N 71 10448

NASA TECHNICAL MEMORANDUM

NASA TM X-58052
November 1970

AXISYMMETRIC STRESSES AND DISPLACEMENTS IN TWO FINITE CIRCULAR CYLINDERS IN CONTACT

By Clarence J. Wesselski

CASE FILE COPY

A Thesis Presented to the
Faculty of the University of Houston
In Partial Fulfillment of
the Requirements for the Degree
Master of Science in Mechanical Engineering

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

ABSTRACT

A boundary value problem of two elastic bodies in contact is considered. The bodies are finite circular cylinders of different dimensions and material constants and are isotropic and homogeneous. They are forced into contact across the plane faces such that the resulting stresses and displacements are axisymmetric. The solution utilizes Love's stress function to generate a family of biorthogonal eigenfunctions for each cylinder. The interrelated Fourier coefficients are expressed implicitly by an infinite system of linear algebraic equations. By truncation, an explicit solution of the Fourier coefficients is obtained.

Two example problems are solved: first, the two cylinders are placed in frictionless contact; second, the two cylinders are placed in bonded contact. In each problem, the other face of each cylinder undergoes a constant displacement with zero shear tractions. Selected numerical results are presented in graphical form.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

(r, θ, z)	Cylindrical coordinates
χ	Love's stress function
$\sigma_r, \sigma_\theta, \sigma_z, \tau$	Non-zero components of the stress tensor in cylindrical coordinates.
u, w	Radial and Axial Displacements, respectively
E	Young's Modulus
ν	Poisson's Ratio
$(\ )_{,x}$	Partial Derivative with respect to x. $(x = r, z)$

$$\nabla^2(\ ) = (\ )_{,rr} + \frac{1}{r}(\ )_{,r} + (\ )_{,zz}$$

δ_{ij}	Kronecker's Delta
η	Radius of upper cylinder
ℓ	Length of lower cylinder
h	Length of upper cylinder
$(\ )_L$	Subscript denoting lower cylinder
$(\ )_U$	Subscript denoting upper cylinder
γ_j	Eigenvalues associated with lower cylinder
λ_j	Eigenvalues associated with upper cylinder

CHAPTER I

THE PROBLEM

In this thesis, a method for solving the contact problem of two finite circular cylinders is presented. The cylinders are forced into contact across the plane faces such that the resulting stresses and displacements are axisymmetric. They are compressed between two rigid, frictionless faces as shown in figure 1.

Both cylinders are assumed to be homogeneous, isotropic, and elastic. Each cylinder may have its own radius, length, Young's modulus, and Poisson's ratio. The curved surfaces will be assumed to be free of tractions.

Two example problems will be solved: first, the two cylinders are placed in frictionless contact with each other; second, the two cylinders are placed in bonded contact.

The solution will be based on Power [1] and Power and Childs [2]. Other solutions for the finite cylinder do exist but are not satisfactory because of numerical properties or approximate schemes for meeting the boundary conditions.

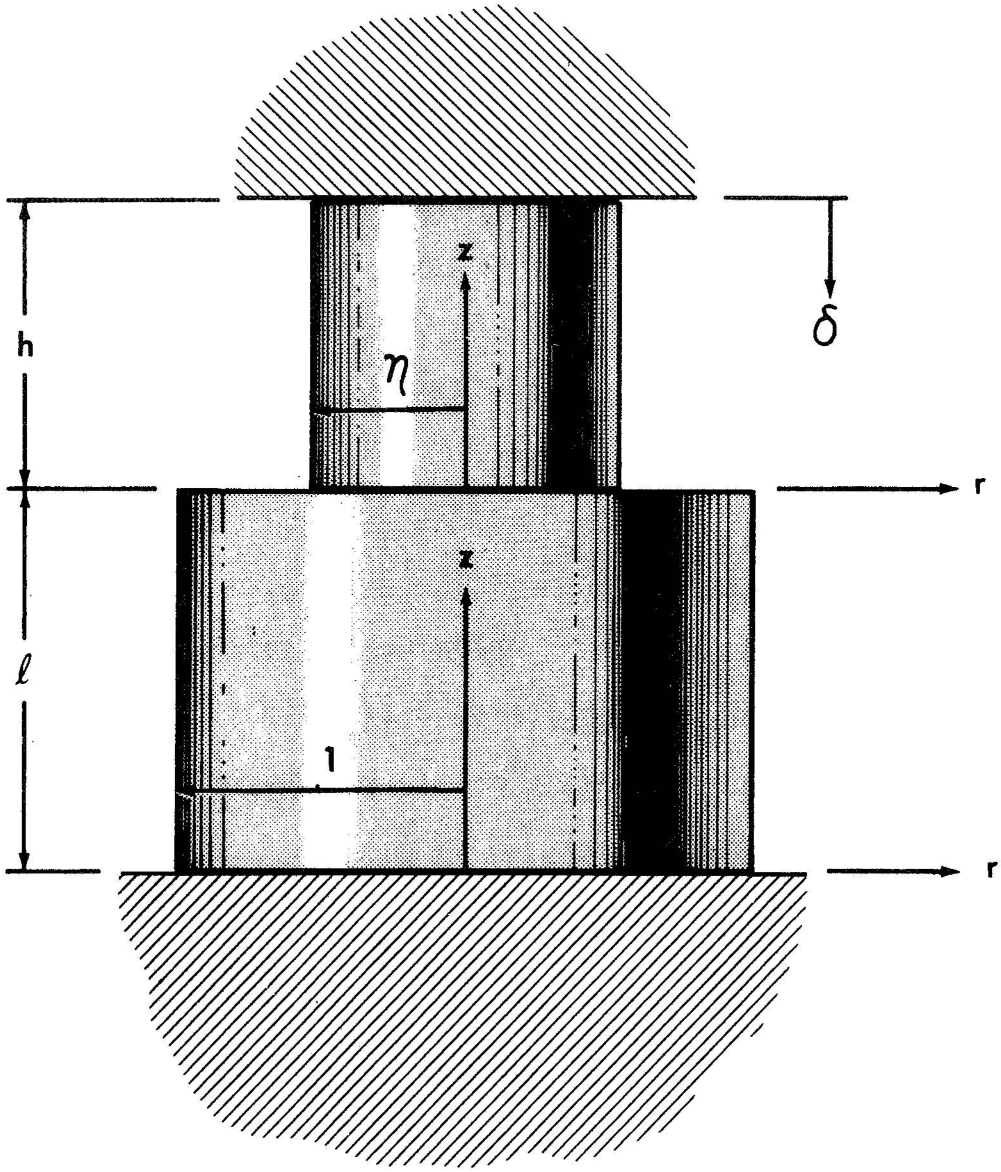

Figure 1. The Two Cylinders in Contact

CHAPTER II

THE FINITE CYLINDER

The solution for a finite cylinder problem, as formulated by Power [1], is based on a Love function [3] in the form of a series of Bessel functions times exponential functions plus an improper Fourier integral. The Love function was taken to be the sum of three solutions

$$\chi = \chi^{0} + \chi^{1} + \chi^{P} \tag{2.1}$$

where χ^{0} is the zero eigenvalue solution; χ^{1} is the self-equilibrated solution associated with the complex eigenvalues; and χ^{P} is the particular solution associated with body forces, which will be neglected.

The governing equation for the homogeneous solutions χ^{0} and χ^{1} is

$$\nabla^{4} \chi = 0 \tag{2.2}$$

Each of the separate solutions satisfies the boundary conditions on the curved surface.

In terms of the Love function, χ, the stresses and displacements that are not zero are given by [3]:

$$\sigma_r = \left[\nu \nabla^2 \chi - \chi_{,rr}\right]_{,z} \tag{2.3a}$$

$$\sigma_\theta = \left[\nu \nabla^2 \chi - \frac{1}{r}\chi_{,r}\right]_{,z} \tag{2.3b}$$

$$\sigma_z = \left[(2-\nu)\nabla^2 \chi - \chi_{,zz}\right]_{,z} \tag{2.3c}$$

$$\tau = \left[(1-\nu)\nabla^2 \chi - \chi_{,zz}\right]_{,r} \tag{2.3d}$$

$$\frac{E u}{1+\nu} = -\chi_{,rz} \tag{2.3e}$$

$$\frac{E w}{1+\nu} = 2(1-\nu)\nabla^2 \chi - \chi_{,zz} \tag{2.3f}$$

The zero eigenvalue solution of the biharmonic equation $\nabla^4 \chi = 0$ is found by separation of variables. A sufficient portion is

$$\chi^0 = Az^3 + Bz^2 + \frac{3\nu}{1-2\nu} Ar^2 z \tag{2.4}$$

For a cylinder of radius R, the stresses and displacements obtained from χ^0 are

$$\sigma_r^0 = \sigma_\theta^0 = \tau^0 \equiv 0 \tag{2.5a}$$

$$\sigma_z^0 = \frac{6(1+\nu)}{1-2\nu} A \tag{2.5b}$$

$$\frac{E u^0}{(1+\nu)R} = -\frac{6\nu}{1-2\nu}\frac{r}{R} A \tag{2.5c}$$

$$\frac{E w^0}{(1+\nu)R} = \frac{6}{1-2\nu}\frac{z}{R} A + 2(1-2\nu)\frac{B}{R} \tag{2.5d}$$

It is more appropriate to call this solution the uniform axial stress solution, as can be seen by equation (2.5b). Equation (2.5d) shows that this solution produces a net axial displacement consisting of two terms. The first term is proportional to z. The second term is the net axial displacement at z = 0 and is considered as a rigid body displacement.

The self-equilibrated solution, X^1, for the finite cylinder is based on Little and Childs [4] solution of the semi-infinite cylinder. This solution, containing only the eigenvalues in the right half of the complex plane, was extended by Power [1] to solve the finite cylinder by using the eigenvalues in the left half of the complex plane and the axial coordinate z- L. This solution for a finite cylinder of radius R, length L, and with self-equilibrated tractions on both ends is

$$\sigma_z^1 = \sum_{j=1}^{\infty} \left[a_{1j} \; e^{-\gamma_j z/R} - a_{2j} \; e^{\gamma_j (z-L)/R} \right] \varphi_1\left(\gamma_j, \frac{r}{R}\right) \qquad (2.6a)$$

$$\tau^1 = \sum_{j=1}^{\infty} \left[a_{1j} \; e^{-\gamma_j z/R} + a_{2j} \; e^{\gamma_j (z-L)/R} \right] \varphi_2\left(\gamma_j, \frac{r}{R}\right) \qquad (2.6b)$$

$$\frac{E u^1}{(1+\nu)R} = \sum_{j=1}^{\infty} \left[a_{1j} \; e^{-\gamma_j z/R} - a_{2j} \; e^{\gamma_j (z-L)/R} \right] \varphi_3\left(\gamma_j, \frac{r}{R}\right) \qquad (2.6c)$$

$$\frac{E w^1}{(1+\nu)R} = \sum_{j=1}^{\infty} \left[a_{1j} \; e^{-\gamma_j z/R} + a_{2j} \; e^{\gamma_j (z-L)/R} \right] \varphi_4\left(\gamma_j, \frac{r}{R}\right) \qquad (2.6d)$$

$$\sigma_r^{1} = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j z/R} - a_{2j}\, e^{\gamma_j (z-L)/R}\right] \varphi_5\!\left(\gamma_j, \frac{r}{R}\right) \tag{2.6e}$$

$$\sigma_\theta^{1} = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j z/R} - a_{2j}\, e^{\gamma_j (z-L)/R}\right] \varphi_6\!\left(\gamma_j, \frac{r}{R}\right) \tag{2.6f}$$

The dimensionless eigenvalues, γ_j , are defined as

$$\gamma_j \stackrel{d}{=} \xi_j R \tag{2.7}$$

where the $\xi_j R$ are the roots of the characteristic equation

$$(\xi R)^2 J_0^2(\xi R) + \left[(\xi R)^2 - 2(1-\nu)\right] J_1^2(\xi R) = 0 \tag{2.8}$$

The eigenfunctions, $\varphi\,(\gamma_j\,,\ r/R)$, are also dimensionless and are given in the Appendix.

The specified boundary values of χ^1 are defined as

$$f_1(r) \stackrel{d}{=} \sigma_z^1(r,0) = \left|\sigma^b - \sigma^o\right|_{z=0} \tag{2.9a}$$

$$h_1(r) \stackrel{d}{=} \tau^1(r,0) = \tau^b\Big|_{z=0} \tag{2.9b}$$

$$g_1(r) \stackrel{d}{=} \frac{E}{1+\nu}\, u^1(r,0) = \frac{E}{1+\nu}\left[u^b - u^o\right]_{z=0} \tag{2.9c}$$

$$K_1(r) \stackrel{d}{=} \frac{E}{1+\nu}\, w^1(r,0) = \frac{E}{1+\nu}\left[w^b - w^o\right]_{z=0} \tag{2.9d}$$

$$f_2(r) \stackrel{d}{=} \sigma^1(r,L) = \left|\sigma^b - \sigma^o\right|_{z=L} \tag{2.9e}$$

$$h_2(r) \stackrel{d}{=} \tau^1(r,L) = \tau^b\Big|_{z=L} \tag{2.9f}$$

$$g_2(r) \stackrel{d}{=} \frac{E}{1+\nu} u'(r, L) = \frac{E}{1+\nu} \left[u^b - u^o \right]_{z=L} \tag{2.9g}$$

$$K_2(r) \stackrel{d}{=} \frac{E}{1+\nu} w'(r, L) = \frac{E}{1+\nu} \left[w^b - w^o \right]_{z=L} \tag{2.9h}$$

The Fourier coefficients are therefore obtained from the integrals

$$a_{1j} = \frac{1}{N(\gamma_j)} \int_0^R \left[W_1\left(\gamma_j, \frac{r}{R}\right) f_1(r) + W_2\left(\gamma_j, \frac{r}{R}\right) h_1(r) + W_3\left(\gamma_j, \frac{r}{R}\right) g_1(r) + W_4\left(\gamma_j, \frac{r}{R}\right) K_1(r) \right] \frac{r}{R} \frac{dr}{R} \tag{2.10a}$$

$$a_{2j} = \frac{1}{N(\gamma_j)} \int_0^R \left[-W_1\left(\gamma_j, \frac{r}{R}\right) f_2(r) + W_2\left(\gamma_j, \frac{r}{R}\right) h_2(r) - W_3\left(\gamma_j, \frac{r}{R}\right) g_2(r) + W_4\left(\gamma_j, \frac{r}{R}\right) K_2(r) \right] \frac{r}{R} \frac{dr}{R} \tag{2.10b}$$

where

$$N(\gamma_j) = 2(1-\nu)\left[2(1-\nu) J_0(\gamma_j) J_1(\gamma_j) - 2\gamma_j J_0^2(\gamma_j) - \gamma_j J_1^2(\gamma_j) \right] \tag{2.11}$$

Explicit forms of the W functions are given in the Appendix.

Only four of the functions (2.9) may be arbitrarily specified as self-equilibrated boundary conditions. The other four are obtained from equations (2.6) evaluated at the appropriate boundary. Equations (2.10) may then be written in the form

$$a_{1j} = G_{1j} + \sum_{K=1}^{\infty} B_{1jK}\, a_{1K} + \sum_{K=1}^{\infty} C_{1jK}\, a_{2K} \qquad (2.12a)$$

$$a_{2j} = G_{2j} + \sum_{K=1}^{\infty} B_{2jK}\, a_{2K} + \sum_{K=1}^{\infty} C_{2jK}\, a_{1K} \qquad (2.12b)$$

where G_{1j} and G_{2j} represent that part of the integral containing the specified boundary conditions on $\boldsymbol{\chi}'$.

It should be noted that, in order to avoid a null self-equilibrated solution, the specified boundary conditions must contain at least one of the functions (2.9) where the uniform axial stress solution does differ from the boundary condition.

CHAPTER III

TWO CYLINDERS IN CONTACT

The solution for the two cylinder contact problem will be found by using the finite cylinder solution applicable to each of the two cylinders. The equations will then be solved simultaneously to yield the Fourier coefficients in accordance with the boundary and continuity conditions. The geometric dimensions of the two cylinders are indicated in figure 1. For the lower cylinder, its eigenvalues and Fourier coefficients will be denoted by γ_j and a_{1j} and a_{2j}, respectively. Similarly, λ_j and a_{3j} and a_{4j} will be used for the upper cylinder. Subscripts $(\ )_L$ and $(\ )_U$ will denote other values for the lower and upper cylinders, respectively.

<u>The Uniform Axial Stress Solution</u>.

Equations (2.5) from the previous chapter will be used. It is convenient to write them in this form:

$$\frac{\sigma^{o}_{zL}}{A_L} = -1 \quad , \qquad \frac{\sigma^{o}_{zU}}{A_L} = -\frac{A_U}{A_L} \tag{3.1a}$$

$$\frac{E_L u^{o}_L}{A} = r\nu_L \quad , \qquad \frac{E_U u^{o}_U}{A_L} = r\nu_U \frac{A_U}{A_L} \tag{3.1b}$$

$$\frac{E_L w^{o}_L}{A_L} = -z + \frac{B_L}{A_L} \quad , \qquad \frac{E_U w^{o}_U}{A_L} = -z\frac{A_U}{A_L} + \frac{B_U}{A_L} \tag{3.1c}$$

The Total Solution.

The total solution will be the sum of the self-equilibrated solution and the uniform axial stress solution. For the lower cylinder, it is

$$\frac{\sigma_{zL}}{A_L} = \sum_{j=1}^{\infty}\left[a_{1j}e^{-\gamma_j z} - a_{2j}e^{\gamma_j(z-\ell)}\right]\varphi_1(\gamma_j, r) - 1 \tag{3.2a}$$

$$\frac{\tau_L}{A_L} = \sum_{j=1}^{\infty}\left[a_{1j}e^{-\gamma_j z} + a_{2j}e^{\gamma_j(z-\ell)}\right]\varphi_2(\gamma_j, r) \tag{3.2b}$$

$$\frac{E_L u_L}{A(1+\nu)} = \sum_{j=1}^{\infty}\left[a_{1j}e^{-\gamma_j z} - a_{2j}e^{\gamma_j(z-\ell)}\right]\varphi_3(\gamma_j, r) + \frac{\nu_L r}{1+\nu_L} \tag{3.2c}$$

$$\frac{E_L W_L}{A_L(1+\nu_L)} = \sum_{j=1}^{\infty}\left[a_{1j}e^{-\gamma_j z} + a_{2j}e^{\gamma_j(z-\ell)}\right]\varphi_4(\gamma_j, r) - \frac{z}{1+\nu_L} \tag{3.2d}$$

Similar equations can be written for the upper cylinder. An example is

$$\frac{\sigma_{zU}}{A_L} = \sum_{j=1}^{\infty}\left[a_{3j}e^{-\lambda_j z/\eta} - a_{4j}e^{\lambda_j(z-h)/\eta}\right]\varphi_1\left(\lambda_j, \frac{r}{\eta}\right) - \frac{A_U}{A_L} \tag{3.3}$$

The Fourier Coefficients.

The following functions will be defined for specifying the boundary values of χ^1 for the lower cylinder:

$$f_1(r) = \sigma'_{zL}(r, 0) \quad , \quad f_2(r) = \sigma'_{zL}(r, \ell) \tag{3.4a}$$

$$g_1(r) = \frac{E_L}{(1+\nu_L)}u'_L(r, 0) \quad , \quad g_2(r) = \frac{E_L}{(1+\nu_L)}u'_L(r, \ell) \tag{3.4b}$$

$$h_1(r) = \tau_L'(r, 0) \quad , \quad h(r) = \tau_L'(r, \ell) \tag{3.4c}$$

$$K_1(r) = \frac{E_L}{(1+\nu_L)} w_L'(r, 0) \quad , \quad K_2(r) = \frac{E_L}{(1+\nu_L)} w_L'(r, \ell) \tag{3.4d}$$

Similarly, these will be defined for the upper cylinder

$$f_3(r) = \sigma_{zu}'(r, 0) \quad , \quad f_4(r) = \sigma_{zu}'(r, h) \tag{3.5a}$$

$$g_3(r) = \frac{E_u}{(1+\nu_u)} u_u'(r, 0) \quad , \quad g_4(r) = \frac{E_u}{(1+\nu_u)} u_u'(r, h) \tag{3.5b}$$

$$h_3(r) = \tau_u'(r, 0) \quad , \quad h_4(r) = \tau_u'(r, h) \tag{3.5c}$$

$$K_3(r) = \frac{E_u}{(1+\nu_u)} w_u'(r, 0) \quad , \quad K_4(r) = \frac{E_u}{(1+\nu_u)} w_u'(r, h) \tag{3.5d}$$

The φ and W functions exhibit an inner biorthogonality in the form

$$\frac{1}{N(\gamma_j)} \int_0^1 \left[W_n(\gamma_j, r)\, \phi_n(\gamma_K, r) + W_{n+2}(\gamma_j, r)\, \phi_{n+2}(\gamma_K, r) \right] r\, dr$$

$$= \frac{1}{2} \delta_{jK} \quad , \quad n = 1, 2 \tag{3.6a}$$

$$\frac{1}{N(\lambda_j)} \int_0^\eta \left[W_n\left(\lambda_j, \frac{r}{\eta}\right) \phi_n\left(\lambda_K, \frac{r}{\eta}\right) + W_{n+2}\left(\lambda_j, \frac{r}{\eta}\right) \phi_{n+2}\left(\lambda_K, \frac{r}{\eta}\right) \right] \frac{r}{\eta} \frac{dr}{\eta}$$

$$= \frac{1}{2} \delta_{jK} \quad , \quad n = 1, 2 \tag{3.6b}$$

Using this property, the Fourier coefficients are obtained from the integrals:

$$a_{1j} = \frac{1}{N(\gamma_j)} \int_0^1 \Big[W_1(\gamma_j, r) f_1(r) + W_2(\gamma_j, r) h_1(r)$$

$$+ W_3(\gamma_j, r) g_1(r) + W_4(\gamma_j, r) K_1(r) \Big] r dr \qquad (3.7a)$$

$$a_{2j} = \frac{1}{N(\gamma_j)} \int_0^1 \Big[-W_1(\gamma_j, r) f_2(r) + W_2(\gamma_j, r) h_2(r)$$

$$- W_3(\gamma_j, r) g_2(r) + W_4(\gamma_j, r) K_2(r) \Big] r dr \qquad (3.7b)$$

$$a_{3j} = \frac{1}{N(\lambda_j)} \int_0^\eta \Big[W_1\left(\lambda_j, \frac{r}{\eta}\right) f_3(r) + W_2\left(\lambda_j, \frac{r}{\eta}\right) h_3(r)$$

$$+ W_3\left(\lambda_j, \frac{r}{\eta}\right) g_3(r) + W_4\left(\lambda_j, \frac{r}{\eta}\right) K_3(r) \Big] r dr \qquad (3.7c)$$

$$a_{4j} = \frac{1}{N(\lambda_j)} \int_0^\eta \Big[-W_1\left(\lambda_j, \frac{r}{\eta}\right) f_4(r) + W_2\left(\lambda_j, \frac{r}{\eta}\right) h_4(r)$$

$$- W_3\left(\lambda_j, \frac{r}{\eta}\right) g_4(r) + W_4\left(\lambda_j, \frac{r}{\eta}\right) K_4(r) \Big] r dr \qquad (3.7d)$$

It can be shown that when the specified boundary conditions pertaining to a two cylinder contact problem and the unspecified conditions are substituted into equations (3.7) and then integrated, the Fourier coefficients can be expressed in the form of the following matrix equations:

$$a_{1j} = G_{1j} + \sum_{K=1}^{\infty} B_{1jK}\, a_{1K} + \sum_{K=1}^{\infty} C_{1jK}\, a_{2K} \tag{3.8a}$$

$$\begin{aligned} a_{2j} = G_{2j} &+ \sum_{K=1}^{\infty} C_{2jK}\, a_{1K} + \sum_{K=1}^{\infty} B_{2jK}\, a_{2K} \\ &+ \sum_{K=1}^{\infty} B_{5jK}\, a_{3K} + \sum_{K=1}^{\infty} C_{5jK}\, a_{4K} \end{aligned} \tag{3.8b}$$

$$\begin{aligned} a_{3j} = G_{3j} &+ \sum_{K=1}^{\infty} B_{3jK}\, a_{3K} + \sum_{K=1}^{\infty} C_{3jK}\, a_{4K} \\ &+ \sum_{K=1}^{\infty} C_{6jK}\, a_{1K} + \sum_{K=1}^{\infty} B_{6jK}\, a_{2K} \end{aligned} \tag{3.8c}$$

$$a_{4j} = G_{4j} + \sum_{K=1}^{\infty} C_{4jK}\, a_{3K} + \sum_{K=1}^{\infty} B_{4jK}\, a_{4K} \tag{3.8d}$$

where G_{mj}, the forcing function vectors, are that part of the integral (3 . 7) which contains the specified boundary conditions on X^1. The matrices B_{njk} and C_{njk} for n = 1 through 4 represent the part of the integral (3 . 7) containing the unspecified conditions on X^1. The unspecified continuity conditions are represented by the matrices B_{5jk}, B_{6jk}, C_{5jk}, C_{6jk}. Unspecified conditions are based on the solution for the stresses and displacements evaluated at the boundary. Generally, the infinite systems of equations (3 . 8) can be satisfactorily approximated by truncation.

As stated by Power [1], this would be true if the matrices are strongly diagonal, which would result in the lower order coefficients being only slightly influenced by the higher order ones. The matrices B_{njk} and C_{njk} for n=1 through 4 does exhibit this characteristic. However, the matrices B_{5jk}, B_{6jk}, C_{5jk}, and C_{6jk} contain eigenfunctions from one cylinder and biorthogonal functions from the other cylinder. It is believed that they are somewhat strongly diagonal depending on the configuration of the cylinders and material constants. A positive proof is beyond the scope of this thesis but it will be assumed that all matrices can be truncated to yield results that are sufficiently accurate.

CHAPTER IV

EXAMPLE PROBLEMS

Two particular problems will be considered:

Problem 1. The cylinders are in frictionless contact with each other. This implies the continuity of normal stresses and normal displacements. In addition, the radial shear stress will be zero at the interface boundaries.

Problem 2. The cylinders are bonded together. This implies that the normal displacements, normal stresses, radial displacements, and radial shearing stresses are continuous through the contact interface.

As a prelude, a short discussion of the indentation of an elastic half space by a rigid, flat-ended cylindrical punch will be made. The solution is concise and well known. It has many similarities to the two cylinder contact problem. For a rigid cylinder of radius η indenting an elastic half space, the contact pressure is given by Sneddon [6] as:

$$\frac{\sigma_z}{p} = \frac{-\eta}{2(\eta^2 - r^2)^{\frac{1}{2}}} \tag{4.1}$$

where p is the average pressure based on the total load, F

$$p = \frac{F}{\pi \eta^2} \tag{4.2}$$

Equation (4.1) shows that the contact pressure increases from $\frac{1}{2}p$ at the center of the punch to infinity at the edge of the punch.

It is expected that a stress singularity of this nature will exist for the two cylinder contact problem. It would occur at the interface for the value $r = \eta$.

Boundary Conditions at the Rigid Faces.

It is convenient to first consider the boundary conditions which are applicable to both problems. It will be recalled that the rigid compression faces are assumed to contact the cylinder ends in a frictionless manner. Moreover, the bottom face remains stationary and the top face is displaced downward an amount equal to δ. Four boundary conditions for the two cylinders are:

$$\tau_L(r,o) = w_L(r,o) = \tau_u(r,h) = 0 \tag{4.3a}$$

$$w_u(r,h) = -\delta \tag{4.3b}$$

The related boundary conditions on χ' are then:

$$h_1(r) = h_4(r) = k_1(r) = k_4(r) = 0 \tag{4.4}$$

Substitution of these specified conditions and the appropriate unspecified conditions into equations (3.7a) and (3.7d) results in

$$G_{1j} = G_{4j} = 0 \tag{4.5a}$$

$$B_{1jk} = B_{4jk} = \tfrac{1}{2}\,\delta_{jk} \tag{4.5b}$$

$$C_{1jk} = -\tfrac{1}{2}\,\delta_{jk}\,e^{-\gamma_k \ell} \tag{4.5c}$$

$$C_{4jk} = -\tfrac{1}{2}\,\delta_{jk}\,e^{-\lambda_k h/\eta} \tag{4.5d}$$

These results are substituted into equations (3.8a) and (3.8d) to yield

$$a_{1k} = -\,e^{-\gamma_k \ell}\,a_{2k} \tag{4.6a}$$

$$a_{4k} = -\,e^{-\lambda_k h/\eta}\,a_{3k} \tag{4.6b}$$

It is apparent that these last two equations can be substituded directly into equations (3.8b) and (3.8c). The results are:

$$D_{2jk}\,a_{2k} = G_{2j} + D_{5jk}\,a_{3k} \tag{4.7a}$$

$$D_{3jk}\,a_{3k} = G_{3j} + D_{6jk}\,a_{2k} \tag{4.7b}$$

where

$$D_{2jk} = \delta_{jk} - B_{2jk} + C_{2jk}\, e^{-\gamma_k \ell} \tag{4.8a}$$

$$D_{3jk} = \delta_{jk} - B_{3jk} + C_{3jk}\, e^{-\lambda_k h/\eta} \tag{4.8b}$$

$$D_{5jk} = B_{5jk} - C_{5jk}\, e^{-\lambda_k h/\eta} \tag{4.8c}$$

$$D_{6jk} = B_{6jk} - C_{6jk}\, e^{-\gamma_k \ell} \tag{4.8d}$$

Substitution of equation (4.7b) into (4.7a) results in this explicit matrix solution for a_{3k}:

$$a_2 = \left[D_2 - D_5 D_3^{-1} D_6\right]^{-1}\left[G_2 + D_5 D_3^{-1} G_3\right] \tag{4.9}$$

The other sets of Fourier coefficients can now be readily found by using equations (4.7b) and (4.6).

<u>The Constants for the Uniform Axial Stress Solution</u>.

The total solution for the axial displacements at the boundaries will first be given. Equations (4.6) and the appropriate value for z will be substituted into equations (3.2d) and (3.13) to give

$$\frac{E_L w_L(r,0)}{A_L(1+\nu_L)} = \frac{B_L}{A_L}\frac{1}{1+\nu_L} \tag{4.10a}$$

$$\frac{E_L w_L(r,\ell)}{A_L(1+\nu_L)} = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} + a_{2j}\right]\varphi_4(\gamma_j, r) - \frac{1}{1+\nu_L} + \frac{B_L}{A_L}\frac{1}{1+\nu_L} \tag{4.10b}$$

$$\frac{E_U w_U(r,0)}{A_L(1+\nu_U)\eta} = \sum_{j=1}^{\infty}\left[a_{3j} + a_{4j}e^{-\lambda_j h/\eta}\right]\varphi_4\left(\lambda_j, \frac{r}{\eta}\right) + \frac{B_U}{A_L}\frac{1}{1+\nu_U}\frac{1}{\eta} \tag{4.10c}$$

$$\frac{E_U w_U(r,h)}{A_L(1+\nu_U)\eta} = -\frac{A_U}{A_L}\frac{1}{1+\nu_U}\frac{h}{\eta} + \frac{B_U}{A_L}\frac{1}{1+\nu_U}\frac{1}{\eta} \tag{4.10d}$$

The boundary condition that $w_L(r, 0) = 0$ is substituted into equation (4.10a) to yield:

$$B_L = 0 \tag{4.11}$$

The force equilibrium condition requires that the following be true

$$\int_0^1 \sigma_{zL} r \, dr = \int_0^\eta \sigma_{zU} r \, dr \tag{4.12}$$

Substitution of equations (3.2a) and (3.3) into (4.12) leads to

$$\frac{A_U}{A_L} = \frac{1}{\eta^2} \tag{4.13}$$

The continuity of displacements at the interface can be ensured by

$$w_L(r, \ell) = w_U(r, 0) \tag{4.14}$$

This equation is valid for all values of $r \leq \eta$. In applications, a finite number of the eigenfunctions will be used and it is then desirable to use the integral of this equation

$$\int_0^\rho w_L(r, \ell)\, r\, dr = \int_0^\rho w_U(r, 0)\, r\, dr \tag{4.15}$$

where $0 < \rho \leq \eta$. Substitution of equations (4.10b) and (4.10c) into (4.15) leads to

$$\frac{B_U}{A_L} = \frac{E_U}{E_L}\left[C - \ell\right] \tag{4.16}$$

where

$$C = \frac{2}{\rho^2}\sum_{j=1}^{\infty}\Bigg\{(1+\nu_L)\gamma_j J_1(\gamma_j)\left[a_{1j}e^{-\gamma_j \ell} + a_{2j}\right]\left[\frac{2\nu_L\rho}{\gamma_j}J_1(\gamma_j\rho) - \rho^2 J_0(\gamma_j\rho)\right.$$

$$\left. + \rho J_1(\gamma_j\rho)\frac{J_0(\gamma_j)}{J_1(\gamma_j)}\right] - (1+\nu_U)\eta^3\lambda_j J_1(\lambda_j)\frac{E_L}{E_U}\left[a_{3j} + a_{4j}e^{-\lambda_j h/\eta}\right]$$

$$\left[\frac{2\nu_U\rho}{\lambda_j\eta}J_1\left(\frac{\lambda_j\rho}{\eta}\right) - \frac{\rho^2}{\eta^2}J_0\left(\frac{\lambda_j\rho}{\eta}\right) + \frac{\rho}{\eta}\frac{J_0(\lambda_j)}{J_1(\lambda_j)}J_1\left(\frac{\lambda_j\rho}{\eta}\right)\right]\Bigg\} \tag{4.17}$$

The constant, A_L, can now be determined. The boundary condition that $w_U(r, h) = -\delta$ and equations (4.13) and (4.16) will be substituted into equation (4.10d) to yield

$$A_L = \frac{E_L\,\delta}{\frac{h\,E_L}{\eta^2 E_U} + \ell - C} \tag{4.18}$$

The zero-eigenvalue solution is therefore found to be

$$\frac{\sigma^{\circ}_{zL}}{A_L} = -1 \quad , \quad \frac{\sigma^{\circ}_{zU}}{A_L} = -\frac{1}{\eta^2} \tag{4.19a}$$

$$\frac{E_L u^{\circ}_L}{A_L} = r\,\nu_L \quad , \quad \frac{E_U u^{\circ}_U}{A_L} = \frac{r\,\nu_U}{\eta^2} \tag{4.19b}$$

$$\frac{E_L w^{\circ}_L}{A_L} = -z \quad , \quad \frac{E_U w^{\circ}_U}{A_L} = -\frac{z}{\eta^2} + \frac{E_U}{E_L}\left(C - \ell\right) \tag{4.19c}$$

Problem 1. Frictionless Contact.

Consider a cylinder of radius 1 as being placed into frictionless contact with an upper cylinder whose radius, η, is equal to or less than 1. For the lower cylinder, the following boundary conditions will be used:

$$\sigma_{zL}(r,\ell) = \sigma_{zU}(r,0), \quad 0 \leq r \leq \eta \tag{4.20a}$$

$$\sigma_{zL}(r,\ell) = 0 \quad , \quad \eta \leq r \leq 1 \tag{4.20b}$$

$$\tau_L(r,\ell) = 0 \tag{4.20c}$$

For the upper cylinder, these boundary conditions will be used:

$$w_U(r,0) = w_L(r,\ell) \tag{4.21a}$$

$$\tau_U(r,0) = 0 \tag{4.21b}$$

The self-equilibrated boundary conditions on $\boldsymbol{X}'$ are then:

$$f_2(r) = \sum_{j=1}^{\infty}\left[a_{3j} - a_{4j}e^{-\lambda_j h/\eta}\right]\varphi_1\left(\lambda_j, \frac{r}{\eta}\right) - \left[\frac{1}{\eta^2} - 1\right],$$

$$0 \leq r \leq \eta \tag{4.22a}$$

$$f_2(r) = 1 \quad , \quad \eta \leq r \leq 1 \tag{4.22b}$$

$$h_2(r) = h_3(r) = 0 \tag{4.22c}$$

$$K_3(r) = \frac{1}{\eta}\frac{E_U}{E_L}\frac{1+\nu_L}{1+\nu_U}\sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} + a_{2j}\right]\varphi_4(\gamma_j, r) \tag{4.22d}$$

The unspecified conditions are

$$g_2(r) = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} - a_{2j}\right]\varphi_3(\gamma_j, r) \tag{4.23a}$$

$$K_2(r) = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} + a_{2j}\right]\varphi_4(\gamma_j, r) \tag{4.23b}$$

$$f_3(r) = \sum_{j=1}^{\infty}\left[a_{3j} - a_{4j}\, e^{-\lambda_j h/\eta}\right]\varphi_1\left(\lambda_j, \frac{r}{\eta}\right) \tag{4.23c}$$

$$g_3(r) = \sum_{j=1}^{\infty}\left[a_{3j} + a_{4j}\, e^{-\lambda_j h/\eta}\right]\varphi_3\left(\lambda_j, \frac{r}{\eta}\right) \tag{4.23d}$$

Substitution of these conditions into the Fourier integrals (3.7b) and (3.7c) result in the following forcing functions and matrices:

$$G_{2j} = \frac{1}{N(\gamma_j)}\left[\frac{1}{\eta^2}\int_0^{\eta} W_1(\gamma_j, r)\, r\, dr - \int_0^{1} W_1(\gamma_j, r)\, r\, dr\right] \tag{4.24a}$$

$$G_{3j} = 0 \tag{4.24b}$$

$$B_{2jk} = \frac{1}{N(\gamma_j)} \int_0^1 \left[W_3(\gamma_j, r)\, \varphi_3(\gamma_k, r) + W_4(\gamma_j, r)\, \varphi_4(\gamma_k, r) \right] r\, dr \tag{4.24c}$$

$$C_{2jk} = e^{-\gamma_k \ell} \frac{1}{N(\gamma_j)} \int_0^1 \left[-W_3(\gamma_j, r)\, \varphi_3(\gamma_k, r) + W_4(\gamma_j, r)\, \varphi_4(\gamma_k, r) \right] r\, dr \tag{4.24d}$$

$$B_{3jk} = \tfrac{1}{2}\, \delta_{jk} \tag{4.24e}$$

$$C_{3jk} = -\tfrac{1}{2}\, \delta_{jk}\, e^{-\lambda_k h/\eta} \tag{4.24f}$$

$$B_{5jk} = \frac{1}{N(\gamma_j)} \int_0^\eta -W_1(\gamma_j, r)\, \varphi_1\left(\lambda_k, \frac{r}{\eta}\right) r\, dr \tag{4.24g}$$

$$C_{5jk} = e^{-\lambda_k h/\eta} \cdot \frac{1}{N(\gamma_j)} \int_0^\eta W_1(\gamma_j, r)\, \varphi_1\left(\lambda_k, \frac{r}{\eta}\right) r\, dr \tag{4.24h}$$

$$B_{6jk} = \frac{1}{N(\lambda_j)} \frac{1}{\eta} \frac{E_U}{E_L} \frac{1+\nu_L}{1+\nu_U} \int_0^\eta W_4\left(\lambda_j, \frac{r}{\eta}\right) \varphi_4(\gamma_k, r)\, \frac{r}{\eta} \frac{dr}{\eta} \tag{4.24i}$$

$$C_{6jk} = \frac{e^{-\gamma_k \ell}}{N(\lambda_j)} \frac{1}{\eta} \frac{E_U}{E_L} \frac{1+\nu_L}{1+\nu_U} \int_0^\eta W_4\left(\lambda_j, \frac{r}{\eta}\right) \varphi_4(\gamma_k, r)\, \frac{r}{\eta} \frac{dr}{\eta} \tag{4.24j}$$

Problem 2. Bonded Contact.

Consider two cylinders as being placed into contact as in problem 1 except in this case they are bonded together. For the lower cylinder, these boundary conditions will be used:

$$\sigma_{zL}(r,\ell) = \sigma_{zU}(r,0)\ , \quad 0 \leq r \leq \eta \qquad (4.25a)$$

$$\tau_L(r,\ell) = \tau_U(r,0)\ , \quad 0 \leq r \leq \eta \qquad (4.25b)$$

$$\sigma_{zL}(r,\ell) = 0 \qquad , \quad \eta \leq r \leq 1 \qquad (4.25c)$$

$$\tau_L(r,\ell) = 0 \qquad , \quad \eta \leq r \leq 1 \qquad (4.25d)$$

For the upper cylinder, these boundary conditions will be used:

$$w_U(r,0) = w_L(r,\ell) \qquad (4.26a)$$

$$u_U(r,0) = u_L(r,\ell) \qquad (4.26b)$$

The self-equilibrated boundary conditions on X' in this case are:

$$f_2(r) = \sum_{j=1}^{\infty}\left[a_{3j} - a_{4j}\, e^{-\lambda_j h/\eta}\right]\varphi_1\left(\lambda_j, \frac{r}{\eta}\right) - \left[\frac{1}{\eta^2} - 1\right],$$

$$0 \leq r \leq \eta \qquad (4.27a)$$

$$f_2(r) = 1\ , \quad \eta \leq r \leq 1 \qquad (4.27b)$$

$$h_2(r) = \sum_{j=1}^{\infty} \left[a_{3j} + a_{4j}\, e^{-\lambda_j h/\eta}\right] \varphi_2\left(\lambda_j, \frac{r}{\eta}\right), \quad 0 \le r \le \eta \tag{4.27c}$$

$$h_2(r) = 0 \quad , \quad \eta \le r \le 1 \tag{4.27d}$$

$$g_3(r) = \frac{1}{\eta}\,\frac{E_U}{E_L}\,\frac{1+\nu_L}{1+\nu_U}\sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} - a_{2j}\right] \varphi_3(\gamma_j, r)$$

$$- \frac{\sigma_{AV}\nu_L r}{(1+\nu_U)\eta}\left[\frac{\nu_U}{\nu_L}\,\frac{E_L}{E_U}\,\frac{1}{\eta^2} - 1\right]\frac{E_U}{E_L} \tag{4.27e}$$

$$K_3(r) = \frac{1}{\eta}\,\frac{E_U}{E_L}\,\frac{1+\nu_L}{1+\nu_U}\sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} + a_{2j}\right] \varphi_4(\gamma_j, r) \tag{4.27f}$$

The unspecified conditions are

$$g_2(r) = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} - a_{2j}\right] \varphi_3(\gamma_j, r) \tag{4.28a}$$

$$K_2(r) = \sum_{j=1}^{\infty}\left[a_{1j}\, e^{-\gamma_j \ell} + a_{2j}\right] \varphi_4(\gamma_j, r) \tag{4.28b}$$

$$f_3(r) = \sum_{j=1}^{\infty}\left[a_{3j} - a_{4j}\, e^{-\lambda_j h/\eta}\right] \varphi_1\left(\lambda_j, \frac{r}{\eta}\right) \tag{4.28c}$$

$$h_3(r) = \sum_{j=1}^{\infty}\left[a_{3j} + a_{4j}\, e^{-\lambda_j h/\eta}\right] \varphi_2\left(\lambda_j, \frac{r}{\eta}\right) \tag{4.28d}$$

Substitution of these conditions into the Fourier integrals as before result in the following forcing functions and matrices:

$$G_{2j} = \frac{1}{N(\gamma_j)}\left[\frac{1}{\eta^2}\int_0^\eta W_1(\gamma_j, r)\, r\, dr - \int_0^1 W_1(\gamma_j, r)\, r\, dr\right] \tag{4.29a}$$

$$G_{3j} = \frac{-\nu_L}{N(\lambda_j)(1+\nu_U)}\,\frac{E_U}{E_L}\left[\frac{\nu_U}{\nu_L}\,\frac{E_L}{E_U}\,\frac{1}{\eta^2} - 1\right]\int_0^\eta W_3\left(\lambda_j, \frac{r}{\eta}\right)\frac{r^2}{\eta^2}\,\frac{dr}{\eta} \tag{4.29b}$$

$$B_{2jk} = \frac{1}{N(\gamma_j)}\int_0^1\left[W_3(\gamma_j, r)\,\varphi_3(\gamma_k, r) + W_4(\gamma_j, r)\,\varphi_4(\gamma_k, r)\right] r\,dr \tag{4.29c}$$

$$C_{2jk} = \frac{e^{-\gamma_k \ell}}{N(\gamma_j)}\int_0^1\left[-W_3(\gamma_j, r)\,\varphi_3(\gamma_k, r) + W_4(\gamma_j, r)\,\varphi_4(\gamma_k, r)\right] r\,dr \tag{4.29d}$$

$$B_{3jk} = \frac{1}{N(\lambda_j)}\int_0^\eta\left[W_1\left(\lambda_j, \frac{r}{\eta}\right)\varphi_1\left(\lambda_k, \frac{r}{\eta}\right) + W_2\left(\lambda_j, \frac{r}{\eta}\right)\varphi_2\left(\lambda_k, \frac{r}{\eta}\right)\right]\frac{r}{\eta}\,\frac{dr}{\eta} \tag{4.29e}$$

$$C_{3jk} = \frac{e^{-\lambda_k h/\eta}}{N(\lambda_j)}\int_0^\eta\left[-W_1\left(\lambda_j, \frac{r}{\eta}\right)\varphi_1\left(\lambda_k, \frac{r}{\eta}\right) + W_2\left(\lambda_j, \frac{r}{\eta}\right)\varphi_2\left(\lambda_k, \frac{r}{\eta}\right)\right]\frac{r}{\eta}\,\frac{dr}{\eta} \tag{4.29f}$$

$$B_{5jk} = \frac{1}{N(\gamma_j)}\int_0^\eta\left[-W_1(\gamma_j, r)\,\varphi_1\left(\lambda_k, \frac{r}{\eta}\right) + W_2(\gamma_j, r)\,\varphi_2\left(\lambda_k, \frac{r}{\eta}\right)\right] r\,dr \tag{4.29g}$$

$$C_{5jk} = \frac{e^{-\lambda_k h/\eta}}{N(\gamma_j)}\int_0^\eta\left[W_1(\gamma_j, r)\,\varphi_1\left(\lambda_k, \frac{r}{\eta}\right) + W_2(\gamma_j, r)\,\varphi_2\left(\lambda_k, \frac{r}{\eta}\right)\right] r\,dr \tag{4.29h}$$

$$B_{6jk} = \frac{1}{N(\lambda_j)}\,\frac{1}{\eta}\,\frac{E_U}{E_L}\,\frac{1+\nu_L}{1+\nu_U}\int_0^\eta\left[-W_3\left(\lambda_j, \frac{r}{\eta}\right)\varphi_3(\gamma_k, r)\right.$$

$$\left. + W_4\left(\lambda_j, \frac{r}{\eta}\right)\varphi_4(\gamma_k, r)\right]\frac{r}{\eta}\,\frac{dr}{\eta} \tag{4.29i}$$

$$C_{6jk} = \frac{e^{-\gamma_k \ell}}{N(\lambda_j)}\,\frac{1}{\eta}\,\frac{E_U}{E_L}\,\frac{1+\nu_L}{1+\nu_U}\int_0^\eta\left[W_3\left(\lambda_j, \frac{r}{\eta}\right)\varphi_3(\gamma_k, r)\right.$$

$$\left. + W_4\left(\lambda_j, \frac{r}{\eta}\right)\varphi_4(\gamma_k, r)\right]\frac{r}{\eta}\,\frac{dr}{\eta} \tag{4.29j}$$

Integration of the functions G_{2j} and G_{3j} leads to

$$G_{2j} = \frac{1}{2\eta\,\gamma_j^2 J_1^2(\gamma_j)\,N(\gamma_j)}\Big[2\nu_L J_1(\gamma_j)\,J_1(\gamma_j\eta) - 2\nu_L\eta\,J_1^2(\gamma_j)$$

$$- \gamma_j\,\eta\,J_1(\gamma_j)\,J_0(\gamma_j\eta) + \gamma_j\,J_0(\gamma_j)\,J_1(\gamma_j\eta)\Big] \qquad (4.30a)$$

$$G_{3j} = -\frac{\nu_L}{\lambda_j^2\,N(\lambda_j)}\;\frac{E_U}{E_L}\left[\frac{\nu_U}{\nu_L}\,\frac{E_L}{E_U}\,\frac{1}{\eta^2} - 1\right] \qquad (4.30b)$$

<u>Discussion of Numerical Results</u>.

To demonstrate some of the aspects of the solution, selected numerical results are presented graphically in Figures 2 through 9. In the graphs which resulted from this numerical study, it is noted that the stresses are given in dimensionless form. Since A_L is the average axial stress in the lower cylinder, the stresses are plotted as ratios σ_z/A_L, τ/A_L, etc. The deflection, δ, has been arbitrarily chosen as

$$\delta = \frac{\ell}{E_L}\left[1 + \frac{h\ E_L\ 1}{\ell\ E_U\ \eta^2}\right] \tag{4.30}$$

The numerical study showed that the series, truncated after 35 terms, exhibited large oscillations for the stresses at the interface due to the discontinuous nature of the boundary conditions at the interface. The amplitudes were greatly reduced by using Fejér's averaging method, as described by Lanczos [5], on the first fifteen terms of the series.

Figures 2 through 6 are for a configuration where $\eta = \frac{1}{2}$ and the material constants and lengths are equal.

The axial stresses as a function of r for problem 1 is shown in Figure 2. The oscillations in the computed results at the interface are not unexpected. The sharp rise as the value $r = \eta$ is approached is seen to have the same characteristic as equation (4.1) for the rigid flat-ended cylinder on an elastic half space. At z = 0.8 for the lower cylinder, it is noted that a small outer portion is in tension.

Figure 3 shows a schematic of the deformed grid for problem 2. The underlying characteristic is that at the interface, very small radial displacements occur compared with the rest of the cylinders. This is due to the restraining action of the free outer annulus of the lower cylinder upper face. The portion of the grid shown dashed is not known exactly because of the lack of enough grid points. The axial stresses at and near the interface is shown in Figure 4. The oscillations are more pronounced than in problem 1; otherwise, the characteristics are the same. Figure 5 shows the shear stresses near the interface and Figure 6 shows the radial and tangential stresses near the interface.

A second configuration has been studied for problem 2. It is an equal diameter case with $E_U = 0.2E_L$. For this case, Figure 7 shows the deformed grid. It is noted that, in view of the different E values, the vertical grid lines are not smooth through the interface. The slight curvature of the deformed interface is caused by the shear stresses transmitted through the interface.

A limited study was then made to determine how the overall axial stiffness is affected by changing the ratio of cylinder radii. The solution from problem 2 was used for this study and results were plotted as dimensionless ratios in order to be applicable to any radius, R_L, for the lower cylinder. The value, η, is then the ratio, R_U/R_L.

Figure 8 represents a stepped cylinder configuration where the length of the lower cylinder was taken to be equal to its diameter. It shows A_L as a function of η. The ordinate, A_L, has been defined in this figure as

$$A_L = F\ell/\pi\delta ER_L/(1 + R_L^2/R_U^2) \qquad (4.31)$$

It is noted that the curve begins and ends with a value of 1.0 and exhibits a minimum value near $R_U/R_L = 0.32$. When $R_U/R_L = 1.0$, the cylinders are in a state of uniform axial stress. For this reason, the curve represents the error factor if the axial stiffness, F/δ, for a stepped cylinder is computed on the uniform axial stress assumption, as commonly done in strength of materials methods. It is seen that, for this configuration, the error can be as much as 7% (near $R_U/R_L = 0.32$).

Figure 9 represents a stiff cylinder bonded to a cylinder having the characteristics of rubber. It shows A_L/δ as a function of η. The ordinate, A_L/δ, is seen to be

$$A_L/\delta = F\ell/\pi\delta E_L R_L^2 \qquad (4.32)$$

Two curves for two typical values of the ratio, ℓ/R_L, is given. For ℓ/R_L=2.0, it is seen that the curve starts at zero, increases to a maximum near the value $R_U/R_L = 0.95$, and then drops slightly as the value $R_U/R_L = 1.0$ is approached. That the curve exhibits a maximum is a paradox which still remains unexplained. For $\ell/R_L = 0.5$, the curve has similar characteristics except that it is not certain if a maximum value is present.

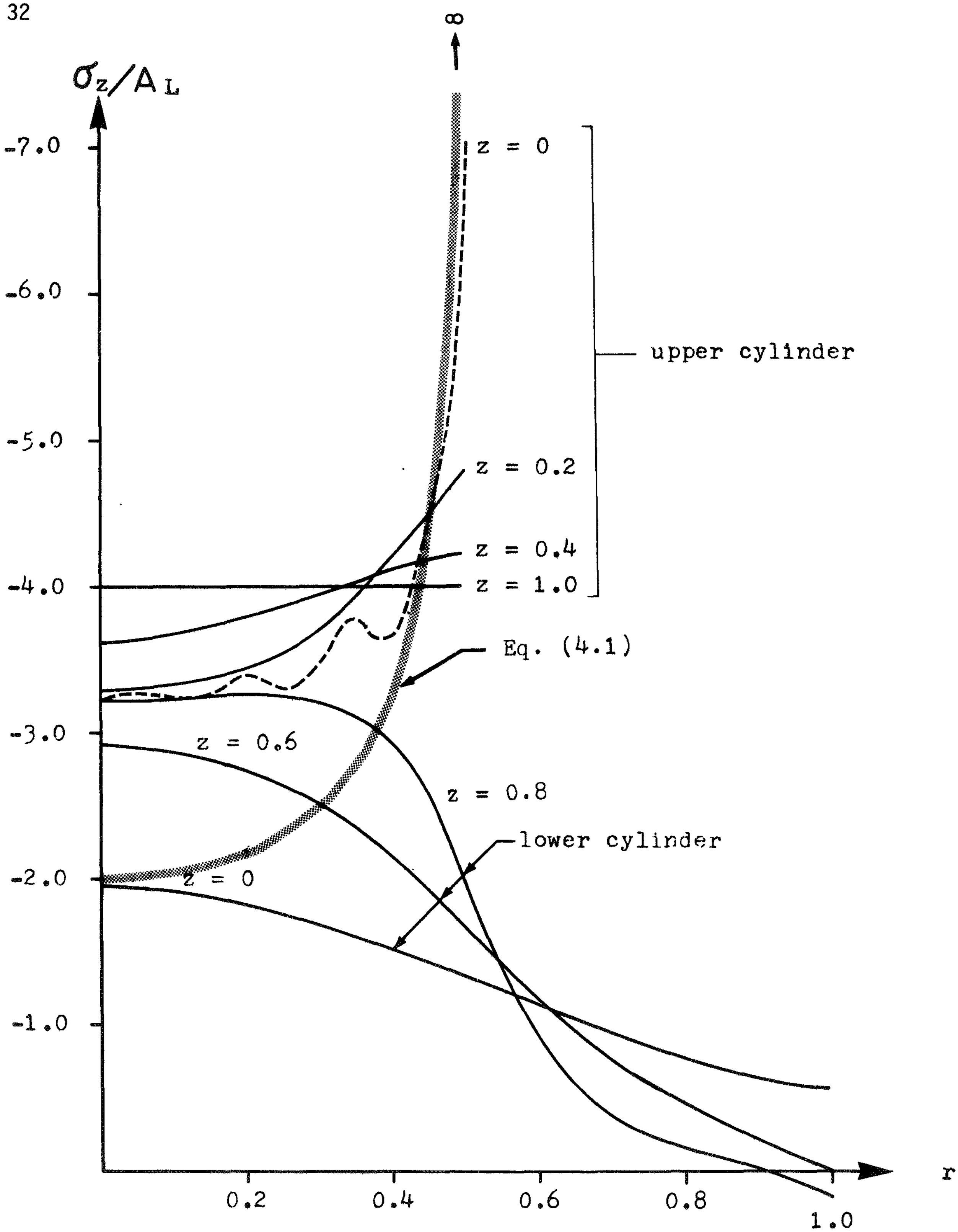

Figure 2. Axial Stresses, Problem 1, $\eta = 0.5$, $h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = E_U = 1.0$

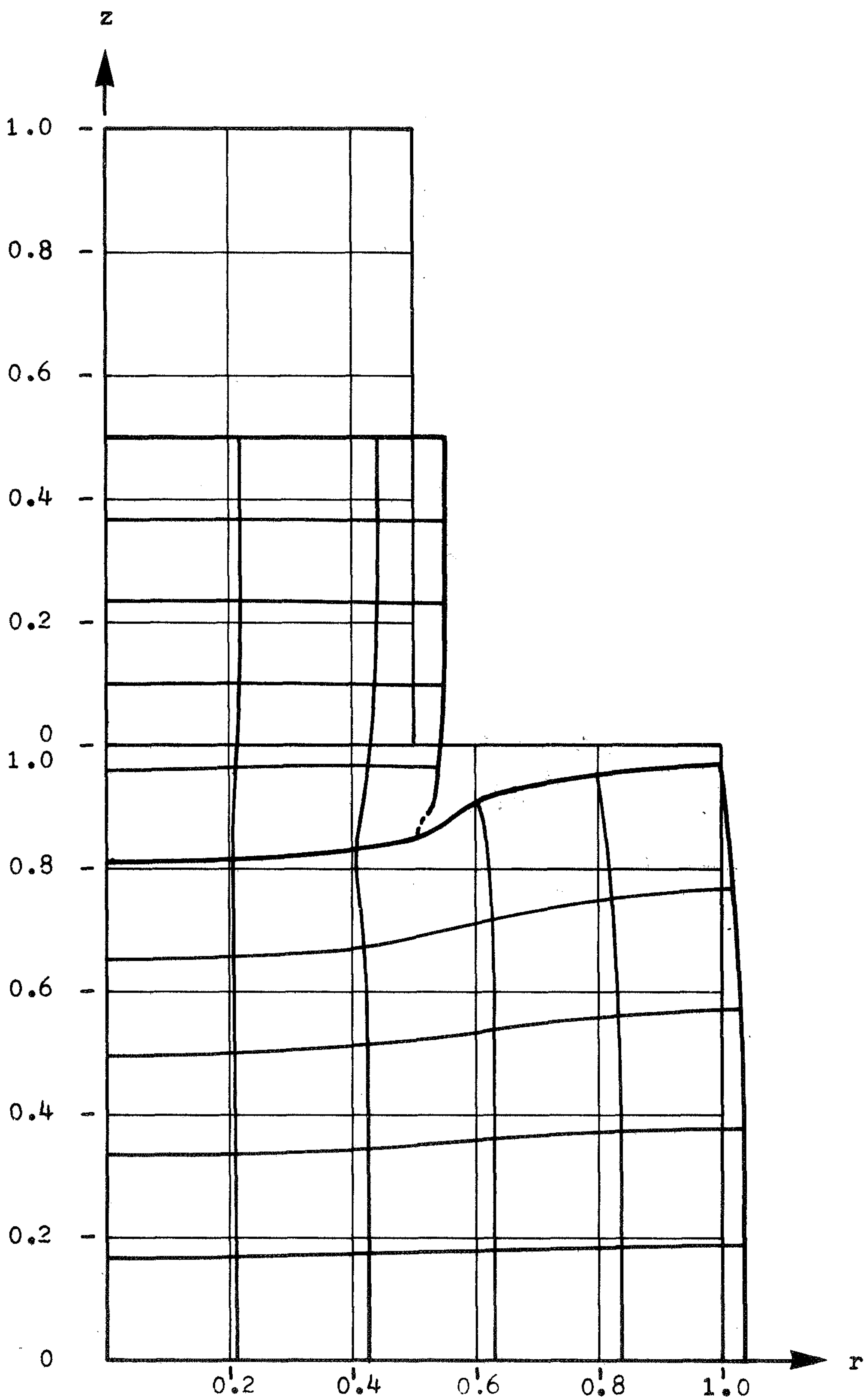

Figure 3. Deformed Grid, Problem 2, $\eta = 0.5$, $h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = E_U = 1.0$

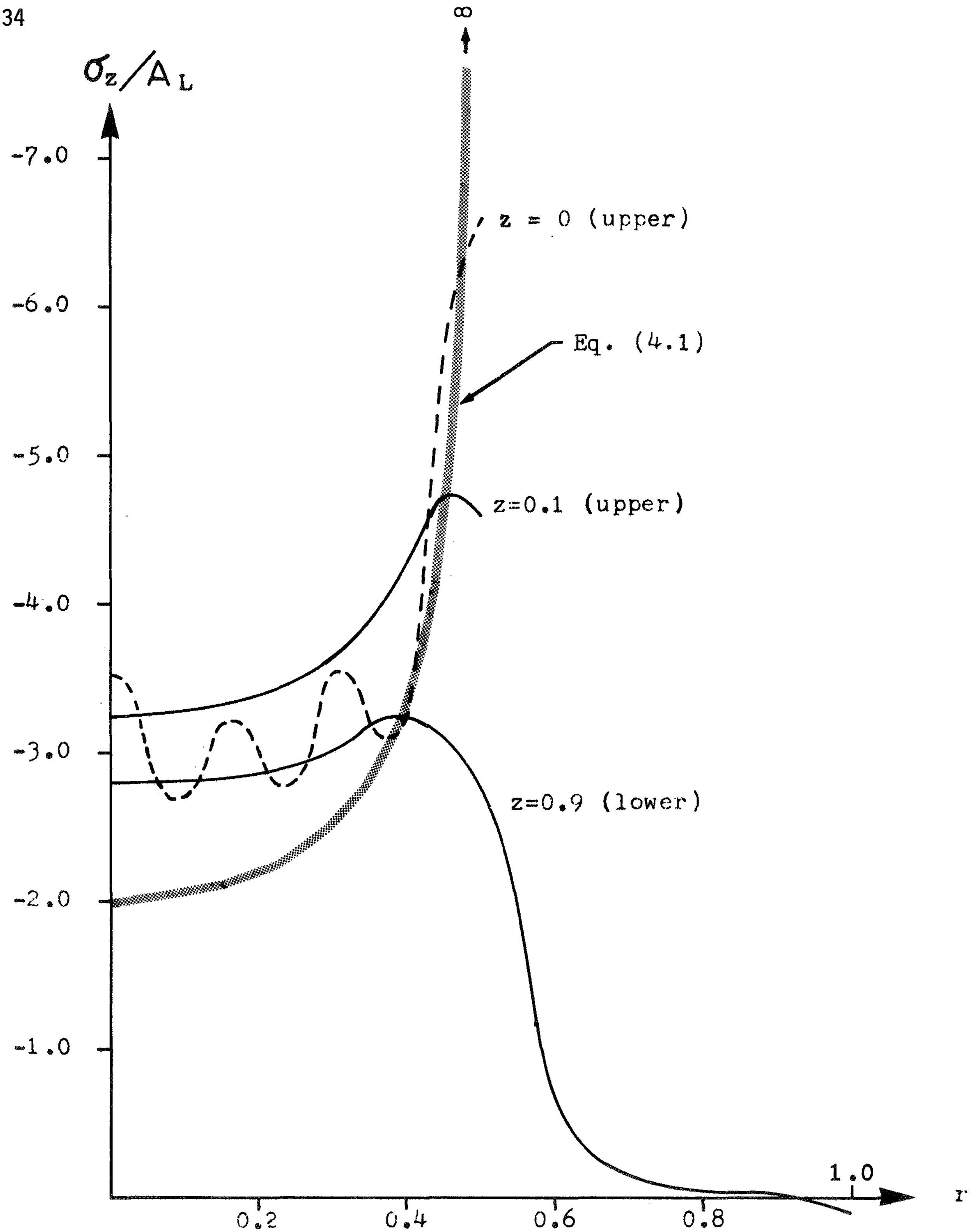

Figure 4. Axial Stresses at and near the Interface, Problem 2, $\eta = 0.5$, $h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = E_U = 1.0$

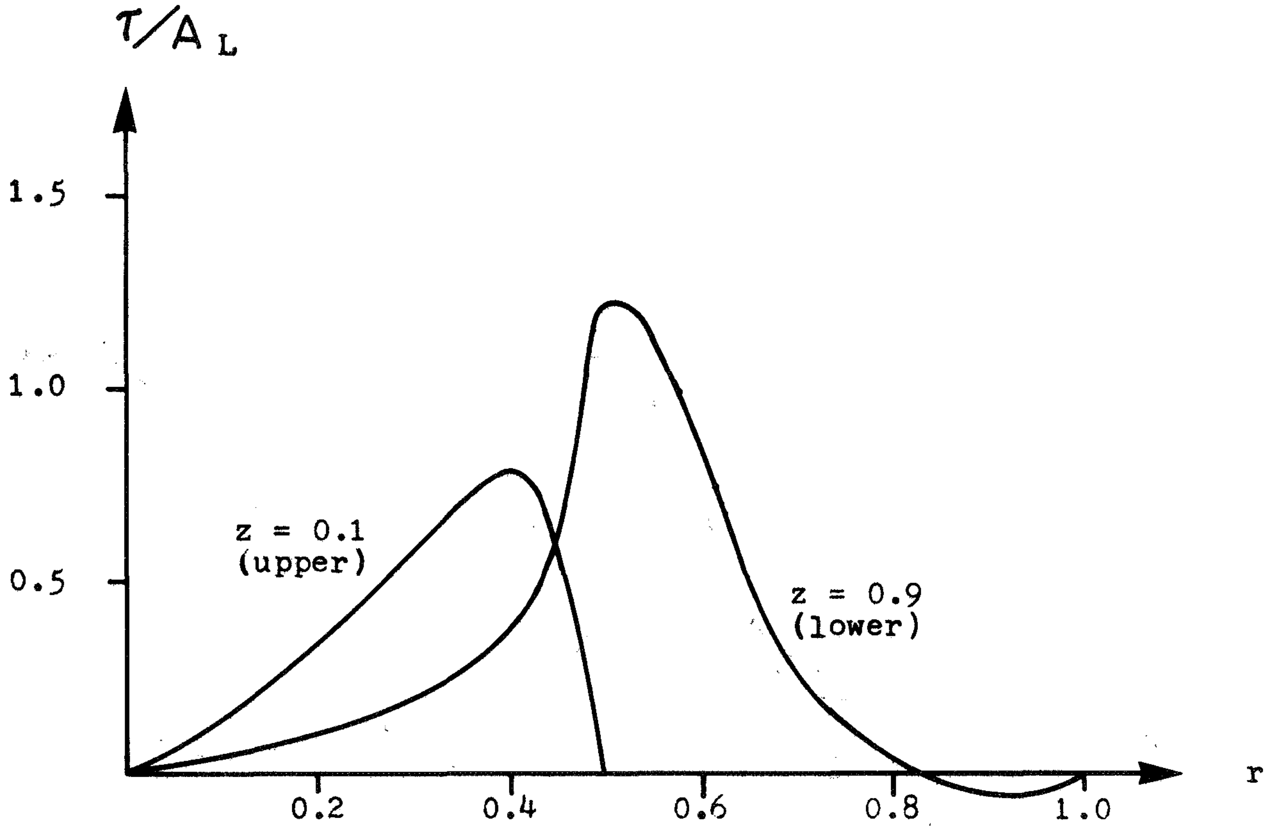

Figure 5. Shear Stresses near the Interface, Problem 2, $\eta = 0.5$, $h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = E_U = 1.0$

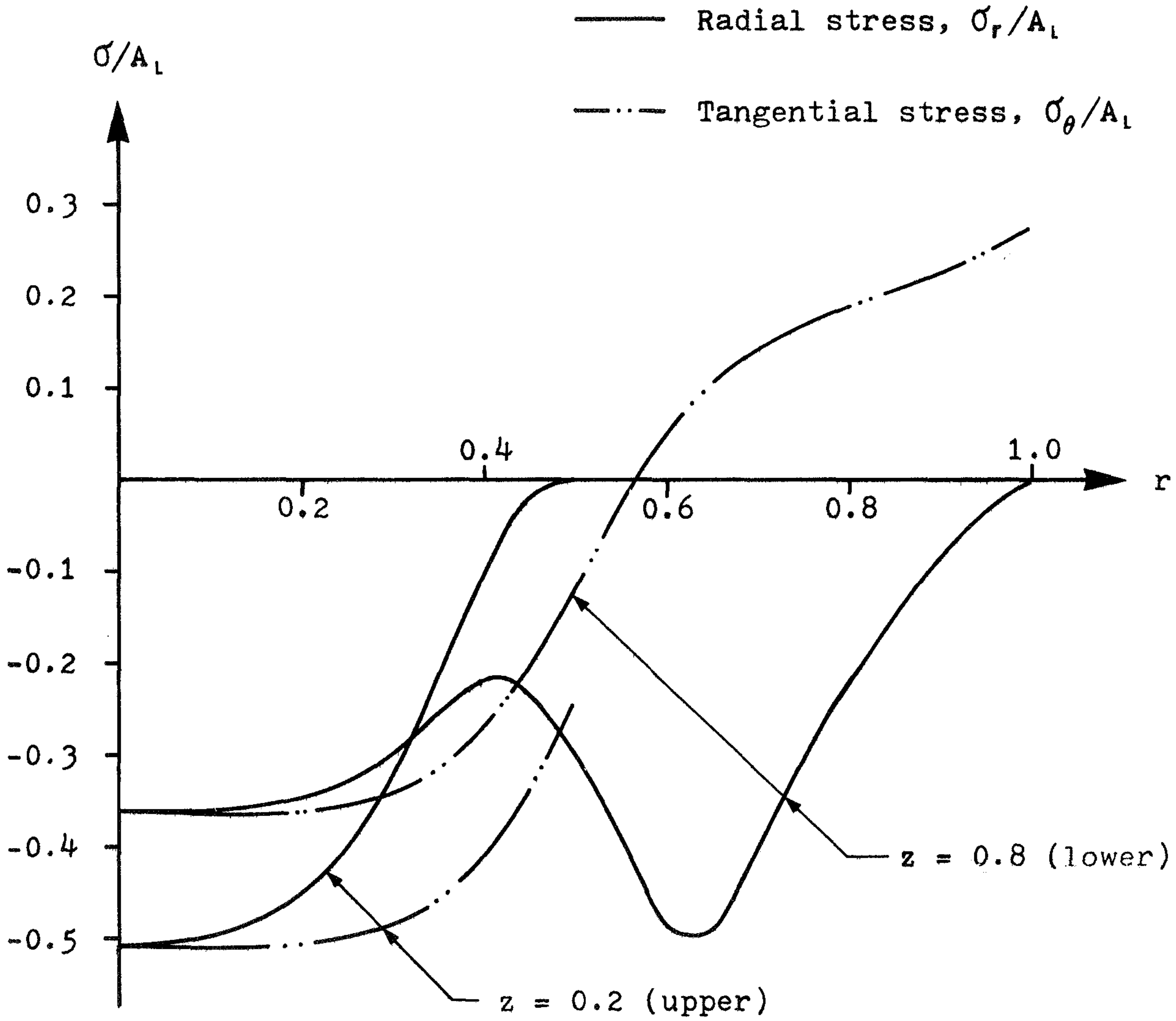

Figure 6. Radial and Tangential Stresses near the Interface, Problem 2, $\eta = 0.5$, $h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = E_U = 1.0$

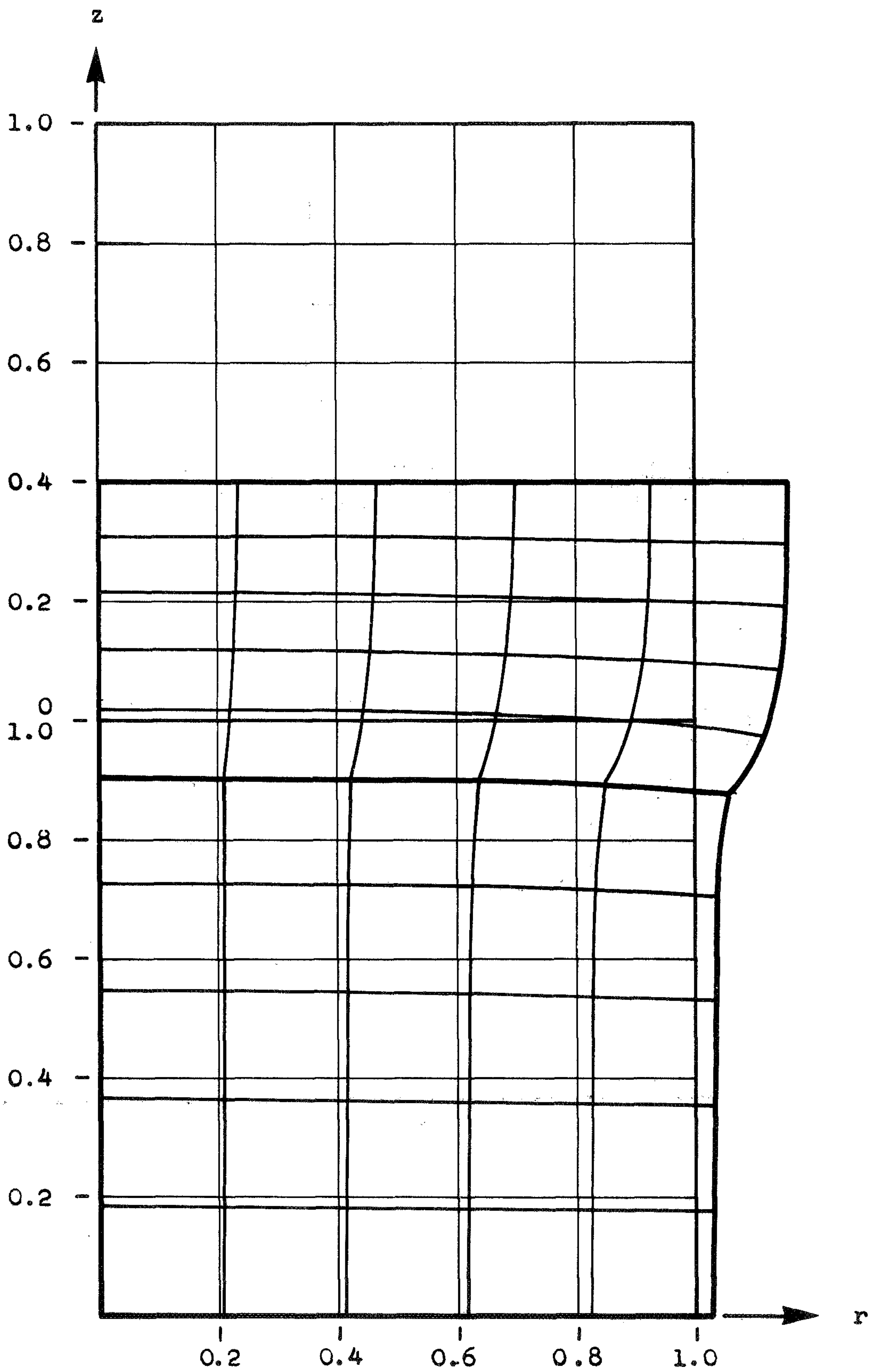

Figure 7. Deformed Grid, Problem 2, $\eta = 1.0$

$h = \ell = 1.0$, $\nu_L = \nu_U = 0.3$, $E_L = 1.0$, $E_U = 0.2$

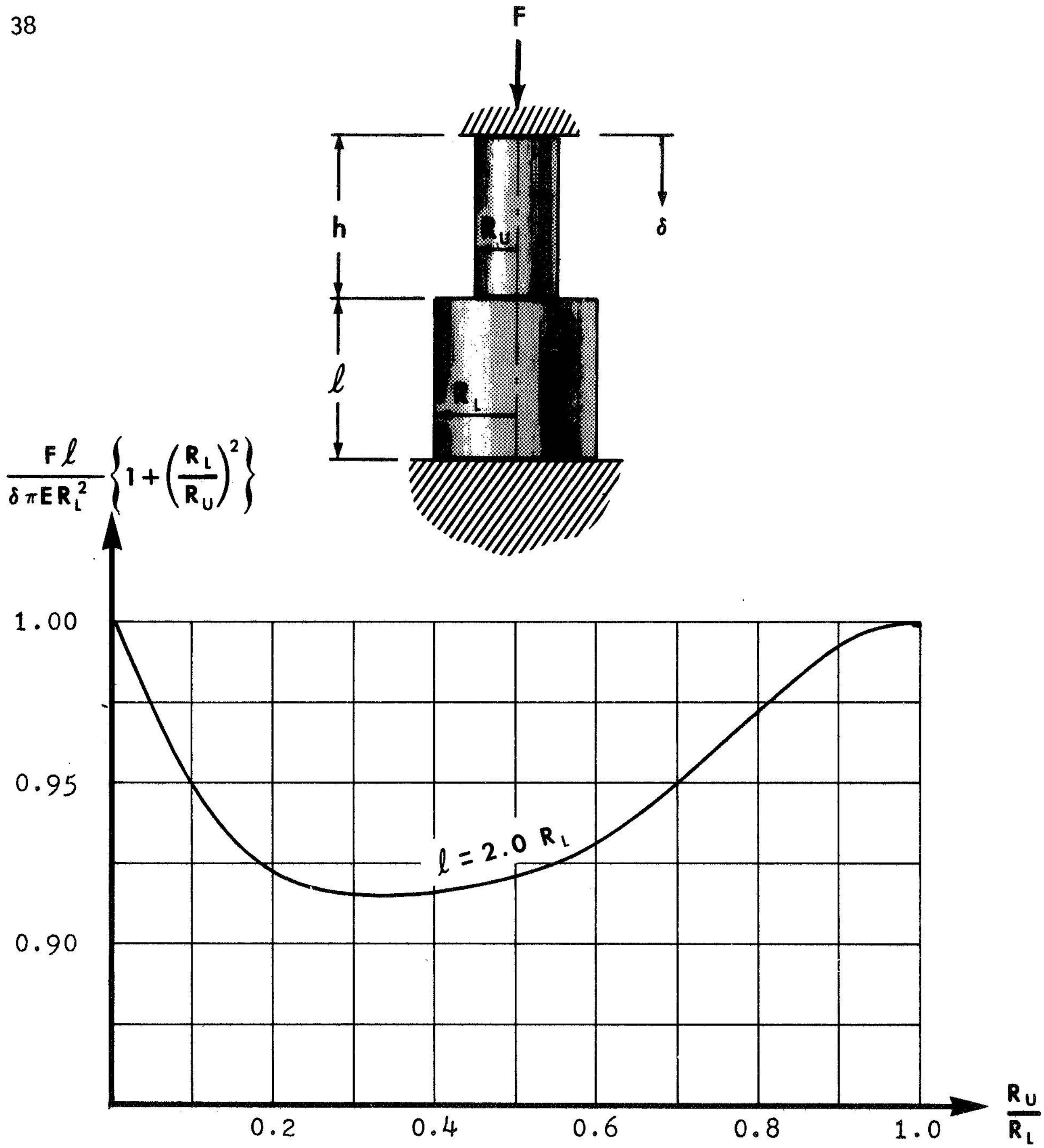

Figure 8. Axial Stiffness, Problem 2, Stepped Cylinder Configuration, $\nu_L = \nu_U = 0.3$, $E_L = E_U$, $\ell = h$

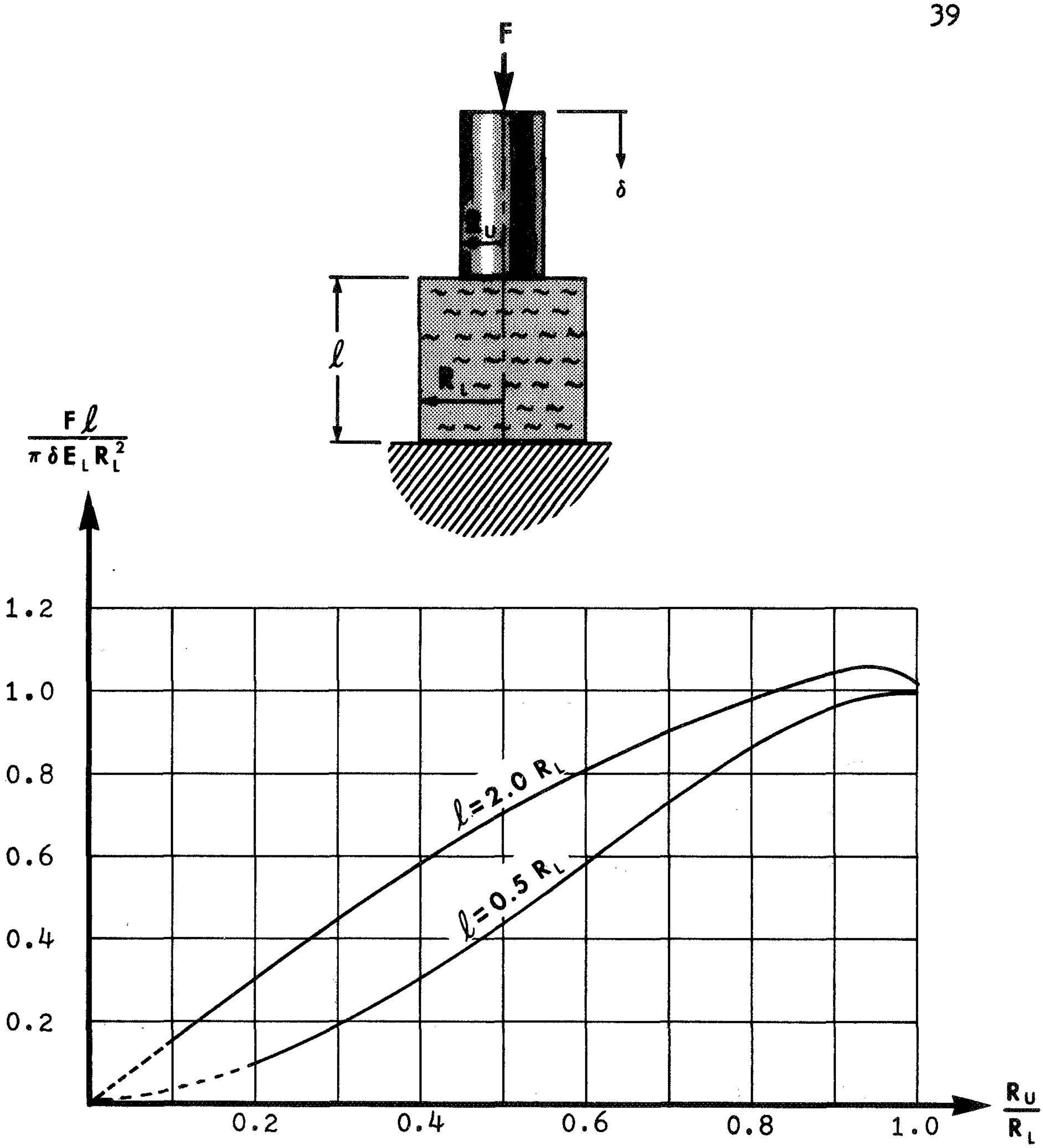

Figure 9. Axial Stiffness, Problem 2, Stiff Upper Cylinder Bonded to a Rubber Cylinder, $\nu_L = 0.5$, $\nu_U = 0.3$, $E_U = 100E_L$

CHAPTER V

SUMMARY AND CONCLUSIONS

The solution to a two cylinder contact problem was achieved by utilizing the known solution of the finite circular cylinder and applying the appropriate boundary and continuity conditions. The interrelated Fourier coefficients were expressed implicitly by an infinite system of linear algebraic equations. By truncation, an explicit solution of the Fourier coefficients was obtained. Two example problems were then solved. It was shown that a part of the uniform axial stress solution is not known until the Fourier coefficients were obtained.

A typical cylinder configuration was chosen for a numerical study of each problem. It was noted that appreciable oscillations in the calculated stresses occurred at the contact interface of the cylinders. These diminished rapidly a short distance away from the interface. The comparison of calculated results with those known for the rigid punch seem reasonable. For this reason, the exact nature of the stress singularity due to the discontinuous nature of the boundary conditions could not be determined.

A definite determination of the validity of this solution is also precluded for this very reason. Since the equilibrium and compatability equations, Hooke's law, and the stress free boundary conditions on the curved surfaces are satisfied implicitly from the eigenfunctions, the solution is valid if the rest of the boundary conditions are met. Those at the lower and upper rigid faces are met identically. At the interface, the boundary conditions are met in an averaged or integrated sense. The numerical results appear to provide useful information.

To demonstrate the practical aspects of the solution, a limited study was made to determine the effect of cylinder radii ratio on the constants, A_L and A_L/δ. It was noted how these constants can be used to obtain the true deflection rate of two cylinders bonded together. Some examples of the applications that this solution could be applied to are: shoulder bolts, integral pistons and rod, vibration and shock isolators, and concrete piers.

The numerical results that were presented were obtained by using the double precision mode on a SDS Sigma 7 computer. Seventeen pairs of eigenvalues were used. Integration of the matrix elements was obtained in closed form.

REFERENCES

1. Power, L. D., "Axisymmetric Stresses and Displacements in a Finite Circular Bar," Doctoral Thesis, University of Houston, Houston, Texas, 1969.

2. Power, L. D., and S. B. Childs, "Axisymmetric Stresses and Displacements in a Finite Circular Bar," to appear International Journal of Engineering Science.

3. Love, A. E. H., "A Treatise on the Mathematical Theory of Elasticity," Dover, New York, 1944, p. 276.

4. Little, R. W. and S. B. Childs, "Elastostatic Boundary Region Problem in Solid Cylinders," Quar. Appl. Math., Vol. 25, No. 3 (Oct., 1967), pp. 261-274.

5. Lanczos, C., Linear Differential Operators, D. Van Nostrand, London, 1961, pp. 71-74.

6. Sneddon, I. N., Fourier Transforms, McGraw-Hill Book Co., Inc., New York, 1951, pp. 458-460.

APPENDIX

TABLE OF FUNCTIONS

The φ functions used in this thesis are:

$$\varphi_1\left(\gamma_j, \frac{r}{R}\right) = \left[2\gamma_j^2 J_1(\gamma_j) - \gamma_j^3 J_0(\gamma_j)\right] J_0\left(\frac{\gamma_j r}{R}\right) - \gamma_j^3 J_1(\gamma_j)\, \frac{r}{R}\, J_1\left(\frac{\gamma_j r}{R}\right)$$

$$\varphi_2\left(\gamma_j, \frac{r}{R}\right) = \gamma_j^3 J_1(\gamma_j)\, \frac{r}{R}\, J_0\left(\frac{\gamma_j r}{R}\right) - \gamma_j^3 J_0(\gamma_j)\, J_1\left(\frac{\gamma_j r}{R}\right)$$

$$\varphi_3\left(\gamma_j, \frac{r}{R}\right) = \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_0(\gamma_j)\right] J_1\left(\frac{\gamma_j r}{R}\right) - \gamma_j^2 J_1(\gamma_j)\, \frac{r}{R}\, J_0\left(\frac{\gamma_j r}{R}\right)$$

$$\varphi_4\left(\gamma_j, \frac{r}{R}\right) = \gamma_j^2 J_1(\gamma_j)\, \frac{r}{R}\, J_1\left(\frac{\gamma_j r}{R}\right) - \left[2(1-\nu)\gamma_j J_1(\gamma_j) - \gamma_j^2 J_0(\gamma_j)\right] J_0\left(\frac{\gamma_j r}{R}\right)$$

$$\varphi_5\left(\gamma_j, \frac{r}{R}\right) = \left[\gamma_j^2 J_1(\gamma_j) + \gamma_j^3 J_0(\gamma_j)\right] J_0\left(\frac{\gamma_j r}{R}\right) + \gamma_j^3 J_1(\gamma_j)\, \frac{r}{R}\, J_1\left(\frac{\gamma_j r}{R}\right)$$

$$- \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_0(\gamma_j)\right] \frac{R}{r}\, J_1\left(\frac{\gamma_j r}{R}\right)$$

$$\varphi_6\left(\gamma_j, \frac{r}{R}\right) = \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_0(\gamma_j)\right] \frac{R}{r}\, J_1\left(\frac{\gamma_j r}{R}\right)$$

$$-(1-2\nu)\, \gamma_j^2 J_1(\gamma_j)\, J_0\left(\frac{\gamma_j r}{R}\right)$$

The W functions are:

$$W_1\left(\gamma_j, \frac{r}{R}\right) = \frac{1}{2\gamma_j J_1^2(\gamma_j)} \left\{ \gamma_j J_1(\gamma_j) \frac{r}{R} J_1\left(\frac{\gamma_j r}{R}\right) - \left[2(1-\nu) J_1(\gamma_j) - \gamma_j J_0(\gamma_j)\right] J_0\left(\frac{\gamma_j r}{R}\right) \right\}$$

$$W_2\left(\gamma_j, \frac{r}{R}\right) = \frac{1}{2\gamma_j J_1^2(\gamma_j)} \left\{ \gamma_j J_1(\gamma_j) \frac{r}{R} J_0\left(\frac{\gamma_j r}{R}\right) - \left[2(1-\nu) J_1(\gamma_j) + \gamma_j J_0(\gamma_j)\right] J_1\left(\frac{\gamma_j r}{R}\right) \right\}$$

$$W_3\left(\gamma_j, \frac{r}{R}\right) = \frac{1}{2\gamma_j J_1^2(\gamma_j)} \left\{ \gamma_j^2 J_0(\gamma_j) J_1\left(\frac{\gamma_j r}{R}\right) - \gamma_j^2 J_1(\gamma_j) \frac{r}{R} J_0\left(\frac{\gamma_j r}{R}\right) \right\}$$

$$W_4\left(\gamma_j, \frac{r}{R}\right) = \frac{1}{2\gamma_j J_1^2(\gamma_j)} \left\{ \left[2\gamma_j J_1(\gamma_j) - \gamma_j^2 J_0(\gamma_j)\right] J_0\left(\frac{\gamma_j r}{R}\right) - \gamma_j^2 J_1(\gamma_j) \frac{r}{R} J_1\left(\frac{\gamma_j r}{R}\right) \right\}$$