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J. P. McDANELL
W. F. POWERS

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THE UNIVERSITY OF MICHIGAN 1817

Department of Aerospace Engineering

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NECESSARY CONDITIONS FOR JOINING OPTIMAL SINGULAR AND NONSINGULAR SUBARCS†

by

J. P. McDanell* and W. F. Powers**
The University of Michigan
Ann Arbor, Michigan

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*Graduate Student, Computer, Information, and Control Engineering Program. National Science Foundation Trainee.

**Assistant Professor, Department of Aerospace Engineering.

ABSTRACT

Necessary conditions for the optimality of junctions between singular and nonsingular subarcs are developed for singular optimal control problems. Previously known necessary conditions concerning the continuity and smoothness of a piecewise analytic optimal control at a junction are clarified and extended. The main result is that the sum of the order of the singular arc and the lowest order time derivative of the control which is discontinuous at the junction must be an odd integer when the strengthened generalized Legendre-Clebsch condition is satisfied. Also, new necessary conditions which do not require an analyticity assumption are developed. These aid in characterizing problems which may possess nonanalytic junctions.

1. Introduction

Optimal control problems in which the control variables appear only linearly admit the possibility of the occurrence of singular extremals. The analysis of such problems is complicated by the fact that the solution, in general, consists of some combination of singular and nonsingular subarcs, the number and sequence of which are not known a priori. If the solution is totally singular, recent results [1], [2] are available to prove optimality in a large number of cases. If the solution is totally nonsingular, it is the familiar bang-bang control generated by a switching function with isolated zeros, as determined by the minimum principle. However, the mathematical characterization of optimal controls which contain both singular and nonsingular subarcs is far from complete.

This paper is concerned with the problem of characterizing the continuity and smoothness properties of the optimal control at a junction between singular and nonsingular subarcs. The analysis was motivated by the preliminary results obtained in this direction by Kelley, Kopp, and Moyer [3] and Johnson [4]. We shall comment on their results in a later section.

2. Problem Statement

The class of problems to which this analysis applies is the following. Determine the scalar control $u^*(t)$, $t \in [t_0, t_f]$ which minimizes the functional

$$J(u) = G(t_f, x(t_f)) + \int_{t_0}^{t_f} [L_0(t,x) + L_u(t,x)u]dt \qquad (2.1)$$

where the system equation is

$$\dot{x} = f_0(t,x) + f_u(t,x)u \qquad (2.2)$$

subject to the constraints

$$|u(t)| \leq K(t), \quad t \in [t_0, t_f] \qquad (2.3)$$

$$\{t_0, x(t_0), t_f, x(t_f)\} \in S \qquad (2.4)$$

Here x is an n-vector and S is a closed subset of R^{2n+2}. The scalar functions f_0, f_u, L_0, L_u are assumed to be analytic in both arguments in a suitable domain, and $K(t)$ is assumed to be continuous and piecewise analytic for $t \in [t_0, t_f]$, i.e., $K(t)$ is permitted to have a finite number of "corners". Of course, the usual case $|u| \leq K$ with K = const. is included as a special case. We restrict attention to a scalar control in order to simplify notation. A similar analysis holds for each component of a vector contr '.

Clearly, the Hamiltonian for this problem is linear in the control, i.e.,

$$H(t,x,\lambda,u) = \lambda^T f_0(t,x) + L_0(t,x) + [\lambda^T f_u(t,x) + L_u(t,x)]u \qquad (2.5)$$

The multiplier equations are then given by

$$\dot{\lambda} = -H_x(t,x,\lambda,u) \qquad (2.6)$$

where H_x is also linear in u. The coefficient of u in (2.5) is called the switching function, which we shall designate as $\phi(t)$, i.e.,

$$\phi(t) \equiv H_u(t, x(t), \lambda(t)) \qquad (2.7)$$

The minimum principle (i.e., Pontryagin's maximum principle in a minimum form) states that for almost every $t \in [t_0, t_f]$ and each u satisfying

$|u| \leqq K(t)$, the optimal control $u^*(t)$ must satisfy

$$H(t,x(t),\lambda(t),u^*(t)) \leqq H(t,x(t),\lambda(t),u) \tag{2.8}$$

Therefore, as is well known, on each open subinterval of $[t_0, t_f]$ there are two distinct possibilities for u^*. Either

$$u^*(t) = -K(t) \operatorname{sgn} \phi(t) \tag{2.9}$$

or

$$\phi(t) \equiv 0 \tag{2.10}$$

Equations (2.9) and (2.10) define, respectively, the nonsingular and singular subarcs of the optimal control.

The class of problems defined above will be called singular control problems, even though only a portion of the total trajectory may be singular.

3. Notation and Definitions

The following definitions will clarify the terminology used in this paper.

DEFINITION 1. A function g is said to be piecewise analytic on an interval (a,b) if it is piecewise continuous on (a,b) and analytic on each subinterval for which g is continuous.

DEFINITION 2. A junction between singular and nonsingular subarcs of the control is said to be a nonanalytic junction if the control is not piecewise analytic in any neighborhood of the junction.

DEFINITION 3. In a singular control problem, let $(d^{2q}/dt^{2q})[H_u(t,x,\lambda)]$ be the lowest order total derivative of H_u in which u appears explicitly. Then the integer q is called the order of the singular arc.

Implicit in Definition 3 is the property that u first appears explicitly in an even order derivative of H_u, i.e., it is correct to refer to q as an integer. For a proof of this property see Robbins [5].

We also need the well known generalized Legendre-Clebsch necessary condition for optimality of singular subarcs [3].

THEOREM. (Generalized Legendre-Clebsch condition) On an optimal singular subarc of order q, it is necessary that

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}}{dt^{2q}} H_u\right] \geqq 0 \tag{3.1}$$

Condition (3.1) hereafter will be called the GLC condition. By the strengthened GLC condition we mean that strict inequality holds in (3.1).

In this paper it will be convenient to consider the lowest order derivative of a function to be its zeroth derivative, by which we mean the function itself. We shall use the notation

$$g^{(0)} \equiv g, \quad g^{(i)} \equiv \frac{d^i g}{dt^i}, \quad i = 1,2,\ldots$$

Also, where the context makes the meaning clear, we shall use u to designate the optimal control instead of u*.

4. The Junction Theorems

As indicated in Section 1, the theory for totally singular and totally nonsingular optimal controls is rather well developed. The main difficulty with singular control problems occurs when both singular and nonsingular subarcs are present. Since a useful sufficient condition for such problems is not available, one is led naturally to the study of necessary conditions which are valid in the neighborhood of a junction between singular and nonsingular subarcs. It is expected that such conditions can be used to eliminate candidate extremals and/or predict beforehand the way in which singular and nonsingular subarcs must be joined, e.g., whether the optimal control is continuous or discontinuous at a junction.

If the optimal control is well-behaved in a neighborhood of a junction, then the following property must hold.

THEOREM 1. Let t_c be a point at which singular and nonsingular subarcs of an optimal control u are joined, and let q be the order of the singular arc. Suppose the strengthened GLC condition is satisfied at t_c, i.e., $(-1)^q(\partial/\partial u)H_u^{(2q)} > 0$, and assume that the control is piecewise analytic in a neighborhood of t_c. Let $u^{(r)}$ $(r \geq 0)$ be the lowest order derivative of u which is discontinuous at t_c. Then $q + r$ is an odd integer.

<u>Proof</u>. Since $H_u^{(2q)}$ is the lowest order time derivative of H_u which contains u explicitly, from (2.2), (2.5), and (2.6) we see that it must have the form

$$H_u^{(2q)}(t,x,\lambda,u) \equiv A(t,x,\lambda) + B(t,x,\lambda)u \qquad (4.1)$$

Define the functions α and β as follows.

$$\alpha(t) \equiv A(t,x(t),\lambda(t)) \qquad (4.2)$$

$$\beta(t) \equiv B(t,x(t),\lambda(t)) \qquad (4.3)$$

From the hypotheses it is clear that α and β are continuous and have at least r continuous derivatives at t_c. The switching function ϕ as defined by (2.7) has exactly $2q + r - 1$ continuous derivatives at t_c.

Let ϵ be a nonzero real number of arbitrarily small magnitude such that $t_c + \epsilon$ is a point on the nonsingular side of t_c and $t_c - \epsilon$ is a point on the singular side. Let u_n and u_s designate the control u on the nonsingular and singular sides of t_c, respectively. By $u_n^{(i)}(t_c)$ and $u_s^{(i)}(t_c)$ we mean the limit as $\epsilon \to 0$ of $u^{(i)}(t_c + \epsilon)$ and $u^{(i)}(t_c - \epsilon)$, respectively.

We wish to expand $\phi(t_c + \epsilon)$ in a Taylor series about t_c. Let $k = 2q + r$. Then $\phi^{(k)}$ will be the lowest order derivative of the switching function ϕ which is discontinuous at t_c, and since $\phi \equiv 0$ on the singular side of t_c, the first nonzero term of the Taylor series will be the term containing $\phi^{(k)}$. Noting that

$$\phi^{(k)} \equiv \frac{d^r}{dt^r}\left[\alpha + \beta u\right] \tag{4.4}$$

we can write

$$\phi(t_c+\epsilon) = \frac{\epsilon^k}{k!}\left[\alpha^{(r)}(t_c) + \sum_{i=0}^{r}\binom{r}{i}\beta^{(r-i)}(t_c)u_n^{(i)}(t_c)\right] + 0(\epsilon^k) \tag{4.5}$$

where Leibniz's formula [6] for differentiation of a product has been used to differentiate βu_n.

On the singular arc

$$\phi^{(2q)} = \alpha + \beta u_s \equiv 0 \tag{4.6}$$

Therefore, $\alpha \equiv -\beta u_s$, and

$$\alpha^{(r)} = \frac{d^r}{dt^r}\left[-\beta u_s\right] = -\sum_{i=0}^{r}\binom{r}{i}\beta^{(r-i)}u_s^{(i)} \tag{4.7}$$

Substituting from (4.7) into (4.5)

$$\phi(t_c+\epsilon) = \frac{\epsilon^k}{k!}\sum_{i=0}^{r}\binom{r}{i}\beta^{(r-i)}(t_c)\left[u_n^{(i)}(t_c) - u_s^{(i)}(t_c)\right] + 0(\epsilon^k) \tag{4.8}$$

If $r > 0$,

$$u_n^{(i)}(t_c) = u_s^{(i)}(t_c), \quad i = 0, \ldots, r-1 \tag{4.9}$$

Therefore, (4.8) becomes

$$\phi(t_c+\epsilon) = \frac{\epsilon^k}{k!}\beta(t_c)[u_n^{(r)}(t_c) - u_s^{(r)}(t_c)] + 0(\epsilon^k) \qquad (4.10)$$

Let $\sigma = -\text{sgn}\,\phi(t_c + \epsilon)$ so that $u_n(t) = \sigma K(t)$. Then recalling that $u_n^{(i)}(t_c) = \lim_{\epsilon\to 0} u_n^{(i)}(t_c + \epsilon)$ we have

$$u_n^{(i)}(t_c) = \sigma K^{(i)}(t_c), \quad i = 0,\ldots,r \qquad (4.11)$$

Now consider the following series expansion on the singular arc.

$$\sigma K(t_c - \epsilon) - u(t_c - \epsilon) = \sum_{i=0}^{r} \frac{(-\epsilon)^i}{i!}\left[\sigma K^{(i)}(t_c) - u_s^{(i)}(t_c)\right] + 0(\epsilon^r) \qquad (4.12)$$

The right hand side of (4.12) can be simplified using (4.9) and (4.11) to obtain

$$\sigma K(t_c - \epsilon) - u(t_c - \epsilon) = \frac{(-1)^r \epsilon^r}{r!}\left[u_n^{(r)}(t_c) - u_s^{(r)}(t_c)\right] + 0(\epsilon^r) \qquad (4.13)$$

Substituting from (4.13) into (4.10) and recalling that $k = 2q + r$, we obtain

$$\phi(t_c+\epsilon) = (-1)^r \frac{\epsilon^{2q} r!}{k!}\beta(t_c)[\sigma K(t_c - \epsilon) - u(t_c - \epsilon)] + 0(\epsilon^k) \qquad (4.14)$$

From the application of the minimum principle on the nonsingular subarc (Equation (2.9)) we have $\sigma = 1$ if $\phi(t_c + \epsilon) < 0$ and $\sigma = -1$ if $\phi(t_c + \epsilon) > 0$. Therefore, the following inequality must hold.

$$(-1)^r \epsilon^{2q} \beta(t_c)[K(t_c - \epsilon) \pm u(t_c - \epsilon)] < 0 \qquad (4.15)$$

From the GLC condition we have

$$(-1)^q \beta(t_c) > 0 \qquad (4.16)$$

Multiplying the left hand side of (4.15) by the positive quantity in (4.16) we obtain

$$(-1)^{q+r} \epsilon^{2q} \beta^2(t_c)[K(t_c - \epsilon) \pm u(t_c - \epsilon)] < 0 \qquad (4.17)$$

Since $|u(t)| \leq K(t)$ for all $t \in [t_0, t_f]$, and the singular arc is tacitly assumed to be interior almost everywhere, the bracketed quantity in (4.17) is strictly positive, regardless of the choice of sign on $\pm u(t_c - \epsilon)$. Also $\epsilon^{2q} > 0$ regardless of the sign of ϵ. Therefore, condition (4.17) reduces to

$$(-1)^{q+r} < 0 \tag{4.18}$$

from which it is clear that q + r is an odd integer. This completes the proof.

Theorem 1 implies the following important corollaries.

COROLLARY 1. In q-even problems, assuming u is piecewise analytic and the strengthened GLC condition is satisfied, the optimal control is continuous at each junction.

COROLLARY 2. In q-odd problems, assuming u is piecewise analytic and the strengthened GLC condition is satisfied, the optimal control either has a jump discontinuity at each junction, or else the singular control joins the boundary smoothly, i.e., with continuous first derivative.

In the corollaries above, especially Corollary 1, which applies to the q-even case, the assumption that u is piecewise analytic is not to be taken lightly. In fact, the authors have not seen or been able to produce a q-even example with a continuous junction, i.e., the junction is usually nonanalytic if q is even.

The conclusions reached by Kelley, Kopp, and Moyer [3] are consistent with those stated in Corollaries 1 and 2, with one important exception—they ruled out the possibility of a continuous junction for q-odd problems. This erroneous conclusion resulted from the claim that continuity of u implies $(\partial/\partial u)H_u^{(2q)} > 0$ which is not true in general, as can be seen from (4.15) (in which $\beta \equiv (\partial/\partial u)H_u^{(2q)}$). That such a junction is realizable will be demonstrated by means of a simple example in a later section.

Theorem 1 requires that the <u>strengthened</u> GLC condition be satisfied at the junction point. While this is the usual and most important case, the possibility exists that the GLC condition is satisfied with equality. To treat this case, note from Definition 3 that for a qth order singular arc the GLC expression $(\partial/\partial u)H_u^{(2q)}$ (i.e., β) cannot be identically zero on the singular subarc. Therefore, in view of our analyticity assumptions, a derivative of some order must be nonzero at the junction point t_c even if $\beta(t_c) = 0$. This leads to the following theorem, which is a generalization of Theorem 1, but is stated separately to avoid obscuring the result for the important case covered by Theorem 1.

THEOREM 2. Let t_c be a point at which singular and nonsingular subarcs of an optimal control u are joined, and let q be the order of the singular arc. Assume that the control is piecewise analytic in a neighborhood of t_c. Let $u^{(r)}$ $(r \geqq 0)$ be the lowest order derivative of u which is discontinuous at t_c, and let $\beta^{(m)}$ $(m \geqq 0)$ be the lowest order derivative of the GLC expression $(\partial/\partial u)H_u^{(2q)} \equiv \beta$ which is nonzero at t_c. Then, (i) if $m \leqq r$, $q + r + m$ is an odd integer; (ii) if $m > r$, $-\text{sgn}[\beta^{(m)}(t_c^+)\beta^{(m)}(t_c^-)] = (-1)^{q+r+m}$.

<u>Proof Outline</u>. The proof is similar to that for Theorem 1; however, in order to obtain a nontrivial term in the Taylor series expansion for $\phi(t_c + \epsilon)$, one must consider higher order terms with the result that (4.15) is replaced by

$$(-1)^r \epsilon^{2q+m} \beta_n^{(m)}(t_c)[K(t_c - \epsilon) \pm u_s(t_c - \epsilon)] < 0 \qquad (4.19)$$

A Taylor series expansion for $\beta(t)$ on the singular arc yields

$$\beta(t_c - \epsilon) = \frac{(-\epsilon)^m}{m!} \beta_s^{(m)}(t_c) + 0(\epsilon^m) . \qquad (4.20)$$

where the subscripts s and n on $\beta^{(m)}(t_c)$ indicate the limit at t_c on the singular and nonsingular sides, respectively. Since $\beta_s^{(m)}(t_c) \neq 0$, from the GLC condition and (4.20) we have

$$(-1)^{q+m} \epsilon^m \beta_s^{(m)}(t_c) > 0 \qquad (4.21)$$

From (4.19) and (4.21) it follows that

$$(-1)^{q+r+m} \beta^{(m)}(t_c^+) \beta^{(m)}(t_c^-) < 0 \qquad (4.22)$$

If $m \leqq r$, $\beta^{(m)}$ is continuous at t_c, in which case (4.22) implies that $q+r+m$ is an odd integer. If $m > r$, $\beta^{(m)}$ may not be continuous at t_c, and the conclusion of Theorem 2 for this case follows.

The main restriction in Theorems 1 and 2 is the assumption that the control is piecewise analytic in a neighborhood of the junction. This hypothesis is usually satisfied on the singular subarc, but not always on the nonsingular subarc. Thus, we are led to consider properties which do not require the assumption of analyticity, as stated in the following theorem. The functions A and B in this theorem are those defined by the identity (4.1).

THEOREM 3. Let u be an optimal control which contains both singular and nonsingular subarcs, where the singular subarcs are of order q. Then, (i) if $H_u^{(2q)} \neq 0$ on the nonsingular side of a junction, the control must be discontinuous; (ii) if $H_u^{(2q)} = 0$ on the nonsingular side of a junction and $B \neq 0$ at the junction, the control must be continuous; (iii) if $A \equiv 0$ and $K(t_c) \neq 0$ at a junction point t_c, the control must be discontinuous.

<u>Proof</u>. Using the same notation as in the proof of Theorem 1, and recalling that $H_u^{(2q)} \equiv 0$ on the singular subarc, we have for case (i)

$$\alpha(t_c) + \beta(t_c)u_n(t_c) \neq 0 = \alpha(t_c) + \beta(t_c)u_s(t_c) \tag{4.23}$$

from which we obtain $u_n(t_c) \neq u_s(t_c)$.

For case (ii) we have

$$\alpha(t_c) + \beta(t_c)u_n(t_c) = 0 = \alpha(t_c) + \beta(t_c)u_s(t_c) \tag{4.24}$$

Since $\beta(t_c) \neq 0$, we must have $u_n(t_c) = u_s(t_c)$.

For case (iii) $A \equiv 0$ implies $u_s \equiv 0$, and since $|u_n(t_c)| = K(t_c) \neq 0$, the control must be discontinuous.

Case (iii) of Theorem 3 may appear to be a rather special case, but it occurs frequently enough to be of interest Note that this result is independent of an even or odd assumption. Because of this, we can couple (iii) with the previous result for q-even problems to obtain the following interesting property.

COROLLARY 3. If q is even, $A \equiv 0$, $K(t_c) \neq 0$, and $\beta(t_c) \equiv B(t_c, x(t_c), \lambda(t_c)) \neq 0$, where t_c is a junction point between optimal singular and nonsingular subarcs, then the junction is nonanalytic.

Proof. Assume the contrary, i.e., that the optimal control is piecewise analytic in a neighborhood of t_c. Then by Corollary 1, the control is continuous at t_c. But, by Theorem 3 (iii) the control is discontinuous, which supplies the necessary contradiction.

In the next section this corollary will be used to predict the nonanalytic junction in the well-known Fuller problem.

5. Example of a Nonanalytic Junction

Consider the Fuller problem [7], which is to minimize

$$J = \frac{1}{2}\int_0^T x_1^2\,dt \tag{5.1}$$

subject to

$$\dot{x}_1 = x_2 , \qquad x_1(0) = \xi_1 \neq 0 \tag{5.2}$$
$$\dot{x}_2 = u , \qquad x_2(0) = \xi_2$$

$$|u| \leq 1 \tag{5.3}$$

where T is fixed. The Hamiltonian, the multiplier equations, and the switching function are given by

$$H = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2}x_1^2 \tag{5.4}$$

$$\dot{\lambda}_1 = -x_1 , \qquad \lambda_1(T) = 0 \tag{5.5}$$
$$\dot{\lambda}_2 = -\lambda_1 , \qquad \lambda_2(T) = 0$$

$$\phi \equiv H_u = \lambda_2 . \tag{5.6}$$

The lowest order derivative of H_u which contains u explicitly is

$$H_u^{(4)} = u \tag{5.7}$$

from which we see that the order of the singular arc is even, namely q = 2. Also, the strengthened GLC condition holds, and $A(t,x,\lambda) \equiv 0$. Thus, we have precisely the conditions of Corollary 3, indicating that any junctions which occur must be nonanalytic junctions.

This problem has been studied thoroughly by Fuller [7] and Johansen [8], and the result is well known. The singular arc is given by

$$u_s = x_1 = x_2 = 0 \tag{5.8}$$

Since $\xi_1 \neq 0$, the initial control must be nonsingular. The nonsingular arc is characterized by the nonlinear differential equation

$$\phi^{(4)} = -\text{sgn}\,\phi \tag{5.9}$$

The solution of (5.9) yields a switching function with an infinite number of zeros such that the ratio of the lengths of successive intervals between zeros is a constant. If T is sufficiently large, the resulting nonsingular (bang-bang) control drives the state to the origin in a finite time t_c, with an infinite number of switches occurring in a neighborhood of t_c, at which point the optimal control becomes singular. The control is clearly discontinuous at the junction point t_c, as it must be according to (iii) of Theorem 3. Even though q is even, Corollary 1 is not violated because the control is not piecewise analytic in a neighborhood of the junction.

The predicted behavior at the junction is useful knowledge for numerical computational schemes, e.g., Jacobson [9] was able to successfully compute bang-bang solutions for this problem with T sufficiently small so that the singular arc did not occur. However, for large T the nonanalytic junction came into play After computing about ten switches, the method became unstable [10].

Note that the optimal control for this "innocent looking", second-order example is measurable but not piecewise continuous. Aside from its physical applicability, the existence of such examples is useful for motivating the assumption of measurable controls in the proof of the minimum principle.

6. Example of a Smooth Junction with q Odd

This example demonstrates not only the realizability of a smooth junction with q odd, but also the dependence of junction phenomena upon boundary conditions. For this case we consider the performance index

$$J = \frac{1}{2}\int_0^T (x_2^2 - x_1^2)dt \tag{6.1}$$

where T = 2.985. The equations of motion and constraints are given by

$$\dot{x}_1 = x_2 , \qquad x_1(0) = 0, \qquad x_1(T) = \sigma_1 \tag{6.2}$$

$$\dot{x}_2 = u , \qquad x_2(0) = 1, \qquad x_2(T) = \sigma_2$$

$$|u| \leqq 1 \tag{6.3}$$

The Hamiltonian and the multiplier equations are given by

$$H = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 \tag{6.4}$$

$$\dot{\lambda}_1 = x_1$$

$$\dot{\lambda}_2 = -\lambda_1 - x_2 \tag{6.5}$$

The switching function and its second derivative are

$$\phi \equiv H_u = \lambda_2 \tag{6.6}$$

$$\ddot{H}_u = -x_1 - u \tag{6.7}$$

so the singular arc is first order, i.e., q is odd. From (6.7) we see that the strengthened GLC condition is satisfied. Setting the right hand side of (6.7) equal to zero and substituting in the equations of motion (6.2), it is readily verified that a singular arc emanating from the initial state (0, 1) is given by

$$u_s = -\sin t$$

$$x_1 = \sin t \tag{6.8}$$

$$x_2 = \cos t$$

If the terminal state (σ_1, σ_2) is chos[illegible] trajector the solution is totally singular, as can be s[illegible] [illegible]icient co[illegible] References [1] and [2]. However, in this pa[illegible] [illegible]cerned v[illegible] tions. Consider the case where $\sigma_1 = 0$, σ_2 [illegible] [illegible] case w[illegible] as a candidate for the optimal control

$$u = \begin{cases} -\sin t, & t \in [0, \frac{\pi}{2} \ldots \\ -1 & ; t \in [\frac{\pi}{2}, \ldots \end{cases}$$

This control is admissible and satisfies all [illegible] conditio[illegible] mality, including that of Theorem 1. There [illegible] at $t_c = \ldots$ [illegible] trol and its first derivative are continuous a[illegible] second [illegible] discontinuous, so we have r = 2, q = 1, and [illegible] integ[illegible]

The authors are unaware of any wor[illegible] cond[illegible] literature which are applicable to this part[illegible] problem [illegible] convex and containing both singular and non[illegible] [illegible]rcs. C[illegible] we employed a gradient type numerical met[illegible] that th[illegible] control is indeed optimal, within the bounds [illegible] [illegible]cal just[illegible] modified conjugate gradient method of Pag[illegible] [illegible]side [1[illegible] with penalty functions to enforce the termi[illegible] [illegible]. The [illegible] shown in Figure 1. The control is continuo[illegible] [illegible] at the [illegible] expected.

It is apparent that the fortuitous occ[illegible] this smoo[illegible] a direct result of our judicious choice of th[illegible] boundary [illegible] fact, to generate this phenomenon, the for[illegible] candidate co[illegible] selected on the basis of intuition; then a c[illegible] point on th[illegible] trajectory was selected as the fixed termin[illegible], and finall[illegible] ing time was taken to be the explicit final ti[illegible]

By changing the terminal state, we [illegible] to genera[illegible] controls, which undoubtedly are the usual ca[illegible], These are sh[illegible] 2 and 3. For these cases r = 0, and the conditions of Theorem [illegible] again. To further emphasize the special cha[illegible] of the smo[illegible] the phase plane trajectories for the controls in Figures 1-2 are [illegible]

7. Conclusion

Necessary conditions for the optimality of junctions between singular and nonsingular subarcs in singular optimal control problems have been developed. Necessary conditions developed previously by Kelley, Kopp, and Moyer [3], which involve an analyticity assumption, have been clarified and extended. The main result in this direction is that the sum of the order of the singular arc and the order of the lowest time derivative of the control which is discontinuous at the junction must be an odd integer when the strengthened generalized Legendre-Clebsch condition is satisfied. Also, new necessary conditions which do not involve an analyticity assumption have been developed. These conditions aid mainly in characterizing problems which may possess nonanalytic junctions.

It should be emphasized that these are local necessary conditions for optimality. Yet, as indicated by the example in Section 6, the point at which a junction occurs is determined mainly by initial and terminal boundary conditions, i.e., by essentially nonlocal information. This means that any junction theory which, for example, might be used to establish criteria for switching between singular and nonsingular arcs in an indirect computational scheme will have to take such nonlocal information into account.

It is becoming increasingly apparent that a close relationship exists between singular problems and bounded state problems [12][13]. In this regard it is interesting to note that the result of Theorem 1 bears some similarity to a result of Jacobson, Lele, and Speyer [13] which identifies certain properties of optimal trajectories associated with odd order state space constraints. Such similarities suggest the possibility of a duality between these two classes of problems which might be profitably exploited.

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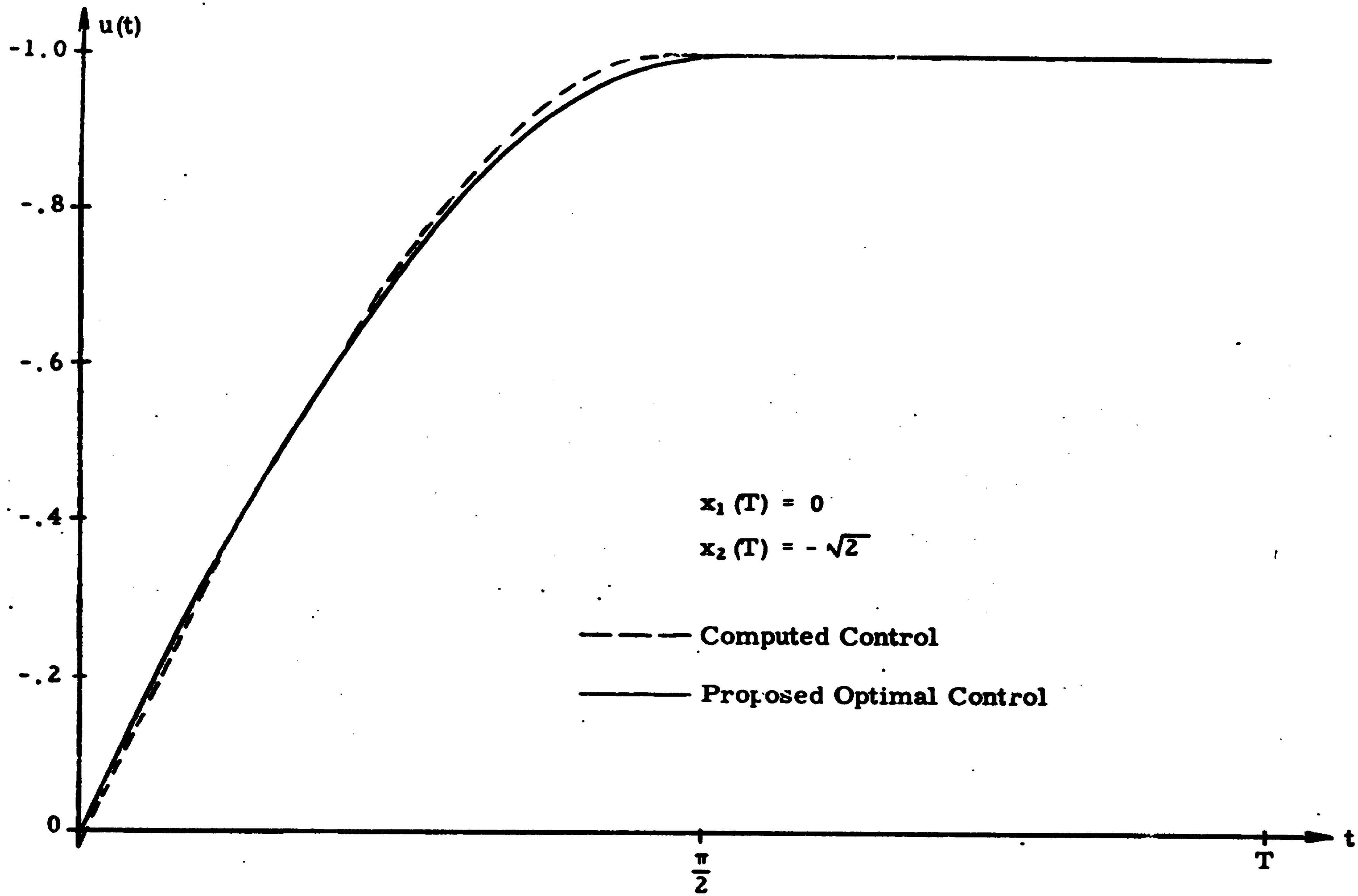

Figure 1. Example of an optimal control which is continuous at a singular-to-nonsingular junction with q odd.

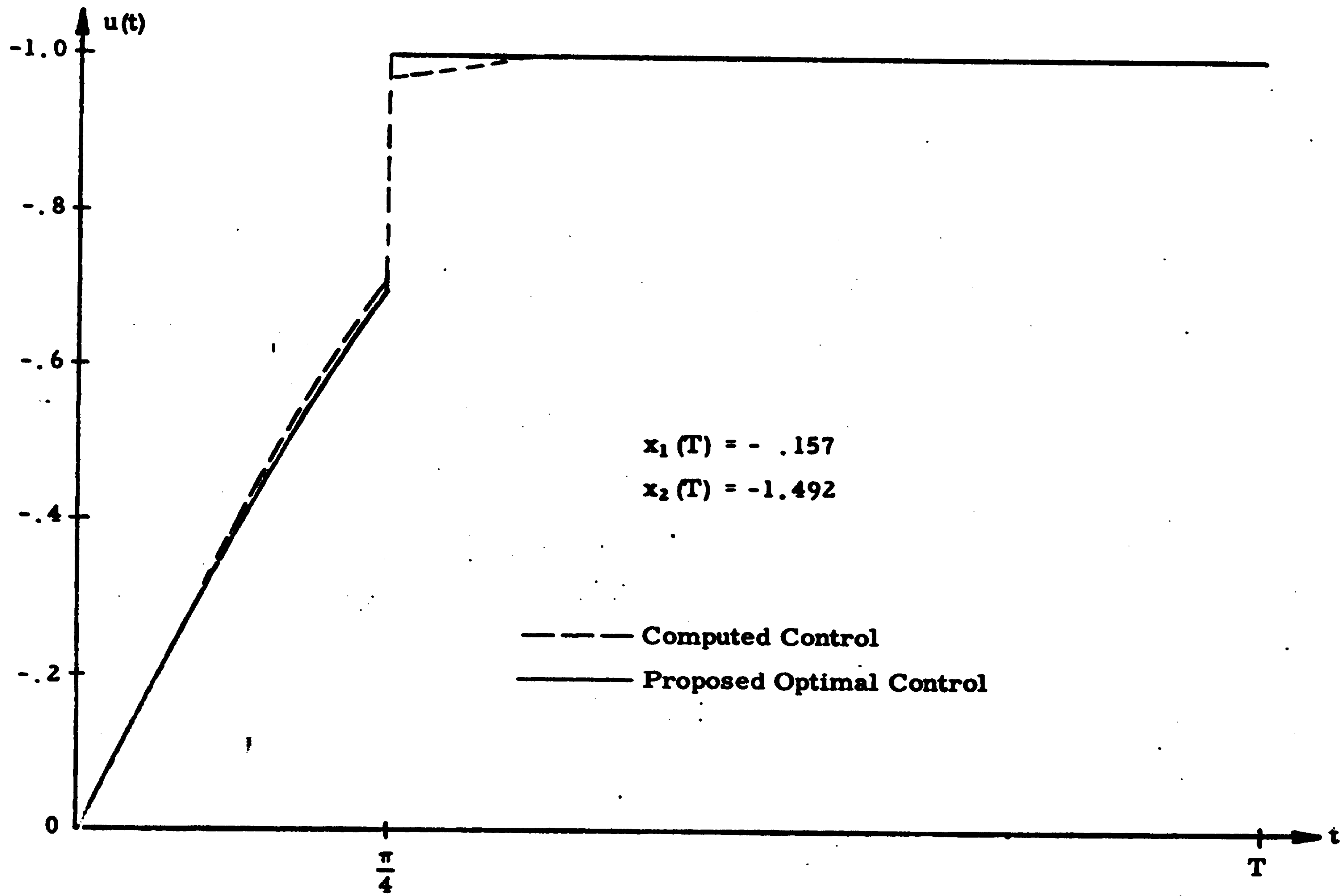

Figure 2. A typical discontinuous junction.

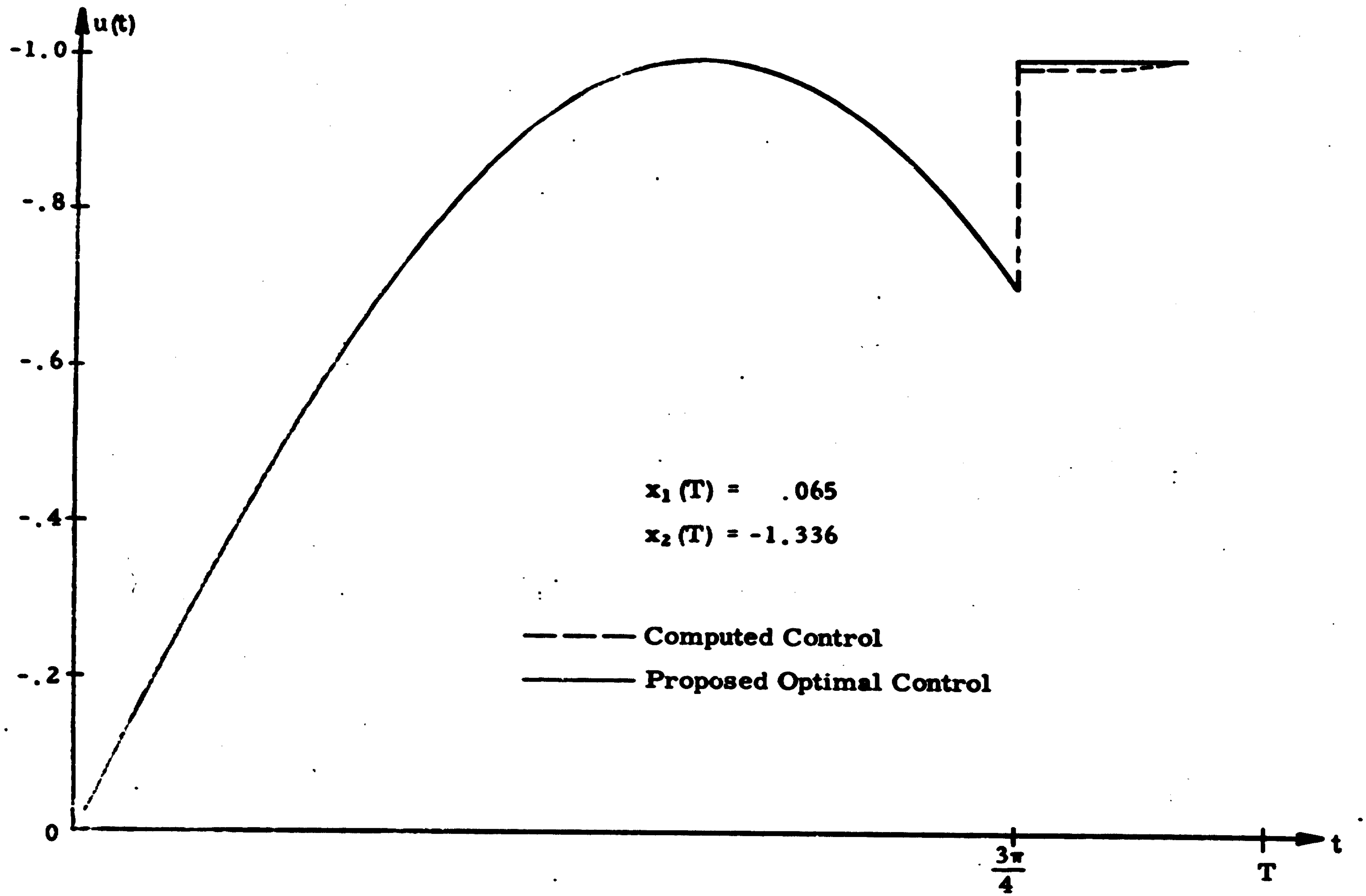

Figure 3. A predominantly singular control with discontinuous junction to a short nonsingular subarc.

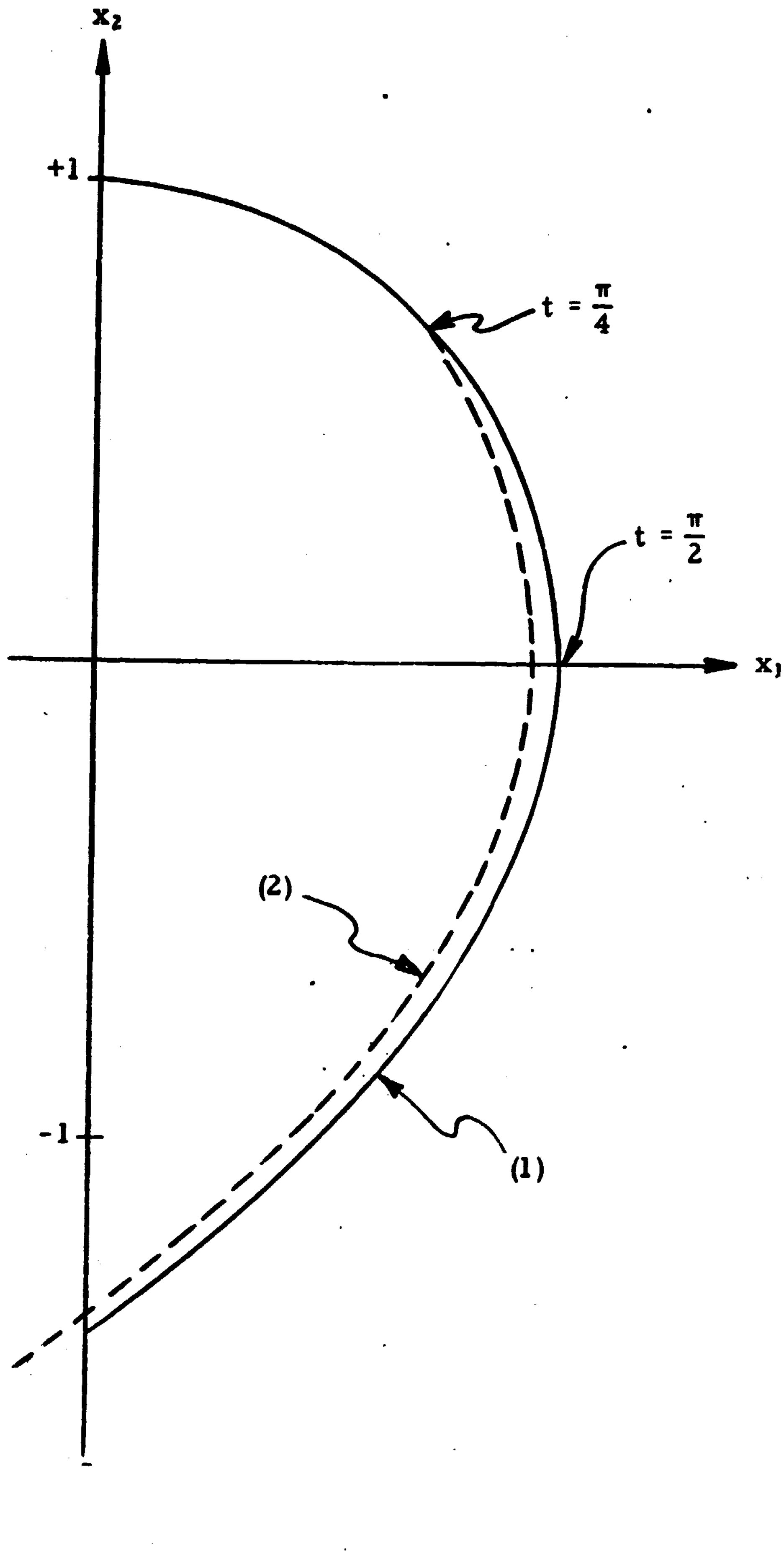

Figure 4. Phase plane trajectories for the controls in Figures 1 and 2.