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USCEE Report 374



UNIVERSITY OF SOUTHERN CALIFORNIA

A UNIFIED THEORY OF COHERENT DIGITAL SYSTEMS WHICH TRACK DOPPLER FREQUENCY

Charles L. Weber

October 1969

Interim Technical Report

Research Sponsored by the U. S. Army Research Office--
Durham, Under Contract DA-ARO-D-31-124-G-1045 and
by the National Aeronautics and Space Administration
Under Contract No. EW 473447.

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ELECTRONIC SCIENCES LABORATORY

1100
Engineering

FACILITY FORM 602

N71-11297

(ACCESSION NUMBER)

40

(PAGES)

CR-111388

(NASA CR OR TMX OR AD NUMBER)

(THRU)

G3

(CODE)

07

(CATEGORY)

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A UNIFIED THEORY OF COHERENT DIGITAL SYSTEMS WHICH TRACK DOPPLER FREQUENCY

Charles L. Weber, Member IEEE*

ABSTRACT

A unified theory from which the design of a large class of coherent digital communication systems can be optimally carried out is presented.

In the design of digital communication systems, the error rate is the criterion which is invariably emphasized. In many digital systems, however, there is relative motion between transmitter and receiver which must be estimated by making use of Doppler frequency information. A new analysis of a general class of coherent digital systems is herein developed, in which the trade-offs that exist between Doppler measurement capability and sub-carrier demodulation error rate are quantitatively presented. The theoretically unrecoverable power loss which exists when employing frequency division multiplexing subcarriers as compared to time division multiplexing is described. The results point out that there is significant parametric dependence of the optimal choice of system parameters on the carrier loop signal-to-noise ratio and the data rate.

* University of Southern California, Los Angeles. This work was supported in part by U. S. Army Contract DA-ARO-D-31-124-G-1045 and in part by NASA Contract Number EW 473447.

A UNIFIED THEORY OF COHERENT DIGITAL SYSTEMS WHICH TRACK DOPPLER FREQUENCY

Charles L. Weber

I. Introduction.

In digital communication systems, there often is the need to maintain continuous Doppler frequency information for range-rate and range estimation as well as the need to maintain the error rate below a specified value. Range-rate and range data contribute to the specification and possibly the eventual control of the trajectory or orbit of the vehicle. In addition to specifying various system parameters, the allocation of the total available transmitted power to the various information bearing subcarrier signals should be carried out based on knowledge of the effect that this choice will have on Doppler tracking capability as well as error rates. The choice of the number of frequency division multiplexing subcarriers as compared to the number of time division multiplexing subcarriers also effects Doppler tracking capability. The goal herein is to provide this design information for a general class of coherent digital communication systems. In an actual design, these results would have to be incorporated with qualitative parameters such as frequency guard bands, guard times, time allotments for synchronization pulses and addressing sequences, etc. These parameters would be estimated based on allowable hardware sophistication and cost.

The fact that the design of voice systems as well as telemetry systems is becoming almost exclusively digital (for a variety of purposes: performance, scrambling capability, etc.) emphasizes the need for such a unified theory.

The type of coherent digital system to be considered is one which transmits the data signals by phase modulating the rf carrier with bi-phase modulated sinewave data subcarriers. The frequencies of the several subcarriers are assumed to have been judiciously chosen so that the spectra of the modulated data are non-overlapping.

For definiteness, the demodulation of the subcarriers will be assumed to be carried out employing squaring loops, Costas loops, etc., so that all necessary subcarrier phase and synchronization reference information is obtained directly from the data signal, thereby eliminating the need for separate sync channels and eliminating the need for placing any power in the subcarrier reference. This has been termed single-channel mechanization with a suppressed subcarrier.¹

II. System Description.

A basic diagram of the general type of digital system to be considered is displayed in Figure 1. In the transmitter portion of the system, the data signals, $\{s_k(t), k = 1, \dots, K\}$, bi-phase modulate the frequency multiplexed subcarrier waveforms $\{\sqrt{2p_k} \cos \omega_k t, k = 1, \dots, K\}$. The input to the carrier phase modulator, $\theta(t)$, is therefore given by

$$\theta(t) = \sum_{k=1}^K \sqrt{2p_k} s_k(t) \cos \omega_k t \quad (1)$$

where $p_k, k = 1, \dots, K$, is the average power in the k^{th} bi-phase modulated subcarrier waveform before carrier phase modulation.

The trend in modern digital communication systems is to employ subcarriers with 100% modulation; that is, there is no residual power at the subcarrier frequency for tracking purposes. The reason for this is that with the advent of squaring loops³, Costas loops⁴, delay-locked loops, etc., coherent phase reference and bit synchronization information can be obtained directly from the 100% modulated data signal. The $\{s_k, k = 1, \dots, K\}$ are each assumed to consist of a sequence of ± 1 's with bit times $\{T_{b_k}, k = 1, \dots, K\}$ respectively.

With this modulation scheme assumed, the output of the phase modulator is given by

$$s(t) = \sqrt{2P} \sin(\omega_c t + \theta(t) + \theta_0) \quad (2)$$

where P is the overall average transmitted power, and θ_0 is some unknown constant reference angle.

The received waveform is then given by

$$y(t) = \sqrt{2P} \sin(\omega_c t + \int_0^t \omega_d(\tau) d\tau + \theta(t) + \theta_0) + n(t) \quad (3)$$

where $n(t)$ is assumed to be white Gaussian noise with one-sided spectral density N_0 watts/hertz, and $\omega_d(\tau)$ represents the Doppler frequency shift due to the relative velocity between transmitter and receiver.

Neglecting frequency shifters, frequency synthesizers, bandpass limiters, etc., the coherent carrier tracking loop generates the reference signal

$$r(t) = \sqrt{2} \cos(\omega_c t + \int^t \omega_d(\tau) d\tau + \hat{\theta}_0(t)) \quad (4)$$

The data bearing waveforms which comprise $\theta(t)$ are assumed to be at frequencies outside the bandwidth of the carrier phase locked loop (PLL). The Doppler frequency is assumed to be varying slowly enough to be within this bandwidth however, so that the carrier PLL is able to track this signal. The output data bearing signal of the carrier tracking loop which goes into the various subcarrier demodulators is thus given by

$$y_0(t) = s_0(t) + n_0(t)$$

where

(5)

$$s_0(t) = \sqrt{P} \sin(\theta(t) + \phi_r(t))$$

$\phi_r(t) \triangleq \theta_0 - \hat{\theta}_0(t)$ is the carrier loop phase error, and the additive noise, $n_0(t)$, has the same statistics⁵ as $n(t)$.

With the Doppler frequency slowly varying, any cycle slipping is attributed solely to the additive noise. No detuning will be assumed to exist between the received carrier and the voltage controlled oscillator (VCO) rest frequency, i.e., steady state operation has been obtained. The approximate steady state

mod 2π probability density function of ϕ_r is then given by^{5,6}

$$p(\phi_r) = \frac{\exp(\alpha_r \cos \phi_r)}{2\pi I_0(\alpha_r)}, \quad -\pi \leq \phi_r \leq \pi, \quad (6)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero, and

$$\alpha_r \triangleq \frac{P_c}{N_0 B_{L_r}} \quad (7)$$

is the signal-to-noise ratio of the carrier tracking loop. In (7), B_{L_r} is the one-sided noise bandwidth of the carrier PLL based on the linear theory^{5,7} of PLL's and P_c is the average power of the received signal at the carrier frequency. The overall signal-to-noise ratio of the received signal is given by

$$\beta_r \triangleq \frac{P_c}{N_0 B_{L_r}}$$

In order to ultimately specify the trade-off between Doppler tracking capability and digital demodulation capability, the distribution of power between the carrier tracking loop and the subcarrier demodulation loops must be specified. To do this $s(t)$ is represented by the series (see, e.g. Lindsey[3])

$$\begin{aligned} s(t) &= \sqrt{2P} Q_m \{ \exp[j(\omega_c t + \theta(t) + \theta_0)] \} \\ &= \sqrt{2P} Q_m \{ \exp j(\omega_c t + \theta_0) \} \prod_{k=1}^K \sum_{m_k=-\infty}^{\infty} (j)^{m_k} \\ &\quad J_{m_k}(\sqrt{2P_k}) \exp[jm_k(\omega_k t + \frac{\pi}{2} s_k(t))] \end{aligned} \quad (8)$$

where $J_{m_k}(\cdot)$ is the Bessel function of order m_k , and $\mathcal{I}_m$ means the imaginary part of. The signal which enters the carrier tracking loop is

$$\sqrt{2P_c} \sin(\omega_c t + \theta_0) + n(t)$$

where the average power in the tracking signal, P_c , is given by the average power of the component of $s(t)$ in (8) at ω_c . This is obtained by setting $m_k = 0$, $k = 1, \dots, K$, from which we obtain the percent of the total power that enters the carrier tracking loop, namely,

$$\frac{P_c}{P} = \prod_{k=1}^K J_0^2(\sqrt{2p_k}) \quad (9)$$

To obtain the power in the subcarriers, $s_0(t)$ is similarly expanded. The average power in the k^{th} subcarrier data signal at the output of the k^{th} extraction filter (see Fig. 1) is obtained by assuming $\phi_r(t)$ is essentially constant over the bit time T_{b_k} of the k^{th} data signal, and setting $m_k = \pm 1$ and $m_{k'} = 0$ for $k' \neq k$. This average power, P_{s_k} , is then given by

$$\frac{P_{s_k}}{P} = 2J_1^2(\sqrt{2p_k}) \prod_{\substack{k'=1 \\ k' \neq k}}^K J_0^2(\sqrt{2p_{k'}}) [E(\cos \phi_r)]^2 \quad (10)$$

where⁸

$$E(\cos \phi_r) = \frac{I_1(\alpha_r)}{I_0(\alpha_r)} \quad (11)$$

The average power, P_{s_k} , of the k^{th} subcarrier in (10) has arbitrarily been defined in terms of the first moment squared of $\cos \phi_r$. It could equally well have been defined in terms of the second moment, i.e., $E(\cos^2 \phi_r)$. The system performance is, as it should be, independent of which definition is chosen. The behavior of both definitions is similar, with the latter always being greater than the former.

For a given ϕ_r , the input signal to the k^{th} subcarrier demodulator is

$$y_k(t) = \sqrt{2P_{s_k}(\phi_r)} \cos(\omega_k t + \frac{\pi}{2} s_k(t)) + n_k(t) \quad (12)$$

where

$$\sqrt{P_{s_k}(\phi_r)} \triangleq \sqrt{P} D_{s_k} \cos \phi_r \quad (13)$$

and where $n_k(t)$ also has the same statistics as $n(t)$. This follows from the fact that the extraction filters $H_k(j\omega)$, $k = 1, \dots, K$, in Figure 1 are normally broadband with respect to the bandwidths of the synchronization tracking loop, the subcarrier phase tracking loop, and the matched filter in the subcarrier demodulator. In (13)

$$D_{s_k} \triangleq \left[2J_1^2(\sqrt{2P_k}) \prod_{\substack{k'=1 \\ k' \neq k}}^K J_0^2(\sqrt{2P_k}) \right]^{1/2} \quad (14)$$

An upper bound on P_{s_k}/P can be obtained by assuming that the power in all other subcarriers is zero ($p_{k'} = 0$ for all $k' \neq k$), and the input SNR is assumed sufficiently high so that $E(\cos \phi_r) \approx 1$. Then

$$\frac{P_{s_k}}{P} \approx 2J_1^2(\sqrt{2p_k})$$

which obtains its maximum value of approximately 0.68 when $p_k \approx 1.7$.

Therefore, with one subcarrier and no sync channels, the maximum power that can be placed into the subcarrier demodulator input is 68% of the total received power. We shall see determine the requisite carrier signal-to-noise ratio to attain this upper bound.

In the important case where all subcarrier channels are afforded the same performance level, we have that

$$p_k = p, \quad k = 1, \dots, K,$$

from which (9) and (10) reduce to

$$\frac{P_c}{P} = [J_0(\sqrt{2p})]^{2K} \quad (15)$$

and

$$\frac{P_{s_k}}{P} = 2J_1^2(\sqrt{2p})[J_0(\sqrt{2p})]^{2K-2} [E(\cos \phi_r)]^2 \quad (16)$$

respectively. Note that p affects $E(\cos \phi_r)$ as well as the other factors in (16). Since P_{s_k}/P is dependent on $E(\cos \phi_r)$, we see that the performance of the digital demodulation of the subcarrier signals is not directly dependent upon the occurrence of cycle slipping events, but only indirectly to the extent that cycle slipping broadens the steady state mod 2π probability density function of the carrier phase error ϕ_r . Cycle slipping, as would be expected, is a fundamental concern in determining Doppler tracking capability.

Plots of these power distributions versus p for various values of β_r and K are displayed in Fig. 2. Note that the maximum value of P_s/P occurs at different values of p . For $K = 1$ (one data channel), for example, when $\beta_r = 1$, the max P_s/P occurs at $p = 0.4$, while when $\beta_r = 100$, the max P_s/P occurs at $p = 1.45$. This overall dependence demonstrates that when designing a bi-phase digital communication system (as well as when designing a PLL) a design point must be picked; that is, an input signal-to-noise ratio, β_r , must be chosen at which the system will be designed to operate optimally. As pointed out by Lindsey [8], note that for sufficiently large K and β_r , the optimal choice is $\tilde{p} \approx 1$. For smaller K and β_r , however, this choice is quite suboptimum.

This dependence of the optimal p on β_r decreases as the number of subcarriers is increased; this is displayed in Figure 3, when the optimal choice of p , $\tilde{p}$, is plotted versus β_r for various K . For that p which maximizes the total power into the subcarriers, we use the notation $\tilde{p}$. In section V, we shall see that $\tilde{p}$ will differ from that value of p , $\hat{p}$, which minimizes the error rate.

It should also be noted $\tilde{p}$ (and $\hat{p}$) take on values which overmodulate the carrier. When $K = 1$ for example, if $p = \frac{\pi^2}{8} \approx 1.23$, the carrier is 100% modulated. In Fig. 3, however, $\tilde{p}$ is greater than 1.23 for all $\beta_r > 35$ when $K = 1$. For $K \geq 2$, there is overmodulation for all β_r which produce reasonable error rates. Therefore, in the design of digital phase-modulated systems, overmodulation is not only good, its optimum in most cases; the exact amount of which is given in Fig. 3, when the criterion is $\max P_s / P$.

The fraction of the total power lost in phase modulation due to noise and distortion can also be easily displayed. This fraction decreases as β_r increases. In Fig. 4, curve 1 is the maximum percent of the total received power that is transferred into the various subcarriers, namely $\left(\frac{P_s}{P}\right)K$, which is plotted versus K at $\tilde{p}$ and for large β_r . When $K = 1$, it is possible to transfer as much as 68% of the total power into the one subcarrier, while when $K = 50$, only 37% of the total power can be transferred into all 50 subcarrier channels. Curve 2 is a plot of the percent of total power which is needed by the carrier tracking loop at $\tilde{p}$ and large β_r . Equivalently stated, $\left(\frac{P_s}{P}\right)K$ is maximized when $\frac{P_c}{P}$ takes on the value given by curve 2 in Fig. 4. Curve 3 in Fig. 4 is the sum of curves 1 and 2. The remainder of the power is lost as a result of the phase modulation process. For one subcarrier, at least 22% of the total power is always lost, while for $K = 50$, at least 36% of the power is always lost.

The performance of the subcarrier demodulators and the Doppler measuring subsystem must next be determined so that their trade-off can be established.

III. Digital Performance - Probability of Error.

The output signal of the k^{th} extraction filter, which is the input signal to the k^{th} subcarrier demodulator, is given by

$$y_k(t) = \sqrt{2P_s(\phi_r)} \cos(\omega_k t + \frac{\pi}{2} s_k(t)) + n_k(t) \quad (17)$$

where $n_k(t)$ has the same statistics as $n(t)$, and where the assumption is maintained that the total available subcarrier power is shared equally among the K frequency multiplexed data channels. The indexing on $P_s(\phi_r)$ has thus been deleted. The data signal $s_k(t)$ fully bi-phase modulates the subcarrier reference signal $\cos \omega_k t$. In order to obtain a simplified description of the trade-off between telemetry performance and Doppler tracking capability, we now invoke the realistic assumption that bit synchronization is obtained directly from the data channel as opposed to placing a certain amount of residual power in the subcarrier reference, and in addition that the synchronization jitter is negligible with respect to phase jitter in the carrier tracking loop. It will be further assumed that the squaring loop, Costas loop, or other method which is employed to obtain coherent subcarrier phase information is functioning with negligible jitter with respect to that in the carrier loop. This is realistic, inasmuch as noise bandwidth of the subcarrier tracking loop can often be made 10^{-3} to 10^{-6} times narrower than that of the carrier tracking loop. Therefore, the subcarrier telemetry demodulation will be assumed to perform like a perfectly coherent system which has perfect synchronization information. The probability of a bit error P_e for such is given by⁵

$$P_e = \int_{-\pi}^{\pi} \operatorname{erfc} \left(\sqrt{\frac{2P_s(\phi_r)T_b}{N_0}} \right) \frac{\exp(\alpha_r \cos \phi_r)}{2\pi I_0(\alpha_r)} d\phi_r \quad (18)$$

where erfc is the complementary error function, defined as

$$\operatorname{erfc}(\mu) \triangleq \int_{\mu}^{\infty} \frac{\exp(-\frac{1}{2}v^2)}{\sqrt{2\pi}} dv$$

The average signal-to-noise ratio per bit, R_s , often called communication efficiency, is defined as the ratio of signal energy per bit to noise spectral density. Therefore

$$R_s \triangleq \frac{P_s T_b}{N_0} = \frac{[E(\sqrt{P_s(\phi_r)})]^2 T_b}{N_0} \quad (19)$$

where T_b is bit time, which has been assumed the same for all digital subcarrier channels. In terms of the parameters which describe the carrier tracking loop, R_s can be expressed as

$$R_s = \beta_r \frac{P_s}{P} \frac{1}{\delta_r} \quad (20)$$

where

$$\delta_r \triangleq \frac{1}{T_b B_{L_r}}$$

is defined as the reciprocal of the product of the bit time and the noise bandwidth of the carrier loop. For fixed carrier loop bandwidth, δ_r is proportional to the bit rate in each channel.

When α_r , the carrier loop SNR, is large, $E(\cos \phi_r) \approx 1$ and P_e is closely approximated by $\text{erfc}(\sqrt{2R_s})$. For low α_r , however, P_e is often significantly higher than $\text{erfc}(\sqrt{2R_s})$. Furthermore, as we will discuss in detail in section V, the minimum value P_e (minimized with respect to p) does not necessarily occur at the maximum value of R_s ($\max P_s / P$). The P_e at $R_{s_{\max}}$ is close enough to $P_{e_{\min}}$ so that when designing a system for minimum P_e , it would generally be acceptable to use $R_{s_{\max}}$. Except at high loop SNR's, consequently, one should never approximate $P_e(R_{s_{\max}})$ by $\text{erfc}(2R_{s_{\max}})$, but should employ the integral representation in (18).

With this description of the digital subcarrier demodulation performance, we next consider Doppler measurement capability. System performance curves and an evaluation of the results are discussed in Section V.

IV. Doppler Measurement Capability.

The signal from which Doppler frequency information is to be obtained, the carrier VCO output, $r(t)$, has the representation

$$r(t) = \sqrt{2} \cos(\omega_c t + \int_0^t \omega_d(\tau) d\tau + \hat{\theta}(t))$$

In a one-way communication link, the carrier frequency is removed by mixing with a noncoherent carrier reference, $\sqrt{2} \sin(\omega_c t)$. In a two-way system, this mixing can be made coherent with a corresponding gain in performance. In fact, the extension of all the ideas herein to two-way systems is straightforward. We shall assume a one-way system, in which case the output of the mixer is (neglecting the double frequency term)

$$c(t) = \cos \Delta(t)$$

where

$$\Delta(t) = \int_0^t \omega_d(\tau) d\tau + \phi_r(t) + \theta_1 \quad (21)$$

The signal $c(t)$ is the input waveform to the Doppler measurement device. For implementation purposes, this input signal may be phase modulated on some intermediate frequency, but this does not alter the performance characteristics that are described here.

There are several methods of obtaining the Doppler frequency information from the signal $c(t)$. Among them are:

- i) use $c(t)$ as the input to a frequency counter,
- ii) frequency demodulate $c(t)$ with a FM discriminator,
- iii) differentiate $c(t)$ and follow this with an envelope detector.

For implementation purposes, $c(t)$ is often formed by mixing $r(t)$ down to an IF (e.g., 1 megahertz) instead of demodulating all the way to a baseband

signal. The analysis and theoretical performance for such is identical to that presented.

All of these schemes are methods of obtaining the instantaneous frequency of $c(t)$, or equivalently of differentiating $\Delta(t)$ in (21). Doppler information may equivalently have been obtained from the output of the carrier PLL filter, but a somewhat cleaner signal is obtained via the output of the VCO.

Whatever the scheme, we shall assume it ideally obtains the instantaneous frequency of $c(t)$, namely $\dot{\Delta}(t)$. Denoting this signal by $d(t)$, we have

$$d(t) \triangleq \dot{\Delta}(t) = \omega_d(t) + \dot{\phi}_r(t) \quad (22)$$

The measurement disturbances in (22) are seen to be $\dot{\phi}_r(t)$. In this initial approach to provide a trade-off between Doppler and error rate, we shall assume $\omega_d(t)$ is sufficiently slowly varying so that it can be assumed constant.

In some cases, nominal Doppler values are known a priori, are stored and subtracted out before the variational Doppler about the nominal value is estimated. For example, in deep space applications, Doppler shifts due to the earth's rotation and the relative velocity of the satellite are approximately known a priori, and can be removed before variations about these nominal values are estimated.

In general, $d(t)$ would be filtered to provide the best estimate, $\hat{\omega}_d(t)$, of $\omega_d(t)$ from $d(t)$. The variance of the error before filtering will be designated $\sigma_{\hat{\omega}_d}^2$. In every case, the fundamental task in acquiring knowledge of Doppler measurement capability is in obtaining necessary statistical information

about $\dot{\phi}_r(t)$, namely, the first and second moments. The obstacle here centers around the fact that, in order to obtain a tractable mathematical model of a PLL, the assumption is generally made that the additive noise is white. In most choices of loop filters, this unfortunately leads to the conclusion that $\dot{\phi}_r(t)$ is also white. This vexation can be partially overcome with the following approach. Let us model the total phase error $\phi_r(t)$ as

$$\phi_r(t) = \int_0^t 2\pi N(\tau) d\tau + \phi_m(t) \quad (23)$$

where $N(\tau)$ is a stochastic process consisting of a sequence of pulses which are each of unit area and of short duration. A pulse is positive whenever the loop slips a cycle in the positive ϕ_r direction and negative whenever the loop slips in the negative ϕ_r direction. The process $\phi_m(t)$ is the mod 2π phase error process⁵. By modeling $\phi_r(t)$ in this manner the error contributions due to cycle-slipping and due to phase jitter between cycle slips are additive. A typical sample function of the $\phi_r(t)$ process is depicted in Fig. 4. At low signal-to-noise ratios, the predominate contribution to overall phase error will be due to cycle slips, while at large signal-to-noise ratios, cycle-slips will occur very rarely, and the predominate variation in $\phi_r(t)$ will be due to the mod 2π phase jitter, $\phi_m(t)$.

Empirical data taken on PLL¹⁰ leads to the conjecture that cycle slipping events in disjoint intervals are statistically independent. This plus the fact that $N(t)$ is the consecution of a jump process, is sufficient to conclude¹¹ that it is a generalized Poisson process.

The cumulative number of cycle slips, $N(t)$, actually can be expressed as

$$N(t) = N_+(t) - N_-(t) \quad (24)$$

where $N_+(t)$ consists of the positive pulses in $N(t)$ and thus represents cycle slippage in the positive $\phi_r(t)$ direction, and similarly for $N_-(t)$ in the negative $\phi_r(t)$ direction. Since steady state operation is assumed, and no detuning is assumed in the carrier VCO, the expected number of cycle slips to the right is equal to that to the left. Hence

$$E[N(t)] = 0$$

The occurrence times of the cycle slip events represented by $N_+(t)$ and $N_-(t)$ are therefore Poissonly distributed and assumed to be statistically independent. The no detuning assumption implies the processes are also identically distributed. Therefore.

$$\sigma_{N_+}^2 = \sigma_{N_-}^2 = E(N_+) = E(N_-)$$

Combining

$$\sigma_N^2 = \sigma_{N_+}^2 + \sigma_{N_-}^2 = E(N_+) + E(N_-) \quad (25)$$

The expected number of cycle slips to the right or left per unit time has been shown to be⁵

$$E(N_+) = E(N_-) = \frac{B_{L_r}}{\pi \alpha_r I_0^2(\alpha_r)} \quad (26)$$

This is the exact result for first order PLL's and an approximation for higher order loops. Since

$$\dot{\phi}_r(t) = 2\pi N(t) + \dot{\phi}_m(t) \quad (27)$$

we can therefore write

$$\begin{aligned} \sigma_{\dot{\phi}_r}^2 &= 4\pi^2 \sigma_N^2 + \sigma_{\dot{\phi}_m}^2 \\ &= \frac{8B_{L_r}}{\alpha_r I_0^2(\alpha_r)} + \sigma_{\dot{\phi}_m}^2 = \sigma_{\hat{\omega}_d}^2 \end{aligned} \quad (28)$$

where the additional assumption has been made that $N(t)$ is statistically independent of $\dot{\phi}_m(t)$.

The remaining task is to determine $\sigma_{\dot{\phi}_m}^2$. As previously indicated, models of PLL's have exclusively assumed the additive disturbance to be white and Gaussian, with the conclusion that $\dot{\phi}$ is also white. Since Doppler measurement is inherently concerned with cycle slipping, all of the various linear theories of PLL's break down when attempting to determine Doppler measurement capability. One way, however, in which realistic statistical information

can be obtained about $\dot{\phi}_m(t)$ is to assume the loop filter of the PLL is of the form

$$F(s) = \frac{1}{\tau s + 1}$$

With this filter, the Fokker-Planck equation for the probability density function of the vector Markov process $(\phi(t), \dot{\phi}(t))$ can be solved¹² in the steady state. The resulting p.d.f. is

$$p(\phi, \dot{\phi}) = \frac{\exp \left(-\frac{1}{2} \frac{\dot{\phi}^2}{\sigma_{\dot{\phi}}^2} + \alpha_r \cos \phi_r \right)}{(2\pi)^{3/2} \sigma_{\dot{\phi}} I_0(\alpha_r)} \quad (29)$$

where the variance of $\dot{\phi}$, $\sigma_{\dot{\phi}}^2$ is given by

$$\sigma_{\dot{\phi}}^2 = \frac{4B_L}{\alpha_r} \quad (30)$$

In (30), α_r is the loop SNR, which we have previously expressed as $\alpha_r = P_c / B_L N_0$, and B_L is the noise bandwidth of the PLL.

At high SNR, the principle disturbance in $\dot{\phi}_r$ is due to $\dot{\phi}_m$, since cycle-slips occur rarely at high SNR. Therefore, at high SNR

$$\sigma_{\dot{\phi}_m}^2 \approx \sigma_{\dot{\phi}}^2 = \frac{4B_L}{\alpha_r} \quad (31)$$

This representation has an additional assumption imbedded; namely, that this result is approximately independent of the structure of the loop filter. These results, namely:

$$\frac{4\pi^2 \sigma_N^2}{B_L} = \frac{8}{\alpha_r I_0^2(\alpha_r)} \quad (32)$$

for the cycle slipping, and

$$\frac{\sigma_{\phi_m}^2}{B_{L_r}} = \frac{4}{\alpha_r} \quad (33)$$

are plotted in Fig. 5.

From (32), since $I_0(\cdot)$ increases exponentially, we see that cycle-slipping decreases exponentially, while phase jitter between cycle-slips decreases only as $(\alpha_r)^{-1}$. Thus, we can qualitatively determine when $\sigma_{\phi_m}^2$ becomes the pre-dominant disturbance in Doppler measurement. The cumulative effect of cycle-slipping and phase-jitter is also shown in Fig. 5 as the dotted line.

V. System Evaluation and Discussion.

The proper choice of system parameters can now be determined from a display of the results of the previous sections. The trade-off between Doppler measurement capability and error rate is displayed in Fig. 7 and 8, for $K = 1$ and 2 respectively. The variation of probability of error and variance of Doppler measurement, normalized by the noise bandwidth of the carrier PLL, B_{L_r} , is plotted as a function of the modulation index parameter, p , for several β_r . At

low data rates (smaller values of δ_r), it is noted that the minima are sharper than at higher data rates. This says that the choice of a modulation index parameter is more sensitive at lower data rates than at higher rates. Examination of Fig. 7d, for example, shows that, on account of this insensitivity at higher rates, a substantial improvement can be obtained in the variance of Doppler measurement, while the corresponding degradation in probability of error is not appreciable. This is not the case at lower rates, as depicted, for example, in Fig. 7a.

The solid lines in Figs. 7 and 8, are curves of constant input signal-to-noise ratio and are therefore design curves. They indicate the cost in performance which is encountered by varying p , for fixed β_r . For these curves, when p increases beyond that which minimizes the error rate, the Doppler variance and the error rate increase. This is due to the fact that too much power is going into distortion terms. The useful regions of these design curves are thus between the values of $p = 0$ and that p which minimizes the probability of error, $\hat{p}$.

The dashed lines in Figs. 7 and 8, on the other hand, are performance curves. They demonstrate how a particular system, i.e., a specific choice of system parameters, varies with signal-to-noise ratio. It is noted that the error rate deteriorates much faster than Doppler variance, and this is independent of the data rate and the number of subcarriers.

In Fig. 9, the optimal value of p in the sense of minimizing the probability of error, $\hat{p}$, is displayed versus β_r for various subcarriers and data rates. The curves demonstrate that the dependence of $\hat{p}$ on signal-to-noise ratio and data rate is quite significant at low rates and low SNR, but that this dependence decreases substantially as the number of subcarriers increases.

In some systems, when the input signal-to-noise ratio drops below a specified threshold, which corresponds to the error rate increasing above a corresponding threshold, the data rate is decreased, thereby keeping the error rate within a specified tolerance. If one is to make optimal use of the system, then, when the error rate is changed, the setting of the modulation index must also be adjusted according to the curves in Fig. 9.

As shown in Fig. 4, the percent of total power lost due to distortion and noise increases as the number of subcarriers is increased. This demonstrates the well known fact, that for a fixed data rate, the error rate decreases as the number of subcarriers is increased. The data rate per subcarrier is proportional to δ_r . Therefore the data rate of the overall system is proportional to $K\delta_r$. The theoretical trade-off that exists in the choice of number of subcarriers is depicted in Figure 10 for $K\delta_r = 100, 500$, and 1000 . For overall fixed data rate, $K\delta_r$, the gain in performance when using one subcarrier versus ten or more subcarriers is almost 3db in signal-to-noise ratio. This is the case for all data rates. This leads to the conclusion that, in general, time multiplexing is preferable to frequency multiplexing.

The effect that the number of subcarriers has on Doppler measurement capability is shown in Fig. 11. The modulation index is set at $p = \hat{p}$, for which normalized Doppler measurement variance is plotted against input signal-to-noise ratio. For fixed β_r , the Doppler variance decreases as the number of subcarriers is increased. This is attributed to the fact that as the number of subcarriers is increased, the fraction of the total power in the carrier is increasing. This is the case whether the design point is chosen so as to maximize total subcarrier power, as described in Fig. 4, or chosen to minimize the error rate, namely $p = \hat{p}$.

In an actual design, the final choice of system parameters requires that these results be incorporated with more qualitative factors such as frequency guard bands between subcarriers, guard times and time allotments for synchronizing pulses and addressing sequences, hardware complexity, etc. These additional considerations may alter the preferences indicated above.*

In the evaluation of Doppler measurement capability, no attempt has been made to filter the waveform containing the Doppler signal. This could be done, based on a specified performance criterion, if sufficient statistics of the Doppler stochastic process, $w_d(t)$ and the disturbance process $\dot{\phi}_r(t)$ are known. The design of the Wiener filter, for example, requires knowledge of second order statistics for both processes, neither of which have thus far been attainable.

*Amplitude modulation of the carrier might be considered preferential to phase modulation, since AM has the distinct advantage of having no distortion power. Linear rf modulators are required for AM, however, but TWT's operate most efficiently in the saturated mode. This inefficiency in the AM system must be considered in an overall system evaluation and comparison.

VI. Concluding Comments.

The theoretical quantitative trade-offs between error-rates, Doppler tracking capability, time division multiplexing and frequency division multiplexing which exist in modern coherent digital systems, are presented in a unified theory. The theoretical gain in performance of time multiplexing over frequency multiplexing is specified. The effect of the system parameters on Doppler measurement capability is described in detail.

At low data rates and low β_r , the assumption that the carrier phase error, ϕ_r , is constant over the bit time gradually breaks down. This would make P_E and $\sigma_{\omega_d}^2$ lower bounds on the actual performance of the system. Since cycle slipping decreases exponentially, however, it is expected that in all design regions of interest, the results of this model will agree excellently with empirical data.

This theory is also directly extendable to the cases where subcarrier phase and bit synchronization information is provided on adjacent subcarriers.

ADKNOWLEDGEMENT

The author is grateful to W. J. Hurd, W. C. Lindsey, P. McAdam, and J. E. Ohlson for helpful suggestions during this study.

REFERENCES

- [1] W. C. Lindsey, "Optimal Design of One-Way and Two-Way Coherent Communication Links," IEEE Trans. on Communication Technology, Vol. COM-14, pp. 418-431, August, 1966.
- [2] J. J. Stiffler, "On the Allocation of Power in Synchronous Binary PSK Communication Systems," Proc. 1964 National Telemetry Conference, June 2-4.
- [3] R. L. Didday and W. C. Lindsey, "Subcarrier Tracking Methods and Communication System Design," IEEE Trans. on Communication Technology, Vol. COM-18, pp. 541-550.
- [4] J. P. Costas, "Synchronous Communications," Proc. IRE, Vol. 44, pp. 1713-1718, December, 1956.
- [5] A. J. Viterbi, Principles of Coherent Communication, McGraw-Hill, 1966.
- [6] W. C. Lindsey, "Nonlinear Analysis and Synthesis of Generalized Tracking Systems," University of Southern California, Electronic Sciences Laboratory, Part I: Report No. USC EE317, December, 1968; Part II: Report No. USC EE342, April, 1969.
- [7] R. C. Tausworthe, "Theory and Practical Design of Phase-Locked Receivers," Jet Propulsion Laboratory, TR No. 32-819, February, 1966.
- [8] W. C. Lindsey, "Design of Block Coded Communication Systems," IEEE Trans. on Communication Technology, Vol. COM-15, No. 14, pp. 525-534, August, 1967.
- [9] C. L. Weber, Elements of Detection and Signal Design, McGraw-Hill, 1968.
- [10] F. J. Charles and W. C. Lindsey, "Some Analytical and Experimental Phase-Locked Loop Results for Low Signal-to-Noise Ratios," Proc. of IEEE, Vol. 54, No. 9, September, 1966.
- [11] J. L. Doob, Stochastic Processes, Wiley, 1953.
- [12] V. I. Tikhonov, "Phase-Locked Automatic Frequency Control Operation in the Presence of Noise," Automatika i Telemekhanika, Vol. 21, pp. 301-309, March, 1960.

FIGURE CAPTIONS

- Figure 1. General System Configuration
- Figure 2. Fraction of Power in Subcarriers, P_s/P , versus Modulation Index Parameter, p .
- Figure 3. Values of p which maximize P_s/P , $\tilde{p}$, versus β_r , for various K
- Figure 4. Power allocations which maximize Total Subcarrier Power
- Figure 5. Typical Sample Function of $\phi_r(t)$
- Figure 6. Doppler Error versus Carrier Signal-to-Noise Ratio
- Figure 7. Error Rate versus Doppler Jitter
- Figure 8. Error Rate versus Doppler Jitter
- Figure 9. Optimum Modulation Index, $\hat{p}$, versus Signal-to-Noise Ratio
- Figure 10. Time Multiplexing versus Frequency Multiplexing for Fixed Data Rates
- Figure 11. Doppler Measurement Variance versus Input Signal-to-Noise Ratio for various K with $p = \hat{p}$.

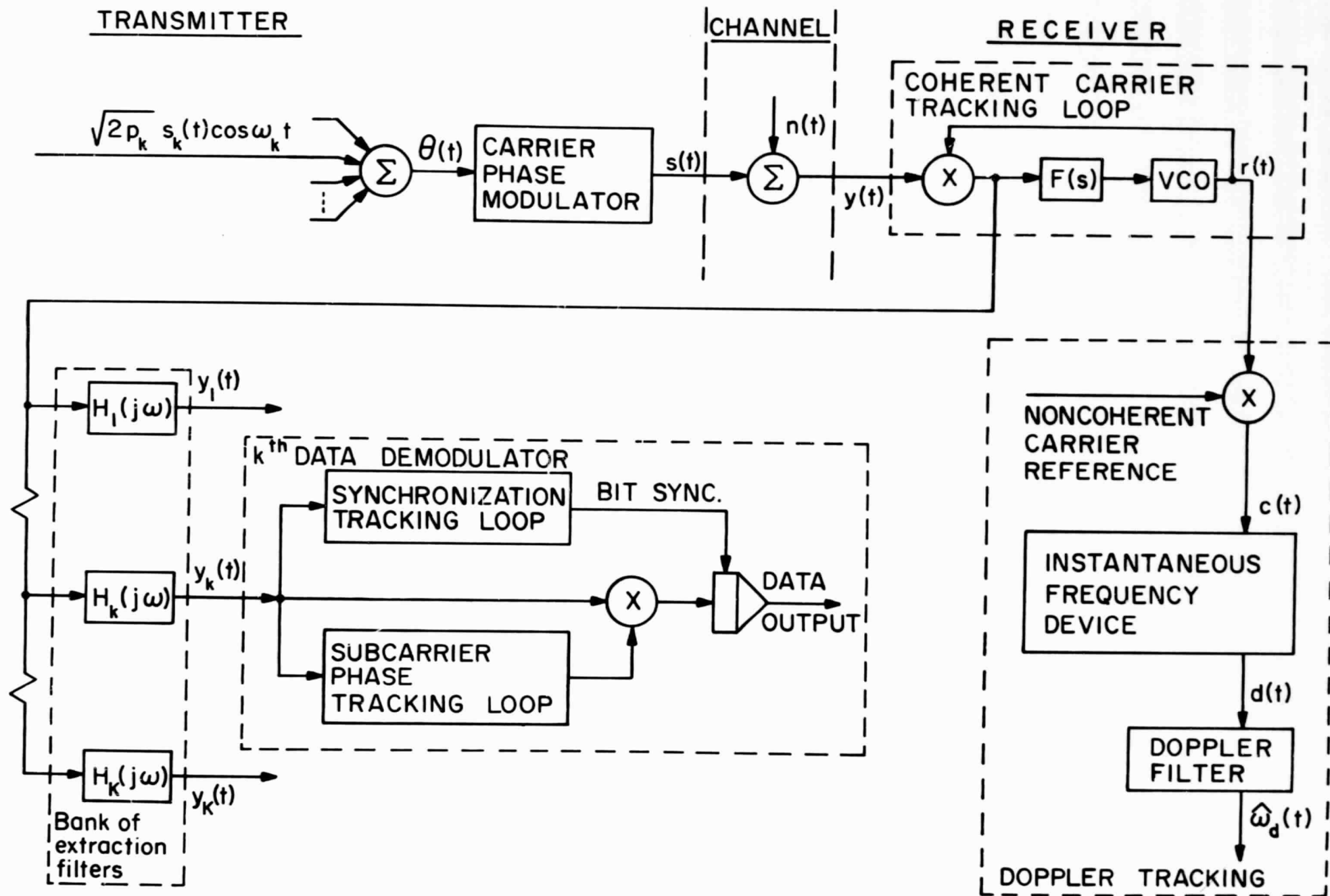


Fig. 1 General System Configuration

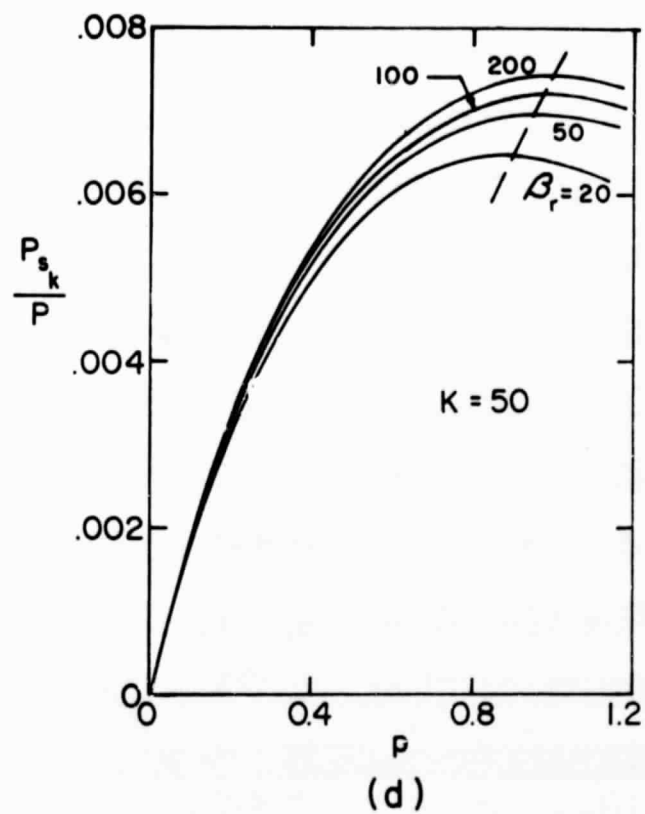
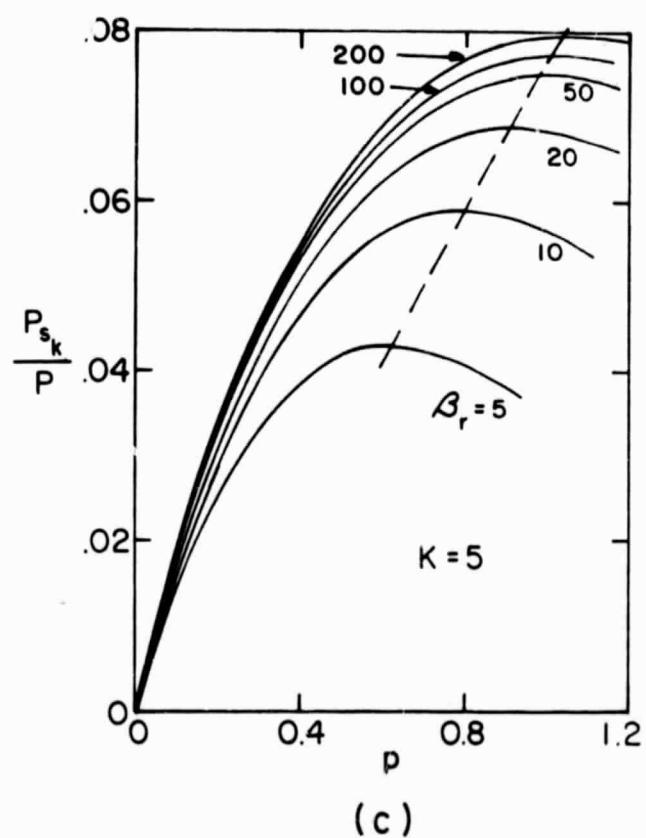
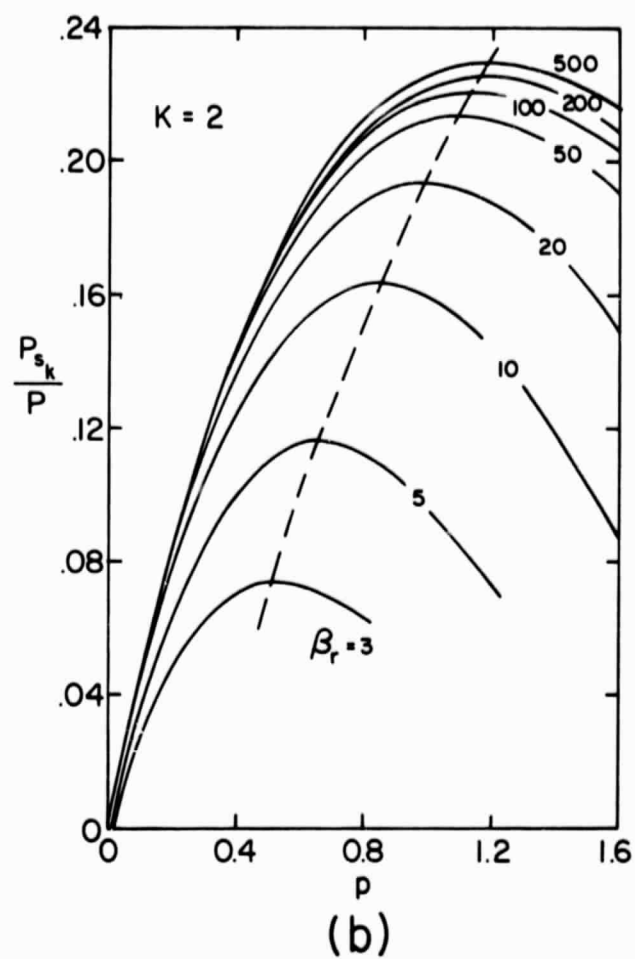
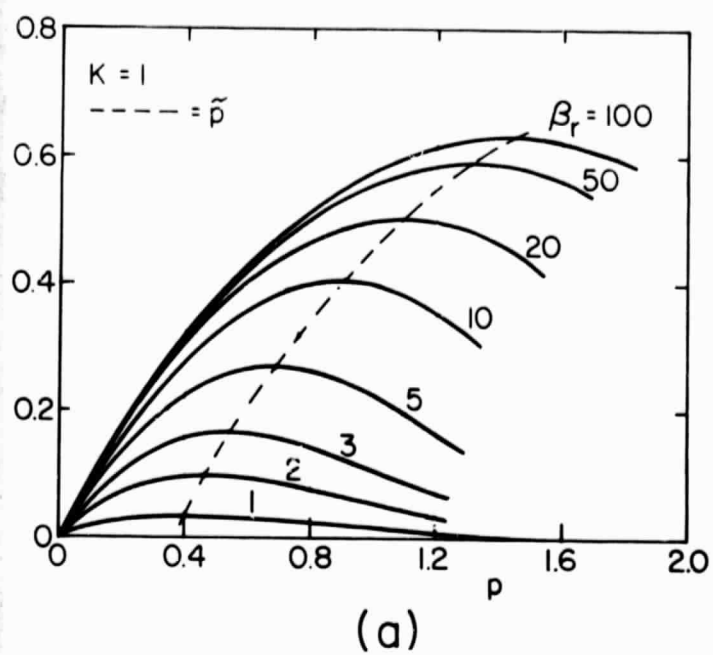


Fig. 2 Fraction of Power in Subcarriers, P_{s_k}/P , versus Modulation Index Parameter, p .

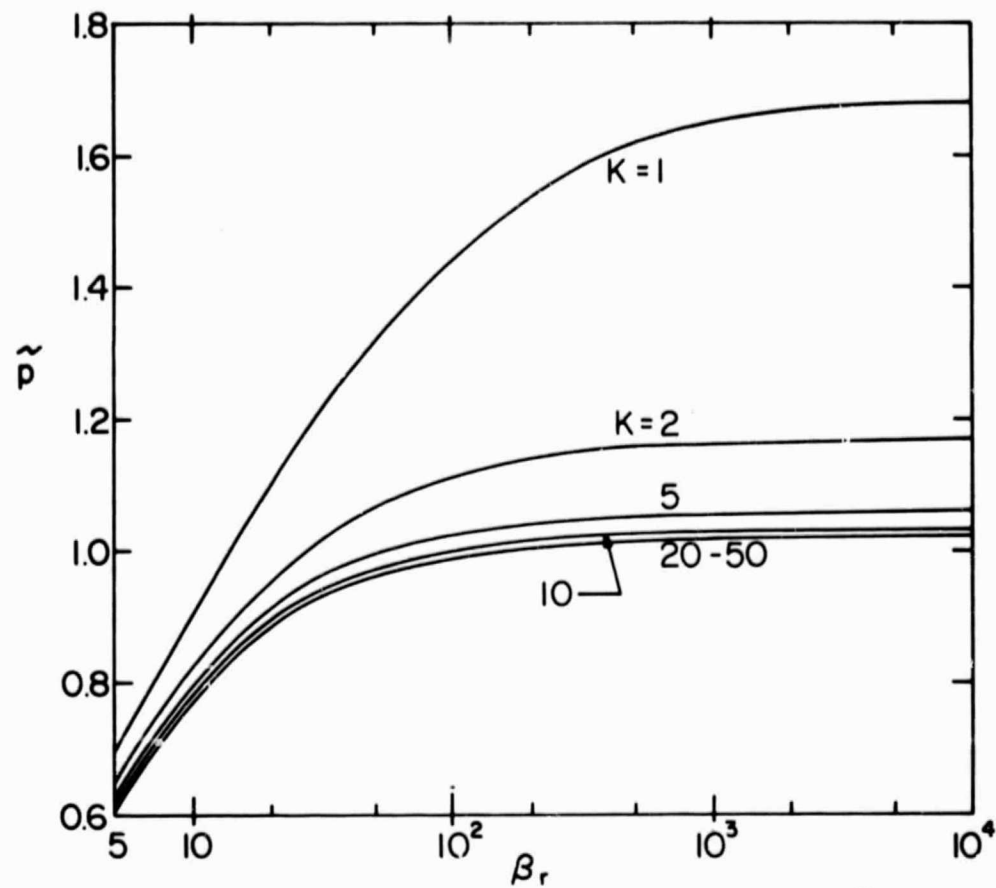


Fig. 3 Values of $\tilde{p}$ which maximize P_s/P , $\tilde{p}$, versus β_r , for various K

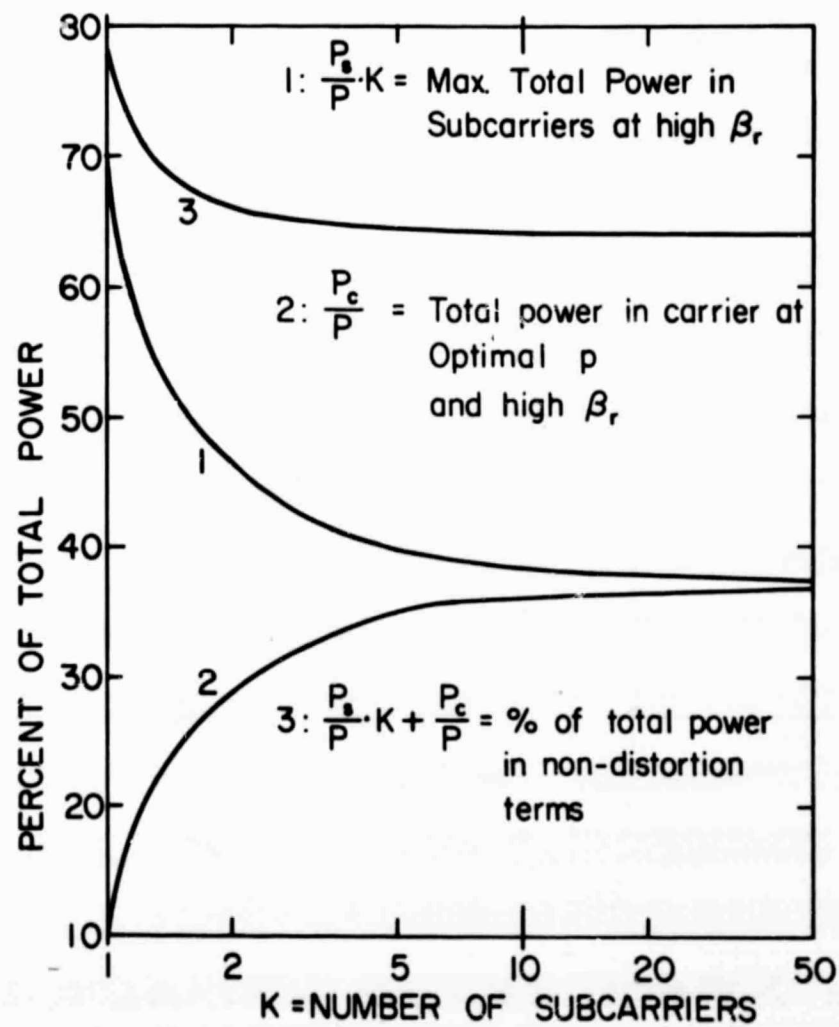


Fig. 4 Power allocations which maximize Total Subcarrier Power

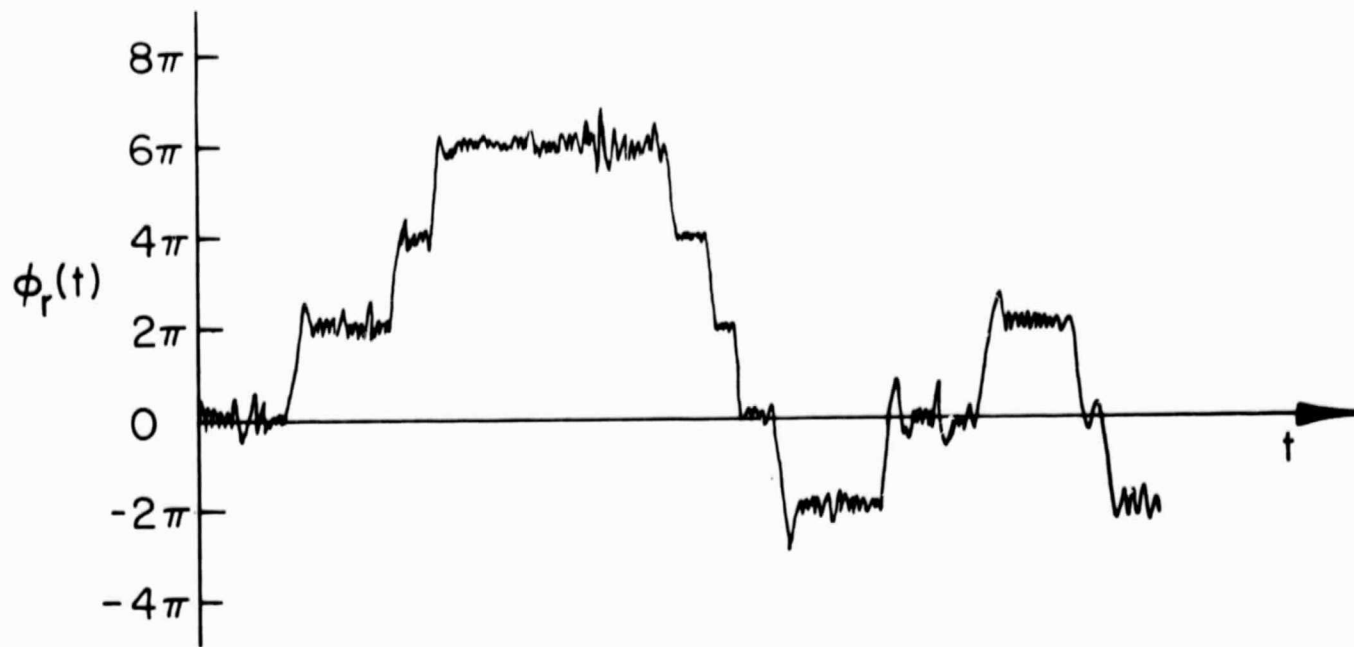


Fig. 5 Typical Sample Function of $\phi_r(t)$

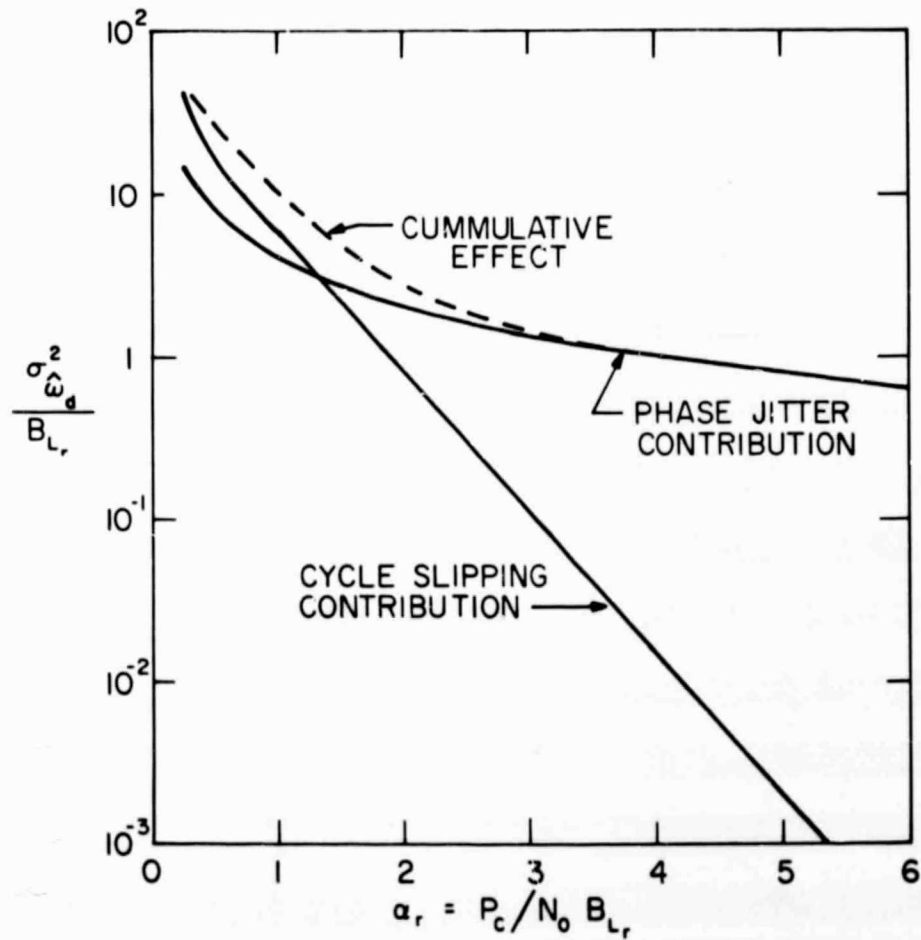
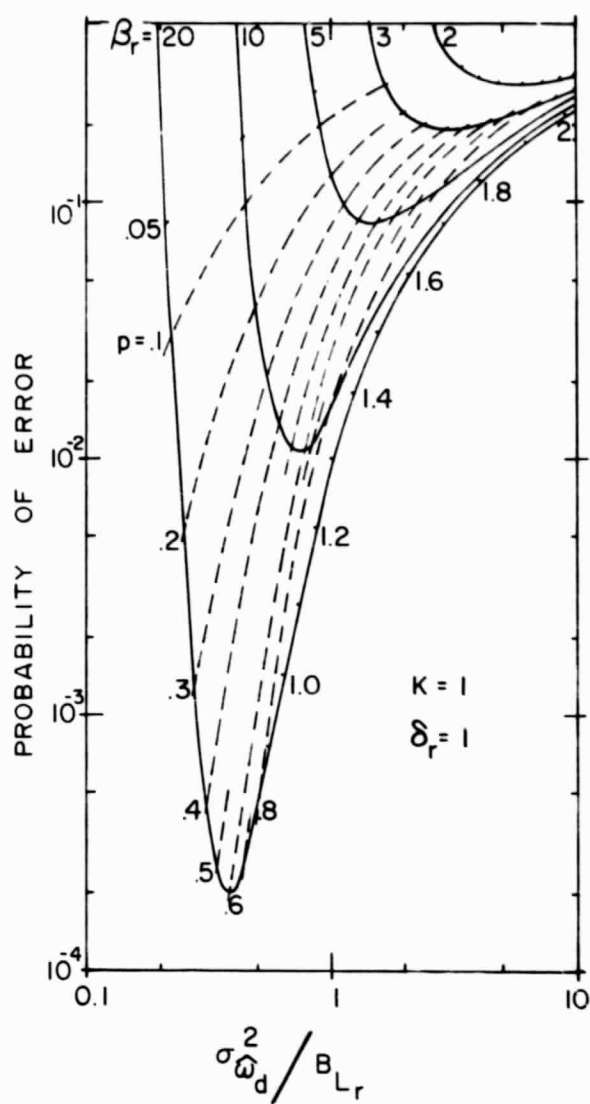
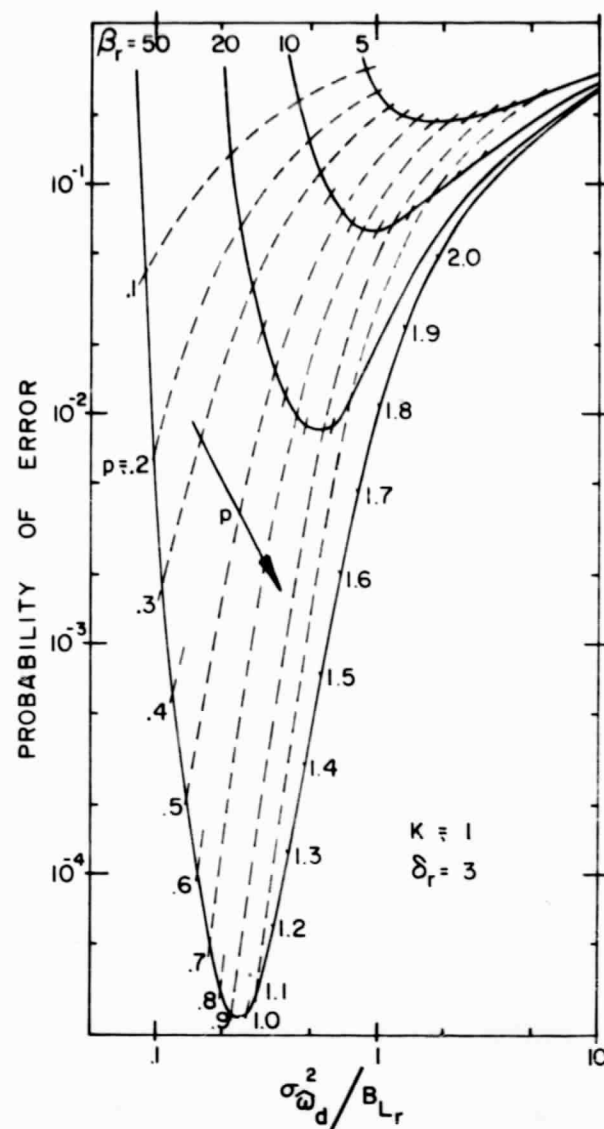


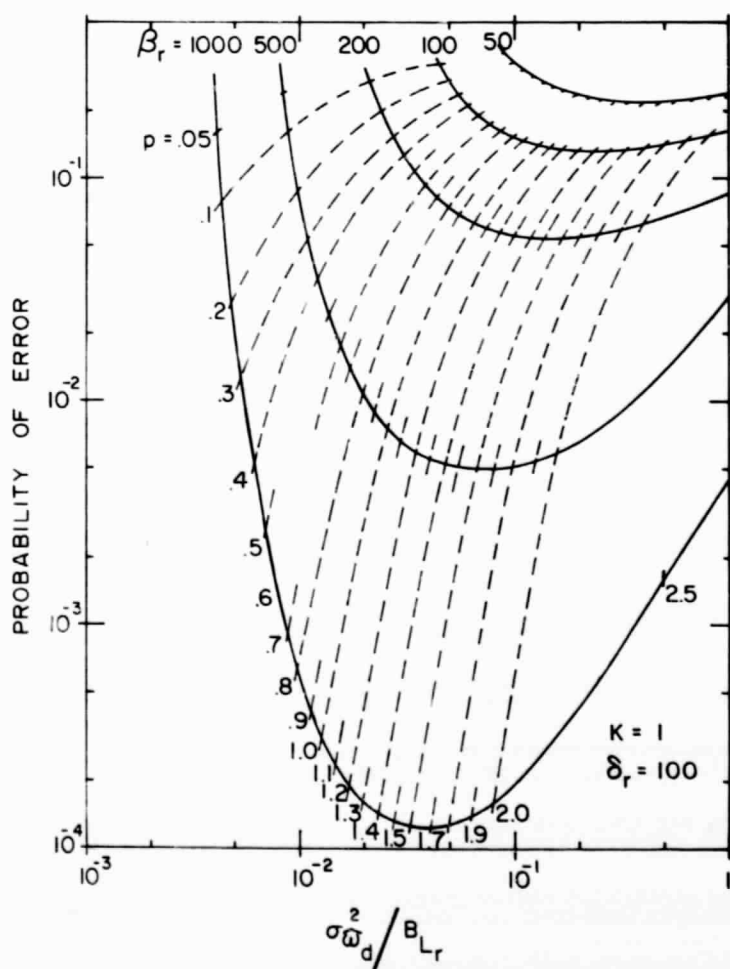
Fig. 6 Doppler Error versus Carrier Signal-to-Noise Ratio



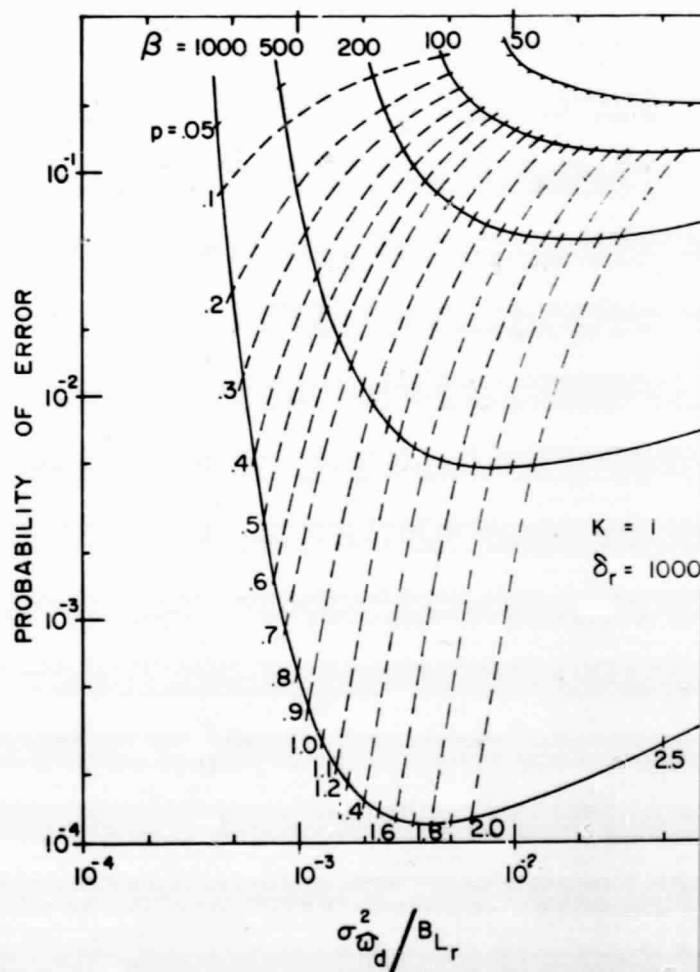
(a)



(b)



(c)



(d)

Fig. 7 Error Rate versus Doppler Jitter

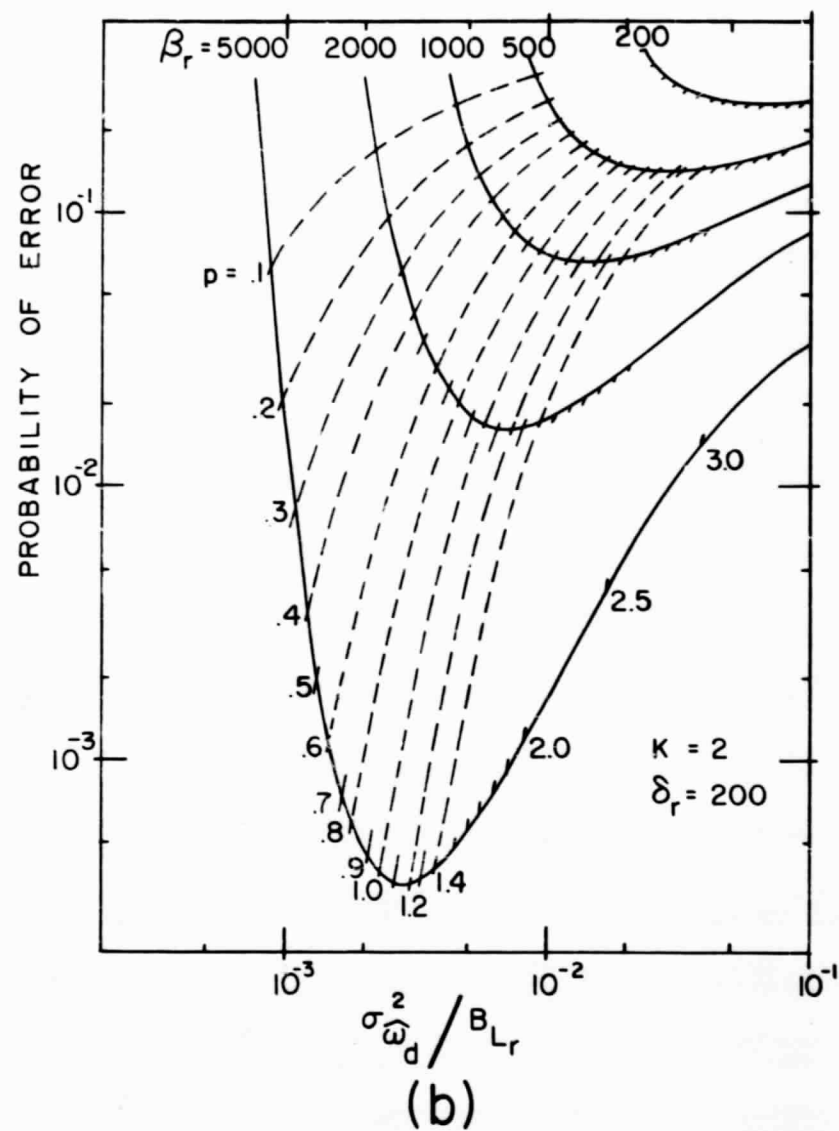
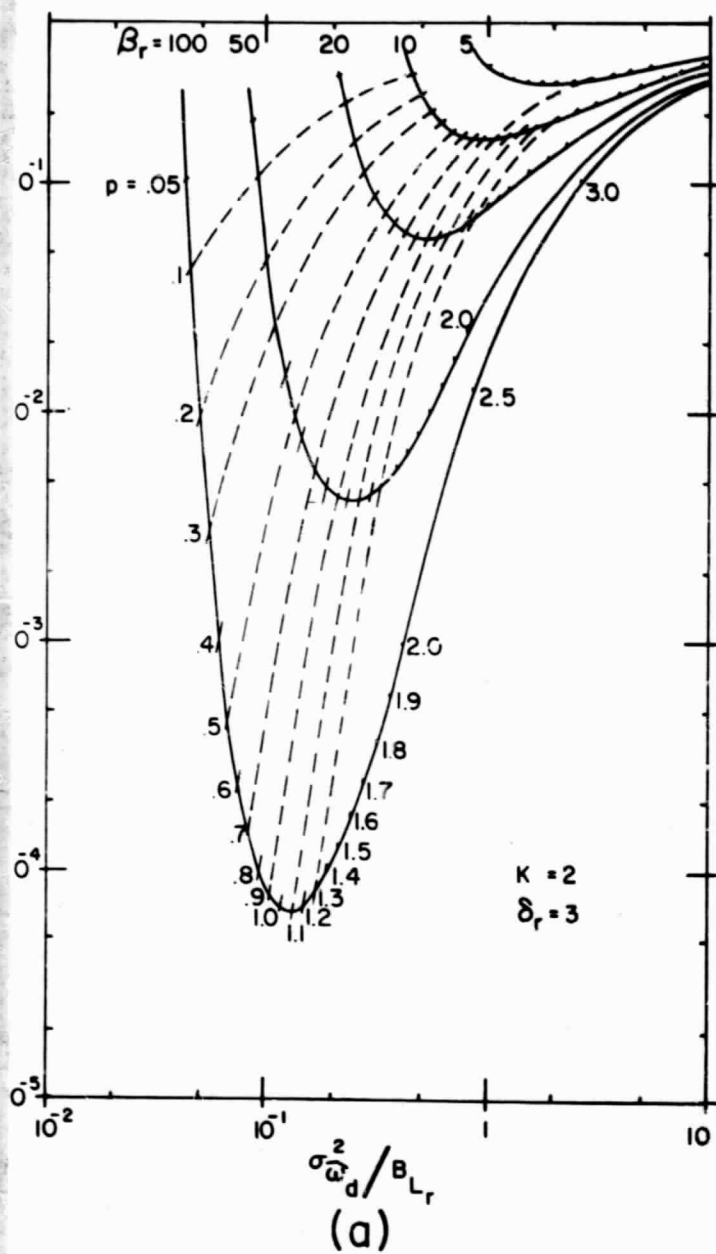


Fig. 8 Error Rate versus Doppler Jitter

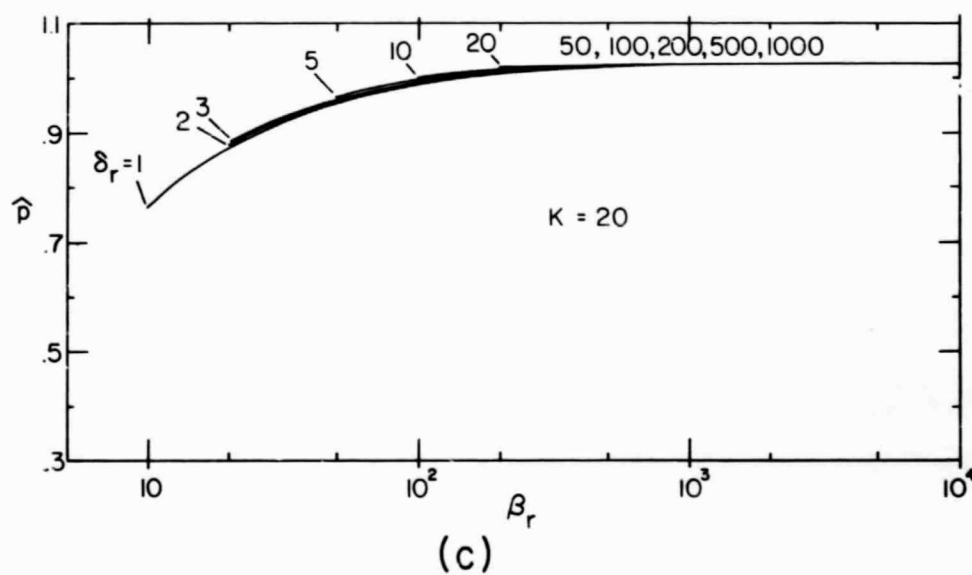
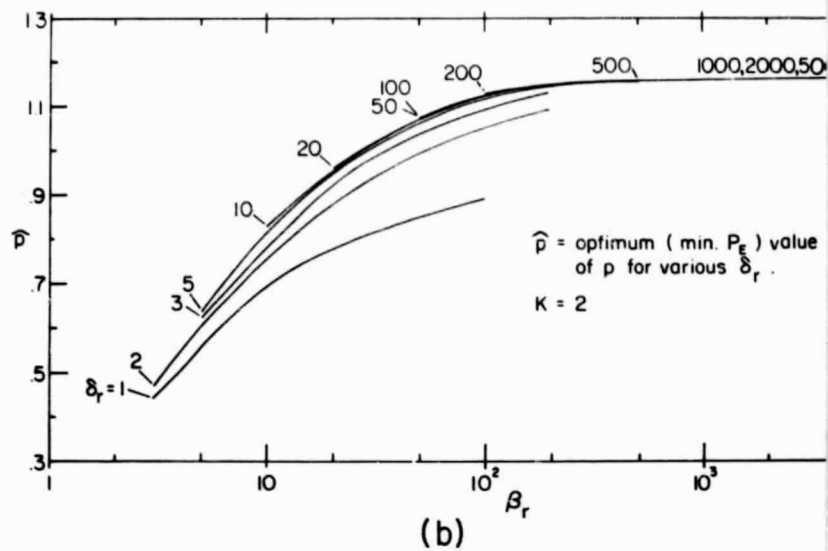
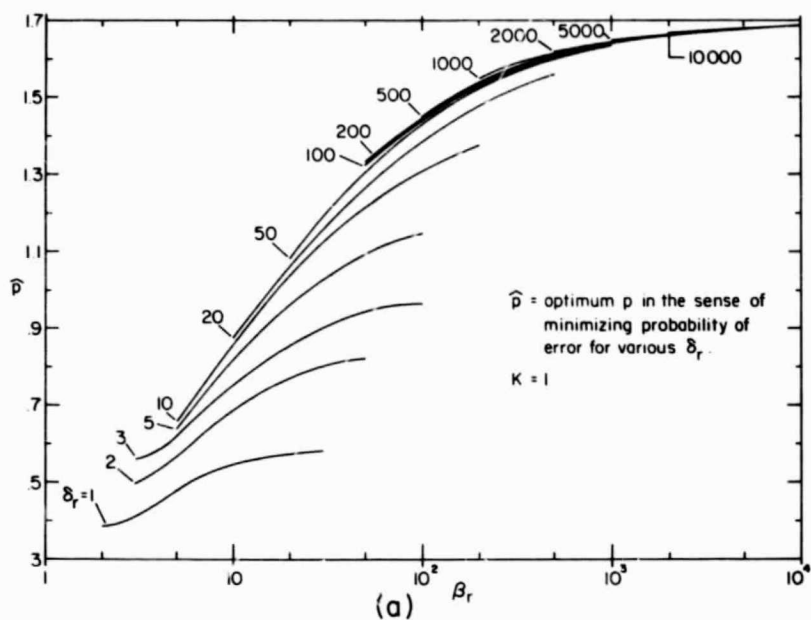
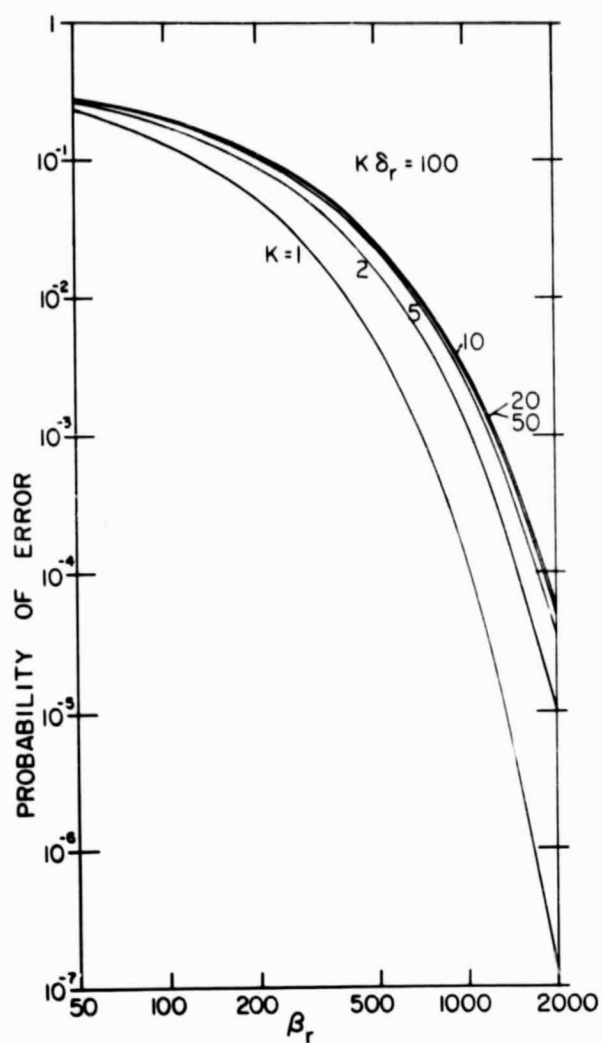
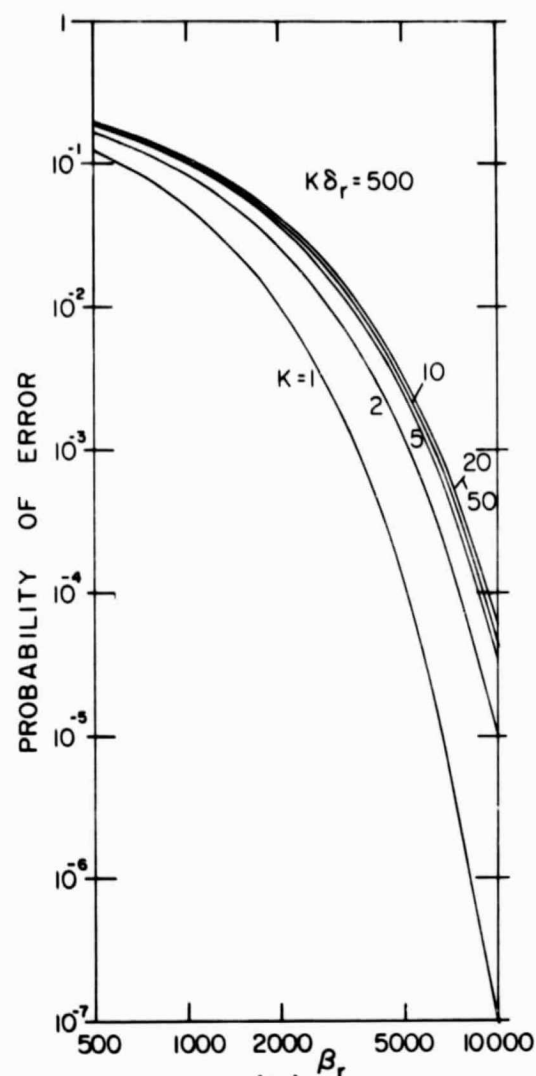


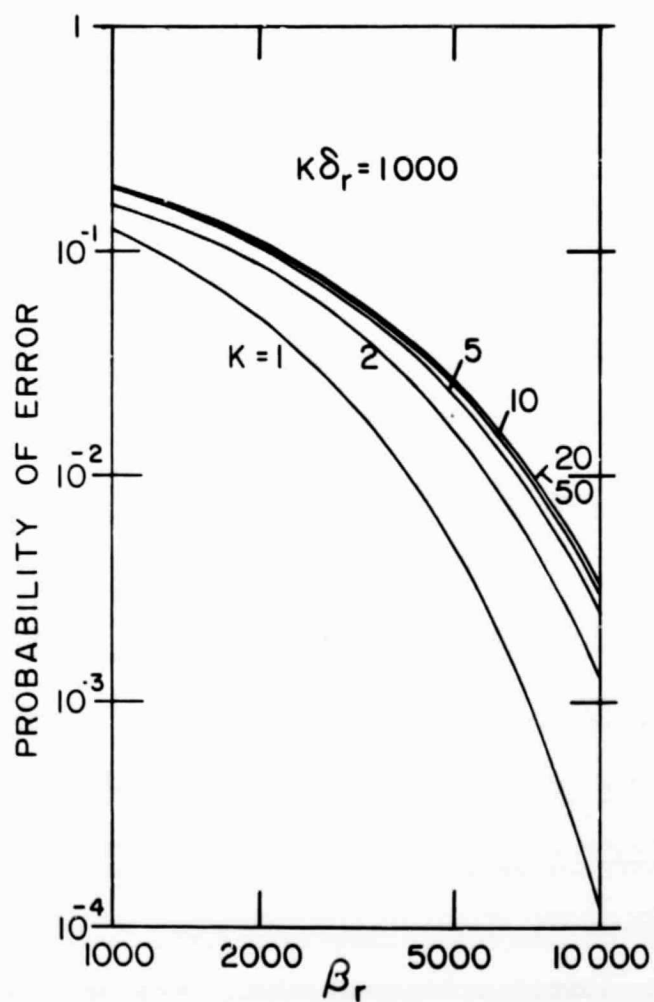
Fig. 9 Optimum Modulation Index, $\hat{p}$, versus Signal-to-Noise Ratio



(a)



(b)



(c)

Fig. 10 Time Multiplexing versus Frequency
Multiplexing for Fixed Data

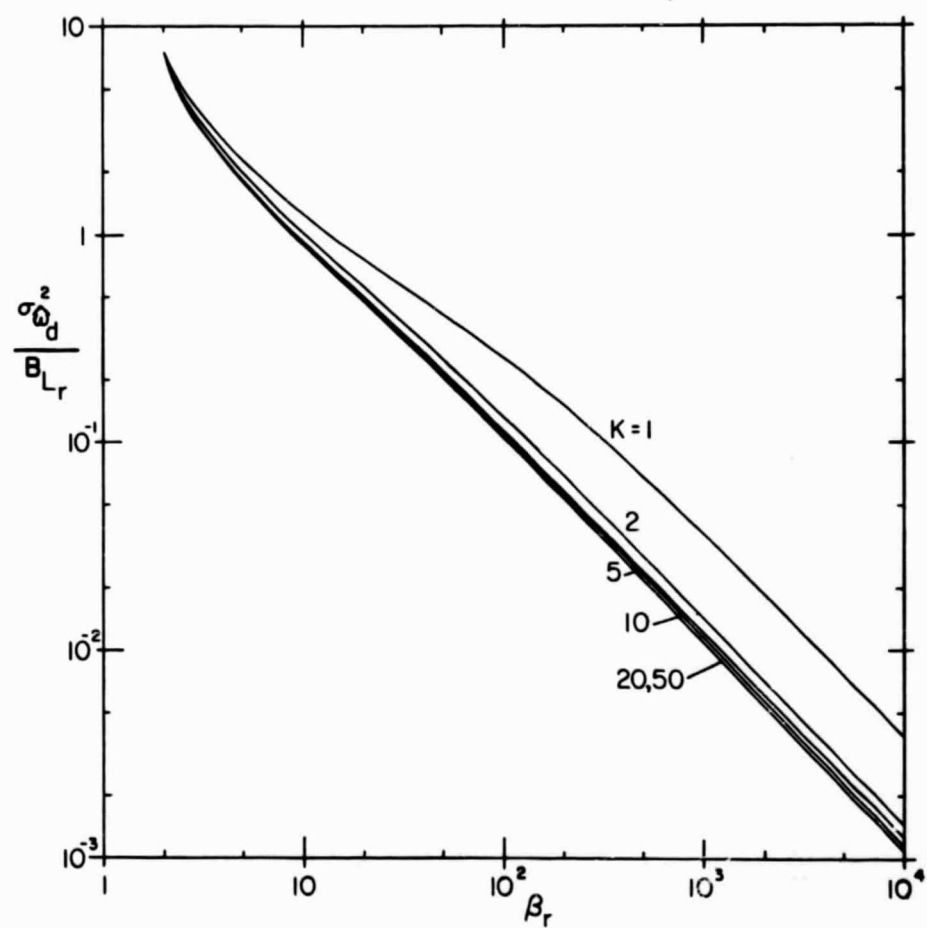


Fig. 11 Doppler Measurement Variance versus Input Signal-to-Noise Ratio for various K with $p = \hat{p}$.

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Electronic Sciences Laboratory University of Southern California Los Angeles, Ca. 90007		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE A Unified Theory of Coherent Digital Systems Which Track Doppler Frequency			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Interim Technical Report			
5. AUTHOR(S) (First name, middle initial, last name) Charles L. Weber			
6. REPORT DATE October 1969	7a. TOTAL NO. OF PAGES 41	7b. NO. OF REFS 12	
8a. CONTRACT OR GRANT NO. DA-AROD-31-124-G-1045	9a. ORIGINATOR'S REPORT NUMBER(S) USCEE 371		
b. PROJECT NO.			
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
d.			
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY U. S. Army Research Office--Durham Box CM, Duke Station, Durham, North Carolina	
13. ABSTRACT A unified theory from which the design of a large class of coherent digital communication systems can be optimally carried out is presented. In the design of digital communication systems, the error rate is the criterion which is invariable emphasized. In many digital systems, however, there is relative motion between transmitter and receiver which must be estimated by making use of Doppler frequency information. A new analysis of a general class of coherent digital systems is herein developed, in which the trade-offs that exist between Doppler measurement capability and sub-carrier demodulation error rate are quantitatively presented. The theoretically unrecoverable power loss which exists when employing frequency division multiplexing subcarriers as compared to time division multiplexing is described. The results point out that there is significant parametric dependence of the optimal choice of system parameters on the carrier loop signal-to-noise ratio and the data rate.			

14.	KEY WORDS	LINK A		LINK B		LINK C	
		ROLE	WT	ROLE	WT	ROLE	WT
	Digital Communication Systems Coherent Digital Systems Frequency Division Multiplexing Time Division Multiplexing						