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CALCULATION OF THE PROFILES OF THE STOKES PARAMETERS AND THE DEGREE OF LINE POLARIZATION: AN APPLICATION OF THE SOLUTIONS OF MOE TO THE UNNO TRANSFER EQUATIONS

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16. ABSTRACT

An outline is given of the solutions of Olav K. Moe to the Unno transfer equations for the Stokes parameters. The application of these solutions to the calculations of the profiles of the Stokes parameters and to the calculations of the degrees of linear and circular polarization using a narrow-band birefringent filter is described in detail. Numerical results for the iron line 5250.22 Å using the photospheric model of Holweger are presented.

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SUMMARY

This report outlines a particular solution to the Unno transfer equations and describes in detail its application to the calculations of the line profiles of the Stokes parameters and to the calculations of the linear and circular polarization integrated over the bandpass of a birefringent filter.

Unno has derived the four transfer equations for the Stokes parameters that describe the state of polarization of a solar absorption line formed in a homogeneous magnetic field. Moe has found a solution to these equations under the assumption that the normalized function describing the variation with wavelength of the line absorption coefficient is independent of optical depth. These solutions must be numerically evaluated using a model of the solar atmosphere that gives the variations of temperature, gas pressure, and electron pressure with optical depth.

In the numerical calculation, three depth-dependent functions are calculated: the source functions, the continuous absorption coefficient at the wavelength of the absorption line K_λ^c, and the line absorption coefficient at line center with zero damping $K_\lambda^l(0)$. The line is assumed to be formed by the mechanism of pure absorption so that the source functions become identical to the Planck distribution function which can be written with an explicit temperature dependence. K_λ^c is calculated in a subprogram whose form is dictated by the model atmosphere. The "fictitious" line absorption coefficient $K_\lambda^l(0)$ is calculated by assuming that the

population density of the lower level of the absorption transition can be calculated from the Saha and Boltzmann equations; this implies the assumption of local thermodynamic equilibrium.

The wavelength-dependent part of the line absorption coefficient, which Moe has assumed to be independent of optical depth, is calculated by assuming contributions from natural, collisional, and thermal broadening mechanisms. It is assumed that these can be treated independently so that the resulting profile is the Voigt function which has been numerically tabulated for different values of the total damping parameter.

The linear and circular polarization at a particular wavelength in the line is obtained from the linear (Q) and circular (V) Stokes parameters. In a magnetograph system using a narrow-band filter, the polarization that is measured is the contribution from all wavelengths in the interval of the bandpass. The actual measured quantities are thus numerically evaluated from the solutions for the Stokes parameters (as described above) and from the transmission characteristics of the filter.

Numerical results are calculated for the Stokes parameters and the degree of linear and circular polarized light for the case of the iron line 5250.22 Å using the photospheric model of Holweger.

INTRODUCTION

Recently Moe [1] has obtained a solution to the Unno [2] transfer equations for the Stokes parameters, assuming only that the magnetic field is homogeneous and that the line absorption coefficient profile is independent of depth. In this paper the applications of this solution are described first for the calculations of the line profiles of the Stokes parameters and, secondly, for the calculations of the degrees of linear and circular polarizations that would be theoretically measured by a real-time solar magnetograph system using a narrow-band birefringent filter.

EQUATIONS DERIVED BY MOE

The transfer equations of Unno [2] as modified by Moe [1] to include contributions to the Stokes parameters from scattering as well as from thermal emission are:

$$\mu \frac{dI}{d\tau_c} = (1+\eta_I)(I - S_I) + \eta_Q Q + \eta_V V$$

$$\mu \frac{dQ}{d\tau_c} = (1+\eta_I)Q + \eta_Q (I - S_Q) \quad , \quad \text{and} \quad (1)$$

$$\mu \frac{dV}{d\tau_c} = (1+\eta_I)V + \eta_V (I - S_V)$$

In these equations I, Q, and V are the Stokes parameters that describe the state of polarization of light of intensity I (in the Unno coordinate system, for a homogeneous magnetic field, the fourth parameter U is zero); $\mu = \cos\theta$ where θ is the angle between the solar radius and the direction of the light beam; τ_c is the optical depth in the continuum; and S_I, S_Q, and S_V are the three source functions for the Stokes parameters. The functions η_I, η_Q, and η_V are defined as follows:

$$\eta_I = \frac{K_\pi}{K_\lambda^c} \frac{\sin^2\psi}{2} + \left(\frac{K_l}{K_\lambda^c} + \frac{K_r}{K_\lambda^c}\right)\left(\frac{1+\cos^2\psi}{4}\right) \quad ,$$

$$\eta_Q = \left[\frac{1}{2}\frac{K_\pi}{K_\lambda^c} - \frac{1}{4}\left(\frac{K_l}{K_\lambda^c} + \frac{K_r}{K_\lambda^c}\right)\right] \sin^2\psi \quad , \quad \text{and} \quad (2)$$

$$\eta_V = \frac{1}{2}\left(\frac{K_r}{K_\lambda^c} - \frac{K_l}{K_\lambda^c}\right)\cos\psi$$

In these equations, ψ is the angle between the magnetic field vector and the line of sight, K_λ^c is the continuous absorption coefficient at the wavelength of the line, and K_π, K_r, and K_l are the absorption coefficients for linearly, right-circularly, and left-circularly polarized light respectively.

It is assumed that these coefficients are identical to the line absorption coefficient $K_\lambda^l(\lambda)$ in the absence of a magnetic field except for the Zeeman displacement $\Delta\lambda_H$ in wavelength; for a simple Zeeman triplet,

$$\pm\Delta\lambda_H = 4.67 \times 10^{-5}\, g\lambda_0^2 H$$

(in centimeters with λ_0 in centimeters), where g is the Lande g factor and H is the strength (in gauss) of the magnetic field. Thus,

$$K_\pi(\lambda) = K_l(\lambda \pm \Delta\lambda_H) = K_r(\lambda \mp \Delta\lambda_H) = K_\lambda^l(\lambda)$$

The line absorption coefficient in the absence of a magnetic field, $K_\lambda^l(\lambda)$, can be written as

$$K_\lambda^l(\lambda) = K_\lambda^l(0)\, H(a, v) \quad ,$$

where $K_\lambda^l(0)$ is the line absorption coefficient in the line center for zero damping, and $H(a, v)$ is the normalized shape function for the line absorption coefficient. $H(a, v)$ depends on the damping parameter "a" and the variable $v = (\lambda - \lambda_0)/\Delta\lambda_D$, where $\Delta\lambda_D$ is the Doppler (or thermal) broadening factor,

$$\Delta\lambda_D = \frac{\lambda_0}{c}\sqrt{\frac{2kT}{M} + \xi^2} \qquad (3)$$

In this expression, c is the speed of light, k is the Boltzmann constant, T is the absolute temperature, M is the mass of the atom being considered, and ξ is the microturbulent velocity of individual elements of the emitting gas. The function $H(a, v)$ is usually assumed to be a folding of a Lorentzian (damping) profile and a Doppler profile,

$$H(a, v) = \frac{a}{\pi}\int_{-\infty}^{+\infty} \frac{e^{-y^2}\, dy}{a^2 + (v - y)^2} \quad ;$$

various authors have described this function. (See Jefferies [3] for a list of references.)

From the above discussion, the following can now be written:

$$\frac{K_\pi}{K_\lambda^c} = \frac{K_\lambda^l(0)}{K_\lambda^c} \cdot H(a, v) \quad ,$$

$$\frac{K_r}{K_\lambda^c} = \frac{K_\lambda^l(0)}{K_\lambda^c} \cdot H\left(a, v \pm \frac{\Delta\lambda_H}{\Delta\lambda_D}\right) \quad , \quad \text{and} \quad (4)$$

$$\frac{K_l}{K_\lambda^c} = \frac{K_\lambda^l(0)}{K_\lambda^c} \cdot H\left(a, v \mp \frac{\Delta\lambda_H}{\Delta\lambda_D}\right)$$

Substituting equations (4) into equations (2) and defining $V_H = \frac{\Delta\lambda_H}{\Delta\lambda_D}$, the following are obtained for the Unno variables η_I , η_Q , and η_V :

$$\eta_I = \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ \frac{1}{2} H(a,v) \sin^2\psi + \left[H\left(a, v \pm V_H\right) + H\left(a, v \mp V_H\right)\right] \cdot \left[\frac{1+\cos^2\psi}{4}\right] \right\} \equiv \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ F_I \right\} \quad ,$$

$$\eta_Q = \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ \left[\frac{1}{2} H(a,v) - \frac{1}{4}\left(H\left(a, v \pm V_H\right) + H\left(a, v \mp V_H\right)\right)\right] \cdot \left[\sin^2\psi\right] \right\} \equiv \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ F_Q \right\} \quad , \quad \text{and}$$

$$\eta_V = \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ \frac{1}{2}\left[H\left(a, v \pm v_H\right) - H\left(a, v \mp v_H\right)\right] \left[\cos\psi\right] \right\}$$

$$\equiv \frac{K_\lambda^l(0)}{K_\lambda^c} \left\{ F_V \right\} \qquad (5)$$

In his solution, Moe assumes that $\frac{K_\lambda^l(0)}{K_\lambda^c}$ is independent of wavelength (which is valid since the continuous absorption varies extremely slowly over the wavelength interval of the absorption line) but dependent on optical depth τ_c, whereas $F_{I,Q,V}$ are independent of depth but are wavelength-dependent. The assumption that $F_{I,Q,V}$ do not vary with depth imposes the constraint that the magnetic field must not vary with depth; i.e., it must be homogeneous. It also means that $H(a,v)$ must also be constant with depth, a condition that implies the damping ("a") and thermal ($\Delta\lambda_D$ in v) broadening parameters cannot vary with depth in the expression for $H(a,v)$. Under these assumptions, Moe finds the following solutions for the emergent intensities of the Stokes parameters:

$$I = \frac{1}{2}(Z_1 + Z_2) \qquad \text{and}$$

$$Q,V = \pm \frac{F_{V,Q}}{D} \cdot G + \frac{F_{Q,V}}{D} \frac{(Z_1 - Z_2)}{2} \qquad , \qquad (6)$$

where

$$Z_{1,2} = \int_0^\infty \left[\left(1 + \eta_I\right) S_I \pm \left(\eta_Q{}^2 S_Q + \eta_V{}^2 S_V\right)\left(\frac{1}{\eta_0 D}\right)\right]$$

$$\times\, e^{-\int_0^{\tau_c} \lambda_{1,2} \frac{d\tau_c'}{\mu}} \frac{d\tau_c}{\mu} \qquad ,$$

$$G = \frac{F_Q F_V}{D} \int_0^\infty \eta_0 \left(S_Q - S_V\right) e^{-\int_0^{\tau_c} \lambda_3 \frac{d\tau_c'}{\mu}} \frac{d\tau_c}{\mu} \quad ,$$

$$\lambda_{1,2} = 1 + \eta_I \pm \eta_0 D \quad ,$$

$$\lambda_3 = 1 + \eta_I \quad ,$$

$$D = \sqrt{F_Q^2 + F_V^2} \quad , \quad \text{and}$$

$$\eta_0 = \frac{K_\lambda^l(0)}{K_\lambda^c} \tag{7}$$

APPLICATION TO THE CASE OF PURE ABSORPTION

If it is assumed that the Fraunhofer line is formed by the mechanism of pure absorption,

$$S_I = S_Q = S_V = B_\lambda(T) \quad ,$$

where $B_\lambda(T)$ is the Planck intensity distribution function. Under this assumption, equations (6) and (7) reduce to the following:

$$I = \frac{1}{2}\left(Z_1 + Z_2\right) \quad ,$$

$$Q,V = \frac{F_{Q,V}}{D} \cdot \left(\frac{Z_1 - Z_2}{2} \right) \quad , \qquad \text{and}$$

$$Z_{1,2} = \int_0^{\infty} \left[\frac{K_\lambda^c + K_\lambda^l(0) \left(F_I \pm D \right)}{K_\lambda^c} \right] B(\tau_c)$$

$$\times e^{-\int_0^{\tau_c} \left[\frac{K_\lambda^c + K_\lambda^l(0) \left(F_I \pm D \right)}{K_\lambda^c} \right] \frac{d\tau_c'}{\mu}} \frac{d\tau_c}{\mu} \qquad (8)$$

In equation (8), the depth-dependent functions are K_λ^c , $K_\lambda^l(0)$, and $B(\tau_c)$; the wavelength (and magnetic field) dependency is found in the functions $F_{I,Q,V}$. To compute Z_1 and Z_2 , and thus the Stokes parameters, the values of K_λ^c , $K_\lambda^l(0)$, and $B(\tau_c)$ must be calculated at each depth in the solar atmosphere; this means a "model atmosphere" must be employed that gives the temperature (T or $\theta = 5040/T$), gas pressure (P_g in dynes per square centimeter), electron pressure (P_e in dynes per square centimeter), and continuous absorption coefficient at a standard wavelength (K_0^c) at various increments of optical depth τ_c . Since τ_c is a function of wavelength, model atmospheres are calculated for a standard wavelength; to change the depth scale to the wavelength being considered, the relation

$$\frac{d\tau_c}{K_\lambda^c} = \frac{d\tau_0}{K_0^c}$$

is used, where τ_0 represents the continuum optical depth at the standard wavelength. The last of equations (8) then becomes

$$Z_{1,2} = \int_0^{\infty} \left[\frac{K_\lambda^c + K_\lambda^l(0)\left(F_I \pm D\right)}{K_0^c} \right] B(\tau_0)$$

$$- \int_0^{\tau_0} \left[\frac{K_\lambda^c + K_\lambda^l(0)\left(F_I \pm D\right)}{K_0^c} \right] \frac{d\tau_0'}{\mu}$$

$$\times \; e^{\displaystyle \frac{d\tau_0}{\mu}} \qquad (9)$$

Finally, for computational purposes, the variable

$$x_0 = \log \tau_0$$

is used rather than τ_0. In this notation the integrals in equation (9) transform into

$$Z_{1,2} = \int_{-\infty}^{+\infty} \left[\frac{K_\lambda^c}{K_0^c} + \frac{K_\lambda^l(0)}{K_0^c}\left(F_I \pm D\right) \right] B(x_0)$$

$$- \int_{-\infty}^{x_0} \left[\frac{K_\lambda^c}{K_0^c} + \frac{K_\lambda^l(0)}{K_0^c}\left(F_I \pm D\right) \right] \frac{10^{x_0'}\, dx_0'}{\mu \, \mathrm{Mod}} \qquad (10)$$

$$\times \; e^{\displaystyle \frac{10^{x_0}\, dx_0}{\mu \, \mathrm{Mod}}} ,$$

where $\mathrm{Mod} = \log_{10} e$.

The intensities of the Stokes parameters are usually expressed relative to the background continuum. The expression for the continuum intensity I_c can be obtained by letting $F_{I,Q,V} = 0$ in equations (10) and setting $I_c = \frac{1}{2}(Z_1 + Z_2)$; the result is

$$I_c = \int_{-\infty}^{+\infty} \frac{K_\lambda^c}{K_0^c} B(x_0) \; e^{-\int_{-\infty}^{x_0} \frac{K_\lambda^c}{K_0^c} \frac{10^{x_0'} dx_0'}{\mu \text{ Mod}}} \; \frac{10^{x_0} dx_0}{\mu \text{ Mod}} \qquad (11)$$

Equations (10) and (11) are the basic formulas from which the line profiles of the Stokes parameters are calculated. Of the terms in these equations, K_λ^c, $K_\lambda^l(0)$, and $B(x_0)$ must be calculated from their known dependence on θ, P_g, and P_e; $K_0^c(x_0)$ is (usually) given with the model.

CALCULATION OF THE PLANCK FUNCTION

The Planck function is given by

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \left(\frac{1}{e^{hc/\lambda kT} - 1} \right)$$

Because our final expressions will be normalized by the continuum intensity I_c, the constant factor $\frac{2hc^2}{\lambda^5}$ can be ignored and the following written for $B_\lambda\left(T(x_0)\right)$:

$$B(x_0) = \frac{1}{e^{\left(2.85476 \times 10^4\right)\left(\theta/\lambda_0\right)} - 1} \quad ,$$

with λ_0 (the central wavelength of the absorption line) expressed in angstroms.

CALCULATION OF K_λ^c

In the construction of a model atmosphere, the continuous absorption coefficient at the standard wavelength, K_0^c, is either directly calculated by assuming a certain number of sources of opacity (such as the negative hydrogen ion H^-, neutral hydrogen H, metals, etc.) or is obtained from tables of K_0^c already calculated by other authors (e.g., Vitense [4], Bode [5]). In the first case, a subprogram must be constructed that calculates K_λ^c using the same opacity sources used in calculating K_0^c (and that calculates values of K_0^c in agreement with those of the model). In the second case, the same tables must be used that were used by the author of the model for the wavelength of the absorption line under study. In the actual calculations described in this paper, one of two methods can be used. The first method calculates from basic physical principles the continuous absorption coefficient with H and H^- as the sources of opacity; it has been found that by varying the helium to hydrogen ratio (B) in this subprogram, the computed values of K_0^c can be made to agree with those of the model in the optical depth regions of interest. This subprogram has been described in detail by the author elsewhere [6]. The second method is a double interpolation (θ, P_e) subprogram for the data of Bode [5] at λ 5250 and is described elsewhere by Wadsworth [7].

CALCULATION OF $K_\lambda^l(0)$

The line absorption coefficient $K_\lambda^l(\lambda)$ per gram of stellar matter, corrected for negative absorptions, is given by

$$K_\lambda^l(\lambda) = N_l^g \, B_{lu} \, h\nu_0 \left(1 - e^{-hc/\lambda_0 kT}\right) \phi(\lambda) \quad ; \qquad (12)$$

here $\phi(\lambda)$ is the normalized profile of the line absorption coefficient. If a combination of natural, collisional, and thermal broadening is assumed,

$$\phi(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\lambda_0^2}{c} \frac{1}{\Delta\lambda_D} H(a, v) \tag{13}$$

For line center, zero-damping

$$H(0, 0) = 1 \quad ,$$

and equations (12) and (13) combine to give

$$K_\lambda^l(0) = N_l^g B_{lu} h\nu_0 \left(1 - e^{-hc/\lambda_0 kT}\right) \frac{\lambda_0^2}{\sqrt{\pi}\, c\Delta\lambda_D} \tag{14}$$

In equation (14), N_l^g is the number of atoms in the lower (l) level of the the line transition per gram (g) of stellar matter, B_{lu} is the Einstein coefficient of absorption, and $h\nu_0$ represents the energy difference between the lower (l) and upper (u) levels. The Einstein coefficient is related to the oscillator strength f (sometimes called the f-value) by the equation

$$h\nu_0 B_{lu} = \frac{\pi e^2}{m_e c} f$$

Using the definition for $\Delta\lambda_D$, equation (3), and the notation

$$\text{STEM} = \left(1 - e^{-hc/\lambda kT}\right) = 1 - e^{-\left(2.85476 \times 10^4\right)\left(\theta/\lambda_0\right)} \quad ,$$

equation (14) becomes

$$K_\lambda^l(0) = \frac{\sqrt{\pi}\, e^2 \lambda_0 f}{m_e c} \quad \frac{N_l^g}{\sqrt{\frac{2kT}{M} + \xi^2}} \qquad \text{(STEM)} \qquad (15)$$

The number of atoms per gram of stellar matter, N_l^g, can be expressed in terms of the number of atoms in the lower level per cubic centimeter, N_l,

$$N_l^g = \frac{N_l}{m_H \epsilon_H N_H} \quad ,$$

where m_H is the atomic mass of hydrogen, ϵ_H is the mean molecular weight of stellar matter relative to hydrogen, and N_H is the number of hydrogen nuclei (atoms, ions) per cubic centimeter. Regrouping the terms in equation (15) gives

$$K_\lambda^l(0) = \frac{\sqrt{\pi}\, e^2 \lambda_0 f}{m_e c\, m_H \epsilon_H} \quad \frac{(N_l/N_H)}{\sqrt{\frac{2kT}{M} + \xi^2}} \qquad \text{(STEM)} \qquad (16)$$

To establish the population density N_l of the lower level, local thermodynamic equilibrium is assumed and the Boltzmann and Saha equations are used. If it is assumed that

N_{El} = total number of nuclei of the element per cubic centimeter,

N_A = the number of atoms of the element per cubic centimeter,

N_i = the number of singly-ionized atoms of the element per cubic centimeter,

N_{ii} = the number of doubly-ionized atoms of the element per cubic centimeter, etc.,

the following can be written (neglecting molecular structure):

$$N_{El} = N_A + N_i + N_{ii} + \ldots \qquad \text{or}$$

$$\frac{N_A}{N_{El}} = \frac{1}{\left[1 + \frac{N_i}{N_A}\left(1 + \frac{N_{ii}}{N_i} + \frac{N_{iii}}{N_{ii}} \cdot \frac{N_{ii}}{N_i} + \ldots\right)\right]}$$

Also, the ratio N_l/N_H can be written as

$$N_l/N_H = \left(\frac{N_l}{N_A}\right)\left(\frac{N_A}{N_{El}}\right)\left(\frac{N_{El}}{N_H}\right)$$

$$= \left[\frac{N_l}{N_A}\right]\left[\frac{1}{1 + \frac{N_i}{N_A}\left(1 + \frac{N_{ii}}{N_i} + \frac{N_{iii}}{N_{ii}} \frac{N_{ii}}{N_i} + \ldots\right)}\right]\left[\frac{N_{El}}{N_H}\right] . \qquad (17)$$

The first term in brackets in equation (17) can be calculated from the Boltzmann relation,

$$\frac{N_l}{N_A} = \frac{g_l}{U_A} e^{-\chi_l/kT} \quad ;$$

g_l is the statistical weight of the lower level, U_A is the partition function of the atom, and χ_l is the excitation potential of the level. If N_l/N_A is designated as N_1, then it can be calculated from the formula

$$N_1 = \frac{g_1}{U_A} e^{-\theta\chi_1/\mathrm{Mod}}$$

where χ_1 is in electron-volts.

The second term in brackets in equation (17) is calculated by repeated applications of the Saha equation,

$$\frac{N_{q+1}}{N_q} \cdot N_e = \left(\frac{2\pi m_e k T}{h^2} \right)^{3/2} \frac{2 U_{q+1}}{U_q} e^{-\chi_q/kT}$$

N_q is the number of atoms in the qth stage of ionization, N_{q+1} is the number in the $(q+1)$ stage, χ_q is the ionization energy from q to $(q+1)$, N_e is the electron number density, and the U's are the respective partition functions. If it is assumed that N_{ii}, N_{iii}, . . . $<< N_i$ (a valid assumption for sunspot and photosphere models for most elements), this second term reduces to $1/N_2$ where

$$N_2 = 1 + \frac{N_i}{N_A} = 1 + \frac{2 U_i}{U_A} \frac{T^{5/2}}{P_e} e^{-(\theta I^* + 0.48)/\mathrm{Mod}}$$

Any effects of reduction of the ionization potential have been included here,

$$I^* = I - 4.98 \times 10^{-4} \theta \sqrt{P_e} \quad ,$$

with P_e equal to the electron pressure in dynes per square centimeter and "I" equal to the first ionization potential of the element in electron-volts.

The third term in equation (17) is given from abundance determinations (A_{El}) ; the assumed abundances used in constructing the model atmosphere should be used here.

Finally, equation (16) is written in the form

$$K_\lambda^l(0) = K \frac{N_1}{N_2} \frac{1}{\sqrt{\frac{2kT}{M} + \xi^2}} \quad \text{(STEM)} \quad , \tag{18}$$

where

$$K = \left(\frac{\sqrt{\pi}\, e^2}{m_e\, m_H\, c} \right) \left(\lambda_0\, f \right) \left(\frac{A_{El}}{\epsilon_H} \right) \tag{19}$$

The "constant" K can be calculated directly for a given line or it can be determined by fitting the calculated profile for zero magnetic field to an experimental profile; the latter procedure has been adopted.

In the calculations of N_1 and N_2, the partition functions U_A and U_i must be calculated as a function of temperature and electron pressure. This is done in this paper by a subprogram that evaluates the partition functions following the method of Traving et al. [8]. This subprogram was adapted from a program given to the author by Olav K. Moe (private communication).

THE MODEL ATMOSPHERE

To evaluate the integrals in equations (10) and (11), it is necessary to be able to calculate K_λ^c, $K_\lambda^l(0)$, and $B(x_0)$ at discrete intervals of x_0; this means that values of θ and P_e (and K_0^c) must be obtained at these intervals. To use any desired increment in x_0, least squares polynomial fits to the $\theta(x_0)$, $P_e(x_0)$, and $K_0^c(x_0)$ data given in various model atmospheres have been made, and these results have been incorporated in three subroutines into the main computer program.

THE FUNCTIONS $F_{I,Q,V}$

The remaining terms in equations (10) and (11) that must be evaluated numerically are the wavelength-dependent functions $F_{I,Q,V}$ defined by equations (5). These functions are determined from the magnetic field inclination ψ and from the functions $H(a, v)$, $H(a, v + v_H)$, and $H(a, v - v_H)$, where $v = (\lambda - \lambda_0)/\Delta\lambda_D$ and $v_H = \Delta\lambda_H/\Delta\lambda_D$ with $\Delta\lambda_D$ having a constant value. The Zeeman splitting factor $\Delta\lambda_H$ is calculated from the assumed magnetic field intensity, and the Voigt functions $H(a, v)$ are calculated from least squares polynomial fits to data tabulated by Hummer [9] for various values of the damping parameter "a".

CALCULATED STOKES PROFILES

Typical results are shown in Figures 1, 2, 3, and 4 for the Stokes parameters for the iron line 5250.22 Å. The constant K was given the value $\log K = 7.914$, and the damping parameter "a" was 0.10; the photospheric model of Holweger [10] was used. In this particular calculation a variable velocity field ξ was used in equation (18); Figure 5 shows this velocity field together with the velocity distribution calculated in Holweger's model. A constant Doppler half-width of 34.0 mÅ was used in the wavelength-dependent functions. The measured profile in Figure 1 was obtained by Dr. John Harvey and the author using the Kitt Peak solar telescope and spectrograph in the photoelectric double-pass mode.

In Figure 2, it is interesting to estimate the inclination angle ψ using the Seares [11] formula,

$$r_\sigma : r_\pi = \frac{1}{4}(1 + \cos^2\psi) : \frac{1}{2}\sin^2\psi \quad ,$$

where $r \equiv 1 - I/I_c$. From Figure 2, $r_\sigma = 1 - 0.6626$ and $r_\pi = 1 - 0.64$;

$$\frac{r_\sigma}{r_\pi} = \frac{0.338}{0.36} = \frac{\frac{1}{4}(1 + \cos^2\psi)}{\frac{1}{2}\sin^2\psi}$$

This gives $\psi = 56.5$ deg compared to the actual value of 60 deg.

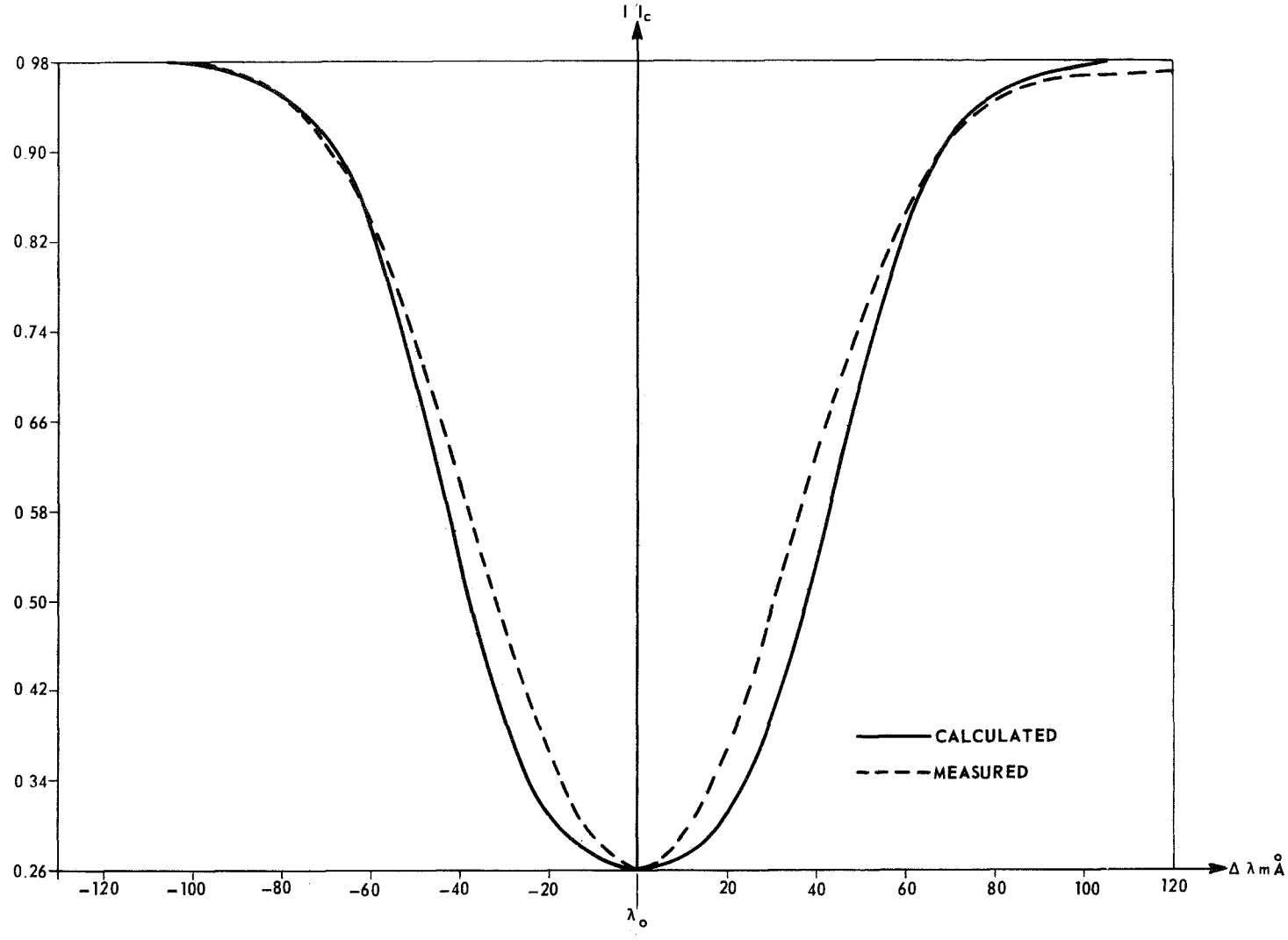

Figure 1. Iron 5250.22 Å line profile in the quiet photosphere.

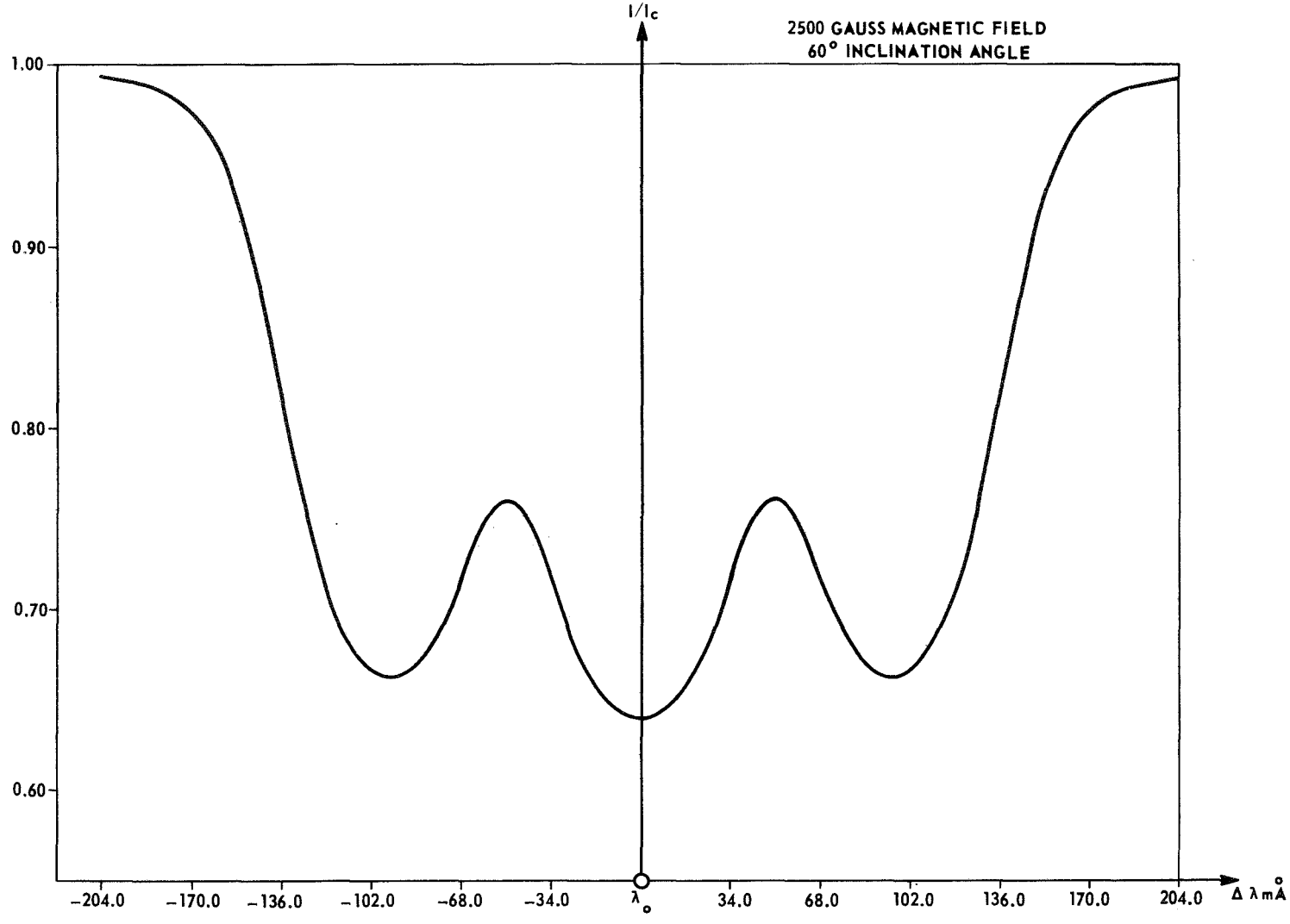

Figure 2. Iron 5250.22 Å line profile in a magnetic field.

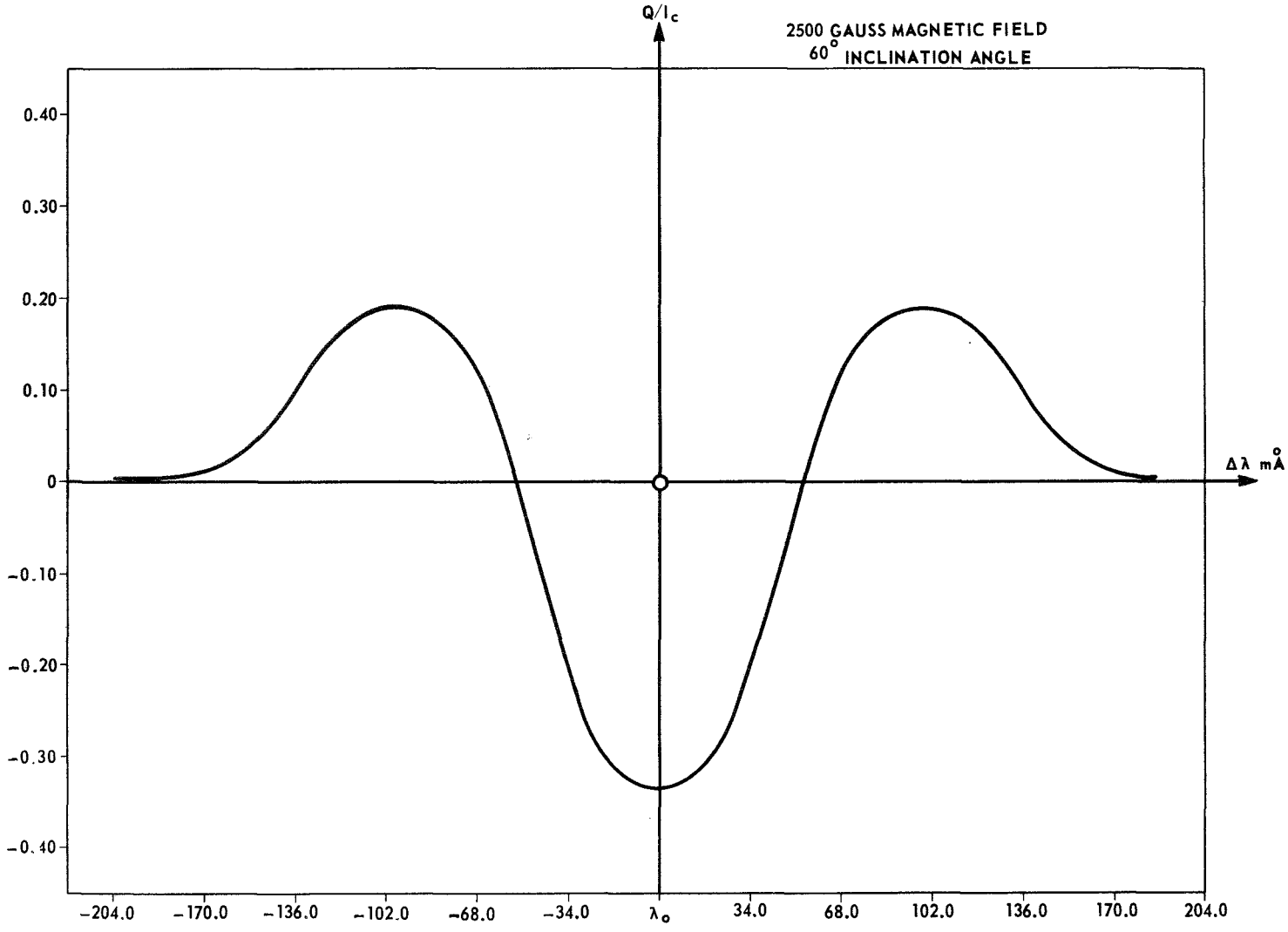

Figure 3. Iron 5250.22 Å Q Stokes parameter profile.

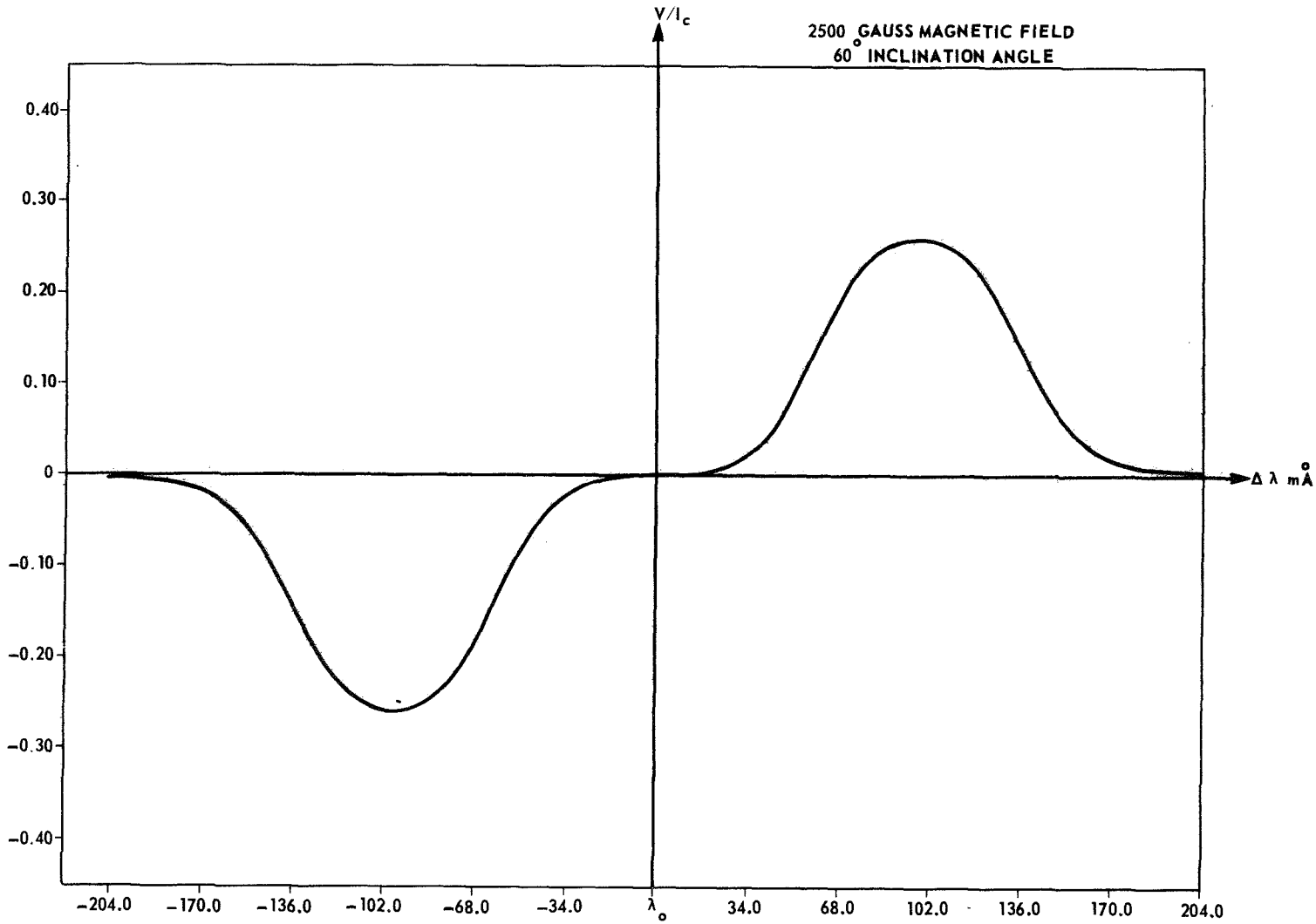

Figure 4. Iron 5250.22 Å V Stokes parameter profile.

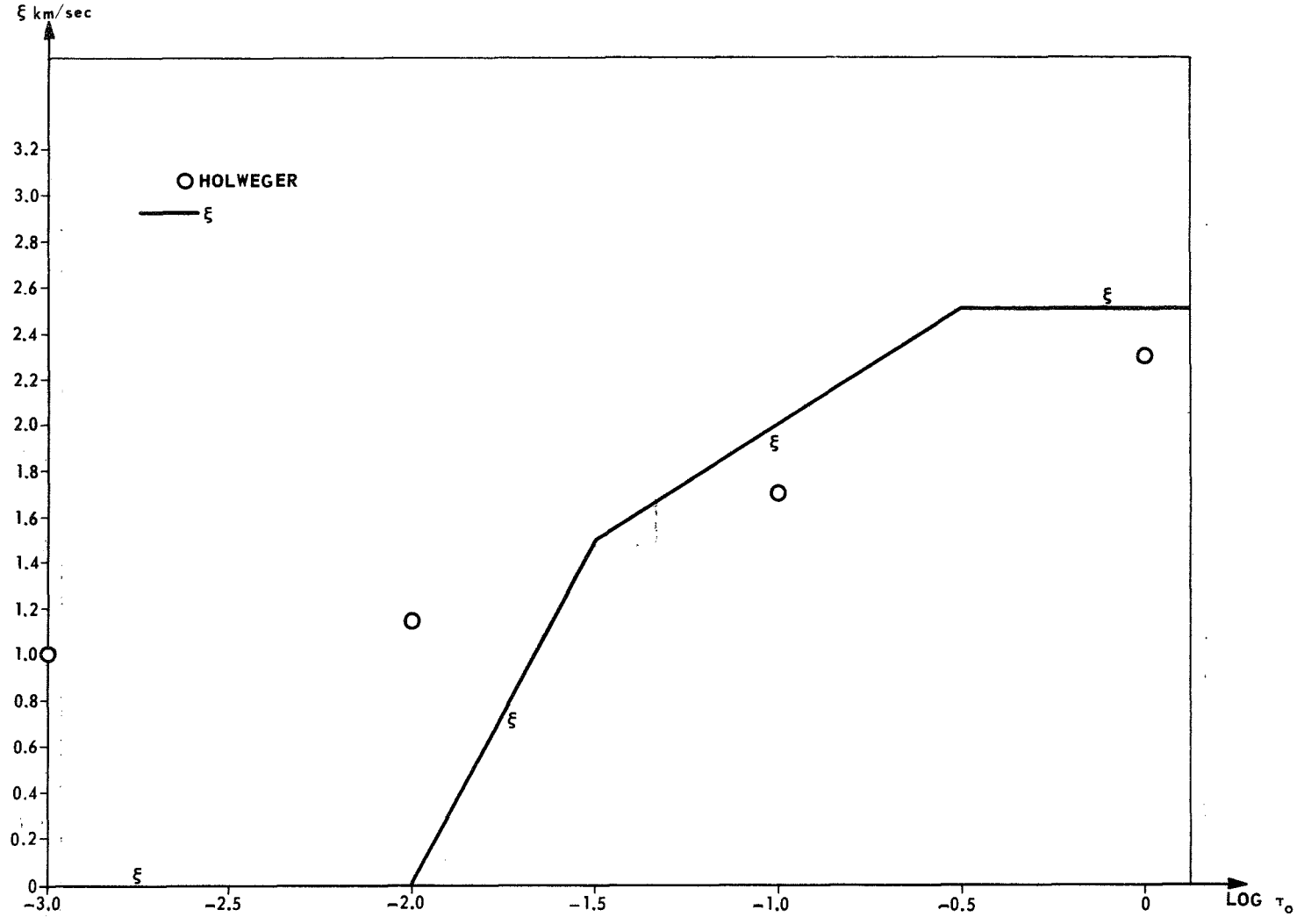

Figure 5. Photospheric velocity field.

CALCULATION OF THE DEGREE OF LINEAR AND CIRCULAR POLARIZATION USING A NARROW-BAND BIREFRINGENT FILTER

A solar magnetograph system, by the successive (or simultaneous) use of various polarizing optics and with a subsequent computer reduction of the resulting data, can provide measurements of the quantities $\frac{Q/I_c}{I/I_c}$ and $\frac{V/I_c}{I/I_c}$, averaged over the bandpass of the light-filtering system used. The transmission characteristics of the filter can be expressed by a function $T(\Delta\lambda, \Delta\lambda_i)$, where $\Delta\lambda_i$ represents $\lambda_i - \lambda_0$ with λ_i being the wavelength at which maximum filter transmission occurs. If the total bandpass of the filtering system is 2Δ, the measured quantities can be mathematically expressed as follows:

$$\overline{P}_{Q,V} = \frac{\int_{\Delta\lambda_i - \Delta}^{\Delta\lambda_i + \Delta} \frac{Q(\Delta\lambda),\ V(\Delta\lambda)}{I_c(\Delta\lambda)} \cdot T\left(\Delta\lambda,\ \Delta\lambda_i\right) d(\Delta\lambda)}{\int_{\Delta\lambda_i - \Delta}^{\Delta\lambda_i + \Delta} \frac{I(\Delta\lambda)}{I_c(\Delta\lambda)} T\left(\Delta\lambda,\ \Delta\lambda_i\right) d(\Delta\lambda)} \tag{20}$$

The integration is generally performed over the variable $v = \Delta\lambda/\Delta\lambda_D$; defining $\delta = \Delta/\Delta\lambda_D$ and $v_i = \Delta\lambda_i/\Delta\lambda_D$, equations (20) become

$$\overline{P}_{Q,V} = \frac{\int_{v_i - \delta}^{v_i + \delta} \frac{Q(v),\ V(v)}{I_c(v)} T\left(v,\ v_i\right) dv}{\int_{v_i - \delta}^{v_i + \delta} \frac{I(v)}{I_c(v)} T\left(v,\ v_i\right) dv} \tag{21}$$

The interest here is in calculating $\overline{P}_{Q,V}$ for a solar magnetograph system using a Zeiss narrow-band birefringent filter. The theoretical transmission profile of such a filter is given by the equation

$$T(\Delta\lambda') = T(0) \prod_{k=0}^{(L-1)} \left[\cos\left(\frac{\pi\Delta\lambda'}{2^k d}\right)\right]^2 \qquad (22)$$

In equation (22), $T(0)$ is the peak transmission, L is the number of filter elements, $d/2$ is that value of $\Delta\lambda'$ where the profile has its first minimum, and $\Delta\lambda'$ is the wavelength distance measured from the peak transmission wavelength. Since all wavelengths ($\Delta\lambda$) are measured relative to the center (λ_0) of the absorption line, $\Delta\lambda'$ is written in terms of $\Delta\lambda$ as follows:

$$\Delta\lambda' = \Delta\lambda - \Delta\lambda_i$$

If the variable v is introduced, equation (22) becomes

$$T\left(v, v_i\right) = T(0) \prod_{k=0}^{(L-1)} \left[\cos\left[\frac{\pi}{2^k}\left(v - v_i\right)\frac{\Delta\lambda_D}{d}\right]\right]^2 . \qquad (23)$$

The actual transmission profile of the Zeiss filter that is to be used in the NRL/MSFC real-time solar magnetograph was measured in December, 1968 by J. Zwiener (private communication) using the Kitt Peak solar telescope and spectrograph; the data were reduced by W. C. Snoddy (private communication). The profile at the central portion of the field of view was found to be slightly asymmetric with a bandpass of 0.122 Å. The asymmetry can be simulated in the theoretical profile by introducing a phase shift into the first term in the product as follows:

$$T\left(v, v_i\right) = T(0)\left[\cos\pi\left[\left(v - v_i\right)\frac{\Delta\lambda_D}{d} + \alpha\right]\right]^2 \times \prod_{k=1}^{(L-1)}\left[\cos\left[\frac{\pi}{2^k}\left(v - v_i\right)\frac{\Delta\lambda_D}{d}\right]\right]^2 . \qquad (24)$$

It was found that a good fit to the measured profile could be obtained for $L = 8$, $\alpha = 0.85$, and $d = 0.2752$ Å; Figure 6 shows this comparison. The introduction of a phase shift α into the formula will shift the peak transmission wavelength from the value v_i by an amount depending on the magnitude of α; the true filter maximum position is obtained by adding a correction term to v_i. For $\alpha = 0.85$, the shift is 0.030 Å and the correction term is $30.0/\Delta\lambda_D$.

NUMERICAL RESULTS

Calculations have been made using equations (21) and (24) for the degree of linear and circular polarization using the same line and parameters as in Figures 1 through 5. One particular application of these equations is in "tuning" the filter across the line; the filter's maximum position v_i is varied through the wavelength region of interest at a few values of H and ψ. These results are of interest in determining optimum filter settings for linear and circular polarization measurements. A typical result is plotted in Figure 7. A simple modification allows the calculation of $\overline{P}_{Q,V}$ at one or two particular filter settings v_i for a large number of values of H and ψ. These results are useful in "calibrating" magnetographs; calculated values of $\overline{P}_{Q,V}$ are plotted versus H at different inclination angles ψ to relate the magnetic field strength and direction on the sun to the measured quantities $\overline{P}_Q$ and $\overline{P}_V$. Figure 8 indicates a typical calibration curve.

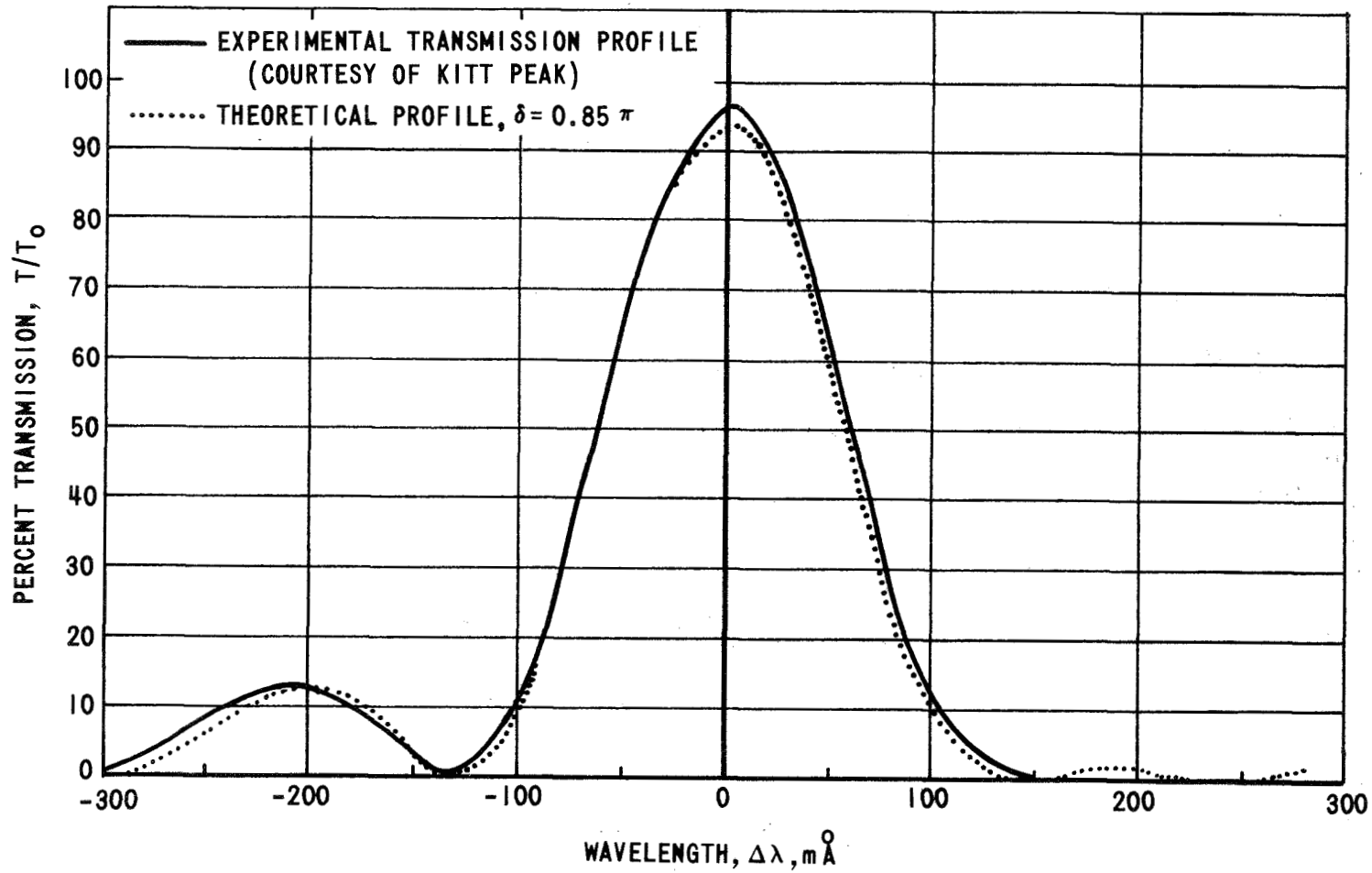

Figure 6. Measured profile of the Zeiss birefringent filter for the iron 5250 line (central part of field of view).

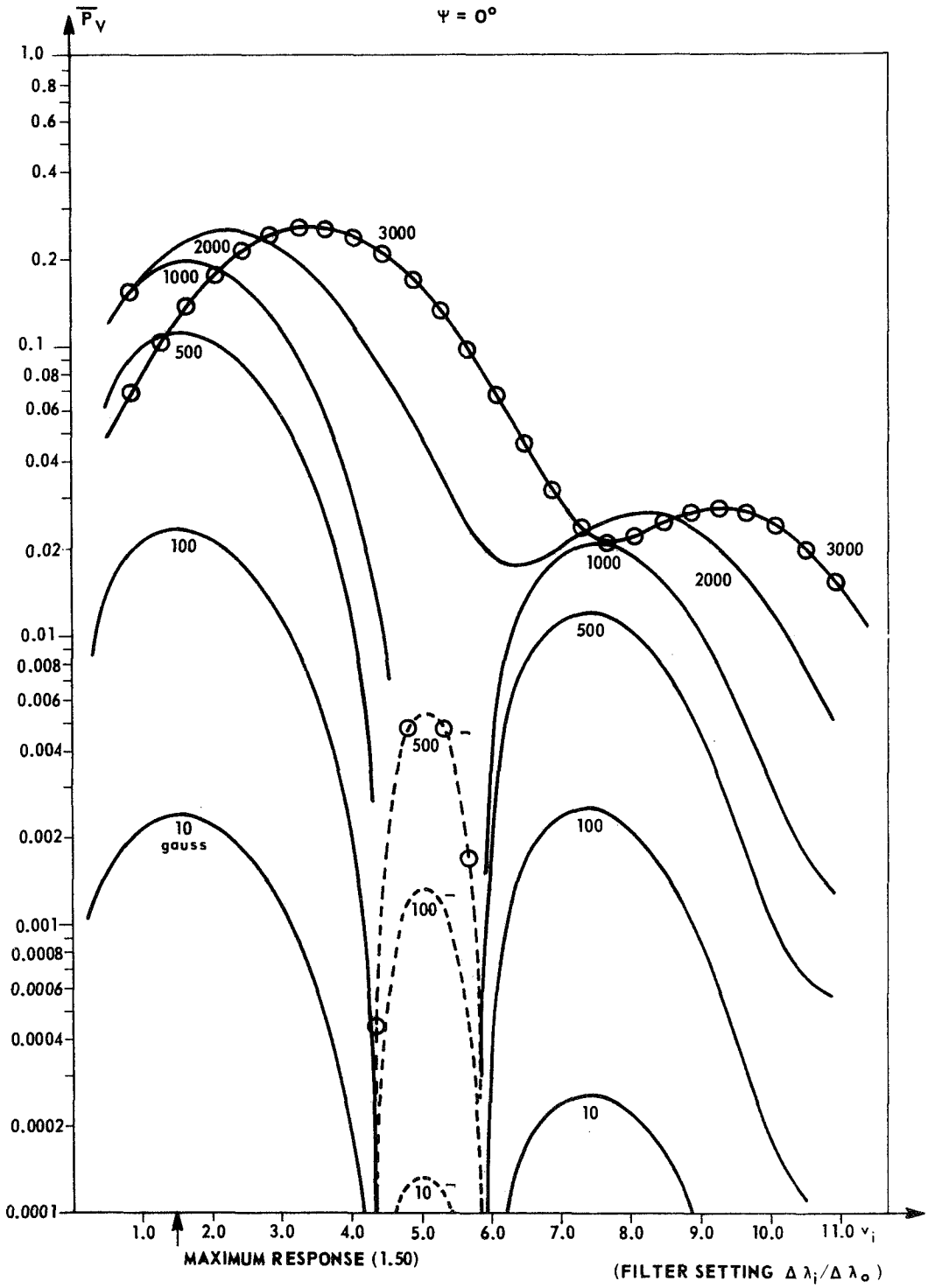

Figure 7. Variation of degree of circular polarization with filter setting.

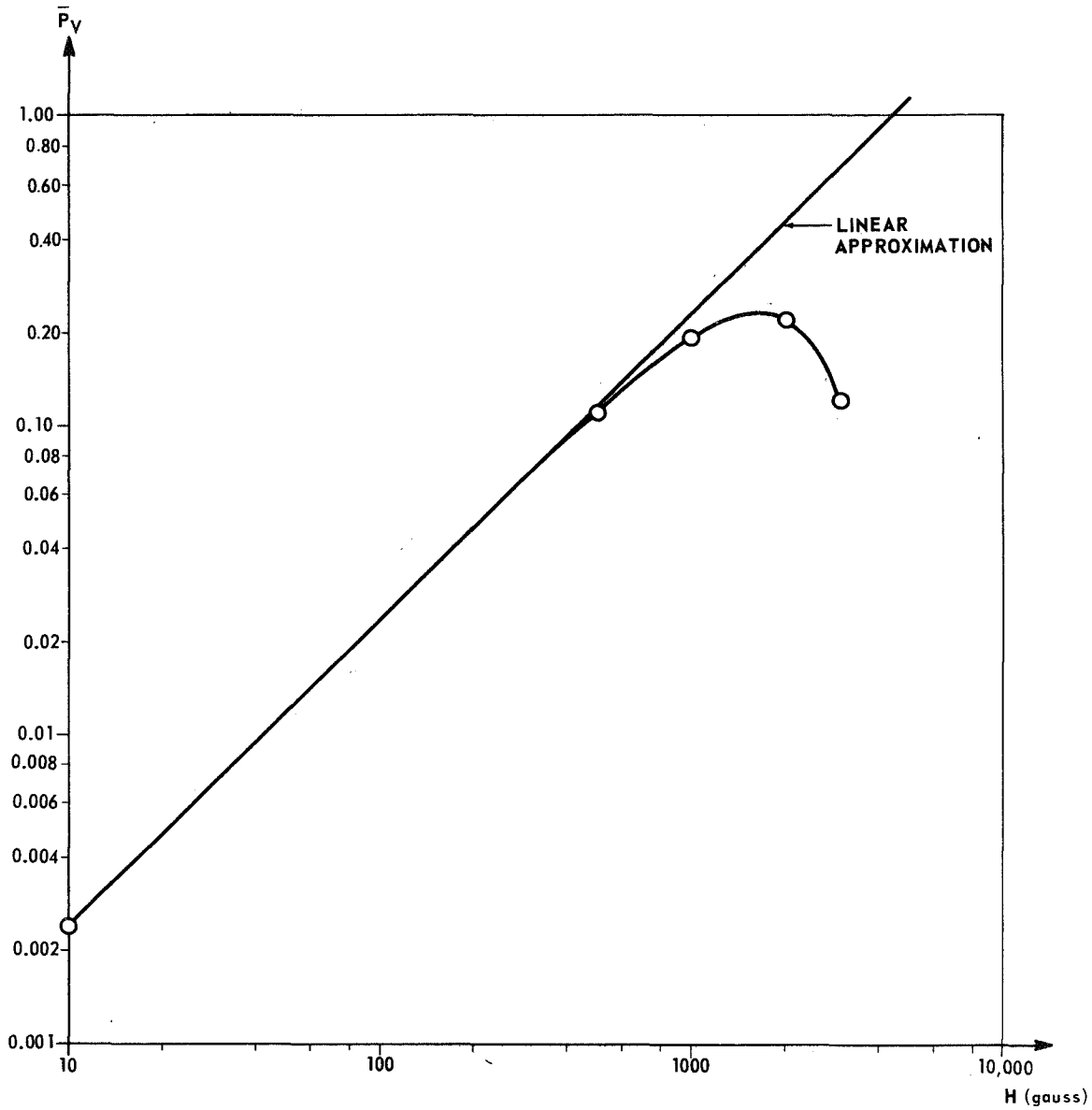

Figure 8. Magnetograph calibration curve, birefringent filter (filter setting at $\Delta\lambda_i = 0.051$ Å, inclination angle = 0 deg).

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CALCULATION OF THE PROFILES OF THE STOKES PARAMETERS AND THE DEGREE OF LINE POLARIZATION: AN APPLICATION OF THE SOLUTIONS OF MOE TO THE UNNO TRANSFER EQUATIONS

By M. J. Hagyard

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This document has also been reviewed and approved for technical accuracy.

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