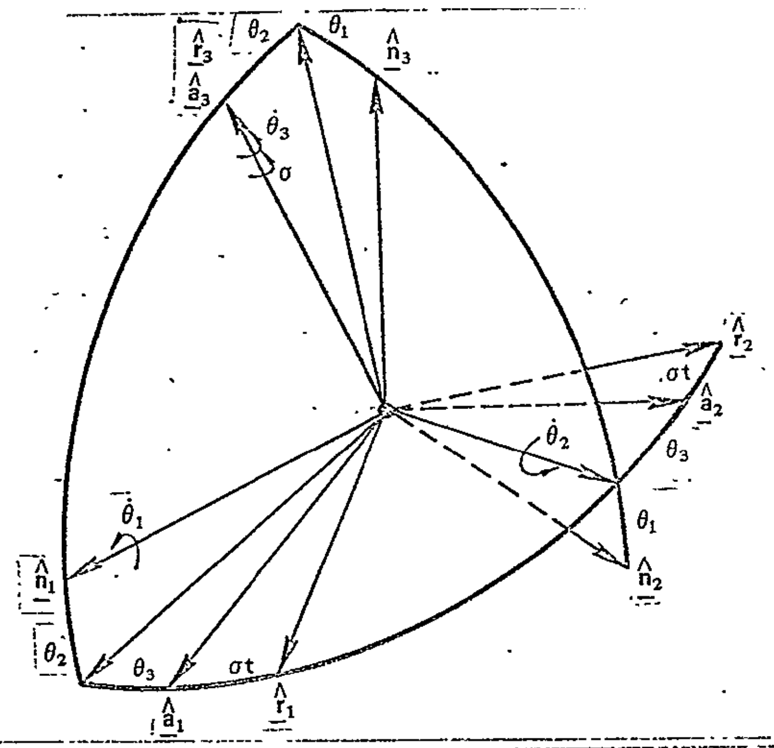
In the above problem statement, it is important to note that the control restraint set Ω is chosen after carefully considering the practical aspects of the problem. It depends on the type of jet used for control, the number of jets used, etc. The target set X_1 is determined from a consideration of the implications of aligning the angular momentum vector $\underline{H}$ to the desired direction $\underline{H}_D$. The cost functional is that appropriate for fuel-optimization and depends on the control restraint set Ω.

4.1 Nature of the Elements of the Control Problem $\{L,\Omega,X_o,X_1,J\}$

In this section, the elements of the fuel-optimal control problem $\{L,\Omega,X_o,X_1,J\}$ are briefly discussed (see Chapter 3 of Appendix 1 for more details).

Plant

The plant which characterizes the dual-spin system is described below (see Figure 4.1 for the definition of the elements of the state vector). The plant is obtained as a special case of a general and powerful formulation suitable for an elastic constant mass system (see



Notes:

1. $\hat{\underline{n}}_\alpha$ form basis for Newtonian frame
2. $\hat{\underline{a}}_\alpha$ form basis for body frame A
3. $\hat{\underline{r}}_\alpha$ form basis for rotor frame
4. Rotation sequence: 1–2–3

Figure 4.1 Coordinate Frames for the Dual-Spin Vehicle

Chapter 2 of Appendix 1).. The equation representing the dual-spin vehicle in the cruise-mode is

$$\underline{x} = A\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

where

$$A = \left[\begin{array}{cc|c} 0 & -\bar{r}\,\sigma & 0 \\ \bar{r}\,\sigma & 0 & \\ \hline & I & 0 \end{array}\right] \tag{4.1}$$

$$B(t)\,\underline{u}(t) = \left[\begin{array}{c} c\sigma t \\ s\sigma t \\ \hline 0 \end{array}\right] K\,u(t)$$

$$\underline{x} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}$$

The coordinate frame used in writing Equation (4.1) is the despun body A and the angular velocities ω_1, ω_2 refer to the indicated components of the angular velocity of frame A relative to the inertial frame N.

Control Restraint Set Ω

The control restraint set suitable for the application at hand depends on the type and the number of jets used. The notion of controllability is used to demonstrate that the minimum number of jets required is one (see Chapter 4 of Appendix 1) and the notion of system normality is used to demonstrate that the preferred location of the jet is the rotor (see Chapter 4 of Appendix 1). The implications of the necessary conditions for optimality (see Chapter 5 of Appendix 1) are used to demonstrate that the preferred type of jet is a one-way jet. Of course, a two-way jet could be used with one side of the jet providing the necessary redundancy. The main advantage of the one-way jet as compared to the two-way jet is that of greater reliability. This follows because the number of switchings (firings) of the one-way jet is one-half that for the two-way jet (see Chapter 5 of Appendix 1). It follows therefore that the control restraint suitable for the problem at hand is

$$\Omega = \{u(t) : 0 \le u(t) \le 1\} \tag{4.2}$$

Initial Set X_o

The initial set X_o consists of the particular pair $(\underline{x}_o, t_o)$ which exists at the time that the relationship

$$||\underline{\theta}|| = \sqrt{< \underline{\theta}, \underline{\theta} >} < \theta_c^{*}$$

is not satisfied. The value θ_c refers to the tolerance on the antenna pointing accuracy; the optimal control sequence must be initiated when the norm of $\underline{\theta}$ becomes as great as θ_c. The value of θ_c is approximately 5 mr near the earth, and 1 mr near Jupiter and beyond. It is tacitly assumed that appropriate sensors for measuring $\underline{x}_o$ would be provided for in the event the optimal scheme were to be implemented.

Target Set X_1

The target set X_1 for the AMCO concept used for the symmetric dual-spin vehicle in the cruise mode is

$$X_1 = \{(\underline{x},t): \ g_j(x(t_1)) = 0, \quad j = 1,2\}^{**} \tag{4.3}$$

where

$$g(\underline{x}(t_1)) = \begin{bmatrix} 1 & 0 & 0 & \bar{r}\sigma \\ 0 & 1 & -\bar{r}\sigma & 0 \end{bmatrix} \underline{x}(t_1)$$

$$\bar{r} = \frac{J_3^R}{I_1}$$

$$\bar{r}\sigma = \bar{\beta}$$

This target set is said to be a smooth two-fold in R^n. Examination of Equation (4.3) reveals that it is a linear manifold and hence is convex. In addition the set X_1 is closed. However, the compactness of X_1 is guaranteed only if the set X_1 is bounded. From practical considerations, it is clear that the control problem would not make any sense if the elements of the state were not bounded. The fact that $\underline{x}(t)$ is bounded

*The components of $\underline{\theta}$ are the attitude angles θ_1, θ_2.

**The components of g refer to the components of $\underline{H}$ transverse to $\underline{H}_D$.

$\forall t$ guarantees that X_1 is bounded and hence is compact.* Stated in another way, the boundedness of the initial state $\underline{x}_o$ and the fact that the optimal control sequence is such that the resulting state $\underline{x}(t) \leq K_1\underline{x}_o$, it follows that the state is bounded $\forall t$ (K_1 is some arbitrary constant).

Cost Functional J

The cost functional J associated with fuel-optimal controller, having determined that the control restraint set appropriate for the application at hand is as given in Equation (4.2)

$$J(u) = \int_{t_o}^{t_1} K\, u(t)dt \tag{4.4}$$

It should be noted that this differs from that usually associated with fuel-optimization of nonspinning vehicles. For spinning vehicles, the spin itself provides the direction of control and hence ensures that the requirement that both positive and negative moments be available for control is satisfied.

4.2 Necessary Conditions for Local Optimality

In this section, the necessary conditions for local optimality are provided. These conditions can be obtained either from the calculus of variations or from Pontryagin's maximum principle (see Chapter 5 of Appendix 1). The necessary conditions

(1) provide information concerning whether the problem is normal or singular,

(2) provide information concerning the nature of the optimal controller so that the most appropriate control restraint set can be selected,

(3) aid in the selection of a computational technique,

*Even for practical problems it is necessary to demonstrate the compactness of the target set X_1 to ensure that there is even a chance that an optimal solution exists.

(4) provide the basis of every computational technique save the gradient method.

The necessary conditions for the symmetric dual-spin vehicle in the cruise mode are:

(1) Hamilton's canonical equations

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},\underline{u},t) = \frac{\partial H}{\partial \underline{p}}(\underline{x},\underline{u},p,t) = A\underline{x}(t) + \underline{b}(t)\, u(t)$$

$$\dot{\underline{p}}(t) = -\frac{\partial H}{\partial \underline{x}} = -A^T \underline{p}(t) \tag{4.5}$$

(2) The boundary conditions

$$\underline{x}(t_o) = \underline{x}_o$$

$$\underline{p}(t_1) = \left[\frac{\partial \underline{g}}{\partial \underline{x}}\right]^T \Bigg|_{t=t_1} \underline{\nu} \tag{4.6}$$

where $\underline{\nu}$ is a constant vector to be determined.

(3) The condition on the Hamiltonian†

$$H^*(t_1^*) = 0 \tag{4.7}$$

(4) The optimality condition

$$u^*(t) = \text{hev}\{q^*(t) - 1\} = \text{hev}\{<\underline{b}(t), \underline{p}^*(t)> - 1\} \tag{4.8}$$

where

$$\text{hev}\{q(t)-1\} = \begin{bmatrix} 0 & \text{if } q(t) < 1 \\ 1 & \text{if } q(t) > 1 \end{bmatrix}$$

The Heaviside function (see Figure 4.2) is appropriate in this case because a one-way jet is used instead of a two-way jet. Conventionally, the fuel-optimal controller is expressed as a dez function (in this case, the control is constrained according to $|u_j| \leq 1$).

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†The Hamiltonian is defined as

$$H = <\underline{p}, \dot{\underline{x}}> - f_o$$

where f_o is the integrand of the cost functional.

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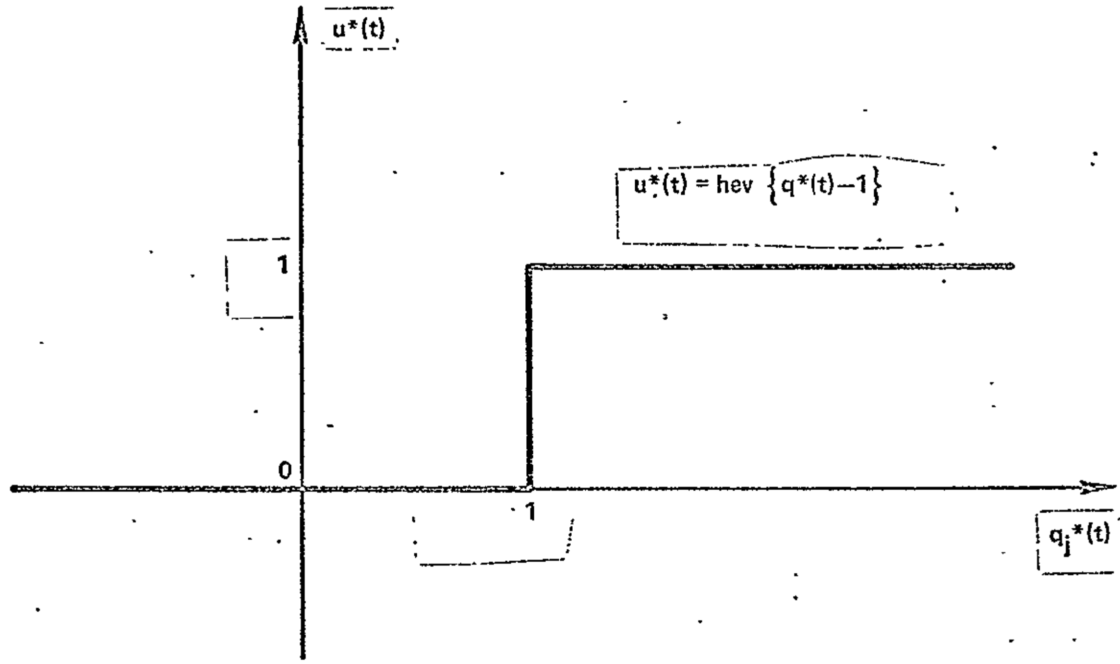

Figure 4.2 The Function u*(t) for a Fuel-Optimal Problem in which a One-Way Jet is Used

4.3 Computational Algorithm

The computational algorithm used in computing the fuel-optimal controller is briefly described in this section (see Chapter 6 of Appendix 1 for more details). The algorithms that are suitable for this task are

(1) the gradient technique,
(2) the generalized Newton-Raphson (GNR) technique or the method of quasilinearization,
(3) the classical Newton-Raphson (CNR) technique.

The advantages and disadvantages of these techniques as well as others are provided in Table 6.3 of Appendix 1. The iterative nature of the computational techniques can be seen by examining Table 6.1 of Appendix 1 which is provided below for convenience (see Table 4.1).

Table 4.1

ITERATIVE NATURE OF COMPUTATIONAL TECHNIQUES

Computational technique	Equations nominal Solution satisfies	Equations iterated on
Direct methods Gradient	State equations Adjoint equations	Boundary conditions Optimality conditions
Second variation Indirect Classical Newton-Raphson	State equations Adjoint equations Optimality conditions	Boundary conditions
Generalized Newton-Raphson Quasilinearization	Boundary conditions Optimality conditions	State equations Adjoint equations

With due consideration to such aspects as

(1) the nature of the optimal controller and the relatively large number of switching times,
(2) the normality of the fuel-optimal problem being studied,
(3) the relatively low dimension of the problem,
(4) the fact that the <u>final</u> rather than the initial adjoint variables are involved,

(5) the theoretical disadvantage concerning the violation of the differentiability hypothesis for both the gradient and GNR techniques (for fuel-optimal problems in which two-way jets are used),*

(6) the ease in which the control constraints are handled in the CNR technique,

(7) the computer storage requirements,

(8) the fact that only very small deviations from the nominal trajectory are allowable,

the CNR algorithm is considered suitable for the determination of the fuel-optimal controller for the symmetric dual-spin vehicle in the cruise mode. Concerning item (8), for the application under consideration, the antenna pointing accuracy requirement is such that the optimal control sequence would be initiated when the pointing error is greater than one mr.** This implies that the values of the state variables must be kept close to the nominal or desired values. This aspect is very important when the CNR technique is used (because of the nature of the iterative scheme).

The use of the CNR method appears appropriate for the problem being investigated. In general, however, the CNR technique is <u>seldom</u> appropriate for optimal control problems.

4.3.1 Iterative Nature of the CNR Algorithm

In this section the iterative nature of the CNR algorithm is briefly discussed. Inherent in this technique is the equation

$$\underline{F}(\underline{y}) = 0 \tag{4.9}$$

which must be solved iteratively for $\underline{y}$. The n and n+1th elements of the vector sequence $\{\underline{y}_n\}$ are related by

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*Initially both two-way jets and one-way jets were considered. In addition, if the spin rate of the vehicle were slow and the torque capacity low, then two-way jets might be desirable for certain problems.

**The pointing accuracy requirement is one mr near Jupiter and beyond; near the earth 5 mr would be allowable.

$$\underline{y}_{n+1} = \underline{y}_n - \left[\frac{\partial F}{\partial \underline{y}}(\underline{y}_n)\right]^{-1} \underline{F}(\underline{y}_n) \qquad (4.10)$$

The iterative procedure consists of the following steps:

(1) guess an initial value of the constant vector $\underline{y}$ and call it $\underline{y}_o$,

(2) using $\underline{y}_o$ solve simultaneously the state equation, the adjoint equation, and the optimality equation and obtain $\underline{F}(\underline{y}_o)$,

(3) evaluate $\partial F/\partial \underline{y}\ (\underline{y}_o)$ numerically and compute its inverse,

(4) obtain $\underline{y}_1$ from Equation (4.10),

(5) repeat the process until $||\underline{F}(\underline{y}_n)|| \leq \delta$.

For the symmetric dual-spin vehicle in the cruise mode, the vectors $\underline{F}$ and $\underline{y}$ have components $(\mathscr{H}, H_1, H_2)^{\dagger}$ and (ν_1, ν_2, t_1), respectively.

4.4 Fuel-Optimal Controller

In this section the numerical results obtained for the problem being investigated are provided (see Figures 7.1 through 7.4 of Appendix 1). The results are obtained for an initial condition of special significance in the fuel-optimal control problem pertaining to the symmetric dual-spin vehicle in the cruise mode. The initial condition refers to components of the initial state vector $\underline{x}_o$; the numerical results are obtained for the case in which the components of $\underline{x}_o$ are (0,0,5 mr,0) where $\theta_1 = 5$ mr refers to the allowable tolerance on the antenna pointing near the earth. The results for this carefully chosen initial condition can be used to estimate the fuel consumption during the cruise mode of the mission.

The optimal controller $u^*(t)$ obtained for the initial condition described above is shown in Figure 7.1 of Appendix 1. For this case,

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$^{\dagger}\mathscr{H}$ is the Hamiltonian and H_1, H_2 are the components of the angular momentum transverse to the desired direction $\hat{\underline{H}}_D$.

the controller is turned on for one half of a rotor revolution and turned off for the other half. The number of switchings involved is 74 and the time used to drive the initial state $\underline{x}_o$ to the target set is 22 seconds. The value of the cost functional

$$J(u) = \int_{t_o}^{t_1} K\, u(t)dt$$

associated with the minimum fuel problem is

$$J(u) = 0.0109 \text{ 1/sec}$$

By using the mass flow properties of the jet used, the amount of fuel consumed in accomplishing the control objective can be computed. That is, by using the relation involving the cost

$$J(u) = \int_{t_o}^{t_1} \frac{M}{I_1} u(t)dt$$

and the relation between the maximum jet thrust F and the specific impulse I_s of the cold gas

$$F = I_s \dot{W},$$

the weight of fuel in pounds is given by

$$\text{Fuel} = \int_{t_o}^{t_1} \dot{W}\, u(t)dt = \frac{I_1}{I_s \times r} J(u) \tag{4.11}$$

For a system having the values

I_1 = 200 slug-ft^2
I_s = 70 sec
r = 4 ft (lever arm, the radius of the rotor)
M/I_1 = 0.001 rad/sec

the relationship between fuel and the cost functional J(u) is

$$W^* = \frac{5}{7} \times J(u)$$

The fuel used in driving the initial state $\underline{x}_o$ to the target set X_1 for the case discussed above is

$$W_c^* = \frac{5}{7} \times (.0109) = .00779 \text{ lb.}$$

For the dual-spin vehicle being studied, the initial antenna misalignment of 5 mr corresponds to an initial transverse component of angular momentum of

$$\Delta h_c = h\theta = 300 \times 5 \times 10^{-3} = 1.5 \text{ ft-lb-sec.}$$

In Figure 7.2 of Appendix 1, the transverse component of the angular momentum vector is shown versus time. It is seen that each firing of the jet reduces the magnitude of the transverse angular momentum. During the off period, the transverse angular momentum is constant. This result is as it should be since $\underline{H}$ is conserved in a torque-free environment. The behavior of the transverse angular momentum depicted in Figure 7.2 allows the fuel consumed for other initial conditions (compatible with the small angle approximation) to be estimated. That is, if a value of Δh is known, the fuel consumed in counteracting this angular impulse is

$$W^* = \frac{\Delta h}{\Delta h_c} \times W_c^* \tag{4.12}$$

It is of interest to compare the estimate of the fuel based on Equation (4.11) with that computed according to the approximate relation

$$\frac{\Delta h}{r} = I_{mp} = \int_{t_o}^{t_1} F\,dt \cong I_s \int_{t_o}^{t_1} \dot{W}\,dt = I_s \times W \tag{4.13}$$

where I_{mp} refers to the total linear impulse, and
Δh refers to the total angular impulse.

From Equation (4.13), an estimate of the weight of fuel consumed in counteracting an angular impulse of Δh is given by

$$\tilde{W} = \frac{\Delta h}{r} \times \frac{1}{I_s} \tag{4.14}$$

In particular if $\Delta h = \Delta h_c$ then

$$\tilde{W} = \frac{1.5}{4} \times \frac{1}{70} = .00536 \text{ lb.}$$

This estimate is smaller than the optimal value; this indicates that the approximation is not sufficiently conservative. The ratio of the two estimates is 1.45, i.e.,

$$W^{*} = 1.45\ \tilde{W} \tag{4.15}$$

Figure 7.3 of Appendix 1 shows the trajectory in angular momentum space. Initially, the transverse components of the angular momentum are

$$(H_1, H_2) = (0, -1.5) \text{ ft-lb-sec.}$$

Each time the jet is turned on, the H_2 component is decreased. The half waves correspond to the on-cycle of the controller. During the off-time, neither H_1 nor H_2 varies.

4.5 Estimates of Fuel-Consumption for the Cruise Mode

The estimate of the fuel required for attitude control during the cruise mode is provided in this section. This estimate depends on estimates of the contribution made by

(1) Solar radiation torque,
(2) Micrometeriods,
(3) Gravity gradient effects,

to the total angular impulse. In addition fuel is required for fine turn control and for tracking the variations in the earth clock angle.

The contribution made to the total angular impulse by solar radiation torques is approximately given by

$$\Delta H_{SR} = \int_o^{t_f} T_D\ dt = \int_o^{t_f} T_o \cdot \frac{1}{R^2}\ dt \tag{4.16}$$

where T_o is the solar radiation torque near the earth's surface,
R is the distance from the sun in a.u.

The distance R can be approximated as

$$R = 1 + Kt = 1 + \frac{R_f - 1}{t_f}\ t \tag{4.17}$$

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where t_f is the final value of t (mission time) and R_f is the distance from the sun at the end of the mission (R_f = 30 a.u. in this analysis). Substitution of Equation (4.17) into Equation (4.18) yields

$$\Delta H_{SR} = T_o \int_o^{t_f} \frac{1}{(1+Kt)^2}\, dt = T_o \left[\frac{t_f}{1+K\, t_f}\right]$$

$$= T_o \frac{t_f}{1 + \frac{R_f - 1}{t_f}\, t_f} = T_o \frac{t_f}{30} \text{ ft-lb-sec.} \tag{4.18}$$

The value of T_o was estimated previously as 1.7×10^{-4} ft-lb (see Section 3) and the value of t_f at Neptune is approximately 3.36×10^8 sec. Substitution of these values of T_o and t_f into Equation (4.18) yields

$$\Delta H_{SR} = 1900 \text{ ft-lb-sec.} \tag{4.19}$$

However, since no fuel is required in counteracting the effects of solar radiation torque about the spin axis, the effective angular impulse that must be counteracted can be taken as

$$\Delta\tilde{H}_{SR} = \frac{1}{2} \times \Delta H_{SR} = 950 \text{ ft-lb-sec.}$$

This value is roughly the same as that given in Reference 1. In Reference 1, the contributions made to the total angular were estimated as

Item	Angular Impulse (ft-lb-sec)
Solar radiation torque (pitch)	700
Earth tracking (pitch)	200
Fine turn control	80
Micrometeriods	60
Gravity gradient	40

The amount of fuel required to counteract such an angular impulse is (from Equation (4.12))

$$W^* = \frac{\Delta h}{\Delta h_c} \times W_c^* = \frac{1100}{1.5} \times (.00779) = 5.7 \text{ lb.}$$

This estimate is slightly less than that calculated by Mankovitch.[2] Mankovitch's estimate is

$$W = 8 \text{ lb.}$$

This implies that only 2.3 lb of fuel could be saved furing the cruise mode for utilizing an optimal control scheme for the pitch-roll motion of the S/C. This comparison is not completely valid because the vehicle considered in this work is not exactly the same as investigated by Mankovitch. Nevertheless, the comparison provides an indication of the fuel savings that could be realized by using an optimal control scheme.

The calculation discussed above does not include the fuel required for spin maintenance nor dues it include that required for large angle turns.

5. FUEL-CONSUMPTION ESTIMATES FOR LARGE ANGLE TURNS

In this section, the estimate of the fuel required for the large angle turns is provided. As stated in Section 2, the large angle turns are required because the desired velocity correction that must be imparted to the vehicle by the midcourse motor is not necessarily in the yaw plane. Hence, the only way the large angle turns could be avoided is by using two midcourse motors.

The duration of the study was not sufficiently great to allow for the determination of the fuel-optimal controller for the large angle turn mode. Nevertheless, an estimate of the fuel required for this task can be computed from

$$\tilde{W} = \frac{\Delta h}{\ell} \times \frac{1}{I_s}$$

where ℓ is the lever arm of the bipropellant system.

In this work it was determined that the use of a bipropellant system located on the despun platform in such a way that the lever arm is maximized is more suitable (in regard to weight savings) than the use of rotor-fixed jets for accomplishing the large angle turns.

The midcourse motor is located on the rotor (along the spin axis), since an autopilot is not required for this location of the motor. The bipropellant system could be located either on the rotor or on the despun portion. In this work, it is arbitrarily assumed that it is located on the despun platform in such a way that the lever arm is maximized.

Since the direction of the thrust can be oriented in the yaw plane by turning the despun portion to the desired direction, the maximum orientation angle for which mass expulsion is required is 180°. However, after the correction has been made the vehicle must then be reoriented to the desired direction. The Δh associated with a 180° turn is

$$\Delta h = h_2 - h_1 = 300-(-300) = 600 \text{ ft-lb-sec}$$

assuming that the stored angular momentum is nominally 300 ft-lb-sec. The angular impulse associated with ten 360° turns is

$$\Delta h = 12000 \text{ ft-lb-sec.}$$

The value is roughly one half that estimated in Reference 1. This is probably due to the fact that the stored angular momentum in Reference 1 is twice as great as that used in this work. As pointed out in Section 3, it is desirable to have as small a value of stored angular momentum as possible for the large angle turn mode. From Equation (4.14), the fuel required for importing a total angular impulse of 12000 ft-lb-sec is

$$\tilde{W} = \frac{12000}{7} \times \frac{1}{250} = 6.86 \text{ lb}$$

for the case in which the lever arm is $\ell = 7$ ft. By scaling this result, according to Equation (4.15), an estimate of the fuel required if an optimal controller were used is

$$\tilde{W}^* = 1.45 \times \tilde{W} = 10 \text{ lb.}$$

According to Reference 4, an estimate of the weight of the bipropellant system (not including the propellant) is 20 lb. Hence the total bipropellant system weight is 30 lb. This estimate is 3 lb less than that given in Reference 2.

6. COMPARISON OF ALTERNATE CONTROL SYSTEMS

In this section, a brief comparison of alternate control systems is provided. The potential methods of attitude control for the Multi-Planet Mission (MPM) were discussed qualitatively in the Phase I report of this study (see Reference 5). In that report, the most suitable methods of attitude control were identified (see Table 6.1) and the relative suitability of various attitude control systems having the most potential for the MPM was qualitatively discussed (see Table 6.2). From Table 6.2, it is seen that the systems having the most potential for the MPM are

(1) the momentum wheel system using mass expulsion for unloading the wheels,
(2) the dual-spin vehicle using reaction jets for attitude control.

The main objective of this phase of the study, the determination of the optimal dual-spin system, has already been discussed (see Sections 4 and 5). A secondary objective is the comparison of the optimal dual-spin system with alternate control schemes (in particular, the momentum wheel system using mass expulsion for unloading the wheel). A qualitative discussion of the attitude control schemes based on

(1) mass expulsion only,
(2) solar radiation for a secondary means of control,
(3) spin stabilization,
(4) CMG's for momentum storage

was provided in Reference 5 and hence will not be repeated here. In this report the momentum wheel system and the dual-spin systems are first qualitatively compared, and then quantitative estimates of the system weight and the power requirements are provided. The estimates of the system weight and power requirements are based on Reference 2.

6.1 Implications of the Mission Requirements on the Momentum Wheel System

In this section, the implications of the mission requirements (see Reference 5) are qualitatively discussed in relation to the

Table 6.1

CLASSIFICATION OF METHODS OF SPACECRAFT CONTROL IN REGARD TO SUITABILITY FOR THE MULTI-PLANET MISSION (MPM)

Item	Representative examples	Suitability: Relatively Unsuitable	Suitability: Suitable
System Categories	Passive	x	
	Semipassive	x	
	Semiactive	x	
	Active		x
	Hybrid		x
Actuation Methods			
Incident momentum	Solar radiation pressure	x	
Interaction with ambient fields	Gravity gradient effect	x	
	Magnetic torque	x	
Expelled momentum			
Gaseous propellant	Cold gas (N_2) reaction jet		x
Solid propellant	Subliming solid (hot tip)	x	
Liquid propellant	Hydrazine plenum		x
Electrochemical	Resistojet	x (flight worthiness not sufficiently demonstrated)	
Internal momentum storage	Reaction wheel		x
	Fluid flywheels	x	
	Reaction sphere	x	
	CMG's		
	Dual-spin vehicles		X
Stabilization Technique	Spin	x	
	Environmentably stabilized	x	

Table 6.2

CLASSIFICATION OF SYSTEMS HAVING MOST POTENTIAL IN REGARD TO THEIR SUITABILITY FOR THE MPM

Potential Systems	Suitability for MPM	
	Relatively Unsuitable	Suitable
o Actuation Methods		
Combination of incident momentum (solar radiation pressure), internal momentum storage (reaction wheels) and expelled momentum (reaction jets).	x	
Expelled momentum (mass expulsion) alone.	x	
Combination of internal momentum storage (reaction wheels) and expelled momentum (reaction jets).		x
Dual-spin vehicle		x
o Stabilization Technique		
Spin stabilized	x	
Environmentally stabilized	x	

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momentum wheel system. The main factors to be considered include

(1) antenna pointing accuracy,

(2) trajectory corrections,

(3) orientation of the planetary instrumentation package,

(4) high reliability.

In regard to the requirement for accurate antenna pointing, no mass expulsion is required to track the earth. In addition, the amount of fuel consumed is not dependent on the deadband size as it is for attitude control systems using mass expulsion only. In achieving the desired antenna pointing accuracy, mass expulsion would be required only in the rare event that the system is subjected to a continuous disturbance.

Concerning the trajectory corrections that are required, no mass expulsion is needed for orienting the S/C to the direction of the desired Δv. This factor is one of the most important considerations in the qualitative comparison of the momentum wheel system and the dual-spin system. An active autopilot is necessary for TVC (Thrust Vector Control).

Concerning the precise orientation of the planetary instrumentation package, no problem areas are expected. In addition, no mass expulsion is needed for counteracting reaction torques.

Concerning the requirement for high reliability, the momentum wheel system is considered adequate since a redundant set of momentum wheels can be easily incorporated.

6.2 Implications of the Mission Requirements on the Dual-Spin System

In this section, the implications of the mission requirements are qualitatively discussed in relation to the dual-spin system. The same factors mentioned in connection with the momentum wheel system are considered.

In regard to the requirement for accurate antenna pointing, mass expulsion is not required for tracking variations in the earth's cone angle, but mass expulsion is required for tracking variations in the earth's clock angle. If the reaction jet is rotor-fixed, only one jet is required for two-axis control. In addition, if the jet is rotor-fixed, leakage torques average out.

Concerning the required trajectory corrections, a relatively large amount of fuel (10 lb) is needed for making the ten large angle turns. However, this value is somewhat conservative since it is based on 360^{o} turns, in some cases, the required turn may be considerably less than this. If the midcourse motor is mounted along the spin axis of the rotor, no autopilot is required. It will be seen later that it is not the weight of the fuel required for the large angle turns that is critical; it is the weight of the bipropellant system as a whole that is the critical factor.

Concerning the precisely oriented planetary instrumentation package, no problem areas are expected. In fact, the dual-spin vehicle is especially suited for missions in which there is a simultaneous requirement for earth communication and planet observation.

Concerning the requirement for high reliability, a problem area exists in that it is difficult to achieve redundancy for a critical component — the spin bearings!

6.3 Comparison of the Weight and Power Requirements Associated With Attitude Control

In this section, estimates of the weight and power requirements associated with attitude control are provided. Estimates of the fuel consumed for the dual-spin vehicle in the cruise mode and that consumed in the large angle turn mode have already been discussed (see Sections 4 and 5). Tables 6.3 and 6.4 (based on Reference 2) provide the weight and power requirements associated with the attitude control task.

Using the fuel optimal controller for the dual-spin vehicle, it is estimated that the fuel weight can be reduced by 2.3 lb during the cruise mode (5.7 lb rather than 8 lb). In addition, the estimated weight of the bipropellant system used for accomplishing the large angle turns is 3 lb less than that given in Table 6.3, (30 lb rather than

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33 lb). Hence, the estimate of the total weight associated with the attitude control task determined in this work is 5.3 lb less than that given in Table 6.3. Assuming the weights of the other items to be those given in Table 6.3, the weight of the optimal dual-spin spacecraft system becomes 143.2 rather than 148.5 lb.

The weights of the momentum wheel systems are given in Table 6.4. Using the weight associated with (option 3), the weight of the momentum wheel system is 17.2 lb less than that for the optimal dual-spin vehicle.

Concerning the power requirements, the continuous power is considerably less for the dual-spin vehicle than for the momentum wheel system (34.5 compared to 91.5 watts).

6.4 Improved Baseline Systems Versus Dual-Spin and Momentum Wheel Systems

This section provides a comparison of the weight and power requirements associated with attitude control (A/C) for the dual-spin, the momentum wheel-gas jet, and the improved baseline systems. The improved baseline systems include the

(1) momentum wheel-hydrazine system
(2) pulsed plasma-hydrazine system.

The use of an hydrazine system instead of gas jets for desaturating the momentum wheels is advantageous because of the accompanying weight savings. The hydrazine used for desaturation is pumped from the midcourse engine supply. Compared to the original baseline system (momentum wheel-gas jet), a weight savings of 12 to 20 lb is realized by using the momentum wheel-hydrazine system. For this reason, this system has replaced the momentum wheel-gas jet system as the baseline system for the MPM.

A proposed baseline system which results in a substantial weight savings is the pulsed plasma — hydrazine system. A pulsed plasma (an electric propulsion system) has recently been flown on the LES 6 Satellite. Although the flight worthiness of this system has not been

Table 6.3

WEIGHT AND POWER REQUIREMENTS FOR THE DUAL-SPIN SYSTEM USING REACTION JETS FOR ATTITUDE

ITEM	WEIGHT (lb)	SIZE	POWER (watts) CONT	PEAK
--ATTITUDE CONTROL ELECTRONICS (REDUNDANT)	5.		4.	
--CANOPUS SENSORS (STANDBY REDUNDANT)	16.		4.	
--SUN SENSORS (REDUNDANT)	1.		1.	
--DESPIN CONTROL ASSEMBLY (REDUNDANT)	30.		6.	
--COLD GAS PRECESSION SYSTEM				
NITROGEN	8.*			
TANKS	12.8	4.8" RAD.		2W
VALVES (2)	1.2			
REGULATORS	2.0			
PLUMBING & MISC	2.5			
--SOLID PROPELL SPINUP & SPIN MAINTENANCE SYSTEM	10.			
--BI-PROPELL PRECESS SYSTEM	33.**			2W
--SCAN PLATFORM ACTUATOR & ELECTRONICS	10.		4.	14
--ANTENNA POINTING ELECTRONICS	4.		3.	
--ACCELEROMETER	2.		2.5	
--GYROS	8.		10.	
--NUTATION DAMPER	3.			
TOTAL:	148.5***		34.5	18

*The estimate of this item obtained in this study is 5.7 lb.

**The estimate of this item obtained in this study is 30 lb.

***The estimate of this item obtained in this study is 143.2 lb.

Table 6.4

WEIGHT AND POWER REQUIREMENTS OF THE MOMENTUM WHEEL SYSTEM USING MASS EXPULSION (GAS JET) FOR UNLOADING THE WHEELS†

ITEM	WEIGHT (lb)	SIZE	POWER (watts) CONT	PEAK
--ATTITUDE CONTROL ELECTRONICS (TRIPLE REDUNDANT)	8		.6	
--CANOPUS SENSORS (TWO-STANDBY REDUNDANT)	16		4	
--SUN SENSORS (REDUNDANT)	1		1	
--MOMENTUM WHEELS (6-Wheels)	30.	6" DIA. x 3" HIGH		2W/WHEEL
--GAS SYSTEM (TRIPLE REDUNDANT)				
*NITROGEN	11.3			
*TANKS (2)	18	5.1" RAD.		
VALVES (12) & THRUSTERS	6.4			2W/AXIS
REGULATORS	2.5			
PLUMBING & MISC	3.1			
--SCAN PLATFORM ACTUATOR & ELECTRONICS	10.		4	14
--ANTENNA POINTING ELECTRONICS (OPTICAL)	4.5		4	
--ACCELEROMETER (ΔV SHUTOFF)	2		2.5	
--GIMBALLED AUTOPILOT ELECTRONICS & ACTUATORS (2 AXES)	6.2		60	
--BYROS (STANDBY REDUNDANT)	12.		10	
TOTAL (OPTIONS 1,2,4)	131.			
*OPTION 3				
NITROGEN	9.5†			
TANKS	15.2			
TOTAL (OPTION 3)	126.		91.5	

†The weight of fuel consumed could conceivably be reduced if a different type of mass expulsion were used.

completely established, nevertheless, it is considered a likely candidate because of the conspicuous weight savings associated with its use. It has been estimated that a weight savings of 48 lbs and a power savings of 11 watts (relative to the momentum wheel-gas jet system) can be realized by using this newly proposed system.

6.4.1 RTG Weight

An important parameter associated with the RTG (radioisotope thermoelectric generator) is the quantity of power (watts) that can be realized per pound. In the development of the RTG, a design goal is that this parameter have a value of 1.5 watts/lb. Hence, the power requirements of a system affect the system weight. The RTG weight and the combined RTG and A/C system weight associated with the various systems are provided in Table 6.5.

Table 6.5

RTG WEIGHT AND COMBINED A/C SYSTEM AND RTG WEIGHT FOR VARIOUS SYSTEMS

	A/C SYSTEM WEIGHT (LB)	RTG WEIGHT (LB)	COMBINED RTG AND A/C SYSTEM WEIGHT (LB)
DUAL-SPIN	143.2	23.	166.†
MOMENTUM WHEEL-GAS JET	126.	61.	187.
MOMENTUM WHEEL-HYDRAZINE	106.	61.	167.
PULSED PLASMA-HYDRAZINE	78.	53.5	131.5

†This weight can potentially be reduced by replacing the cold gas and bipropellant systems by a hydrazine system.

7. RECOMMENDATIONS FOR FURTHER STUDY

In this section, recommendations for further study are provided. One of the main objectives of this study was to determine to what extent optimization techniques could be used to evaluate the relative merits of attitude control systems. In this vein in order to determine the accurate weight of the fuel required for attitude control, further study would include the

(1) determination of the fuel-optimal controller for the large-angle case,

(2) determination of the optimal parameters for the dual-spin system,

(3) examination of additional initial conditions.

It is expected, however, that the weight of the fuel saved by using a fuel-optimal controller throughout the entire mission will not significantly change the results provided in Table 6.5.

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APPENDIX A

ABSTRACT

The dynamics and fuel-optimal control aspects of a class of dual-spin vehicles appropriate for deep-space missions are investigated. A dual-spin spacecraft typically consists of a spinning rotor providing sufficient stored angular momentum for stabilization, and a despun portion which provides a platform for an antenna and planetary encounter instrumentation. This dissertation deals primarily with the cruise mode of the mission; in this mode, the fuel-optimal controller has the task of maintaining the desired orientation of the rotor spin axis relative to inertial space, so that the antenna can be directed toward its target by an electric motor of a single-axis control system.

The linearized rotational equations of motion for the class of dual-spin vehicles of concern are developed. These equations characterize the plant or the control process (S). The other constituents of the optimal control problem include the class of admissible controllers Δ, the control restraint set Ω, the initial set X_o, the target set X_1, and the cost functional J; these elements are systematically discussed with emphasis on the practical aspects. The nature of the control restraint set Ω and the target set X_1 are determined by carefully considering the physical and practical aspects of the problem. Once the fuel-optimal control problem $\{S, \Delta, \Omega, X_o, X_1, J\}$ is formulated, a solid theoretical framework is provided. The cornerstones of this foundation consist of such concepts as controllability, normality, and their connection with the existence and uniqueness of the fuel-optimal controller. The necessary conditions for local optimality are obtained by using the calculus of variations and are verified by appealing to the maximum principle. Sufficient conditions for optimality are also discussed.

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The computational algorithm used for determining the fuel-optimal controller belongs to the family of the so-called indirect methods in general and to the class of Newton-Raphson techniques, in particular. The basic achievement of this dissertation is in the introduction of practically motivated innovations in the selection of the target set X_1 and the control restraint set Ω. The result is a substantial improvement over previous "fuel-optimal" controllers, both in terms of cost in fuel and ease of implementation.

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CONTENTS (Continued)

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FIGURES

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FIGURES (Continued)

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TABLES

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NOMENCLATURE

S	system of nonlinear differential equations representing the plant
Δ	class of admissible controllers
Ω	control restraint set
X_o	initial set
X_1	target set
J	cost functional
$\bar{J}$	Jordan canonical form
$\underline{H}$	angular momentum vector
$\underline{H}_D$	desired angular momentum vector
$\underline{x}$	state vector
$\underline{\omega}$	angular velocity vector
$\underline{g}$	constraint vector representing the transverse components of angular momentum in inertial space
$\underline{u}$	control vector
$\forall$	for all
$\ni$	such that
$\rangle$	column vector
$\langle$	row vector
$\langle , \rangle$	scalar product
$\rangle \langle$	dyad product
$\mathbb{J}^P$	inertia dyadic relative to an arbitrary point P fixed in the body
$\mathbb{I}^c$	inertia dyadic relative to the center of mass
$\mathbb{E}$	identity dyadic
$\underline{\mu}$	mass moment
$\underline{M}^c$	applied moment about the center of mass

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$\bar{r}$	$\frac{J_3^R}{I_1}$, ratio of inertia of the rotor about its spin axis and the system transverse inertia
r	$\frac{I_1 - I_3}{I_1}$, ratio of system inertias
σ	rotor speed relative to the despun portion
β	$\bar{r}\,\sigma = \frac{J_3^R}{I_1}\,\sigma$, ratio of rotor angular momentum and transverse inertia of the system
$\underline{\theta}$	antenna orientation angles θ_1, θ_2
C^1	class of continuous functions having continuous first partial derivatives
$R^n x R^m x R$	cartesian product of spaces R^n, R^m, and R
A	system matrix
B	control matrix
L	system of linear equations representing the plant
Φ	transition matrix
Ψ	adjoint transition matrix
$\Vert\ \Vert$	norm
M	modal matrix
$C(t_o, t_1)$	controllability matrix
Λ	diagonal matrix having as its elements the eigenvalues of the system matrix A
$K(t_1)$	set of attainability
$\underline{q}$	vector switching function
DEZ	vector deadzone function
HEV	vector Heaviside function
$\overline{K(t_1)}$	closure of the set of attainability
$\partial\Omega$	boundary of the control restraint set

ess sup	essential supremum
a. e.	almost everywhere
$\underline{p}$	adjoint vector
$\underline{\nu}$	Lagrange multipliers associated with right end constraints
H	Hamiltonian
E	Weierstrass function
glb	greater lower bound
δJ	first variation of cost functional
$\delta^2 J$	second variation of cost functional
$\mathcal{H}$	Hilbert space
B-space	Banach space

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Section 1
INTRODUCTION

1.1 Dual-Spin Spacecraft

This dissertation deals with the determination of the fuel-optimal controller for a class of dual-spin spacecraft appropriate for deep-space missions. A dual-spin vehicle has been the subject of several recent investigations (see, e.g., [1]). Considerable attention has been focused on the important question of the attitude stability of these vehicles (see, e.g., [2] - [5]). Typically, a dual-spin vehicle consists of a spinning rotor which provides sufficient stored angular momentum for stabilization, a despun portion which provides a platform for planetary encounter instrumentation, a nutation damper, and an antenna for tracking the target. The dual-spin vehicle is especially suited for applications requiring simultaneous earth communication and planet observation (see Likins and Larson [6]).

1.2 Fuel-Optimization

Because of its great practical importance, the notion of fuel-optimization has received considerable attention in the aerospace industry (see Refs. [7] and [8]). However, the application of optimization techniques specifically to attitude control problems is still relatively rare (see Refs. [9] and [10]). There have been even fewer studies which have dealt with the fuel-optimal control of spinning vehicles.

1.3 Fuel-Optimal Control of Spinning Vehicles

Sohoni and Guild [11] investigate the fuel-optimal control of the spin axis of a spinning symmetric vehicle. Athans and Debs [12] investigate the analytical aspects of the problem concerning the control of the angular velocity of a spinning symmetric vehicle. Porcelli investigates the sub-fuel-optimal control of both a symmetric spinning vehicle [13] and a symmetric dual-spin vehicle [14]. In his work, optimization techniques are not utilized; instead, an intuitive and graphical approach is used.

1.4 Fuel-Optimal Control Using an Angular Momentum Control Concept

None of the preceding studies makes use of a properly placed nutation damper, although such a device is an integral part of every dual-spin spacecraft. In a sense, the previous studies are not practically oriented. Since the objective of this dissertation is the application of optimization techniques to a meaningful and practical problem, a nutation damper is included in the control problem formulation. The use of a nutation damper results not only in a practical implementation but has a considerable effect on the optimal control concept. It not only affects the theoretical aspects of the problem formulation and the computational algorithm, but more importantly it affects the amount of fuel required to achieve the control objective.

The inclusion of a nutation damper in the control concept leads to what is called a hybrid control scheme. The hybrid control scheme consists of both an active phase and a passive phase. During the active phase, the angular momentum vector $\underline{H}$ is aligned to the desired direction in inertial space $\underline{H}_D$. The notion of aligning $\underline{H}$ to $\underline{H}_D$ is called an angular momentum control (AMCO) concept in this work. During the passive phase, the nutation damper is used to complete the task of aligning the antenna axis to $\underline{H}_D$. In previous investigations, an active phase is used to accomplish the entire control objective. The notion of aligning the spin axis (antenna axis) to the desired direction in inertial space by using an active controller is called a spin axis control (SACO) concept in this work.

The differences between the practical concept (AMCO) and the previously investigated concept (SACO) can be discerned by examining Figure 1. 1. Figure 1. 1 illustrates the torque-free motion of a symmetric body. The trace swept out by the angular velocity vector on the energy ellipsoid is called the polhode and that swept out on the invariant plane is called the herpolhode [15]. The motion is characterized by the rolling of the body cone on the space cone without slip. For the SACO concept, it is necessary to apply a moment (control) in order to align the spin axis $\underline{a}_3$, the angular velocity vector $\underline{\omega}$ and the desired angular momentum vector $\underline{H}_D$. In the AMCO concept, the angular momentum vector $\underline{H}$ is aligned to the desired direction in inertial space $\underline{H}_D$ during the active phase. The alignment of the bearing axis to the desired direction $\underline{H}_D = \underline{H}$ is accomplished during the passive phase by the properly designed nutation damper.

In regard to the optimal control problem, the significant difference between the AMCO and SACO concepts is the target set X_1. For the SACO concept, the target set consists of a fixed point in R^n (in particular, the null vector) that is, the target set is given by

$$X_1 = \{(\underline{x}, t) : \underline{x}(t_1) = \underline{x}_1 = \underline{0},\ t_1 \text{ free but finite}\}$$

On the other hand, the target set for the AMCO concept is a smooth two-fold in R^n and is given by

$$X_1 = \{(\underline{x}, t) : g_1(\underline{x}(t_1)) = 0 \text{ and } g_2(\underline{x}(t_1)) = 0,\ t_1 \text{ free}\}$$

where $g_1(\underline{x})$ and $g_2(\underline{x})$ refer to the components of the inertial angular momentum transverse to $\underline{H}_D$, the desired angular momentum vector.

Another important difference between the optimal control problem formulated in this work and that discussed by previous investigators

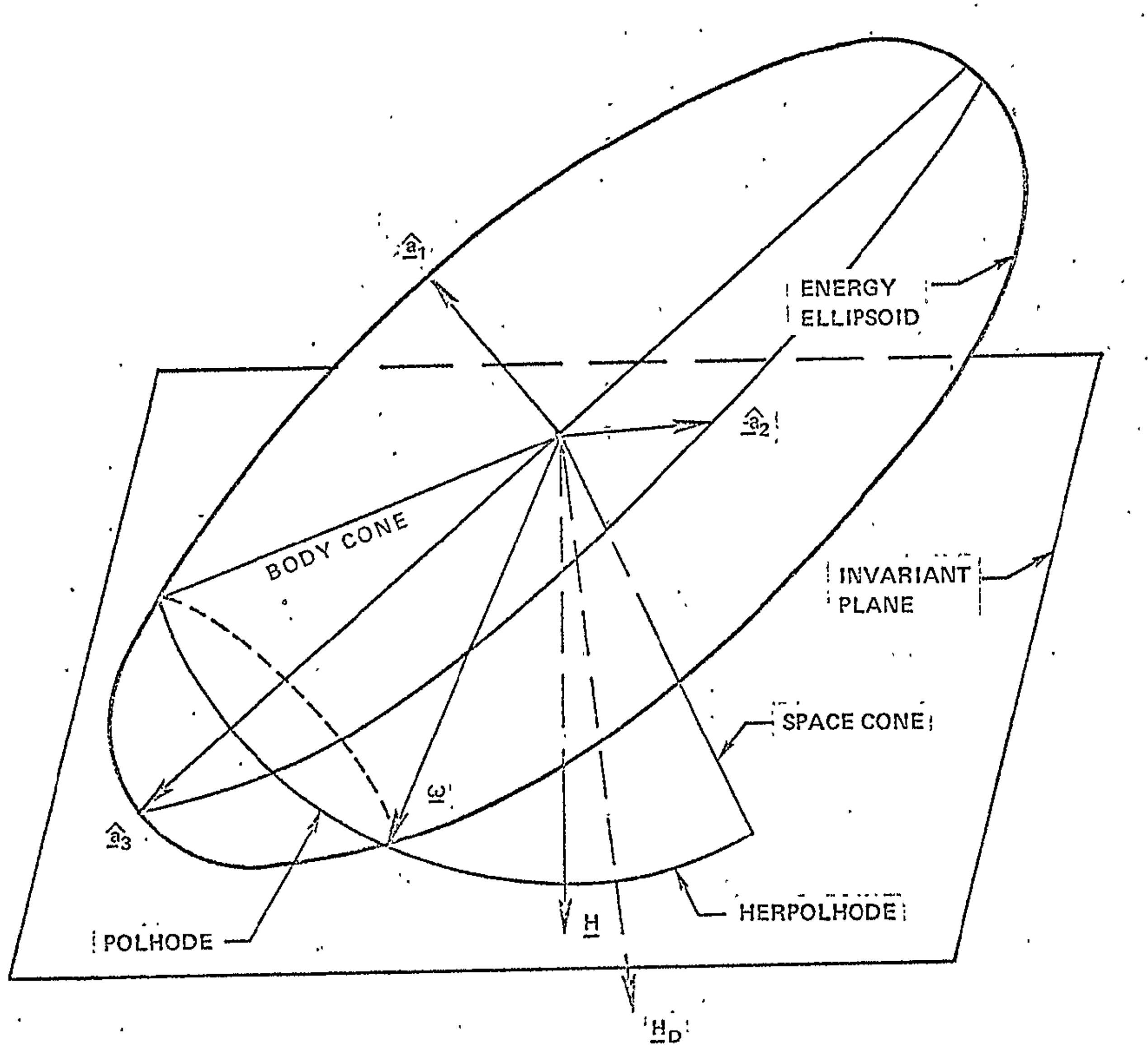

Figure 1-1 Illustration Showing Torque—Free Motion of Symmetric Spinning Body

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is in the nature of the control restraint set Ω. The control restraint set should be chosen primarily on the basis of practical rather than mathematical considerations. Previous investigators have invariably used the compact convex control restraint set

$$\Omega = \{\underline{u}(t) : |u_j(t)| \leq 1 \quad \forall j\}$$

in formulating the fuel-optimal attitude control problem for spinning vehicles. This set is mathematically correct and even physically appropriate for the attitude control of nonspinning bodies. However, when the vehicle is spinning, the spin rate itself provides the means for satisfying the requirement that both positive and negative moments be available for control. The compact convex control restraint set used in this work is given by

$$\Omega = \{\underline{u}(t) \; : \; 0 \leq u_j(t) \leq 1 \quad \forall j\}$$

It will be seen later that this seemingly slight difference has some significant practical implications.

1.5 Scope of the Dissertation

The fuel-optimal control problem formulated in this work evolved from the previously cited investigations. The use of a fuel-optimal controller for a ballistic spacecraft for deep-space missions is extremely important since only a limited amount of fuel can be carried. The determination of the fuel-optimal controller for a class of dual-spin vehicles, including symmetric spinning vehicles as special cases, is one of the primary aims of this work. The comparison of the AMCO concept introduced in this work with the SACO concept studied by other investigators is of special concern.

An equally important objective of this work is to demonstrate the utility of optimization theory in the preliminary design of competitive spacecraft configurations. By determining the nature of the optimal controller for various control restraint sets Ω and for various target sets X_1, the most practical design can be chosen. The use of optimization theory for such purposes is not accompanied by the customary limitations concerning computer storage and computer speed.

Although practical rather than theoretical considerations are emphasized in this dissertation, nevertheless, a solid theoretical framework is provided. This framework emphasizes the structure of a general optimal control problem and provides the machinery for attacking a general problem even though only the fuel-optimal controller for a dual-spin spacecraft is determined herein.

Chapter 2 provides a powerful and extremely useful development concerning the rotational motion of an arbitrary elastic constant mass system. With slight modifications, these equations would be appropriate for variable mass systems as well. The rotational equations of motion for the dual-spin spacecraft and the symmetric spinning vehicle are obtained as special cases of the general result. The damper terms are discussed for completeness even though they enter into the control problem only during the passive phase. Such effects are of great importance when the question of stability is being considered (see Likins [4]) but are insignificant in comparison to the relatively large applied control torques. Nevertheless, the damper plays a significant role in the AMCO concept.

In Chapter 3, the formulation of the fuel-optimal control problem pertaining to a class of dual-spin spacecraft is discussed. This chapter sets the stage for the ensuing discussion of the long step-by-step procedure for determining the fuel-optimal controller.

The concepts of Chapter 4 are extremely important in the determination of the fuel-optimal controller not only because they strongly affect the computational procedure but also because they affect the interpretation of the computational results. For example, if it cannot be proved or demonstrated that a control problem is normal, then special attention must be given to the possibility that a singular optimal solution exists. The investigation of the notions of Chapter 4 constitutes the first step in the determination of the optimal controller.

Chapter 5 provides the necessary conditions for optimality. For completeness, the necessary conditions for optimality for the problems of interest are developed using the calculus of variations rather than just stated. Consequently, the treatment tends to be self-contained. The necessary conditions obtained from Pontryagin's Maximum Principle under weaker differentiability assumptions are stated without proof. In addition to providing a discussion of the necessary conditions, Chapter 5 also provides some sufficient conditions for optimality.

Chapter 6 provides a discussion of the computational algorithm used to determine the fuel-optimal controller. The algorithm chosen in this work belongs to the family of indirect methods in general and to the class of Newton-Raphson techniques, in particular. This algorithm is considered suitable for the application at hand, especially when due concern is given to the practical considerations. In general, the selection of a computational algorithm is made after considering the nature of the optimal controller, the practical implications, the simplicity of the formulation and implementation, and estimates of the computer storage requirements, convergence sensitivity, and convergence time. Ideally, it would be desirable to compare such factors as computer storage requirements, convergence sensitivity, and convergence time associated with the Newton-Raphson algorithm with those associated with the use of several other algorithms. In this way, the most suitable algorithm for this class of problems could be determined. Such an undertaking, however, is not within the scope of the present work.

Chapter 7 provides a summary of the significant results obtained from this study. The most important results are those that pertain to the comparison of the angular momentum control concept with the spin axis control concept. Chapter 7 also provides the conclusions drawn from this study.

Section 2

ROTATIONAL EQUATIONS OF MOTION FOR A DUAL-SPIN VEHICLE

This chapter provides the rotational equations of motion for a general elastic (constant mass) body. The rotational equations of motion for a dual-spin vehicle and a spinning vehicle are then obtained as special cases of the general result. The derivation provided below is based solely on fundamental notions of Newtonian mechanics.

2.1 Rotational Equations of Motion for a General Flexible Body

In this section, the rotational equations of motion for a general flexible body are obtained. The results are obtained by straightforwardly manipulating the basic definitions of Newtonian mechanics.

In this development, the following well-known vector relationship will be frequently used [16]

$$^{R_1}\frac{d}{dt}\,\underline{q} = {}^{R_2}\frac{d}{dt}\,\underline{q} + {}^{R_1}\underline{\omega}^{R_2} \times \underline{q} \tag{2-1}$$

where $\underline{q}$ represents any vector

R_1, R_2 represent any Euclidean frames having the same origin

$^{R_1}\frac{d}{dt}\,\underline{q}$, $^{R_2}\frac{d}{dt}\,\underline{q}$ refer to time derivatives of $\underline{q}$ relative to frames R_1, R_2, respectively

$^{R_1}\underline{\omega}^{R_2}$ refers to the angular velocity of frame R_2 relative to R_1.

When frame R_1 is the Newtonian frame, and R_2 is some body-fixed frame, the time derivatives and the angular velocity vector will sometimes be written as

$$
{}^{N}\frac{d}{dt}\,\underline{q} = \dot{\underline{q}}
$$

$$
{}^{R_2}\frac{d}{dt}\,\underline{q} = \mathring{\underline{q}}
$$

$$
{}^{N}\underline{\omega}^{R_2} = \underline{\omega}
$$

for notational convenience. The notation first introduced by Dirac [17] to distinguish between column and row vectors will occasionally be used. Dirac used the symbols $>$ and $<$ to represent column and row vectors, respectively. Using this notation, the common operations known as inner and outer products become $<\ ,\ >$ and $>\,<$, respectively. The inner product $<\ ,\ >$ is used frequently in this work; sometimes it is defined on the space E^n while at other times it is defined on a Hilbert space. When the space is E^n the elements are underscored to make it clear that the elements are vectors in E^n (the underscoring is, of course, redundant).

The outer product $\underline{u} >< \underline{v}$ defined on a finite-dimensional space has as its matrix representation

$$
\begin{pmatrix}
u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\
u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\
\vdots & & & \\
u_n v_1 & u_n v_2 & \cdots & u_n v_n
\end{pmatrix}
$$

A result similar to that of Equation (2-1) applies to the operation $\underline{u} >< \underline{v}$; this result is obtained from the chain rule for differentiation and is given by

$$
\begin{aligned}
{}^{N}\frac{d}{dt}(\underline{u} > < \underline{v}) &= \dot{\underline{u}} > < \underline{v} + \underline{u} > < \dot{\underline{v}} \\
&= (\mathring{\underline{u}} + \underline{\omega} \times \underline{u}) > < \underline{v} + \underline{u} > < (\mathring{\underline{v}} + \underline{\omega} \times \underline{v}) \\
&= \mathring{\underline{u}} > < \underline{v} + \underline{u} > < \mathring{\underline{v}} + \underline{\omega} \times \underline{u} > < \underline{v} - \underline{u} > < \underline{v} \times \underline{\omega} \\
&= {}^{A}\frac{d}{dt}(\underline{u} > < \underline{v}) + \underline{\omega} \times \underline{u} > < \underline{v} - \underline{u} > < \underline{v} \times \omega \qquad (2\text{-}2)
\end{aligned}
$$

In particular, when the inertia dyadic $\mathcal{J}$ is involved, the result is

$$
{}^{N}\frac{d}{dt}\mathcal{J} = {}^{A}\frac{d}{dt}\mathcal{J} + \underline{\omega} \times \mathcal{J} - \mathcal{J} \times \underline{\omega} \qquad (2\text{-}3)
$$

Since operations involving vectors and operations involving dyads can be represented as matrices, the matrix representation of these operations are used frequently in this chapter. For example, the inertia dyadic $\mathcal{J}$ written as a sum of dyads is

$$
\mathcal{J} = J_{\alpha\beta}\, \hat{\underline{e}}_{\alpha} > < \hat{\underline{e}}_{\beta}
$$

where $\hat{\underline{e}}_{\alpha}$, $\hat{\underline{e}}_{\beta}$ belong to the space E_3. A typical dyad $\hat{\underline{e}}_1 > < \hat{\underline{e}}_3$ is simply

$$
\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} (0\ 0\ 1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Hence, the matrix representation of the dyadic $\mathcal{J}$ is

$$
\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}
$$

Other common operations involving vectors that are frequently represented by matrices in this chapter are $\underline{\omega} \times \underline{q}$ and $\underline{\omega} \times (\underline{\omega} \times \underline{q})$. The operation $\underline{\omega} \times \underline{q}$ (where $\underline{\omega}$ and $\underline{q}$ are vectors in E_3) can be written as

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

and the operation $\underline{\omega} \times (\underline{\omega} \times \underline{q})$ can be written as

$$\begin{bmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1 \omega_2 & \omega_1 \omega_3 \\ \omega_2 \omega_1 & -(\omega_1^2 + \omega_3^2) & \omega_2 \omega_3 \\ \omega_3 \omega_1 & \omega_3 \omega_2 & -(\omega_1^2 + \omega_2^2) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

2.1.1 Newtonian Approach

In this section the rotational equations of motion are obtained from the so-called "Newtonian" approach. Later the same result is obtained by recasting the equations in such a way that the "kinematical" nature of the problem is revealed. The notion of relative angular momentum plays a central role in this development.

<u>Definition</u> Relative Angular Momentum

The relative angular momentum in a frame R with respect to any point P is given by (see Figure 2.1)

$$^{R}\underline{H}^{P} = \int \underline{r}_p \times \frac{^{R}d}{dt} \underline{r}_p \, dm \tag{2-4}$$

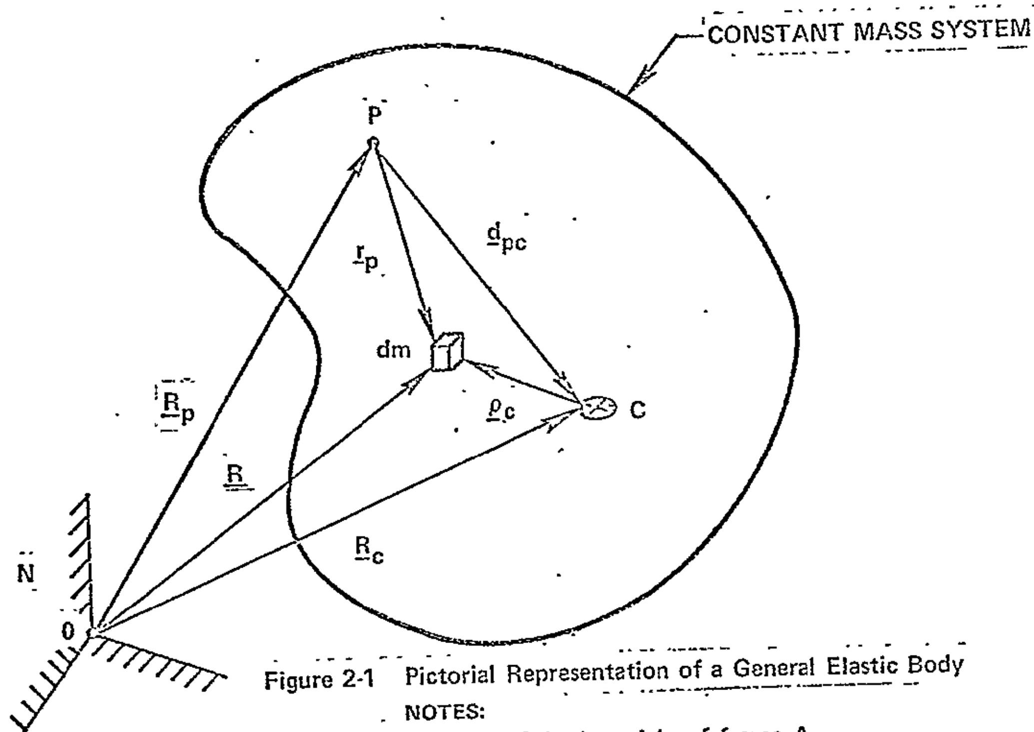

Figure 2-1 Pictorial Representation of a General Elastic Body

NOTES:

1. Point P is the origin of frame A
2. Point C is the center of mass of the body

In particular, when the point P is the center of mass C and frame R is the Newtonian frame, the angular momentum is given by

$$^{N}\underline{H}^{c} = \int \underline{\rho}_c \times {}^{N}\frac{d}{dt}\,\underline{\rho}_c \, dm \tag{2-5}$$

Equation (2-5) is the starting point for the derivation.

For convenience, introduce a frame A containing the point P which coincides with the nominal or undeformed center of mass and participates in the motion of the corresponding point of the material system, if such exists. Unlike the center of mass (point C), point P is fixed in frame A. The vector $\underline{\rho}_c$ expressed in terms of vectors measured relative to point P is given by

$$\underline{\rho}_c = \underline{r}_p - \underline{d}_{pc} \tag{2-6}$$

Substituting Equation (2-6) into Equation (2-5) and differentiating relative to the Newtonian frame yields

$$\dot{\underline{H}}^c = \int \left(\underline{r}_p - \underline{d}_{pc}\right) \times \left(\ddot{\underline{r}}_p - \ddot{\underline{d}}_{pc}\right) dm \tag{2-7}$$

Expanding Equation (2-7) yields

$$\dot{\underline{H}}^c = \int \underline{r}_p \times \ddot{\underline{r}}_p \, dm - \underline{d}_{pc} \times \int \ddot{\underline{r}}_p \, dm + \underline{d}_{pc} \times M \ddot{\underline{d}}_{pc} - \int \underline{r}_p \, dm \times \ddot{\underline{d}}_{pc} \tag{2-8}$$

The temporal operator $\frac{d^2}{dt}$ ( ) and the distribution operator $\int ( )\, dm$ are clearly commutative and, hence,

$$\int \ddot{\underline{r}}_p \, dm = \frac{d^2}{dt^2} \int \underline{r}_p \, dm \tag{2-9}$$

By definition, the first moment relative to the point P is

$$\underline{\mu}^P \equiv \int \underline{r}_p \, dm = M \underline{d}_{pc} \tag{2-10}$$

It follows, therefore, that,

$$\begin{aligned} -\underline{d}_{pc} \times \int \ddot{\underline{r}}_p \, dm &= -\underline{d}_{pc} \times \ddot{\underline{\mu}}^P \\ -\int \underline{r}_p \, dm \times \ddot{\underline{d}}_{pc} &= -\underline{\mu}^P \times \ddot{\underline{d}}_{pc} \\ \underline{d}_{pc} \times M \ddot{\underline{d}}_{pc} &= \underline{\mu}^P \times \ddot{\underline{d}}_{pc} \end{aligned} \tag{2-11}$$

Substituting the results of Equation (2-11) into Equation (2-8) yields

$$\dot{\underline{H}}^c = \int \underline{r}_p \times \ddot{\underline{r}}_p \, dm + \underline{d}_{pc} \times (-M \ddot{\underline{d}}_{pc}) \tag{2-12}$$

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The integral term can be rewritten as

$$\int \underline{r}_p \times \ddot{\underline{r}}_p \, dm = {}^N\frac{d}{dt} \int \underline{r}_p \times \dot{\underline{r}}_p \, dm \tag{2-13}$$

Using the results of Equation (2-1) in Equation (2-13) yields

$$\int \underline{r}_p \times \ddot{\underline{r}}_p \, dm = {}^N\frac{d}{dt} \int \underline{r}_p \times \left[\mathring{\underline{r}}_p + \underline{\omega} \times \underline{r}_p \right] dm \tag{2-14}$$

Expanding Equation (2-14) yields

$$\int \underline{r}_p \times \ddot{\underline{r}}_p \, dm = {}^N\frac{d}{dt} \int \underline{r}_p \times \mathring{\underline{r}}_p \, dm + {}^N\frac{d}{dt} \int \underline{r}_p \times (\underline{\omega} \times \underline{r}_p) dm \tag{2-15}$$

Combining all the terms, the resulting equation is

$$\dot{\underline{H}}^c = {}^N\frac{d}{dt} \int \underline{r}_p \times \mathring{\underline{r}}_p \, dm + {}^N\frac{d}{dt} \int \underline{r}_p \times (\underline{\omega} \times \underline{r}_p) dm + \underline{d}_{pc} \times (-M \ddot{\underline{d}}_{pc}) \tag{2-16}$$

From the definition of relative angular momentum, it follows that

$${}^N\frac{d}{dt} \int \underline{r}_p \times \mathring{\underline{r}}_p \, dm = {}^N\frac{d}{dt} \, {}^A\underline{H}^P = {}^A\dot{\underline{H}}^P$$

The second integral term of Equation (2-16) can be expanded as

$${}^N\frac{d}{dt} \int \left[< \underline{r}_p , \underline{r}_p > \underline{\omega} > - \underline{r}_p > < \mathring{\underline{r}}_p , \underline{\omega} > \right] dm \tag{2-17}$$

This form is convenient since it leads naturally to the notion of dyads and to the definition of the inertia dyadic. Rewriting Equation (2-17) as an operation on the vector $\underline{\omega}$ yields

$${}^N\frac{d}{dt} \left[\int \left(< \underline{r}_p , \underline{r}_p > \mathbb{E} - \underline{r}_p > < \underline{r}_p \right) dm , \underline{\omega} > \right] \tag{2-18}$$

From the definition of the inertia dyadic, Equation (2-18) can be written as

$$ {}^{N}\frac{d}{dt}\left(\mathcal{J}^{P} \cdot \underline{\omega}\right) $$

The final equation is thus

$$ \dot{\underline{H}}^{c} = {}^{A}\dot{\underline{H}}^{P} + {}^{N}\frac{d}{dt}\left(\mathcal{J}^{P} \cdot \underline{\omega}\right) + \underline{d}_{pc} \times \left(-M\,\ddot{\underline{d}}_{pc}\right) \tag{2-19} $$

From the definition of $\underline{H}^{c}$, it follows that

$$ \dot{\underline{H}}^{c} = \int \underline{\rho}_{c} \times \ddot{\underline{\rho}}_{c}\, dm = \int \underline{\rho}_{c} \times \left(\ddot{\underline{R}} - \ddot{\underline{R}}_{c}\right) dm \tag{2-20} $$

Substituting the relationship

$$ \ddot{\underline{R}}\, dm = \underline{dF} + \underline{df} $$

where $\underline{dF}$, $\underline{df}$ refer to differential external and internal forces, respectively, into Equation (2-20) yields

$$ \begin{aligned} \dot{\underline{H}}^{c} &= \int \underline{\rho}_{c} \times (\underline{dF} + \underline{df}) - \int \underline{\rho}_{c}\, dm \times \ddot{\underline{R}}_{c} \\ \dot{\underline{H}}^{c} &= \underline{M}^{c} - \underline{\mu}^{c} \times \ddot{\underline{R}}_{c} + \int \underline{\rho}_{c} \times \underline{df} \end{aligned} \tag{2-21} $$

It follows from the definition of the mass moment about the center of mass that $\underline{\mu}^{c} \equiv 0$. Hence, for a system in which the moment about the center of mass due to the internal forces is zero (one that obeys Newton's third law), the following relationship holds

$$ \dot{\underline{H}}^{c} = \underline{M}^{c} \tag{2-22} $$

Equations (2-19) and (2-22) represent the desired rotational equations of motion for any constant mass system. The term ${}^{A}\dot{\underline{H}}^{P}$ when expanded becomes

$$ \begin{aligned} {}^{A}\dot{\underline{H}}^{P} &= {}^{N}\frac{d}{dt}\int \underline{r}_{p} \times \mathring{\underline{r}}_{p}\, dm = {}^{A}\frac{d}{dt}\int \underline{r}_{p} \times \mathring{\underline{r}}_{p}\, dm + \underline{\omega} \times \int \underline{r}_{p} \times \mathring{\underline{r}}_{p}\, dm \\ {}^{A}\dot{\underline{H}}^{P} &= {}^{A}\mathring{\underline{H}}^{P} + \underline{\omega} \times {}^{A}\underline{H}^{P} \end{aligned} \tag{2-23} $$

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From the results of Equation (2-3) the term ${}^N\frac{d}{dt}\left(\mathcal{J}^P \cdot \underline{\omega}\right)$ becomes

$${}^N\frac{d}{dt}\left(\mathcal{J}^P \cdot \underline{\omega}\right) = \overset{\circ}{\mathcal{J}}{}^P \cdot \underline{\dot{\omega}} + \underline{\omega} \times \mathcal{J}^P \cdot \underline{\omega} - \mathcal{J}^P \times \underline{\omega} \cdot \underline{\omega} \tag{2-24}$$

Note, however, that the term $\mathcal{J}^P \times \underline{\omega} \cdot \underline{\omega}$ is identically zero as can be seen by examining the following identity

$$\mathcal{J}^P \times \underline{\omega} \cdot \underline{\omega} = J_{\alpha\beta}\, \underline{\hat{e}}_\alpha > < \underline{\hat{e}}_\beta \times \underline{\omega}, \cdot \underline{\omega} > \tag{2-25}$$

Collecting all the results together the final expanded equation is

$$\begin{aligned} \underline{M}^c = {} & \overset{\circ}{\mathcal{J}}{}^P \cdot \underline{\omega} + \underline{\omega} \times \mathcal{J}^P \cdot \underline{\omega} + \int \underline{r}_p \times \overset{\circ\circ}{\underline{r}}_p \, dm \\ & + \underline{\omega} \times \int \underline{r}_p \times \overset{\circ}{\underline{r}}_p \, dm \\ & + \underline{d}_{pc} \times \left[- M \left\{ \overset{\circ\circ}{\underline{d}}_{pc} + 2\underline{\omega} \times \overset{\circ}{\underline{d}}_{pc} + \underline{\dot{\omega}} \times \underline{d}_{pc} + \underline{\omega} \times (\underline{\omega} \times \underline{d}_{pc}) \right\} \right] \end{aligned} \tag{2-26}$$

2.1.2 Kinematical Approach

In this section, the same result obtained in the previous section is obtained in such a way that the "kinematical" nature of the problem is more evident. Starting with the relative angular momentum with respect to point P, a relationship between $\underline{H}^P$ and $\underline{H}^c$ is easily obtained and is given by

$$\begin{aligned} {}^N\underline{H}^P &= \int \underline{r}_p \times \underline{\dot{r}}_p \, dm = \int \left(\underline{\rho}_c + \underline{d}_{pc}\right) \times \left(\underline{\dot{\rho}}_c + \underline{\dot{d}}_{pc}\right) dm \\ {}^N\underline{H}^P &= \underline{H}^c + \underline{d}_{pc} \times M\, \underline{\dot{d}}_{pc} \end{aligned} \tag{2-27}$$

where

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$$\int \underline{\rho}_c \times \dot{\underline{\rho}}_c \, dm \equiv \underline{H}^c$$

$$\underline{d}_{pc} \times \int \dot{\underline{\rho}}_c \, dm = \underline{d}_{pc} \times \frac{d}{dt} \int \underline{\rho}_c \, dm = 0$$

$$\int \underline{\rho}_c \, dm \times \dot{\underline{d}}_{pc} = 0$$

Examination of the term $\underline{H}^P$ reveals

$$^N\underline{H}^P = \int \underline{r}_p \times \dot{\underline{r}}_p \, dm = \int \underline{r}_p \times \left(\overset{\circ}{\underline{r}}_p + \underline{\omega} \times \underline{r}_p \right) dm = {}^A\underline{H}^P + \int \underline{r}_p \times \left(\underline{\omega} \times \underline{r}_p \right). \tag{2-28}$$

As shown previously,

$$\int \underline{r}_p \times \left(\underline{\omega} \times \underline{r}_p \right) = \mathcal{J}^P \cdot \underline{\omega} \tag{2-29}$$

The term $\mathcal{J}^P \cdot \underline{\omega}$ can be viewed as the angular momentum of the total system rotating with the angular velocity of frame A, $^N\underline{H}^A$. Combining the results of Equations (2-27) through (2-29) provides

$$^N\underline{H}^c = {}^N\underline{H}^P - \underline{d}_{pc} \times M \dot{\underline{d}}_{pc} = {}^N\underline{H}^A + {}^A\underline{H}^P - \underline{d}_{pc} \times M \dot{\underline{d}}_{pc} \tag{2-30}$$

Taking the time derivative of Equation (2-30) yields the desired result

$$^N\dot{\underline{H}}^c = \underline{M}^c = {}^N\dot{\underline{H}}^A + {}^A\dot{\underline{H}}^P - \underline{d}_{pc} \times M \ddot{\underline{d}}_{pc}$$

$$^N\dot{\underline{H}}^c = \frac{^N d}{dt} \left(\mathcal{J}^P \cdot \underline{\omega} \right) + {}^A\dot{\underline{H}}^P - \underline{d}_{pc} \times M \ddot{\underline{d}}_{pc} \tag{2-31}$$

Equation (2-31) is a convenient and concise statement of the general result. For the special case in which the elastic body is rigid and the point P is the center of mass, the result is

$$\underline{M}^c = \frac{^N d}{dt} \left(\mathcal{I}^c \cdot \underline{\omega} \right) \tag{2-32}$$

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2.2 Rotational Equations of Motion for A Dual-Spin Vehicle

In this section, the rotational equations of motion for the dual-spin vehicle (see Figure 2-2) are provided. They are obtained by appropriately interpreting the general result given in Equation (2-31). The dual-spin vehicle being considered here is essentially the same as that discussed by Likins [4]. As shown in Figure 2-2, the vehicle consists of

(1) an asymmetrical portion†.

(2) a mass-spring-dashpot damper

(3) a symmetrical rotor

The frame A is chosen as that frame established by the asymmetrical body. The rotor axis is assumed to be that established by the bearings which permit relative rotation of the two primary bodies. The mass centers of the two primary bodies are constrained against relative translation. It is also assumed that a closed-loop control system governs the behavior of the motor used to maintain the rotor speed or the single axis control of the despun platform.

2.2.1 Evaluation of the Term $\frac{{}^N d}{dt}\left({}^A \underline{H}^P\right)$

Both the rotor and the damper contribute to the term $\frac{{}^N d}{dt}\left({}^A \underline{H}^P\right)$. The contribution to ${}^A \underline{H}^P$ made by the rotor is

$$ {}^A \underline{H}^P_R = J_3^R \, \sigma \, \hat{\underline{a}}_3 \tag{2-33} $$

where

$\sigma \hat{\underline{a}}_3$ = the angular velocity of the rotor relative to the frame A

J_3^R is the axial moment of inertia of the rotor.

The contribution to ${}^A \underline{H}^P$ made by the damper mass m is

†The equations of motion are obtained for the general case in which the despun body is asymmetrical even though a symmetric vehicle is of concern in this work.

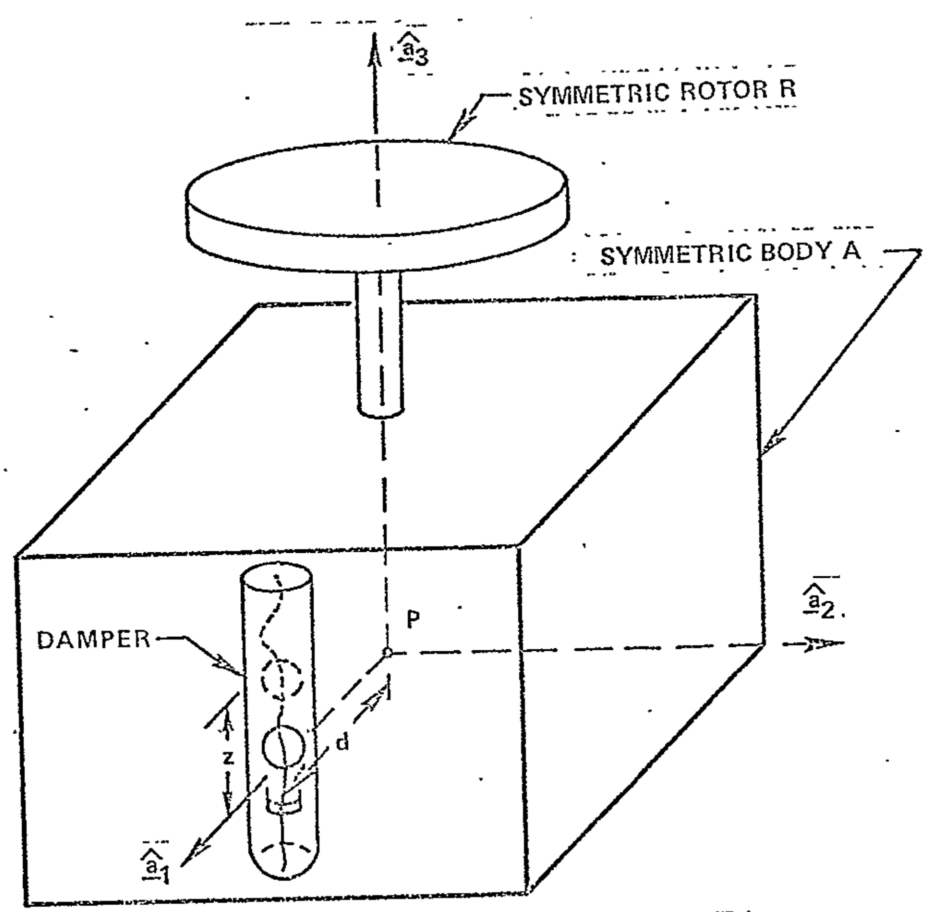

Figure 2-2 Pictorial Representation of a General Dual–Spin Vehicle

NOTE :

1: The damper mass is constrained to move in the a_3 direction and is located relative to the point P by

$r_p = d\ a_1 + z\ a_3$

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$$ {}^{A}\underline{H}^{P}_{\text{Damper}} = \left(d\,\hat{\underline{a}}_1 + z\,\hat{\underline{a}}_3\right) \times m\,{}^{A}\frac{d}{dt}\left(d\,\hat{\underline{a}}_1 + z\,\hat{\underline{a}}_3\right) $$

$$ = \left(d\,\hat{\underline{a}}_1 + z\,\hat{\underline{a}}_3\right) \times m\,\dot{z}\,\hat{\underline{a}}_3 $$

$$ = -m\,d\,\dot{z}\,\hat{\underline{a}}_2 \qquad (2\text{-}34) $$

Differentiating and collecting the terms of Equations (2-33) and (2-34) yields

$$ {}^{N}\frac{d}{dt}\left({}^{A}\underline{H}^{P}\right) = {}^{A}\frac{d}{dt}\left({}^{A}\underline{H}^{P}\right) + \underline{\omega} \times {}^{A}\underline{H}^{P} $$

$$ = {}^{A}\frac{d}{dt}\left(J_3^R\,\sigma\,\hat{\underline{a}}_3 - m\,d\,\dot{z}\,\hat{\underline{a}}_2\right) + \underline{\omega} \times \left(J_3^R\,\sigma\,\hat{\underline{a}}_3 - m\,d\,\dot{z}\,\hat{\underline{a}}_2\right) $$

$$ = J_3^R\,\dot{\sigma}\,\hat{\underline{a}}_3 - m\,d\,\ddot{z}\,\hat{\underline{a}}_2 + J_3^R\,\sigma\left\{-\omega_1\,\hat{\underline{a}}_2 + \omega_2\,\hat{\underline{a}}_1\right\} $$

$$ - m\,d\,\dot{z}\left\{\omega_1\,\hat{\underline{a}}_3 - \omega_3\,\hat{\underline{a}}_1\right\} \qquad (2\text{-}35) $$

The term ${}^{N}\frac{d}{dt}\left({}^{A}\underline{H}^{P}\right)$ written as a column matrix is

$$ \left\{\begin{matrix} J_3^R\,\sigma\,\omega_2 + m\,d\,\dot{z}\,\omega_3 \\ -m\,d\,\ddot{z} - J_3^R\,\sigma\,\omega_1 \\ J_3^R\,\dot{\sigma} - m\,d\,\dot{z}\,\omega_1 \end{matrix}\right\} \qquad (2\text{-}36) $$

2.2.2 Evaluation of the term $\underline{d}_{pc} \times \left(-M\,\ddot{\underline{d}}_{pc}\right)$

Only the damper contributes to the term $\underline{d}_{pc} \times \left(-M\,\ddot{\underline{d}}_{pc}\right)$. Since the damper mass m is constrained to move in the direction $\hat{\underline{a}}_3$, it follows that

$$ M\,\underline{d}_{pc} = m\,z\,\hat{\underline{a}}_3 \qquad (2\text{-}37) $$

where M = the total system mass.

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From Equation (2-37), the vector $\underline{d}_{pc}$ is given by

$$\underline{d}_{pc} = \frac{m}{M} z \hat{\underline{a}}_3 \tag{2-38}$$

and $\overset{\circ}{\underline{d}}_{pc}$, $\overset{\circ\circ}{\underline{d}}_{pc}$ are given by

$$\overset{\circ}{\underline{d}}_{pc} = \frac{m}{M} \dot{z} \hat{\underline{a}}_3$$

$$\overset{\circ\circ}{\underline{d}}_{pc} = \frac{m}{M} \ddot{z} \hat{\underline{a}}_3 \tag{2-39}$$

Substituting Equation (2-39) in the expanded form of $\underline{d}_{pc} \times \left(-M \ddot{\underline{d}}_{pc}\right)$ yields

$$\underline{d}_{pc} \times \left(- M \ddot{\underline{d}}_{pc}\right) =$$

$$- \frac{m}{M} z \hat{\underline{a}}_3 \times \Big[m \ddot{z} \hat{\underline{a}}_3 + 2 \underline{\omega} \times m \dot{z} \hat{\underline{a}}_3 + \dot{\underline{\omega}} \times m z \hat{\underline{a}}_3 + \underline{\omega} \times \left(\underline{\omega} \times m z \hat{\underline{a}}_3\right)\Big] \tag{2-40}$$

Simplifying Equation (2-40) and writing the term $\underline{d}_{pc} \times \left(- M \ddot{\underline{d}}_{pc}\right)$ as a column matrix yields

$$\begin{Bmatrix} \frac{(mz)^2}{M} \omega_2 \omega_3 - 2 \frac{m^2}{M} z \dot{z} \omega_1 - \frac{(mz)^2}{M} \dot{\omega}_1 \\ - \frac{(mz)^2}{M} \omega_1 \omega_3 - 2 \frac{m^2}{M} z \dot{z} \omega_2 - \frac{(mz)^2}{M} \dot{\omega}_2 \\ 0 \end{Bmatrix} \tag{2-41}$$

2.2.3 Evaluation of the term $\frac{^N d}{dt} \left(\mathcal{J}^P \cdot \underline{\omega}\right)$

The term $\frac{^N d}{dt} \left(\mathcal{J}^P \cdot \underline{\omega}\right)$ is the time rate of change of the angular momentum of the total system rotating with the angular velocity of frame A. The inertia matrix for the total system about point P is designated by J, the principal axis moments of inertia of the undeformed system about point P are designated I_α. The equation for this matrix J is

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$$J = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & J_3 \end{bmatrix} + m \begin{bmatrix} z^2 & 0 & -dz \\ 0 & z^2 & 0 \\ -dz & 0 & 0 \end{bmatrix} \tag{2-42}$$

The term $\mathcal{J}^P \cdot \underline{\omega}$ is equivalent to the matrix

$$\begin{bmatrix} I_1 & 0 & \\ & I_2 & \\ 0 & & I_3 \end{bmatrix} \{\omega\} + m \begin{bmatrix} z^2 & 0 & -dz \\ 0 & z^2 & 0 \\ -dz & 0 & 0 \end{bmatrix} \{\omega\} \tag{2-43}$$

The form of the contribution made by the term involving I_α is well known and is given by (Euler's equation). In matrix notation the term $\frac{{}^N d}{dt} [I] \underline{\omega}$ becomes

$$\begin{Bmatrix} I_1 \ \dot{\omega}_1 - (I_2 - I_3)\, \omega_2 \ \omega_3 \\ I_2 \ \dot{\omega}_2 - (I_3 - I_1)\, \omega_1 \ \omega_3 \\ I_3 \ \dot{\omega}_3 - (I_1 - I_2)\, \omega_1 \, \omega_2 \end{Bmatrix} \tag{2-44}$$

The remaining term due to the damper mass is given by $\frac{{}^N d}{dt}\left(\mathcal{J}_m^P \cdot \underline{\omega}\right) = \mathring{\mathcal{J}}_m \cdot \underline{\omega} + \underline{\omega} \times \mathcal{J}_m \cdot \underline{\omega} + \mathcal{J}_m \cdot \dot{\underline{\omega}}$. In matrix notation this is written as

$$m \begin{bmatrix} 2\, z\dot{z} & 0 & -d\dot{z} \\ 0 & 2z\dot{z} & 0 \\ -d\dot{z} & 0 & 0 \end{bmatrix} \{\omega\} + m \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} z^2 & 0 & -dz \\ 0 & z^2 & 0 \\ -dz & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$+ m \begin{bmatrix} z^2 & 0 & -dz \\ 0 & z^2 & 0 \\ -dz & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} \tag{2-45}$$

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After performing the matrix multiplications indicated in Equation (2-45), the matrix representation of the term ${}^{N}\frac{d}{dt}\left(\mathcal{J}_m \cdot \underline{\omega}\right)$ becomes

$$m\begin{Bmatrix} 2z\dot{z}\,\omega_1 - d\dot{z}\,\omega_3 - z^2\,\omega_2\omega_3 - dz\,\omega_1\omega_2 + z^2\,\dot{\omega}_1 - dz\,\dot{\omega}_3 \\ 2z\dot{z}\,\omega_2 + z^2\,\omega_1\omega_3 - dz\,\omega_3^2 + dz\,\omega_1^2 + z^2\,\dot{\omega}_2 \\ -d\dot{z}\,\omega_1 - z^2\,\omega_1\omega_2 + dz\,\omega_2\omega_3 + z^2\,\omega_1\omega_2 - dz\,\dot{\omega}_1 \end{Bmatrix} \tag{2-46}$$

With the combination of the results of Equations (2-44) and (2-46), the term ${}^{N}\frac{d}{dt}\left(\mathcal{J}^P \cdot \underline{\omega}\right)$ becomes

$$\begin{aligned} {}^{N}\frac{d}{dt}\left(\mathcal{J}^P \cdot \underline{\omega}\right) = {} & \hat{\underline{a}}_1\Big[\left(I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3\right) + m\big(2z\dot{z}\,\omega_1 - d\dot{z}\,\omega_3 - z^2\omega_2\omega_3 - dz\,\omega_1\omega_2 \\ & + z^2\,\dot{\omega}_1 - dz\,\dot{\omega}_3\big)\Big] \\ & + \hat{\underline{a}}_2\Big[\left(I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3\right) + m\big(2z\dot{z}\,\omega_2 + z^2\,\omega_1\omega_3 - dz\,\omega_3^2 + dz\,\omega_1^2 \\ & + z^2\,\dot{\omega}_2\big)\Big] \\ & + \hat{\underline{a}}_3\Big[\left(I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2\right) + m\left(-d\dot{z}\,\omega_1 + dz\,\omega_2\omega_3 - dz\,\dot{\omega}_1\right)\Big] \end{aligned} \tag{2-47}$$

2.2.4 Rotational Equations

The combined terms of Equations (2-36), (2-41) and (2-47) represent three of the five equations required to characterize the motion for the system. These scalar equations are

$$\begin{aligned} M_1 = {} & \left[I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 + J_3^R\,\sigma\,\omega_2\right] \\ & + \Big[2m\,z\dot{z}\left(1 - \frac{m}{M}\right)\omega_1 - m\,z^2\left(1 - \frac{m}{M}\right)\omega_2\,\omega_3 \\ & + m\,z^2\left(1 - \frac{m}{M}\right)\dot{\omega}_1 - m\,d\,z\,\dot{\omega}_3 - m\,d\,z\,\omega_1\omega_2\Big] \end{aligned} \tag{2-48}$$

$$M_2 = \left[I_2\,\dot{\omega}_2 - (I_3 - I_1)\,\omega_1\omega_3 - J_3^R\sigma\omega_1 \right]$$

$$+ \left[-m\,d\ddot{z} + m\,z^2\left(1 - \frac{m}{M}\right)\dot{\omega}_2 + m\,z^2\left(1 - \frac{m}{M}\right)\omega_1\omega_3 \right.$$

$$\left. + 2\,m\,z\,\dot{z}\left(1 - \frac{m}{M}\right)\omega_2 + m\,d\,z\left(\omega_1^2 - \omega_3^2\right) \right]$$

$$M_3 = \left[I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 + J_3^R\dot{\sigma} \right]$$

$$+ \left[-2\,m\,d\,\dot{z}\,\omega_1 + m\,d\,z\,\omega_2\omega_3 - m\,d\,z\,\dot{\omega}_1 \right]$$

To obtain a complete set of equations in the five unknowns (ω_1, ω_2, ω_3, σ, z), the preceding equations must be supplemented by some internal specification of the behavior of the rotor and the damper mass.

Supplementary Rotor Equation

The supplementary equation for the rotor is

$$M_R = J_3^R(\dot{\omega}_3 + \dot{\sigma}) \tag{2-49}$$

where M_R is the rotor torque about the $\hat{\underline{a}}_3$ axis.

The torque M_R is due to the combination of bearing friction and the applied motor torque. In the present application, the motor torque would be determined by a closed loop control system designed to maintain the desired rate ω_3.

Supplementary Damper Equation

The damper mass motion is governed by the equation

$$\underline{F} \cdot \hat{\underline{a}}_3 = m\,\ddot{\underline{d}}_{CQ} \cdot \hat{\underline{a}}_3 \tag{2-50}$$

where $\ddot{\underline{d}}_{CQ}$ is the inertial acceleration of the mass point Q

$\underline{d}_{CQ}$ is the position vector of the mass m located relative to the center of mass of the system (point C).

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The component of the force applied to the damper in the $\hat{\underline{a}}_3$ direction is that due to the spring and dashpot. Consequently, Equation (2-50) becomes

$$-kz - c\dot{z} = m\,\hat{\underline{a}}_3 \cdot \ddot{\underline{d}}_{CQ} = m\,\hat{\underline{a}}_3 \cdot \frac{{}^{N}d^2}{dt^2}\left\{\underline{d}_{CP} + \underline{d}_{PQ}\right\} \tag{2-51}$$

The position vectors $\underline{d}_{CP}$, $\underline{d}_{PQ}$ are given by

$$\begin{aligned} \underline{d}_{CP} &= -\underline{d}_{PC} = -\frac{m}{M} z\,\hat{\underline{a}}_3 \\ \underline{d}_{PQ} &= d\,\hat{\underline{a}}_1 + z\,\hat{\underline{a}}_3\,; \end{aligned} \tag{2-52}$$

the vector $\underline{d}_{CQ}$ is given by

$$\underline{d}_{CQ} = \underline{d}_{CP} + \underline{d}_{PQ} = d\,\hat{\underline{a}}_1 + z\left(1 - \frac{m}{M}\right)\hat{\underline{a}}_3 \tag{2-53}$$

The term $\ddot{\underline{d}}_{pc}$ is evaluated according to

$$\ddot{\underline{d}}_{CQ} = \overset{\circ\circ}{\underline{d}}_{CQ} + 2\underline{\omega} \times \overset{\circ}{\underline{d}}_{CQ} + \dot{\underline{\omega}} \times \underline{d}_{CQ} + \underline{\omega} \times (\underline{\omega} \times \underline{d}_{CQ}) \tag{2-54}$$

The $\hat{\underline{a}}_3$ component of $\ddot{\underline{d}}_{CQ}$ is obtained from the following partitioned matrix equation

$$\left\{\begin{array}{c} \\ \hline \ddot{d}_3 \end{array}\right\} = \left\{\begin{array}{c} \\ \hline \ddot{z}\left(1-\frac{m}{M}\right) \end{array}\right\} + 2\left[\begin{array}{ccc} & & \\ \hline -\omega_2 & \omega_1 & 0 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \dot{z}\left(1-\frac{m}{M}\right) \end{array}\right\} + \left[\begin{array}{ccc} & & \\ \hline -\dot{\omega}_2 & \dot{\omega}_1 & 0 \end{array}\right]\left\{\begin{array}{c} d \\ 0 \\ z\left(1-\frac{m}{M}\right) \end{array}\right\}$$

$$- \left[\begin{array}{ccc} & & \\ \hline -\omega_3\omega_1 & -\omega_3\omega_2 & \left(\omega_1^2 + \omega_2^2\right) \end{array}\right]\left\{\begin{array}{c} d \\ 0 \\ z\left(1-\frac{m}{M}\right) \end{array}\right\} \tag{2-55}$$

Combining Equations (2-51) and (2-55) yields the desired result

$$0 = m\left(1 - \frac{m}{M}\right)\ddot{z} + c\dot{z} + kz - m\,d\dot{\omega}_2 + m\,d\omega_1\omega_3 - m\left(1 - \frac{m}{M}\right)z\left(\omega_1^2 + \omega_2^2\right) \tag{2-56}$$

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Equation (2-48), (2-49), and (2-56) provide the set of five scalar equations which governs the motion of the system.

2.3 Simplified Rotational Equations of Motion for Dual-Spin Vehicles

In this section, the rotational equations of motion for a specific class of vehicles designed to perform specific mission objectives are obtained. As stated in Section 1, the dual-spin vehicle was selected for the deep-space mission because of the requirement for simultaneous earth communication and planet observation. In this dissertation, it is assumed that the $\underline{a}_3$ axis is an axis of symmetry. The main function of the control system is to maintain the desired orientation of the angular momentum vector during the cruise mode and to reorient the angular momentum vector during the large angle turn mode. In comparison with the applied moments, the torques due to the presence of the damper can be neglected. Under these assumptions the approximate rotational equations become

$$\frac{M_1}{I_1} = \dot{\omega}_1 - \frac{I_1 - I_3}{I_1}\,\omega_2\omega_3 + \frac{J_3^R\sigma}{I_1}\,\omega_2$$

$$\frac{M_2}{I_1} = \dot{\omega}_2 - \frac{I_3 - I_1}{I_1}\,\omega_1\omega_3 - \frac{J_3^R\sigma}{I_1}\,\omega_1 \qquad \text{2-56)}$$

$$\frac{M_3}{I_3} = \dot{\omega}_3 + \frac{J_3^R}{I_3}\,\dot{\sigma}$$

$$\frac{M_R}{J_3^R} = \dot{\omega}_3 + \dot{\sigma}$$

As stated previously, a closed loop control system for the $\underline{a}_3$ axis will ensure that the angular velocity about that axis is maintained at the desired value. Consequently, in this work, the control of the angular velocities ω_1 and ω_2 and the angles θ_1, θ_2 is of chief concern (see Figure 2-3).

2.3.1 Rotational Equations of Motion in Terms of Attitude Angles (Symmetric Vehicle)

In this section the rotational equations of motion for a symmetric dual-spin vehicle are obtained in terms of the attitude angles $\theta_1, \theta_2, \theta_3$. In terms of θ_1, θ_2, θ_3 the angular velocity ${}^N\underline{\omega}^A$ expressed in the $\underline{a}_\alpha$ basis can be obtained directly from Figure 2-3; expressed as a column matrix this result is

$$\left\{{}^N\omega^A\right\} = \begin{Bmatrix} \dot{\theta}_1 C\theta_2 C\theta_3 + \dot{\theta}_2 S\theta_3 \\ \dot{\theta}_2 C\theta_3 - \dot{\theta}_1 S\theta_3 C\theta_2 \\ \dot{\theta}_3 + \dot{\theta}_1 S\theta_2 \end{Bmatrix} \tag{2-57}$$

The Euler rates expressed as a function of ω_1, ω_2, ω_3 are given by

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} \dfrac{C\theta_3}{C\theta_2} & \dfrac{-S\theta_3}{C\theta_2} & 0 \\ S\theta_3 & C\theta_3 & 0 \\ -C\theta_3 \tan\theta_2 & -S\theta_3 \tan\theta_2 & 1 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{2-58}$$

Assuming that the angular rate of the despun body about the $\hat{\underline{a}}_3$ axis is precisely controlled to its desired small and constant value, it follows that $\dot{\omega}_3 \equiv 0$. Combining the results of Equations (2-56) through (2-58) one may obtain a set of equations for $\omega_1, \omega_2, \theta_1, \theta_2, \theta_3$ as follows

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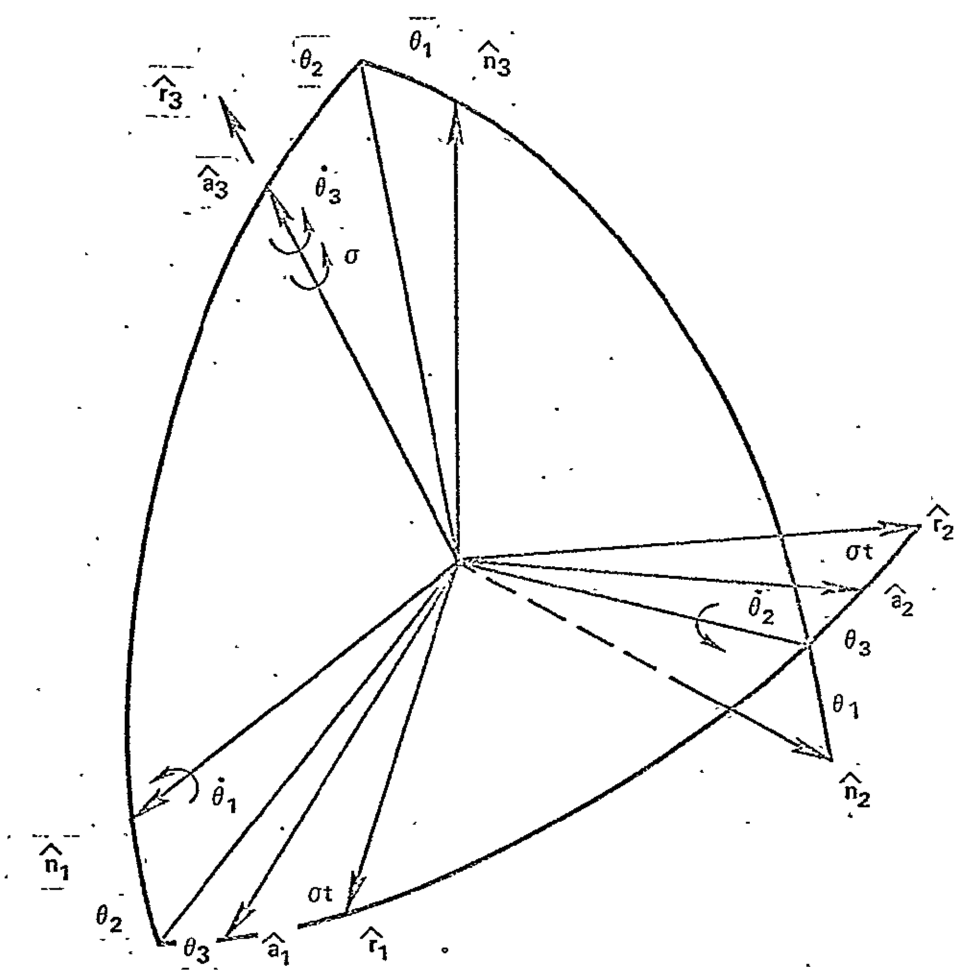

Figure 2-3 Coordinate Frames for the Dual-Spin Vehicle

NOTES:

1. n_α from basis for Newtonian frame
2. a_α from basis for body frame A
3. r_α from basis for rotor frame
4. Rotation sequence: 1–2–3

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$$
\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \left[\begin{array}{cc|c} 0 & \left[\frac{I_1 - I_3}{I_1}\omega_3 - \frac{J_3^R \sigma}{I_1}\right] & \\ -\left[\frac{I_1 - I_3}{I_1}\omega_3 - \frac{J_3^R \sigma}{I_1}\right] & 0 & 0 \\ \hline \frac{C\theta_3}{C\theta_2} & \frac{S\theta_3}{C\theta_2} & \\ S\theta_3 & C\theta_3 & 0 \\ -C\theta_3 \tan\theta_2 & -S\theta_3 \tan\theta_2 & \end{array}\right] \begin{pmatrix} \omega_1 \\ \omega_2 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} + \left[\begin{array}{c} \frac{M_1}{I_1} \\ \frac{M_2}{I_1} \\ \hline 0 \\ 0 \\ \omega_3 \end{array}\right] \tag{2-59}
$$

For notational convenience, Equation (2-59) can be written as

$$
\underline{\dot{x}} = A(\underline{x}, t)\, \underline{x}(t) + B(t)\, \underline{u}(t) = \underline{f}(\underline{x}, \underline{u}, t) \tag{2-60}
$$

where $\underline{x}$ can be considered the state and $\underline{u}$ the control. In general, the rotational equations of motion are characterized by a set of non-linear time-varying ordinary differential equations. The control $\underline{u}$ is produced by appropriately located jets. For this formulation†, a singularity would exist in the direction cosine formulation when $\theta_2 = (2n+1)\frac{\pi}{2}$ for $n = 0, \pm 1, \pm 2, \ldots$. However, if it is known that the allowed range for θ_2 does not include these values, this potential problem is of no concern.

2.3.2 Small Angle Case (Symmetric Vehicle)

In this section, the rotational equations of motion for the small angle case are obtained. Because of the precise pointing requirements for the mission being considered, deviations from the desired orientation are accurately represented by small angles. The small angle case is the one of chief concern in this dissertation. The linearized rotational

†By using a 3-2-1 sequence, the potential singularity problem can be avoided for the problems being studied in this work.

equations, under the assumption of small angles ($S\theta_j = \theta_j$, $C\theta_j = 1$ for $j = 1, 2, 3$), become

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \left[\begin{array}{cc|c} 0 & -\frac{J_3^R \sigma}{I_1} & \\ \frac{J_3^R \sigma}{I_1} & 0 & 0 \\ \hline 1 & 0 & \\ 0 & 1 & 0 \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} \frac{M_1}{I_1} \\ \frac{M_2}{I_1} \\ \hline 0 \\ 0 \end{Bmatrix} \tag{2-61}$$

2.3.3 Rotational Equations of Motion for the Small Angle Case as Expressed in the Coordinates of the Rotor Frame (Symmetric Vehicle)

In the preceding sections, the equations of motion were expressed in the coordinates of the "despun" frame A. In this section, the equation corresponding to Equation (2-61) is obtained for the case in which the reference frame is the rotor. In rotor coordinates, Equation (2-31) becomes

$$\underline{M}^c = {}^R\frac{d}{dt}(\mathcal{J}^P \cdot \underline{\omega}) + {}^N\underline{\omega}^R \times (\mathcal{J}^P \cdot \underline{\omega}) + {}^R\frac{d}{dt}({}^A\underline{H}^P) + {}^N\underline{\omega}^R \times {}^A\underline{H}^P \tag{2-62}$$

Since the axis $\hat{\underline{a}}_3$ is an axis of symmetry, $\mathcal{J}^P$ is constant in the rotor frame as well as in the A frame. The vectors ${}^A\underline{H}^P$, ${}^N\underline{\omega}^R$, and ${}^N\underline{\omega}^A$ are given by

$$\begin{aligned} {}^A\underline{H}^P &= J_3^R\,\sigma\,\hat{\underline{a}}_3 = J_3^R\,\sigma\,\hat{\underline{r}}_3 \\ {}^N\underline{\omega}^R &= {}^N\underline{\omega}^A + \sigma\,\hat{\underline{a}}_3 \\ {}^N\underline{\omega}^A &= {}^N\underline{\omega}^R - \sigma\,\hat{\underline{r}}_3 \end{aligned} \tag{2-63}$$

Examination of Equations (2-62) and (2-63) indicates that Equation (2-62) can be written as

$$\underline{M}^c = \left[{}^R\frac{d}{dt}\left(\mathcal{J}^P \cdot {}^N\underline{\omega}^R\right) + {}^N\underline{\omega}^R \times \left(\mathcal{J}^P \cdot {}^N\underline{\omega}^R\right) + {}^R\frac{d}{dt}\left({}^A\underline{H}^P\right) + {}^N\underline{\omega}^R \times {}^A\underline{H}^P \right]$$
$$+ {}^R\frac{d}{dt}\left(\mathcal{J}^P \cdot (-\sigma\, \hat{\underline{r}}_3)\right) + {}^N\underline{\omega}^R \times \mathcal{J}^P \cdot (-\sigma\, \hat{\underline{r}}_3). \qquad (2\text{-}64)$$

The bracketed term in Equation (2-64) has the same form as that of Equation (2-56). Hence, the equations written relative to the rotor frame are

$$\frac{M_1}{I_1} = \left[{}^N\dot{\omega}^R_1 + \frac{(I_3 - I_1)}{I_1}\, {}^N\omega^R_2\, {}^N\omega^R_3 + \frac{J^R_3\,\sigma}{I_1}\, {}^N\omega^R_2 \right] - \frac{I_3\, {}^N\omega^R_2\,\sigma}{I_1}$$

$$\frac{M_2}{I_1} = \left[{}^N\dot{\omega}^R_2 - \frac{(I_3 - I_1)}{I_1}\, {}^N\omega^R_1\, {}^N\omega^R_3 - \frac{J^R_3\,\sigma}{I_1}\, {}^N\omega^R_1 \right] + \frac{I_3\, {}^N\omega^R_1\,\sigma}{I_1} \qquad (2\text{-}65)$$

$$\frac{M_3}{I_3} = \left[{}^N\dot{\omega}^R_3 + \frac{J^R_3}{I_3}\,\dot{\sigma} \right] - \frac{I_3\,\dot{\sigma}}{I_3}$$

$$\frac{M_R}{J^R_3} = {}^N\dot{\omega}^R_3 + \dot{\sigma}$$

For the small angle case, the linearized equations for the control of ${}^N\omega^R_1$, ${}^N\omega^R_2$, θ_1, θ_2 are given by

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$$
\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \left[\begin{array}{cc|c} 0 & -\dfrac{J_3^R - I_1}{I_1}\sigma & 0 \\ \dfrac{J_3^R - I_1}{I_1}\sigma & 0 & \\ \hline C\,\sigma t & -S\,\sigma t & 0 \\ S\,\sigma t & C\,\sigma t & \end{array} \right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} \dfrac{M_1}{I_1} \\ \dfrac{M_2}{I_1} \\ \hline 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}66)
$$

where the superscripts on $\underline{\omega}$ are dropped for notational convenience, and where the nominal value of $\dot{\omega}_3$ has been selected as zero to correspond to the case of interest.

2.3.4 Rotational Equations of Motion for a Spinning Symmetric Vehicle (Small Angle Case)

The rotational equations of motion for a spinning symmetric vehicle are obtained as a special case of those obtained for a dual-spin vehicle. These equations can be obtained by interpreting the results provided in the last section, viz., the case in which the equations are expressed in the coordinates of the rotor frame. If the "despun" portion of the dual spin vehicle is spun up to the rotor speed, a spinning symmetric vehicle results. Hence, substituting I_3 (the moment of inertia of the total system about the spin axis) for J_3^R (the moment of inertia of the rotor about the spin axis) and ω_3 for σ in Equation (2-66), provides the rotational equations of motion

ηS

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & \frac{I_1 - I_3}{I_1}\omega_3 & & 0 \\ -\frac{I_1 - I_3}{I_1}\omega_3 & 0 & & \\ \hline C\omega_3 t & -S\omega_3 t & & 0 \\ S\omega_3 t & C\omega_3 t & & \end{array}\right] \begin{pmatrix} \omega_1 \\ \omega_2 \\ \hline \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} \frac{M_1}{I_1} \\ \frac{M_2}{I_1} \\ \hline 0 \\ 0 \end{pmatrix} \qquad (2\text{-}67)$$

Although this equation is correct, it has the relative disadvantage that the system matrix A is time-varying. However, it is felt intuitively that such a system can be characterized by a set of linear time-invariant differential equations,

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} \qquad (2\text{-}68)$$

The problem is thus to define a new $\underline{x}$ which results in the desired property. In particular, a more suitable rotation sequence is sought. Geometrically, it is clear that if the attitude angles ϕ_1, ϕ_2, ϕ_3 of Figure 2-4 are chosen instead of the attitude angles of Figure 2-3, then the desired result is achieved (keeping in mind that ϕ_1, ϕ_2 are small angles but ϕ_3 is, in general, large). In this case, the relationship between the rates $\dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3$ and the angular velocity of the despun body relative to the basis vectors $\hat{\underline{b}}_\alpha$ can be obtained by inspection of Figure 2-4; the desired relationship is

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \left[\begin{array}{c|cc} 1 & 0 & -S\phi_2 \\ \hline 0 & C\phi_1 & C\phi_2\, S\phi_1 \\ 0 & -S\phi_1 & C\phi_1\, C\phi_2 \end{array}\right] \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} \qquad (2\text{-}69)$$

The rates $\dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3$ are thus given by

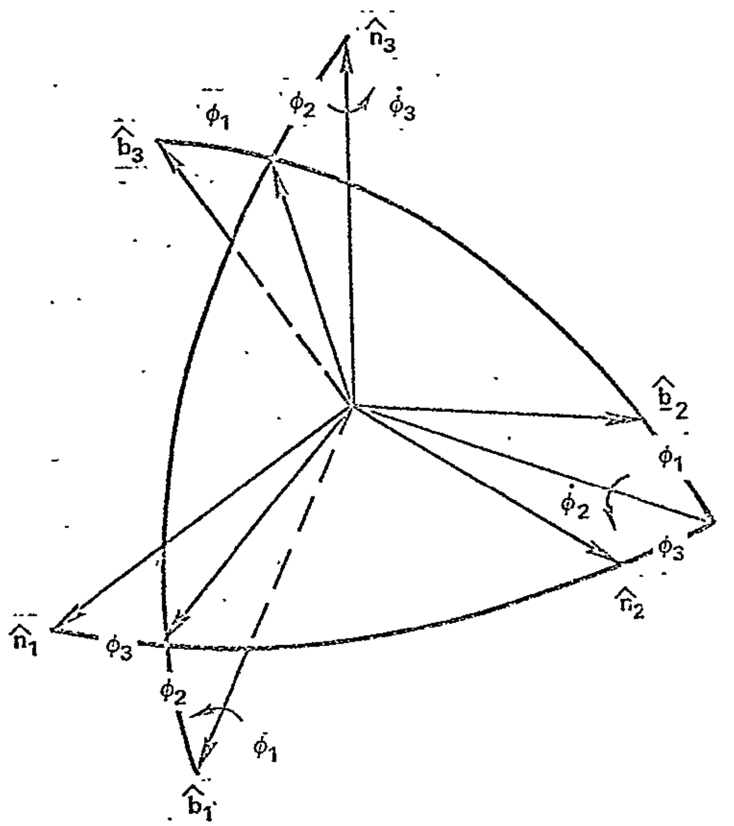

Figure 2-4 Coordinate Frames for a Spinning Symmetric Vehicle

NOTES:

Rotation sequence 3–2–1
Newtonian frame N with basis n_α
Body frame B with basis b_α

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$$
\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix}
\left[\begin{array}{c|cc}
1 & 0 & -S\phi_2 \\ \hline
0 & C\phi_1 & C\phi_2 S\phi_1 \\
0 & -S\phi_1 & C\phi_1 C\phi_2
\end{array}\right]^{-1}
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}
\tag{2-70}
$$

Because of the presence of the identity and the null matrices in Equation (2-70), the inverse can be immediately obtained. The notion of obtaining the inverse of a matrix by partitioning is used frequently in this work. In general, the inverse of the partitioned matrix T is given by

$$
\left[\begin{array}{c|c}
T_{11} & T_{12} \\ \hline
T_{21} & T_{22}
\end{array}\right]^{-1}
=
\left[\begin{array}{c|c}
\left[T_{11} - T_{12}T_{22}^{-1}T_{21}\right]^{-1} & -T_{11}^{-1}T_{12}\left[T_{22} - T_{21}T_{11}^{-1}T_{12}\right]^{-1} \\ \hline
-T_{22}^{-1}T_{21}\left[T_{11} - T_{12}T_{22}^{-1}T_{21}\right]^{-1} & \left[T_{22} - T_{21}T_{11}^{-1}T_{12}\right]^{-1}
\end{array}\right]
\tag{2-71}
$$

Using Equation (2-71) the inverse of the matrix in Equation (2-70) becomes (noting that $T_{21} = 0$ and $T_{11} = T_{11}^{-1} = I$)

$$
\left[\begin{array}{c|c}
I & -T_{12}T_{22}^{-1} \\ \hline
0 & T_{22}^{-1}
\end{array}\right]
=
\left[\begin{array}{c|cc}
1 & S\phi_1 \tan\phi_2 & C\phi_1 \tan\phi_2 \\ \hline
0 & C\phi_1 & -S\phi_1 \\
0 & \dfrac{S\phi_1}{C\phi_2} & \dfrac{C\phi_1}{C\phi_2}
\end{array}\right]
\tag{2-72}
$$

The final expression for $\dot{\phi}_1$ and $\dot{\phi}_2$ is given by

$$
\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}
=
\begin{bmatrix} 1 & 0 & 0 & \omega_3 \\ 0 & 1 & -\omega_3 & 0 \end{bmatrix}
\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \end{Bmatrix}
\tag{2-73}
$$

The rotational equations of motion in terms of the newly defined $\underline{x}$ become

$$
\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \hline \dot{\phi}_1 \\ \dot{\phi}_2 \end{Bmatrix} = \left[\begin{array}{cc|cc} 0 & \frac{I_1 - I_3}{I_1}\,\omega_3 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\frac{I_1 - I_3}{I_1}\,\omega_3 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{I}} & 0 & \omega_3 \\ & & -\omega_3 & 0 \end{array} \right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \phi_1 \\ \phi_2 \end{Bmatrix} + \begin{Bmatrix} \frac{M_1}{I_1} \\ \frac{M_2}{I_1} \\ \hline 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}74)
$$

Equation (2-74) is in agreement with the equivalent result provided in Reference [11]; in Reference [11], since dual-spin vehicles are of no concern, the result is obtained by using Euler's equation.

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Section 3

FORMULATION OF THE FUEL-OPTIMAL CONTROL PROBLEM

This chapter provides a discussion of the formulation of the optimal control problem. The general structure introduced in this chapter provides a convenient framework in which the fuel-optimal control problem concerning the dual-spin spacecraft can be imbedded.

3.1 Statement of the Optimal Control Problem

An optimal control problem is characterized by the composite of the following elements

(1) the plant or process S

(2) the initial set X_o containing the initial state $\underline{x}_o$ and the initial time t_o

(3) the target set X_1 containing the final state $\underline{x}_1$ and the final time t_1

(4) the class of admissible controllers Δ

(5) the control restraint set Ω

(6) the cost functional or performance index J

Briefly, then an optimal control problem is completely specified in terms of the composite

$$\{S, \Delta, \Omega, X_o, X_1, J\} .$$

A statement of the general optimal control problem is as follows. Given

(1) the dynamical system S having the state equations

$$\dot{x}_i = f_i(\underline{x}, \underline{u}, t) \qquad i = 1, \ldots, n$$

where $\underline{x} \in R^n$, $\underline{u} \in R^m$, $t \in R^1$

and $\dot{x}_i$ belongs to the class of continuous functions having continuous first partial derivatives with respect to $\underline{x}$, $\underline{u}$, t, i.e.,

$f_i \in C^1$ in $R^n \times R^m \times R^1$

(2) the initial state $\underline{x}_o$ and initial time t_o

(3) the class of admissible controllers Δ

(4) the control restraint set Ω,

the problem is to find the controller $\underline{u}(t) \subset \Omega$ which

(i) takes $\underline{x}_o$ to X_1 such that the pair $\left(\underline{x}(t_1),\ t_1\right) \in \dot{X}_1$ or equivalently the pair $\left(\phi\left(t_1;\ u(t_o, t_1],\ \underline{x}_o\right),\ t_1\right) \in X_1$

where ϕ is a vector function (transition function) which maps the cartesian product space $R^1 \times R^m \times R^n$ into R^n, i. e.,

$$\phi : R^1 \times R^m \times R^n \to R^n$$

(ii) minimizes the cost functional $J(\underline{x}_o,\ to,\ \underline{u})$ where $J(\underline{x}_o,\ t_o,\ \underline{u})$ maps the cartesian product space $R^n \times R^1 \times R^m$ into $\hat{R}^1$

As used here $\hat{R}^1$ refers to the extended real number system defined by [18]

$$\hat{R}^1 = R^1 \cup \{-\infty,\ +\infty\}$$

The control function $\underline{u}(t)$ which accomplishes the task described above is called the optimal controller $\underline{u}^*(t)$.†

3.2 Formulation of the Fuel-Optimal Control Problem for the Dual-Spin Spacecraft

In this section, a specific optimal control problem for a specific class of vehicles is discussed. In this work, the fuel optimal control of dual-spin spacecraft and spinning vehicles will be treated in detail.

†A controller $\underline{u}^*(t)$ belonging to the admissible class Δ is called optimal relative to the cost functional $J(\underline{x}_o,\ to,\ \underline{u}(t))$ if the relation

$$J\left(\underline{x}_o,\ to,\ \underline{u}^*(t)\right) \le J\left(\underline{x}_o,\ to,\ \underline{u}(t)\right)$$

is satisfied $\forall\ \underline{u}(t) \in \Delta$.

Once formulated, the problem under consideration can be compared to similar efforts discussed in the literature. To be specific this work will be compared with published works dealing with the fuel optimal attitude control of spinning vehicles.

3.2.1 Class of Admissible Controllers Δ

In this work, the class of admissible controllers is taken as those functions $u(t)$ which are measurable† on various intervals $t \in [t_o, t_1]$ and which steer the initial state $\underline{x}_o$ to the target set X_1. An important example of a measurable function is one that is piecewise continuous.

3.2.2 Control Restraint Set Ω

In this section, the nature of control restraint sets Ω which could conceivably be applied to the fuel-optimal control of dual-spin and spinning vehicles is discussed. Later, it will be seen that the control restraint set Ω for a particular problem should be carefully chosen. Many text books and journal publications give the erroneous impression that the control restraint set Ω which applies to the fuel-optimal attitude control problem in which magnitude limited jets are used is <u>always</u> given by

$$\Omega = \{\underline{u}(t) : |u_j(t)| \leq 1 \qquad j = 1, 2, \ldots, m\}$$

The "optimal" solution which results from the use of a model which is not "best" from a physical and practical pint of view should be cautiously interpreted.

---

†A real-valued function $u(t)$ defined on a real interval $\mathcal{a}$ is called measurable if for all real α and β, the set

$$\{t \,|\, t \in \mathcal{a} \text{ and } \alpha < u(t) < \beta\} \text{ is measurable.}$$

In general, the control restraint set Ω is defined in terms of the functions $u(t)$ (belonging to the class of admissible controllers Δ) which satisfy some appropriate constraints. Frequently (and perhaps too frequently), the control restraint set Ω is assumed to be that described above. In this case Ω is known as an m-cube and is both compact and convex.†

For the problems of interest in this work, the controllers $\underline{u}(t)$ are defined as

$$\underline{u}(t) = \frac{\underline{M}}{I_1}$$

where $\underline{M}$ is the applied moment (control) and I_1 is the transverse moment of inertia. The control restraint set Ω depends on

(1) the type of reaction jets used

(2) the number of reaction jets used to generate the applied moment $\underline{M}$.

Later, it will be seen that the number of jets needed for control depends on their location.

---

†A topological space X is said to be compact if every open covering $\mathcal{U}$ of X has a finite subcovering, that is, if there is a finite collection

$$\{O_1, O_2, \ldots, O_n\} \subset U \ni X = \bigcup_{i=1}^{N} O_i$$

In particular if the space $X \subset R^n$ then X is compact if it is closed and bounded.

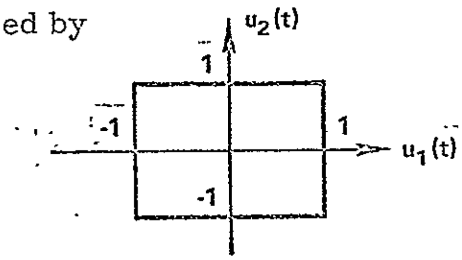
A subset K of a vector space X is said to be convex if whenever it contains x and y, it also contains $\lambda x + (1-\lambda)y$ for $0 \leq \lambda \leq 1$.

Magnitude limited two-way jet

For magnitude limited two-way jets the control restraint set Ω is given by

$$\Omega = \{ \underline{u}(t) : | u_j(t) | \leq 1 \qquad j = 1, 2, \ldots, m \}$$

For this case, the components of $\underline{u}(t)$ are independent of each other and belong to a hypercube. For the case in which $m = 2$, Ω is geometrically depicted by



In this work, a two-way jet is designated by . In a two-way jet, either side of the jet can be separately activated by a valve.

Magnitude limited one-way jet

Magnitude limited one-way jets have a control restraint set characterized by

$$\Omega = \{ \underline{u}(t) : 0 \leq u_j(t) \leq 1 , \qquad j = 1, \ldots, n \}$$

This type of jet is represented by  and is a special case of the two-way jet.

Gimballed-jet

A gimballed-jet is one in which both the magnitude of thrust and its direction are controlled. The salient feature of a gimballed-jet is that it can simultaneously produce torques about two axes rather than one as in the case of a fixed-jet. The components of $\underline{u}(t)$ for a gimballed-jet are dependent and belong to a smooth hypersphere. The control restraint set Ω is given by

$$\Omega = \{ \underline{u}(t) : \| \underline{u}(t) \| \leq 1 \}$$

where $\| \underline{u}(t) \|$ is the norm of the vector $\underline{u}(t)$

Rate-limited controller

In order to more realistically consider the inertia of the control mechanism, the controller is sometimes assumed to have a limited rate of variation. In this case, instantaneous switching is eliminated. The admissible controllers belong to the class of absolutely continuous† functions on various finite time intervals with $0 \leqq t \leqq t_1$. In this case the control restraint set is defined by

$$\Omega = \{ \underline{u}(t) : | \dot{u}_j(t) | \leq 1 \ \forall j \text{ a.e.}, \ u_j(t) \text{ are measurable, and } \underline{u}(0) = \underline{u}(t_1) = \underline{0} \}$$

3.2.3 Discussion of the Plant S

In this section the plant or process S is discussed. The rotational equations of motion for both the dual-spin vehicle and the spinning vehicle are derived in Chapter 2. As shown there, the plant can in general, be characterized by

$$\text{(S)} \quad \dot{\underline{x}} = A(\underline{x}, t)\, \underline{x}(t) + B(t)\, \underline{u}(t) \tag{3-1}$$

This characterization accurately represents the large angle turn mode of the deep-space mission. In the phase of the mission of chief interest in this work, the cruise phase, the plant can accurately be represented by

---

†A real-valued function u(t) defined on [a, b] is said to be absolutely continuous on [a, b] if given an $\epsilon > 0 \ \exists \delta > 0 \ni$

$$\sum_i | u(t_i') - u(t_i) | < \epsilon \text{ for every finite collection } \{(t_i, t_i')\} \text{ of}$$

nonoverlapping intervals with $\sum_i |t_i' - t_i| < \delta$ [19].

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$$(L) \qquad \underline{\dot{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \tag{3-2}$$

with an intial state $\underline{x}_o$ at time t_o. Hence, for the problems of chief concern, the plant is characterized by a linear time-varying set of ordinary differential equations.

The exact representation of (L) depends on the location of the jets. As mentioned previously, the number of jets required to accomplish the control objective depends on their location. In this section, the forms of the matrices A(t), B(t) are given for various jet locations.

Plant for the Dual-Spin Vehicle Using Rotor-Fixed Jets

When the rotational equations of motion are expressed in the "despun" frame (as was done in Chapter 2) and the jets are rotor-fixed the plant for the cruise phase is given by

$$(L) \qquad \underline{\dot{x}}(t) = A\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

or

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \hline \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \left[\begin{array}{cc|c} 0 & -\beta & 0 \\ \beta & 0 & \\ \hline I & & 0 \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \theta_1 \\ \theta_2 \end{Bmatrix} + \left[\begin{array}{c|c} C\,\sigma t & -S\,\sigma t \\ S\,\sigma t & C\,\sigma t \\ \hline 0 & 0 \end{array}\right] \begin{Bmatrix} u_1 \\ \hline u_2 \end{Bmatrix} \tag{3-3}$$

$$\text{where} \quad \beta = \frac{J_3^R\,\sigma}{I_1} \doteq \bar{r}\,\sigma$$

$$I = \text{the identity matrix } (2 \times 2).$$

Clearly, if only one rotor fixed jet (e.g., the one generating u_1), is used then the matrix B(t) becomes

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$$B(t) = \begin{Bmatrix} C\sigma t \\ S\sigma t \\ 0 \\ 0 \end{Bmatrix}$$

Plant for the Dual-Spin Vehicle Using Jets Located on the "Despun Body

If the jets are located on the despun portion of the spacecraft (S/C), the plant is given by

$$\text{(L)} \qquad \underline{\dot{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

or

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \hline \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \left[\begin{array}{cc|c} 0 & -\beta & \\ \beta & 0 & 0 \\ \hline & I & 0 \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \theta_1 \\ \theta_2 \end{Bmatrix} + \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (3\text{-}4)$$

In this case the system (L) is time-invariant. Optimal control theory for such systems has been intensively studied. It will be shown later when the notion of controllability is discussed that two jets would be required for this case.

Plant for the Dual-Spin Vehicle with the Equations of Motion Expressed in the Rotor Frame

For the case in which the linearized equations of motion are expressed in the rotor frame and the jets are rotor-fixed, the plant is given by

$$\underline{\dot{x}} = A(t)\,\underline{x}(t) + B\,\underline{u}(t)$$

or

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$$
\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \hdashline \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \left[\begin{array}{cc:c} 0 & -\gamma & \\ & & 0 \\ \gamma & 0 & \\ \hdashline C\sigma t & -S\sigma t & \\ & & 0 \\ S\sigma t & C\sigma t & \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hdashline \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{bmatrix} I \\ \hdashline 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} . \qquad (3\text{-}5)
$$

where $\underline{\omega} = {}^{N}\underline{\omega}^{R}$,

$$
\gamma = \frac{J_3^R - I_1}{I_1} \sigma
$$

In this case the matrix A is time-varying and B is time-invariant. If only one jet is used, the time-invariant matrix B becomes

$$
B = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

This formulation has no advantages over that given by Equation (3-2). In fact, it has the disadvantage that the transition matrix cannot be evaluated as easily. Although the solution is given by

$$
\underline{x}(t) = \phi(t)\, \underline{x}_o + \phi(t) \int_{t_o}^{t_1} \phi^{-1}(\tau)\, B(\tau)\, \underline{u}(\tau)\, d\tau , \qquad (3\text{-}6)
$$

where $\phi(t)$ is the transition matrix satisfying

$$
\begin{aligned} \dot{\phi}(t) &= A(t)\phi(t) \\ \phi(0) &= I \quad , \end{aligned} \qquad (3\text{-}7)
$$

the fact that the matrix product

$$
A(t_1)\; A(t_2)
$$

is not commutative implies that the transition matrix is not simply

$$\phi(t) = e^{At}$$

as it was for the formulation given by Equation (3-2).

Plant for a Symmetric Spinning Vehicle

The plant for a spinning vehicle can be viewed as a special case of that for a dual-spin vehicle. A spinning vehicle is of special interest in this work because the fuel-optimal spin-axis control (SACO) of such a vehicle has been discussed in the literature [11]. By comparing the results given in [11] with the new results obtained for the same vehicle when the angular momentum control (AMCO) concept is used, the relative merits of the new concept can be ascertained. The rotational equations for this vehicle are provided in Equation (2-74). The plant can be represented by the time-invariant linear system

$$\text{(L)} \qquad \dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{3-8}$$

or

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \hline \dot{\phi}_1 \\ \dot{\phi}_2 \end{Bmatrix} = \left[\begin{array}{cc|cc} 0 & r\,\omega_3 & & \\ -r\omega_3 & 0 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{} & 0 & \omega_3 \\ \multicolumn{2}{c|}{I} & -\omega_3 & 0 \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \hline \phi_1 \\ \phi_2 \end{Bmatrix} + \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\text{where} \quad r = \frac{I_1 - I_3}{I_1} .$$

3.2.4 Initial Set X_o and Target Set X_1

In this section, the initial set X_o and the target set X_1, as they pertain to the fuel-optimal control of a dual-spin S/C and a spinning vehicle are discussed. As stated in the introduction it is the target set X_1 and the control restraint set Ω which distinguish the present work from that which has been discussed in the literature. In Reference [11], the fuel-optimal control problem for a spinning symmetric vehicle is discussed. In that case, the target set X_1 is given by

$$X_1 = \{(\underline{x}, t): \underline{x}(t_1) = 0, \ t_1 \ \text{free}\}.$$

This type of fuel-optimal problem can be conveniently classified as one of spin-axis-control (SACO).

In this dissertation, emphasis is placed on the practical aspects of the control problem. It is not considered practical, especially when the objective of the control problem is to minimize fuel, to drive the final state to zero (or even to a small neighborhood of zero -- i.e., the dead band region). Instead, the notion of angular momentum control (AMCO) is introduced. In this concept, the angular momentum vector $\underline{H}$ is oriented to its desired direction in inertial space by applying the control $\underline{u}$ in such a way that fuel is minimized. A properly designed damper then aligns the spin axis with the angular momentum vector. In this case the target set X_1 is given by

$$X_1 = \left\{(\underline{x}(t), t): g_1\left(\underline{x}(t_1), t_1\right) = g_2\left(\underline{x}(t_1), t_1\right) = 0\right\}$$

where

$$g_1(\underline{x}, t) = \underline{H} \cdot \underline{\hat{n}}_1$$

$$g_2(\underline{x}, t) = \underline{H} \cdot \underline{\hat{n}}_2$$

and $\hat{n}_3$ parallels the desired angular momentum vector $\underline{H}_D$.

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In this case, the target set is a smooth 3-fold† in $R^n \times R^1$ (if the functions g_1 and g_2 are not explicit functions of time, then the target set is a smooth 2-fold in R^n). It is noted that the transverse components of the angular momentum in inertial space, $g_1(\underline{x})$ and $g_2(\underline{x})$, are smooth functions. The set of points for which $g_j(\underline{x}) = 0$ is said to be a smooth hypersurface. Hence, the target set can be viewed as the intersection of the smooth hypersurfaces associated with $g_1(\underline{x})$ and $g_2(\underline{x})$. For the dual-spin vehicle, the functions $g_1(\underline{x})$ and $g_2(\underline{x})$ are given by

$$\underline{g}(\underline{x}) = \begin{bmatrix} 1 & 0 & 0 & \bar{r}\sigma \\ 0 & 1 & -\bar{r}\sigma & 0 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = C\,\underline{x} \tag{3-8}$$

$$\text{where } \bar{r} = \frac{J_3^R}{I_1}$$

$$\text{and } \bar{r}\sigma = \beta \quad .$$

Hence, the target set X_1 for the AMCO is convex and closed. Note that as defined in Equation (3-8) it is not compact since it is unbounded. However, when due consideration is given to the fact that both $\underline{\theta}$ and $\underline{\omega}$ are bounded, then the set X_1 is compact. The target set X_1 in terms of the $\omega_1 - \theta_2$ and $\omega_2 - \theta_1$ planes is depicted in Figure 3-1.

The initial set X_o consists of the pair $(\underline{x}_o, t_o)$ where $\underline{x}_o$ is the initial state and t_o is the initial time. The initial state, for the problems

---

†The set of points X_1 defined by

$$X_1 = \{\underline{x} : g_j(\underline{x}) = 0 \qquad j = 1, 2, \ldots, n-k\}$$

is said to be a smooth k-fold in R^n if for every point $\underline{x}_o \in X_1$ the n-k vectors $\frac{\partial g_j}{\partial \underline{x}}(\underline{x}_o)$ are linearly independent [20].

of interest in this work, is that which exists at the time the optimal control sequence is initiated. The components of $\underline{x}_o$ are sensed by appropriate sensors and when the initial state is such that the antenna pointing error is too large, the optimal control sequence is initiated. Hence, the elements of the initial state provide a criterion for initiating the optimal control sequence.

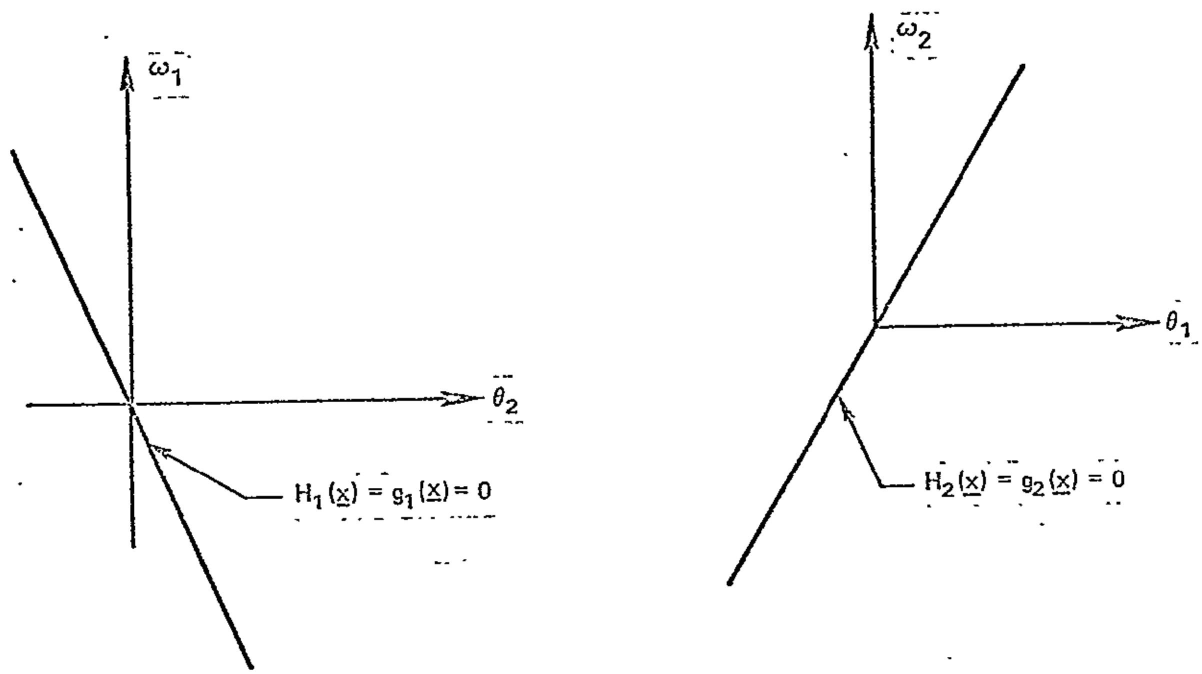

Figure 3-1 Illustration of the Target Set for the AMCO Concept

3.2.5 Cost Functional

The cost functional $J(\underline{x}_o, to, \underline{u})$ is, in general, the quantitative criterion for the efficiency of the controllers $\underline{u}(t)$ on $t_o \leq t \leq t_1$ in the class Δ. In this work, the cost functional is related to the fuel used in driving the initial state $\underline{x}_o$ to the target set X_1. Hence, the cost functional is given by

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$$J\left(\underline{x}_o, t_o, \underline{u}\right) = \int_{t_o}^{t_1} h_o\left(\underline{u}(t), t\right) dt \tag{3-9}$$

where $h_o\left(\underline{u}(t), t\right)$ is related to the flow of fuel. For convenience, $J(\underline{x}_o,$ to, $\underline{u})$ will frequently be written as $J(\underline{u})$. The exact form of $h\left(\underline{u}(t), t\right)$ depends on the control restraint set Ω. The form of $h_o\left(\underline{u}(t), t\right)$ for various control restraint sets Ω of interest for the dual-spin S/C is discussed below.

Magnitude-Constrained Controller

The control restraint most commonly used in regard to attitude control problems is given by

$$\Omega = \{\underline{u}(t): |u_j(t)| \leq 1 \quad j = 1, \ldots, m\}$$

The cost functional corresponding to this Ω for the fuel-optimal problem is given by

$$J(\underline{x}_o, to, \underline{u}) = \int_{t_o}^{t} \sum_{i=1}^{m} K_i |u_i(t)| dt \tag{3-10}$$

A special case of the above category which is especially suited for the dual-spin vehicle is given by

$$\Omega = \{\underline{u}(t) : 0 \leq u_j(t) \leq 1, \quad j = 1, \ldots, m\}$$

In this case, the cost functional becomes

$$J(\underline{x}_o, to, \underline{u}) = \int_{t_o}^{t_1} \sum_{j=1}^{m} K_j u_j(t) dt \tag{3-11}$$

Norm-Constrained Controller

As mentioned in Section 3.2.2, the use of a gimballed jet results in a control restraint set Ω given by

$$\Omega = \{ \underline{u}(t) : \| \underline{u}(t) \| \leq 1 \}$$

The cost functional corresponding to this Ω is

$$J(\underline{x}_o, \text{ to}, \underline{u}) = \int_{t_o}^{t} K \| \underline{u} \| dt. \qquad (3\text{-}12)$$

3.2.6 Statement of the Fuel-Optimal Control Problem for Systems Being Studied

Having discussed the notions which make up an optimal control problem, the fuel-optimal control problem for the systems being studied can now be explicitly and briefly stated. In this dissertation, the cruise phase of a deep space mission is of prime concern. In this phase, the task is to maintain the precise inertial orientation of the rotor axis. Loosely, the control problem is to find the optimal controller $\underline{u}^*(t)$ which drives the initial state $\underline{x}_o$ to the target set (defined by requiring that the transverse components of the angular momentum vector in inertial space be zero) in such a way that a minimum amount of fuel is expended. The optimal control sequence is initiated when the condition

$$\| \underline{\theta} \| \geq \theta_c \qquad (3\text{-}13)$$

where $\underline{\theta}$ has components θ_1, θ_2

θ_c is based on the required antenna pointing accuracy

is satisfied and is terminated when the condition

$$\| \underline{H}_T \| \leq \epsilon \qquad (3\text{-}14)$$

where $\underline{H}_T$ has components $\underline{H} \cdot \underline{n}_1$, $\underline{H} \cdot \underline{n}_2$

ϵ represents some arbitrarily small positive number

is satisfied.

More precisely, the control problem $\{L, \Delta, \Omega, X_o, X_1, J\}$ of prime concern is the following. Given

(1) the dynamical system L

$$(L) \quad \underline{\dot{x}} = A\,\underline{x}(t) + B(t)\,\underline{u}(t) \,. \tag{3-15}$$

(2) the initial state $\underline{x}_o$ and the corresponding initial time t_o

(3) the class of admissible controllers Δ

(4) the control restraint set Ω (to be carefully selected based on practical considerations),

the problem is to find the controller $\underline{u}(t) \subset \Omega$ which

(a) takes $\underline{x}_o$ to X_1 such that the pair

$$\left(\underline{x}(t_1), t_1\right) \in X_1 \tag{3-16}$$

$$\text{where } X_1 = \left\{(\underline{x}, t) : g_j(\underline{x}) = 0 \quad j = 1, 2\right\}$$

(b) minimizes the cost functional $J(\underline{u})$.

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Section 4

CONTROLLABILITY, NORMALITY, EXISTENCE, AND UNIQUENESS

In this chapter, the related concepts of controllability and normality and their connection with the existence, and uniqueness of optimal controllers is discussed. These notions are of great importance when the computational aspects of the optimal control problem are of concern. They indicate whether a problem is well posed, whether a unique optimal controller can be found, and even indicate to an extent what computational approach should be taken to determine the optimal controller. As in the preceding chapters, the general theory is first stated and then applied to the specific problems of interest.

4.1 Controllability and Normality

In this section, the notions of controllability and normality are discussed. The notions of controllability, popularized by Kalman [22], provides a convenient means for determining whether a linear control problem is well-posed. Normality is closely related to controllability but is a stronger property in that normality implies controllability but not vice versa. The notion of normality plays a key role in existence theory pertaining to optimal solutions of linear systems. After these concepts are defined, some pertinent theorems are stated and applied to the problems of interest in this work.

4.1.1 Controllability

In this work, the concept of controllability is used to establish if the linear control problems being studied are well-posed. If the system is linear, the very first step in the determination of the optimal controller involves an investigation of this concept. It is demonstrated in this section that the nature of the notion of controllability is not merely mathematical but is practical as well. In fact, controllability

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can aid in the determination of the location and number of jets required for control.

Controllability is defined as follows [21]. If the system

$$\text{(L)} \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) = \underline{f}(t, \underline{u}, \underline{x})$$

with initial state $\underline{x}(t_o) = \underline{x}_o$ admits a solution such that

$$\underline{x}(T) = \underline{\phi}(T;\ \underline{u}(t_o, T],\ \underline{x}_o) = \underline{0}$$

for some finite $T > t_o$ and for some measurable $\underline{u}$ for each $\underline{x}_o \in R^n$, then the system (L) is said to be completely controllable. In this definition, the vector function $\underline{\phi}$ is such that

$$\underline{\phi} : R \times R^m \times R^n \rightarrow R^n$$

and satisfies

$$\underline{\phi}\left(t_o;\ \underline{u}(t_o, t_o],\ \underline{x}_o\right) = \underline{x}_o$$

$$\frac{d}{dt}\,\underline{\phi}\left(t\,;\ \underline{u}(t_o, t],\ \underline{x}_o\right) = \underline{f}\left[t,\ \underline{u}(t),\ \underline{\phi}\left(t;\ u(t_o, t],\ \underline{x}_o\right)\right]$$

<u>Controllability for Time-Varying Systems</u>

For a time-varying linear system, a computational check on controllability involving only the transition matrix $\Phi(t, \tau)$ and the matrix B(t) is given by the following theorem due to Kalman [22].

<u>Theorem</u>. The Linear System

$$\text{(L)} \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

is completely controllable iff for every t there exists a $t_1 > t$ such that the $n \times n$ symmetric matrix

$$C(t, t_1) = \int_t^{t_1} \Phi(t_1, \tau)\, B(\tau)\, B^T(\tau)\, \Phi^T(t_1, \tau)\, d\tau \qquad (4\text{-}1)$$

is positive definite.

Because of the nature of the controllability matrix $C(t, t_1)$, the requirement that it be positive definite is equivalent to the requirement that it be nonsingular or that its determinant be nonzero.

For the dual-spin vehicle, the nature of the system

$$(L) \qquad \underline{\dot{x}} = A(t)\, \underline{x}(t) + B(t)\, \underline{u}(t) \tag{4-2}$$

for various control restraint sets and for various jet locations was discussed in Section 3.2. When the jets are rotor-fixed, the matrix A is time-invariant and the matrix B is time varying. For such a system the transition matrix $\phi(t, t_o)$ can be conveniently computed by using some fundamental results pertaining to the spectral theory of operators. The notions of this theory which are used in obtaining $\phi(t, t_o)$ include

(1) the spectral representation of the operator A

(2) the Jordan canonical form for the operator A

(3) functions of the operator A

Spectral Representation

The spectral representation of a simple operator A is given in terms of its eigenvalues λ_i, its eigenvectors x_i, and the eigenvectors y_i of the adjoint† operator A^*. That is, any vector $\underline{x}$ expressed in terms of its eigenvectors x_i is represented by

$$x = \alpha_i\, x_i\,, \tag{4-3}$$

the vector A x is represented by

$$Ax = \lambda_i\, \alpha_i\, x_i\,, \tag{4-4}$$

and the scalars α_i are represented by

†A linear operator L^* is said to be the adjoint of L if for all x, y belonging to the domain of L, $\langle y, Lx \rangle = \langle L^* y, x \rangle$.

$$\alpha_i = \langle x, y_i \rangle \tag{4-5}$$

When the operator L is the matrix A, then the adjoint operator L^* is simply A^T. Using Equations (4-3) through (4-5), the vectors x and Ax can be written as

$$x = \sum_i x_i \langle y_i, x \rangle$$

$$Ax = \sum_i \lambda_i x_i \langle y_i, x \rangle$$

In terms of dyads these results are

$$\begin{aligned} A &= \sum_i \lambda_i x_i \rangle\langle y_i \\ x\rangle &= \sum_i x_i \rangle\langle y_i, x \rangle \end{aligned} \tag{4-6}$$

It follows that the operator

$$\sum_i x_i \rangle\langle y_i$$

is the identity operator. In addition, the eigenvectors of A and those of A^* form a biorthogonal set, that is

$$\langle x_i, y_j \rangle = \delta_{ij} \tag{4-7}$$

The representation of A given in Equation (4-6) immediately suggests that

$$A = M \; \Lambda \; M^{-1}$$

where the eignevectors x_i are the columns of M, the eigenvectors y_i are the rows of M^{-1}, and the eigenvalues λ_i are the elements of the diagonal matrix Λ. In the general case, the eigenvalues are not distinct and the diagonal matrix Λ is replaced by the Jordan canonical form $\overline{J}$.

Jordan Canonical Form

The following theorems [17] provide important features of the Jordan canonical form.

Theorem. Every matrix A can be transformed into its Jordan canonical form $\bar{J}$ by means of a similarity transformation.

Theorem. Let A be an arbitrary matrix with x_i as its right eigenvectors or right generalized eigenvectors. Let M be the matrix whose columns are the vectors x_i, then the matrix

$$\bar{J} = M^{-1} AM$$

is the Jordan canonical form and the rows of M^{-1} are the left eigenvectors or left generalized eigenvectors† of A.

Hence, when the ordinary eigenvectors do not span the space, the notion of a generalized eigenvector is introduced. The generalized eigenvector x_k belongs to the null space of the operator $(L-\lambda_o I)^k$. Repeated applications of $(L - \lambda_o I)$ to a generalized eigenvector of rank r generates a chain of r generalized eigenvectors. In this way, an optimal basis for the operator L is created and relative to this basis the operator L has the Jordan canonical form.

For the dual-spin system, the matrix A is given by

$$A = \left[\begin{array}{cc|cc} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \tag{4-8}$$

† A vector x_k for which $(L-\lambda_o I)^{k-1} x_k \neq 0$, but for which $(L-\lambda_o I)^k x_k = 0$ is called a generalized eigenvector or an eigenvector of rank k corresponding to the eigenvalue λ_o.

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The eigenvalues are computed from the characteristic equation. The determinant of A is conveniently evaluated by partitioning the matrix A. That is,

$$\left|\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right| = |A_{11}| \; |A_{22} - A_{21} A_{11}^{-1} A_{12}| \quad \text{if } |A_{11}| \neq 0 \tag{4-9}$$

The eigenvalues are given by

$$\lambda_1 = 0, \; \lambda_2 = 0, \; \lambda_3 = j\beta, \; \lambda_4 = -j\beta$$

For the repeated eigenvalue $\lambda = 0$, the generalized eigenvector x_k is that vector for which

$$\left(A - \lambda_o I\right)^k x_k = 0$$

The null space for $(A - \lambda I)\,x$ is determined from the relation

$$\begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} -\beta\xi_2 \\ \beta\xi_1 \\ \xi_1 \\ \xi_2 \end{pmatrix}$$

that is, the null space is given by

$$\mathcal{N}_1 = \{x : \xi_1 = 0 \text{ and } \xi_2 = 0\}$$

Similarly, the null space for

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$(A - \lambda_o I)^2 x$ is determined from

$$\begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\beta\xi_2 \\ \beta\xi_1 \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\beta^2\xi_1 \\ -\beta^2\xi_2 \\ -\beta\ \xi_2 \\ \beta.\ \xi_1 \end{pmatrix}$$

hence, the null space for $(A - \lambda_o I)^2$ is given by

$$\mathcal{N}_2 = \{\underline{x} : \xi_1 = 0 \text{ and } \xi_2 = 0\} = \mathcal{N}_1$$

Since the null spaces $\mathcal{N}_1$ and $\mathcal{N}_2$ are identical, there are no ones in the super diagonal and the Jordan canonical form is given by the diagonal matrix

$$\overline{J} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & j\beta & \\ & & & -j\beta \end{bmatrix} = \Lambda$$

The generalized eigenvectors of rank 1 associated with $\lambda = 0$ are

$$x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvaectors corresponding to the distinct eigenvalues $\lambda = j\beta$ and $\lambda = -j\beta$ are obtained from the definition of an eigenvector and are

$$x_1 = \begin{pmatrix} -\beta \\ j\beta \\ j \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda = j\beta$$

$$x_2 = \begin{pmatrix} \beta \\ j\beta \\ j \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = -j\beta$$

Hence, the modal matrix M is given by

$$M = \left[\begin{array}{cc|cc} -\beta & \beta & 0 & 0 \\ j\beta & j\beta & 0 & 0 \\ \hline j & j & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse of M is easily computed by partitioning; that is,

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} (M_{11} - M_{12} M_{22}^{-1} M_{21})^{-1} & -M_{11}^{-1} M_{12} (M_{22} - M_{21} M_{11}^{-1} M_{12})^{-1} \\ \hline -M_{22}^{-1} M_{21} (M_{11} - M_{12} M_{22}^{-1} M_{21})^{-1} & (M_{22} - M_{21} M_{11}^{-1} M_{12})^{-1} \end{array}\right] \tag{4-10}$$

with $M_{12} = [0]$

$M_{22} = [I]$

The inverse of M is given by

$$M^{-1} = \left[\begin{array}{cc|c} \frac{1}{-2\beta} & \frac{1}{2j\beta} & \\ \frac{1}{2\beta} & \frac{1}{2j\beta} & 0 \\ \hline 0 & -1/\beta & \\ \frac{1}{\beta} & 0 & I \end{array}\right]$$

As a check, it is seen that, indeed

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$$\bar{J} = M^{-1} A M \quad ,$$

as it should.

Functions of An Operator

The transition matrix $\phi(t, \tau)$ can now be easily obtained since the Jordan canonical form and the matrices M and M^{-1} are known. The operator A is simple, that is, every eigenvalue of A is an eigenvalue of A^*, all eigenvectors of A and A^* are of rank one, and the eigenvectors of A or A^* span the space. For a simple operator, the spectral representation is given by

$$x = \sum_i x_i \langle y_i, x \rangle$$

$$Ax = \sum_i \lambda_i x_i \langle y_i, x \rangle$$

It follows that functions of the operator A are given by

$$A^2 x = \sum_i \lambda_i^2 x_i \langle y_i, x \rangle$$

$$A^n x = \sum_i \lambda_i^n x_i \langle y_i, x \rangle$$

$$q(A)x = \sum_i x_i \, q(\lambda_i) \langle y_i, x \rangle$$

where $q(\lambda)$ is any polynomial in λ.

Extending this notion of a function of an operator to analytic functions $f(\lambda)$ by using the power series for $f(\lambda)$ it follows that

$$f(A)\, x = \sum_i x_i f(\lambda_i) \langle y_i, x \rangle \tag{4-11}$$

The controllability of the system at time t_o can be determined by examining the determinant of the controllability matrix

$$C(t_o, t_1) = \int_{t_o}^{t_1} \phi(t_o, t)\, B(t)\, B^T(t)\, \phi^T(t_o, t)\, dt \tag{4-12}$$

The controllability matrix can be easily evaluated analytically by using the fundamental properties of the transition matrix. Some of the useful properties of transition matrices are the following:

$$\begin{aligned}
\phi(t, t_o) &= \phi(t)\phi^{-1}(t_o) \\
\phi(t_o, t_o) &= I \\
\phi^{-1}(t, t_o) &= \phi(t_o, t) = \phi(t_o)\ \phi^{-1}(t) \\
\phi(t_o, t) &= \phi(t_o, t_1)\ \phi(t_1, t) \\
\phi^T(t_o, t) &= \left[\phi(t_o)\ \phi^{-1}(t)\right]^T = \phi^{-1}(t)^T\, \phi(t_o)^T
\end{aligned} \tag{4-13}$$

The transition matrix for a time-invariant system has the additional properties

$$\begin{aligned}
\phi^{-1}(t) &= \phi(-t) \\
\phi(t + \tau) &= \phi(t)\, \phi(\tau)
\end{aligned}$$

Hence, the controllability matrix $C(t_o, t_1)$ is given by

$$C(t_o, t_1) = \phi(t_o)\left[\int_{t_o}^{t_1} \phi(-t)\, B(t)\, B^T(t)\, \phi(-t)^T dt\right]\phi(t_o)^T \tag{4-14}$$

From Equation (4-13), it is seen that

$$\phi(t_o) = I$$

and, hence, Equation (4-14) reduces to

$$C(t_o, t_1) = \int_{t_o}^{t_1} \Phi(-t)\ B(t)\ B^T(t)\ \Phi(-t)^T dt \tag{4-15}$$

The controllability matrix can now be evaluated for the cases in which

(1) two rotor-fixed jets are used, i. e.,

$$B(t) = \begin{bmatrix} C\sigma t & -S\sigma t \\ S\sigma t & C\sigma t \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4-16}$$

(2) one rotor-fixed jet is used, i. e.,

$$B(t) = \begin{bmatrix} C\sigma t \\ S\sigma t \\ 0 \\ 0 \end{bmatrix} \tag{4-17}$$

It is known that for a linear time-invariant system of differential equations

$$\text{(L)} \quad \dot{\underline{x}} = A\,\underline{x}(t)$$

$$\underline{x}(t_o) = \underline{x}_o \,,$$

the solution is given by

$$\underline{x}(t) = e^{At}\,\underline{x}_o = \phi(t)\,\underline{x}_o$$

where $e^{At} = \phi(t) =$ the transition matrix.

The function e^{At} is calculated from Equation (4-11) as

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$$e^{At} = M\, e^{\Lambda t}\, M^{-1} = \phi(t) \tag{4-18}$$

where $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & e^{\lambda_3 t} & \\ 0 & & & e^{\lambda_4 t} \end{bmatrix}$

Expanding Equation (4.18) yields the transition matrix for the dual-spin vehicle

$$\phi(t) = \left[\begin{array}{cc|c} C\beta t & -S\beta t & \\ S\beta t & C\beta t & 0 \\ \hline \frac{1}{\beta}\, S\beta t & -\frac{1}{\beta}\,(1 - C\beta t) & \\ \frac{1}{\beta}\,(1 - C\beta t) & \frac{1}{\beta}\, S\beta t & I \end{array}\right] \tag{4-19}$$

Two Rotor-Fixed Jets

Substituting Equation (4-19) into Equation (4-14) and integrating yields the analytical expression for the controllability matrix for the two rotor-fixed jet case.

The controllability matrix is given by

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$$
C(t_o, t_1) = \left[\begin{array}{cc|cc}
t & 0 & \frac{1}{\beta^2}(C\beta t - 1) & -\frac{t}{\beta} + \frac{S\beta t}{\beta^2} \\
0 & t & \frac{t}{\beta} - \frac{S\beta t}{\beta^2} & \frac{1}{\beta^2}(C\beta t - 1) \\
\hline
\frac{1}{\beta^2}(C\beta t - 1) & \frac{t}{\beta} - \frac{S\beta t}{\beta^2} & \frac{2}{\beta}\left(\frac{t}{\beta} - \frac{S\beta t}{\beta^2}\right) & 0 \\
-\frac{t}{\beta} + \frac{S\beta t}{\beta^2} & \frac{1}{\beta^2}(C\beta t - 1) & 0 & \frac{2}{\beta}\left(\frac{t}{\beta} - \frac{S\beta t}{\beta^2}\right)
\end{array} \right]_{t = t_1} \tag{4-20}
$$

Evaluating the determinant according to the relation

$$
\left| \begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right| = |C_{11}| \left| C_{22} - C_{21} C_{11}^{-1} C_{12} \right|
$$

yields

$$
\left| C(t_o, t_1) \right| = \left[\frac{t}{\beta} \left(\frac{t}{\beta} - \frac{S\beta t}{\beta^2} \right) - \frac{2}{\beta^4} (1 - C\beta t) \right]^2_{t = t_1}
$$

It is clear that the determinant is greater than zero for all $t_1 > 0$; at $t_1 = 0$, the determinant is identically zero. Hence, the dual-spin system using two rotor-fixed jets is completely controllable at t_o. This result is in agreement with that obtained by using a digital computer in evaluating the determinant.

<u>One Rotor-Fixed Jet Case</u>

Repeating the above procedure for the one rotor-fixed jet case, it was determined that again the dual-spin system using one rotor-fixed jet is completely controllable at t_o. This result was also corroborated by that obtained by using a digital computer.

Controllability for Time-Invariant Systems

For time-invariant systems, the computational effort involved in determining whether the system is completely controllable is significantly reduced. In this work, time-invariant systems are of interest since

(1) if the jets are located on the "despun" body, the system is characterized by a time-invariant system of differential equations

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$$

(2) the symmetric spinning vehicle is characterized by a time-invariant plant.

Computational techniques for determining whether a time-invariant system is controllable are provided by the following theorems.

Theorem. The time-invariant system

$$(L) \quad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

with $\underline{x} \in R^n$, $\underline{u} \in R^m$ and having distinct eigenvalues is completely controllable iff there are no zero rows in the matrix

$$M^{-1} B,$$

where the rows of the matrix M^{-1} are simply the left generalized eigenvectors of the operator A or equivalently the right generalized eigenvectors of the adjoint operator A^*.

Theorem. The time-invariant system (L) $\dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$ is completely controllable iff the $n \times nm$ matrix

$$G = \left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots, \ldots, A^{n-1} B \right]$$

has rank n.

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Jets Located on "Despun" Body

For the dual-spin vehicle, with the jets located on the despun portion, the system is characterized by

$$(L) \qquad \dot{\underline{x}} = A\, \underline{x}(t) + B(t)\, \underline{u}$$

The spectral representation for the operator A has already been determined. The matrix B is given by

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{when two jets are used}$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{when one jet is used}$$

The matrix M^{-1} B for the two-jet case is given by

$$\left[\begin{array}{cc|cc} \frac{1}{-2\beta} & \frac{1}{2j\beta} & 0 & 0 \\ \frac{1}{2\beta} & \frac{1}{2j\beta} & 0 & 0 \\ \hline 0 & -\frac{1}{\beta} & 1 & 0 \\ \frac{1}{\beta} & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] = \left[\begin{array}{cc} \frac{1}{-2\beta} & \frac{1}{2j\beta} \\ \frac{1}{2\beta} & \frac{1}{2j\beta} \\ \hline 0 & -\frac{1}{\beta} \\ \frac{1}{\beta} & 0 \end{array}\right] \qquad (4\text{-}21)$$

Since there are no zero rows of M^{-1} B, the system is completely controllable.

For the one-jet case, the matrix M^{-1} B is given by

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$$
\left[\begin{array}{cc|c}
\frac{1}{-2\beta} & \frac{1}{2j\beta} & \\
 & & 0 \\
\frac{1}{2\beta} & \frac{1}{2j\beta} & \\
\hline
0 & -1/\beta & \\
 & & I \\
\frac{1}{\beta} & 0 &
\end{array}\right]
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix} \frac{1}{-2\beta} \\ \frac{1}{2\beta} \\ 0 \\ \frac{1}{\beta} \end{bmatrix}
\qquad (4\text{-}22)
$$

Since there is one zero row, the dual-spin vehicle using one jet on the despun portion is not controllable. In this problem, this conclusion can be confirmed by intuitive reasoning. Intuitively, it is felt that if the jet is rotor-fixed, only one is required, but if the jets are fixed to the despun body, two are required. The ideal location of the jets for this work has thus been determined. In the sequel, the type of jet to be used will be determined.

Controllability of a Symmetric Spinning Vehicle

The controllability for the time-invariant system

$$(L) \qquad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

representing a symmetric spinning vehicle is determined in exactly the same way as it was for the dual-spin vehicle with the jets mounted on the despun portion. The matrix A is given by

$$
A = \begin{bmatrix}
0 & \frac{I_1 - I_3}{I_1}\omega_3 & 0 & 0 \\
-\frac{I_1 - I_3}{I_1}\omega_3 & 0 & 0 & 0 \\
1 & 0 & 0 & \omega_3 \\
0 & 1 & -\omega_3 & 0
\end{bmatrix}
\qquad (4\text{-}23)
$$

Its eigenvalues are

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$$j\,\frac{I_1 - I_3}{I_1}\,\omega_3\,,\quad -j\,\frac{I_1 - I_3}{I_1}\,\omega_3,\ j\,\omega_3,\ -j\,\omega_3;$$

the modal matrix is

$$M = \begin{bmatrix} -j\,\omega_3\ \dfrac{I_3}{I_1} & j\,\omega_3\ \dfrac{I_3}{I_1} & 0 & 0 \\[2ex] \omega_3\ \dfrac{I_3}{I_1} & \omega_3\,\dfrac{I_3}{I_1} & 0 & 0 \\[2ex] j & -j & j & -j \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad (4\text{-}24)$$

the inverse of the modal matrix is

$$M^{-1} = \begin{bmatrix} \dfrac{j\,I_1}{2\,\omega_3 I_3} & \dfrac{I_1}{2\,I_3\,\omega_3} & 0 & 0 \\[2ex] \dfrac{-j}{2\,\omega_3}\ \dfrac{I_1}{I_3} & \dfrac{I_1}{2\,I_3\,\omega_3} & 0 & 0 \\[2ex] \dfrac{I_1}{2j\,\omega_3\,I_3} & -\dfrac{I_1}{2\,I_3\,\omega_3} & \dfrac{1}{2j} & \dfrac{1}{2} \\[2ex] -\dfrac{I_1}{2j\,\omega_3\,I_3} & -\dfrac{I_1}{2\,I_3\,\omega_3} & \dfrac{1}{-2j} & \dfrac{1}{2} \end{bmatrix} \qquad (4\text{-}25)$$

and the matrix $M^{-1}\,B$ is

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$$M^{-1}B = \frac{1}{2(1-r)\omega_3} \begin{bmatrix} j\,\epsilon^{-jr\omega_3 t} & e^{-jr\omega_3 t} \\ -j\,\epsilon^{jr\omega_3 t} & \epsilon^{jr\omega_3 t} \\ -j\,\epsilon^{-jr\omega_3 t} & \epsilon^{-jr\omega_3 t} \\ j\,\epsilon^{jr\omega_3 t} & -\epsilon^{jr\omega_3 t} \end{bmatrix} \tag{4-26}$$

where $r = \dfrac{I_1 - I_3}{I_1}$.

Hence, for the two-jet case, the system is completely controllable since there are no zero rows of the matrix $M^{-1}B$. If only one jet is used the matrix $M^{-1}B$ is simply the first column of that given in Equation (4-26), i.e.,

$$M^{-1}B = \frac{1}{2(1-r)\omega_3} \begin{pmatrix} j\,\epsilon^{-jr\omega_3 t} \\ -j\,\epsilon^{jr\omega_3 t} \\ -j\,\epsilon^{-jr\omega_3 t} \\ j\,\epsilon^{jr\omega_3 t} \end{pmatrix} \tag{4-27}$$

It is seen that a symmetric spinning vehicle using only <u>one</u> jet is completely controllable. This result is also in agreement with that obtained intuitively. Since the body-mounted jet rotates relative to an inertial frame, the requirement for both negative and positive torques is automatically met.

<u>Relation Between Controllability and Classical Vibration Theory</u>

It is interesting to note that the notion of controllability has an analogous counterpart in structural dynamics. In fact, it can be said that the notion originated in classical vibration theory. Consider the linear system

(L) $\underline{\dot{x}} = A\,\underline{x}(t) + B(t)\,\underline{u}(t)$ (4-28)

The spectral representation for the operator A was previously given as

$$A = \sum_i \lambda_i\, x_i > < y_i$$

$$x > = \sum_i < x,\ y_i > x_i = \sum_i \alpha_i\, x_i = \sum_i x_i\, n_i$$

The measure members α_i of the vector $\underline{x}$ relative to the basis defined by the eigenvectors are termed the normal coordinates. That is

$$x = \sum_i n_i\, x_i = (n_1, n_2, \ldots, n_n)$$

where $n_i \equiv < x,\ y_i >$

The relation between normal coordinates and the spectral theory of the operator A has thus been established. The result

$$x = \sum_i x_i\, n_i$$

is equivalent to

$$\underline{x} = M\,\underline{n} \tag{4-29}$$

In terms of the spectral representation of the operator A, the matrix A is given by

$$A = M\,\Lambda\,M^{-1} \tag{4-30}$$

for the distinct eigenvalue case.

Substituting Equations (4-29) and (4-30) into Equation (4-28) yields

$$\frac{d}{dt}(M\,\underline{n}) = [M\,\Lambda\,M^{-1}]\,M\,\underline{n} + B(t)\,\underline{u} \tag{4-31}$$

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Simplifying Equation (4-31) yields

$$\underline{\dot{n}} = \Lambda \underline{n} + M^{-1} B(t) \underline{u}(t) \tag{4-32}$$

The solution for the vector $\underline{n}$ is

$$\underline{n}(t) = \epsilon^{\Lambda t} \underline{n}(0) + \int_0^t e^{\Lambda(t-\tau)} M^{-1} B(\tau) \underline{u}(\tau) d\tau \tag{4-33}$$

If the j^{th} row of $M^{-1} B(t)$ is zero, the j^{th} coordinate n_j is unaffected by the input and the system is uncontrollable.

Knowing $\underline{n}$, the vector $\underline{x}$ becomes

$$\underline{x}(t) = M \underline{n}(t) = M e^{\Lambda t} M^{-1} \underline{x}(0) + \int_0^t M e^{\Lambda(t-\tau)} M^{-1} B(\tau) \underline{u}(\tau) d\tau \tag{4-34}$$

Equation (4-34) can be conveniently written as

$$\underline{x}(t) = \sum_i < y_i, \underline{x}(0) > e^{\lambda_i t} x_i + \int_0^t \sum_i < y_i, B \underline{u}(\tau) > e^{\lambda_i (t-\tau)} x_i d\tau \tag{4-35}$$

In Equation (4-35), the initial condition response is seen to be a weighted sum of the modes $e^{\lambda_i t} x_i$, that is

$$\sum_i < y_i, \underline{x}(0) > e^{\lambda_i t} x_i \quad ;$$

the forced response is

$$\int_0^t \sum_i < y_i, B \underline{u}(\tau) > e^{\lambda_i (t-\tau)} x_i d\tau .$$

The amount of excitation of the i^{th} mode due to the forcing function is

$$\int_0^t < y_i, B \underline{u}(\tau) > e^{\lambda_i (t-\tau)} x_i d\tau \tag{4-36}$$

The vector $B \underline{u}(\tau)$ can be rewritten as

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$$B\,\underline{u}(\tau) = \sum_j \underline{u}_j\, \underline{b}_j' \tag{4-37}$$

where $\underline{b}_j$ are the columns of B.

Substituting Equation (4-37) into Equation (4-36) yields the amount of excitation of the i^{th} mode due to the forcing function and is given by

$$\int_0^t \sum_j < y_i, \underline{b}_j > \underline{u}_j(\tau)\, e^{\lambda_i (t-\tau)} x_i \, d\tau$$

If the scalar product $< y_i, b_j >$ is zero for the i^{th} mode for all j, then the input is not coupled to that mode and cannot excite or control that mode. The criterion for controllability (for the case of distinct eigenvalues) is that the scalar products $< y_i, \underline{b}_j >$ do not vanish for all j. In vibration theory, the scalar product $< y_i, \underline{b}_j >$ is analogous to the participation factor. Hence, for time-invariant systems having distinct eigenvalues, the notion of controllability is essentially a generalization of the participation factor of classical vibration theory.

4.1.2 Normality

In this section, the notion of normality and its relationship to controllability are discussed.

In the next section the connection between normality and the existence of optimal solutions is discussed. The term normality has several connotations and is used differently by various authors. In this work, the notions of

(1) normal systems

(2) normal problems

(3) normality conditions

are discussed. The definitions of these terms and the definition of the set of attainability are given below.

A linear time-invariant system

$$(L) \quad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

with $\underline{x} \in R^n$, $\underline{u} \in R^m$ is said to be normal if each of the systems

$$\dot{\underline{x}} = A\,\underline{x}(t) + \underline{b}_1\,u_1$$
$$\dot{\underline{x}} = A\,\underline{x}(t) + \underline{b}_2\,u_2$$
$$\vdots$$
$$\dot{\underline{x}} = A\,\underline{x}(t) + \underline{b}_m\,u_m$$

is completely controllable. The vectors $\underline{b}_j$ are the columns of the matrix B. If a system is normal then it is controllable with respect to each component of the control and, hence, it is completely controllable. The normality condition is defined as follows. Consider the time-invariant system

$$(L) \quad \dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} + \underline{v}$$

with convex polyhedral restraint set $\Omega \subset R^m$ and initial state $\underline{x}_o \in R^n$. Let a nonzero vector along an edge of Ω be designated as $\underline{w}$. The normality condition is that the vectors

$$B\,\underline{w},\ A\,B\,\underline{w}, \ldots, A^{n-1}\,B\,\underline{w}$$

must be linearly independent for each nonzero vector $\underline{w}$.

Problem normality is defined as follows. Consider the linear control process

$$(L) \quad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) + \underline{v}(t)$$

with restraint set Ω and initial state $\underline{x}_o$ at time t_o. The problem $(L, \Omega, \underline{x}_o, t_o, t_1)$ is said to be normal in case any two controllers

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$u_1(t)$ and $u_2(t)$ on $t_o \leqq t \leqq t_1$ which steer $\underline{x}_o$ to the same boundary point P_1 belonging to the set of attainability at time t_1 must be equal almost everywhere. The set of attainability is defined as follows.

Consider the linear control process

$$(L) \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) + \underline{v}(t)$$

with restraint set Ω, initial state $\underline{x}_o$, and controllers $\underline{u}(t) \subset \Omega$ on $[t_o, t_1]$. The set of attainability $K(L, \Omega, \underline{x}_o, t_o, t_1)$ is the set of all endpoints $\underline{x}(t_1)$ in R^n. For notational convenience the set of attainability is written as $K(t_1)$.

It should be noted that system normality does not imply problem normality. This is as expected since in determining system normality the control restraint set Ω and the set of attainability are not considered.

Having defined the notion of normality, some theorems in which it is used can now be stated.

Theorem [24]. Consider the linear control process in R^n

$$(L) \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) + \underline{v}(t)$$

with compact restraint set Ω and initial state $\underline{x}_o$ at time t_o. The control problem $(L, \Omega, \underline{x}_o, t_o, t_1)$ is normal iff the following uniqueness property holds: for each nontrivial solution of

$$\dot{\underline{p}}(t) = -A^T(t)\,\underline{p}(t)$$

and for any two controllers $\underline{u}_1(t)$ and $\underline{u}_2(t) \subset \Omega$ satisfying

$$< \underline{p}(t), B(t)\,\underline{u}_1(t) > = < \underline{p}(t), B(t)\,\underline{u}_2(t) > = \sup_{u \in \Omega} < \underline{p}(t), B(t)\,\underline{u} >$$

almost everywhere, the extremal controllers $\underline{u}_1(t)$ and $\underline{u}_2(t)$ are the same; that is,

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$$\underline{u}_1(t) = \underline{u}_2(t) \text{ a.e. on } t \in [t_0, t_1].$$

In addition, if the problem is normal, and if Ω contains more than one point, the set of attainability is strictly convex; hence, $K(t_1)$ is a compact convex set with nonempty interior.

The above theorem shows the intimate connection between normality and uniqueness and also the relationship between normality and the set of attainability.

Although quadratic cost functionals are not of concern in this work, it is interesting to note that the normality conditions which guarantee the uniqueness of extremal controllers steering $(0, \underline{x}_0)$ to the boundary point of $K(t_1)$ are automatically satisfied for linear control processes with integral quadratic cost criteria.

System Normality for Problems of Interest in this Work

For a time-varying system there are no useful computational techniques for determining whether the linear system is normal. For a linear time-invariant system however, the normality condition can be used. Hence, for the dual-spin system with rotor-fixed jets, there is no way to determine a priori whether the system is normal. The normality condition can be applied to the dual-spin system with the jets on the despun portion and to the symmetric spinning vehicle.

Dual-Spin System with Jets Located on Despun Body

The matrices A and B for the dual-spin system have already been given; for the case in which the jets are located on the despun body they are

$$A = \begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The normality condition implies that the rank of each of the following matrices must be four in order for the system to be normal:

$$G_1 = \left[\underline{b}_1 \;\vdots\; A\,\underline{b}_1 \;\vdots\; A^2\,\underline{b}_1 \;\vdots\; A^3\,\underline{b}_1 \right]$$

$$G_2 = \left[\underline{b}_2 \;\vdots\; A\,\underline{b}_2 \;\vdots\; A^2\,\underline{b}_2 \;\vdots\; A^3\,\underline{b}_2 \right]$$

where $\underline{b}_1$ and $\underline{b}_2$ are the columns of B

The matrices G_1 and G_2 are given by

$$G_1 = \begin{bmatrix} 1 & 0 & -\beta^2 & 0 \\ 0 & \beta & 0 & -\beta^3 \\ 0 & 1 & 0 & -\beta^2 \\ 0 & 0 & \beta & 0 \end{bmatrix} \; ; \; G_2 = \begin{bmatrix} 0 & -\beta & 0 & \beta^3 \\ 1 & 0 & -\beta^2 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 1 & 0 & -\beta^2 \end{bmatrix} .$$

The rank of both G_1 and G_2 is three. Hence the dual-spin vehicle using jets located on the despun body is not a normal system.

Symmetric Spinning Vehicle

For the symmetric spinning vehicle the matrices A and B are

$$A = \begin{bmatrix} 0 & r\omega_3 & 0 & 0 \\ -r\omega_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & \omega_3 \\ 0 & 1 & -\omega_3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the matrices G_1 and G_2 are

$$G_1 = \begin{bmatrix} 0 & -(r\omega_3)^2 & 0 & (r\omega_3)^4 \\ -r\omega_3 & 0 & (r\omega_3)^3 & 0 \\ 1 & 0 & -\omega_3^2[1+r+r^2] & 0 \\ 0 & -\omega_3(1+r) & 0 & \omega_3^3[1+r+r^2+r^3] \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & r\omega_3 & 0 & -(r\omega_3)^3 \\ 1 & 0 & -(r\omega_3)^2 & 0 \\ 0 & 0 & \omega_3(1+r) & 0 \\ 0 & 1 & 0 & -\omega_3^2(1+r+r^2) \end{bmatrix}$$

Since the rank of both G_1 and G_2 is four, the system characterizing the symmetric spinning vehicle is normal.

Problem Normality

A sufficient condition for a restricted class of fuel-optimal problems to be normal is now stated.

Theorem[20]. Consider the control problem (L, Ω, X_0, X_1, J) with the system

$$(L) \qquad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

$$\underline{x} \in R^n, \; \underline{u} \in R^m ,$$

with the control restraint set Ω

$$\Omega = \left\{ \underline{u} : |u_j(t)| \le 1 \;\; \forall j \right\},$$

with the initial state $\underline{x}_o$ at time t_o, with the target set consisting of a fixed endpoint $\underline{x}_1$ and fixed final time T, and with the cost functional

$$J(\underline{u}) = \int_{t_o}^{T} \sum_j |u_j(t)|\,dt \; .$$

A sufficient condition for this fuel-optimal problem to be normal is that

$$\det\left(G_j^T A^T\right) \neq 0 \qquad j = 1, 2, \ldots, m$$

On the other hand, for this problem to be singular it is necessary that

$$\det\left(G_j^T A^T\right) = 0 \qquad \text{for some } j$$

Thus, if this system is normal and if the matrix A is nonsingular then the problem is normal.

Using this theorem, it is seen that the spin-axis control concept applied to the symmetric spinning vehicle results in a normal fuel-optimal problem for the fixed-time case. Note, however, that this statement does not hold for the free-time problem. The theorem also indicates that if the spin-axis control concept were used for the

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dual-spin vehicle with the jets on the despun portion, then the fuel-optimal fixed-time problem would necessarily be singular. For this system, the fact that

$$|G_1| = 0$$

implies that the system is not normal and the fact that

$$|A| = 0$$

implies that there is at least one stage of integration. Actually there are two stages of integration since the eigenvalue $\lambda = 0$ has a multiplicity of two.

A useful geometric property of a more general fuel-optimal normal problem is given below. This definition is appropriate when the angular momentum control (AMCO) concept is used. Consider the problem $\{S, \Omega, X_o, X_1, J\}$ with the system

$$(S) \quad \dot{\underline{x}} = \underline{f}\left(\underline{x}(t),\ t\right) + B\left(\underline{x}(t),\ t\right)\underline{u}(t)\ ,$$

$$\underline{x} \in R^n\ ,\ \underline{u} \in R^m$$

with the control restraint set Ω

$$\Omega = \{\underline{u} : |u_j(t)| \le 1,\ j = 1, 2, \ldots, m\}$$

with the smooth target set

$$X_1 = \{(\underline{x},\ t) : g_i[\underline{x}, t] = 0 \quad , \quad i = 1, 2, \ldots, n-k\}\ ,$$

with the initial set

$$X_o = \left\{ (\underline{x}, t) : \begin{array}{ll} \underline{x}(t_o) = x_o & \text{fixed} \\ t_o \ \text{fixed} & \end{array} \right\}$$

and cost functional

$$J(u) = \int_{t_o}^{t_1} \sum_j |u_j(t)| dt \quad .$$

Suppose that in the interval $[t_o, t_1]$ for the free time case or in the interval $[t_o, T]$ for the fixed-time case, there is a countable set of times $\tau_{1j}, \tau_{2j} \cdots$ (switch times) such that

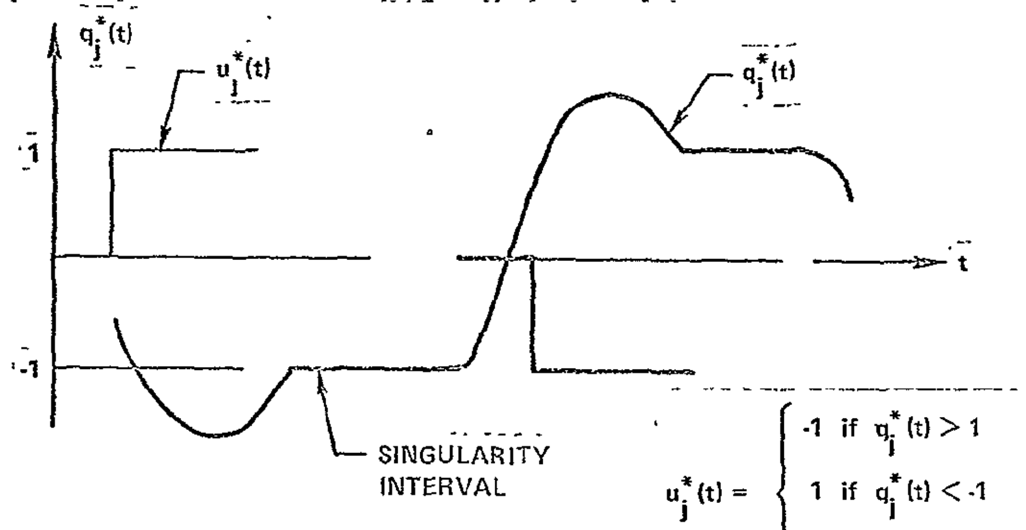
$$| q_j^*(t) | = | < \underline{b}_j \left(\underline{x}(t), t\right) , \underline{p}^*(t) > | = 1$$

iff $t = \tau_{\gamma j}$ $\forall j = 1, 2, \ldots, m$, then the fuel-optimal problem is normal. In this definition, the vectors $\underline{b}_j$ are the columns of the matrix B, and the vector $\underline{p}^*(t)$ is the adjoint (costate) vector.

A similar definition can be stated for a singular fuel-optimal problem. For the problem described above, suppose that in the interval $[t_o, t_1^*]$ or in the interval $[t_o, T]$ there are one or more subintervals $[T_1, T_2]$ such that

$$|q_j^*(t)| = | < \underline{b}_j \left(\underline{x}(t), t\right) , \underline{p}^*(t) > | = 1 \quad \forall t \in [T_1, T_2] \, .$$

Then, the problem is said to be singular, and the intervals $[T_1, T_2]$ are singularity intervals (see Figure 4-1).



$$u_j^*(t) = \begin{cases} -1 & \text{if } q_j^*(t) > 1 \\ 1 & \text{if } q_j^*(t) < -1 \\ 0 & \text{if } |q_j^*(t)| < 1 \end{cases}$$

Figure 4-1 Illustration of Singular Condition for Controller

4.2 Existence and Uniqueness of Optimal Solutions

The topics dealing with the existence and uniqueness of optimal solutions are of great importance and can, of course, have a tremendous effect on the computational aspect of the problem. In fact, the study of the existence of optimal solutions often leads to new and better computational algorithms. Because of the mathematical complexity of these topics and because of the limited scope of this dissertation, only those aspects that appear to have a direct effect on the computational procedure for obtaining the fuel-optimal controller for the dual-spin and spinning vehicles are discussed. At the same time, however, those notions which are fundamental in the proof of a general existence proof are noted. Included among these are the notions of the compactness of the set of attainability and uniform bounds on the response.

In practice, some of the conditions required to guarantee the existence and uniqueness of an optimal controller are not always satisfied. Nevertheless, the necessary conditions obtained from either the maximum principle or from the calculus of variations are used to find the extremal controllers in the so-called indirect method for finding the optimal solution. The indirect methods and the direct methods are discussed in Chapter 6.

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An extremal controller is one which satisfies the necessary conditions for optimality. If it can be shown that an optimal control exists and in addition that the extremal control is unique then the unique extremal controller is the unique optimal control. For the general nonlinear system, however, it is difficult to prove that an optimal solution exists and even more difficult to prove that the extremal control is unique. The procedure for finding the optimal solution (if it exists) for this case entails an examination of the cost functional associated with the computationally-determined extremal controllers. The extremal controller which results in a minimum value of the cost functional is then considered the optimal solution. A global search is required, perforce, to find all the extremal controllers; this ensures that the solution termed "optimal" is indeed optimal and not merely "locally-optimal."

4.2.1 Existence of Optimal Controllers

Existence theory for systems represented by ordinary differential equations has been extensively studied (Reference [25] through [35]). The theorems presented in this section are based primarily on [24], [34], and [35]. The proofs of the theorems stated herein can be found in the cited references. In this work, emphasis is placed on the determination of the applicability of the available existence theorems for the fuel-optimal control of dual-spin and spinning vehicles. First, a general theorem for linear systems is stated. Next the basic existence theorems for nonlinear systems are given.

Linear Systems

In this section, some existence theorems which are applicable for linear systems with general integral cost criteria are provided. For linear systems the notions which are fundamental for the existence of optimal solutions include

(1) convexity of the integrand of the cost functional

(2) problem normality (compactness of the set of attainability)

A theorem concerning the compactness of the set of attainability is now stated [24].

Theorem. Consider the linear system

$$(L) \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

with compact convex restraint set Ω, initial state $\underline{x}_o$ at time t_o, and controllers $\underline{u}(t)$ on $t \in [t_o, t_1]$. Then the set of attainability $K(t_1)$ is compact and convex and varies continuously with t_1 for $t_1 \geq t_o$.

The following theorem provides the hypotheses necessary for the existence of optimal controllers for linear systems with general integral cost criteria.

Theorem. Consider the system

$$(L) \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

with the integral cost functional

$$J(\underline{u}) = \psi\left(\underline{x}(T)\right) + \int_{t_o}^{T}\left[f_o(\underline{x}, t) + h_o(t, \underline{u})\right] dt$$

Assume that $A(t)$, $B(t)$ are real continuous matrices on the fixed finite interval $[t_o, T]$, that $\psi(\underline{x})$, $f_o(t, t)$, $h_o(t, \underline{u})$ are continuous for all values of their arguments for $\underline{x} \in R^n$ and $\underline{u} \in R^m$, that $f_o(t, \underline{x})$ and $h_o(t, \underline{u})$ are convex functions for each fixed value of $t \in [t_o, T]$, that the controller $\underline{u}(t)$ on $t \in [t_o, T]$ belongs to a compact convex restraint set $\Omega \subset R^m$, and that the problem $\{L, \Omega, X_o, X_1, J\}$ is normal. Then there exists an optimal controller.

It is noted that there is no mention of the compactness of the target set in this theorem. However, the assumption of problem normality

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is sufficient, in this case, to guarantee that the set of attainability $K(T)$ in R^n is a strictly convex compact set with nonempty interior. It is also noted that this theorem applies to fixed-time problems. Hence, the theorem, cannot be applied to time-optimal problems. Existence theorems for time-optimal problems are more prevalent than for fuel-optimal problems and are not discussed in this work. It is not to be assumed that existence theorems can be proven only for the fixed final time case. In the general existence theorems for nonlinear systems, the final time is allowed to be free.

Applicability of the Existence Theorem to the Fuel-Optimal Control of Dual-Spin and Spinning Vehicles

First, it is noted that for the fuel-optimal problems of concern in this work both free and fixed final time are being considered. The existence theorem applies only to the fixed -time case. This fact illustrates that linear fixed-time problems are more attractive from a theoretical point of view than free-time problems. This point will be re-enforced when the question of uniqueness is discussed. Of course, a free-time problem can be treated computationally as several fixed time problems. The particular fixed-time problem which minimizes the cost functional is considered the optimal solution to the free-time problem.

The hypotheses concerning $A(t)$, $B(t)$, $f_o(t, x)$ and $h_o(t, u)$ are satisfied for the fuel-optimal problems being considered. In particular, the cost functionals being considered are

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j |u_j(t)| dt$$

$$J(\underline{u}) = \int_{t_o}^{t_1} \|u\| dt$$

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j u_j(t) dt \quad ;$$

The integrands of each of these cost functionals are convex. The hypothesis that the control restraint set be compact and convex is also satisfied; the sets Ω of interest in this work include

$$\Omega = \{\underline{u} : | u_j(t) | \leq 1 \quad \forall j\}$$

$$\Omega = \{\underline{u} : 0 \leq u_j(t) \leq 1 \ \forall j\}$$

$$\Omega = \{\underline{u} : \|u\| \leq m\}$$

The crucial hypothesis is that concerning the normality of the problem. Problem normality was considered in Section 4-1. The conclusions drawn considering normality are repeated here for convenience:

System and Control Concept	Problem Normality
Spin axis control of symmetric spinning body using either one or two jets	Yes
Spin axis control of dual-spin vehicle with jets located on despun-portion	No
Angular momentum control of symmetric spinning vehicle	Depends on nature of the switching function

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Angular momentum control of dual-spin vehicle	Depends on nature of the switching function

Using this theorem the following statements can be made

(1) the existence of the fuel-optimal controller for the SACO concept applied to a symmetric spinning vehicle in which the final time is fixed is assured

(2) no conclusions concerning the existence of the fuel-optimal controller for the AMCO concept can be drawn until the nature of the switching function is investigated

(3) the theorem does not assure the existence of the fuel-optimal controller when the SACO concept is used for the dual-spin vehicle with the jets located on the despun body

Nonlinear Systems

In this section, the basic existence theorems for nonlinear systems are stated. It is noted that certain notions are fundamental in any general existence proof. For linear systems the notion of problem normality in conjunction with a compact convex restraint set resulted in a compact set of attainability $K(t_1)$. For nonlinear systems the notion of a uniform bound in conjunction with a compact convex restraint set Ω results in a compact $\overline{K(t_1)}$.† The definition of a uniform bound, the statement of a theorem relating a uniform bound and $\overline{K(t_1)}$, and two existence theorems (one due to Markus and Lee [24], the other due to Neustadt [34] are provided below. Consider a nonlinear process

$$(S) \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \qquad \text{in } R^n$$

where $\underline{f}$ is in C^1 in R^{n+m+1} and where the admissible controllers

†$\overline{K(t_1)}$ refers to the closure of the set of attainability $K(t_1)$.

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$\underline{u}(t)$ defined on $[t_o, T]$ constitute a certain family of measurable m-vector functions. Assume the initial point $\underline{x}_o$ lies in a given compact initial set X_o in R^n and that the respone $\underline{x}(t; \underline{x}_o, t_o) = \underline{x}(t)$ for $\underline{u}(t) \in \mathfrak{F}$ exists on $[t_o, T]$. Suppose for each $\underline{u}(t) \in \mathfrak{F}$ there is a bound

$$|x_i(t)| < b$$

and

$$\left|f_i\left(\underline{x},\ t,\ \underline{u}(t)\right) + \left|\frac{\partial f_i}{\partial x_k}\left(\underline{x},\ t,\ \underline{u}(t)\right)\right| \leq m(t)\right.$$

for $i, k = 1, 2, \ldots, n$ with $\int_{t_o}^{T} m(t)dt < \infty$, then $\underline{u}(t)$ admits a bound for the response. If, in addition, the bound b and the integrable function $m(t)$ can be chosen independently of the controller $\underline{u}(t) \in \mathfrak{F}$, then the problem $\{S, \mathfrak{F}, X_o, X_1\}$ has a uniform bound.

A theorem relating the notion of a uniform bound and the compactness of the closure of the set of attainability is stated below.

Theorem. Consider the nonlinear process

$$(S) \qquad \dot{\underline{x}} = \underline{f}(\underline{x},\ \underline{u},\ t) \text{ in } C^1 \text{ in } R^{n+m+1}$$

with initial state $\underline{x}_o$ at time t_o and admissible control family $\mathfrak{F}$ on $[t_o, T]$. Assume the process $\{S, \mathfrak{F}, X_o, X_1\}$ has a uniform bound. Then $\overline{K(t_1)}$ is a compact, continuously varying set in R^n for $t \in [t_o, T]$. A general existence theorem (sometimes called the basic existence theorem) for nonlinear systems is stated below. The theorem applies to minimax problems as well as to problems with inequality constraints on the state.

Theorem. Consider the nonlinear process in R^n

$$(S) \qquad \dot{\underline{x}} = \underline{f}(\underline{x},\ \underline{u},\ t) \text{ in } C^1 \text{ in } R^{n+m+1}$$

The problem is as follows:

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(1) the initial and target sets $X_o(t)$ and $X_1(t)$ are nonempty compact sets varying continuously in R^n for all t in the basic prescribed compact interval $t \in [\tau_o, \tau_1]$

(2) the control restraint set $\Omega(\underline{x}, t)$ is a nonempty compact set varying continuously in R^m for $(\underline{x}, t) \in R^n \times [\tau_o, \tau_1]$

(3) the state constraints are (possibly vacuous) $h_1(x) \geq 0, \ldots h_r(x) \geq 0$, a finite or infinite family of constraints, where $h_1, \ldots, h_r$ are real continuous functions on R^n

(4) the family $\mathfrak{F}$ of admissible controllers consists of all measurable functions $\underline{u}(t)$ on various time intervals $t \in [t_o, t_1]$ in $[\tau_o, \tau_1]$ such that each $\underline{u}(t)$ has a response $\underline{x}(t)$ on $t \in [t_o, t_1]$ steering $\underline{x}(t_o) \in X_o(t_o)$ to $\underline{x}(t_1) \in X_1(t_1)$ and $\underline{u}(t) \in \Omega(\underline{x}, t)$ and $h_j\left(\underline{x}(t)\right) \geq 0 \quad j = 1, \ldots, r$

(5) the cost for each $\underline{u} \in \mathfrak{F}$ is

$$J(\underline{u}) = \psi\left(\underline{x}(t_1)\right) + \int_{t_o}^{t_1} f_o\left(\underline{x}(t), \underline{u}(t), t\right) dt + \max_{t \in [t_o, t_1]} \gamma\left(\underline{x}(t)\right)$$

where $f_o \in C^1$ in R^{n+m+1}, and $\psi(\underline{x})$ and $\gamma(\underline{x})$ are continuous in R^n.

Assume

a) the family $\mathfrak{F}$ of admissible controllers is not empty

b) there exists a uniform bound

$$|x(t)| \leq b \quad \text{on} \quad t \in [t_o, t_1]$$

for all responses $\underline{x}(t)$ to controllers $\underline{u} \in \mathfrak{F}$

c) the extended velocity set

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$$\hat{V}(x,t) = \{f_o(x,\underline{u},t),\ \underline{f}(\underline{x},\underline{u},t) \mid \underline{u} \in \Omega(\underline{x},t)\}$$

is convex in R^{n+1} for each fixed $(\underline{x},t)$. Then, there exists an optimal controller $\underline{u}^*(t)$ on $t_o^* \le t \le t_1^*$ in $\mathfrak{F}$ minimizing $J(\underline{u})$.

A useful corollary to this theorem which applies to the case in which the control enters linearly in both the plant and the integrand of the cost functional is provided below.

Corollary. Consider the process

$$\text{(S)} \qquad \dot{\underline{x}} = A(x,t) + B(x,t)\,\underline{u}$$

in R^n with cost

$$J(u) = \psi\left(\underline{x}(t_1)\right) + \int_{t_o}^{t_1} A_o\left(\underline{x}(t),t\right) + B_o(\underline{x},t)\,\underline{u}(t)dt$$

$$+ \operatorname*{ess\,sup}^{\dagger}_{t\in[t_o,t_1]} \gamma\left(\underline{x}(t),\ \underline{u}(t)\right)$$

where the matrices A, B, A_o, B_o are in C^1 in R^{n+1}, $\psi(\underline{x})$ and $\gamma(\underline{x},\ \underline{u})$ are continuous in R^{n+m}, and $\gamma(\underline{x},\underline{u})$ is a convex function of $\underline{u}$ for each fixed $\underline{x}$. Assume that the restraint set $\Omega(\underline{x},t)$ is compact and convex for all $(\underline{x},t)$. Then, hypothesis (c) of the preceding theorem is satisfied. If the problem is defined by (1) through (4) of the preceding theorem and if hypotheses (a) and (b) are assumed, then an optimal control $\underline{u}^*(t)$ on $t \in [t_o^*,\ t_1^*]$ exists.

This corollary is appropriate for determining the existence of the fuel-optimal controller for the dual-span vehicle in the large-angle turn mode. Previously, it was shown that the plant for this case is

$$\text{(S)} \qquad \dot{\underline{x}} = A(\underline{x},t) + B(t)\,\underline{u}(t)$$

Note also that when one one-way rotor fixed jet is used the integrand of the cost functional is

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†The term ess sup refers to the essential supremum.

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$$A_o\left(\underline{x}(t),\ t\right) + B_o\left(\underline{x}(t),\ t\right)\underline{u}(t) = u(t) \in R^1$$

and the control restraint set $\Omega \subset R^1$ is

$$\Omega = \left\{u(t) :\ 0 \leq u(t) \leq 1\right\}$$

The final existence theorem due to Neustadt is interesting in that no convexity hypotheses are required. It applies to the restricted class of problems in which both the plant and the integrand of the cost function are linear in $\underline{x}$ and nonlinear in $\underline{u}$.

Theorem. Consider the process in R^n

$$\text{(S)} \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + B(t, \underline{u})$$

where $A(t)$ and $B(t, \underline{u})$ are continuous in R^{1+m}. The problem is as follows:

1. The initial and target sets $X_o(t)$ and $X_1(t)$ are nonempty compact sets varying continuously in R^n for all t in the basic prescribed compact interval $t \in [\tau_o, \tau_1]$.

2. The control restraint set $\Omega(t)$ is a nonempty compact set varying continuously in R^m for $t \in [\tau_o, \tau_1]$.

3. The integral constraints (possibly vacuous)

$$\int_{t_o}^{t_1} h_j\left(t,\ \underline{u}(t)\right) dt \geq 0 \qquad \text{for } j = 1, 2, \ldots, r$$

   where h_j are real continuous functions in R^{1+m}

4. The family $\mathcal{F}$ of admissible controllers consists of all measurable functions $\underline{u}(t)$ on various time intervals $t_o \leq t \leq t_1$ in $[\tau_o, \tau_1]$ such that each $\underline{u}(t)$ has a response $\underline{x}(t)$ on $t_o \leq t \leq t_1$ steering $\underline{x}(t_o) \in X_o(t_o)$ to $\underline{x}(t_1) \in X(t_1)$, such that the restraint $\underline{u}(t) \subset \Omega(t)$ in $[t_o, t_1]$, and such that the integral constraints are satisfied.

5. The cost of each $\underline{u} \in \mathfrak{F}$ is

$$J(\underline{u}) = \psi\left(\underline{x}(t_1)\right) + \int_{t_0}^{t_1} A_0(t)\,\underline{x}(t) + B_0\left(\underline{t}, \underline{u}(t)\right) dt$$

where $\psi(\underline{x})$, $A_0(t)$, $B_0(t, u)$ are continuous in all $(\underline{x}, \underline{u}, t)$.

Assume that the set $\mathfrak{F}$ of admissible controllers is not empty. Then there exists an optimal controller $\underline{u}^*(t)$ on $[t_0^*, t_1^*]$ in $\mathfrak{F}$ which minimizes $J(\underline{u})$.

4.2.2 Uniqueness of Optimal Controllers

In this section, the question dealing with the uniqueness of the optimal controller for linear systems is examined (for a more detailed discussion of this topic see Reference [36] through [38]). The hypotheses guaranteeing the uniqueness of an optimal controller are, as expected, more stringent than for existence and the classes of problems for which uniqueness can be demonstrated are more restricted.

The lack of uniqueness of the extremal controllers is naturally undesirable because of the increased computational effort involved in obtaining the unique optimal controller. Note, however, that the nonuniqueness of the extremal controllers does not imply the nonuniqueness of the optimal controller, but the nonuniqueness of the optimal controller does imply the nonuniqueness of the extremal controllers. In addition, the uniqueness of the optimal controller does not imply the uniqueness of the extremal controller.

It is possible that in certain instances, nonuniqueness of the optimal controls is not necessarily a curse. In Reference [20] it is pointed out that if the nonunique optimal controllers are examined carefully, it may be possible to find one that has definite practical advantages.

Just as in the case of existence, the uniqueness theorems provided in this work pertain to problems having a general integral cost and to fuel-optimal problems in particular. Similar theorems apply to the time-optimal problem but are not included in this work. The following theorem describes a problem for which the extremal controllers are unique [24].

Theorem. Consider the linear process

$$\text{(S)} \qquad \dot{\underline{x}}(t) = A(t)\, \underline{x}(t) + B(t)\, \underline{u}(t)$$

with the integral cost functional

$$J(\underline{u}) = \psi\left(\underline{x}(T)\right) + \int_{t_o}^{T} \left[f_o(t, \underline{x}) + h_o(t, \underline{u}) \right] dt$$

Assume that $A(t)$, $B(t)$ are real continuous matrices on $[t_o, T]$, that $\psi(\underline{x})$, $f_o(t, \underline{x})$, and $h_o(t, \underline{u})$ are continuous for all values of their arguments for $\underline{x} \in R^n$ and $\underline{u} \in R^m$, that $f_o(t, \underline{x})$ is convex for each fixed value of $t \in [t_o, T]$, that $h_o(t, \underline{u})$ is strictly convex for each t, and that the restraint set Ω is compact and convex in R^m. Assume that the problem $\{L, \Omega, x_o, t_o, T\}$ is normal. Then any two extremal controllers steering $(0, \underline{x}_o)$ to the same boundary point of $\hat{K}$† must coincide almost everywhere. Moreover, a unique optimal controller exists.

Note that this theorem applies only to fixed final time problems. The normality hypothesis was previously discussed in relation to existence. The crucial hypothesis in this theorem is that requiring $h_o(t, \underline{u})$ to be strictly convex. The integrand of the cost

† In this theorem, $\hat{K}$ refers to the totality of all response endpoints

$$\left(x_o(T), \underline{x}(T) \right),$$

that is, the set of attainability for the augmented response.

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functional which is appropriate for fuel-optimal problems is convex but is not strictly convex; hence, the theorem cannot be applied to the problems of interest in this work. However, if the linear process is time-invariant, the following theorem is applicable [37].

Theorem. Consider the linear process

$$\text{(L)} \qquad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t) \;, \quad \underline{x} \in R^n \;, \quad \underline{u} \in R^m$$

with cost functional

$$J(\underline{u}) = \int_{t_o}^{T} \sum_j |u_j(t)|\,dt \;,$$

with compact convex restraint set $\Omega \subset R^m$

$$\Omega = \{\underline{u} : |u_j(t)| \leq 1 \quad \forall j = 1, 2, \ldots, m\},$$

with initial state $\underline{x}_o$ at t_o, and with the final state $\underline{x}_1$ corresponding to the final time $t = T$. Assume that A and B are nxn and nxm constant matrices, respectively, and that the problem $\{L, \Omega, X_o, X_1, J\}$ is normal. Then any two extremal controllers steering $(t_o, \underline{x}_o)$ to $(T, \underline{x}_1)$ must be same for all $t \in [0, T]$.

This theorem indicates that if the plant is linear and time-invariant and if the fixed final time fuel-optimal problem is normal, then the extremal controllers are unique. The relationship between uniqueness of the extremal controllers and the reduced computational effort involved in determining the optimal controller has already been mentioned. Of the problems of concern in this work, this theorem applies only to the case in which

(1) the spin axis control concept is used for the symmetric spinning vehicle

(2) the final time is fixed

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It follows that if the final time is free and all other conditions are satisfied, then the theorem is still applicable if the free-time problem is treated as several fixed-time problems.

Section 5

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

In this chapter, the necessary and sufficient conditions for optimality are provided. Although necessary conditions for general control problems are known both from the calculus of variations and from the maximum principle, nevertheless, the necessary conditions for optimality are developed by using the calculus of variations for the problems of interest in this work. This tends to make this treatment concerning the fuel-optimal control of dual-spin and spinning vehicles self-contained. The necessary conditions obtained from the maximum principle are stated, compared to those obtained from the calculus of variations approach, and applied to the problems of interest in this work.

General sufficiency conditions obtainable from each of the major approaches to the optimal control problem are provided. The applicability of these theorems for the problems of interest in this work is discussed.

5.1 Necessary Conditions for Optimality

In this section, the necessary conditions for optimality are provided. These conditions are obtained both from the calculus of variations approach and from Pontryagin's maximum principle. The choice of one approach over the other, is a matter of personal taste. Each approach has some distinct advantages.

5.1.1 Calculus of Variations Approach

First, the necessary conditions for a weak extremal are obtained. Then these conditions are strengthened by finding an additional necessary condition for a strong extremal. It will be seen that the necessary conditions for a strong extremal result in a version of the maximal principle (see Hestenes [39]). The set of necessary conditions obtained in this section are for local optimality.

Necessary Conditions for a Weak Extremal

The necessary conditions for a weak extremal are obtained by using some fundamental notions of analysis; included among these are

(1) the notion of a derivative

(2) the notion of an extremum

(3) a Taylor series expansion (TSE) of a vector function

An elegant definition of a derivative due to Caratheodory is as follows [18]. Consider a function f

$$f : V \to C$$

where V is a neighborhood of $b \in R$ and C is the complex plane. The function f is said to possess a derivative at b provided that there exists

$$g : V \to C$$

continuous at b such that

$$f(x) - f(b) = g(x)(x-b) \quad , \qquad x \in V .$$

A similar extremely useful definition of the derivative of a function f is given below [39]. A function f is said to be differentiable at a point x_o if it is defined on a neighborhood of x_o and there is a linear function $f'(x_o, h)$ such that

$$\lim_{h \to 0} \frac{f(x_o + h) - f(x_o) - f'(x_o, h)}{|h|} = 0 \tag{5-1}$$

$$\text{where } f'(x_o, h) = \left[\frac{\partial f(x_o)}{\partial x}\right]$$

A generalization of this definition which is extremely useful when the computational aspects of the optimal control problem are of concern, is now given [40]. Let X, Y be Banach spaces. Let

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T be an operator with a domain D in X and a range in Y. The operator T is called Fréchet-differentiable at the point $f \in D$ if there exists a bounded linear operator T' with a domain in D and a range in Y, and a real function $\epsilon(\|h\|)$ with the property

$$\|T(f+h) - T(f) - T'_{(f)} h\| \le \epsilon(\|h\|)\ \|h\| \qquad \text{for } \|h\| \le h_o$$

$$\text{and where} \quad \lim_{\|h\| \to 0} \epsilon(\|h\|) = 0 \ . \tag{5-2}$$

Equation (5-2) can be rewritten as

$$\lim_{\|h\| \to 0} \frac{\|T(f+h) - T(f) - T'_{(f)} h\|}{\|h\|} = 0 \tag{5-3}$$

It can now be seen that the definition given by Equation (5-1) is a special case of Equation (5-3) where the norm is taken to be the absolute value and the operator T is simply the function f. The Fréchet derivative is sometimes called a strong derivative in contrast to the Gateau† or weak derivative [41].

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†A Gateau derivative is defined as follows. Let P be an operation mapping an open subset E of a Banach space (B-space) X into a subset F of another B-space Y. Consider a fixed element $\underline{x}_o \in E$ and suppose that there exists a linear operation U

$$U : X \to Y$$

and such that for every $\underline{x} \in X$

$$\lim_{t \to 0} \frac{P(\underline{x}_o + t\,\underline{x}) - P(\underline{x}_o)}{t} = U(\underline{x})$$

The linear operation U is then said to be a Gateau or weak derivative of the operation P at $\underline{x}_o$, that is

$$U = P'(\underline{x}_o)$$

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The preceding definitions of the Frechet derivative of a function and an operator can now be applied to the functional $J(\underline{u})$. In this development, $\underline{u}$ and $\underline{\Delta u}$ are assumed either to belong to the space of continuous functions $C(t_o, t_1)$ defined on a closed interval $[t_o, t_1]$ or to belong to the space $D_1(t_o, t_1)$ consisting of all functions defined on an interval $[t_o, t_1]$ which are continuous and have continuous first derivatives. The norms for the spaces $C(t_o, t_1)$ and $D_1(t_o, t_1)$ for the vector $\underline{\Delta u}(t) = \underline{u}_1(t) - \underline{u}_2(t)$ are

$$\|\underline{u}_1 - \underline{u}_2\|_o = \sup_t \sum_i |\underline{u}_1^{i}(t) - \underline{u}_2^{(i)}(t)| \text{ for the space } C(t_o, t_1) \quad (5\text{-}4)$$

$$\|\underline{u}_1 - \underline{u}_2\|_1 = \sup_t \sum_i |\underline{u}_1^{(i)}(t) - \underline{u}_2^{(i)}(t)| + \sup_t \sum_i |\dot{\underline{u}}_1^{(i)}(t) - \dot{\underline{u}}_2^{(i)}(t)|$$

$$\text{for the space } D_1(t_o, t_1) \qquad (5\text{-}5)$$

The vectors $\underline{x}$ and $\underline{\Delta x}$ are assumed to belong to the space $D_1(t_o, t_1)$. The functional $J(\underline{u})$ is said to be differentiable at $\underline{u}$ if there exists a continuous linear functional $\delta J(\underline{h}, \underline{u})$ with a domain in $C(t_o, t_1)$ and a range in R, and a real function $\epsilon(\|h\|)$ with the property

$$J(\underline{u} + \underline{h}) - J(\underline{u}) - \delta J(\underline{h}, \underline{u}) \le \epsilon(\|h\|)\ \|h\|$$

$$\text{for } \|h\| \le h_o \text{ and where } \lim_{\|h\| \to 0} \epsilon(\|h\|) = 0 \qquad (5\text{-}6)$$

or equivalently with the property

$$\lim_{\|h\| \to 0} \frac{J(\underline{u} + h) - J(\underline{u}) - \delta J(\underline{h}, \underline{u})}{\|h\|} = 0$$

The expression $J(\underline{u} + \underline{h}) - J(\underline{u})$ is termed the increment in $J(\underline{u})$. The continuous linear functional $\delta J(\underline{h}, \underline{u})$ which maps the space $C(t_o, t_1)$ into R is called the strong differential (the first variation) of $J(\underline{u})$ at $\underline{u}$. It follows that the increment of a differentiable functional $J(\underline{x}, \underline{u})$ is given by

$$\Delta J(\underline{x},\ \underline{u};\ \underline{\Delta} u,\ \underline{\Delta x}) = J(\underline{u} + \underline{\Delta} u,\ \underline{x} + \underline{\Delta} x) - J(\underline{u},\ \underline{x}) \tag{5-7}$$

$$= \delta J(\underline{x},\ \underline{u};\ \underline{\Delta} x,\ \underline{\Delta} u) + \epsilon_1(\|\underline{\Delta} x\|_1)\|\underline{\Delta} x\|_1 + \epsilon_2(\|\underline{\Delta} u\|_0)\|\underline{\Delta} u\|_0$$

for the case in which $\underline{u} \in C(t_0, t_1)$. An easily proved theorem providing a necessary condition for the differentiable functional $J(\underline{x}, \underline{u})$ to have an extremum is now stated [42].

Theorem. A necessary condition for the differentiable functional $J(\underline{x},\ \underline{u})$ to have an extremum for $\underline{u} = \hat{\underline{u}}$ is that its first variation vanish for $\underline{u} = \hat{\underline{u}}$, i. e., that

$$\delta J(\underline{h},\ \underline{u}) = 0$$

for $\underline{u} = \hat{\underline{u}}$ and all admissible $\underline{h}$.

The definitions of weak and strong extremums are now given. The functional $J(\underline{u})$ has a weak extremum for $\underline{u} = \tilde{\underline{u}}$ provided there exists a positive ϵ such that

$$J(\underline{u}) - J(\tilde{\underline{u}}) \geq 0$$

for all $\underline{u}$ in the domain of definition of the functional which satisfy the condition

$$\|\underline{u} - \tilde{\underline{u}}\|_1 < \epsilon$$

The functional $J(\underline{u})$ has a strong extremum for $\underline{u} = \hat{\underline{u}}$ provided there exists a positive ϵ such that

$$J(u) - J(\hat{u}) > 0$$

for all $\underline{u}$ in the domain of definition of the functional which satisfy the condition

$$\|\underline{u} - \hat{\underline{u}}\|_0 < \epsilon$$

It is noted that every strong extremum is simultaneously a weak extremum since if

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$$\| \underline{u} - \hat{\underline{u}} \|_1 < \epsilon \text{, then } \| \underline{u} - \tilde{\underline{u}} \|_0 < \epsilon \quad \text{a fortiori;}$$

hence, if $J(\hat{\underline{u}})$ is an extremum with respect to all $\underline{u}$ such that $\| \underline{u} - \hat{\underline{u}} \|_0 < \epsilon$ then $J(\hat{\underline{u}})$ is clearly an extremum with respect to all $\underline{u}$ such that $\| \underline{u} - \hat{\underline{u}} \|_1 < \epsilon$.

The calculus of variations approach to the optimal control problem formulated in Chapter 3 can now be discussed. Given the system

$$\begin{array}{ll} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) & \text{with} \\ \underline{x}_o, t_o \text{ fixed} & \text{and} \\ t_1 \text{ free} & \end{array}$$

subject to the end constraints

$$\underline{g}(\underline{x}_1, t_1) = \underline{0} \ ,$$

the problem is to find the necessary conditions for the first variation of the functional

$$J(\underline{x}, t_o, \underline{u}) = \int_{t_o}^{t_1} f_o(\underline{x}, \dot{\underline{x}}, \underline{u}, t)\, dt$$

to vanish. By appending the constraints to the functional J through the use of Lagrange multipliers $\underline{\nu}$, $\underline{p}$ the new functional becomes

$$\tilde{J}(\underline{x}_o, t_o, \underline{u}) = \langle \underline{\nu}, \underline{g}(\underline{x}_1, t_1) \rangle + \int_{t_o}^{t_1} \left[f_o(\underline{x}, \underline{x}, \underline{u}, t) + \langle \underline{p}, \underline{f} - \dot{\underline{x}} \rangle \right] dt$$

The so-called "multiplier rule" ensures that the minimization of the new functional is equivalent to the minimization of the original one subject to the constraints. A rigorous treatment of this is given by Hestenes [39]. It is clear, however, that the terms added to the original function are identically zero, i.e.,

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$$< \underline{\nu},\ \underline{g}(\underline{x}_1,\ t_1') > = 0$$

$$< \underline{p},\ \dot{\underline{x}} - \underline{f} > = 0$$

since $\underline{g}(\underline{x}_1,\ t_1) = 0$ and $\dot{\underline{x}} - \underline{f} = 0$. Rewriting Equation (5-8) as

$$\tilde{J}(\underline{x},\ t_o,\ \underline{u}) = < \underline{\nu},\ \underline{g}(\underline{x}_1,\ t_1) > + \int_{t_o}^{t_1} \left\{ \left[f_o(\underline{x}, \dot{\underline{x}}, \underline{u}, t) - < \underline{p}, \underline{f} > \right] + < \underline{p}, \dot{\underline{x}} > \right\} dt \tag{5-9}$$

and defining the bracketed term as the negative of the Hamiltonian yields

$$\tilde{J}(\underline{x},\ \underline{u}, t_o) = < \underline{\nu},\ \underline{g}(\underline{x}_1, t_1) > + \int_{t_o}^{t_1} \left[-H(\underline{x}, \underline{u}, \underline{p}, t) + < \underline{p},\ \dot{\underline{x}} > \right] dt \tag{5-10}$$

The definition of the Hamiltonian as used above is the same as that given in classical mechanics; in classical mechanics, the Hamiltonian is given by

$$H(\underline{q},\ \underline{p}, t) = - L(\underline{q}, \dot{\underline{q}}, t) + < \underline{p},\ \dot{\underline{q}} >$$

where $L(\underline{q},\ \dot{\underline{q}},\ t)$ is the Lagrangian, the components of $\underline{p}$ are the generalized momenta, and the components of $\underline{q}$ are the generalized position coordinates. Hence, the function $f_o(\underline{x},\ \dot{\underline{x}},\ \underline{u}, t)$ plays the role of the Lagrangian and the Lagrange multipliers $\underline{p}$ play the role of the generalized momenta. The increment in $\tilde{J}(\underline{u},\ \underline{x})$ is given by

$$\Delta\tilde{J}(\underline{u}, \underline{x}; \underline{\Delta u},\ \underline{\Delta x}) = \tilde{J}(\underline{u} + \underline{\Delta u},\ \underline{x} + \underline{\Delta x}, t_1 + \delta t_1) - \tilde{J}(\underline{u},\ \underline{x}) \tag{5-11}$$

$$= \delta\tilde{J}(\underline{u}, \underline{x}; \underline{\Delta u},\ \underline{\Delta x}) + \epsilon_1(\|\underline{\Delta x}\|_1)\ \|\underline{\Delta x}_1\| + \epsilon_2(\|\underline{\Delta u}\|_0)\ \|\underline{\Delta u}\|_0$$

The term $\tilde{J}(\underline{u} + \underline{\Delta u},\ \underline{x} + \underline{\Delta x}, t_1 + \delta t_1)$ is given by

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$$\tilde{J}(\underline{u}+\underline{\Delta u},\ \underline{x}+\underline{\Delta x},\ t_1+\delta t_1) =$$

$$<\underline{\nu},\ \underline{g}(\underline{x}_1+\underline{\Delta x}_1,\ t_1+\delta t_1)> +$$

$$\int_{t_o}^{t_1+\delta t_1} - H(\underline{x}+\underline{\Delta x},\ \underline{u}+\underline{\Delta u},\ \underline{p},t)+<\underline{p},\ \dot{\underline{x}}+\underline{\Delta \dot{x}})> dt \qquad (5\text{-}12)$$

Expressing the term $H(\underline{x}+\underline{\Delta x},\ \underline{u}+\underline{\Delta u},\ \underline{p},\ t)$ as a TSE about $(\underline{x}, \underline{u}, \underline{p}, t)$, expressing the term $\underline{g}(\underline{x}_1+\underline{\Delta x}_1,\ t_1+\delta t_1)$ as a TSE about $(\underline{x}_1, t_1)$ and solving for the first variation $\delta J(\underline{u},\ \underline{x};\ \underline{\Delta u},\ \underline{\Delta x})$ yields

$$\delta J(\underline{u},\ \underline{x};\underline{\Delta u},\underline{\Delta x}) = \int_{t_o}^{t_1}\left[<-\frac{\partial H}{\partial \underline{x}},\ \underline{\Delta x}> + <-\frac{\partial H}{\partial \underline{u}},\ \underline{\Delta u}> + <\underline{p},\ \underline{\Delta \dot{x}}>\right] dt \qquad (5\text{-}13)$$

$$+<\underline{\nu},\ \frac{\partial g}{\partial \underline{x}_1}\underline{\Delta x}_1> + <\underline{\nu},\ \frac{\partial g}{\partial t_1}\ \delta t_1>$$

$$-\left[H(\underline{x},\ \underline{u},\ \underline{p},t) - <\underline{p},\ \dot{\underline{x}}>\right]\Big|_{t_1} \delta t_1$$

The term $\int_{t_o}^{t_1} <\underline{p},\ \underline{\Delta \dot{x}}> dt$ can be integrated by parts to yield

$$\int_{t_o}^{t_1} <\underline{p},\ \underline{\Delta \dot{x}}> dt = <\underline{p},\underline{\Delta x}>\Big|_{t_o}^{t_1} - \int_{t_o}^{t_1} <\dot{\underline{p}},\ \underline{\Delta x}> dt \qquad (5\text{-}14)$$

Substituting Equation (5-14) into Equation (5-13) yields

$$\delta J(\underline{u},\ \underline{x},\ t_1;\ \underline{\Delta u},\ \underline{\Delta x},\ \delta t_1) =$$

$$\int_{t_o}^{t_1}\left[<-\frac{\partial H}{\partial \underline{x}} - \dot{\underline{p}},\ \underline{\Delta x}> + <-\frac{\partial H}{\partial \underline{u}},\ \underline{\Delta u}>\right] dt$$

$$+<\underline{p},\ \underline{\Delta x}>\Big|_{t_1} + <\underline{\nu},\frac{\partial g}{\partial \underline{x}_1}\ \underline{\Delta x}_1> + <\underline{\nu},\frac{\partial g}{\partial t_1}\ \delta t_1> \qquad \text{5-15)}$$

$$-\left[H[\underline{x},\ \underline{u},\ \underline{p},\ t] - <\underline{p},\ \dot{\underline{x}}>\right]\Big|_{t_1} \delta t_1$$

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The terms $\underline{\Delta x}_1$ and $\underline{\Delta x}(t_1)$ are related by

$$\underline{\Delta x}_1 = \underline{\Delta x}(t_1) + \underline{\dot{x}}(t_1)\ \delta t_1 \tag{5-16}$$

Hence, the term $< \underline{p},\ \underline{\Delta x} > \Big|_{t_1}$ becomes

$$\begin{aligned} < \underline{p}(t_1),\ \underline{\Delta x}_1 - \underline{\dot{x}}(t_1)\ \delta t_1 > &= < \underline{p}(t_1),\ \underline{\Delta x}_1 > \\ &\quad - < \underline{p}(t_1),\ \underline{\dot{x}}(t_1) > \delta t_1 \end{aligned} \tag{5-17}$$

Substituting Equation (5-17) into Equation (5-15) yields

$$\begin{aligned} \delta J = &\int_{t_o}^{t_1} \left[< - \frac{\partial H}{\partial \underline{x}} - \underline{\dot{p}},\ \underline{\Delta x} > + < - \frac{\partial H}{\partial \underline{u}},\ \underline{\Delta u} > \right] dt \\ &+ \left[- H[\underline{x}, \underline{u}, \underline{p}, t] \Big|_{t_1} + < \underline{\nu}, \frac{\partial g}{\partial t_1} > \right] \delta t_1 \\ &+ < \left(\frac{\partial g}{\partial \underline{x}_1} \right)^T \underline{\nu} + \underline{p}(t_1),\ \underline{\Delta x}_1 > \end{aligned} \tag{5-18}$$

Hence, the necessary conditions for δJ to vanish are given by

1) $\underline{\dot{p}} = - \dfrac{\partial H^\dagger}{\partial \underline{x}}$ (one of Hamilton's canonical equations)

2) $\dfrac{\partial H}{\partial \underline{u}} = 0$ (optimality condition)

3) $\underline{p}(t_1) = - \left(\dfrac{\partial g}{\partial \underline{x}_1} \right)^T \underline{\nu}$ (transversality condition) (5-19)

4) $- H(t_1) + < \nu, \dfrac{\partial g}{\partial t_1} > = 0$ (boundary condition)

5) $g(\underline{x}_1, t_1) = 0$ (end constraint)

6) $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t) \equiv \dfrac{\partial H}{\partial \underline{p}}$ (differential constraint, the other canonical equation)

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†The Hamiltonian is defined as $H = < \underline{p},\ \underline{\dot{x}} > - f_o$

Weierstrass E Condition

An additional condition is required in order to have a strong extremum. The additional necessary condition is known as the Weierstrass E condition (a convexity condition). The Weierstrass E-function of the functional

$$J(\underline{x}, \underline{u}) = \int_a^b F(t, \underline{x}, \dot{\underline{x}}, \underline{u})dt$$

is given by

$$E(t, \underline{x}, \dot{\underline{X}}, \dot{\underline{x}}, \underline{U}, \underline{u}) =$$

$$F(t, \underline{x}, \dot{\underline{X}}, \underline{u}) - F(t, \underline{x}, \dot{\underline{x}}, \underline{u}) - < \dot{\underline{X}} - \dot{\underline{x}}, \frac{\partial F}{\partial \dot{\underline{x}}} >$$

$$+ F(t, \underline{x}, \dot{\underline{x}}, \underline{U}) - F(t, \underline{x}, \dot{\underline{x}}, \underline{u}) - < \underline{U} - \underline{u}, \frac{\partial F}{\partial \underline{u}} >$$

where $\dot{\underline{X}}$ refers to some arbitrary $\dot{\underline{x}}$

$\underline{U}$ refers to some arbitrary $\underline{u}$.

The property

$$E(t, \underline{x}, \dot{\underline{X}}, \dot{\underline{x}}, \underline{U}, \underline{u}) \geq 0 \tag{5-21}$$

for arbitrary finite vectors $\dot{\underline{X}}$, $\underline{U}$ is known as the E-condition. The Weierstrass E function for the functional

$$\hat{J}(\underline{u}, \underline{x}) = < \underline{\nu}, \underline{g}(\underline{x}_1, t_1) > + \int_{t_o}^{t_1} \left[-H(t, \underline{x}, \underline{u}, \underline{p}) + < \underline{p}, \dot{\underline{x}} > \right] dt$$

is given by

$$E(t, \underline{x}, \dot{\underline{x}}, \underline{u}, \dot{\underline{X}}, \underline{U}) =$$

$$- H(t, \underline{x}, \underline{U}, \underline{p}) + H(t, \underline{x}, \underline{u}, \underline{p}) - < \underline{U} \quad \underline{u}, - \frac{\partial H}{\partial \underline{u}} >$$

$$+ < + \underline{p}, \dot{\underline{X}} > - < + \underline{p}, \dot{\underline{x}} > - < \dot{\underline{X}} - \dot{\underline{x}}, + \underline{p} >$$

The E-condition for this functional is simply

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$$- H(t, \underline{x}, \underline{U}, \underline{p}) + H(t, \underline{x}, \underline{u}, \underline{p}) \geqq 0 \tag{5-22}$$

or equivalently

$$H(t, \underline{x}, \underline{u}, \underline{p}) \geq H(t, \underline{x}, \underline{U}, \underline{p}) \tag{5-23}$$

Equation (5-23) represents a form of the celebrated maximum principle.

In summary, it is noted that the first order necessary conditions for a strong extremum are of two types; these types are designated as A and B for convenience. The conditions belonging to Type A are <u>independent</u> of the nature of the endpoints (i. e., fixed, free, or constrained) and independent of the nature of the initial and final time (i. e., fixed or free time); those belonging to Type B are dependent on the nature of the endpoints and on the nature of the initial and final time. The necessary conditions belonging to Type A include

(1) Hamilton's Canonical Equations

$$\dot{\underline{p}} = - \frac{\partial H}{\partial \underline{x}}$$

$$\dot{\underline{x}} = \underline{f} = \frac{\partial H}{\partial \underline{p}}$$

(2) The Weierstrass E Condition

$$H(t, \underline{x}, \underline{u}, \underline{p}) \geq H(t, \underline{x}, \underline{U}, \underline{p})$$

(3) A condition which the extremal controller must satisfy $\dfrac{\partial H}{\partial \underline{u}} = 0.$

The conditions belonging to Type B include

(1) that which describes behavior of the Hamiltonian evaluated along the extremal

(2) those which the Lagrange multipliers $\underline{p}$ must satisfy (transversality condition)

The necessary conditions (Type B) for the problems of interest in this dissertation are provided in Table 5.1.

It is noted that in this development it was tacitly assumed that

1) $H(\underline{x}, \underline{u}, \underline{p}, t)$ is sufficiently differentiable in both $\underline{x}$ and $\underline{u}$ and hence, that $\frac{\partial H}{\partial \underline{u}}$ is defined.

In many practical problems these assumptions do not hold. In addition, when the control $\underline{u}$ is constrained according to

$$|\underline{u}_j(t)| \leq 1 \quad \forall j$$

the problem becomes more difficult, albeit tractable. For such problems, Pontryagin's maximum principle provides a convenient technique for obtaining the necessary conditions for local optimality. It will be seen in the next section that the maximum principle requires relatively weak differentiability assumptions.

5.1.2 Pontryagin's Maximum Principle

In this section, Pontryagin's maximum principle is stated. The proof of the maximum principle is given in many of the recent books on optimal control theory (e.g., [24], [43], and [44]) and will not be repeated here. In the maximum principle, the notions of the weak and strong extremals that were introduced in the calculus of variations approach are no longer used. The maximum principle in conjunction with the associated transversality conditions provides a necessary criterion which the optimal controller $\underline{u}^*(t)$ must satisfy.

A theorem which applies to autonomous systems is first stated. The results which are applicable to the general nonautonomous systems are then obtained by treating t as an additional spatial coordinate.

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Table 5-1

FIRST ORDER NECESSARY CONDITIONS (TYPE B) FOR A STRONG EXTREMUM OBTAINED FROM THE CALCULUS OF VARIATIONS

Note 1. Plant (differential equation constraint), $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$.

Initial state $\underline{x}_o$ and initial time t_o, fixed.

Cost functional, $J(u) = \int_{t_o}^{t_1} f_o(\underline{x}, \dot{\underline{x}}, \underline{u}, t)dt$

Hamiltonian, $H(\underline{x}, \underline{p}, \underline{u}, t) \equiv \langle \underline{p}, \dot{\underline{x}} \rangle - f_o(\underline{x}, \dot{\underline{x}}, \underline{u}, t)$

Note 2. Necessary conditions (Type A) include

1) Hamiltons canonical equations : $\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{x}}$; $\dot{\underline{x}} = \frac{\partial H}{\partial \underline{p}}$

2) Weierstrass E condition : $H(\underline{x}, \underline{p}, \underline{u}, t) \geq H(\underline{x}, \underline{p}, \underline{U}, t)$

3) $\frac{\partial H}{\partial \underline{u}} = 0$

Nature of right end	Hamiltonian	Lagrange multipliers $\underline{p}$
$\underline{x}_1$ fixed, t_1 fixed	No condition	No condition
$\underline{x}_1$ fixed, t_1 free	$H(t_1) = 0$	No condition
$\underline{x}_1$ free, t_1 fixed	No condition	$\underline{p}(T) = 0$
$\underline{x}_1$ free, t_1 free	$H(t_1) = 0$	$\underline{p}(t_1) = 0$
$\underline{g}(\underline{x}_1, t_1) = \underline{0}$, t_1 fixed	No condition	$\underline{p}(T) = -\left[\frac{\partial g}{\partial \underline{x}_1}\Big\vert_{t=T}\right]^T \underline{\nu}$
$\underline{g}(\underline{x}_1, t_1) = \underline{0}$, t_1 free	$-H(t_1) + \langle \underline{\nu}, \frac{\partial g}{\partial t_1} \rangle = 0$	$\underline{p}(t_1) = -\left[\frac{\partial g}{\partial \underline{x}_1}\Big\vert_{t=t_1}\right]^T \underline{\nu}$

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Autonomous Systems

Consider the autonomous control process

$$\text{(S)} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

with $\underline{f}(\underline{x}, \underline{u})$ and $\frac{\partial f}{\partial \underline{x}}(\underline{x}, \underline{u})$ continuous in R^{n+m}. The initial and target sets X_o and $X_1 \subset R^n$ and the nonempty control restraint set $\Omega \subset R^m$. Let the class of admissible controllers Δ consist of the bounded measurable functions $\underline{u}(t) \subset \Omega$ on some finite interval $0 \le t \le t_1$. The response associated with the controller $\underline{u}(t)$ is $\underline{x}(t, \underline{x}_o)$; the controller $\underline{u}(t)$ transfers $\underline{x}(0, \underline{x}_o) = \underline{x}_o \in X_o$ to $\underline{x}(t_1, \underline{x}_o) = \underline{x}_1 \in X_1$. The cost functional associated with the contro... $\underline{u}(t)$ on $t \in [0, t_1]$ in Δ with response $\underline{x}(t)$ is given by

$$J(\underline{u}) = \int_{t_o}^{t_1} f_o\left(\underline{x}(t), \underline{u}(t)\right) dt$$

where f_o and $\frac{\partial f_o}{\partial \underline{x}}$ are continuous in R^{n+m}.†

An augmented system is formed by introducing the integrand of the cost functional f_o as an additional state equation.

The augmented state is denoted as $\hat{\underline{x}}$ and is given by

$$\hat{\underline{x}} = \begin{pmatrix} x_o \\ \underline{x} \end{pmatrix}.$$

the augmented adjoint vector $\hat{\underline{p}}$ is given by

$$\hat{\underline{p}} = \begin{pmatrix} p_o \\ \underline{p} \end{pmatrix}$$

the augmented system $(\hat{S})$ is given by

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† Note that it is not necessary to assume that $\frac{\partial f_o}{\partial \underline{u}}$ exists as it was in the calculus of variations.

$$(\hat{S})\quad \hat{\underline{\dot{x}}} = \begin{pmatrix} \dot{x}_o \\ \underline{\dot{x}} \end{pmatrix} = \hat{\underline{f}} = \begin{pmatrix} f_o(\underline{x}, \underline{u}) \\ \underline{f}(\underline{x}, \underline{u}) \end{pmatrix} \tag{5-24}$$

the augmented adjoint system $(\hat{Q})$ is given by

$$(\hat{Q})\quad \begin{aligned} \dot{p}_o &= -\left\langle \frac{\partial \hat{\underline{f}}}{\partial x_o}(\underline{x}(t), \underline{u}(t)), \hat{\underline{p}} \right\rangle = 0 \\ \underline{\dot{p}} &= -\left[\frac{\partial \hat{\underline{f}}}{\partial \underline{x}}(\underline{x}(t), \underline{u}(t))\right]^T \hat{\underline{p}} \end{aligned} \tag{5-25}$$

or

$$\dot{\hat{\underline{p}}} = \begin{bmatrix} 0 \cdots\cdots\cdots 0 \\ -\left[\frac{\partial \hat{\underline{f}}}{\partial \underline{x}}(\underline{x}(t), \underline{u}(t))\right]^T \end{bmatrix} \hat{\underline{p}} = -\left[\frac{\partial \hat{\underline{f}}}{\partial \hat{\underline{x}}}(\underline{x}(t), \underline{u}(t))\right]^T \hat{\underline{p}}$$

and the augmented Hamiltonian function is defined as

$$H(\hat{\underline{x}}, \hat{\underline{p}}, \underline{u}) = \langle \hat{\underline{p}}, \hat{\underline{f}}(\underline{x}, \underline{u}) \rangle \tag{5-26}$$

This definition (when p_o is taken as -1) corresponds exactly to that used in classical mechanics. The function $M(\hat{\underline{p}}, \hat{\underline{x}})$ is defined as

$$M(\hat{\underline{p}}, \hat{\underline{x}}) = \sup_{\underline{u} \in \Omega} H(\hat{\underline{x}}, \hat{\underline{p}}, \underline{u}) \tag{5-27}$$

A theorem for autonomous systems can now be stated [24].

<u>Theorem.</u> The control problem $(S, \Delta\ \Omega, X_o, X_1, J)$ being considered is

(S) : $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u})$

Ω : $\Omega \subset R^m = \{\underline{u}(t) : \underline{u}(t)$ are bounded and measurable on various intervals $0 \leqq t \leqq t_1\}$

Δ : all admissible controllers which steer some initial point of X_o to a final point in X_1

J : $J(\underline{u}) = \int_{t_o}^{t_1} f_o(\underline{x}(t), \underline{u}(t))dt$

If $\underline{u}^*(t)$ on $0 \le t \le t^*$ is optimal in Δ with augmented response $\underline{x}^*(t)$, then there exists a nontrivial augmented adjoint response $\hat{\underline{p}}(t)$ satisfying

$$\dot{\hat{\underline{p}}} = \begin{pmatrix} \dot{p}_o \\ \dot{\underline{p}} \end{pmatrix} \begin{pmatrix} 0 \\ -\left[\frac{\partial \hat{\underline{f}}}{\partial \underline{x}}\right]^T \hat{\underline{p}} \end{pmatrix} = -\left[\frac{\partial \hat{\underline{f}}}{\partial \hat{\underline{x}}}\right]^T \hat{\underline{p}}$$

such that $H(\hat{\underline{x}}^*, \hat{\underline{p}}^*, \underline{u}^*) = M(\hat{\underline{x}}^*, \hat{\underline{p}}^*)$ a.e., and $M(\hat{\underline{x}}^*, \hat{\underline{p}}^*) \equiv 0$ and $p_o^* \le 0$ everywhere on $0 \le t \le t_1^*$. In addition, if X_o and X_1 (or just one of them) are manifolds with tangent spaces T_o and T_1 at $\underline{x}^*(0)$ and $\underline{x}^*(t_1^*)$, then $\hat{\underline{p}}^*(t)$ can be selected to satisfy the transversality conditions at both ends (or at just one end)

$\underline{p}^*(0)$ is orthogonal to T_o

$\underline{p}^*(t_1^*)$ is orthogonal to T_1

Similar results (see Table 5-2) apply to

(1) the case in which target set X_1 is all R^n (the free-endpoint problem)

(2) the case in which the time duration is fixed and finite), i.e., the controllers $\underline{u}(t)$ are defined on $0 \le t \le T$

(3) the case in which the fixed time duration is infinite.

The necessary conditions for local optimality are seen to be of two types (A and B). Type A consists of those conditions which are independent of the initial and target sets X_o, X_1; Type B consists of those conditions which are dependent on the initial and target sets. The necessary conditions belonging to Type A include

(1) Hamilton's canonical equations

Table 5-2

NECESSARY CONDITIONS (TYPE B) FOR LOCAL OPTIMALITY FOR AUTONOMOUS SYSTEMS (OBTAINED FROM THE MAXIMUM PRINCIPLE)

Notes:

1. The necessary conditions denoted as Type A which are common to all autonomous systems are not included.

2. The system (S) is given by

$$(S) \quad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}),$$

the cost functional is given by

$$J(u) = \int_{t_o}^{t_1} f_o(\underline{x}, \underline{u})dt$$

3. The initial set consists of the fixed initial state $\underline{x}_o$ and the fixed initial time t_o.

Target Set X_1	Hamiltonian, $H(\hat{\underline{x}}^*, \hat{\underline{p}}^*, \underline{u}^*)$	Adjoint vector $\underline{p}^*$
$\{(\underline{x}, t) : \underline{x}_1$ fixed, t_1 fixed$\}$	$H^* =$ constant	No condition
$\{(\underline{x}, t) : \underline{x}_1$ fixed, t_1 infinite$\}$	$H^* = 0$	No condition
$\{(\underline{x}, t) : \underline{x}_1$ fixed, t_1 free$\}$	$H^* = 0$	No condition
$\{(\underline{x}, t) : \underline{x}_1$ free, t_1 free$\}$	$H^* = 0$	$\underline{p}^*(t_1^*) = \underline{0}$
$\{(\underline{x}, t) : g_i(\underline{x}) = 0$, t_1 fixed, $i = 1, 2, \ldots, n-k\}$	$H^* =$ constant	$\underline{p}^*(T) = -\left[\frac{\partial g}{\partial \underline{x}^*}\left(\underline{x}^*(T)\right)\right]^T \underline{\nu}$
$\{(\underline{x}, t) : g_i(\underline{x}) = 0$, t_1 free, $i = 1, 2, \ldots, n-k\}$	$H^* = 0$	$\underline{p}^*(t_1^*) = -\left[\frac{\partial g}{\partial \underline{x}^*}\left(\underline{x}^*(t_1^*)\right)\right]^T \underline{\nu}$

$$\dot{\hat{\underline{p}}}^*(t) = -\left[\frac{\partial \hat{\underline{f}}}{\partial \hat{\underline{x}}^*}\left(\underline{x}^*(t), \underline{u}^*(t)\right)\right]^T \hat{\underline{p}}^*(t) = -\frac{\partial H}{\partial \hat{\underline{x}}^*}\left(\hat{\underline{x}}^*(t), \hat{\underline{p}}^*(t), \underline{u}^*(t)\right)$$

$$\dot{\hat{\underline{x}}}^*(t) = \hat{\underline{f}}\left(\underline{x}^*(t), \underline{u}^*(t)\right) = \frac{\partial H}{\partial \hat{\underline{p}}^*}$$

(2) the so-called "optimality condition" which characterizes the controller $\underline{u}^*(t)$

$$M\left(\hat{\underline{x}}^*(t), \hat{\underline{p}}^*(t)\right) = H\left(\hat{\underline{x}}^*(t), \hat{\underline{p}}^*(t), \underline{u}^*(t)\right)$$

(3) $p_o \leq 0$

The conditions belonging to Type B (see Table 5-1) include

(1) those which pertain to the adjoint vector $\hat{\underline{p}}^*(t)$ (the transversality conditions)

(2) that which describes the behavior of the Hamiltonian along the optimal trajectory

Nonautonomous System

The results for the most general nonlinear nonautonomous process are obtained immediately by introducing the time t as an additional spatial coordinate, i.e.,

$$t = x_{n+1}$$

For this case, the system (S) is given by

$$\text{(S)} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \tag{5-28}$$

where $\underline{f} : R^n \times R^m \times R^1 \to R^n$

and $\underline{f} \in C^1$ in R^{n+1+m}

and the cost functional is given by

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$$J(\underline{u}) = \int_{t_o}^{t_1} f_o\left(\underline{x}(t),\ \underline{u}(t),\ t\right) dt \tag{5-29}$$

where

$$f_o:\ R^n \times R^m \times R^1 \rightarrow R^1$$

$$\text{and } f_o \in C^1 \text{ in } R^{n+1+m}$$

The time augmented response $\underline{\tilde{x}}(t)$ corresponding to $\underline{u}(t)$ is

$$\underline{\tilde{x}}(t) = \begin{pmatrix} x_o(t) \\ \underline{x}(t) \\ x_{n+1}(t) \end{pmatrix}$$

and represents the solution to the time augmented system $(\tilde{S})$

$$(\tilde{S}) \qquad \dot{\underline{\tilde{x}}} = \underline{\tilde{f}}\,(\underline{\tilde{x}}\,,\ \underline{u}) \tag{5-30}$$

$$\text{or} \qquad \begin{aligned} \dot{x}_o &= f_o(\underline{x},\ x_{n+1},\ \underline{u}) \\ \dot{\underline{x}} &= \underline{f}(\underline{x},\ x_{n+1},\ \underline{u}) \\ \dot{x}_{n+1} &= 1 \end{aligned}$$

The time-augmented adjoint system based on $\underline{u}(t)$ and $\underline{\tilde{x}}(t)$ is

$$(\tilde{A}) \qquad \dot{\underline{\tilde{p}}} = -\left[\frac{\partial \tilde{f}}{\partial \underline{\tilde{x}}}\left(\underline{\tilde{x}}(t),\ \underline{u}(t)\right)\right]^T \underline{\tilde{p}}$$

$$\text{or} \qquad \dot{p}_o = -< \frac{\partial f_o}{\partial x_o}\,;\ \underline{\tilde{p}} > \;\equiv\; < \underline{0},\ \underline{\tilde{p}} > = 0 \tag{5-31}$$

$$\dot{\underline{p}} = -\left[\frac{\partial \tilde{f}}{\partial \underline{x}}\left(\underline{x}(t),\ \underline{u}(t),\ t\right)\right]^T \underline{\tilde{p}}$$

$$\dot{p}_{n+1} = -< \frac{\partial \tilde{f}}{\partial x_{n+1}}\ ,\ \underline{\tilde{p}} >$$

The time augmented Hamiltonian is

$$\tilde{H}(\tilde{\underline{x}}, \tilde{\underline{p}}, \underline{u}) = < \tilde{\underline{p}}, \tilde{\underline{f}}(\underline{x}, x_{n+1}, \underline{u}) > \tag{5-32}$$

The function $\tilde{M}(\tilde{\underline{x}}, \tilde{\underline{p}})$ is defined as

$$\tilde{M}(\tilde{\underline{x}}, \tilde{\underline{p}}) = \sup_{u \in \Omega} \tilde{H}(\tilde{\underline{x}}, \tilde{\underline{p}}, \underline{u})> \tag{5-33}$$

It is seen that

$$\tilde{\underline{x}} = \begin{pmatrix} \hat{\underline{x}} \\ x_{n+1} \end{pmatrix} ; \tilde{\underline{p}} = \begin{pmatrix} \hat{\underline{p}} \\ p_{n+1} \end{pmatrix}$$

$$\tilde{H}(\tilde{\underline{x}}, \tilde{\underline{p}}, \underline{u}) = < \hat{\underline{p}}, \hat{\underline{f}} > + p_{n+1} f_{n+1} = H(\hat{\underline{x}}, \hat{\underline{p}}, \underline{u}, t) + p_{n+1} \tag{5-34}$$

$$\tilde{M}(\tilde{\underline{x}}, \tilde{\underline{p}}) = M(\hat{\underline{x}}, \hat{\underline{p}}, t) + p_{n+1} \tag{5-35}$$

The theorem for the nonautonomous case can now be stated [24].

<u>Theorem.</u> Consider the control problem $(S, \Delta\ \Omega, X_o, X_1, J)$ where

1) the process (S) is given by

$$(S) \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

2) the class of admissible functions Δ are all the bounded measurable controllers $\underline{u}(t) \subset \Omega \subset R^m$ on various finite intervals $t_o \leq t \leq t_1$ which steer points of X_o to X_1 (fixed endpoint, free time)

3) the cost functional is

$$J(\underline{u}) = \int_{t_o}^{t_1} f_o\Big(\underline{x}(t), \underline{u}(t), t\Big) dt$$

If $\underline{u}^*(t)$ on $t_o^* \leq t \leq t_1^*$ with time-augmented response $\tilde{\underline{x}}^*(t)$ is optimal in Δ, <u>then</u> there exists a nontrivial time-augmented adjoint response $\tilde{\underline{p}}^*(t)$ of $\tilde{A}$ such that

$$\tilde{H}\Big(\tilde{\underline{x}}^*(t), \tilde{\underline{p}}^*(t), \underline{u}^*(t)\Big) = \tilde{M}\Big(\tilde{\underline{x}}^*(t), \tilde{\underline{p}}^*(t)\Big) \text{ a.e.} \tag{5-36}$$

and

$$\tilde{M}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t)\right) \equiv 0 \ , \ p_o^* \leq 0 \tag{5-37}$$

everywhere on $t_o^* \leq t \leq t_1^*$.

These conclusions are equivalent to

$$H\left(\hat{\underline{x}}^*(t),\ \hat{\underline{p}}^*(t),\ \underline{u}^*(t), t\right) = M\left(\hat{\underline{x}}^*(t),\ \hat{\underline{p}}^*(t), t\right) \text{ a. e.} \tag{5-38}$$

and

$$M\left(\hat{\underline{x}}^*(t),\ \hat{\underline{p}}^*(t), t\right) \equiv \int_{t_o^*}^{t} < \hat{\underline{p}}(s), \frac{\partial \hat{\underline{f}}}{\partial t}\left(\underline{x}^*(s),\ \underline{u}^*(s), s\right) > ds \tag{5-39}$$

The transversality conditions are

$$p_{n+1}^*(t_o^*) = p_{n+1}^*(t_1^*) = 0 \tag{5-40}$$

so

$$M\left(\hat{\underline{x}}^*(t_1^*),\ \hat{\underline{p}}^*(t_1^*),\ t_1^*\right) = 0 \tag{5-41}$$

If X_o and X_1 (or just one of them) are manifolds in R^n with tangent spaces T_o and T_1 at $\underline{x}_o^*$ and $\underline{x}_1^*$, respectively, then $\tilde{\underline{p}}^*(t)$ can be selected to satisfy the additional transversality conditions (or just one of them)

$$\begin{aligned} &\underline{p}^*(t_o^*) \text{ is orthogonal to } T_o \\ &\underline{p}^*(t_1^*) \text{ is orthogonal to } T_1 \end{aligned} \tag{5-42}$$

Similar results apply to the nonautonomous cases in which (see Table 5-3)

(1) the initial time t_o is fixed $(t_o = t_o^*)$

(2) the initial and target sets are time varying

(3) the time duration is fixed and finite

5.1.3 Application of the Necessary Conditions for Local Optimality to the Dual-Spin and Spinning Vehicles

In this section, the general first order necessary conditions obtained in the preceding sections are applied to the systems being studied.

Table 5-3

NECESSARY CONDITIONS (TYPE B) FOR LOCAL OPTIMALITY FOR NONAUTONOMOUS SYSTEMS (OBTAINED FROM THE MAXIMUM PRINCIPLE)

Note 1. The necessary conditions (Type A) which are common to all nonautonomous systems are not included.

Note 2. The system (S) is given by $\dot{\underline{x}} = \underline{f}\left(\underline{x}(t), \underline{u}(t), t\right)$

The cost function J(u) is given by $J(u) = \int_{t_o}^{t_1} f_o\left(\underline{x}(t), \underline{u}(t), t\right) dt$

Note 3. The initial state $\underline{x}_o$ at time t_o is fixed.

Target Set X_1	Hamiltonian $\tilde{H}^*(t_1^*) = H^*(t_1^*) + p^*_{n+1}(t_1^*)$	Adjoint vector, $\underline{p}^*(t)$
$X_1 = \{(\underline{x}, t) : \underline{x}_1 \text{ fixed}, t_1 \text{ fixed}\}$	No useful condition	No condition
$X_1 = \{(\underline{x}, t) : \underline{x}_1 \text{ fixed}, t_1 \text{ free}\}$	$H^*(t_1^*) = 0$	No condition on $\underline{p}^*(t)$; $p_{n+1}(t_1^*) = 0$
$X_1 = \{(\underline{x}, t) : \underline{x}_1 \text{ free}, t_1 \text{ fixed}\}$	No useful condition	$\underline{p}^*(T) = 0$
$X_1 = \{(\underline{x}, t) : \underline{x}_1 \text{ free}, t_1 \text{ free}\}$	$H^*(t_1^*) = 0$	$\underline{p}^*(t_1^*) = 0$; $p_{n+1}(t_1^*) = 0$
$X_1 = \{(\underline{x}, t) : g_i(\underline{x}, t) = 0, t_1 \text{ fixed}$ $i = 1, 2, \ldots, n-k\}$	No useful condition	$\underline{p}^*(T) = -\left[\dfrac{\partial \underline{g}}{\partial \underline{x}}\Big\|_{\substack{t=T \\ \underline{x}=\underline{x}^*}}\right]^T \underline{\nu}$
$X_1 = \{(\underline{x}, t) : g_i(\underline{x}, t) = 0, t_1 \text{ free}$ $i = 1, 2, \ldots, n-k\}$	$H^*(t_1^*) = -p^*_{n+1}(t_1^*) = +\left[\dfrac{\partial \underline{g}}{\partial t}\Big\|_{t_1^*}\right]^T \underline{\nu}$	$\underline{p}^*(t_1^*) = -\left[\dfrac{\partial \underline{g}}{\partial \underline{x}}\Big\|_{t=t_1^*}\right]^T \underline{\nu}$ $p_{n+1}(t_1^*) = -\left[\dfrac{\partial \underline{g}}{\partial t}\right]^T_{t=t_1^*} \underline{\nu}$

As seen in Tables 5-1 and 5-3, the results obtained from the calculus of variations approach are identical to those obtained from the maximum principle, as expected.

Target-Dependent Necessary Conditions

The target set X_1 which applies to the angular momentum control (AMCO) concept was given in Equation (3-8). For the dual-spin vehicle the convex target sets X_1 of concern are given by (both fixed and free final time t_1 are considered)

$$X_1 = \left\{ \left(x(t), t \right) : \ \underline{g}(\underline{x}, t) \Big|_{t=t_1} = 0, \ t_1 \text{ free} \right\}$$

$$X_1 = \left\{ \left(x(t), t \right) : \ \underline{g}(\underline{x}, t) \Big|_{t=t_1} = 0, \ t_1 \text{ fixed} \right\} \tag{5-43}$$

$$\text{where } \ \underline{g}(\underline{x}, t) = \left[\begin{array}{c|cc} I & 0 & \beta \\ & -\beta & 0 \end{array} \right] \underline{x} = C \, \underline{x}$$

$$\beta = \frac{J_3^R \sigma}{I_1} = \bar{r} \sigma$$

$$I = 2 \times 2 \text{ identity matrix}$$

The target set for a spinning symmetric vehicle is also of interest because the angular momentum control concept (AMCO) is later applied to this vehicle. The results obtained by using the AMCO concept for this vehicle will later be compared with those obtained by using the SACO concept (the SACO concept was studied in [11].) The target sets X_1 which apply to the spinning symmetric vehicle (when the AMCO concept is used) are

$$X_1 = \left\{ (\underline{x}, t) : \ \underline{g}(\underline{x}, t) \Big|_{t=t_1} = 0 \ , \ t_1 \ \text{ free} \right\}$$

$$X_1 = \left\{ (\underline{x}, t) : \ \underline{g}(\underline{x}, t) \Big|_{t=t_1} = 0, \ t_1 \ \text{ fixed} \right\} \tag{5-44}$$

where $\underline{g}(\underline{x}, t) = \underline{g}(\underline{x}) = \left[\begin{array}{c|cc} I & 0 & r\omega_3 \\ & -r\omega_3 & 0 \end{array}\right] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \end{Bmatrix}$

The target set X_1 which was used in [11] and which applies to the SACO concept is given by

$$X_1 = \{(\underline{x}, t) : \underline{x}_1 = \underline{0} \ , \ t_1 \ \text{free}\} \tag{5-45}$$

This target set is of interest in this work only in that the results obtained when this X_1 is assumed will be compared with those obtained when the X_1 given by Equation (5-44) is assumed. However, since the results provided in [11] are not complete enough for a thorough comparison, the SACO concept will also be simulated.

The remaining target sets discussed in Table 5-3, viz. the free-end point problems are clearly unsuitable for the control problem being studied. The necessary conditions corresponding to the specific target sets discussed above are provided in Table 5-4. As seen in Table 5-4, the Hamiltonian evaluated at the optimal conditions when t_1 is free vanishes, i. e.,

$$H\left(\underline{x}^*(t_1^*), \ \underline{u}^*(t_1^*), \ \underline{p}^*(t_1^*), \ t_1^*\right) = H^*(t_1^*) \equiv 0$$

When the final time t_1 is fixed, however, there is no useful condition on $H^*(T)$, i. e., none except for the fact that H^* must satisfy its definition,

$$H^* = \langle \underline{p}^*, \underline{f}^* \rangle - f_o^*$$

For the angular momentum control (AMCO) concept, $\underline{p}^*(t_1^*)$ is transversal to the smooth 2-fold in R^n (where n = 4) at $\underline{x}_1^*(t_1^*)$, that is, $\underline{p}^*(t_1^*)$ can be represented as a linear combination of the linear

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independent vectors $\left.\frac{\partial g_i(\underline{x})}{\partial \underline{x}(t)}\right|_{\underline{x}(t)=\underline{x}_1(t_1)}$, i= 1, 2. Alternatively, it can be said that $\underline{p}^*(t_1^*)$ is orthogonal to the tangent plane $T_1(\underline{x}_1(t_1^*))$ of the manifold $X_1(t_1^*)$, i.e.,

$$< \underline{p}^*(t_1^*),\ \underline{x} - \underline{x}_1^*(t_1^* > \equiv 0 \ , \qquad \underline{x} \in T_1(\underline{x}_1(t_1^*))$$

For the spin axis control (SACO) concept, however, there is no condition on the adjoint vector $\underline{p}^*(t_1^*)$. It will be seen later that this condition (or its absence) has a pronounced effect on the computational aspects of the control problem. The parameters ν_1, ν_2 that appear in Table 5-4 are arbitrary constants (Lagrange multipliers); they are determined so that the defining relationships for the manifold X_1 are satisfied, i.e.,

$$\underline{g}(\underline{x}) = 0$$

<u>Hamilton's canonical equations</u>

In this section, Hamilton's canonical equations for the systems being studied are provided. In general, these equations are given by

$$\begin{aligned} \dot{\underline{p}} &= - \frac{\partial H}{\partial \underline{x}} = - \left[\frac{\partial \underline{f}(\underline{x}, \underline{u}, t)}{\partial \underline{x}}\right]^T \underline{p} \\ \dot{\underline{x}} &= \underline{f}(\underline{x}, \underline{u}, t) = \frac{\partial H}{\partial \underline{p}} \end{aligned} \tag{5-46}$$

$$\text{where } H(\underline{x}, \underline{u}, \underline{p}, t) \equiv < \underline{p},\ \dot{\underline{x}} > - f_o(\underline{u}(t), t) \ .$$

For the cruise mode of the deep-space mission, the equations for the dual-spin vehicle are

Table 5-4

TARGET-DEPENDENT NECESSARY CONDITIONS FOR LOCAL OPTIMALITY FOR THE SYMMETRIC DUAL-SPIN AND SYMMETRIC SPINNING VEHICLES

NOTE: $\beta = \dfrac{J_3^R \sigma}{I_1} = \bar{r}\sigma$; $\bar{\beta} = \dfrac{I_1 - I_3}{I_1}\,\omega_3 = r\omega_3$

Control Concept	Target Set X_1	Necessary Conditions			
		Dual-Spin Vehicle		Spinning Vehicle	
		Hamiltonian, H^*	Adjoint Vector, $\underline{p}^*$	Hamiltonian, H^*	Adjoint Vector, $\underline{p}^*$
AMCO	$g(\underline{x})\big\vert_{t=T} = 0$ $t_1 = T$ (fixed)	None	$\underline{p}^*(T) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\beta \\ \beta & 0 \end{bmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$	None	$\underline{p}^*(T) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\bar{\beta} \\ \bar{\beta} & 0 \end{bmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$
	$g(\underline{x})\big\vert_{t=t_1} = 0$ t_1 free	$H^*(t_1^*) = 0$	$\underline{p}^*(t_1^*) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\beta \\ \beta & 0 \end{bmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$	$H^*(t_1^*) = 0$	$\underline{p}^*(t_1^*) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\bar{\beta} \\ \bar{\beta} & 0 \end{bmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$
SACO	$\underline{x}_1(t_1) = 0$ $t_1 = T$ (fixed)	None	None	None	None
	$\underline{x}_1(t_1) = 0$ t_1 free	$H^*(t_1^*) = 0$	None	$H^*(t_1^*) = 0$	None

(Q) $\underline{\dot{p}}(t) = -[A]^T \underline{p}(t)$

(L) $\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) = A\ \underline{x}(t) + B(t)\ \underline{u}(t)$

$$\text{where } A = \left[\begin{array}{cc|c} 0 & -\beta & \\ \beta & 0 & 0 \\ \hline & I & 0 \end{array}\right] \quad ; \quad \underline{x} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} \tag{5-47}$$

$$\beta = \frac{J_3^R}{I_1}\ \sigma = \bar{r}\sigma$$

$$B(t) = \left[\begin{array}{cc} c\sigma t & -s\sigma t \\ s\sigma t & c\sigma t \\ \hline & 0 \end{array}\right] \quad \text{if two jets are used}$$

$$B(t) = \begin{bmatrix} c\sigma t \\ s\sigma t \\ 0 \\ 0 \end{bmatrix} \quad \text{if one jet is used}$$

For the spinning symmetric vehicle, the equations are

(Q) $\underline{\dot{p}}(t) = -[A]^T \underline{p}(t)$

(L) $\underline{\dot{x}}(t) = A\ \underline{x}(t) + B\ \underline{u}(t)$

where

A, B, and $\underline{x}$ are given by

$$A = \left[\begin{array}{cc|cc} 0 & r\omega_3 & & \\ -r\omega_3 & 0 & & 0 \\ \hline & & 0 & \omega_3 \\ & I & -\omega_3 & 0 \end{array}\right] ; \quad \underline{x} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \end{pmatrix}$$

(5-48)

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$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{if two jets are used}$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{if one jet is used}$$

From Equations (5-47) and (5-48), it is seen that both the adjoint system ($\mathcal{Q}$) and the plant (L) are time-invariant for the symmetric spinning vehicle while in the case of the dual-spin vehicle, the adjoint system ($\mathcal{Q}$) is time-invariant but the plant (L) is time-varying.

Optimality Condition

In this section, the nature of the optimal controller for the dual-spin and spinning vehicles is determined by examining the "optimality" condition. The optimality condition, in general, is given by

$$\tilde{H}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t),\ \underline{u}^*(t)\right) = \tilde{M}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t)\right) \quad \text{a. e.}$$

$$\text{where } \tilde{M}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t)\right) \equiv \sup_{\underline{u}\in\Omega} \tilde{H}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t),\ \underline{u}(t)\right)$$

$$\tilde{H}\left(\tilde{\underline{x}}(t),\ \tilde{\underline{p}}(t),\ \underline{u}(t)\right) = H\left(\underline{x}(t),\ \underline{p}(t),\ \underline{u}(t), t\right) + p_{n+1}(t) \tag{5-49}$$

$$H\left(\underline{x}(t), \underline{p}(t), \underline{u}(t), t\right) \equiv \left< \underline{p}(t), \underline{f}\left(\underline{x}(t), \underline{u}(t), t\right)\right> - f_o\left(\underline{x}(t), \underline{u}(t), t\right)$$

It follows, therefore, that

$$\sup_{\underline{u}\in\Omega} \tilde{H}\left(\tilde{\underline{x}}^*(t),\ \tilde{\underline{p}}^*(t),\ \underline{u}(t)\right) = \sup_{\underline{u}\in\Omega} H\left(\underline{x}^*(t),\ \underline{p}^*(t),\ \underline{u}(t), t\right) \tag{5-50}$$

Hence, the optimal controller $\underline{u}^*(t)$ must be such that it maximizes the Hamiltonian. As shown previously, the function $\underline{f}\left(\underline{x}(t), \underline{u}(t), t\right)$ is given by

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$\underline{f}\left(\underline{x}(t),\ \underline{u}(t), t\right) = A\ \underline{x}(t) + B(t)\,\underline{u}(t)$ for the dual-spin vehicle in the cruise mode

$$\underline{f}\left(\underline{x}(t),\ \underline{u}(t), t\right) = A\left(\underline{x}(t), t\right)\ \underline{x}(t) + B(t)\ \underline{u}(t) = \underline{f}\,(\underline{x}, t) + B(t)\ \underline{u}(t) \quad \text{for the dual-spin vehicle in the large-angle turn mode} \tag{5-51}$$

$\underline{f}\left(\underline{x}(t),\ \underline{u}(t)\right) = A\ \underline{x}(t) + B\ \underline{u}(t)$ for the spinning symmetric vehicle

Hence, for the most general system being discussed the Hamiltonian is given by

$$H = \langle \underline{p},\ \underline{f}(\underline{x}, t) \rangle + \langle \underline{p},\ B(t)\,\underline{u}(t) \rangle - f_o\left(\underline{x}(t),\ \underline{u}(t), t\right) \tag{5-52}$$

and $\underline{u}^*(t)$ must be such that

$$H^*\left(\underline{u}^*(t),\ \underline{p}^*(t),\ \underline{x}^*(t), t\right) = \sup_{\underline{u}(t)\in\Omega}\left[\langle \underline{p}^*(t),\ B(t)\,\underline{u}(t) \rangle - f_o\left(\underline{x}^*(t), \underline{u}(t), t\right)\right] \tag{5-53}$$

That is, the optimal controller $\underline{u}^*(t)$ is that particular $\underline{u}(t) \in \Omega$ which maximizes the part of the Hamiltonian which is a function of $\underline{u}(t)$. Equation (5-53) indicates that the nature of the optimal controller is dependent on the control restraint set Ω and the function f_o.

In obtaining the fuel-optimal solution, it is often advantageous to examine the time-optimal solution as well. This is true because the fuel-optimal solution does not exist unless the final time involved in the fuel-optimization problem (T for the fixed-time problem and t_1^* for the free-time problem) is greater than the time-optimal solution t^*. In addition, in many cases the most appropriate cost functional is neither time nor fuel but a combination of both. For these reasons, the optimality condition will be examined for the cases in which (see Table 5-5)

$f_o\left(\underline{u}(t), t\right) \equiv 1$, time-optimal problem

$f_o\left(\underline{u}(t), t\right) = h\left(\underline{u}(t), t\right)$, fuel-optimal problem

$f_o\left(\underline{u}(t), t\right) = k + h\left(\underline{u}(t), t\right)$, combination time-fuel optimal problem

Table 5-5

NATURE OF THE OPTIMAL CONTROLLER† (OBTAINED FROM THE OPTIMALITY CONDITION)

Control restraint set Ω	Integrand $f_o\left(\underline{u}(t), t\right)$ of cost functional	Optimal Controller
$\Omega = \{\underline{u}(t) : \lvert u_j(t)\rvert \le 1 \quad j=1,\ldots m\}$	$f_o = \sum_j \lvert u_j(t)\rvert$	$\underline{u}^*(t) = \mathrm{DEZ}\,\{B^T(t)\,\underline{p}^*(t)\}$, fuel-optimal
	$f_o = 1$	$\underline{u}^*(t) = \mathrm{SGN}\,\{B^T(t)\,\underline{p}^*(t)\}$, time-optimal
	$f_o = K = \sum_j \lvert u_j(t)\rvert$	$\underline{u}^*(t) = \mathrm{DEZ}\,\{B^T(t)\,\underline{p}^*(t)\}$, combination of fuel and time
$\Omega = \{\underline{u}(t) : 0 \le u_j(t) \le 1 \quad j=1,2,\ldots m\}$	$f_o = \sum_j u_j(t)$	$\underline{u}^*(t) = \mathrm{HEV}\,\{B^T(t)\,\underline{p}^*(t) - \underline{e}\}$, fuel-optimal
	$f_o = 1$	$\underline{u}^*(t) = \mathrm{HEV}\,\{B^T(t)\,\underline{p}^*(t)\}$, time-optimal
	$f_o = K + \sum_j \lvert u_j(t)\rvert$	$\underline{u}^*(t) = \mathrm{HEV}\,\{B^T(t)\,\underline{p}^*(t) - \underline{e}\}$, combination of fuel and time

†The plant is

(S) $\quad \dot{\underline{x}} = \underline{f}(\underline{x}, t) + B(t)\,\underline{u}(t)$

and the cost functional is

$$J(\underline{u}) = \int_0^{t_1} f_o\left(\underline{u}(t), t\right)dt$$

Magnitude-Limited Case (two-way jet)

The control restraint set Ω for the various types of jets has already been discussed (see Chapter 3). For the two-way jet the control restraint set is

$$\Omega = \{\underline{u} : |u_j(t)| \leq 1 \quad \forall j\} \tag{5-54}$$

The nature of the optimal controller for the functions $f_o\left(\underline{u}(t), t\right)$ of interest is discussed in this section. The optimal controller must be chosen to maximize the expression

$$\overline{H} = < \underline{p}^*(t), B(t)\ \underline{u}(t) > - f_o\left(\underline{u}(t), t\right) \tag{5-55}$$

where $\overline{H}$ refers to the part of H that is a function of $\underline{u}$.

Fuel-Optimal Problem

The optimality condition is examined below for the case in which

$$f_o\left(\underline{u}(t), t\right) = \sum_j^m |u_j(t)|$$

The term $B(t)\ \underline{u}(t)$ can be conveniently written as

$$B(t)\ \underline{u}(t) = \sum_j u_j\ \underline{b}_j \tag{5-56}$$

where $\underline{b}_j$ refers to the j^{th} column of B(t).

Substituting Equation (5-56) into Equation (5-55) yields

$$\begin{aligned} \overline{H} &= < \underline{p}^*(t), \sum_j u_j\ \underline{b}_j > - \sum_j |u_j(t)| \\ &= \sum_j \left[u_j < \underline{p}^*(t), \underline{b}_j > - |u_j(t)|\right] \end{aligned} \tag{5-57}$$

By defining the scalar product $< \underline{p}^*(t), \underline{b}_j >$ as $q_j^*(t)$, it follows that

$$\underline{q}^*(t) = B^T(t)\,\underline{p}^*(t)$$

and that

$$\overline{H} = \sum_j \left[u_j(t)\,q_j^*(t) - |u_j(t)|\right] \tag{5-58}$$

The function $\overline{H}$ is maximized relative to $u_j(t)$ if

$$\begin{aligned} u_j^*(t) &= 0 \quad \text{for } |q_j^*(t)| \le 1 \\ u_j^*(t) &= -1 \quad \text{for } q_j^*(t) < -1 \qquad j=1,\ldots m \\ u_j^*(t) &= 1 \quad \text{for } q_j^*(t) > 1 \end{aligned} \tag{5-59}$$

The optimal controller $\underline{u}^*(t)$ is thus given by (see Figure 5.1)

$$\underline{u}^*(t) = \mathrm{DEZ}\{\underline{q}^*(t)\} = \begin{pmatrix} \mathrm{dez}\ q_1^*(t) \\ \mathrm{dez}\ q_2^*(t) \\ \vdots \\ \mathrm{dez}\ q_m^*(t) \end{pmatrix} = \begin{pmatrix} \mathrm{dez} < \underline{b}_1(t), \underline{p}^*(t) > \\ \vdots \\ \mathrm{dez} < \underline{b}_m(t), \underline{p}^*(t) > \end{pmatrix} \tag{5-60}$$

$$= \mathrm{DEZ}\{B^T(t)\,\underline{p}^*(t)\}$$

$$\text{where dez } q_i^*(t) = \begin{cases} 0 & \text{for } |q_i^*(t)| < 1 \\ 1 & \text{for } q_i^*(t) > 1 \\ -1 & \text{for } q_i^*(t) < -1 \end{cases}$$

Equation (5-60) is general in that it applies to nonlinear systems which are linear in $\underline{u}(t)$ and to all linear systems for which

$$\Omega = \{\underline{u} : |u_j(t)| \le 1 \ \forall j\}$$

$$f_o(\underline{x}, \underline{u}, t) = f_o(\underline{u}) = \sum_j |u_j(t)|$$

Time-Optimal Case

In this section the nature of the time-optimal controller is determined. The time-optimal controller is actually a special case of the fuel-optimal problem. Using the relations

$$f_o\left(\underline{u}(t), t\right) = 1$$

$$B(t)\ \underline{u}(t) = \sum_j u_j \underline{b}_j$$

in Equation (5-57) yields

$$\overline{H} = \sum_j u_j < \underline{p}^*(t),\ \underline{b}_j > = \sum_j u_j(t)\ q_j^*(t) \tag{5-61}$$

The $u_j(t)$ which maximizes $\overline{H}$ is given by

$$u_j^*(t) = \mathrm{sgn}\left\{q_j^*(t)\right\} \qquad j = 1, \ldots m \tag{5-62}$$

The optimal controller $\underline{u}^*(t)$ is thus given by (see Figure 5-1)

$$\underline{u}^*(t) = \mathrm{SGN}\{B^T(t)\ \underline{p}^*(t)\} = \left\{\begin{bmatrix} \mathrm{sgn} < \underline{b}_1(t),\ \underline{p}^*(t) > \\ \mathrm{sgn} < \underline{b}_2(t),\ \underline{p}^*(t) > \\ \vdots \\ \mathrm{sgn} < \underline{b}_m(t),\ \underline{p}^*(t) > \end{bmatrix}\right. \tag{5-63}$$

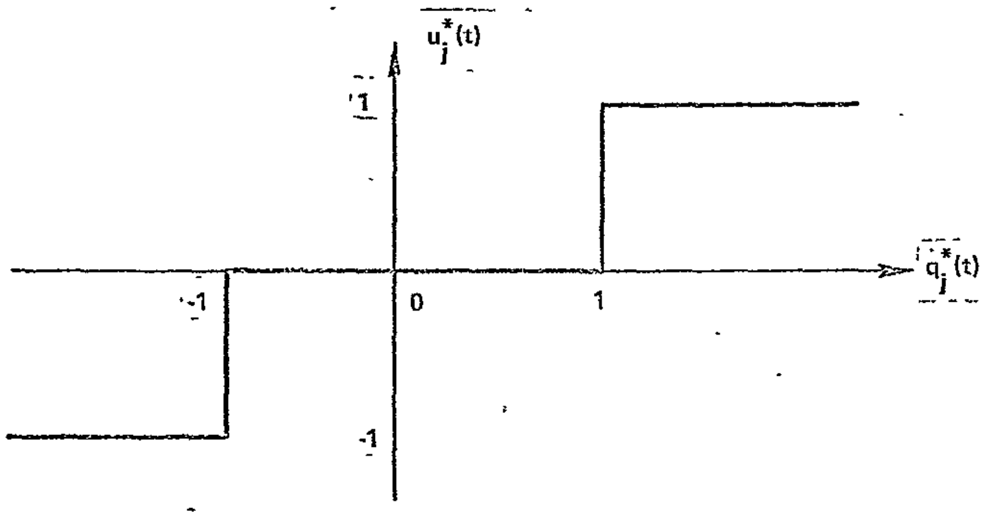
Combination Time-Fuel Problem

For the case in which the integrand of the cost functional is a combination of fuel and time, the function

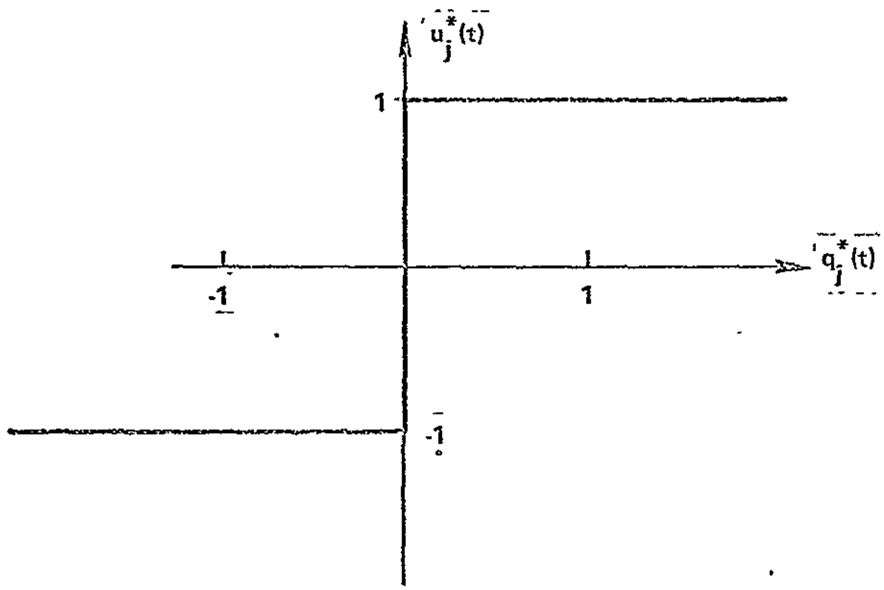
$f_o\left(\underline{u}(t)\,;\, t\right)$ is given by

$$f_o\left(\underline{u}(t), t\right) = K + \sum_j |u_j(t)| \tag{5-64}$$

The shape of the optimal controller for this case is the same as that for the fuel-optimal problem since $\overline{H}$ is the same for the two cases. In this case, the optimal solution is the one which minimizes



(a) $u_j^*(t) = \text{dez}\, q_j^*(t)$, fuel—optimal problem



(b) $u_j^*(t) = \text{sgn}\, q_j^*(t)$, time—optimal problem

Figure 5-1 The Function $u_j^*(t)$ for Fuel and Time Optimal Problems in which Two—Way Jets are Used

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$$J(\underline{u}) = k\, t_1^* + F \quad , \tag{5-65}$$

$$\text{where } F = \int_{t_o}^{t_1} \sum_j K_j \left| u_j(t) \right| dt$$

k = some positive constant

t_1^* = final time t_1 for the optimal solution

Magnitude-Limited Case (One-Way Jet, $\overset{\downarrow}{\nabla}$)

Intuitively, it is felt that since a two-way jet is essentially two one-way jets back-to-back, the optimal controller for the one-way jet case should be characterized by Equation (5-59) with the modification that

$$u_j^*(t) = 0 \quad \text{for} \quad q_j^*(t) < -1$$

It is shown below that this is indeed the case.

For the magnitude-limited one-way jet case, the control restraint set Ω is

$$\Omega = \left\{ \underline{u}(t) : \ 0 \le u_j(t) \le 1 \qquad j = 1, 2, \ldots m \right\} \tag{5-66}$$

The functions $f_o\left(\underline{u}(t), t\right)$ corresponding to the fuel-optimal, the time-optimal, and the optimal for a combination of fuel and time for this case are given by

$$f_o\left(\underline{u}(t), t\right) = \sum_j u_j(t) \qquad , \text{ fuel-optimal}$$

$$f_o\left(\underline{u}(t), t\right) = 1 \qquad , \text{ time-optimal}$$

$$f_o\left(\underline{u}(t), t\right) = k + \sum_j u_j(t) \qquad , \text{ combination of fuel and time}$$

Fuel-Optimal Case

The portion of the Hamiltonian that depends on $\underline{u}(t)$ is immediately obtained from Equation (5-55) and is given by

$$\overline{H} = \sum_j \left[u_j(t)\, q_j^*(t) - u_j(t)\right] = \sum_j u_j(t) \left[q_j^*(t) - 1\right] \tag{5-68}$$

The $u_j(t)$ which maximizes $\overline{H}$ is given by

$$\begin{aligned} u_j^*(t) &= 0 \quad \text{for} \quad q_j^*(t) < 1 \\ u_j^*(t) &= 1 \quad \text{for} \quad q_j^*(t) > 1 \end{aligned} \tag{5-69}$$

Because of its similarity to the Heaviside function $h(t - \xi)$, the controller satisfying Equation (5.69) is designated by (see Figure 5.2)

$$\begin{aligned} \underline{u}^*(t) = \mathrm{HEV}\{\underline{q}^*(t) - \underline{e}\} &\equiv \begin{pmatrix} \mathrm{hev}[<\underline{b}_1(t),\ \underline{p}^*(t)> - 1] \\ \vdots \\ \vdots \\ \mathrm{hev}[<\underline{b}_m(t),\ \underline{p}^*(t)> - 1] \end{pmatrix} \\ &= \mathrm{HEV}\{B^T(t)\ \underline{p}^*(t) - \underline{e}\} \end{aligned} \tag{5-70}$$

$$\text{where} \quad \underline{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

The control restraint set Ω and the function f_o given by Equation (5-66) and Equation (5-67) have not previously been considered relevant to the fuel-optimal attitude control problem. Hence, the HEV function (unlike the DEZ function) introduced in this work is new. The DEZ function is used by Athans and Falb [20].

It will be seen later that the use of the control set Ω and the function f_o given by

$$\Omega = \{\underline{u}(t) : 0 \le u_j(t) \le 1 \quad \forall j\}$$

$$f_o = \sum_j u_j(t)$$

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is especially suitable for spinning spacecraft both from a practical point of view (considering such items as reliability and the number of jet firings) and from a computational point of view.

It is also noted that even though a few fuel-optimal control studies pertaining to spinning vehicles have been conducted (e. g., [11] through [14]), the invesitgators invariably and perhaps unwarily (from a practical point of view) chose the control set Ω and the function $f_o\left(\underline{u}(t)\right)$ given by

$$\Omega = \{\underline{u}(t) : |u_j(t)| \leq 1 \quad j\}$$

$$f_o\left(\underline{u}(t)\right) = \sum_j |u_j(t)|$$

Time-Optimal Case

For the case in which $f_o\left(\underline{u}(t), t\right) \equiv 1$, the portion of the Hamiltonian which is a function of $\underline{u}(t)$ is

$$\overline{H} = \sum_j u_j(t)\, q_j^*(t) \tag{5-71}$$

The function $\overline{H}$ is maximized when

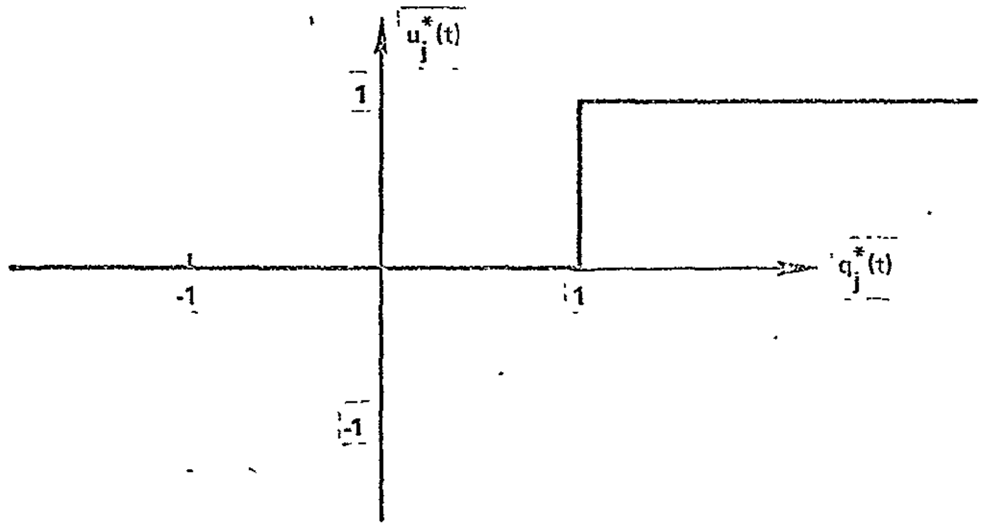
$$u_j^*(t) = \begin{cases} 0 & \text{for } q_j^*(t) < 0 \\ 1 & \text{for } q_j^*(t) > 0 \end{cases}$$

and hence, the time-optimal controller is given by (see Figure 5-2)

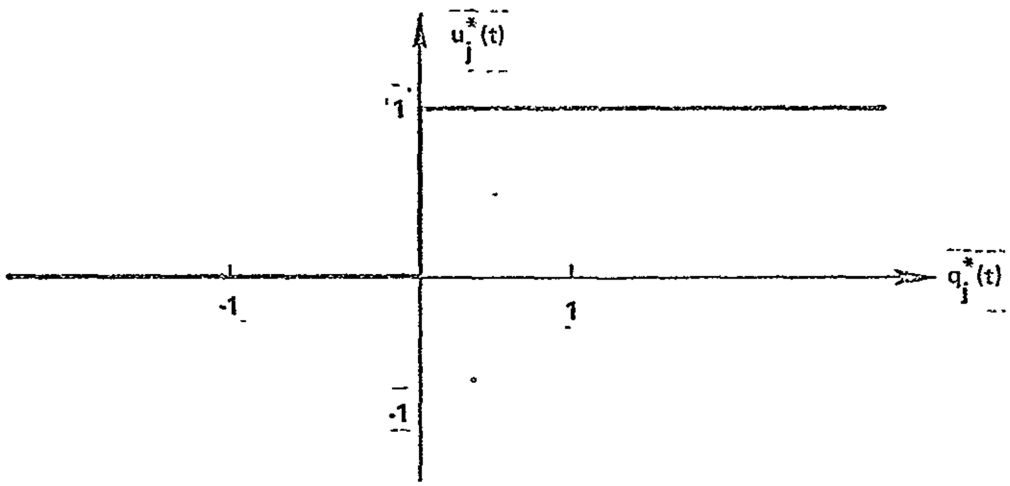
$$\underline{u}^*(t) = \text{HEV}\left\{\underline{q}^*(t)\right\} = \text{HEV}\left\{B^T(t)\, \underline{p}^*(t)\right\} \tag{5-72}$$

Norm-Limited Controller

Norm-limited controllers will not be treated in depth in this work, but it is of interest to note some of their chief characteristics. It was pointed out previously that the control restraint set Ω given by



(a) $u_j^*(t) = \text{hev}\left(q_j^*(t) - 1\right)$; fuel–optimal problem



(b) $u_j^*(t) = \text{hev} \; q_j^*(t)$; time–optimal problem

Figure 5-2 The Function $u_j^*(t)$ for Fuel and Time Optimal Problems in which One–Way Jets are Used

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$$\Omega = \left\{ \underline{u}(t) : \| \underline{u}(t) \| \leq 1 \right\} \tag{5-73}$$

is an accurate respresentation of the gimballed-jet. In addition, this constraint set could be used as an approximation of that for the magnitude-limited case. It is clear that if the constraint

$$\| \underline{u}(t) \| \leq 1$$

is satisfied, the constraint

$$| u_j(t) | \leq 1 \quad \forall j$$

is satisfied a fortiori. Hence, the norm-limited constraint set (a hypersphere) can be viewed as a smoothed magnitude-limited constraint set (a hypercube).

In this section, it is shown that the optimal controllers belonging to the smooth control restraint set discussed above are smooth. Later, the smoothness property will be discussed in relation to sufficient conditions for optimality and in relation to the existence of optimal controllers.

In this work, the norm-limited case is discussed for a time-optimal problem.

Time-Optimal Case

For the norm-limited time-optimal control problem in which

$$\Omega = \left\{ \underline{u}(t) : \| \underline{u}(t) \| \leq 1 \right\}$$
$$f_o\left(\underline{u}(t), t\right) = 1$$

The function $\overline{H}$ is given by [from Equation (5-55)]

$$\overline{H} = \langle B(t)\, \underline{u}(t),\ \underline{p}^*(t) \rangle = \langle \underline{u}(t),\ B^T(t)\, \underline{p}^*(t) \rangle$$

The controller which maximizes $\overline{H}$ is given by

$$\underline{u}^*(t) = \frac{B^T(t)\, \underline{p}^*(t)}{\left\| B^T(t)\, \underline{p}^*(t) \right\|} \qquad (5-74)$$

assuming that

$$\left\| B^T(t)\, \underline{p}^*(t) \right\| \neq 0$$

If the relation

$$\left\| B^T(t)\, \underline{p}^*(t) \right\| \equiv 0$$

is satisfied, then no information concerning $\underline{u}^*(t)$ can be obtained (the singular case). A distinguishing property of Equation (5-74) is that the components $u_j^*(t)$ are continuous functions of time and are in general, smooth functions of the state.

Controller Having a Limited Rate of Variation

In section 3.2, a smooth controller having a limited rate of variation was briefly discussed. It was stated that a constraint having such a property allows the inertia of the control system to be realistically modeled. Although this type of controller will not be studied in detail in this work, it is of interest to note that the maximum principle is applicable. A result that applies to a specific system in which a rate-limited controller is used is given below. Consider the linear autonomous process in R^n

$$\text{(L)} \qquad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t)$$

The problem is to find the optimal controller $\underline{u}^*(t)$ in R^m which steers the initial state $\underline{x}_o$ to the target state $\underline{x}_1$ in minimum time. The admissible controllers are those functions $\underline{u}(t)$ which are absolutely continuous on various finite time durations $0 \leq t \leq t_1$ which satisfy the constraints

(1) $\underline{u}(t) \subset \Omega$ where Ω is a given closed convex set containing the origin of R^m

(2) $\underline{u}(t) = \dot{\underline{u}}(t)$ are measurable and

$|v_j(t)| \leq 1 \quad \forall j = 1, \ldots m \qquad$ a. e.

(3) $\underline{u}(0) = \underline{u}(t_1) = \underline{0}$

It can be shown [84] that the optimal controller $\underline{u}^*(t)$ on $0 \leq t \leq t^*$ for this problem is such that either

$\underline{u}^*(t) \in \partial\Omega$ (the boundary of the set Ω)

or

$| \dot{u}_j(t) | = 1 \qquad \forall j = 1, \ldots m$

at almost every instant.

A controller having this property is said to be a pang-bang controller. The intervals of $0 \leq t \leq t_1^*$ for which

$| \dot{u}_j(t) | = 1$ are known as pang

and those for which

$\underline{u}^*(t) \in \partial\Omega$

are known as bang.

5.1.4 Functional Analysis Approach

In this section, elementary functional analysis is used to determine the nature of the fuel-optimal controller for a restricted class of problems of interest in this work. This section is a digression and is included to show the power and utility of functional analysis in solving certain classes of optimal control problems. Functional analysis will be used again in Chapter 6 in discussing the computational algorithms.

In many applications it is desirable to drive the transverse components of angular velocity to zero while maintaining a constant spin velocity about the third axis. Typical applications for such an objective include

(1) a manned space station in which a constant artificial gravity is established by maintaining a constant spin velocity and by driving the transverse angular velocities to zero

(2) a spinning reentry vehicle with a spin velocity appropriate for aerodynamic stability

Special Fuel-Optimal Problem

In this section, the problem of nulling the transverse components of angular velocity of a symmetric vehicle so as to minimize fuel consumption is considered. The equations governing the behavior of ω_1 and ω_2 are given by (from Equation (2-74))

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} = \begin{bmatrix} 0 & \frac{I_1 - I_3}{I_1}\omega_3 \\ -\frac{I_1 - I_3}{I_1}\omega_3 & 0 \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{5-75}$$

or

$$\underline{\dot{x}} = A\,\underline{x} + B\,\underline{u}$$

It is immediately noticed that the matrix A is skew-symmetric, that is

$$A = -A^T \tag{5-76}$$

It is now shown that any system (time-varying or time-invariant) having the property given by Equation (5-76) is norm-invariant. The dynamical system

$$(S) \qquad \underline{\dot{x}}(t) = \underline{f}(\underline{x}, t) + \underline{u}(t))$$

is said to be norm-invariant if the solution $\underline{x}(t)$ of the homogeneous system satisfies the property

$$\frac{d}{dt}\,\|\underline{x}(t)\| = 0$$

for all $\underline{x}(t)$ and all t [20]. The derivative of the norm of $\underline{x}(t)$ is given by

$$\frac{d}{dt}\ \| \underline{x}(t) \| = \frac{d}{dt} \sqrt{< \underline{x}(t), \cdot \underline{x}(t) >} = \frac{< \underline{x}(t),\ \dot{\underline{x}}(t) >}{\sqrt{< \underline{x}(t),\ \underline{x}(t) >}} = \frac{< \underline{x}(t),\ \dot{\underline{x}}(t) >}{\| \underline{x}(t) \|}$$

The homogeneous system corresponding to Equation (5-75) is

$$\text{(L)} \qquad \dot{\underline{x}}(t) = A\ \underline{x}(t) \tag{5-77}$$

The system given by Equation (5-77) is norm-invariant if and only if

$$< \underline{x}(t),\ A\ \underline{x}(t) > = 0 \tag{5-78}$$

From Equation (5-78) it follows also that

$$< A^T\ \underline{x}(t),\ \underline{x}(t) > = < \underline{x}(t),\ A^T\ \underline{x}(t) > = 0 \tag{5-79}$$

Combining Equations (5-78) and (5-79) yields

$$< (A + A^T)\ \underline{x}(t),\ \underline{x}(t) > = 0 \tag{5-80}$$

It follows, therefore, that for the norm-invariant system

$$\text{(L)} \qquad \dot{\underline{x}} = A\ \underline{x}$$

the matrix A must satisfy the relation

$$A = -A^T,$$

that is, A is skew-symmetric. In addition, the homogeneous norm-invariant system of Equation (5-75), viz.

$$\dot{\underline{x}} = A\ \underline{x}(t) \tag{5-81}$$

is self-adjoint. The adjoint system is given by

$$\dot{\underline{x}} = -A^T\ \underline{x}(t) \tag{5-82}$$

Since the matrix A is skew symmetric it follows that the adjoint system is identical to the plant and, hence, the system is self-adjoint.

The optimal controller for norm-invariant systems can be found by a direct method, viz. by using the Schwarz inequality. For the norm-invariant systems [Equation (5-75)], the norm of $\underline{x}$ satisfies the equation

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$$\frac{d}{dt}\|\underline{x}(t)\| = \frac{<\dot{\underline{x}}(t),\ \underline{x}(t)>}{\|\underline{x}(t)\|} = \frac{<A\,\underline{x}(t),\ \underline{x}(t)>}{\|\underline{x}(t)\|} +$$

$$+\frac{<\underline{u}(t),\ \underline{x}(t)>}{\|\underline{x}(t)\|} = <\underline{u}(t), \frac{\underline{x}(t)}{\|\underline{x}(t)\|}> \qquad (5\text{-}83)$$

The Schwarz inequality applied to the term $<\underline{u}(t), \frac{\underline{x}(t)}{\|\underline{x}(t)\|}>$ implies that

$$\left| <\underline{u}(t), \frac{\underline{x}(t)}{\|\underline{x}(t)\|}> \right| \le \|u(t)\| \left\| \frac{\underline{x}(t)}{\|\underline{x}(t)\|} \right\| = 1 \qquad (5\text{-}84)$$

Straightforward manipulations of Equations (5-83) and (5-84) yield the optimal controller for the problem $(L_{NI}, \Delta, \Omega, X_o, X_1, J)$ where

(L_{NI})	$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{u}(t)$, norm-invariant
Δ	class of admissible controllers consisting of the smooth functions $\underline{u}(t) \subset \Omega$
$\Omega =$	$\{\underline{u}(t) : \|\underline{u}(t)\| \le 1\}$
$X_o =$	$\{(\underline{x}, t) : \underline{x}_o = \underline{\xi}, t_o = 0\}$
$X_1 =$	$\{(\underline{x}, t) : \underline{x}(t_1) = 0, t_1 \text{ free}\}$.
$J(u) =$	$\int_o^{t_1} \|\underline{u}(t)\| dt$

Integrating Equation (5-83) yields

$$\|\underline{x}(t)\| = \|\xi\| + \int_o^t <\underline{u}(\tau), \frac{\underline{x}(\tau)}{\|\underline{x}(\tau)\|}> d\tau \qquad (5\text{-}85)$$

Evaluating Equation (5-85) at $t = t_1$ yields

$$\|\xi\| = -\int_o^{t_1} <\underline{u}(t), \frac{\underline{x}(t)}{\|\underline{x}(\tau)\|}> dt \qquad (5\text{-}86)$$

Taking the absolute value of Equation (5-86) yields

$$\left| \|\xi\| \right| = \|\xi\| = \left| -\int_0^{t_1} < \underline{u}(t), \frac{\underline{x}(t)}{\|x(t)\|} > dt \right| \leq \int_0^{t_1} \left| < \underline{u}(t), \frac{x(t)}{\|x(t)\|} > \right| dt \tag{5-87}$$

Using the Schwarz inequality in Equation (5-87) yields

$$\|\xi\| \leq \int_0^{t_1} \|\underline{u}(t)\| \left\| \frac{\underline{x}(t)}{\|\underline{x}(t)\|} \right\| dt = \int_0^{t_1} \|\underline{u}(t)\| dt = J(\underline{u}) \tag{5-88}$$

Hence, the norm of the initial state is the greatest lower bound of the cost functional, i. e.,

$$\|\xi\| = \text{glb } J(\underline{u}) \tag{5-89}$$

It follows that the optimal control is that for which

$$J(\underline{u}) = \|\xi\|$$

It is easily shown that the optimal controller is given by [20]

$$\underline{u}^*(t) = -\alpha(t) \frac{\underline{x}^*(t)}{\|x^*(t)\|} \tag{5-90}$$

where $\alpha(t)$ belongs to the set $\overline{\mathcal{Q}}$. The set $\overline{\mathcal{Q}}$ is the set of non-negative scalar functions $\alpha(t)$ defined as

$$\overline{\mathcal{Q}} = \{\alpha(t) : 0 \leq \alpha(t) \leq 1 \quad \forall t \text{ and for every } \alpha(t)$$
$$\exists\, T_\alpha \ni \int_0^{T_\alpha} \alpha(t)\, dt = \|\xi\|\}$$

Substituting Equation (5-90) into Equation (5-87) yields

$$\|\xi\| = -\int_0^{T_\alpha} -\alpha(t) < \frac{\underline{x}^*(t)}{\|\underline{x}^*(t)\|}, \frac{\underline{x}^*(t)}{\|\underline{x}^*(t)\|} > dt$$

$$\|\xi\| = \int_0^{T_\alpha} \alpha(t)\, dt = \int_0^{T_\alpha} \|\underline{u}^*(t)\| dt = J^*(\underline{u}) \tag{5-91}$$

where T_α is the time required to force $\underline{\xi}$ to $\underline{0}$ given an $\alpha(t)$.

The distinguishing features of the optimal controller specified by Equation (5-90) are that

(1) the controllers are smooth

(2) the control is oppositely directed to the state vector

(3) the control represents a feedback solution

5.2 Sufficiency Conditions

In this section, three theorems concerning sufficient conditions for optimality are stated. More detailed treatments of this subject are provided in Reference [45] through [51]. The sufficiency conditions presented here are representative of those obtainable from each of the three approaches to the optimal control problem.

Sufficient Conditions Obtained from the Maximum Principle

For an important class of systems, the maximum principle provides not only the necessary but also the sufficient conditions for optimality. The theorem presented here is due to Lee [47]. The control problem $(S, \Delta, \Omega, X_o, X_1, J)$ for which the theorem applies is now described. The system (S) is given by

$$(S) \qquad \dot{\underline{x}} = A(t)\,\underline{x} + \underline{h}\left(\underline{u}(t), t\right) \quad ,$$

the class of admissible controllers Δ is such that

> Δ = all bounded measurable m-vector functions on the fixed finite duration $t_o \leq t \leq T$ which steer the initial state $\underline{x}_o$ the target set X_1,

the control restraint set Ω is such that

$$\Omega = \text{a nonempty set} \subset R^m ,$$

the initial set X_o is such that

$$X_o = \left\{ \left(\underline{x}(t), t\right) : \underline{x}_o \text{ fixed, } t_o \text{ fixed} \right\} ,$$

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the target set X_1 is such that

$$X_1 = \text{a closed convex set}^{\dagger},$$

the cost functional $J(u)$ is given by

$$J(\underline{u}) = \int_{t_o}^{T} \left[f_o\left(\underline{x}(t), t\right) + h_o\left(\underline{u}(t), t\right)\right] dt$$

Theorem. Consider the control process in R^n

$$(S) \qquad \dot{\underline{x}} = A(t)\,\underline{x}(t) + \underline{h}(\underline{u}, t)$$

with the initial state $\underline{x}_o$ and the closed convex target set $X_1 \subset R^n$. The cost functional corresponding to an admissible controller $\underline{u}(t)$ on $t_o \leq t \leq T$ lying in the restraint at $\Omega \subset R^m$ is defined by

$$J(\underline{u}) = \int_{t_o}^{T} \left[f_o\left(\underline{x}(t), t\right) + h_o\left(\underline{u}(t), t\right)\right] dt$$

$$\text{with } \dot{x}_o(t) = f_o\left(\underline{x}(t), t\right) + h_o\left(\underline{u}(t), t\right), \quad x_o(t_o) = 0$$

The quantities $f_o\left(\underline{x}(t), t\right)$, $\frac{\partial f_o}{\partial \underline{x}}(\underline{x}(t), t)$, $h_o(\underline{u}(t), t)$, $A(t)$, and $\underline{h}(\underline{u}(t), t)$ are assumed continuous in all $(\underline{x}(t), \underline{u}(t), t)$ in R^{n+m+1}. The function $f_o(\underline{x}(t), t)$ is assumed to be convex in $\underline{x}(t)$ for each fixed $t \in [t_o, T]$. If the controller $\underline{u}^*(t)$ with response $\hat{\underline{x}}^*(t) = \left(x_o^*(t), \underline{x}^*(t)\right)$ satisfies the maximum principle, then $\underline{u}^*(t)$ is an optimal controller achieving the minimal cost

$$J(\underline{u}^*) = x_o^*(T)$$

As stated previously, the target set for the angular momentum control concept is given by

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† Note that compactness of the target set is not required for the sufficiency theorem but it is for existence.

$$X_1 = \{(\underline{x}, t) : \; g_1[\underline{x}(t_1)] = 0 \;, \; g_2[\underline{x}(t_1)] = 0\}$$

$$\text{where } \; \underline{g}[\underline{x}(t_1)] = C\,\underline{x}(t_1)$$

$$C = \text{a time-invariant matrix} \; .$$

In this case, X_1 is a linear manifold and is thus convex (every linear manifold is convex). It was stated previously that X_1 is also compact (closed and bounded). It is noted that the theorem applies only to fixed-time problems. Both fixed and free time problems are being considered in this work. The theorem is useful for both the angular momentum control concept and the spin axis control concept when

(1) the cruise mode (small angles) is being considered

(2) the final time t_1 is fixed

The theorem is not applicable to the large angle turn mode because then the plant is nonlinear in $\underline{x}$. Of course, in the problems of interest in this work since the plant is linear in $\underline{u}$ and the integrand of the cost functional is linear in $\underline{u}_j(t)$ or $|u_j(t)|$ depending on the type of jet being used, special attention must be paid to the possibility of the existence of singular optimal controllers.

Sufficient Conditions Obtained from the Calculus of Variations

In this section, a sufficiency theorem involving the calculus of variations is presented. This theorem involves the notion of the second variation of the cost functional and provides the sufficient conditions for local optimality. The notion of the second variation has no counterpart in the maximum principle. The elegant theory involving the second variation of the cost functional is important not only because it provides sufficient conditions for optimality but also because it provides a computational technique as well [85].

Previously it was stated that the cost functional $J(\underline{x})$ is said to be differentiable if the increment of $J(\underline{x})$ can be written as

$$\Delta J(\underline{x};\underline{\Delta x}) \equiv J(\underline{x}+\underline{\Delta x}) - J(\underline{x}) = \delta J(\underline{x};\underline{\Delta x}) + \epsilon_1 \|\underline{\Delta x}\|_1 \qquad (5\text{-}92)$$

where $\delta J(\underline{x};\underline{\Delta x})$ is a linear functional known as the first variation (first differential)

Analogously, the functional $J(\underline{x})$ is said to be twice differentiable if its increment can be written as

$$\begin{aligned}\Delta J(\underline{x};\underline{\Delta x}) &= J(\underline{x}+\underline{\Delta x}) - J(\underline{x})\\ &= \delta J(\underline{x};\underline{\Delta x}) + \delta^2 J(\underline{x};\underline{\Delta x}) \qquad (5\text{-}93)\\ &\quad + \epsilon \|\underline{\Delta x}\|_1^2\end{aligned}$$

where $\delta^2 J$ is a quadratic functional known as the second variation (second differential). The form of $\delta^2 J$ is easily obtained by expressing ΔJ in a TSE. Given the functional

$$J(\underline{x}) = \int_o^t f_o(\underline{x}, \underline{\dot{x}}, t) ,$$

the increment of J is given by

$$\Delta J(\underline{x};\underline{\Delta x}) = J(\underline{x}+\underline{\Delta x}) - J(\underline{x}) = \int_o^t f_o(\underline{x}+\underline{\Delta x}, \underline{\dot{x}}+\underline{\Delta \dot{x}}, t)\,dt - \int_o^t f_o(\underline{x}, \underline{\dot{x}}, t)\,dt$$

$$\begin{aligned}= \int_o^t \Big[&< \frac{\partial f_o}{\partial \underline{x}}, \underline{\Delta x} > + < \frac{\partial f_o}{\partial \underline{\dot{x}}}, \underline{\Delta \dot{x}} > + \frac{1}{2} < \underline{\Delta x}, \frac{\partial^2 f_o}{\partial \underline{x}^2} \underline{\Delta x} > \\ &+ \frac{1}{2} < \underline{\Delta \dot{x}}, \frac{\partial^2 f_o}{\partial \underline{\dot{x}}^2} \underline{\Delta \dot{x}} > + < \underline{\Delta x}, \frac{\partial^2 f_o}{\partial \underline{x} \partial \underline{u}} \underline{\Delta u} > \qquad (5\text{-}94)\\ &+ \epsilon_1 < \underline{\Delta x}, \underline{\Delta x} > + \epsilon_2 < \underline{\Delta \dot{x}}, \underline{\Delta \dot{x}} > + \epsilon_3 < \underline{\Delta x}, \underline{\Delta \dot{x}} > \Big] dt\end{aligned}$$

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From Equations (5-93) and (5-94) the second variation is given by

$$\delta^2 J(\underline{x}, \underline{\Delta x}) = \frac{1}{2} \int_0^t \begin{pmatrix} \underline{\Delta x} \\ \underline{\Delta \dot{x}} \end{pmatrix}^T \begin{bmatrix} \dfrac{\partial^2 f_o}{\partial \underline{x}^2} & \dfrac{\partial^2 f_o}{\partial \underline{x} \partial \underline{\dot{x}}} \\ \dfrac{\partial^2 f_o}{\partial \underline{\dot{x}} \partial \underline{x}} & \dfrac{\partial^2 f_o}{\partial \underline{\dot{x}}^2} \end{bmatrix} \begin{pmatrix} \underline{\Delta x} \\ \underline{\Delta \dot{x}} \end{pmatrix} dt \tag{5-95}$$

If the functional J is given by $J(\underline{u}, \underline{x}) = \int_0^t f_o(\underline{u}, \underline{x}) dt$, the second variation $\delta^2 J$ becomes

$$\delta^2 J(\underline{x}, \underline{u}; \underline{\Delta x}, \underline{\Delta u}) = \frac{1}{2} \int_0^t \begin{pmatrix} \underline{\Delta x} \\ \underline{\Delta u} \end{pmatrix}^T \begin{bmatrix} \dfrac{\partial^2 f_o}{\partial \underline{x}^2} & \dfrac{\partial^2 f_o}{\partial \underline{x} \partial \underline{u}} \\ \dfrac{\partial^2 f_o}{\partial \underline{u} \partial \underline{x}} & \dfrac{\partial^2 f_o}{\partial \underline{u}^2} \end{bmatrix} \begin{pmatrix} \underline{\Delta x} \\ \underline{\Delta u} \end{pmatrix} dt \tag{5-96}$$

Before stating the sufficiency theorem the following easily proven lemma is stated [42].

<u>Lemma</u> A necessary condition for the functional $J(\underline{u}, \underline{x})$ to have a minimum for $\underline{x} = \underline{x}^*$, $\underline{u} = \underline{u}^*$ is that

$$\delta^2 J(\underline{x}, \underline{u}, \underline{\Delta x}, \Delta u) \geq 0$$

for $\underline{x} = \underline{x}^*$, $\underline{u} = \underline{u}^*$ and all admissible $\underline{\Delta x}$, $\underline{\Delta u}$.

This condition is clearly satisfied if the symmetric matrix

$$\begin{bmatrix} \dfrac{\partial^2 f_o}{\partial \underline{x}^2} & \dfrac{\partial^2 f_o}{\partial \underline{x} \partial \underline{u}} \\ \dfrac{\partial^2 f_o}{\partial \underline{u} \partial \underline{x}} & \dfrac{\partial^2 f_o}{\partial \underline{u}^2} \end{bmatrix}$$

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is positive semidefinite. With this definition of the notion of the second variation the sufficiency theorem can now be stated [24].

Theorem. Consider the autonomous process in R^n

$$\text{(S)} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

with initial state $\underline{x}(0) = \underline{x}_o$ and cost functional

$$J(\underline{u}, \underline{x}) = \int_o^T f_o(\underline{x}, \underline{u})\,dt$$

Assume that $f_o(\underline{x}, \underline{u})$, $\underline{f}(\underline{x}, \underline{u})$ are in C^2 in R^{n+m}. The admissible controllers are those bounded measurable functions $\underline{u}(t)$ defined on the fixed finite interval $[0, T]$ which satisfy the restraint

$$\underline{u}(t) \subset \Omega \subset R^m$$

The controller $\underline{u}^*(t)$ is optimal if $\underline{u}^*(t)$ is such that

(1) the first order necessary conditions for a local minimum are satisfied, viz.

i) $\dfrac{\partial H}{\partial \underline{u}} \left(\underline{p}^*(t), \underline{u}^*(t), \underline{x}^*(t)\right) = 0 \qquad$ a. e.

where $H \equiv -f_o(\underline{x}, \underline{u}) + \langle \underline{p}, \underline{f} \rangle$

ii) $\dot{\underline{x}} = \dfrac{\partial H}{\partial \underline{p}} \left(\underline{x}(t), \underline{p}(t), \underline{u}^*(t)\right) \quad , \quad \underline{x}(0) = \underline{x}_o$

$\dot{\underline{p}} = -\dfrac{\partial H}{\partial \underline{x}} \left(\underline{x}(t), \underline{p}(t), \underline{u}^*(t)\right), \quad \underline{p}(T) = 0$

(2) the symmetric matrix involved in the expression for the second variation is positive definite, i. e.,

$$\left[\begin{matrix} \dfrac{\partial^2 f_o}{\partial \underline{x}^2} & \dfrac{\partial^2 f_o}{\partial \underline{x} \partial \underline{u}} \\ \\ \dfrac{\partial^2 f_o}{\partial \underline{u} \partial \underline{x}} & \dfrac{\partial^2 f_o}{\partial \underline{u}^2} \end{matrix}\right]_{\substack{\underline{x} = \underline{x}^*(t) \\ \underline{u} = \underline{u}^*(t)}} \quad \text{is positive definite}$$

(3) either of the two following conditions holds along $\left(\underline{x}^*(t), \underline{u}^*(t)\right)$

(i) $\dfrac{\partial^2 f}{\partial \underline{x}^2} = \dfrac{\partial^2 f}{\partial \underline{x} \partial \underline{u}} = \dfrac{\partial^2 f}{\partial \underline{u}^2} = 0$

(ii) $\dfrac{\partial f_o}{\partial \underline{x}} = 0$

It is immediately apparent that this theorem cannot be used for the problems of interest in this work. The theorem is typically applicable for cost functions which are quadratic in both $\underline{x}$ and $\underline{u}$. In this work, the functions $f_o(\underline{u}, t)$ under consideration include

$$f_o(\underline{u}, t) = \sum_j K_j \left| u_j(t) \right|$$

$$f_o(\underline{u}, t) = \sum_j K_j u_j(t)$$

$$f_o(\underline{u}, t) = \| \underline{u} \|$$

For these functions, the following observations are apparent

(1) the matrices $\dfrac{\partial^2 f_o}{\partial \underline{x}^2} = \dfrac{\partial^2 f_o}{\partial \underline{x} \partial \underline{u}} \equiv 0$

(2) the matrix defining the second variation is not positive definite

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(3) the function $f_o(\underline{u}', t) = \sum_j K_j |u_j(t)|$ does not belong to C^2 in R^{n+m} and, hence, the operation $\frac{\partial^2 f_o}{\partial \underline{u}^2}$ cannot be performed.

It is also noted that the theorem as stated here applies only to autonomous systems for which the final time is fixed and finite.

Sufficient Conditions Obtained from Dynamic Programming

In this section, a sufficiency theorem based on dynamic programming is provided. The theorem is applicable only if a feedback solution $\underline{u}^o$ exists. Previously it was noted that if the control constraint Ω given by

$$\Omega = \{\underline{u} : \|\underline{u}\| \leq 1\}$$

were assumed appropriate, then in certain cases a smooth feedback solution exists. The advantages of having a feedback solution are well-known to the practicing control engineer and considerable effort is expended in an attempt to obtain feedback solutions. In fact, in many cases, a feedback law obtained by appropriate approximations is more useful than an exact open loop control law. The control process in R^n

$$\text{(S)} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

with restraint set $\Omega \subset R^m$, with the cost functional

$J(\underline{u}) = \int_{t_o}^{t_1} f_o\left(\underline{u}(t), \underline{x}(t), t\right) dt$, and with the Hamiltonian

$$H^o(\underline{x}, \underline{p}, t) \equiv \sup_{u \in \Omega} H(\underline{x}, \underline{p}, \underline{u}, t) \equiv H(\underline{x}, \underline{p}, \underline{u}^o\left(\underline{x}, \underline{p}, (t), t\right)$$

The close analogy between Hamilton's canonical equations and the Hamiltonian of analytical mechanics and Pontryagin's maximum principle has already been noted. A similar analogy exists between the Hamilton-Jacobi theory of analytical mechanics and Bellman's theory of dynamic programming.

The quintessence of the Hamilton Jacobi theory of classical mechanics concerns such notions as canonical transformations and their generating function [52]. The distinguishing feature of the Hamilton-Jacobi approach is that the problem of solving the entire system of canonical equations is reduced to the problem of solving one partial differential quotation. Consider the functional $J(\underline{x}) = \int_{t_o}^{t_1} f_o(\underline{x}, \dot{\underline{x}}, t)\,dt$ defined in a region R and define $\bar{S}(\underline{x}, t)$ as the functional $J(\underline{x})$ evaluated along the extremal joining the points

$$A = (t_o, \underline{x}_o)$$

$$B = (t_1, \underline{x}_1)$$

The quantity $\bar{S}$ is a singled-valued function of the points A and B. If point A is fixed and point B is variable, then $\bar{S}$ is a single-valued function of the coordinates of point B, i.e.,

$$\bar{S} = \bar{S}(t_1, \underline{x}_1) = J^*(t_1, \underline{x}_1) \tag{5-97}$$

By definition, the increment of $\bar{S}$ is given by

$$\begin{aligned} \Delta\bar{S} &= \bar{S}(t_1 + dt_1, \underline{x}_1 + \underline{\Delta x}_1) - \bar{S}(t_1, \underline{x}_1) \\ &= J(\gamma^*) - J(\gamma) \end{aligned} \tag{5-98}$$

where γ = the extremal going from the point A to the point $(t_1, \underline{x}_1)$

γ^* = the extremal going from the point A to the point $(t_1 + dt_1, \underline{x}_1 + \underline{\Delta x}_1)$

From Equation (5-98), it follows that

$$d\bar{S} = \delta J \tag{5-99}$$

The first variation δJ is given by

$$\delta J = \int_{t_o}^{t_1} \left[< \frac{\partial f_o}{\partial \underline{x}}, \underline{\Delta x} > + < \frac{\partial f_o}{\partial \underline{\dot{x}}}, \underline{\Delta \dot{x}} > \right] dt + f_o(\underline{x}, \underline{\dot{x}}, t) dt_1 \tag{5-100}$$

Simplifying Equation (5-100) yields

$$\delta J = \int_{t_o}^{t_1} < -\frac{d}{dt} \frac{\partial f_o}{\partial \underline{\dot{x}}} + \frac{\partial f_o}{\partial \underline{x}}, \underline{\Delta x} > dt +$$

$$+ < \frac{\partial f_o}{\partial \underline{\dot{x}}}, \underline{\Delta x} > \Bigg|_{t_o}^{t_1} + f_o(\underline{x}, \underline{\dot{x}}, t) \delta t_1 \tag{5-101}$$

Since the Euler-Lagrange equations

$$-\frac{d}{dt} \frac{\partial f_o}{\partial \underline{\dot{x}}} + \frac{\partial f_o}{\partial \underline{x}} = 0 \tag{5-102}$$

are satisfied for an extremal, the general variation becomes

$$\delta J = - H(\underline{x}, \underline{p}, t) \Big|_{t=t_1} \delta t_1 + < \underline{p}(t_1), \underline{\Delta x}_1 > \tag{5-103}$$

$$\text{where } H \equiv - f_o(\underline{x}, \underline{\dot{x}}, t) + < \underline{p}, \underline{\dot{x}} >$$

$$\underline{p} \equiv - \frac{\partial f_o}{\partial \underline{\dot{x}}} .$$

From Fquation (5-97), an additional expression for $d\bar{S}$ is

$$d\bar{S} = \frac{\partial \bar{S}}{\partial t_1} \delta t_1 + < \frac{\partial \bar{S}}{\partial \underline{x}_1}, \underline{\Delta x}_1 > \tag{5-104}$$

Equating the results of Equations (5-103) and (5-104) yields

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$$\frac{\partial \bar{S}}{\partial t_1} = - H(t_1)$$

$$\frac{\partial \bar{S}}{\partial \underline{x}_1} = \underline{p}(t_1) \qquad (5\text{-}105)$$

It follows from Equation (5-120) that

$$\frac{\partial \bar{S}}{\partial t_1} + H\left(t_1, \underline{x}_1, \frac{\partial \bar{S}}{\partial \underline{x}_1}\right) = 0 \qquad (5\text{-}106)$$

where $\bar{S}$ and H are expressed in terms of the coordinates of point B. In classical mechanics, the partial differential equation

$$\frac{\partial \bar{S}}{\partial t} + H\left(t, \underline{x}, \frac{\partial \bar{S}}{\partial \underline{x}}\right) = 0 \qquad (5\text{-}107)$$

is known as the Hamilton-Jacobi equation and $\bar{S}$ is known as the generating function.

The sufficiency theorem related to the Hamilton-Jacobi equation may now be stated [24]

Theorem. Consider the control process in R^n

(S) $\quad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$

with initial state $\underline{x}_o$ and target set $X_1 \subset R^n$. The admissible controllers are all bounded measurable functions $\underline{u}(t)$ on $[t_o, T]$ with values in the control restraint set $\Omega \subset R^m$ which steer the response $\underline{x}(t)$ from $\underline{x}(t_o) = \underline{x}_o$ to $\underline{x}(T) \in X_1$. The cost functional is

$$J(\underline{u}) = \psi\left(\underline{x}(T)\right) + \int_{t_o}^{T} f_o\left(\underline{x}(t), \underline{u}(t), t\right) dt$$

where ψ $\underline{f}$, f_o are in C^1 in all arguments.

Assume that there exists a feedback control $\underline{u}^o(\underline{x}, \underline{p}, t)$ in C^1 in R^{n+n+1} such that

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$$H^o(\underline{x}, \underline{p}, t) \equiv H\left(\underline{x}, \underline{p}, \underline{u}^o(\underline{x}, \underline{p}, t), t\right)$$

If $\overline{S}(\underline{x}, t) \in C^2$ for $\underline{x} \in R^n$, $t \leqq T$ is a solution of the H-J equation

$$\frac{\partial \overline{S}}{\partial t} + H^o\left(\underline{x}, \frac{\partial \overline{S}}{\partial \underline{x}}, t\right) = 0$$

$$\overline{S}(\underline{x}, T) = -\psi(\underline{x}) \qquad \text{for} \quad \underline{x} \in X_1$$

then the control law $\underline{\overline{u}}(\underline{x}, t) = \underline{u}^o\left(\underline{x}, \frac{\partial \overline{S}}{\partial \underline{x}}, t\right)$ which determines a a response $\overline{\underline{x}}(t)$ - steering $(\underline{x}_o, t_o)$ to the target set at $t = T$ is an optimal controller provided it lies in Ω and has cost

$$J\left(\underline{\overline{u}}(t)\right) = -\overline{S}(\underline{x}_o, t_o)$$

Before commenting on the applicability of this theorem for the problems of concern in this work, a theorem concerning a restricted class of control problems for which a feedback control can be determined is given [24]. Consider the minimal-time problem of steering a given initial state $\underline{x}_o$ to the target set $X_1 \subset R^n$. Assume that the control restraint set $\Omega \subset R^n$ is compact and is diffeomorphic† for each fixed $\underline{x} \in R^n$ with the velocity set

$$V = \{\underline{f}(\underline{x}, \underline{u}) \mid \underline{u} \in \Omega\}$$

Theorem. Consider the autonomous control process in R^n

$$\text{(S)} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \ \text{ in } C^1 \text{ in } R^{n+n}$$

with compact restraint set $\Omega \subset R^n$ and cost functional $J(u) = \int_{t_o}^{t_1} dt$. For each $\underline{x} \in R^n$, the velocity set is defined as

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†The map

$$\Omega \rightarrow V: \ \underline{u} \rightarrow \underline{f}(\underline{x}, \underline{u})$$

is said to be diffeomorphic if it is 1-to-1 onto V and is in C^1 with a nonvanishing Jacobian determinant.

$$V(x) = \{ \underline{f}(\underline{x}, \underline{u}) | \underline{u} \in \Omega \}$$

If for each $\underline{x} \in R^n$

(i) there is a diffeomorphism of Ω onto V

$$\Omega \rightarrow V : \underline{u} \rightarrow \underline{f}(\underline{x}, \underline{u})$$

(ii) $V(x)$ is a strictly convex body in R^n with a smooth C^2 boundary manifold ∂V having positive Gaussian curvature,

then there exists a smooth feedback control

$$\underline{u}^o(\underline{p}, \underline{x}) \text{ in } C^1 \quad \text{for } \underline{p} \neq 0, \ \underline{x} \in R^n$$

describing the unique point in Ω where

$$H^o(\underline{p}, \underline{x}) = \sup_{\underline{u} \in \Omega} [-1 + < \underline{p}, \underline{f}(\underline{x}, u) >]$$

This sufficiency theorem does not apply to those problems being considered in this work for which the cost functional is given by

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j |u_j|$$

or

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j u_j(t) dt$$

This follows because the extremal controller $\underline{u}^*(t)$ is neither a feedback solution nor does it belong to C^1. However, if the control restraint set Ω given by

$$\Omega = \{ \underline{u} : \| \underline{u} \| \leq 1 \} \tag{5-108}$$

were used, the theorem could conceivably be used. Recall that the smooth control constraint given by Equation (5-108) is appropriate

(1) when a gimballed-jet is used

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(2) as an approximation of the restraint set

$$\Omega = \{ \underline{u}(t) : | u_j(t) | \leq 1 \quad \forall j \}$$

Besides providing a sufficient condition for optimality the theorem demonstrates a geometric property that was conspicuous neither in the calculus of variations approach nor in the maximum principle. This geometric property is that

(1) the adjoint vector $\underline{p}(t)$ corresponding to an optimal trajectory is the gradient of the optimal performance index (cost functional), i.e.,

$$\underline{p}^*(t) = \frac{\partial J^*}{\partial \underline{x}^*}$$

(2) the optimal Hamiltonian H^* is equal to the negative time rate of change of the optimal performance index, i.e.,

$$H^* = -\frac{\partial J^*}{\partial t}$$

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Section 6

COMPUTATIONAL ALGORITHMS

In this chapter, the computational algorithms that could be used for determining the fuel-optimal controller are categorized (see Table 6-2). A few of the algorithms considered most suitable for the problems of interest in this work are briefly described. It is emphasized, that a thorough treatment of this topic is not an objective of this dissertation. It is also emphasized that the algorithm to be selected in this work is not necessarily the most suitable for an arbitrary class of fuel-optimal problems but is suitable for the particular class of problems of interest. Almost all the algorithms mentioned in this chapter are potential candidates for the application at hand; however, appropriate modifications would have to be made to some of them.

There are two types of optimization problems that are commonly encountered, these types include

(1) the general optimization problem

(2) the parameter optimization problem.

In the general optimization problem, a nonlinear two-point boundary value problem must be solved. In the unconstrained parameter optimization problem, the task is to determine the values of say m parameters which minimize some appropriately chosen performance index.

6.1 Parameter Optimization

A class of optimization problems of great practical importance and of relevance in this work is that associated with parameter optimization. In this type of problem, the task is to determine the values of say m parameters which minimize some appropriately chosen performance index. This task is considerably simpler than the general optimization problem. However, many of the computational techniques used for this class of problems can be modified and used for the general optimization problem. Some of the techniques which have proven worthiness in attacking parameter optimization problems include

(1) relaxation search [53]
(2) random search [54]
(3) direct climbing [55].

Of these techniques, the category that is currently used most frequently is that associated with direct-climbing. In the direct climbing technique, an n-dimensional search problem is converted into a series of unidimensional searches. Frequently employed search procedures include the direct elimination and the polynomial approximation categories. The techniques belonging to the direct elimination category include [55]

(1) dichotomous search
(2) Fibonacci search
(3) Golden Section search.

The climbing techniques are usually classified as

(1) first-order gradient
(2) second-order gradient.

The most common technique belonging to the first-order gradient classification is the well-known method of steepest ascent (descent). The second-order gradient techniques overcome some of the convergence problems associated with the first-order gradient technique. Algorithms belonging to the second-order gradient technique include

(1) PARTAN [56]

(2) FLETCHER-POWELL DESCENT METHOD [57]

(3) ACCELERATED GRADIENT METHOD [58]

(4) HESTENES-STIEFEL CONJUGATE GRADIENT [59]

(5) DAVIDON VARIABLE METRIC [60].

Of these, the most promising ones are (4) and (5).

In many practical applications, it is desirable not only to determine the optimal controller but also to determine the optimal values of certain system parameters. Hence, the parameter optimization problem is coupled with the general optimization problem in this case. According to Cicala [61], it is frequently convenient to handle these parameters as initial conditions. That is, the parameters e_i are characterized as initial values of additional state variables x_i defined as

$$\dot{x}_i = 0$$

$$x_i(t_o) = e_i \qquad i = n+1, \ldots\ldots$$

6.2 General Optimization Problem

The general optimization problem is defined in terms of the nonlinear plant (S)

$$(S) \qquad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \ ;$$

the cost functional $J(u)$

$$J(\underline{u}) = \psi\left(\underline{x}(t_1), t_1\right) + \int_{t_o}^{t_1} f_o(\underline{x}, \underline{u}, t)\,dt \ ;$$

the target set X_1 associated with either a fixed-endpoint, a free-endpoint, or a constrained-endpoint and the final time t_1 (fixed or free), the initial set X_o, and the control restraint set Ω. In general, both the state variables and the control variables are constrained. In the problems of interest in this work, however, only the control

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variables are constrained. An important consideration affecting the selection of the algorithm to be used in this work is the relative ease in which the constraints on the control variables are handled in the various algorithms.

6.2.1 Classification of Computational Methods for the General Optimization Problem

Although the various computational algorithms for the general optimization problems do not fall automatically into distinct categories, nevertheless, it is convenient to classify them (see Table 6.2). Frequently, the computational algorithms are categorized as [63].

(1) direct methods

(2) indirect methods.

In the direct methods, the equations of motion and the appropriate terminal conditions are used as the starting point and an attempt is made to maximize or minimize the cost functional without using the necessary conditions.

In the indirect methods, the necessary conditions for optimality are used as the starting point and an attempt is made to satisfy these conditions by using an iterative approach. Advantages and disadvantages of these techniques (based on Reference [62], [63], and [64]) are listed in Table 6.3.

In the general nonlinear two-point boundary-value problem, the task is to find

(a) the n state variables $\underline{x}(t)$

(b) the n influence functions $\underline{p}(t)$ (sometimes called the adjoint or costate variables)

(c) the m control variables $\underline{u}(t)$

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to satisfy simultaneously the

(1) the n state equations involving $\underline{x}$ and $\underline{u}$

(2) the n adjoint equations involving, $\underline{p}$, $\underline{x}$, and $\underline{u}$

(3) the m optimality conditions involving $\underline{p}$, $\underline{x}$, and $\underline{u}$

(4) the boundary conditions involving $\underline{x}$ and $\underline{p}$.

A characteristic which distinguishes the various algorithms pertains to the equations which the nominal solution satisfies and the equations which are iterated on (see Table 6. 1).

Direct Methods

The direct methods have been applied successfully to many practical problems. As will be seen later, the same statement cannot be made for the indirect methods. In 1960 Kelley [65] applied the direct method to a control problem; he called his method the Gradient Method. Since then several modifications have been suggested to cope with the inherent disadvantage of extremely slow convergence in the neighborhood of an optimal solution (see, e. g., Reference [66] and [67]). The implementation of the gradient technique has varied widely because the proper step size in the control space is not well defined. The gradient direction, however, is well defined. Because of this arbitrariness in the implementation of the gradient method, considerable care is required in selecting the proper control step size to avoid violating the linearity constraints imposed on the problem. This suggests that there is a certain amount of art involved in the successful application of these techniques. Indeed, this statement applies to all known computational techniques for solving optimization problems.

The direct methods are categorized by Bryson and Ho [64] as

(1) first order gradient methods (method of steepest descent)

(2) second order gradient method

Table 6-1

ITERATIVE NATURE OF COMPUTATIONAL TECHNIQUES

Computational Technique	Equations Nominal Solution Satisfied	Equations Iterated On
Direct Methods		
Gradient	State equations	Boundary Conditions
Second Variation	Adjoint equations	Optimality Conditions
Indirect Methods		
Classical Newton-Raphson	State Equations Adjoint Equations Optimality Conditions	Boundary Conditions
Generalized Newton-Raphson (Quasilinearization)	Boundary Conditions Optimality Conditions	State Equations Adjoint Equations

(3) second variation method

(4) conjugate-gradient method.

The gradient methods were developed to surmount the "initial guess" problem associated with the classical Newton-Raphson technique to be discussed later. First-order gradient methods reportedly show great improvement in the first few iterations, but have poor convergence characteristics as the optimal solution is reached. Second order gradient methods have improved convergence characteristics as the optimal solution is approached, but may have starting difficulties associated with choosing a convex nominal solution.

The basis of the steepest descent or the first order gradient technique is to determine the estimates of the control variables $\underline{u}(t)$ which minimize the cost functional. The previous estimate $\underline{u}^i$ is updated according to

$$\underline{u}^{i+1} = \underline{u}^i - k \frac{\partial J(\underline{u}^i)}{\partial \underline{u}}$$

where k is some small positive constant. The min-H algorithms are modified first-order gradient methods which were developed to improve the convergence characteristics, included among these are

(1) Halkin's method of convex ascent [68]

(2) Gottlieb's min-H strategy [69].

In Reference [64], a second order gradient technique in which both $\frac{\partial J}{\partial \underline{u}}$ and $\left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1}$ must be computed is discussed. The estimate $\underline{u}^i$ is updated according to

$$\underline{u}^{i+1} = \underline{u}^i - \left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1} \frac{\partial J}{\partial \underline{u}}(\underline{u}^i)$$

A method related to the second order gradient method known as the method of second variation [85] is based on the calculus of variations. A significant advantage of this method is that the step size is automatically determined, thus eliminating the independent search procedure needed in the gradient method. An additional advantage is that the penalty functions (linear or quadratic depending on the problem) associated with the terminal conditions are not needed in the final stages of the computational procedure. Thus, the undetermined constants associated with the penalty function terms are eliminated.

The conjugate-gradient method [70] attempts to combine the advantages of both the first-order gradient and the second-order gradient methods. Initially, the algorithm behaves like a first-order method and as the iteration progresses, it behaves like a second-order method. One of its chief advantages is that it is not necessary to compute $\left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1}$. Fundamental to the conjugate-gradient method are such ideas as

(1) the conjugate property of a sequence of directions $n_1, n_2, \ldots, n_n$ relative to $\frac{\partial^2 J}{\partial \underline{u}^2}$, that is

$$< n_i, \frac{\partial^2 J}{\partial \underline{u}^2} \; n_j > = 0^\circ \qquad i \neq j$$

(2) determination of the optimum in each of the conjugate directions n_i by making a sequence of one-dimensional searches.

Since this technique has not been extended to constrained control problems, it will not be discussed further.

It is expected, however, that this extension will be made because the method appears very promising.

Table 6-2

CLASSIFICATION OF COMPUTATIONAL METHODS FOR GENERAL OPTIMIZATION PROBLEMS

Computational Technique	Remarks
Direct Methods	
• First-order gradient or method of steepest descent (ascent)	• Developed to overcome initial guess problem associated with classical indirect method • Shows great improvement in the first few iterations • Update estimate of $\underline{u}^{(i)}$ by $\underline{u}^{i+1} = \underline{u}^{(i)} - k \frac{\partial J}{\partial \underline{u}} (\underline{u}^{i})$
Min-H algorithms	
Gottlieb's min-H Strategy	• Gottlieb's method seeks to satisfy optimality conditions
Halkin's method of convex ascent	• Improved convergence characteristics near optimal solution • Require nominal solution to be convex, i.e., $\frac{\partial^2 H}{\partial \underline{u}^2} > 0$ for minimization problem
• Second-order gradient	• Improved convergence characteristics near optimal solution • Update estimate $\underline{u}^{(i)}$ by $\underline{u}^{(i)} = \underline{u}^{(i)} - \left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1} \frac{\partial J}{\partial \underline{u}} (\underline{u}^{(i)})$

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Table 6-2 (Continued)

Computational Technique	Remarks
Direct Methods	
• Second variation method	• Developed to improve iteration technique and eliminate shortcomings of gradient method • Based on theory of second variation of the calculus of variations • Find $\Delta \underline{u}^{(i)}$ which minimizes $\Delta J = \delta J + \delta^2 J$
• Conjugate-gradient	• Combines advantages of first-order and second-order gradient methods • Initially behaves like a first-order method, as iteration proceeds it behaves like a second-order method • Not necessary to compute $\left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1}$ • Currently can be applied to unconstrained control problems but possibly can be extended for general problems

Table 6-2 (Continued)

Computational Method	Remarks
Indirect	
Perturbation Methods	
Method of Perturbation Functions	
• Classical Newton-Raphson	• Difficult to guess appropriate initial values for the adjoint variables • Success depends on the dimension of the problem • Terminal conditions are very sensitive to variations in the initial adjoint variables • Trajectories are determined by integrating nonlinear equations of motion
Method of Adjoint Functions	• Same as method of perturbation functions except the system of equations adjoint to the system equations are integrated backwards
Quasilinearization Methods	• Developed to overcome problems associated with the classical Newton-Raphson technique • Rapid convergence near the optimum • Succession of nonhomogeneous linear two-point boundary value problems are solved until state and adjoint equations are satisfied

Indirect Methods

Indirect methods have not been used as frequently for solving optimal control problems as the direct methods, despite the fact that they were introduced earlier. Hestenes [71], as early as 1949, applied a calculus of variations formulation to the study of time-optimal solutions to the fixed endpoint problem. The differential variations method proposed by Hestenes for generating the numerical solutions is considered the forerunner of the recently popularized quasilinearization methods. The lack of success of the classical indirect method (Newton-Raphson) is attributed to the sensitivity of the terminal conditions to variations in the initial adjoint variables.

The indirect methods are categorized [63] as

(1) perturbation methods

(2) quasilinearization methods.

In the perturbation methods, the reference trajectory is generated by integrating the nonlinear differential equations of motion. In the quasilinearization methods, the linearized differential equations of motion are used to generate the reference trajectory. The perturbation methods are further subdivided as

(1) method of perturbation functions (classical Newton-Raphson)

(2) method of adjoint functions.

The main difference between these two perturbation methods is that in the method of adjoint functions, the set of equations adjoint to the system equations (Hamilton's canonical equations) is integrated backwards while in the method of perturbation functions, the system equations are integrated forwards.

The quasilinearization methods include several methods, which although essentially the same, are known by various names; included in this category are

(1) Hestenes' method of differential variations [71]

(2) Bellman and Kalaba's quasilinearization [72]

(3) McGill and Kenneth's generalized Newton-Raphson [73].

Kalaba [74] studied the convergence characteristics of the fixed end condition problem from a theoretical point of view. Long [75], Conrad [76] and Lewallan [77] have extended the generalized Newton-Raphson method to handle variable final time problems. In addition, Lewallen [77] extended the generalized Newton-Raphson technique so that it can handle general terminal conditions.

In the quasilinearization methods, the reference solution is obtained by integrating the linearized form of the system equations. The coefficients used to generate a new reference trajectory are obtained from the previous reference trajectory. Under appropriate conditions, the successive solution of the linearized equations converges to the solution of the original set of nonlinear equations. Being linear, the boundary conditions can be satisfied on each iteration. Note, however, that the optimality condition $\frac{\partial H}{\partial \underline{u}} = \underline{0}$ is satisfied only when convergence occurs.

Although the quasilinearization methods have not been extensively used, all indications are that the methods appear promising.

6.2.2 Brief Description of Most Suitable Algorithms for the Problems of Interest in This Work

In this section, brief descriptions of the method of steepest descent, the generalized Newton-Raphson method, and the classical Newton-Raphson techniques are provided. It is felt that these techniques are the most appropriate for the problems of interest in this work.

Table 6-3

ADVANTAGES AND DISADVANTAGES OF VARIOUS COMPUTATIONAL ALGORITHMS

Computation Technique	Advantages	Disadvantages
Direct		
First-order gradient	• Conceptually simple • Easy to program • Seeks out relative minima rather than stationary solutions • Control constraints are easily implemented • Useful for starting a solution • Possible to determine singular solutions with this algorithm	• Penalty functions required for terminal boundary conditions • Poor convergence characteristics near optimal solution • Step size determined by an independent search procedure
Second-order gradient	• Conceptually simple • Fast convergence near optimum • Seeks out relative minima rather than stationary solutions	• Relatively difficult to program • Must compute $\left[\frac{\partial^2 J}{\partial \underline{u}^2}\right]^{-1} = J^{-1}_{\underline{uu}}$ • Initially $J^{-1}_{\underline{uu}}$ may not exist or resemble value near optimum • Requires nominal solution to be convex • Method must be modified to handle control constraints
Second Variation	• Penalty functions not required in final phase for satisfying terminal boundary conditions • Quadratic convergence near optimum • Step size automatically determined • Improvement in each step • Convergence not contingent upon a good starting function	• Control constraints cannot be easily handled • Relatively difficult to program • Method seeks out stationary value rather than relative minima

Table 6-3 (Continued)

Computational Algorithm	Advantages	Disadvantages
Direct		
Conjugate-gradient	• Conceptually simple	• Has not been extended to apply to the general control problem
	• Good convergence characteristics	• Control constraints are not easily handled
	• Not necessary to compute $J_{\underline{u}\underline{u}}^{-1}$ as in the second-order gradient method	
Indirect		
Classical Newton-Raphson (CNR)	• Conceptually simple	• Terminal boundary conditions very sensitive to variations in initial adjoint variables
	• Easy to program	
	• Control constraints are easily implemented	• Success depends on dimension of problem
	• Neighboring optimal solutions obtained in addition to optimum	• Method seeks out stationary values rather than relative minima
Quasilinearization Methods	• Convergence of method is quadratic near optimum	• Method seeks out stationary values rather than relative minima
	• Step size is automatically determined	• Relatively difficult to program
	• Potential savings in computer running time	
	• Penalty functions are not required for the treatment of terminal boundary conditions	

Ideally, it would be desirable to determine the fuel-optimal controller by using each of these methods and to compare such factors as convergence characteristics, computer running time, sensitivity to starting function, etc. Despite the importance of such a task, it is not within the scope of the present work and must be delayed until a future time.

Method of Steepest Descent

The method of steepest descent is important not only as a computational technique in its own right but also because many of the available algorithms for computing optimal controllers are based on it. For example, a typical modification to the steepest descent technique centers around the notion of using the gradient in an appropriate space (one that ensures that the gradient exists for all elements in the space). In another modification, the geometry in the function space is changed by introducing a locally linear transformation; this modification results in the Newton-Raphson method. The mathematical theory of this method is discussed in Reference [78] through [81] and in an Appendix of [24].

In this section, the method of steepest descent for functionals defined on a function space is discussed. The results for the case in which the space is R^n are obtained as a special case of the general results. Fundamental notions involved in this development include such concepts as

(1) arc length in a Banach space
(2) Frechet derivatives
(3) the Riesz representation theorem.

Let $u(t)$ be a smooth function in a real complete normed linear space (i. e., a real Banach space or simply a real B-space). The arc length s is given by

$$s = \int_0^t \| u'(\sigma) \| d\sigma \qquad \text{where } u'(\sigma) = \frac{\partial u}{\partial \sigma}(\sigma) . \tag{6-1}$$

If the curve is parameterized by arc length, it follows that the norm of $u'(t)$ is unity, that is,

$$\| u'(t) \| = 1$$

For the case in which the real B-space is the space of square-integrable functions on $[0, 1]$, that is $L_2[0, 1]$, and the function $u(s,\sigma)$ with $0 \leq \sigma \leq 1$ is a smooth curve in $L_2[0, 1]$, then the following is true

$$\int_0^1 \left[\frac{\partial u}{\partial s}(s, \sigma) \right]^2 d\sigma = 1 \tag{6.2}$$

where s = arc length, $\frac{\partial u}{\partial s}(s,\sigma) = u'(s)$.

The notion of a Frechet derivative has already been discussed in reference to the differentiability of the cost functional J. For convenience, this definition is given again. Let $\mathcal{H}$ be a real complete space having a scalar product (that is, $\mathcal{H}$ is a Hilbert space). Suppose u_o, $h \in \mathcal{H}$ and let J be a function such that

$$J : \mathcal{H} \to R$$

where R denotes the set of real numbers. The function J has a Frechet (strong) differential $J'(u)h$, if there is a continuous linear functional (the Frechet derivative of J at u_o) $J'(u_o)$ on $\mathcal{H}$ such that

$$|J(u_o+h) - J(u_o) - J'(u_o)h| = 0(\|h\|)^\dagger$$

† A function $g(h)$ is said to be $0(\|h\|)$ as $\|h\| \to 0$ if $\lim_{\|h\| \to 0} \frac{|g(h)|}{\|h\|} = 0$.

as $\|h\| \to 0$ with $\|h\| \doteq \langle h, h \rangle^{1/2}$. Moreover, if the expression

$$DJ(u_o, h) = \frac{\partial J}{\partial \lambda}(u_o + \lambda h) = \lim_{\lambda \to 0} \frac{J(u_o + \lambda h) - J(u_o)}{\lambda}$$

exists, it is called the Gateau (weak) differential of J at u_o or the directional derivative of J at u_o. If the Gateau differential has certain properties then the Frechet and the Gateau derivatives are equal, that is

$$DJ(u_o, h) = J'(u_o)h$$

The following theorem due to Luisternik [82] provides the conditions which must be satisfied in order for the two derivatives to be equal (Luisternik uses the term weak derivative while Kantorovich [41] uses the term Gateau derivative).

<u>Theorem.</u> [24] If the Gateau derivative $DJ(u_o, h)$ exists in $\|u-u_o\| \leq \alpha$, $\alpha > 0$ and if it is uniformly continuous in u and continuous in h then the Frechet differential exists and

$$J'(u)\,h = DJ(u, h)$$

Higher order Frechet derivatives are defined in an analogous manner. Let $\mathcal{H}^*$ denote the B-space of continuous linear functionals on $\mathcal{H}$ with norm $|\;\;|_1$. If J(u) has a Frechet derivative, then J(u) is said to have a second Frechet differential $J''(u_o)\,h$ if

$$|J(u_o + h) - J(u_o) - J''(u_o)h|_1 = 0\,(\|h\|)$$

as $\|h\| \to 0$. The term $J''(u_o)$ is a continuous linear operator from $\mathcal{H}$ into $\mathcal{H}^*$ and is the second Frechet derivative of J at u_o.

Suppose J has two continuous Frechet derivatives in $\mathcal{H}$. Since the first Frechet derivative is a linear functional on $\mathcal{H}$, it follows from the Riesz representation theorem [17] that

$$J'(u)h = \langle \frac{\partial J}{\partial u}(u),\ h \rangle \qquad h \in \mathcal{H} \tag{6-3}$$

where $\frac{\partial J}{\partial u}(u)$ is a unique element in $\mathcal{H}$. The element $\frac{\partial J}{\partial u}(u)$ is called the gradient of J at u. This agrees with the finite dimensional definition of a gradient if $\mathcal{H} = R^n$, that is

$$J'(\underline{u})\ \underline{h} = \langle \frac{\partial J}{\partial \underline{u}}(\underline{u}),\ \underline{h} \rangle$$

and

$$\nabla J(\underline{u}) = \frac{\partial J}{\partial \underline{u}}(\underline{u})$$

where $\underline{u},\ \underline{h} \in R^n$

Since $J''(u)h$ is also a continuous linear functional on $\mathcal{H}$, it follows that

$$\left(J''(u)h \right) h = \langle \mathcal{H}_J(u)\ h, h \rangle \tag{6-4}$$

where $\mathcal{H}_J(u)$ is a continuous linear operator on H known as the Hessian of J at u.

With the concepts of arc length in a B-space, Frechet derivatives, and the Riesz representation theorem established the method of steepest descent can now be stated for functions defined on a Hilbert space $\mathcal{H}$. Let J be a real-valued function defined on $\mathcal{H}$ with one continuous Frechet derivative, let $u_1 \in \mathcal{H}$ and let γ be a smooth curve in $\mathcal{H}$ passing through u_1. Parameterizing the curve by arc length, it follows that

$$\| u'(s) \|^2 = 1$$

and

$$\frac{dJ}{ds} = \lim_{\Delta s} \frac{J[u(s+\Delta s)] - J[u(s)]}{\Delta s} = \langle \frac{\partial J}{\partial u}(u(s)), u'(s) \rangle$$

The direction of steepest descent is found by minimizing

$$< \frac{\partial J}{\partial u}(u_1),\ u'(0) > ,$$

subject to

$$\|u'(0)\|^2 = 1$$

and the path of steepest descent is found by solving the differential equation

$$\frac{du}{d\sigma} = - \frac{\partial J}{\partial u}(u) \quad , \quad u(0) = u_1 \in \mathcal{H} \tag{6-5}$$

$$\sigma \geqq 0$$

The solution is a function with values in $\mathcal{H}$ and along this path $J\left(u(\sigma)\right)$ is decreasing since

$$\frac{dJ}{d\sigma} = < \frac{\partial J}{\partial u}, \frac{du}{d\sigma} > = - \left\| \frac{\partial J}{\partial u}(u(\sigma)) \right\|^2 < 0 \tag{6-6}$$

if $\frac{\partial J}{\partial u} \neq 0$.

The following theorem due to Rosenbloom [83] indicates under what circumstances a unique solution based on the steepest descent technique exists and also indicates the convergence characteristics of the algorithm.

Theorem. Let J have two continuous Frechet derivatives on a convex domain D of $\mathcal{H}$. Suppose the sphere S(u) defined by

$$S(u) = \{u \mid u \in \mathcal{H},\ \|u - u_o\| \leq \frac{a}{A}\}$$

where $a = \left\| \frac{\partial J}{\partial u}(u_o) \right\|$ and A (chosen later) is contained in D. Moreover, assume that

$$< \mathcal{H}_J(u)\, v, v > \ \geq A\, \|v\|^2$$

for $u \in D$, $v \in \mathcal{H}$, $A > 0$ and fixed, and that $u(\sigma)$ satisfies the equation for the path of steepest descent, that is,

$$\frac{du}{d\sigma}(\sigma) = -\frac{\partial J}{\partial u}\Big(u(\sigma)\Big) \qquad \text{for } \sigma \geq 0,\ u(0) = u_o \in D$$

Then

a. $\lim_{\sigma \to \infty} u(\sigma) = u_\infty$ exists in $\mathcal{H}$ (that is, the minimizing element, is unique and the path of steepest descent u exists).

b. $\lim_{\sigma \to \infty} J\Big(u(\sigma)\Big) = c$ exists for $\sigma \geq 0$

c. $\|u(\sigma) - u_\infty\| \leq \frac{a}{A} \exp(-A\sigma)$ and

$$0 \leq J(u(\sigma)) - c \leq \frac{a^2}{2A} \exp(-2A\sigma)$$

and $\forall\ u \in D \qquad J(u) \geq c + \frac{A}{2}\|u - u_\infty\|^2$

For the control problem, it is frequently necessary to minimize a function J on $\mathcal{H}$ subject to the side condition that $g = 0$. In this case, the path of steepest descent is defined by

$$\frac{du}{d\sigma} = -\frac{\partial J}{\partial u} + \lambda(u)\frac{\partial g}{\partial u} \tag{6-7}$$

$$\lambda(u) = \frac{\left< \frac{\partial J}{\partial u}, \frac{\partial g}{\partial u} \right>}{\| \partial g / \partial u \|^2}$$

provided that $\|\frac{\partial g}{\partial u}\| \neq 0$. With this choice of path, it follows that

$$\frac{dJ}{d\sigma} = -\left\{\frac{\|\frac{\partial J}{\partial u}\| \ \|\frac{\partial g}{\partial u}\| - \left< \frac{\partial J}{\partial u}, \frac{\partial g}{\partial u} \right>}{\|\frac{\partial g}{\partial u}\|^2}\right\} < 0$$

$$\frac{dg(u)}{d\sigma} = 0 \tag{6-8}$$

Thus far it has been shown how the steepest descent path can be constructed based on the gradient information of the functional J(u). In practice, however, a discrete version of the differential equation representing the path of steepest descent is implemented, that is, the value of u at the k+1th step is taken as

$$u_{k+1} = u_k - \rho_k \nabla J(u_k) \qquad \text{for } \rho_k > 0 \quad k = 0, ., ...$$

Application to Fuel-Optimal Problems

It is of interest to note that the existence theorem stated above could be directly applied to the fuel-optimal problem in which the cost functional is given by

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j u_j(t)\, dt \quad ;$$

it cannot be directly applied, however, to the case in which the cost functional is given by

$$J(u) = \int_{t_o}^{t_1} \sum_j k_j |u_j|\, dt \tag{6-10}$$

since the hypothesis that J have two continuous Frechet derivatives is violated.

It was demonstrated in this work that the cost functional

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j u_j(t)\, dt \tag{6-11}$$

is appropriate for the fuel-optimal control of spinning and dual-spin vehicles even though in the literature the cost functional for such problems is always taken as

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j |u_j(t)|\, dt$$

It is possible that even though the differentiability hypothesis is violated, a solution could still be found for the cost functional given in Equation (6-10). That is, the notion of a Gateau derivative could be used in lieu of the Fréchet derivative and could be evaluated numerically.

Generalized Newton-Raphson Method

In this section, the rudimentary notions concerning the generalized Newton-Raphson Method are discussed. As in the method of steepest descent, it is convenient to discuss this method for functions defined on a function space. Kantorovich [41] was one of the first to discuss this technique. Kenneth and McGill [73] applied it to an optimal control problem.

Let $\mathcal{P}$ be a nonlinear operation mapping a B-space X into another B-space Y. Consider the task of finding a zero of the equation

$$\mathcal{P}(x) = 0 \tag{6-12}$$

Kantorovich [41] showed that under certain conditions the solution x^* to this equation can be obtained from the sequence $\{x_n\}$ defined by

$$x_1 = x_o - [\mathcal{P}'(x_o)]^{-1}\mathcal{P}(x_o)$$

$$x_2 = x_1 - [\mathcal{P}'(x_1)]^{-1}\mathcal{P}(x_1)$$

$$x_{n+1} = x_n - [\mathcal{P}'(x_n)]^{-1}\mathcal{P}(x_n) \tag{6-13}$$

In this technique it is assumed that the inverse of the Frechet derivative $\mathcal{P}'(x)$ exists, that the initial guess x_o is sufficiently close to the solution x^*, and that the operator $\mathcal{P}$ is bounded. These conditions are summarized in the fundamental theorem of Kantorovich [41].

Theorem. Consider the operation $\mathcal{P}$ defined above and suppose it is defined on the open sphere $\{x \in X \mid \|x-x_o\| < r\}$ and has a continuous second derivative on the closed sphere $\{x \in X \mid \|x-x_o\| \leq r_o\}$. Assume that

(1) the linear operation $\Gamma_o = [\mathcal{P}'(x_o)]^{-1}$ exists

(2) $\| \Gamma_o(\mathcal{P}(x_o)) \| \leq \eta$

(3) $\| \Gamma_o \mathcal{P}''(x) \| \leq k$

If $h = k\eta \leq \frac{1}{2}$ and $r \geq r_o = \frac{1 - \sqrt{1-2h}}{h} \eta$ then the sequence $\{x_n\}$ defined by Newton's method

$$x_{n+1} = x_n + [\mathcal{P}'(x_n)]^{-1} \mathcal{P}(x_n)$$

converges to the solution x^* of the equation $\mathcal{P}(x) = 0$. Moreover, the solution will be unique provided that the following condition is satisfied

$$\text{for } h < \frac{1}{2} \qquad r < r_1 = \frac{1 + \sqrt{1-2h}}{h} \eta$$

$$\text{for } h = 1/2 \qquad r \leq r_1$$

Furthermore, the speed of convergence is characterized by the inequality

$$\| x^* - x_n \| \leq \frac{1}{2^n} (2h)^{2^n} \frac{\eta}{h}, \quad n = 0, 1, \ldots.$$

Note that the differentiability hypothesis of this theorem is violated for fuel-optimal problems in which the cost functional $J(\underline{u})$ is given by

$$J(\underline{u}) = \int_{t_o}^{t_1} \sum_j k_j |u_j(t)| dt$$

The implication of this has already been discussed in reference to the method of steepest descent.

This theorem immediately suggests a method for solving two-point boundary value problems. Consider the system

(S) $\quad \underline{\dot{x}} = \underline{f}(\underline{x}, t)$ in C^1 in R^{n+1}

with fixed endpoints $\underline{x}_o$, $\underline{x}_1$. If the system equations are rewritten as

$$\frac{d}{dt}\underline{x} - \underline{f}(\underline{x}, t) = 0 = \mathcal{P}(x)$$

where $\mathcal{P}$ is defined <u>formally</u> as

$$\mathcal{P} = \frac{d}{dt} - \underline{f} ,$$

then heuristically, the following sequence is obtained

$$\underline{x}_{n+1} = \underline{x}_n - \left[\frac{d}{dt} - \frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_n)\right]^{-1}\left[\frac{d}{dt}\underline{x}_n - \underline{f}(\underline{x}_n, t)\right] \qquad (6\text{-}14)$$

By formally applying the operator $\left[\frac{d}{dt} - \frac{\partial \underline{f}}{\partial \underline{x}}(x_n)\right]$ to both sides of Equation (6-14), the following iterative sequence results

$$\frac{d}{dt}\underline{x}_{n+1} = \left[\frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_n)\right](\underline{x}_{n+1} - \underline{x}_n) + \underline{f}(\underline{x}_n, t) \qquad (6\text{-}15)$$

with the boundary conditions as previously prescribed. Since the matrix $\frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_n)$ is the Jacobian matrix of the system, the system described by Equation (6-15) is linear and thus can be readily solved.

It has been shown previously that the standard optimal control problem involves

(1) n system differential equations, $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t) = \frac{\partial H}{\partial \underline{p}}$

(2) n adjoint differential equations, $\underline{\dot{p}} = -\frac{\partial H}{\partial \underline{x}}$

(3) 1 differential equation describing the cost functional

(4) r algebraic equations of the form

$\underline{G}(\underline{x}, \underline{u}, t) = 0$

(5) boundary conditions

Designating the 2n+1 differential equations corresponding to (1) through (3) as

$$\dot{\underline{X}} = \underline{F}(\underline{X}, \underline{u}, t) = \underline{F}(\underline{z}, t)$$

and the r algebraic equations as

$$\underline{G}(\underline{x}, \underline{u}, t) = 0 = \underline{G}(\underline{z}, t) ,$$

it follows that the generalized Newton-Raphson sequence is given by (assuming $\underline{u}$ can be determined as a function of $\underline{x}$)

$$\dot{\underline{X}}_{n+1} = \left[\frac{\partial \underline{F}}{\partial \underline{X}}(\underline{z}_n, t) \right] (\underline{X}_{n+1} - \underline{X}_n) + \underline{F}(\underline{z}_n, t) \qquad (6\text{-}16a)$$

$$\underline{0} = \left[\frac{\partial \underline{G}}{\partial \underline{X}} (\underline{X}_n, t) \right] (\underline{X}_{n+1} - \underline{X}_n) + \underline{G}(\underline{z}_n, t) \qquad (6\text{-}16b)$$

$$n = 0, 1, 2. \dots$$

The computational procedure entails the following steps

(1) guess a trial solution $\underline{X}_o$ satisfying the boundary conditions (at least as many of them as possible)

(2) obtain $\underline{z}_o$ from Equation (6-16b) using the initial guess $\underline{X}_o$

(3) substitute $\underline{z}_o$ into Equation (6-16a) and obtain the estimate $\underline{X}_1$

(4) obtain $\underline{z}_1$ from Equation (6-16b) using the updated state $\underline{X}_1$

(5) repeat the process until

$$\|\underline{X}_{n+1} - \underline{X}_n\| \leq \delta$$

where the norm could be taken as

$$\|\underline{X}_{n+1} - \underline{X}_n\| = \sum_i \max_{t \in [t_o, t_1]} \left| \underline{X}^{(i)}_{n+1}(t) - \underline{X}^{(i)}_n(t) \right|$$

In this method some of the necessary conditions for optimality were used so singular trajectories and solutions cannot be found with it. Recall that, singular solutions could be found with the method of steepest descent since none of the necessary conditions were used. Kenneth and McGill [73] show that this method can be used for

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problems in which there are inequality constraints on the state and on the control. An important factor to consider in potential applications of this technique is that pertaining to computer storage requirements. Depending on the application, the storage requirements could be relatively great.

Classical Newton-Raphson Technique

The classical Newton-Raphson (CNR) technique is a special case of the generalized Newton-Raphson (GNR) technique. Important differences between the two include such items as

(1) the CNR technique uses the optimality conditions while the GNR technique doesn't

(2) the reference trajectories are generated by integrating the nonlinear equations in the CNR technique while in the GNR technique they are generated from the linearized equations

(3) in the CNR technique the boundary conditions are iterated on while in the GNR technique each estimate satisfies the boundary conditions

Using Kantorovich's result, the solution to the equation

$$\mathcal{P}(x) = 0$$

for the case in which the B-spaces X and Y are each in R^n and the operator $\mathcal{P}$ is simply a vector function $\underline{F}$, it follows that the solution to

$$\underline{F}(\underline{x}) = 0 \tag{6-17}$$

is obtained from the sequence

$\underline{x}_{n+1} = \underline{x}_n - \left[\frac{\partial \underline{F}}{\partial \underline{x}}(\underline{x}_n)\right]^{-1} \underline{F}(\underline{x}_n)$. Hence, if the optimal control problem can be cast in the form of Equation (6-17), then the solution can be obtained iteratively. It is easily shown that even the most general optimal control problem with inequality constraints on both

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the state and the control variables can be readily cast in this form. The necessary conditions for the problems of interest in this work were previously obtained and were designated as Type A or Type B; Type A included the necessary conditions common to any optimization problem regardless of the target set and Type B included those that were target-set dependent. In the CNR technique, only those conditions designated as Type B are used to form the vector $\underline{F}$; the vector $\underline{y}$ is composed of either the initial adjoint variables $\underline{p}(o)$ and the final time t_1 for the fixed-end point free-final time problem or the Lagrange multipliers $\underline{\nu}$ due to the presence of the end-constraints and the final time t_1 for the constrained right end problem. The control problem now has the form

$$\underline{F}(\underline{y}) = 0$$

and the solution $\underline{y}$ is obtained from

$$\underline{y}_{n+1} = \underline{y}_n - \left[\frac{\partial \underline{F}}{\partial \underline{y}} (\underline{y}_n) \right]^{-1} \underline{F}(\underline{y}_n) \tag{6-18}$$

The iterative procedure consists of the following steps

(1) guess an initial value of the constant vector $\underline{y}$ and call it $\underline{y}_o$

(2) using $\underline{y}_o$ solve simultaneously

the state equation $\quad \dot{\underline{x}} = \underline{f} = \dfrac{\partial H}{\partial \underline{p}}$

the adjoint equation $\quad \dot{\underline{p}} = -\dfrac{\partial H}{\partial \underline{x}}$

the optimality equation $\quad \underline{u}(t) = \underline{u}(\underline{x}, \underline{p}, t)$

and obtain $\underline{F}(\underline{y}_o)$

(3) evaluate $\dfrac{\partial F}{\partial \underline{y}}(\underline{y}_o)$ numerically and compute its inverse

(4) obtain $\underline{y}_1$ from Equation (6-18)

(5) repeat the process until

$$\| \underline{F}(\underline{y}_n) \| \leq \delta$$

where $\| \quad \|$ is some appropriate norm.

The method is conceptually simple and intuitively appealing. The control constraints are automatically taken into account because only extremal controllers are allowed. The problem areas are encountered in steps (2) and (3). The simultaneous solution of the state equations, the adjoint equations, and the optimality conditions is not as straightforward as it appears when the control function $\underline{u}(t)$ is the on-off type and matters are even worse when the control function takes on the values -1, 0, 1. In step (3), an accurate determination of $\left[\frac{\partial F}{\partial \underline{y}} (\underline{y}_n) \right]^{-1}$ is difficult because the vector $\underline{F}$ is extremely sensitive to perturbations in $\underline{y}_n$ (this is especially true if $\underline{y}_n$ is based on the initial adjoint vectors). However, the method can be made to work satisfactorily and perhaps it may even be the best method for certain problems. As stated previously, there is a great deal of art involved in any of the computational methods.

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Section 7

RESULTS AND CONCLUSIONS

The preceding chapters provided a discussion of a methodology which is appropriate for handling a large class of optimal control problems. In essence, the preceding chapters describe a step-by-step procedure which can be followed in the determination of the optimal controller. Briefly, these steps include

(1) the derivation of the equations of motion for the systems being investigated (see Chapter 2)

(2) the formulation of the optimal control problem $\{S, \Delta, \Omega, X_0, X_1, J\}$ (see Chapter 3)

(3) an investigation of such concepts as controllability, normality, and the existence and uniqueness of optimal solutions (see Chapter 4)

(4) the determination of the necessary conditions for local optimality and an investigation of sufficiency conditions (see Chapter 5)

(5) a categorization and qualitative comparison of the various computational algorithms and a selection of the most suitable algorithm for the problems of interest (see Chapter 6).

In this chapter, the final aspects in the determination of the fuel-optimal controller for the problems formulated in this work are discussed; these aspects pertain to

(1) the selection of a suitable computational algorithm

(2) the numerical determination of the fuel-optimal controller

(3) the evaluation of the relative merits of the proposed control concept.

The evaluation of the relative merits of the angular momentum control concept is of special importance in this work. As stated previously,

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the emphasis in this dissertation is on practical rather than theoretical considerations. In fact, the entire problem formulation is based on such practical considerations as

(1) the importance of fuel-optimal attitude control for deep-space missions using a ballistic spacecraft

(2) the relative advantages of a dual-spin vehicle when compared to a spinning vehicle

(3) the advantages of the implementation of a properly placed nutation damper

(4) the physical significance of controlling the angular momentum vector rather than the spin axis

(5) the use of the minimum number of jets for achieving the control objective, the most appropriate type of jet (i.e., one-way, two-way, gimballed, etc.), and the most appropriate jet location.

In this chapter, the results of all the preceding chapters are synthesized so that the final steps can be efficaciously executed. In the preceding chapters, various control restraint sets, various jet locations, and various cost functionals were considered and their effects on the optimal control problem were noted. Now, only the most appropriate control restraint set, the most appropriate jet location, and the most appropriate cost functional are considered in the numerical determination of the fuel-optimal controller.

Before executing the final steps, the results obtained in the preceding chapters are summarized. Next, a discussion of the results obtained in this chapter is provided. Finally, the conclusions drawn from the study are given.

7.1 Summary of Results Pertaining to the Theoretical Aspects of the Control Problem

In this section, the results that were obtained in the preceding chapters concerning controllability, normality, the control restraint set, the existence and uniqueness of the optimal solutions, and the necessary and sufficient conditions for optimality are summarized.

7.1.1 Controllability and Normality

It was shown in Chapter 4 that controllability can aid not only in the determination of the number of jets required for the control objective but also in the determination of the most suitable jet location. The use of either one or two jets resulted in a completely controllable system for the spinning symmetric vehicle. For the dual-spin vehicle, if the jets are fixed to the despun body two are required for complete controllability; if the jets are rotor-fixed, only one jet is required for complete controllability. Concerning system normality, the system characterizing the symmetric spinning vehicle is normal when either one or two jets are used. Concerning the dual-spin vehicle, the system is singular when either one or two jets are fixed to the despun body. If either one or two rotor-fixed jets are used, the system is time-varying and the normality condition does not apply.

The connection between problem normality and the existence and uniqueness of the optimal controller was discussed in Chapter 4. Concerning problem normality, when the final time is fixed, the fuel-optimal control problem in which the spin axis control concept is used for the symmetric spinning vehicle is normal. On the other hand, the fuel-optimal control problem in which the angular momentum control concept is used for the dual-spin vehicle with jets fixed to the despun body is singular. However, if rotor-fixed jets are used, problem normality depends on the behavior of the switching function

$$q^{*}(t) = < \underline{p}^{*}(t), \underline{b}(t) >$$

The above considerations reveal that only one body-mounted jet is required for the symmetric spinning vehicle and that only one rotor-fixed jet is required for the dual-spin vehicle. The fact that the system representing the dual-spin vehicle with the jets fixed to the despun body is singular is especially important. There is still little known about the existence of singular optimal solutions. In addition, only the gradient computational algorithm could be used for the determination of the singular optimal controller. Singular solutions are not uncommon when the system is linear in $\underline{u}$ and the Hamiltonian is linear in $|\underline{u}|$ or $\underline{u}$. Note that if the necessary conditions alone were used (or at least a computational algorithm which makes use of the necessary conditions were programmed) and no attention was paid to the important notion of problem normality, then no computational results could be obtained regardless of the mathematical elegance of the algorithm used.

7.1.2 Existence and Uniqueness of the Fuel-Optimal Controller

In Chapter 4, it was stated that for a linear time-varying system, the fundamental hypotheses that are used in proving the existence of the optimal solution include

(1) problem normality

(2) compactness and convexity of the control restraint set Ω

(3) convexity of the integrand of the cost functional.

It was also noted that (1) and (2) imply that the set of attainability K(T) is a strictly convex compact set with nonempty interior. It was determined that a fuel-optimal controller exists for the fixed final time case in which the spin axis control (SACO) concept is used

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for the symmetric spinning vehicle. It was noted in Chapter 4 that the stated existence theorem is not applicable when the angular momentum control (AMCO) concept is used because problem normality has not been demonstrated (yet).

Concerning nonlinear systems, it was stated that the fundamental notions needed in demonstrating the existence of the optimal controller include

(1) the existence of a uniform bound on the response $\underline{x}(t)$ to controllers $\underline{u} \in \mathcal{J}$ (where $\mathcal{J}$ is the family of admissible controllers)

(2) the compactness of the control restraint set Ω

(3) the compactness of the initial and target sets

(4) the convexity of the extended velocity set

$$V(\underline{x}, t) = \left\{ f_o(\underline{x}, \underline{u}, t), \underline{f}(\underline{x}, \underline{u}, t) \,\middle|\, \underline{u} \in \Omega(\underline{x}, t) \right\}$$

where f_o is the integrand of the cost functional and $\underline{f}$ is the function defining the plant

(5) suitable continuity characteristics of f_o, viz. $f_o \in C^1$ in R^{n+m+1}

In regard to the uniqueness of the optimal controller, for linear time-varying systems, the hypotheses that were required to demonstrate uniqueness include

(1) problem normality

(2) compactness and convexity of the control restraint set Ω

(3) convexity of $f_o(\underline{x}, t)$ and <u>strict</u> convexity of $h_o(\underline{u}, t)$ where

$$J(\underline{u}) = \int_{t_o}^{T} \left[f_o(\underline{x}, t) + h_o(\underline{u}, t) \right] dt$$

Because of (3), the theorem cannot be applied to the problems of interest in this work. However, for a time-invariant system, the strict convexity of $h_o(\underline{u}, t)$ can be relaxed, i.e., the convexity of $h_o(\underline{u}, t)$ is sufficient. In this case, it can be shown that a unique fuel-optimal controller exists for the fixed final time case in which the spin axis control concept is used for a symmetric spinning vehicle.

7.1.3 Necessary and Sufficient Conditions for Local Optimality

The necessary conditions for local optimality were developed in Chapter 5. It should be noted that if the method of steepest descent (or some other gradient method) is used in determining the optimal controller, the necessary conditions are not needed. Nevertheless, it is felt that it is desirable to have all the facts concerning the optimal controller available before choosing a computational technique.

Necessary Conditions

It was shown in Chapter 5 that if a smooth control restraint set

$$\Omega = \{\underline{u} : \|\underline{u}\| \leq 1\}$$

associated with a gimballed jet were used in formulating the control problem, then in certain cases, a smooth feedback controller could be obtained. A feedback solution is almost always more desirable than an open loop controller and in practice, considerable effort is expended in an attempt to obtain a feedback solution. The connection between the smoothness of the control restraint set Ω, the associated smoothness of the integrand of the cost functional and the sufficiency conditions was also noted. Despite the theoretical niceties of a smooth control restraint set associated with a gimballed jet, the use of a gimballed jet is not recommended for the present problem.[†]

[†]The use of the smooth control restraint as an approximation to the set associated with magnitude-limited jets could prove to be very useful, especially if the feedback solution can be easily determined. This notion should be investigated in the future.

Note that a high performance servo is required when a gimballed jet is used. For the mission being investigated, extremely high reliability is an important mission requirement. Hence, the theoretical advantages of a gimballed jet are in significant when the practical considerations are weighed. Note also that gimballed-jets would be more appropriately used for non-spinning vehicles. Hence, the potential jet types have been reduced to two, viz.,

(1) a one-way jet having the control restraint set $\Omega = \{u(t): 0 \leq u(t) \leq 1\}$

(2) a two-way jet having the control restraint set

$$\Omega = \{ u(t) : |u(t)| \leq 1 \} .$$

The necessary conditions for local optimality for the control problem $\{L, \Omega, X_0, X_1, J\}$ where the system is

$$\text{(L)} \qquad \dot{\underline{x}} = A\,\underline{x}(t) + \underline{b}(t)\,u(t),$$

the compact convex control restraint set is either

$$\Omega = \{ u(t) : 0 \leq u(t) \leq 1 \}$$

or

$$\Omega = \{ u(t) : |u(t)| \leq 1 \} ,$$

the initial set X_0 consists of a fixed $\underline{x}_o$ and a fixed t_o, the target set X_1 is

$$X_{1_{AMCO}} = \{ (\underline{x}, t) : g_j \left(\underline{x}(t_1) \right) = 0 \quad j = 1, 2, \; t_1 \text{ free} \}$$

or

$$X_{1_{SACO}} = \{ (\underline{x}, t) : \underline{x}_1(t_1) = \underline{x}_1 = \underline{0}, \; t_1 \text{ free} \} ,$$

and the cost functional $J(u) = \int_{t_o}^{t_1} f_o(\underline{x}, \underline{u}, t)\, dt$ is

$$J(u) = \int_{t_o}^{t_1} K\, u(t)\, dt \qquad \text{for the one-way jet}$$

or

$$J(u) = \int_{t_o}^{t_1} K \left| u(t) \right| dt \quad , \text{ for the two-way jet}$$

are stated below. Regardless of the jet type or the target set, Hamilton's canonical equations are given by

$$\underline{\dot{x}} = \frac{\partial H}{\partial \underline{p}} = A \, \underline{x}(t) + \underline{b}(t) \, u(t)$$

$$\underline{\dot{p}} = - \frac{\partial H}{\partial \underline{x}} = - A^T \, \underline{p}(t)$$

$$\text{where } H \equiv < \underline{p}, \, \underline{\dot{x}} > - f_o(\underline{x}, \, \underline{u}, t)$$

Regardless of the control concept, the optimality condition is given by

$$u^*(t) = \text{dez} \left\{ < \underline{p}^*(t) \, , \, \underline{b}(t) \; > \right\} = \text{dez } q^*(t)$$

and

$$u^*(t) = \text{hev} \left\{ < \underline{p}^*(t) \, , \, \underline{b} \; (t) \; > - 1 \right\} = \text{hev } \left\{ q^* \, (t) \; -1 \right\}$$

for the two-way jet and the one-way jet respectively. Only the boundary conditions are target set dependent. For the AMCO concept, the boundary conditions are

$$\underline{x}(t_o) = \underline{x}_o$$

$$\underline{p}^*(t_1^*) = \left[\frac{\partial \underline{g}}{\partial \underline{x}} (\underline{x}(t_1)) \right]^T \underline{\nu}$$

$$H^*(t_1^*) = 0$$

while for the SACO concept the boundary conditions are the same xcept there is no condition on the adjoint variables, that is, the oundary conditions are

$$\underline{x}(t_o) = \underline{x}_o$$

$$H^*(t_1^*) = 0$$

It might appear that the seemingly slight difference in the boundary conditions would have no significant affect on the computational algorithm. Later, it will be shown that this difference can have a tremendous affect depending on the computational technique used.

The final selection of the most appropriate jet type will be made after carefully considering the practical and computational implications of the nature of the extremal controllers†

$u^*(t) = \text{dez } q^*(t)$ for the two-way jet

$u^*(t) = \text{hev}\{q^*(t) - 1\}$ for the one-way jet

The importance of the necessary conditions can now be appreciated; they

(1) provide information concerning whether the problem is normal or singular

(2) provide information concerning the nature of the optimal controller so that the most appropriate control restraint set can be selected

(3) aid in the selection of a computational technique

(4) provide the basis of every computational technique save the gradient method

Sufficient Conditions for Local Optimality

In Chapter 5, sufficient conditions based on

(1) the maximum principle

(2) the calculus of variations

(3) dynamic programming

were discussed. It was noted that sufficiency conditions based on the maximum principle were the most appropriate for the problems of

†Intuitively, it is anticipated that the one-way jet is the most appropriate. Nevertheless, optimization theory is used to confirm this feeling.

interest in this work. Concerning the control problem $\{S, \Omega, X_0, X_1, J\}$ in which

(1) the plant is linear in $\underline{x}$ and nonlinear in $\underline{u}$, i.e.,

$$\dot{\underline{x}} = A(t)\,\underline{x}(t) + \underline{h}(\underline{u}, t)$$

(2) the target set is closed and convex

(3) the initial set consists of the fixed point $\underline{x}_o$ and the fixed initial time t_o

(4) the cost functional $J(\underline{u})$ is

$$J(\underline{u}) = \int_{t_o}^{T} \left[f_o(\underline{x}, t) + h_o(\underline{u}, t) \right] dt$$

(5) $f_o(\underline{x}, t)$, $h_o(\underline{u}, t)$, $\frac{\partial f_o}{\partial \underline{x}}$, $h(\underline{u}, t)$ and $A(t)$ are continuous in all $(\underline{x}, \underline{u}, t)$ in R^{n+m+1}

(6) $f_o(\underline{x}, t)$ is convex in $\underline{x}$ for each fixed $t \in [t_o, T]$,

it was stated in Chapter 5 that if a controller $\underline{u}^*(t)$ satisfying the maximum principle is found, then it is an optimal controller. All the conditions of this theorem are satisfied for the fixed final time case for all the problems of interest in this work. Note, however, that in this work, the function $\underline{h}(\underline{u}, t)$ is linear in u and $h_o(\underline{u}, t)$ is linear in u or $|u|$ depending on the jet type. Hence, special attention must be given to the possibility of the existence of singular optimal controllers.

7.1.4 Computational Algorithms

In Chapter 6, the various computational algorithms were categorized and qualitatively compared. It was stated that the algorithms most suited for the problems of interest in this work include

(1) the gradient methods

(2) the generalized Newton-Raphson (GNR) method

(3) the classical Newton-Raphson (CNR) method.

It was noted that in both the method of steepest descent and in the generalized Newton-Raphson technique, that the existence of the second Frechet derivative of the cost functional is assumed. For fuel-optimal problems in which two-way jets are used, the cost functional

$$J(u) = \int_{t_o}^{t_1} \sum_j K_j \left| u_j(t) \right| dt$$

does not have a second Frechet derivative. Hence, at least from a theoretical point of view, the violation of the differentiability hypothesis is important. It was also noted that if a one-way jet were used, the differentiability hypothesis would not be violated.

It was stated that the control constraints are most easily handled in the CNR technique but that they can be handled in the gradient and GNR techniques. Concerning the determination of singular optimal solutions, only the gradient technique is suitable.

7.2 Selection of a Computational Algorithm for the Determination of the Fuel-Optimal Controller

In this section, the results obtained in Chapters 2 through 6 and summarized in the preceding section are used in the selection of a suitable computational algorithm for the class of fuel-optimal problems involved in this work. In general, the selection of a computational algorithm depends to a great extent on the nature of the specific problem being investigated.

The values of the system parameters appropriate for the fuel-optimal control of the dual-spin vehicle being investigated are

rotor speed, $\sigma = 10 \dfrac{\text{rad}}{\text{sec}}$

ratio of jet torque capacity and transverse inertia , $k = 0.001 \ \frac{1}{\sec^2}$

ratio of stored angular momentum and transverse inertia , $\beta = 1.5 \ \frac{\text{rad}}{\text{sec}}$

Of these parameters, the one that has the most effect on the computational algorithm is the rotor speed σ. This is because the extremal controller is given by

$$\begin{aligned} &\text{either} \quad u^*(t) = \text{dez} \ \langle \underline{b}(t), \ \underline{p}^*(t) \rangle \quad \text{for a two-way jet} \\ &\text{or} \quad u^*(t) = \text{hev} \ \{\langle \underline{b}(t), \ \underline{p}^*(t) \rangle - 1\} \quad \text{for a one-way jet} \end{aligned} \tag{7-1}$$

where the vector $\underline{b}(t)$ is given by

$$\underline{b}(t) = \begin{pmatrix} \cos \sigma t \\ \sin \sigma t \\ 0 \\ 0 \end{pmatrix}$$

Using the necessary conditions for local optimality, the nature of the switching function can be determined. From Equation (7-1), the switching function $q^*(t)$ is given by

$$q^*(t) = \langle \underline{b}(t), \ \underline{p}^*(t) \rangle$$

Using the boundary conditions on $\underline{p}^*(t)$ and the adjoint transition matrix $\Psi(t, \tau)$ it follows that

$$q^*(t) = \begin{bmatrix} \nu_1 & \nu_2 \end{bmatrix} \begin{pmatrix} c\sigma t \\ s\sigma t \end{pmatrix} \tag{7-2}$$

Considering the geometric implications of a singular control problem (see Chapter 4) and the result of Equation (7-2), it follows that the dual-spin vehicle using one rotor-fixed jet is normal. In addition the number of switchings and the switching times are obtained by examining the functions

$$u^*(t) = \text{dez}\left\{\underset{\sim}{\nu}^T \begin{pmatrix} c\sigma t \\ s\sigma t \end{pmatrix}\right\} \qquad \text{for a two-way jet}$$

$$u^*(t) = \text{hev}\left\{\underset{\sim}{\nu}^T \begin{pmatrix} c\sigma t \\ s\sigma t \end{pmatrix} - 1\right\} \qquad \text{for a one-way jet} \tag{7-3}$$

The number of switchings is proportional to the rotor speed σ and the switching times are simply the zeros of the transcendental equations equations

$$\left.\begin{aligned} q^*(t) - 1 = 0 \\ q^*(t) + 1 = 0 \end{aligned}\right\} \qquad \text{for the two-way jet}$$

$$q^*(t) - 1 = 0 \qquad \text{for the one-way jet} \tag{7-4}$$

Note that this result agrees with the intuitive notion that a two-way jet is actually two one-way jets back-to-back and, hence, the average number of switchings for a one-way jet should be one half that for a two-way jet. This result is of great practical significance because if a two-way jet were used the probability of jet failure would increase with the increased number of jet firings. The high reliability required for long-duration missions is one of the most critical mission requirements.

The number of switchings associated with the extremal controller is an important factor to be considered in selecting a computational algorithm. For example, in time-optimal problems in which the extremal controllers turned out to be bang-bang, it has been proposed by several investigators that the optimal controller can be conveniently determined by treating the switching points as parameters and, hence, converting the problem into a parameter optimization problem. This method is unthinkable for the problem at hand!

For example, if two two-way jets were used (this jet combination is the one previous investigators actually used in fuel-optimal control problems involving spinning vehicles), the number of switchings for the case in which the rotor speed is 100 rpm would be in excess of 300. Treating the switching times as parameters for this type of problem is ill-advised indeed! Even if the optimal controller were successfully determined by such a technique, its implementation would be impractical.

The boundary condition for the adjoint variables $\underline{p}(t)$ has already been used in determining the switching function $q^*(t)$. An examination of the necessary conditions for both the AMCO and SACO concepts reveals that this boundary condition is the feature which distinguishes one concept from the other. For the AMCO concept, the boundary condition is given by

$$\underline{p}^*(t_1^*) = \left[\frac{\partial \underline{g}}{\partial \underline{x}} (\underline{x}(t_1) \right]^T \quad \underline{\nu} = \begin{Bmatrix} \nu_1 \\ \nu_2 \\ -\beta\nu_2 \\ \beta\nu_1 \end{Bmatrix} \tag{7-5}$$

Hence, the final value of the adjoint vector is completely specified in terms of the two unknown constants ν_1 and ν_2. This reduction in the dimension of the problem can be very significant depending on the algorithm used. An examination of Equation (7-5) reveals that if the CNR technique were used, the final rather than the initial adjoint variables would be iterated on. This could be extremely important because the sensitivity of the terminal conditions to variations in the initial adjoint variables is actually the only inherent disadvantage of the CNR technique. It may be expected that the sensitivity of the terminal conditions to perturbations in the final adjoint variables (i.e., their values at t_1) will not be too great.

With due consideration to such aspects as

(1) the nature of the optimal controller and the large number of switching times

(2) the normality of the fuel-optimal problem

(3) the relatively low dimension of the problem

(4) the fact that the final rather than the initial adjoint values are involved

(5) the theoretical disadvantage concerning the violation of the differentiability hypothesis for both the gradient and GNR techniques (for fuel-optimal problems in which two-way jets are used)

(6) the ease in which the control constraints are handled in the CNR technique

(7) the computer storage requirements

(8) the fact that only very small deviations from the nominal trajectory are allowable,

the CNR algorithm is considered suitable for the determination of the fuel-optimal controller for the dual-spin vehicle in which the AMCO concept is used. Concerning item (8), for the application under consideration, the antenna pointing accuracy requirement is such that the optimal control sequence would be initiated when the pointing error is greater than one milliradian.† This implies that the values of the state variables must be kept relatively close to the nominal or desired values. This aspect is very important when the classical Newton-Raphson (CNR) technique is used because of the nature of the iterative scheme.

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†The pointing accuracy requirement is one milliradian near Jupiter and beyond, in the vicinity of the earth, 5 milliradians would be allowable.

It is noted that the usé of the CNR method appears appropriate for the problems under investigation. In general, however, this method is very <u>seldom</u> appropriate for optimal control problems.

7.3 Determination of the Fuel-Optimal Controller for the Dual-Spin Vehicle Using the Angular Momentum Control (AMCO) Concept

In this section, the fuel-optimal controller for the dual-spin vehicle using the AMCO concept is determined for a specific initial state $\underline{x}_o$. The necessary conditions for local optimality and the CNR algorithm have alread been discussed. The only items that need further discussion are

(1) the method for determining the switching times

(2) the initial state $\underline{x}_o$.

7.3.1 Switching Times

It was previously shown that the switching times for this problem are the zeros of the transcendental equation

$$q^*(t) - 1 = 0 = \nu_1 \cos \sigma t + \nu_2 \sin \sigma t - 1 \qquad (7\text{-}6)$$

for the case in which one one-way jet is used.

A convenient technique for determining the zeros of this equation is the method of Regula Falsi. Since this method is a well-known technique of numerical analysis, it will not be discussed in this work (see, e.g., Reference [40]).

7.3.2 Initial State

The boundary condition on the system equations

$$\underline{x}(t_o) = \underline{x}_o$$

has not been used yet. As stated previously, the initial value of the state $\underline{x}_o$ and the time t_o are the elements of the initial set X_o.

The initial state $\underline{x}_o$ refers to the state $\underline{x}(t)$ existing at the time the optimal control sequence is initiated at time t_o. The optimal control sequence is initiated when the antenna pointing error becomes excessive, that is, when the condition

$$\|\underline{\theta}\| \leq \theta_c \tag{7-7}$$

is not satisfied. Nominally the initial state is such that

$$\underline{x}_o = \underline{0}$$

but due to the presence of solar radiation torques† and other disturbances torques, the angular momentum vector (and hence, the antenna axis) drifts away from the desired direction. When the condition of Equation (7-7) is not satisfied, the elements ω_1, ω_2, θ_1, θ_2 are sensed (observed) and are used to define the initial set X_o. The initial state used in the numerical work is

$$\underline{x}_o = \begin{pmatrix} 0 \\ 0 \\ 0.005 \\ 0 \end{pmatrix}$$

where $\theta_1 = 5$ milliradians refers to the maximum allowable antenna pointing error in the vicinity of the earth.

7.3.3 Iterative Procedure

In essence, the unknowns ν_1, ν_2, t_1 are determined iteratively until the terminal conditions are satisfied. That is, the vector $\underline{y}$ is determined iteratively to satisfy the equation

†See Likins and Larson [6] for a discussion of the external environment relevant to deep-space missions.

$$\underline{F}(\underline{y}) = 0 ,$$

where $\qquad \underline{y}^T = (\nu_1, \nu_2, t_1) \qquad$ (7-8)

$$\underline{F}^T = (H, H_1, H_2)$$

In Equation (7-8), H_1 and H_2 are the transverse components of angular momentum vector in inertial space, H is the Hamiltonian, t_1 is the final time, and ν_1 and ν_2 are constants arising because of the transversality condition.

7.3.4 Fuel-Optimal Controller

The optimal controller $u^*(t)$ obtained for the initial condition previously described is shown in Figure 7-1. During each revolution of the rotor, the controller is turned on for one half of the revolution and turned off for the other half. The number of switchings involved is 74 and the time to drive the initial state $\underline{x}_o$ to the target set is 22 seconds. The value of the cost functional

$$J(u) = \int_{t_o}^{t_1} K\, u(t)\, dt$$

associated with the minimum fuel problem is

$$J(u) = 0.0109 \ \frac{1}{\text{sec}}$$

By using the mass flow properties of the jet used, the amount of fuel consumed in accomplishing the control objective can be computed. The relation between the fuel weight W and the cost functional J(u) is

$$W = \frac{I_1}{I_s x\, r}\ J(u)$$

where r is the jet lever arm and I_s is the specific impulse of the jet.

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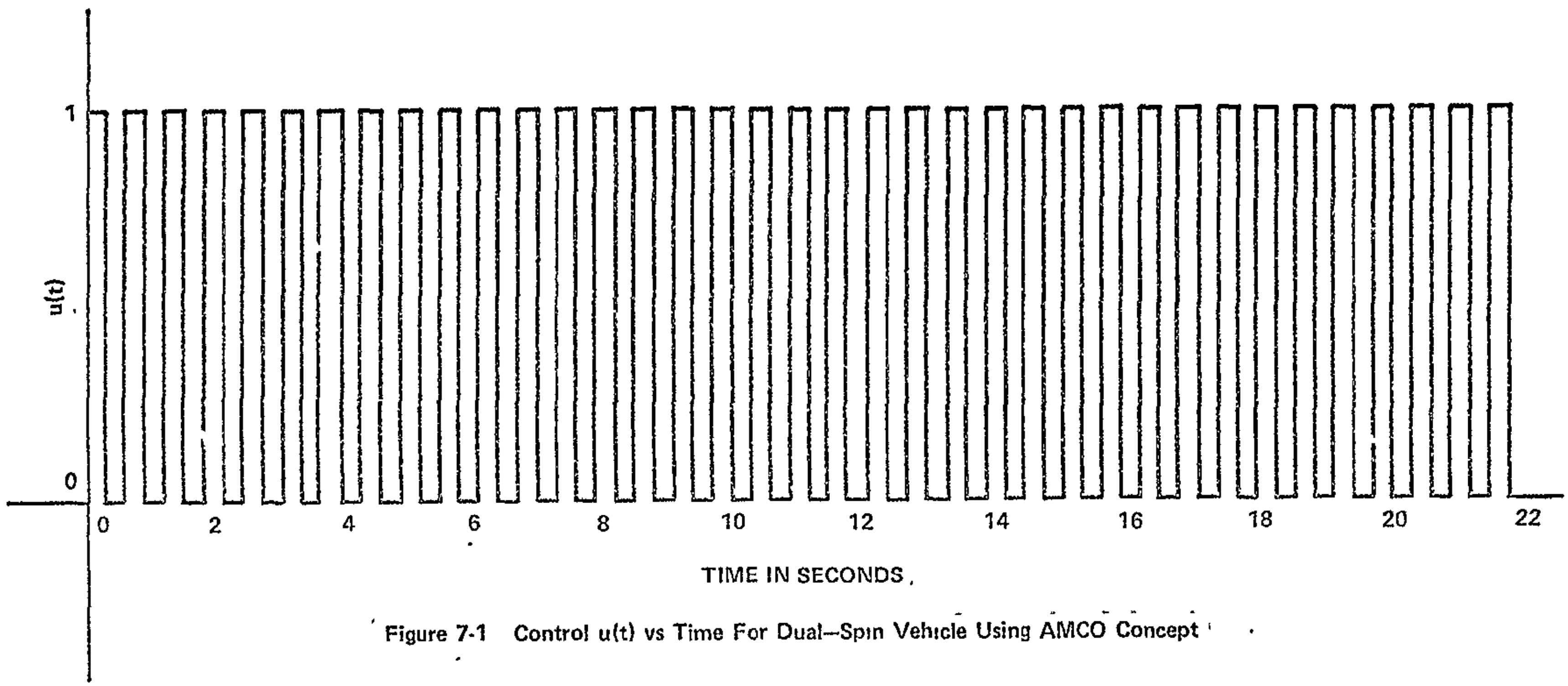

Figure 7-1 Control u(t) vs Time For Dual–Spin Vehicle Using AMCO Concept

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The fuel consumed in accomplishing the task described above is only 0.008 lb.

In Figure 7-2, the transverse component of the angular momentum vector $\underline{H}_T$ is shown vs. time. It is seen that each firing of the jet reduces the magnitude of the transverse angular momentum. During the off period, the transverse angular momentum is constant. This result is as it should be since $\underline{H}$ is conserved in a torque-free environment. Figure 7-3 shows the trajectory in angular momentum space and Figure 7-4 shows the antenna angles. Initially, the transverse components of the angular momentum were

$$(H_1, H_2) = (0, -1.5) \text{ft \# sec}$$

Each time the jet is turned on, the H_2 component is decreased. The half waves correspond to the on-cycle of the controller. During the off time, $\underline{H}$ is constant and neither H_1 nor H_2 varies.

7.4 Evaluation of the Relative Merits of the Angular Momentum Control Concept

In this section, the relative merits of the angular momentum control concept are evaluated. The fuel-optimal control problem for a spinning symmetric vehicle was investigated in Reference [11], in that reference the concept termed SACO in this work was used. The nature of the SACO concept has been previously discussed in this work so that it can be compared with the AMCO concept. By comparing the results of the two concepts when applied to the same problem, the relative merits of the AMCO concept can be determined, meeting one of the main objectives of this dissertation.
The problem that will be solved is that which was studied in Reference [11], viz., the determination of the fuel-optimal controller for a symmetric spinning vehicle. In order to have a meaningful comparison, the same initial conditions, the same system parameters,

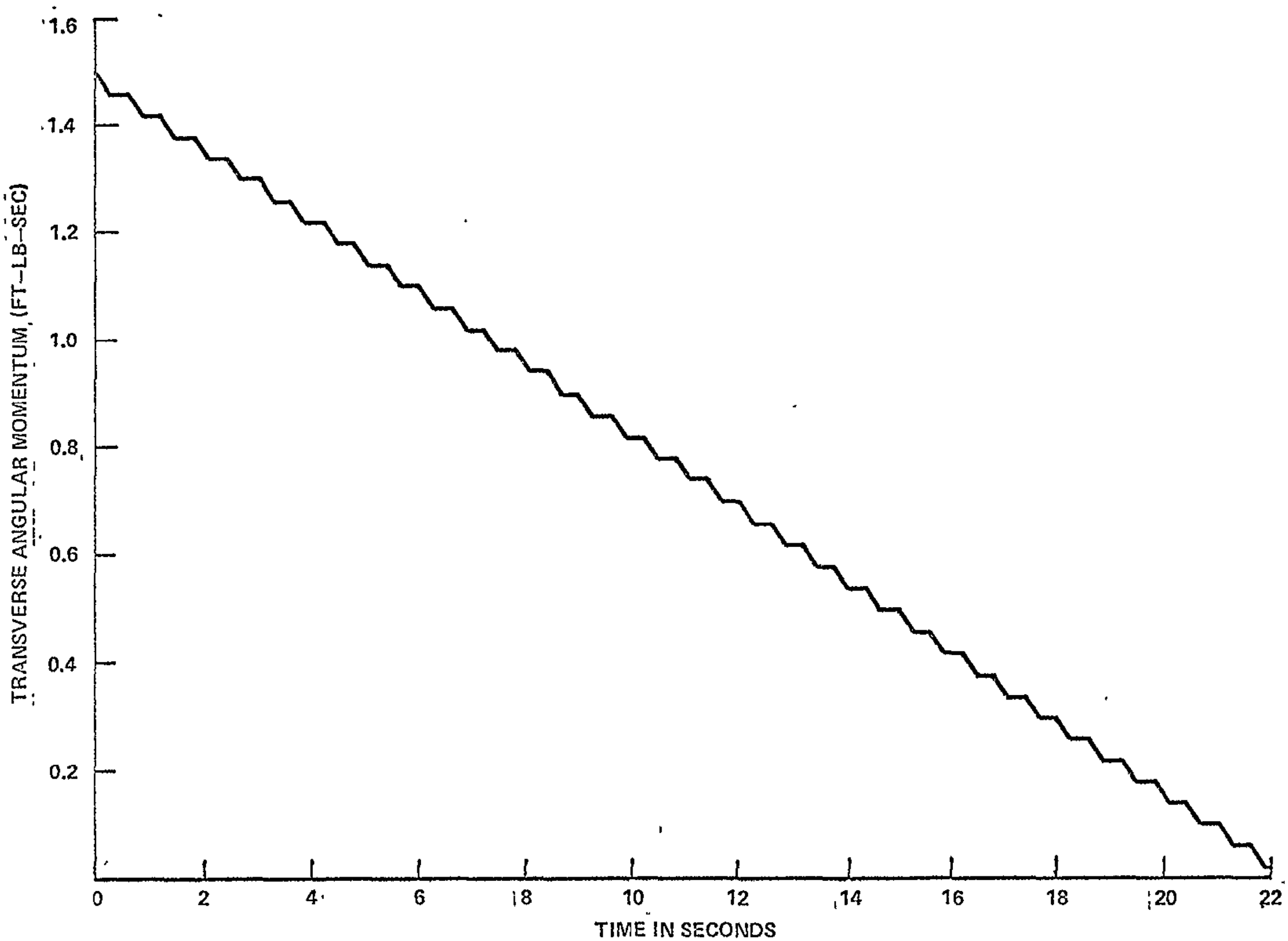

Figure 7-2 *Transverse Angular Momentum for Dual–Spin Vehicle Using AMCO Concept*

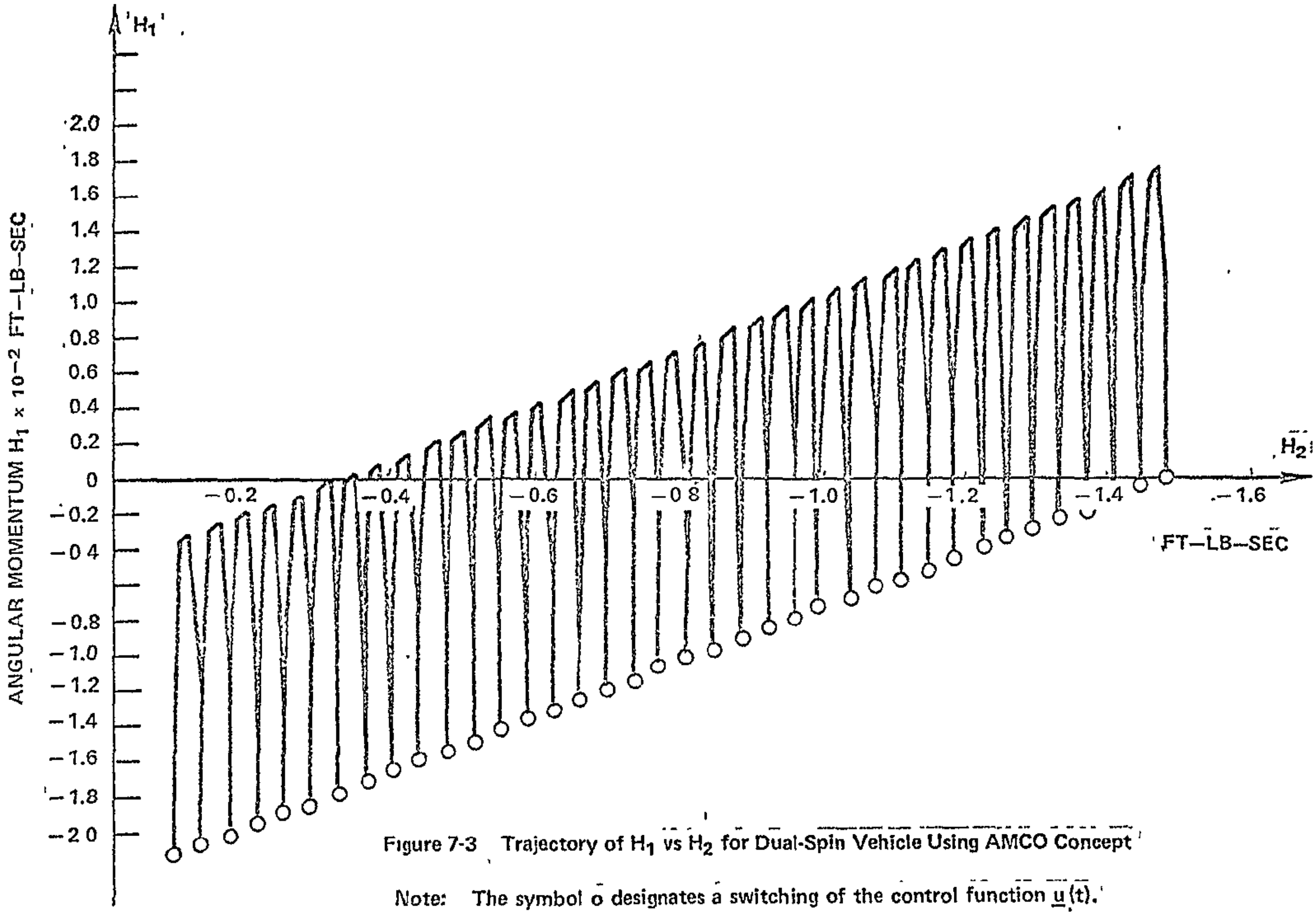

Figure 7-3 Trajectory of H_1 vs H_2 for Dual-Spin Vehicle Using AMCO Concept

Note: The symbol o designates a switching of the control function $\underline{u}(t)$.

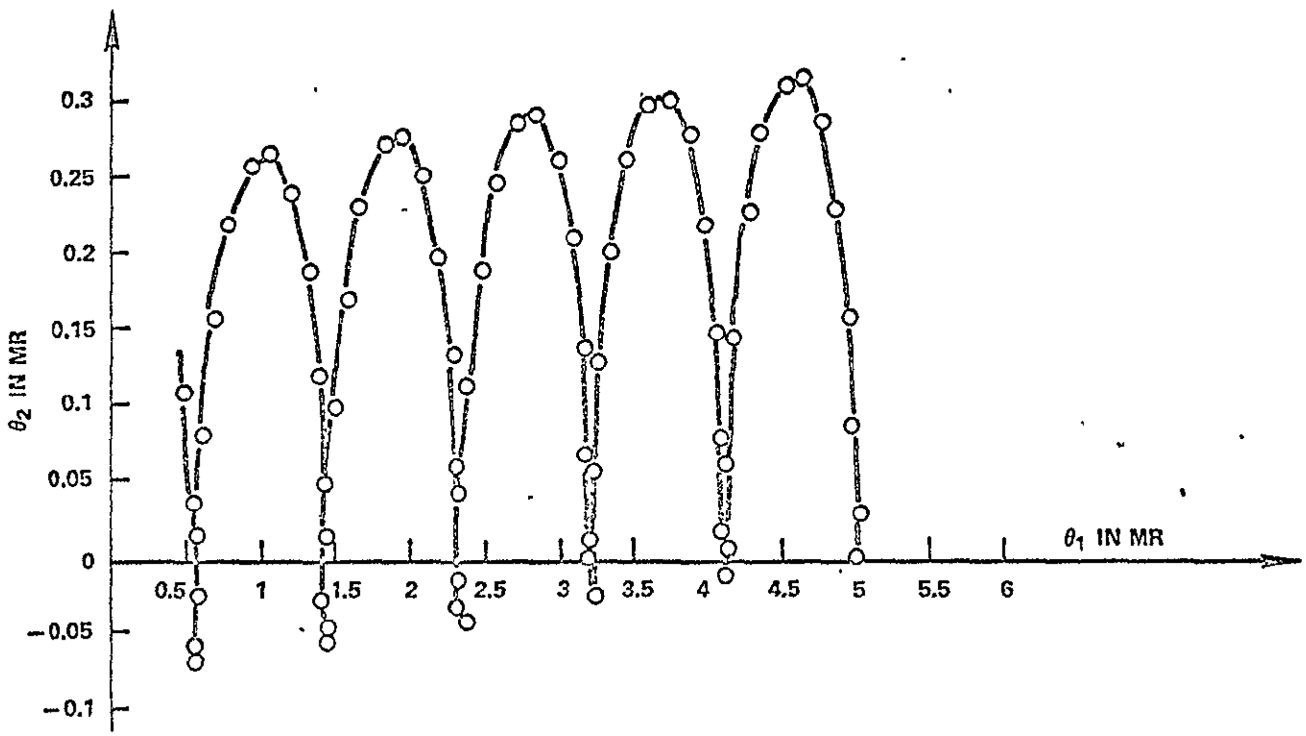

Figure 7-4 Trajectory of Antenna Angles θ_1, θ_2 for Dual Spin Vehicle Using AMCO Concept

and the same number and type of jets will be used in the numerical work.

Control Problem

The control problem $\{L, \Omega, X_0, X_1, J\}$ being studied is as follows:

(1) the system (L) is given by

$$(L) \qquad \dot{\underline{x}} = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad \text{in } C^1 \text{ in } R^{n+m}$$

(2) the control restraint set $\Omega \subset R^m$ is given by

$$\Omega = \{\underline{u}(t) : |u_j(t)| \leq 1 \quad j = 1, 2\}$$

(3) the initial set X_0 consists of the pair $(\underline{x}_o, t_o)$ where $\underline{x}_o$ is the initial state, i.e.,

$$X_0 = \{(\underline{x}, t) : \underline{x}(t_o) = \underline{x}_o \,,\; t_o \text{ fixed}\}$$

(4) the target sets X_1 for the two concepts are

$$X_{1_{SACO}} = \{(\underline{x}, t) : \underline{x}(t_1) = \underline{0}, t_1 \text{ free}\}$$

$$X_{1_{AMCO}} = \{(\underline{x}, t) : g_j(\underline{x}(t_1)) = 0, \; j = 1, 2, \; t_1 \text{ free}\}$$

(5) the cost functional $J(\underline{u})$ is given by

$$J(\underline{u}) = \int_{t_o}^{t_1} \left[\sum_{j=1}^{2} K_j \left| u_j(t) \right| \right] dt$$

Necessary Conditions for Local Optimality

The necessary cpnditions for optimality for both concepts have already been obtained. The only difference between the necessary conditions for the two concepts is that in the AMCO concept, the adjoint vector at the final time t_1^* is orthogonal to the tangent plane $(T_1(\underline{x}(t_1^*))$ of the manifold $X_1(t_1^*)$; in the SACO concept, there is no condition on the adjoint vector. The necessary conditions are repeated below for convenience:

Hamilton's canonical equations, $\dot{\underline{x}} = \frac{\partial H}{\partial \underline{p}} = A\,\underline{x}(t) + B\,\underline{u}(t)$

$$\dot{\underline{p}} = \frac{-\partial H}{\partial \underline{x}} = -A^T\,\underline{p}(t)$$

Optimality condition, $\underline{u}^*(t) = \text{DEZ}\,\{B^T \cdot \underline{p}^*(t)\}$†

Boundary conditions,

SACO	AMCO
$\underline{x}(t_o) = \underline{x}_o$	$\underline{x}(t_o) = \underline{x}_o$
$H^*(t_1^*) = 0$	$H^*(t_1^*) = 0$
	$\underline{p}^*(t_1^*) = \left[\frac{\partial g}{\partial \underline{x}}\right]^T_{t=t_1} \underline{\nu}$

Iterative Procedure

The CNR algorithm is used for both concepts. As stated previously, in this algorithm, the equation

$$\underline{F}(\underline{y}) = \underline{0}$$

is solved iteratively for the vector $\underline{y}$. The vectors $\underline{F}$ and $\underline{y}$ have already been discussed; the elements of these vectors for the two concepts are repeated below for convenience:

Vector	AMCO	SACO
$\underline{F}$	$\underline{F}^T = (H, H_1, H_2)$	$\underline{F}^T = (H, \underline{x}(t_1)^T)$
$\underline{y}$	$\underline{y}^T = (\nu_1, \nu_2, t_1)$	$\underline{y}^T = (\underline{p}(o)^T, t_1)$

†Two two-way jets are used in this comparison because they were used in [11].

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A very significant difference in the iterative procedure for the two concepts is that in the SACO concept, the initial adjoint vector $\underline{p}(0)$ is iterated on while in the AMCO concept, the constants $\underline{\nu}$ are iterated on. The sensitivity of the conditions $\underline{F}$ to perturbations in the initial adjoint variables is extremely great. This problem was discussed previously and is, in general, a characteristic of the CNR technique. In the AMCO concept, the sensitivity of $\underline{F}$ due to perturbations in $\underline{\nu}$ is <u>considerably</u> less. This follows because $\underline{\nu}$ is related to the final adjoint variables $\underline{p}(t_1)$. Another significant feature of the AMCO concept is the reduction in the dimension of the problem. When the CNR technique is used, the probability of success and the computer running time are inversely proportional to the dimension of the problem.

<u>Fuel-Optimal Controller</u>

In this section, the results obtained by using the two concepts for a specific initial state are provided. The initial state is that which was used in [11] and is given by (note that $\phi_1 \equiv \phi$ and $\phi_2 \equiv \theta$)

$$\underline{x}_o = \begin{pmatrix} \omega_1(0) \\ \omega_2(0) \\ \phi_2(0) \\ \phi_1(0) \end{pmatrix} = \begin{pmatrix} -0.01 \text{ rad/sec} \\ 0.008 \text{ rad/sec} \\ 0.1 \text{ rad} \\ 0.05 \text{ rad} \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \theta_o \\ \phi_o \end{pmatrix}$$

Before interpreting the results obtained for each concept, the nature of the switching function $\underline{q}^*(t)$ and the implication of the terminal conditions are examined. Just as in vibration theory, it is convenient to express the response of the system in terms of its modes. The modes of the system were previously determined as a by-product of the spectral theory of the operator A (the system matrix). The nature of the switching function and the implications of the terminal conditions for the dual-spin vehicle are also examined.

Symmetric Spinning Vehicle

The switching function $\underline{q}^*(t)$ was previously defined as

$$\underline{q}^*(t) = B^T \underline{p}^*(t) \tag{7-9}$$

For the SACO concept, Equation (7-9) becomes

$$\underline{q}^*(t) = \begin{pmatrix} p_1^*(t) \\ p_2^*(t) \end{pmatrix} = \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \end{bmatrix} \underline{p}(0) \tag{7-10}$$

where Ψ is the adjoint transition matrix $\left[\begin{array}{c|c} \Psi_{11} & \Psi_{12} \\ \hline \Psi_{21} & \Psi_{22} \end{array}\right]$,

for the AMCO concept, Equation (7-9) becomes

$$\underline{q}^*(t) = \begin{bmatrix} \Psi_{11}(t-t_1) & \Psi_{12}(t-t_1) \end{bmatrix} \left[\frac{\partial \underline{g}}{\partial \underline{x}}\right]^T \underline{\nu} \tag{7-11}$$

Representing $\underline{q}^*(t)$ in terms of the function space having as its basis the modes of the system, yields

$$\underline{q}^*(t)_{SACO} = \left[\begin{array}{c|c|c|c} p_1 + \dfrac{p_3}{\omega_3(r-1)} & p_2 - \dfrac{p_4}{\omega_3(r-1)} & \dfrac{-p_3}{\omega_3(r-1)} & \dfrac{p_4}{\omega_3(r-1)} \\ p_2 - \dfrac{p_4}{\omega_3(r-1)} & -\left(p_1 + \dfrac{p_3}{\omega_3(r-1)}\right) & \dfrac{p_4}{\omega_3(r-1)} & \dfrac{p_3}{\omega_3(r-1)} \end{array}\right] \begin{pmatrix} cr\omega_3 t \\ sr\omega_3 t \\ c\omega_3 t \\ s\omega_3 t \end{pmatrix} \tag{7-12}$$

$$\underline{q}^*(t)_{AMCO} = \left[\begin{array}{c|c|c|c} \nu_1 \dfrac{2r-1}{r-1} & \nu_2\left[\dfrac{2r-1}{r-1}\right] & -\nu_1 \dfrac{r}{r-1} & -\nu_2 \dfrac{r}{r-1} \\ \nu_2 \dfrac{2r-1}{r-1} & -\nu_1 \dfrac{2r-1}{r-1} & -\nu_2 \dfrac{r}{r-1} & \nu_1 \dfrac{r}{r-1} \end{array}\right] \begin{pmatrix} cr\omega_3 t \\ sr\omega_3 t \\ c\omega_3 t \\ s\omega_3 t \end{pmatrix} \tag{7-13}$$

An examination of Equations (7-12) and (7-13) reveal that for the general case the nature of the switching function for both concepts is the same. However, for the AMCO concept, if the vehicle parameters are chosen such that

$$r = \frac{I_1 - I_3}{I_1} = \frac{1}{2}$$

then the modes $cr\omega_3 t$ and $sr\omega_3 t$ are not involved in the representation of $\underline{q}^*(t)$. In general, then the number of switch points are expected to be approximately the same and to depend primarily on the spin rate ω_3.

Terminal Conditions

Using the modal response of the system, the terminal condition

$$\underline{x}(t_1) = 0$$

for the SACO concept can be written as

$$\omega_1 = 0$$
$$\omega_2 = 0$$

$$0 = -\frac{1}{\omega_3(1-r)}\,\omega_1 + \Bigg[\left(\theta_o + \frac{\omega_{10}}{\omega_3(1-r)}\right) c\,\omega_3 t + \left(\frac{\omega_{20}}{\omega_3(1-r)} - \phi_o\right) s\omega_3 t$$
$$+ \int_{t_o}^{t_1} \frac{1}{\omega_3(1-r)} \left[c\omega_3(t-\tau)\,u_1(\tau) + s\omega_3(t-\tau)\,u_2(\tau)\right] d\tau\Bigg] \tag{7-14}$$

$$0 = \frac{1}{\omega_3(1-r)}\,\omega_2 + \Bigg[\left(\phi_o - \frac{\omega_{20}}{\omega_3(1-r)}\right) c\omega_3 t + \left(\phi_o + \frac{\omega_{10}}{\omega_3(1-r)}\right) s\omega_3 t$$
$$+ \int_{t_o}^{t_1} \frac{1}{\omega_3(1-r)} \left(s\omega_3(t-\tau)\,u_1(\tau) - c\omega_3(t-\tau)\,u_2(\tau)\right) d\tau\Bigg]$$

For the AMCO concept, the terminal condition that

$$\underline{H}_T = 0$$

can be written as

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$$0 = \left(1 - \frac{r}{1-r}\right)\omega_1 + r\omega_3 \left[\left(\theta_o + \frac{\omega_{10}}{\omega_3(1-r)}\right) c\omega_3 t + \left(\frac{\omega_{20}}{\omega_3(1-r)} - \phi_o\right) s\omega_3 t \right.$$

$$\left. + \int_{t_o}^{t_1} \frac{1}{\omega_3(1-r)} \Big(c\omega_3(t-\tau)\, u_1(\tau) + s\omega_3(t-\tau)\, u_2(\tau)\Big) d\tau \right] \qquad (7\text{-}15)$$

$$0 = \left(1 - \frac{r}{1-r}\right)\omega_2 - r\omega_3 \left[\left(\phi_o - \frac{\omega_{20}}{\omega_3(1-r)}\right) c\omega_3 t + \left[\phi_o + \frac{\omega_{10}}{\omega_3(1-r)}\right] s\omega_3 t \right.$$

$$\left. + \int_{t_o}^{t_1} \frac{1}{\omega_3(1-r)} \Big(s\omega_3(t-\tau)\, u_1(\tau) - c\omega_3(t-\tau)\, u_2(\tau)\Big) d\tau \right]$$

It is convenient to represent these terminal conditions in terms of the modes defined in the complex plane rather than the real plane. Letting $\omega_1 + i\omega_2$ be ω^*, the terminal conditions for the SACO concept become

$$0 = \omega^* = \omega_o^* e^{-ir\omega_3 t} + \int_{t_o}^{t_1} e^{-ir\omega_3(t-\tau)} \Big(u_1(\tau) + i\, u_2(\tau)\Big) d\tau \qquad (7\text{-}16a)$$

$$0 = -\frac{1}{\omega_3(1-r)}\,\bar{\omega}^* + \left[\left\{\left(\theta_o + \frac{\omega_1(0)}{\omega_3(1-r)}\right) - i\left(\frac{\omega_2(0)}{\omega_3(1-r)} - \phi_o\right)\right\} e^{i\omega_3 t} \right.$$

$$\left. + \int_{t_o}^{t_1} \frac{e^{i\omega_3(t-\tau)}}{\omega_3(1-r)} \Big(u_1(\tau) - i\, u_2(\tau)\Big) d\tau \right] \qquad (7\text{-}16b)$$

where $\bar{\omega}^*$ refers to the complex conjugate of ω^*.

For the AMCO concept, the terminal condition reduces to

$$0 = \left(1 - \frac{r}{1-r}\right)\omega^* + r\omega_3 \left[\left\{\left(\theta_o + \frac{\omega_1(0)}{\omega_3(1-r)}\right) + i\left(\frac{\omega_2(0)}{\omega_3(1-r)} - \phi_o\right)\right\} e^{-i\omega_3 t} \right.$$

$$\left. + \int_{t_o}^{t_1} \frac{e^{-i\omega_3(t-\tau)}}{\omega_3(1-r)} \Big(u_1(\tau) + i\, u_2(\tau)\Big) d\tau \right] \qquad (7\text{-}17)$$

The nutation damper can effectively damp out the motion characterized by

$$\omega^* = \omega_o^* e^{-ir\omega_3 t}$$

Effectively, in the AMCO concept, the vehicle is designed so that no fuel is required for damping out the response

$$\omega^* = \omega_o^* e^{-ir\omega_3 t}$$

In the SACO concept, however, fuel is required to damp out the initial condition response

$$\omega^* = \omega_o^* e^{-ir\omega_3 t}$$

and in addition fuel is required to satisfy the constraint given in Equation (7-16b). Since the modes $\epsilon^{ir\omega_3 t}$ and $\epsilon^{i\omega_3 t}$ have different frequencies, the excitation (control) which damps one to zero does not, in general, simultaneously damp the other to zero. This implies that more fuel is required to satisfy Equation (7.16b) than is required to satisfy only the bracketed part of Equation (7.16b). If the vehicle parameters are properly chosen, the terminal constraint to be satisfied for the AMCO concept is simply the complex conjugate of the bracketed part of Equation (7.16b). That is, if $r = \frac{1}{2}$ then no fuel is required to damp out the initial response

$$\omega^* = \omega_o^* \epsilon^{-ir\omega_3 t}$$

because this term drops out of Equation (7.17). Hence, if $r = \frac{1}{2}$, then it is expected that for general initial conditions less fuel will always be used for the AMCO concept than for the SACO concept. Note, however, if initially $\omega^* = 0$ then the two sets of constraints are essentially the same and the same amount of fuel would be used for each. This agrees with the intuitive notion that fuel savings is achieved by not burning fuel to damp out the motion

$$\omega^* = \omega_o^* \epsilon^{-ir\omega_3 t}$$

which can effectively be damped out by the nutation damper. In the comparison of the two concepts the <u>same</u> vehicle studied in [11] will be used, viz., that for which

$$r = \frac{1}{2}$$

As noted above, this choice of r is especially desirable for the AMCO concept.

It is noted that for general initial conditions in which $r \neq \frac{1}{2}$, the relative advantages of the inclusion of the nutation damper are not as pronounced.

Even though numerical results are provided only for the case in which two two-way jets are used, it is of interest to determine if the same conclusion holds for the case in which one-way jets are used. It is clear from the above discussion that the <u>type</u> of jet did not enter into the analysis. Hence, the same conclusions would hold if two one-way jets were used. If only one jet is used the equation stated above still holds except $u_2 \equiv 0$. That is, for the SACO concept in which one jet is used, the terminal conditions are

$$0 = \omega^* = \omega_o e^{-ir\omega_3 t} + \int_{t_o}^{t_1} e^{-ir\omega_3(t-\tau)} u_1(\tau)d\tau$$

$$0 = -\frac{1}{\omega_3(1-r)}\bar{\omega}^* + \left[\left\{\left(\theta_o + \frac{\omega_1(0)}{\omega_3(1-r)}\right) - i\left(\frac{\omega_2(0)}{\omega_3(1-r)} - \phi_o\right)\right\} e^{i\omega_3 t} + \int_{t_o}^{t_1} \frac{e^{i\omega_3(t-\tau)}}{\omega_3(1-r)} u_1(\tau)d\tau\right] \quad (7\text{-}18)$$

For the AMCO concept in which one jet is used, the terminal condition is

$$0 = \left(1 - \frac{r}{1-r}\right)\omega^{*} + r\omega_3 \left[\left[\left(\theta_o + \frac{\omega_1(0)}{\omega_3(1-r)}\right) + i\left(\frac{\omega_2(0)}{\omega_3(1-r)} - \phi_o\right)\right] e^{-i\omega_3 t} + \int_{t_o}^{t_1} \frac{e^{-i\omega_3(t-\tau)}}{\omega_3(1-r)} u_1(\tau)\,d\tau\right] \tag{7-19}$$

An examination of Equations (7-18) and (7-19) reveals that the conclusions drawn for the two-jet case also hold for the case in which one jet is used.

Dual-Spin Vehicle

It is also of interest to examine the nature of the switching function and the implications of the terminal conditions for the dual-spin vehicle. Using the same procedure as discussed above, the switching function is given (with two jets assumed for generality) by

$$\underline{q}(t)_{SACO} = \begin{bmatrix} p_1(0) - \dfrac{p_4(0)}{\bar{r}\sigma} & p_2(0) + \dfrac{p_3(0)}{\bar{r}\sigma} & \dfrac{p_4(0)}{\bar{r}\sigma} & -\dfrac{p_3(0)}{\bar{r}\sigma} \\ p_2(0) + \dfrac{p_3(0)}{\bar{r}\sigma} & -\left(p_1(0) - \dfrac{p_4(0)}{\bar{r}\sigma}\right) & -\dfrac{p_3(0)}{\bar{r}\sigma} & -\dfrac{p_4(0)}{\bar{r}\sigma} \end{bmatrix} \begin{pmatrix} c\sigma(1-\bar{r})t \\ s\sigma(1-\bar{r})t \\ c\sigma t \\ s\sigma t \end{pmatrix}$$

$$\underline{q}(t)_{AMCO} = \begin{bmatrix} \nu_1 & \nu_2 \\ \nu_2 & -\nu_1 \end{bmatrix} \begin{pmatrix} c\sigma t \\ s\sigma t \end{pmatrix} \tag{7-20}$$

$$\text{where } \bar{r} = \frac{J_3^R}{I_1}$$

An examination of Equation (7-20) reveals that the number of switchings depends on the rotor speed σ. It is also noted that the modes $c\sigma(1-\bar{r})t$ and $s\sigma(1-\bar{r})t$ are not used in representing $\underline{q}$ in the AMCO concept.

Concerning the terminal conditions, for the SACO concept the condition $\underline{x}(t_1) = 0$ is given by

$$0 = \omega^* = \omega_o^* e^{i\bar{r}\sigma t} + \int_{t_o}^{t_1} e^{i[\bar{r}\sigma(t-\tau)+\sigma\tau]} \left[u_1(\tau) + i\, u_2(\tau)\right] d\tau$$

$$0 = -\frac{1}{\bar{r}\sigma}\, \omega^* + \left[\left(\theta_2(0) \frac{\omega_1(0)}{\bar{r}\sigma}\right) - i \left(\theta_1(0) - \frac{\bar{\omega}_2(0)}{\bar{r}\sigma}\right)\right. \tag{7-21}$$

$$\left. + \frac{1}{\bar{r}\sigma} \int_{t_o}^{t_1} e^{i\sigma\tau} \left[u_1(\tau) + i\, u_2(\tau)\right]\right] d\tau \;,$$

while for the AMCO concept, the terminal condition is given by

$$0 = \bar{r}\sigma \left[\left(\theta_2(0) + \frac{\omega_1(0)}{\bar{r}\sigma}\right) - i \left(\theta_1(0) - \frac{\omega_2(0)}{\bar{r}\sigma}\right)\right.$$

$$\left. + \frac{1}{\bar{r}\sigma} \int_{t_o}^{t_2} \epsilon^{i\sigma\tau} \left(u_1(\tau) + i(u_2(\tau)\right) d\tau\right] \tag{7-22}$$

An examination of Equations (7-21) and (7-22) reveals that conclusions similar to those drawn for the symmetric vehicle can be drawn for the dual-spin vehicle. An important difference is that for the dual-spin vehicle, regardless of the vehicle parameters less fuel is used for the AMCO concept than for the SACO concept for all initial conditions except those for which $\omega^* \equiv 0$. When $\omega^* \equiv 0$, the two concepts are identical. This conclusion is in complete agreement with the notion that the nutation damper damps out the term

$$\omega^* = \omega_o^* \; \epsilon^{i\bar{r}\sigma t}$$

and if the complex mode $\epsilon^{i\bar{r}\sigma t}$ is not excited (that is $\omega_o^* \equiv 0$) then no fuel is required to damp it out and, hence, no fuel savings can be realized. These conclusions hold also for the case in which one jet is used. These results can be obtained by setting $u_2(\tau) \equiv 0$ in Equations (7-21) and (7-22).

The foregoing analysis is extremely important because it shows the relationship between the expected fuel savings, the vehicle parameters, and the initial conditions. The use of the function space having as its basis the modes of the system, proved to be an invaluable tool in this analysis. Note that these conclusions could not have been reached as easily by using the computer.

Numerical Results for the Symmetric Spinning Vehicle

Figures 7-5 through 7-7 provide the results for the SACO concept. Figure 7-5 shows the nature of the optimal controller $\underline{u}^*(t)$ and the curves ω_1 versus time and ω_2 versus time. It is noted each component of $\underline{u}^*(t)$ switches eight times. Figure 7-6 shows the curves θ versus time and ϕ versus time. Close examination of Figure 7-5 and 7-6 reveals that each switching of the optimal controller can be explained; that is, the results are consistent with the terminal conditions that must be satisfied. The terminal conditions were previously given as (recall that $\phi_1 \equiv \phi$ and $\phi_2 \equiv \theta$)

$$\omega^* = \omega_o^* \, \epsilon^{-ir\omega_3 t} + \int_{t_o}^{t_1} \epsilon^{-ir\omega_3(t-\tau)} \left[u_1(\tau) + i\, u_2(\tau)\right] d\tau = 0 \qquad (7\text{-}23a)$$

$$-\frac{1}{\omega_3(1-r)}\,\bar{\omega}^* + \left[\left\{\left(\theta_o + \frac{\omega_{10}}{\omega_3(1-r)}\right) - i\left(\frac{\omega_2(0)}{\omega_3(1-r)} - \phi_o\right)\right\}\epsilon^{i\omega_3 t} + \int_{t_o}^{t_1} \frac{e^{i\omega_3(t-\tau)}}{\omega_3(1-r)}\left[u_1(\tau) - i\, u_2(\tau)\right] d\tau\right] = 0 \qquad (7\text{-}23b)$$

Initially the switchings are such that the bracketed term of Equation (7-23b) is decreased. This is accomplished by applying a torque to damp the amplitude of ϕ. These initial switchings tend to get the

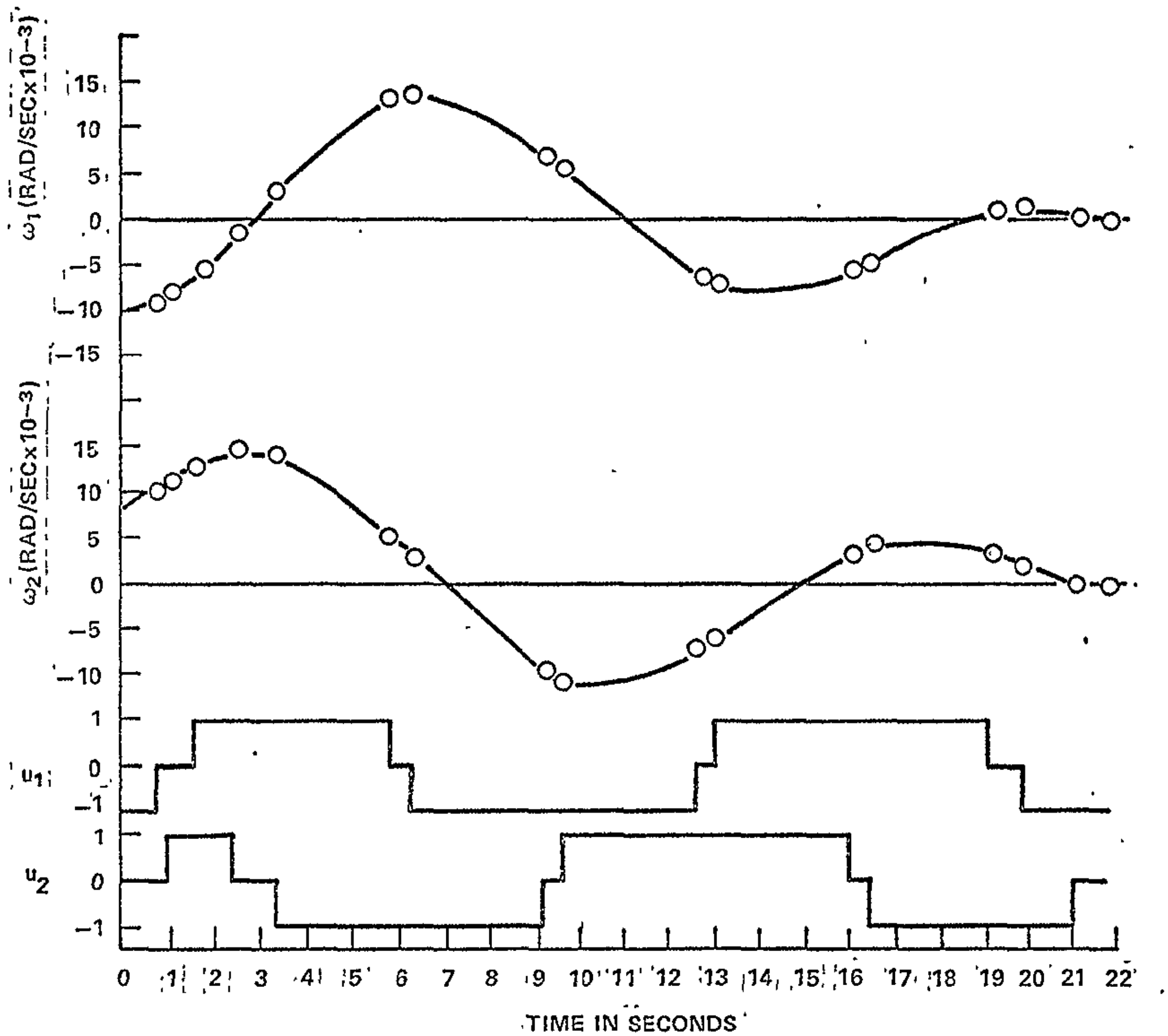


Figure 7-5 Control $\underline{u}$ and Transverse Angular Velocity $\underline{\omega}$ for Symmetric Spinning Vehicle Using SACO Concept

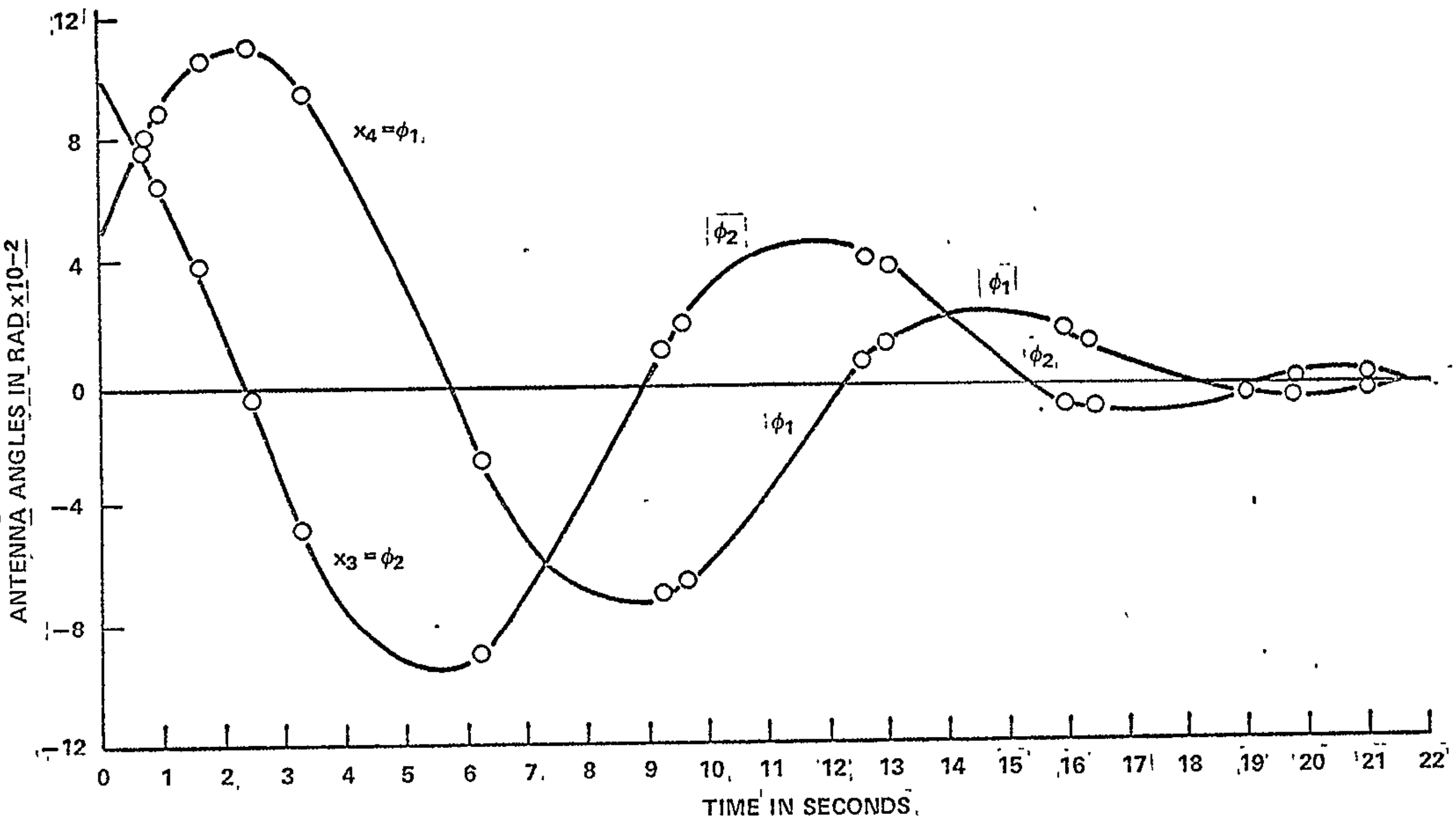
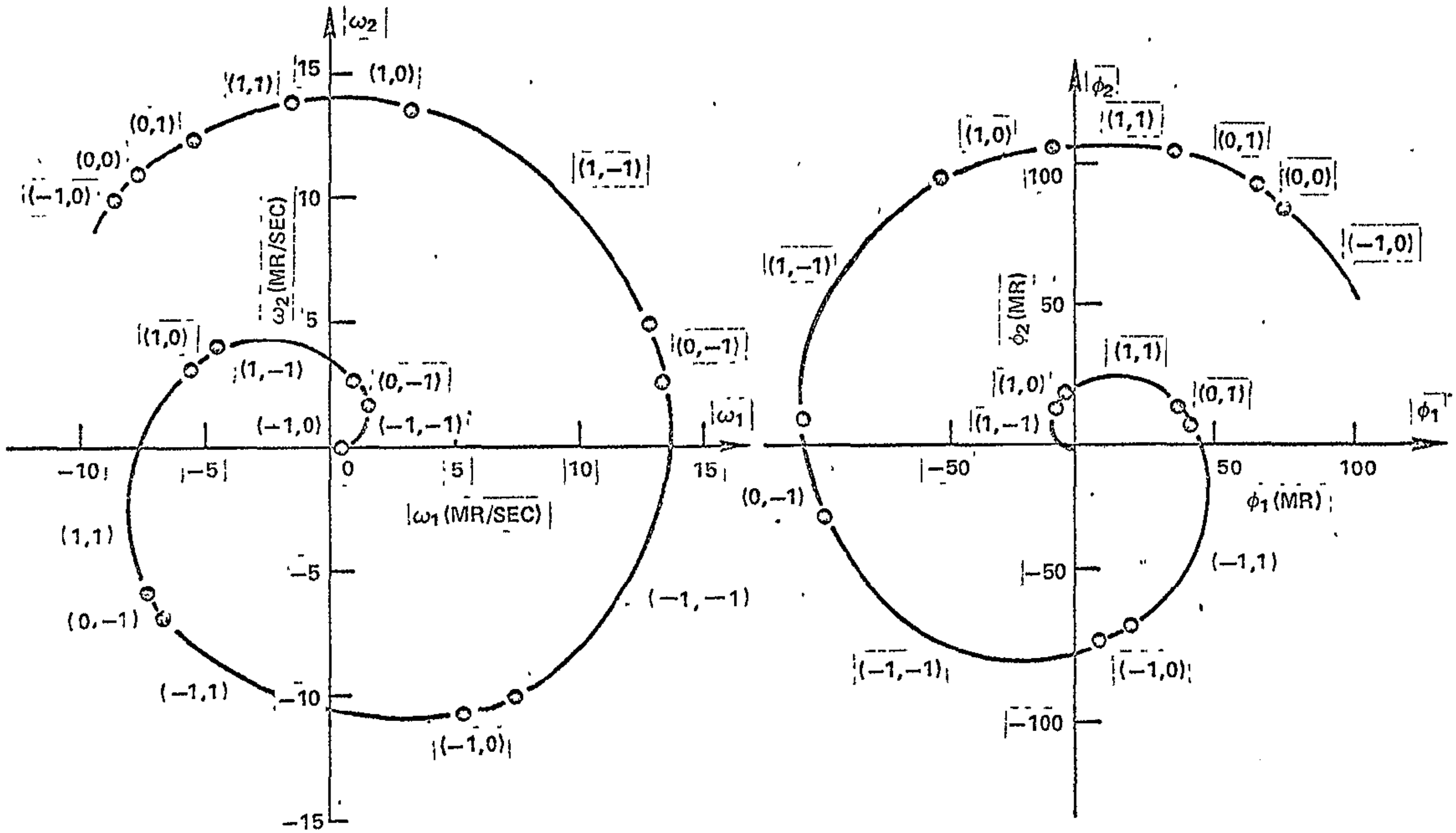


Figure 7-6 Antenna Angles vs Time for Symmetric Spinning Vehicle Using SACO Concept



Note: (−1,1) means $u_1 = -1$, $u_2 = 1$, etc.

Figure 7-7 Trajectories of ω_2 vs ω_1 and ϕ_2 vs ϕ_1 for Symmetric Spinning Vehicle Using SACO Concept modes in phase so that later they can be damped to zero simultaneously. Once the modes are approximately in phase, there is a 1 : 1 correspondence between the jet firings and the times the components ω_1 and ω_2 reach their maximum amplitudes, that is, to reduce the amplitude of ω_1 a torque is applied slightly before ω_1 reaches its crest. The same remark holds for ω_2 - - the jet designated u_2 is fired slightly before ω_2 reaches its crest. The notion of damping the sinusoid slightly before it reaches its crest is intuitively appealing.

As seen in Figure 7-5 the off-times of the controllers are very short. Fuel, of course, can be saved only during the times the jets are off. Figure 7-6 illustrates the time history of the antenna pointing errors for the SACO concept. Figure 7-7 shows the phase plane plots of ω_2 versus ω_1 and ϕ_2 versus ϕ_1. In Reference [11], the equivalent of Figure 7-7 is provided. The results obtained in this work support those given in [11]. An important observation concerning the SACO concept is that the jet firings are not in 1 : 1 correspondence with the spin rate of the vehicle because the response is given in terms of the complex modes $\epsilon^{-ir\omega_3 t}$ and $\epsilon^{i\omega_3 t}$.

AMCO Results

The results for the AMCO concept are provided in Figures 7-8 through 7-14. Figure 7-8 shows the controller $\underline{u}^*(t)$ and the switching function $\underline{q}^*(t)$. It is immediately noted that each controller is off almost 50% of the time implying that fuel is consumed during these times. Figure 7-9 provides the time histories of the transverse components of the angular momentum vector and the magnitude of the transverse angular momentum. Examination of Figures 7-8 and 7-9 reveals that the jet firings are in 1 : 1 correspondence with the times the components of the transverse angular momentum reach

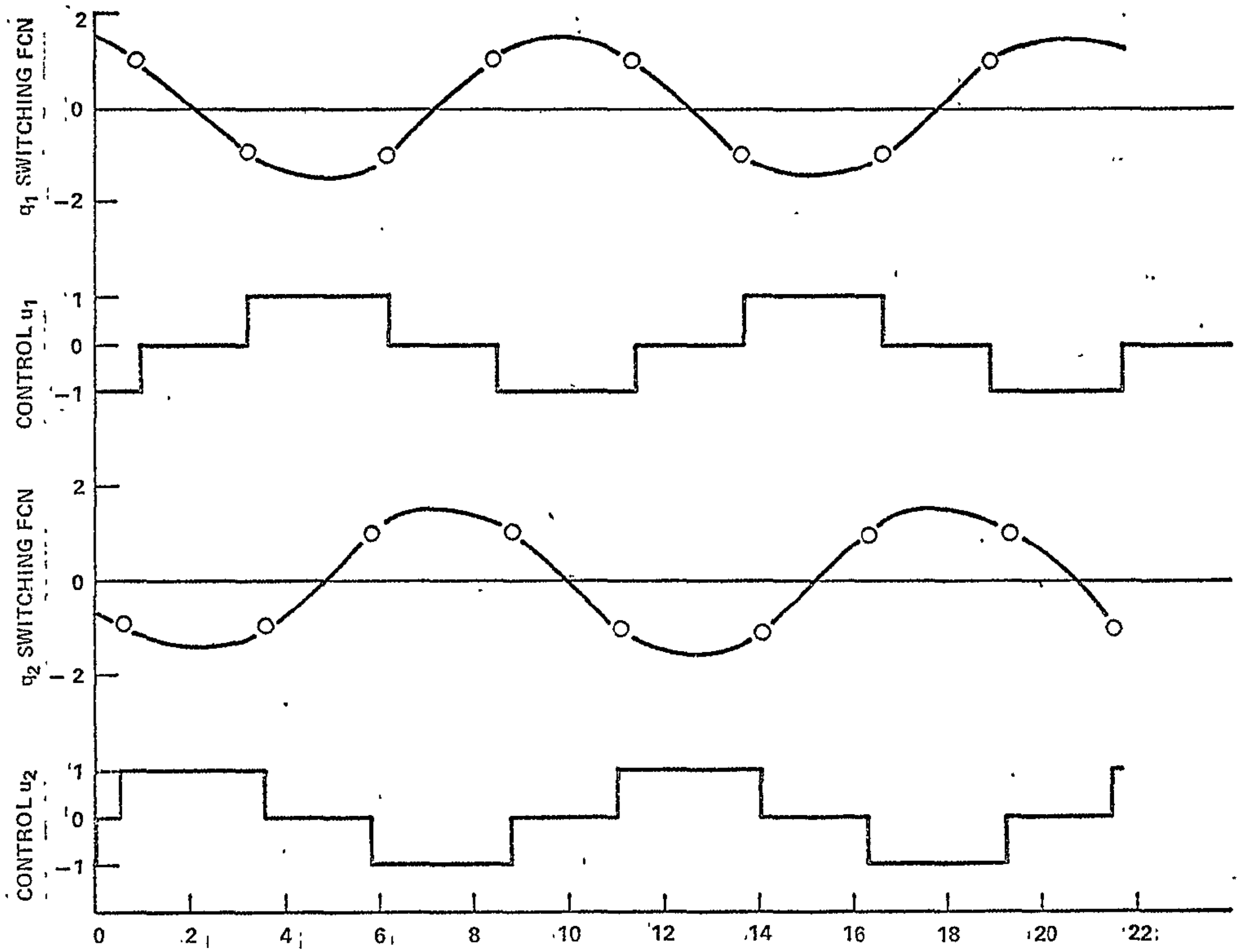


Figure 7-8 Switching Function $\underline{q}$ and Control $\underline{u}$ vs Time for Symmetric Spinning Vehicle Using AMCO Concept I

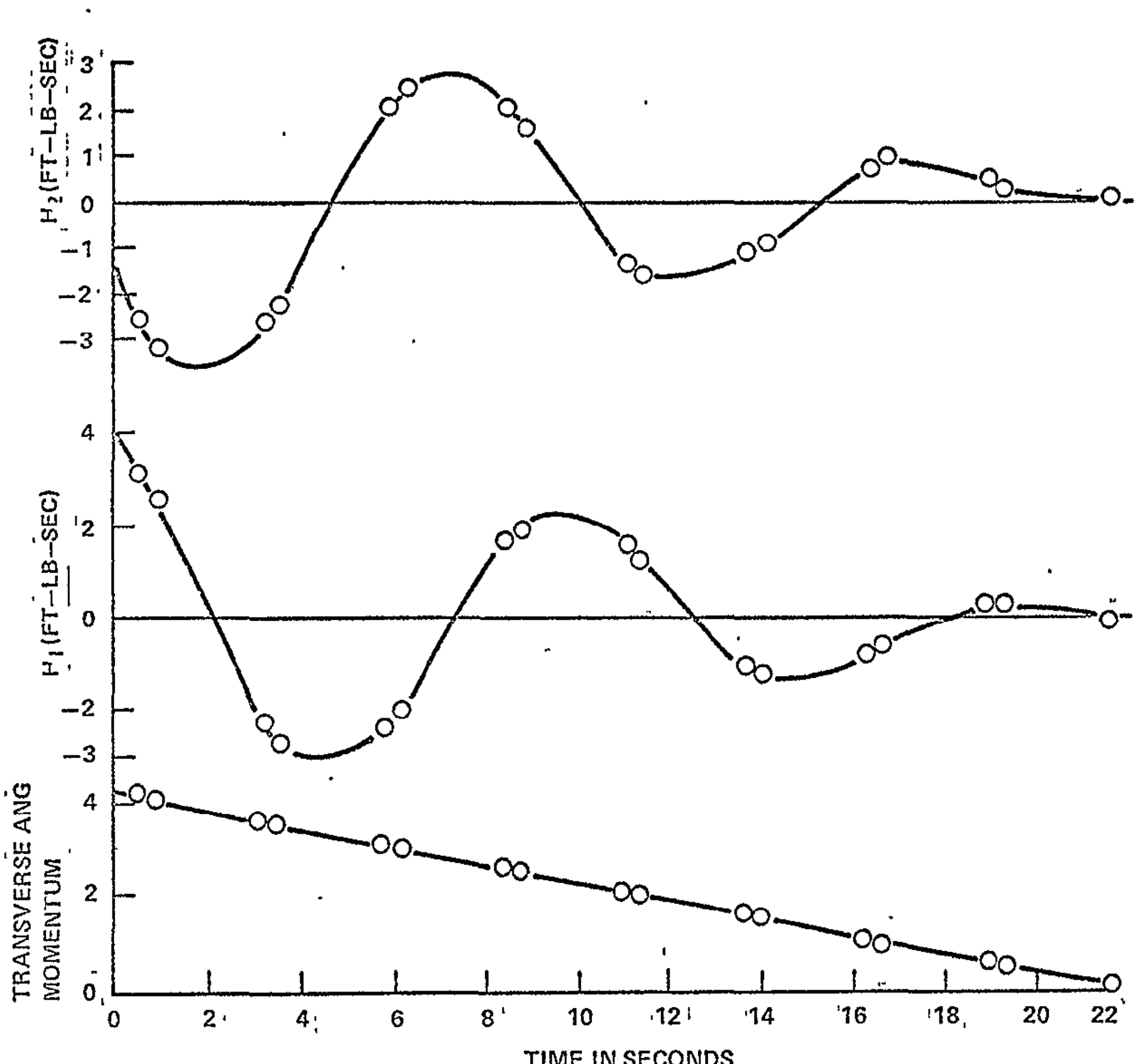


Figure 7-9 Transverse Angular Momentum vs Time for Symmetric Spinning Vehicle Using AMCO Concept

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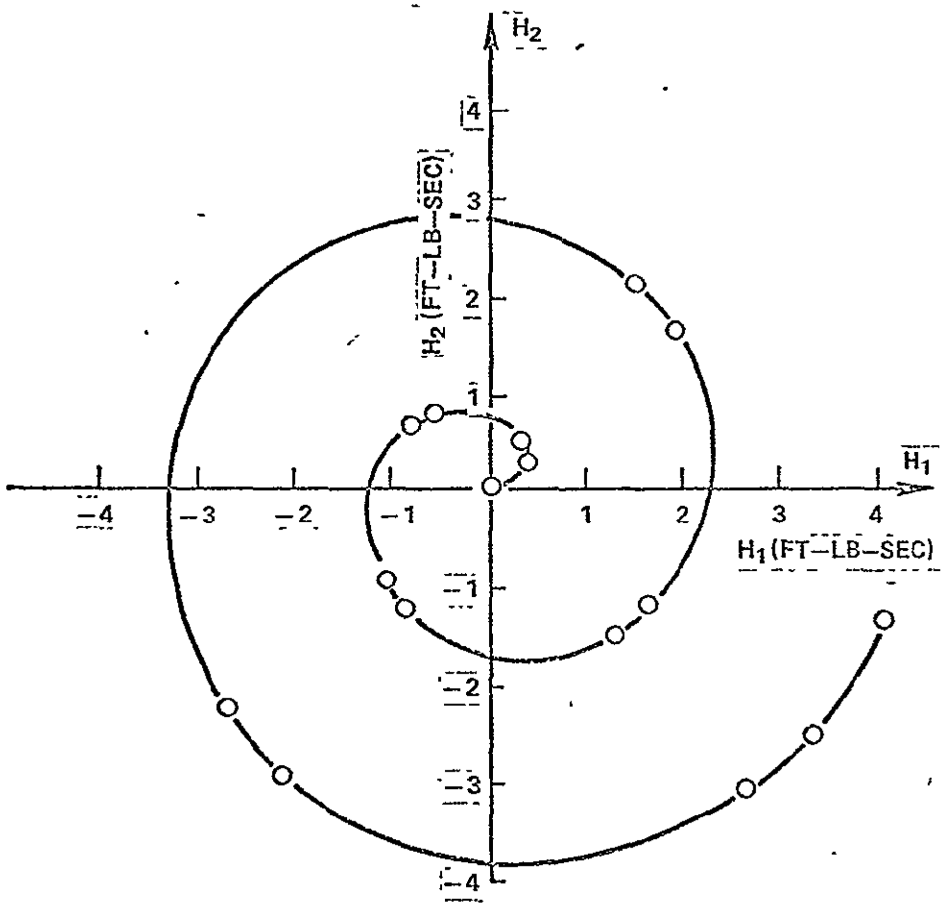

Figure 7-10 Transverse Components of Angular Momentum H_2 vs H_1 for Symmetric Spinning Vehicle Using AMCO Concept

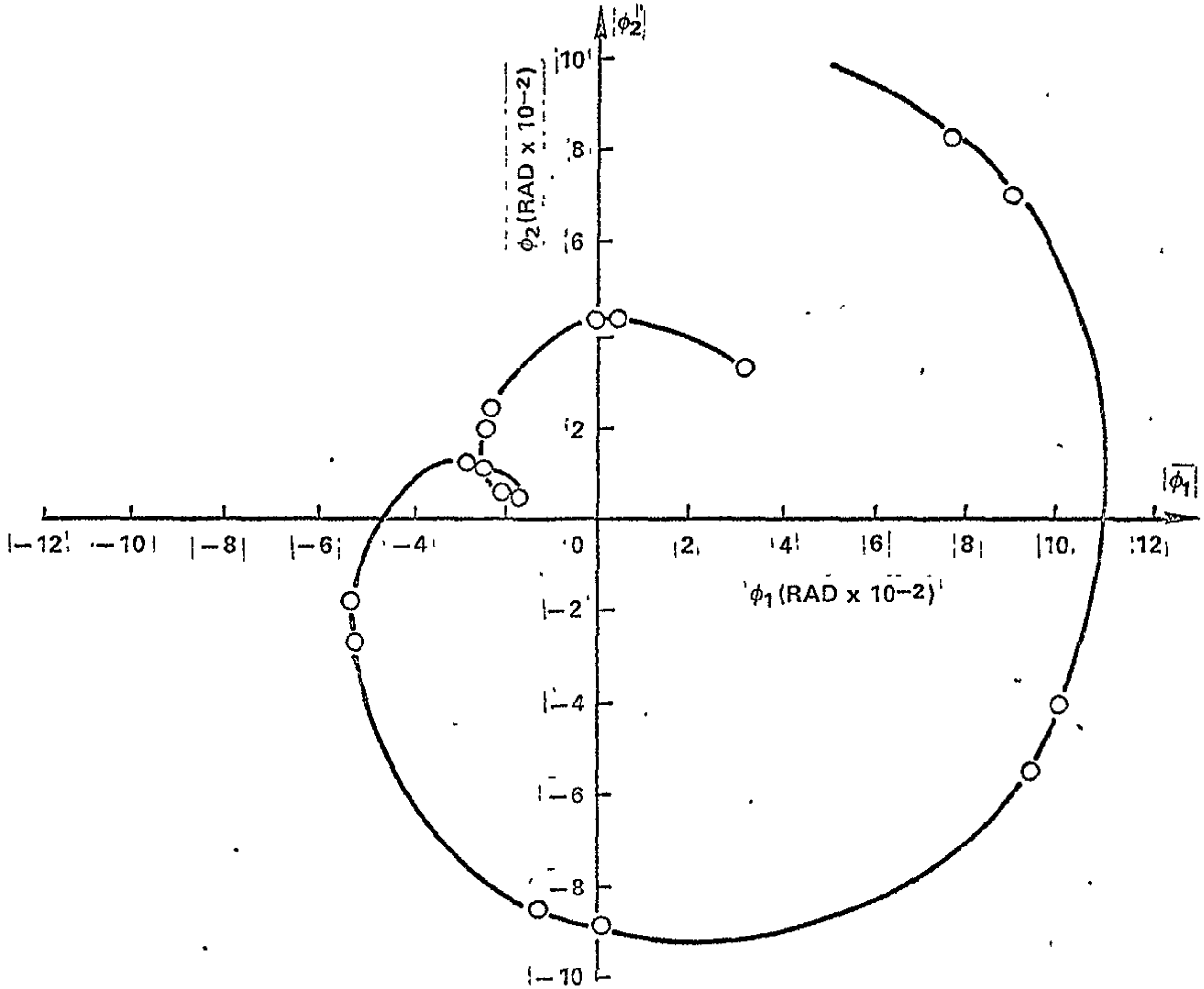

Figure 7-11 Antenna Angles ϕ_2 vs ϕ_1 for Symmetric Spinning Vehicle Using AMCO Concept

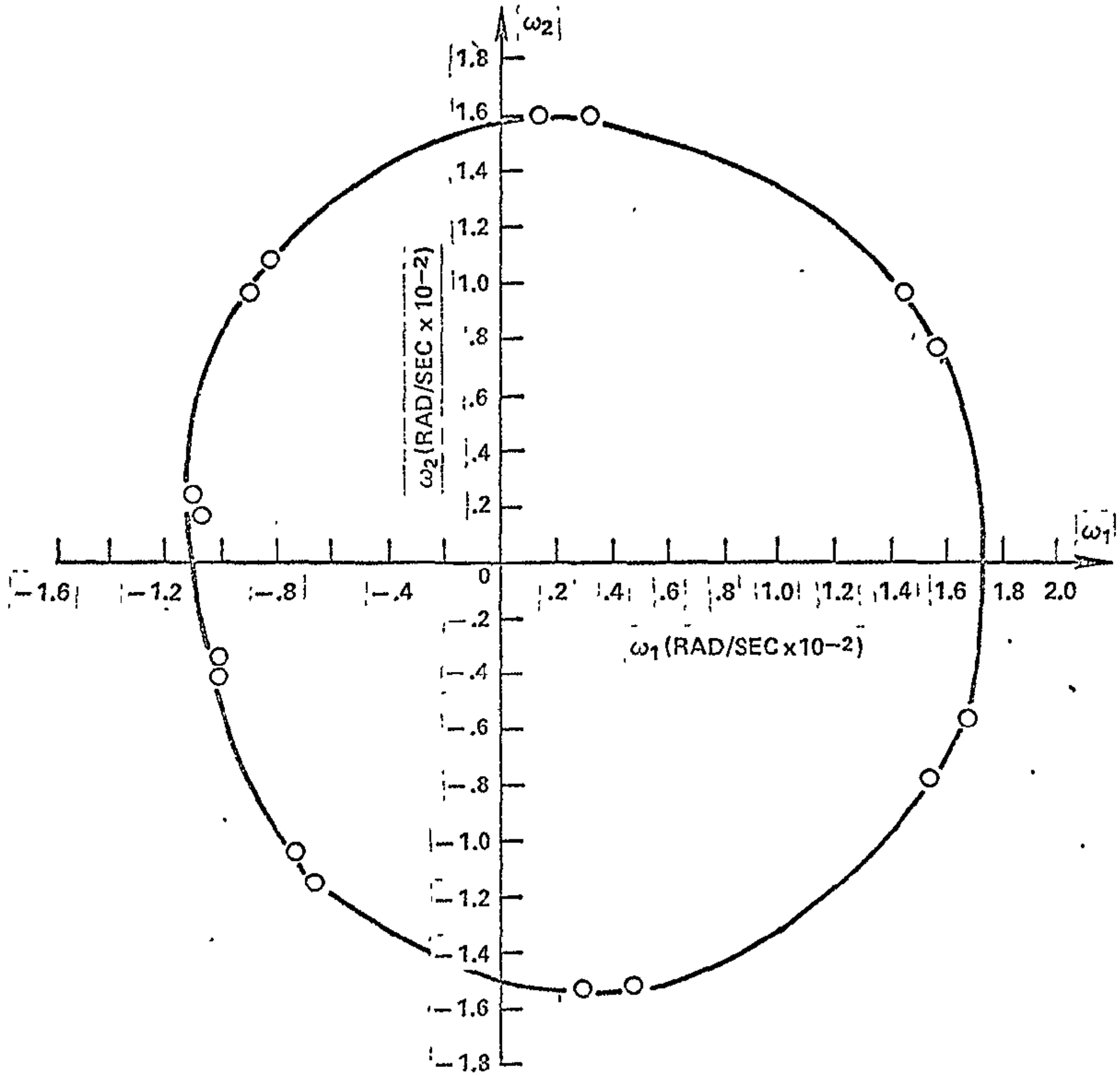

Figure 7-12 Trajectory of ω_2 vs ω_1 for Symmetric Spinning Vehicle Using AMCO Concept

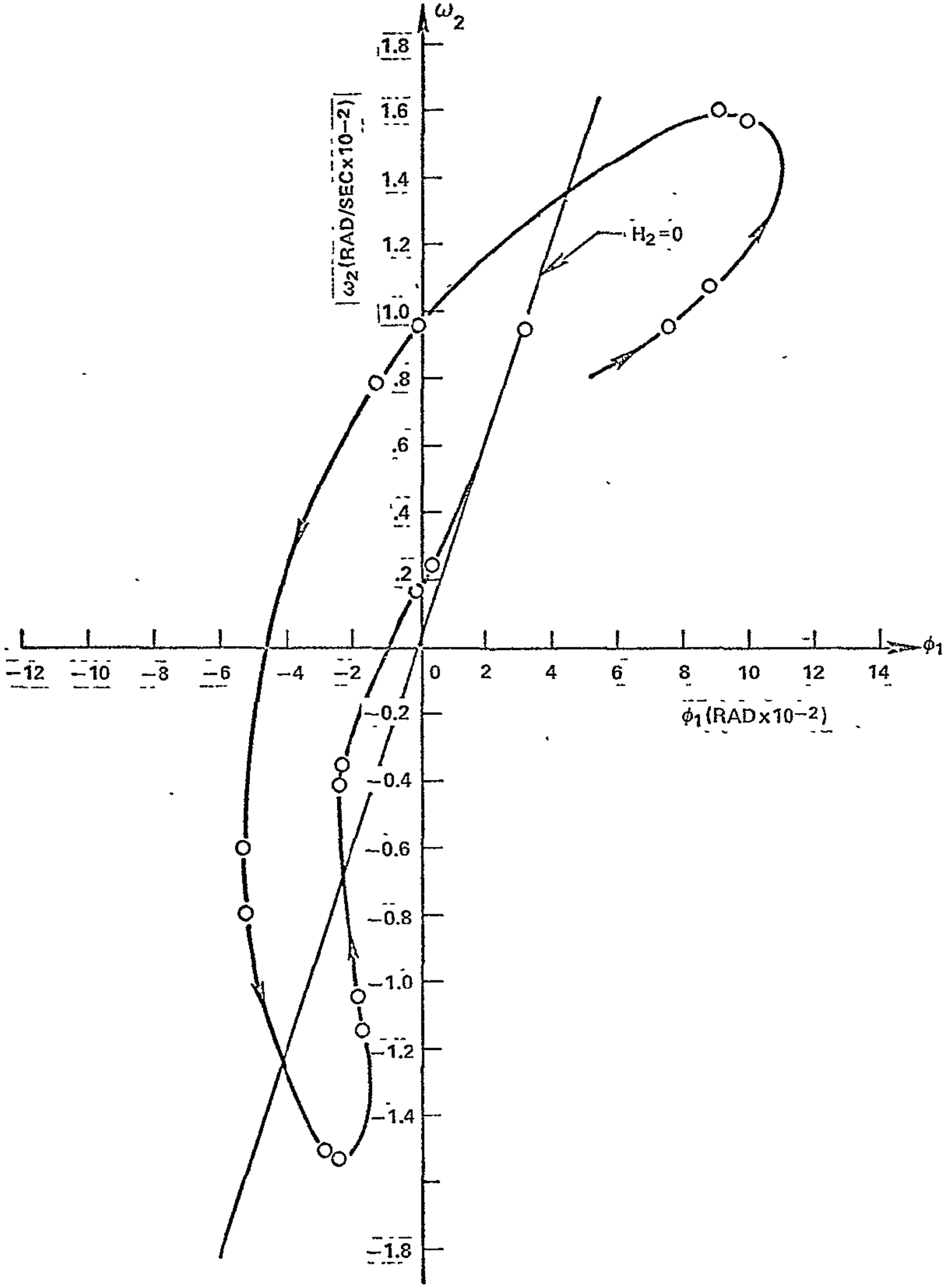

Figure 7-13 Trajectory of ω_2 vs ϕ_1 for Symmetric Spinning Vehicle Using AMCO Concept

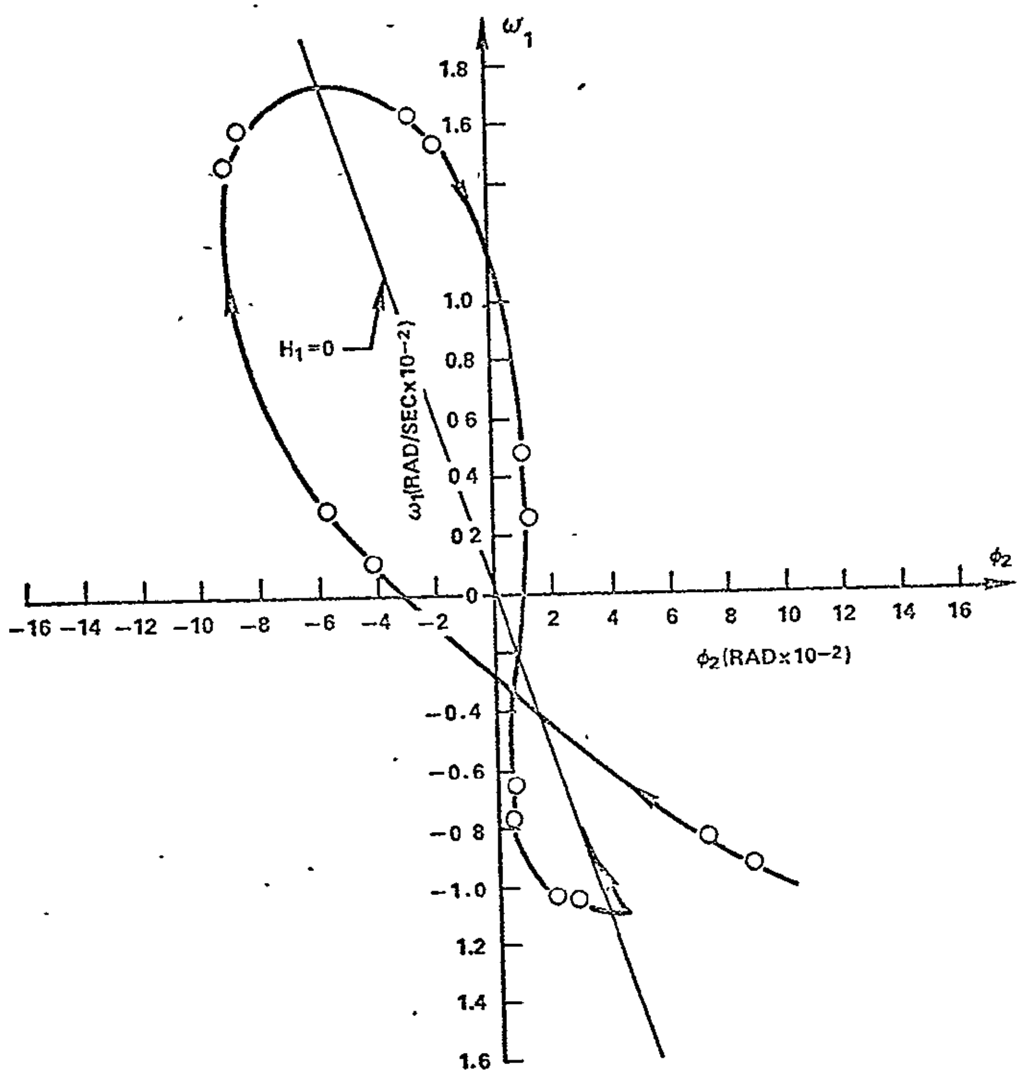

Figure 7-14 Trajectory of ω_1 vs ϕ_2 for Symmetric Spinning Vehicle Using AMCO Concept

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their crests. Also, since only the modes of the system having a frequency equal to the spin rate are excited, the jet firings are in 1 : 1 correspondence with the spin rate. That is, effectively, two impulses are applied each revolution of the body to damp out H_1 and two impulses are applied each revolution to damp out H_2. The impulses are applied slightly before the sinusoids reach their crests. This suggests that it may be possible to devise an advantageous suboptimal method of synthesizing the optimal controller.† This result is especially interesting because the notion of a two-impulse scheme has been used by the attitude control engineer in the past without giving any attention to the question of optimality. Another observation that can be made from Figures 7-8 and 7-9 is that the controllers are never off simultaneously. This implies that $\underline{H}$ cannot be constant after the optimal control sequence is initiated. The implication of this is that the transverse angular momentum is a strictly monotonically decreasing function. Note that this would not be true if only one jet were used as was seen previously when the dual-spin vehicle results were examined. This suggests that two jets can accomplish the task in less time than one jet. This assertion was demonstrated for the symmetric spinning body by using only one jet to accomplish the control objective. This fact implies that there is a trade-off between the number of jets used and the time taken to accomplish the control task. If the spin rate is sufficiently great, then only one jet would be appropriate. If the body is slowly-spinning and the thrust capacity of the jet cannot be increased, then it may be desirable to use two one-way jets. However, if a jet having a greater thrust capacity were used, one jet would be suitable even for slowly-spinning vehicles.

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†This aspect will be considered in more detail at a later time.

Figure 7-10 shows the trajectory in angular momentum space. The initial condition $(H_1, H_2) = (4, -1.4)$ ft # sec is driven to $(-0.017, 0.016)$ ft # sec. The trajectory has a spiral shape. Figure 7-11 shows the antenna angles; the initial angles $(\phi_2, \phi_1) = (10, 5) \times 10^{-2}$ rad are driven to $(3.1, 3) \times 10^{-2}$ rad. Figure 7-12 shows the plot of ω^*. The plot of ω^* is circular as it should be since in the AMCO concept no fuel is used to damp out this response. Recall that it was stated previously that the trace swept out on the energy ellipsoid by the tip of the angular velocity vector is circular for a symmetric spinning body in a torque-free environment.

Figures 7-12 and 7-13 show how the target is approached. The target set consists of the lines shown in Figures 7-12 and 7-13. It is noted that the ω_1-ϕ_2 and ω_2-ϕ_1 trajectories approach these lines tangentially at the final time.

The important conclusions drawn from the comparison of the two concepts are that

(1) 35% less fuel is used for the AMCO concept than for the SACO concept

(2) the fact that the jet firings associated with the AMCO concept are in 1 : 1 correspondence with the spin rate while in the SACO concept they are not implies that there is a strong likelihood that the synthesis of a sub-optimal controller for the AMCO concept would be considerably simpler than that for the SACO concept.†

7.5 Conclusions

In this section, some concluding remarks concerning this work are provided. Some conclusions concerning the importance of such theoretical notions as controllability, normality, existence, and

†This possibility should be examined at a later time.

uniqueness and their effects on the computational algorithm have already been given (see Section 7.1). The importance of these items is evidenced by the facts that controllability aided in the determination of the number of jets required and normality aided in the determination of the location of these jets. In this section, the conclusions deal with the practically motivated innovations, introduced in this work in the formulation of the fuel-optimal control problems, for a class of dual-spin spacecraft.

The inclusion of the nutation damper as a passive means of control led to what is called in this work an angular momentum control (AMCO) concept. The distinguishing feature of the AMCO concept is that the target set is a smooth 2-fold in R^n rather than a fixed point. When this concept is compared to the more conventional formulation (called a spin axis control (SACO) concept in this work), some dramatic differences are noted. The comparison entailed the determination of the fuel-optimal controller for a symmetric spinning vehicle for each concept. The practically motivated scheme (AMCO concept) used 35% less fuel than the SACO concept in achieving the identical control objective. This result is startling in that the solution obtained from each method is called the "fuel-optimal" controller. This experiment dramatically illustrates the importance of practical considerations in formulating an optimal control problem.

By expressing the terminal constraints associated with each concept in terms of the modes of the system, it was shown that if the symmetric spinning vehicle is designed such that

$$r = \frac{I_1 - I_3}{I_1} = \frac{1}{2}$$

then for all meaningful initial conditions, fuel savings can be realized by using the AMCO concept rather than the SACO concept. For a dual-spin vehicle, the statement is true regardless of the inertia characteristics of the system.

Another important consequence of the use of the AMCO concept concerns the computational aspects of the determination of the fuel-optimal controller. The number of variables to be iterated on in using the AMCO concept is two less than that required for the SACO concept. This results in a reduction in computer running time. In addition, the convergence characteristics of the AMCO concept are better than for the SACO concept. This is due to the fact that in using the classical Newton-Raphson algorithm, the final adjoint variables are iterated on in the AMCO concept while the initial adjoint variables are iterated on in the SACO concept.

Another practically motivated innovation introduced in this work pertains to the choice of the suitable control restraint set. For spinning vehicles, the use of the control restraint set

$$\Omega = \{ \underline{u} \; : \; 0 \le u_j(t) \le 1 \quad \forall j \}$$

has a distinct advantage over that which is customarily used for fuel-optimal problems, viz.,

$$\Omega = \{ \underline{u} \; : \; |u_j(t)| \le 1 \quad \forall j \}$$

These control sets are associated with one-way and two-way jets, respectively. By using the necessary conditions of optimality, the nature of the optimal controller for each Ω can be obtained. Examination of the optimal controllers for each Ω reveals that if the vehicle is spinning sufficiently rapidly then the number of firings (switchings) associated with the one-way jet for the problems

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studied in this work is significantly less than that associated with the use of a two-way jet. This fact has a considerable bearing on the reliability of the system. An increase in the number of jet firings is accompanied by an increase in the probability of jet failure. Although such factors as reliability are difficult to incorporate into an optimal control problem formulation, nevertheless, in the final analysis, they must be given due consideration.

One of the important conclusions drawn from this work is that optimization theory can be used very effectively in the preliminary design of competitive spacecraft. By determining the optimal configurations of various systems that are considered potential candidates for the given task, and then factoring in such factors as cost in dollars, reliability, weight, power requirements, etc., the most suitable system can be selected. It is important to recognize that it is not always advantageous to implement the optimal controller. Nevertheless, the optimal scheme provides a very good standard for judging the performance of the scheme that is implemented. The writer feels that this latter application of optimal control techniques has a promising future.

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