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RESEARCH IN THE THEORY
OF
MAGNETOHYDRODYNAMIC TURBULENCE

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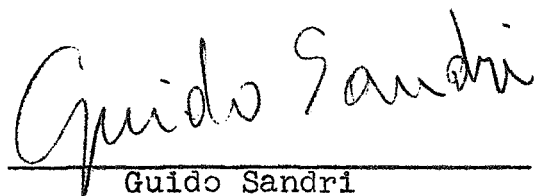
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MAGNETOHYDRODYNAMIC TURBULENCE

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I. INTRODUCTION

The purpose of our research is that of understanding the behavior of a plasma (on a microscopic scale) when it is unstable and therefore potentially turbulent, but in which collisional effects cannot be neglected. Species-exchange collisions require the understanding of the interaction of at least three particles to be described realistically. In recent work, we have shown that the classical kinetic theory developed by Bogolubov and others, which is largely based on the "adiabatic assumption," fails to give a proper asymptotic characterization of the two-particle correlation function on time scales of the order of the relaxation time of the system. The reason for this failure is that in the adiabatic approach the slow variation of the velocity distribution function is neglected in its effect on the correlation function. This very plausible assumption is, in fact, limited to times which are small compared to the relaxation time. The consequences of the failure are physically quite drastic; in particular, the adiabatic correlation function has an infinite spatial range, while the nonadiabatic correlation has a finite spatial range of the order of the mean free path. This fact has very mild consequences with regard to the kinetic behavior of a stable system in leading order because of the strong cutoff offered by the interaction potential. On the other hand, the oversight of nonadiabatic effects has dramatic consequences for a system which is unstable or for a system in which strong collisional effects have to be taken into account. In this report, we show how the adiabatic behavior can be corrected to give a full description of the two-particle correlation. The main result of our analysis is that the inclusion of three-body collisions damps the peak value of the two-particle correlation, which in leading order has the form of a wave travelling in the direction of the relative velocity of the particle pair. Furthermore, this nonadiabatic effect allows

us to remove the divergent terms that appear in the Bogolubov expansion when the kinetic condition is developed. We are thus enabled to obtain the kinetic equation to higher order and prove explicitly the H-theorem for the velocity distribution function, including three-body effects.

Two main ingredients of our analysis are: (i) full use of the nonadiabatic theory and (ii) a technique recently developed to separate the secular part of the perturbation functions from the regular part. Here, for illustrative purposes, we treat a stable system, neglecting ion recoil, but with inclusion of three-body collisions. We will then extend our analysis to the unstable case.

In the text, we carry out a "hybrid" calculation in which we isolate the main problems to be solved by using time scales but omitting the use of space scales explicitly. In a series of appendices, the hard core of the nonadiabatic treatment is developed. Furthermore, with the help of an exactly soluble model, we check the accuracy of our asymptotic expansion.

II. PRELIMINARY COMPARISONS

We consider the first three equations of the BBGKY hierarchy in the F form for the Lorentz gas. We have for the rate of change of the one-particle distribution function

$$\frac{\partial F_1^1}{\partial t} = \epsilon \int dx_1 I_1 F_1^2 \quad (2.1)$$

The two-particle distribution function satisfies

$$\frac{\partial F_1^2}{\partial t} + v \cdot \nabla F_1^2 = \epsilon I_1 F_1^2 + \epsilon \int dx_2 I_2 F_{12}^3 \quad (2.2)$$

Similarly, the three-particle distribution function satisfies

$$\frac{\partial F_{12}^3}{\partial t} + v \cdot \nabla F_{12}^3 = \epsilon(I_1 + I_2)F_{12}^3 + \epsilon \int dx_3 I_3 F_{123}^4 \quad (2.3)$$

We first show the connection with the correlation equations which are introduced by

$$F^1 = F \quad (2.4)$$

$$F_1^2 = F + G_1 \quad (2.5)$$

$$F_{12}^3 = F + G_1 + G_2 + H_{12} \quad (2.6)$$

$$F_{123}^4 = F + G_1 + G_2 + G_3 + H_{12} + H_{13} + H_{23} + K_{123} \quad (2.7)$$

Substitution of Eqs. (2.4)-(2.7) into Eqs. (2.1)-(2.3) yields the following three basic equations

$$\frac{\partial F}{\partial t} = \epsilon \int dx_1 I_1 G_1 \quad (2.8)$$

$$\frac{\partial G_1}{\partial t} + v \cdot \nabla G_1 = \epsilon I_1 F + \epsilon I_1 G_1 + \epsilon \int dx_2 I_2 H_{12} \quad (2.9)$$

$$\frac{\partial H_{12}}{\partial t} + v \cdot \nabla H_{12} = \epsilon(I_1 G_2 + I_2 G_1) + \epsilon(I_1 + I_2)H_{12} + \epsilon \int dx_3 I_3 K_{123} \quad (2.10)$$

We shall now proceed to reorder in order to facilitate the calculation in the correlation form and compare with Ramanathan and Sandri. The proper ordering for weak coupling is given by

$$G = G^0 + \epsilon G^1 = \epsilon g \quad (2.11)$$

$$H = H^0 + \epsilon H^1 + \epsilon^2 H^2 = \epsilon^2 h \quad (2.12)$$

$$K = K^0 + \epsilon K^1 + \epsilon^2 K^2 + \epsilon^3 K^3 = \epsilon^3 k \quad (2.13)$$

Inserting Eqs. (2.11)-(2.13) into Eqs. (2.8)-(2.10), we obtain

$$\frac{\partial F}{\partial t} = \epsilon^2 \int dx_1 I_1 g_1 \quad (2.14)$$

$$\frac{\partial g_1}{\partial t} + v \cdot \nabla g_1 = I_1 F + \epsilon I_1 g_1 + \epsilon^2 \int dx_2 I_2 h_{12} \quad (2.15)$$

$$\frac{\partial h_{12}}{\partial t} + v \cdot \nabla h_{12} = I_1 g_2 + I_2 g_1 + \epsilon(I_1 + I_2)h_{12} + \epsilon^2 \int dx_3 I_3 k_{123} \quad (2.16)$$

We now write the equations for the G formulae in successive powers of ϵ . We notice that the following equations correspond to successively higher powers of ϵ , the weak-coupling parameter. We find the following sets

$$\frac{\partial F^{10}}{\partial \tau_0} = 0 \quad (2.17)$$

$$\frac{\partial F^{11}}{\partial \tau_0} + \frac{\partial F^{10}}{\partial \tau_1} = 0 \quad (2.18)$$

$$\frac{\partial F^{12}}{\partial \tau_0} + \frac{\partial F^{11}}{\partial \tau_1} + \frac{\partial F^{10}}{\partial \tau_2} = \int dx_1 I_1 g_1^0 \quad (2.19)$$

$$\frac{\partial F^{13}}{\partial \tau_0} + \frac{\partial F^{12}}{\partial \tau_1} + \frac{\partial F^{11}}{\partial \tau_2} + \frac{\partial F^{10}}{\partial \tau_3} = \int dx_1 I_1 g_1^1 \quad (2.20)$$

$$\frac{\partial F^{14}}{\partial \tau_0} + \frac{\partial F^{13}}{\partial \tau_1} + \frac{\partial F^{12}}{\partial \tau_2} + \frac{\partial F^{11}}{\partial \tau_3} + \frac{\partial F^{10}}{\partial \tau_4} = \int dx_1 I_1 g_1^2 \quad (2.21)$$

In order to give a full comparison with Ramanathan and Sandri, we note the following simplification in the ordering

$$\begin{aligned}\frac{\partial F}{\partial t} &= \epsilon^2 \int dx_1 I_1 g_1 \\ &= \eta \int dx_1 I_1 g_1\end{aligned}\quad (2.22)$$

Ramanathan and Sandri then use η as a parameter. The time scales that they use are given by

$$\chi_0 = t \quad (2.23)$$

$$\chi_1 = \epsilon^2 t = \eta t \quad (2.24)$$

$$\chi_2 = \epsilon^4 t = \eta^2 t \quad (2.25)$$

The expansion of the unknown functions are given by

$$F = F^{(0)} + \eta F^{(1)} + \eta^2 F^{(2)} = F^{(0)} + \epsilon^2 F^{(1)} + \epsilon^4 F^{(2)} \quad (2.26)$$

$$g = g^{(0)} + \eta g^{(1)} + \eta^2 g^{(2)} = g^{(0)} + \epsilon^2 g^{(1)} + \epsilon^4 g^{(2)} \quad (2.27)$$

The equations that are obtained then are in successive powers of η rather than ϵ , namely

$$\frac{\partial F^{(0)}}{\partial \chi_0} = 0 \quad (2.28)$$

$$\frac{\partial F^{(1)}}{\partial \chi_0} + \frac{\partial F^{(0)}}{\partial \chi_1} = \int dx_1 I_1 g_1^{(0)} \quad (2.29)$$

$$\frac{\partial F^{(2)}}{\partial \chi_0} + \frac{\partial F^{(1)}}{\partial \chi_1} + \frac{\partial F^{(0)}}{\partial \chi_2} = \int dx_1 I_1 g_1^{(1)} \quad (2.30)$$

We notice that Eq. (2.28) corresponds to (2.17); equation (2.29) corresponds to (2.19), which is the second-order theory in weak-coupling expansion; Eq. (2.30) corresponds to Eq. (2.21) which is the fourth-order theory in the weak-coupling expansion. The work of Ramanathan and Sandri and the weak-coupling expansion will cease being the same after the ϵ^2 (or the η) term; namely, Eqs. (2.19) and (2.29) can be compared, but at the next level the equations will be different because the direct interaction term has been ordered in terms of η rather than ϵ . We shall call this ordering "unphysical ordering" for the purposes of the present discussion which is completely centered on the weak-coupling expansion. It does remain puzzling that the calculation of Ramanathan and Sandri fails. The interpretation that one might give to these results is that in view of an unphysical ordering of the direct interaction term, the model equations used by Ramanathan and Sandri do not have solutions. This rather strange possibility cannot be disregarded at the moment.

We now give a comparison of the weak-coupling scheme, as will be used in the hybrid calculation of this paper with the weak-coupling scheme that is used in the simple initial value problem. Equations (2.17)-(2.21) are rewritten

$$\frac{\partial F^{10}}{\partial \tau_0} = 0 \quad (2.31)$$

$$\frac{\partial F^{11}}{\partial \tau_0} + \frac{\partial F^0}{\partial \tau_1} = 0 \quad (2.32)$$

$$\frac{\partial F^{12}}{\partial \tau_0} + \cancel{\frac{\partial F^{11}}{\partial \tau_1}} + \frac{\partial F^{10}}{\partial \tau_2} = \int dx_1 I_1 g_1^0 \quad (2.33)$$

$$\frac{\partial F^{13}}{\partial \tau_0} + \cancel{\frac{\partial F^{12}}{\partial \tau_1}} + \cancel{\frac{\partial F^{11}}{\partial \tau_2}} + \frac{\partial F^{10}}{\partial \tau_3} = \int dx_1 I_1 g_1^1 \quad (2.34)$$

$$\frac{\partial F^{14}}{\partial \tau_0} + \cancel{\frac{\partial F^{13}}{\partial \tau_1}} + \frac{\partial F^{12}}{\partial \tau_2} + \frac{\partial F^{11}}{\partial \tau_3} + \frac{\partial F^{10}}{\partial \tau_4} = \int dx_1 I_1 g_1^2 \quad (2.35)$$

In these equations, the simple initial value problem eliminates the diagonal that contains F^{11} as indicated by the dashed line and also eliminates the entire column that is associated with time derivatives with respect to the τ_1 scale as also shown by the dashed line. We see that the second-order theory forces immediately the Fokker-Planck result and the third-order theory does not allow for any leeway in identifying the τ_3 derivative of the leading-order term. The purpose of the hybrid calculation is to relieve that excessive restriction. We notice that the calculation of Ramanathan and Sandri would be extremely similar to the simple initial value problem if there had not been a rearrangement in the ordering of the direct interaction term; in particular, the middle term of Eq. (2.30) would be identified with the middle term of Eq. (2.35) and similarly for the two side terms. Therefore, the theory of Ramanathan and Sandri is still within the scheme of the simple initial value problem.

III. THE TIME-SCALE EQUATIONS

We now set up the problem of the F formulation and first give the complete set of equations to be solved. For the one-body distribution function, we have the following set of equations

$$\frac{\partial F^{10}}{\partial \tau_0} = 0 \quad F^{10} = f(\vec{\tau}_1) \quad (3.1)$$

$$\frac{\partial F^{11}}{\partial \tau_0} + \frac{\partial F^{10}}{\partial \tau_1} = \int dx_1 I_1 F_1^{20} \quad (3.2)$$

$$\frac{\partial F^{12}}{\partial \tau_0} + \frac{\partial F^{11}}{\partial \tau_1} + \frac{\partial F^{10}}{\partial \tau_2} = \int dx_1 I_1 F_1^{21} \quad (3.3)$$

$$\frac{\partial F^{13}}{\partial \tau_0} + \frac{\partial F^{12}}{\partial \tau_1} + \frac{\partial F^{11}}{\partial \tau_2} + \frac{\partial F^{10}}{\partial \tau_3} = \int dx_1 I_1 F_1^{22} \quad (3.4)$$

$$\frac{\partial F^{14}}{\partial \tau_0} + \frac{\partial F^{13}}{\partial \tau_1} + \frac{\partial F^{12}}{\partial \tau_2} + \frac{\partial F^{11}}{\partial \tau_3} + \frac{\partial F^{10}}{\partial \tau_4} = \int dx_1 I_1 F_1^{23} \quad (3.5)$$

For the two-particle distribution function which includes an electron and an ion, the complete set of equations is

$$\frac{\partial F_1^{20}}{\partial \tau_0} + v \cdot \nabla F_1^{20} = 0 \quad (3.6)$$

$$\frac{\partial F_1^{20}}{\partial \tau_1} + \frac{\partial F_1^{21}}{\partial \tau_0} + v \cdot \nabla F_1^{21} = I_1 F_1^{20} + \int dx_2 I_2 F_{12}^{30} \quad (3.7)$$

$$\frac{\partial F_1^{20}}{\partial \tau_2} + \frac{\partial F_1^{21}}{\partial \tau_1} + \frac{\partial F_1^{22}}{\partial \tau_0} + v \cdot \nabla F_1^{22} = I_1 F_1^{21} + \int dx_2 I_2 F_{12}^{31} \quad (3.8)$$

$$\frac{\partial F_1^{20}}{\partial \tau_3} + \frac{\partial F_1^{21}}{\partial \tau_2} + \frac{\partial F_1^{22}}{\partial \tau_1} + \frac{\partial F_1^{23}}{\partial \tau_0} + v \cdot \nabla F_1^{23} = I_1 F_1^{22} + \int dx_2 I_2 F_{12}^{32} \quad (3.9)$$

Finally, the three-body distribution function, including an electron and two ions, is represented by

$$\frac{\partial F_{12}^{30}}{\partial \tau_0} + v \cdot \nabla F_{12}^{30} = 0 \quad (3.10)$$

$$\frac{\partial F_{12}^{30}}{\partial \tau_1} + \frac{\partial F_{12}^{31}}{\partial \tau_0} + v \cdot \nabla F_{12}^{31} = (I_1 + I_2) F_{12}^{30} + \int dx_3 I_3 F_{123}^{40} \quad (3.11)$$

$$\frac{\partial F_{12}^{30}}{\partial \tau_2} + \frac{\partial F_{12}^{31}}{\partial \tau_1} + \frac{\partial F_{12}^{32}}{\partial \tau_0} + v \cdot \nabla F_{12}^{32} = (I_1 + I_2) F_{12}^{31} + \int dx_3 I_3 F_{123}^{41} \quad (3.12)$$

In the next section, we shall study in detail these equations in successive orders in ϵ .

IV. THE SEMIADIABATIC CALCULATION

We now describe in detail the hybrid calculation, order-by-order.

A. The Zero-order Theory.

The leading order is readily discussed. We obtain

$$F^{10} = f(\vec{\tau}_1) \quad (4.1)$$

$$F^{s0} = f(\vec{\tau}_1) \quad (4.2)$$

B. The First-order Theory.

We use extensively the following lemma

$$\int dx I(x) f(\vec{\tau}_1) = 0 \quad (4.3)$$

We then have

$$\frac{\partial F^{11}}{\partial \tau_0} + \frac{\partial F^{10}}{\partial \tau_1} = 0 \quad (4.4)$$

Therefore, we conclude that

$$\frac{\partial F^{10}}{\partial \tau_1} = \frac{\partial f}{\partial \tau_1} = 0 \quad , \quad f = f(\vec{\tau}_2) \quad (4.5)$$

$$\frac{\partial F^{11}}{\partial \tau_0} = 0 \quad , \quad F^{11} = F^{11}(\vec{\tau}_1) \quad (4.6)$$

We can therefore write

$$\frac{\partial F^{s0}}{\partial \tau_1} = \frac{\partial f(\vec{\tau}_2)}{\partial \tau_1} \quad (4.7)$$

These equations are useful to establish the higher-order velocity distribution functions. The relevant equations are, in fact, using the lemma given above,

$$\frac{\partial F_1^{21}}{\partial \tau_0} + \mathbf{v} \cdot \nabla F_1^{21} = I_1 f \quad (4.8)$$

$$\frac{\partial F_{12}^{31}}{\partial \tau_0} + \mathbf{v} \cdot \nabla F_{12}^{31} = (I_1 + I_2) f \quad (4.9)$$

$$\frac{\partial F_{123}^{41}}{\partial \tau_0} + \mathbf{v} \cdot \nabla F_{123}^{41} = (I_1 + I_2 + I_3) f \quad (4.10)$$

The initial conditions to be applied can be visualized as follows. We consider the expansion in the electron-ion velocity distribution given by

$$F^2 = F^2(\tau_0, \tau_1, \vec{\tau}_2) = F^{20}(\vec{\tau}_0) + \epsilon F^{21}(\vec{\tau}_0) + \epsilon^2 F^{22}(\vec{\tau}_0) + O(\epsilon^3) \quad (4.11)$$

Equating order-by-order, we then obtain

$$F^2(0, 0, \vec{\tau}_2) = F^1(\vec{\tau}_2) \quad (4.12)$$

$$F^{20}(0,0,\vec{\tau}_2) = F^{10}(\vec{\tau}_2) = f(\vec{\tau}_2) \quad (4.13)$$

$$F^{21}(0,0,\vec{\tau}_2) = F^{11}(\vec{\tau}_2) \quad (4.14)$$

Therefore, we conclude (using the first-order equations)

$$F_1^{21}(\vec{\tau}_0) = F^{11} + \xi I_1 f(\vec{\tau}_2) \quad (4.15)$$

$$F_{12}^{31}(\vec{\tau}_0) = F^{11} + \xi(I_1 + I_2)f(\vec{\tau}_2) \quad (4.16)$$

$$F_{123}^{41}(\vec{\tau}_0) = F^{11} + \xi(I_1 + I_2 + I_3)f(\vec{\tau}_2) \quad (4.17)$$

C. Second-order Theory.

The second-order theory is somewhat more complicated. First of all, we obtain

$$\begin{aligned} \frac{\partial F^{12}}{\partial \tau_0} + \frac{\partial F^{11}}{\partial \tau_1} + \frac{\partial F^{10}}{\partial \tau_2} &= \int dx_1 I_1 F_1^{21} \\ &= \int dx_1 I_1 F^{11} + \int dx_1 I_1 \xi I_1 f(\vec{\tau}_2) \end{aligned} \quad (4.18)$$

This equation can be rewritten as

$$\begin{aligned} \frac{\partial F^{12}}{\partial \tau_0} &= - \frac{\partial F^{11}(\vec{\tau}_1)}{\partial \tau_1} - \frac{\partial f(\vec{\tau}_2)}{\partial \tau_2} + \int dx_1 I_1 \xi I_1 f(\vec{\tau}_2) \\ &\sim_{\tau_0 \uparrow} - \frac{\partial F^{11}(\vec{\tau}_1)}{\partial \tau_1} - \frac{\partial f(\vec{\tau}_2)}{\partial \tau_2} + \int dx_1 I_1 \xi^* I_1 f(\vec{\tau}_2) \end{aligned} \quad (4.19)$$

Therefore, we conclude that the transient is given by

$$\frac{\partial F^{12}}{\partial \tau_0} = - \int dx_1 I_1 \int_{\tau_0}^{\infty} I_1(\xi_1 - v\lambda) \cdot d\lambda f(\vec{\tau}_2) \quad (4.20)$$

while we are going to obtain for the rest

$$\begin{aligned} \frac{\partial F^{11}}{\partial \tau_1} &= - \frac{\partial f}{\partial \tau_2} + \int dx_1 I_1 \xi^* I_1 f(\vec{\tau}_2) \\ &\sim - \frac{\partial f}{\partial \tau_2} + \int dx_1 I_1 \xi^* I_1 f(\vec{\tau}_2) \end{aligned} \quad (4.21)$$

We conclude, therefore, that the correction satisfies

$$\frac{\partial F^{11}}{\partial \tau_1} = 0 \quad , \quad F^{11} = F^{11}(\vec{\tau}_2) \quad (4.22)$$

while the main part of the distribution satisfies

$$\frac{\partial f(\vec{\tau}_2)}{\partial \tau_2} = \int dx_1 I_1 \xi^* I_1 f(\vec{\tau}_2) \quad (4.23)$$

Quadrature then yields

$$F^{12}(\vec{\tau}_0) = F^{12}(0, \vec{\tau}_1) - \int_0^{\tau_0} d\mu \int dx_1 I_1 \int_{\mu}^{\infty} I(\xi_1 - v\lambda) d\lambda f(\vec{\tau}_2) \quad (4.24)$$

We now consider the second-order two-body velocity distribution function. This satisfies

$$\begin{aligned} \frac{\partial F_1^{22}}{\partial \tau_0} + \frac{\partial F_1^{21}}{\partial \tau_1} + \frac{\partial F_1^{20}}{\partial \tau_2} + v \cdot \nabla F_1^{22} &= I_1 F_1^{21} + \int dx_2 I_2 F_{12}^{31} \\ &= I_1 F^{11} + I_1 \xi I_1 f(\vec{\tau}_2) + \\ &+ \int dx_2 I_2 \xi (I_1 + I_2) f(\vec{\tau}_2) \end{aligned} \quad (4.25)$$

We calculate separately two terms given by the time scales as

$$\frac{\partial F^{21}}{\partial \tau_1} = \frac{\partial F^{11}}{\partial \tau_1} + \xi I_1 \frac{\partial f(\vec{\tau}_2)}{\partial \tau_1} = 0 \quad (4.26)$$

$$\frac{\partial F^{20}}{\partial \tau_2} = \frac{\partial f(\vec{\tau}_2)}{\partial \tau_2} = \int dx_1 I_1 \xi^* I_1 f(\vec{\tau}_2) \quad (4.27)$$

We then can write

$$\begin{aligned} \frac{\partial F_1^{22}}{\partial \tau_0} + L_1 \xi^* I_2 f(\vec{\tau}_2) + v \cdot \nabla F_1^{22} = \\ = I_1 F^{11}(\vec{\tau}_2) + I_1 \xi I_1 f(\vec{\tau}_2) + \\ + L_2 \xi I_2 f(\vec{\tau}_2) + L_2 \xi I_1 f(\vec{\tau}_2) \end{aligned} \quad (4.28)$$

We now notice the following lemma

$$\begin{aligned} L_2 \xi I_1 f(\vec{\tau}_2) &= \int dx_2 \nabla \phi(\xi_2) \cdot \frac{\partial}{\partial v} \int_0^{\tau_0} \nabla \phi(\xi_1 - v\lambda) d\lambda \cdot \frac{\partial f(\vec{\tau}_2)}{\partial \vec{v}} \\ &= 0 \end{aligned} \quad (4.29)$$

which is based on the simple fact

$$\int d\xi_2 \vec{\nabla} \phi(\xi_2) \equiv 0 \quad (4.30)$$

always valid for a spherically symmetric potential. We thus conclude that in second order the two-body velocity distribution is

given by

$$F_1^{22} = F_1^{12}(\tau_0, \vec{\tau}_1) + \zeta I_1 F_1^{11}(\vec{\tau}_2) + \zeta I_1 \zeta I_1 f(\vec{\tau}_2) + [L_2 \zeta I_2 - L_2 \zeta^* I_2] f(\vec{\tau}_2) \quad (4.31)$$

We now consider, still in second order, the three-body velocity distribution function. Two preliminary lemmas yield

$$\frac{\partial F^{30}}{\partial \tau_2} = \frac{\partial f(\vec{\tau}_2)}{\partial \tau_2} = L_2 \zeta^* I_2 f(\vec{\tau}_2) \quad (4.32)$$

$$\frac{\partial F^{31}}{\partial \tau_1} = \frac{\partial F^{11}(\vec{\tau}_2)}{\partial \tau_1} + \zeta(I_1 + I_2) \frac{\partial f(\vec{\tau}_2)}{\partial \tau_1} = 0 \quad (4.33)$$

We can then write the equation for the three-body velocity distribution function as

$$\begin{aligned} \frac{\partial F_{12}^{33}}{\partial \tau_0} + v \cdot \nabla F_{12}^{32} &= - L_2 \zeta^* I_2 f(\vec{\tau}_2) \\ &+ (I_1 + I_2) F_1^{11}(\vec{\tau}_2) + (I_1 + I_2) \zeta(I_1 + I_2) f(\vec{\tau}_2) \\ &+ \int dx_3 I_3 \zeta(I_1 + I_2 + I_3) f(\vec{\tau}_2) \\ &= - L_2 \zeta^* I_2 f(\vec{\tau}_2) + I_{12} F_1^{11}(\vec{\tau}_2) + I_{12} \zeta I_{12} f(\vec{\tau}_2) \\ &+ L_3 \zeta I_3 f(\vec{\tau}_2) \end{aligned} \quad (4.34)$$

We conclude that asymptotically for large values of τ_0 , we have two main second-order results, namely, the two- and three-body velocity distributions as

$$F_1^{22} \sim F_1^{12*} + \zeta^* I_1 F_1^{11}(\vec{\tau}_2) + \zeta^* I_1 \zeta^* I_1 f(\vec{\tau}_2) \quad (4.35)$$

$$F_{12}^{32} \sim F^{12*} + \xi^* I_{12} F^{11}(\vec{\tau}_2) + \xi^* I_{12} \xi^* I_{12} f(\vec{\tau}_2) \quad (4.36)$$

We notice that the simple initial value problem would omit the two first terms in each of these expressions. We remark that we have made some asymptotic assumptions that are unproved and should be proved for short-range interactions. The three statements are

$$L_2(\xi I_2 - \xi^* I_2) f(\vec{\tau}_2) \xrightarrow{\tau_0 \rightarrow \infty} 0 \quad (4.37)$$

$$\xi I_1 \xi I_1 f(\vec{\tau}_2) \xrightarrow{\tau_0 \rightarrow \infty} \xi^* I_1 \xi^* I_1 f(\vec{\tau}_2) \quad (4.38)$$

$$\xi I_{12} \xi I_{12} f(\vec{\tau}_2) \xrightarrow{\tau_0 \rightarrow \infty} \xi^* I_{12} \xi^* I_{12} f(\vec{\tau}_2) \quad (4.39)$$

D. Third-order Theory.

First of all, we consider the one-body velocity distribution function. This satisfies

$$\begin{aligned} \frac{\partial F^{13}}{\partial \tau_0} + \frac{\partial F^{12}}{\partial \tau_1} + \frac{\partial F^{11}(\vec{\tau}_2)}{\partial \tau_2} + \frac{\partial f(\vec{\tau}_2)}{\partial \tau_3} &= \int dx_1 I_1 \xi I_1 F^{11}(\vec{\tau}_2) \\ &+ \int dx_1 I_1 \xi I_1 \xi I_1 f(\vec{\tau}_2) \\ &+ \int dx_1 I_1 L_2 [\xi I_2 - \xi^* I_2] f(\vec{\tau}_2) \end{aligned} \quad (4.40)$$

For large τ_0 , we can write

$$\frac{\partial F^{12*}}{\partial \tau_1} + \frac{\partial F^{11}(\vec{\tau}_2)}{\partial \tau_2} + \frac{\partial f(\vec{\tau}_2)}{\partial \tau_3} = \int dx_1 I_1 \xi^* I_1 F^{11}(\vec{\tau}_2) + \int dx_1 I_1 \xi^* I_1 \xi^* I_1 f(\vec{\tau}_2) \quad (4.41)$$

as well as

$$\frac{\partial F^{12*}}{\partial \tau_1} = 0 \quad (4.42)$$

We thus conclude for the transient

$$\begin{aligned} \frac{\partial F^{13}}{\partial \tau_0} = & \int dx_1 I_1 (\xi - \xi^*) I_1 F_{11}(\vec{\tau}_2) \\ & + \int dx_1 I_1 (\xi I_1 \xi - \xi^* I_1 \xi^*) I_1 f(\vec{\tau}_2) \\ & + \int dx_1 I_1 L_2 (\xi - \xi^*) I_2 f(\vec{\tau}_2) \end{aligned} \quad (4.43)$$

We are now interested in the decomposition corresponding to Eq. (4.41). We notice that since for large τ_1 several quantities are τ_1 independent, we can write

$$\frac{\partial F^{11}(\vec{\tau}_2)}{\partial \tau_2} + \frac{\partial f(\vec{\tau}_2)}{\partial \tau_3} = c_1 F^{11}(\vec{\tau}_2) + \int L dx I \xi^* I \xi^* I f(\vec{\tau}_2) \quad (4.44)$$

We now notice that the collision integral on the right-hand side of (4.44) has the form C_2 of the Ramanathan-Sandri theory. We cannot therefore assume that this term is associated with the τ_3 derivative of f . We rather are forced to set

$$\frac{\partial F^{11}}{\partial \tau_2} = c_1 F^{11} + c_2 f(\vec{\tau}_2) \quad (4.45)$$

$$\frac{\partial f(\vec{\tau}_2)}{\partial \tau_3} = 0 \quad (4.46)$$

Equation (4.46) is very similar to the closure proposed in the old "Foundation paper." We now notice that this kinetic description is compatible with the approach to equilibrium of the system.

In fact, we have from the Fokker-Planck equation

$$f \xrightarrow{\tau_2 \uparrow} \langle f \rangle \quad (4.47)$$

Furthermore, a general property guarantees

$$c_2 \langle f \rangle = 0 \quad (4.48)$$

We can therefore conclude that

$$F^{11} \xrightarrow{\tau_2 \uparrow} \langle F^{11} \rangle \quad (4.49)$$

Since we have the sum of the distribution functions as defined by their expansion, we conclude

$$F^1 = F^{10} + \epsilon F^{11} = f + \epsilon F^{11} \xrightarrow{\tau_2 \uparrow} \langle F^1 \rangle \quad (4.50)$$

This shows that the third-order theory does seem to be compatible with the approach to equilibrium of the velocity distribution function.

We now consider the two-body velocity distribution in third order. We begin with three calculational lemmas which are given by

$$\frac{\partial F^{20}}{\partial \tau_3} = \frac{\partial f}{\partial \tau_3} = 0 \quad (4.51)$$

$$\frac{\partial F_1^{21}}{\partial \tau_2} = \frac{\partial F^{11}}{\partial \tau_2} + \xi I_1 \frac{\partial f}{\partial \tau_2} = c_1 F^{11} + c_2 f + \xi I_1 c_1 f \quad (4.52)$$

$$\frac{\partial F^{22}}{\partial \tau_1} = \frac{\partial F^{12}}{\partial \tau_1} + 0 \underset{\tau_0 \uparrow}{\sim} 0 \quad (4.53)$$

The third-order two-body equation then can be written as

$$\begin{aligned}
 \frac{\partial F_1^{23}}{\partial \tau_0} + v \cdot \nabla F_1^{23} = & I_1 F^{12} + I_1 \xi I_1 F^{11} + I_1 \xi I_1 \xi I_1 f(\tau_2) \\
 & + I_1 L_2 (\xi - \xi^*) I_2 f + \\
 & + \int dx_2 I_2 \left\{ F^{12} + \xi I_{12} F^{11} + \xi I_{12} \xi I_{12} f + \right. \\
 & \left. + L_3 (\xi - \xi^*) I_{3f} \right\} - C_1 F^{11} - C_2 f - \xi I_1 C_1 f \quad (4.54)
 \end{aligned}$$

For large values of τ_0 we can therefore write

$$\begin{aligned}
 \frac{\partial F_1^{23}}{\partial \tau_0} + v \cdot \nabla F_1^{23} \underset{\tau_0 \uparrow}{\sim} & I_1 F^{12*} + I_1 \xi^* I_1 F^{11} + I (\xi^* I)^2 f + \\
 & + \int dx_2 I_2 \xi^* (I_1 \xi^* I_2 + I_2 \xi^* I_1) f - \xi^* I_1 C_1 f \quad (4.55)
 \end{aligned}$$

We notice that we have obtained an expression that is considerably more complicated than the usual adiabatic expression. Furthermore, the supersecular term that occurs in the initial value problem is the last one in Eq. (4.55). This term would lead to a divergence because of a product of ξ^* upon integration of Eq. (4.55). We here invoke heuristically the nonadiabatic treatment of the correlation, remembering that we are discussing the electron-ion correlation in third order. The ξ^* that appears in Eq. (4.55) is going to be smoothed by the nonadiabatic term; and, furthermore, the integration of it through Eq. (4.55) itself would also be smoothed. We expect, therefore, to have a regular function for F^{23} once the adiabatic treatment is invoked. The problem that remains is that of completely splitting the one-body equation, using this two-body distribution function. We remember that we have

$$\frac{\partial F^{11}}{\partial \tau_2} = C_1 F^{11} + C_2 f \quad (4.56)$$

where we have rewritten

$$C_2 = L(\xi^* I)^2 f \quad (4.57)$$

We notice that because f satisfies the Fokker-Planck equation, the detailed behavior in the decomposition of the one-body equation can be carried out; in particular, we have

$$f(\tau_2) = \sum_{\ell m} y_{\ell m} f_{\ell m}(v) e^{-\lambda_{\ell m} \tau_2} \quad (4.58)$$

We are now in a position to propose a number of possibilities. It is, for example, likely that

$$\frac{\partial F^{12}}{\partial \tau_2} = C_1 F^{12} + C_2 F^{11} + C_3 f \quad (4.59)$$

where we have set

$$C_3 = L(\xi^* I)^3 \quad (4.60)$$

together with

$$\frac{\partial F^{13}}{\partial \tau_1} = 0 \quad (4.61)$$

This would leave for the last two functions which are unknown

$$\begin{aligned} \frac{\partial F^{11}}{\partial \tau_3} + \frac{\partial f}{\partial \tau_4} &= K_2 f - L_1 \xi^* \xi^* I_1 C_1 f = C_4 f \\ &= \int dx_1 I_1 \xi^* f dx_2 I_2 \xi^* (I_1 \xi^* I_2 + I_2 \xi^* I_1) f \\ &\quad - \int dx_1 I_1 \xi^* \xi^* I_1 \int dx_2 I_2 \xi^* I_2 f \end{aligned} \quad (4.62)$$

The major open problem is that of determining simultaneous compatibility of the equations and, at the same time, guaranteeing that the entropy principle follows. We notice that a very tempting set of equations is unsatisfactory; in particular, if we set

$$\frac{\partial F^{11}}{\partial \tau_2} = C_1 F^{11} + C_2 f \quad (4.63)$$

$$\frac{\partial F^{11}}{\partial \tau_3} = C_4 f \quad , \quad \frac{\partial f}{\partial \tau_3} = 0 \quad (4.64)$$

we would find upon cross differentiation

$$\frac{\partial^2 F^{11}}{\partial \tau_3 \partial \tau_2} = C_1 C_4 f \quad (4.65)$$

$$\frac{\partial^2 F^{11}}{\partial \tau_2 \partial \tau_3} = C_4 C_1 f \quad (4.66)$$

These equations will become compatible if and only if

$$[C_4, C_1] = 0 \quad (4.67)$$

In order that this last equation be correct, a calculation of the commutator must be carried out along the lines of the Ramanathan-Sandri theory. Until that is carried out, we cannot properly split the one-body equation. It is fair to conclude that the hybrid calculation leaves open the hope that a splitting which is compatible can be achieved. It is much harder to hope that the resulting collision integral would have the definiteness that is required by the entropy theorem. However, we see that the splitting suggested in Eqs. (4.56) and (4.59) tends to avoid the definiteness of the successive collision integrals because it is sufficient to know the behavior given in (4.58) in order to guarantee that the solutions of (4.56) and (4.59) will reach zero

after sufficiently long times. We notice that they will reach zero on the τ_2 scale which seems rather satisfactory. The role of C^4 remains still to be determined. It is not inconceivable to insert C_4 , together with C_3 , into Eq. (4.59). This would satisfy the H-theorem, though the relaxation time would not be affected directly.

APPENDIX A. SMALL MOMENTUM TRANSFER IN THREE-BODY COLLISIONS

In Fourier space ($\phi_i = \phi(k_i)$) we have

$$\frac{\partial h_{12}}{\partial t} + i(k_1 + k_2) \cdot v h_{12} = \phi_1 i k_1 \cdot \frac{\partial}{\partial v} g_2 + \phi_2 i k_2 \cdot \frac{\partial}{\partial v} g_1 \quad (A1)$$

$$g_i = \frac{1}{i k_i \cdot v + \epsilon} \phi_i i k_i \cdot \frac{\partial F}{\partial v} \quad (A2)$$

Then the correction to g due to h is

$$\delta g_1 = \int dk_2 \phi_2 i k_2 \cdot \frac{\partial}{\partial v} h_{12} \quad (A3)$$

and the contribution to the collision integral (three-body leading term in momentum transfer) is

$$k_2 F = \int dk_1 dk_2 J_{12} \frac{1}{i(k_1 + k_2) \cdot v_+} J_{12}^+ F \quad (A4)$$

$$J_{12} = \left[i k_1 \cdot \frac{\partial}{\partial v} \frac{1}{i k_2 \cdot v_+} i k_2 \cdot \frac{\partial}{\partial v} + 1 \longleftrightarrow 2 \right] \phi_1 \phi_2 \quad (A5)$$

This would be positive definite (i.e., there would be an H-theorem) if the h propagator were $\delta((k_1 + k_2) \cdot v)$ without the $\mathcal{P}$ part. This δ seems to follow naturally in the nonadiabatic treatment (e.g., the Appendix of J. Math. Phys., "Model for the Derivation of Kinetic Theory.")

APPENDIX B. ADIABATIC TREATMENT WITH VANISHING INITIAL PERTURBATIONS

We examine the adiabatic treatment in the g formalism.

It is important to study carefully the Lorentz gas because:
 (1) the kinetic equation can be solved (being linear); (2) the transients and correlations can be calculated explicitly;
 (3) therefore, it must be possible to go beyond lowest order.

A point brought to clear light by the calculation of Klimas and Sandri on the matrix exponential is the need to pay careful attention to initial conditions. In the kinetic theory calculation, this has never been done to full satisfaction. In this Appendix, we will consider what we call the "very simple initial value problem"; other initial value problems will be considered later. We consider the following condition

$$f^k(\tau_0=0) = 0 \quad , \quad k > 0 \quad (B1)$$

$$g(\tau_0=0) = 0 \quad (B2)$$

We shall study the theory order-by-order. The equations of relevance are (2.14), (2.15) and (2.16) of Section II.

Zero order. We determine the following quantities

$$f^0(\tau_0) \quad , \quad g^0(\tau_0) \quad , \quad h^0(\tau_0) \quad (B3)$$

The results are the following

$$\frac{\partial f^0}{\partial \tau_0} = 0 \quad (B4)$$

$$g_1^0 = \zeta I_1 f^0 \quad (B5)$$

$$h_{12}^0 = \zeta (I_1 \zeta I_2 + I_2 \zeta I_1) f^0 \quad (B6)$$

First-order theory. We determine the following quantities

$$f^0(\tau_1) , \quad f^1 , \quad g^1(\tau_0) \quad (B7)$$

The results are the following

$$\frac{\partial f^0}{\partial \tau} = 0 \quad (B8)$$

$$f^1 = 0 \quad (B9)$$

$$g^1 = \xi I \xi I f^0 \quad (B10)$$

Second-order theory. We determine the following three quantities

$$f^0(\tau_2) , \quad f^2 , \quad g^2(\tau_0) \quad (B11)$$

The kinetic condition is given by

$$\frac{\partial f^0}{\partial \tau_2} = L \xi^* I f^0 = L_2 f^0 = \frac{\alpha}{v^3} \hat{v}^2 f^0 \quad (B12)$$

The transient is given by

$$\begin{aligned} \frac{\partial f^2}{\partial \tau_0} &= L \left(\xi_{\tau_0} - \xi^* \right) I f^0 \\ &= \int dx I (\xi - \xi^*) I f^0 \end{aligned} \quad (B13)$$

Quadrature of (B13) yields

$$\begin{aligned}
\psi^2 &= \int_0^{\tau_0} \delta\lambda L(\xi_\lambda - \xi^*) I f^0 \\
&= \xi L(\xi - \xi^*) I f^0
\end{aligned} \tag{B14}$$

We now calculate g^2 . We need three lemmas and they are given by

$$\begin{aligned}
-\frac{\partial g_1^0}{\partial \tau_2} &= -\xi I_1 \frac{\partial f^0}{\partial \tau_2} \\
&= -\xi I_1 \int dx_2 I_2 \xi^* I_2 f^0
\end{aligned} \tag{B15}$$

$$I f^2 = I \int_0^{\tau_0} d\lambda L(\xi_\lambda - \xi^*) I f^0 \tag{B16}$$

$$\int dx_2 I_2 h_{12}^0 = \int dx_2 I_2 \xi (I_1 \xi I_2 + I_2 \xi I_1) f^0 \tag{B17}$$

The result for g^2 is then

$$\begin{aligned}
g^2 &= -\xi^* \xi^* I_1 C_2 f^0 \\
&+ \xi^* I \int_0^\infty d\lambda L(\xi_\lambda - \xi^*) I f^0 \\
&+ \xi^* \int dx_2 I_2 \xi^* (I_1 \xi^* I_2 + I_2 \xi^* I_1) f^0
\end{aligned} \tag{B18}$$

Third-order theory. We determine two quantities

$$f^0(\tau_3), \quad f^3 \tag{B19}$$

The one-body equation reduces to

$$\frac{\partial f^0}{\partial \tau_3} + \frac{\partial f^3}{\partial \tau_0} = \int dx I (\xi I)^2 f^0 = \int dx I \xi I \xi I f^0 \tag{B20}$$

Therefore, we obtain

$$\frac{\partial f^0}{\partial \tau_3} = \int dx I(\xi^* I)^2 f^0 = c_3 f^0 \quad (B21)$$

$$\frac{\partial f^3}{\partial \tau_0} = \int dx I(\xi I \xi - \xi^* I \xi^*) I f^0 \quad (B22)$$

We can perform formal quadratures to yield

$$f^0 = e^{c_2 \tau_2 + c_3 \tau_3} f^0(\tau_4) \quad (B23)$$

$$f^3 = \xi L(\xi I \xi - \xi^* I \xi^*) I f^0 \quad (B24)$$

Fourth-order theory. We determine two quantities

$$f^0(\tau_4) \quad , \quad f^4 \quad (B25)$$

We notice that

$$\frac{\partial f^2}{\partial \tau_2} = \xi L(\xi - \xi^*) I L \xi^* I f^0 \quad (B26)$$

We then conclude

$$\begin{aligned} \frac{\partial f^0}{\partial \tau_4} + \frac{\partial f^4}{\partial \tau_0} = & - \xi L(\xi - \xi^*) I L \xi^* I f^0 \\ & + L \xi^* I \xi^* L(\xi - \xi^*) I f^0 \\ & - L \xi^{*2} I c_2 f^0 \\ & + K_2 f^0 \end{aligned} \quad (B27)$$

APPENDIX C. ADIABATIC TREATMENT

We use the g formalism to discuss a modification of the adiabatic treatment given in Appendix B. Three major facts enter the successful construction of higher-order equations: (1) the use of proper conditions; in fact, neglect of the initial conditions is disastrous in simple matrix exponential examples; (2) a technique that allows for a proper splitting off of the secular part of a perturbation. We shall call the construction given by Klimas and Sandri the "Klimas-Sandri construction;" and (3) the use of nonadiabatic correlations which allows, through an understanding of their wave nature and finite range, correct decomposition of the secular part of the perturbation.

The modified expansion will be based on the following choices. We assume

$$f^k \xrightarrow{\tau_0 \rightarrow \infty} 0, \quad k > 0 \quad (C1)$$

$$g(\tau_0=0) = 0 \quad (C2)$$

The problem studied, therefore, is very similar to the simple initial value problem; however, the perturbations in the distribution are assumed to disappear for times long compared to the duration of a collision. Assumption (C1) is a boundary condition which resembles very closely the boundary condition used by Enskog to obtain an expansion of the Boltzmann equation that leads to the hydrodynamic equations. This boundary condition has to be compared to Eq. (B1); the different physics in the two choices corresponds to the fact that we normalize the velocity distribution in Appendix B in such a way that initially it is all put into the zero order; while in Appendix C, the Enskog approach, we normalize it so that it is all put in the zero order at the onset of the true kinetic regime.

Zero order. We determine three quantities

$$f^0(\tau_0) , \quad g^0 , \quad h^0 \quad (C3)$$

The results are identical to the results of the simple initial value problem given in Appendix B; they are

$$\frac{\partial f^0}{\partial \tau_0} = 0 \quad (C4)$$

$$g^0 = \xi I f^0 \quad (C5)$$

$$h_{12}^0 = \xi (I_1 \xi I_2 + I_2 \xi I_1) f^0 \quad (C6)$$

First-order theory. We determine three quantities

$$f^0(\tau_1) , \quad f^1 , \quad g^1 \quad (C7)$$

Again, we have a situation very similar to that of Appendix B. We obtain

$$\frac{\partial f^0}{\partial \tau_1} = 0 \quad (C8)$$

$$f^1 = 0 \quad (C9)$$

$$g^1 = \xi I \xi I f^0 = (\xi I)^2 f^0 \quad (C10)$$

We see that because of the simplicity of first-order theory, at this stage of the calculation, choosing the Enskog boundary condition or the simple initial value condition on the perturbations of the velocity distribution is equivalent.

Second order. The second-order equation is given by

$$\frac{\partial f^0}{\partial \tau_2} + \frac{\partial f^2}{\partial \tau_0} = \int dx I \xi I f^0 \quad (C11)$$

The equation is decomposed as

$$\frac{\partial f^0}{\partial \tau_2} = \int dx I \xi^* I f^0 = c_2 f^0 \quad (C12)$$

$$\frac{\partial f^2}{\partial \tau_0} = - \int_{\tau_0}^{\infty} d\lambda \int dx I(x) I(x-v\lambda) f^0 \quad (C13)$$

where we have introduced the notation

$$c_2 = \int_0^{\infty} d\lambda K(\lambda) \quad (C14)$$

We also have a useful representation for the kernel of the collision integral; namely,

$$K(\lambda) = \int dx I(x) I(x-v\lambda) \quad (C15)$$

This kernel will also allow us to express a transient, f^2 , in a useful way. Integration of Eq. (C13) yields

$$\begin{aligned} f^2 &= f^2(0) - \int_0^{\tau_0} d\mu \int_{\mu}^{\infty} d\lambda \int dx I(x) I(x-v\lambda) f^0 \\ &= f^2(0) - \int_0^{\tau_0} d\mu \int_{\mu}^{\infty} d\lambda K(\lambda) f^0 \end{aligned} \quad (C16)$$

By means of an integration by parts, we can put Eq. (C16) in a very neat form

$$f^2 = f^2(0) - \tau_0 \int_{\tau_0}^{\infty} d\lambda K(\lambda) f^0 - \int_0^{\tau_0} \lambda d\lambda K(\lambda) f^0 \quad (C17)$$

The integration by parts is discussed as follows. We consider the quantity

$$J = \int_0^{\tau_0} d\mu \int_{\mu}^{\infty} d\lambda K(\lambda) \quad (C18)$$

We now notice that we have

$$u = \int_{\mu}^{\infty} d\lambda K(\lambda) \quad (C19)$$

$$du = -K(\mu) d\mu \quad (C20)$$

For J , we can therefore write the following

$$\begin{aligned} J &= \int_0^{\tau_0} dv u = uv \Big|_0^{\tau_0} - \int_0^{\tau_0} v du \\ &= \mu \int_{\mu}^{\infty} d\lambda K(\lambda) \Big|_0^{\tau_0} + \int_0^{\tau_0} \mu K(\mu) d\mu \\ &= \tau_0 \int_{\tau_0}^{\infty} d\lambda K(\lambda) + \int_0^{\tau_0} \lambda d\lambda K(\lambda) \end{aligned} \quad (C21)$$

Inserting the result into Eq. (C16) proves Eq. (C17).

The usefulness of the expression given by Eq. (C17) lies in the fact that the second term in Eq. (C17) can be shown to vanish for large values of τ_0 independently of the potential, of course, assuming that the potential has a finite range. The proof of this useful lemma is as follows. We consider the quantity

$$M = \lim_{\tau_0 \rightarrow \infty} \tau_0 \int_{\tau_0}^{\infty} d\lambda K(\lambda) f^0 \quad (C22)$$

The limit now will be studied with L'Hospital's rule; we find successively

$$\begin{aligned}
M &= \lim_{\tau_0 \rightarrow \infty} \frac{1}{\tau_0 - 1} \int_{\tau_0}^{\infty} d\lambda K(\lambda) f^0 \\
&= \lim_{\tau_0 \rightarrow \infty} \frac{-K(\tau_0) f^0}{-\frac{1}{\tau_0^2}} \\
&= \lim_{\tau_0 \rightarrow \infty} \tau_0^2 K(\tau_0) f^0 \\
&= \lim_{\tau_0 \rightarrow \infty} \tau_0^2 \int dx I(x) I(x - v\tau_0) f^0 = 0 \quad (C23)
\end{aligned}$$

The last statement is made because the interaction I cuts off with a finite range of zero; therefore, the overlap indicated in the integral will go to zero rapidly for finite velocities. It may be worth remarking that in this theorem, as in most theorems used in weak coupling, finite velocity is needed so that the theory is not fully uniform in velocity space. Returning to Eq. (C17), we can write the asymptotic form of the f^2 function as

$$f^2 \underset{\tau_0}{\sim} f^2(0) - \int_0^{\infty} \lambda d\lambda K(\lambda) f^0 \quad (C24)$$

With the help of the notation

$$C' = \int_0^{\infty} \lambda d\lambda K(\lambda) \quad (C25)$$

we then have

$$f^2(0) = C' f^0 \quad (C26)$$

The operator C' is a second-order operator with regards to velocity derivatives in B just as the Fokker-Planck operator.

It does not have constant coefficients and it depends on the potential through the overlap integral which is the same that appears in the Fokker-Planck quantity C^2 . We thus see that in second-order theory we can express, by means of the single kernel K that appears in Eq. (C25), both the kinetic part and the transient part of the velocity distribution function.

In order to impose the Enskog boundary condition which is what we call the modified initial value problem, we choose

$$f^2(0) = C' f^0 \quad (C27)$$

We then clearly have

$$f^2 \underset{\tau_0 \uparrow}{\sim} 0 \quad (C28)$$

in agreement with the postulated boundary condition, Eq. (C1). The final expression in the modified simple initial value problem for the transient is then given by

$$f^2 = \int_{\tau_0}^{\infty} \lambda d\lambda K(\lambda) f^0 - \tau_0 \int_{\tau_0}^{\infty} d\lambda K(\lambda) f^0 \quad (C29)$$

This quantity has been studied carefully by Ramanathan and Sandri. We shall see later why they did not succeed in obtaining compatible equations. The reasons are: (1) neglect of the initial conditions of f^1 and (2) a lack of a technique for properly splitting the second part of the perturbation later developed by Klimas and Sandri.

We now calculate the quantity g^2 ; we need two simple lemmas, given by

$$\frac{\partial g^1}{\partial \tau_1} = 0 \quad (C30)$$

$$\frac{\partial g^0}{\partial \tau_2} = \xi_I \frac{\partial f^0}{\partial \tau_2} = \xi_{IC_2} f^0 \quad (C31)$$

The assumption will be made that

$$\xi I f^2 \xrightarrow{\tau_0} 0 \quad (C32)$$

We then obtain

$$g^2 \sim (\xi^* I)^3 f^0 + \int dx_2 I_2 \xi^* (I_1 \xi^* I_2 + I_2 \xi^* I_1) f^0 - \xi^{*2} I C_2 f^0 \quad (C33)$$

We shall use this expression to obtain the third-order theory in the modified adiabatic treatment.

Third-order theory. We obtain readily

$$\frac{\partial f^1}{\partial \tau_2} = 0, \quad \frac{\partial f^2}{\partial \tau_1} = 0 \quad (C34)$$

We then have for the third-order equation

$$\frac{\partial f^0}{\partial \tau_3} + \frac{\partial f^3}{\partial \tau_0} = \int dx I (\xi I)^2 f^0 \quad (C35)$$

We split the equations as follows

$$\frac{\partial f^0}{\partial \tau_3} = c_3 f^0 \quad (C36)$$

$$\frac{\partial f^3}{\partial \tau_0} = \int dx I (\xi I \xi - \xi^* I \xi^*) I f^0 \quad (C37)$$

with

$$c_3 = \int dx I (\xi^* I)^2 \quad (C38)$$

The modified adiabatic treatment is forced to lump the entire asymptotic behavior of the operator Γ^3 on f^0 because of the first of equations (C34). This is one of the critical points in

the new treatment; release of a boundary condition that destroys f^1 is essential to have the freedom of separating the operator Γ^3 to a proper C_3 part and Δ_3 . Since C_3 does not commute with C_2 , Eq. (C36) is suspect and, in fact, incorrect.

Fourth-order theory. In fourth order, we again readily obtain, with the modified adiabatic treatment,

$$\frac{\partial f^1}{\partial \tau_3} = 0, \quad \frac{\partial f^3}{\partial \tau_1} = 0 \quad (C39)$$

We also have

$$\frac{\partial f^2}{\partial \tau_2} \tau_0 = 0 \quad (C40)$$

from inspection of Eq. (C29). The fourth-order theory then resembles, in the modified adiabatic treatment, the third-order theory because we have to split the perturbation term into two parts; namely, we have

$$\frac{\partial f^0}{\partial \tau_4} + \frac{\partial f^4}{\partial \tau_0} = \int dx I g^2 \quad (C41)$$

It is clearly forced upon us to choose

$$\frac{\partial f^0}{\partial \tau_4} = \int dx I g^{2*} = C_4 f^0 \quad (C42)$$

if we do not want to have f^4 simply secular. C_4 is then readily calculated from the expression (C33) to be

$$C_4 = \int dx I (\xi^* I)^3 + K_2 - L \xi^{*2} I C_2 \quad (C43)$$

This expression is designed to show the connection with the work carried out with Goldberg because K^2 is precisely the collision integral proposed by Goldberg and Sandri. The modification given by the first term in Eq. (C43) would be a correction to the small

momentum transfer in the two-body effects, while the last term in (C43) is simply divergent. It is the supersecular term that appears in leading order of the weakly-coupled expansion. Equation (C42), besides being incompatible with (C38) and (C12), defines an ill-defined integral which is the last term in Eq. (C43).

APPENDIX D. LORENTZ GAS TO THIRD ORDER IN THE SEMIADIABATIC TREATMENT WITH g FORMALISM

We now discuss the semiadiabatic treatment to third order in the g formalism, rather than the f formalism, in order to have a clearer comparison of terms. It is worth remarking that the semiadiabatic treatment to third order will actually coincide with the full nonadiabatic treatment. It is, therefore, the first testing ground of the new type of calculation proposed in this work.

Zero order. In zero order, we have

$$\frac{\partial f^0}{\partial \tau_0} = 0 \quad (D1)$$

The two-particle velocity distribution function satisfies the incompressible continuity equation with source

$$\frac{\partial g^0}{\partial \tau_0} + \mathbf{v} \cdot \nabla g^0 = I f^0 \quad (D2)$$

A solution can be written in various forms; for example, we can write

$$\begin{aligned} g^0(\mathbf{x}, \tau_0) &= \int_0^{\tau_0} d\lambda I(\mathbf{x} - \mathbf{v}\lambda) f^0 \\ &= \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{v}\lambda) \cdot \frac{\partial f^0}{\partial \mathbf{v}} \end{aligned} \quad (D3)$$

First-order theory. In first order, we have

$$\frac{\partial f^0}{\partial \tau_1} + \frac{\partial f^1}{\partial \tau_0} = 0 \quad (D4)$$

Integration of this equation with respect to τ_0 gives

$$f_1 = -\tau_0 \frac{\partial f^0}{\partial \tau_1} + f^1(0) \quad (D5)$$

In order to avoid a secularity in the perturbation, f_1 , we set

$$\frac{\partial f^0}{\partial \tau_1} = 0 \quad (D6)$$

We thus conclude

$$f_1 = f_1(\vec{\tau}_1) \quad (D7)$$

We shall indicate with a vector argument a function that depends on all the higher scales. Equation (D7) is crucial to allow for the proper splitting of the third-order equation. In other words, we shall not adopt any of the boundary conditions used in Appendices B and C; we will, however, keep zero initial conditions on the correlation function, as indicated in Eq. (D3) above. The first-order equation yields

$$\frac{\partial g^1}{\partial \tau_0} + v \cdot \frac{\partial g^1}{\partial \vec{x}} = I f^1 + I g^0 - \frac{\partial g^0}{\partial \tau_1} \quad (D8)$$

Using Eq. (D3), we conclude

$$\frac{\partial g^0}{\partial \tau_1} = 0 \quad (D9)$$

We can carry out the integration by noting

$$If^1 = If^1(\tau_1) \quad (D10)$$

$$\begin{aligned} Ig^0 &= I \int_0^{\tau_0} d\lambda I(x-v\lambda) f^0 \\ &= \frac{\partial \phi}{\partial x} - \frac{\partial}{\partial v} \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \frac{\partial f^0}{\partial v} \quad (D11) \end{aligned}$$

Then our final result for the first-order correlation function is

$$g^1 = \int_0^{\tau_0} d\lambda I(x-v\lambda) f^1 + \int_0^{\tau_0} d\lambda e^{-k(\tau_0-\lambda)} I \int_0^{\lambda} d\lambda^1 \frac{\partial \phi}{\partial x}(x-v\lambda^1) \cdot \frac{\partial f^0}{\partial v} \quad (D12)$$

Second-order theory. The complete second-order equation can be written as

$$\begin{aligned} \frac{\partial f^0}{\partial \tau_2} + \frac{\partial f^1}{\partial \tau_1} + \frac{\partial f^2}{\partial \tau_0} &= \int dx Ig^0 \\ &= \int dx I \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \frac{\partial f^0}{\partial v} \\ &= \int_0^{\tau_0} d\lambda \int dx \frac{\partial \phi}{\partial x}(x) - \frac{\partial}{\partial v} \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \frac{\partial f^0}{\partial v} \\ &= \int_0^{\tau_0} d\lambda K_2(v, \lambda) f^0(\tau_2) \quad (D13) \end{aligned}$$

where we have introduced

$$K_2(v, \lambda) = \frac{\partial}{\partial v} \int \frac{\partial \phi}{\partial x}(x) \frac{\partial \phi}{\partial x}(x-v\lambda) dx \frac{\partial}{\partial v} \quad (D14)$$

We introduce the decomposition

$$\int_0^{\tau_0} d\lambda K_2(v, \lambda) f^0(\tau_2) = \int_0^{\infty} d\lambda K_2(v, \lambda) f^0(\tau_2) - \int_{\tau_0}^{\infty} d\lambda K_2(v, \lambda) f^0(\tau_2) \quad (D15)$$

We now examine the situation for large τ_0 and obtain

$$\frac{\partial f^0}{\partial \tau_2} + \frac{\partial f^1}{\partial \tau_1} = c_2 f^0 \quad (D16)$$

$$\frac{\partial f^2}{\partial \tau_0} = - \int_{\tau_0}^{\infty} d\lambda K_2(v, \lambda) f^0(\tau_2) \quad (D17)$$

where

$$c_2 = \int_0^{\infty} d\lambda K_2(v, \lambda) \quad (D18)$$

Using a technique developed in Appendix C, the transient can be written as

$$\begin{aligned} f^2(\tau_0) &= f^2(0) - \int_0^{\tau_0} d\mu \int_{\mu}^{\infty} d\lambda K_2(v, \lambda) f^0(\tau_2) \\ &= f^2(0) - \left\{ \mu \int_{\mu}^{\infty} d\lambda K_2(v, \lambda) f^0 \right\}_0^{\tau_0} + \int_0^{\tau_0} \mu d\mu K_2(v, \mu) f^0 \\ &= f^2(0) - \tau_0 \int_{\tau_0}^{\infty} d\lambda K_2(v, \lambda) f^0 = \int_0^{\tau_0} \lambda d\lambda K_2(v, \lambda) f^0 \quad (D19) \end{aligned}$$

The asymptotic value of the transient is given by

$$f^2(\infty) = f^2(0) - \int_0^{\infty} \lambda d\lambda K_2(v, \lambda) f^0 = f^2(0) - c'_2 f^0 \quad (D20)$$

The operator c'_2 is given by

$$\begin{aligned}
c_2' &= \int_0^\infty \lambda d\lambda K_2(v, \lambda) \\
&= \int_0^\infty \lambda d\lambda \frac{\partial}{\partial v} T(v, \lambda) \frac{\partial}{\partial v} \\
&= \frac{\partial}{\partial v} \int_0^\infty \lambda d\lambda T(v, \lambda) \frac{\partial}{\partial v}
\end{aligned} \tag{D21}$$

where we have introduced

$$T(\vec{v}, \lambda) = \int \frac{\partial \phi}{\partial x}(x) \frac{\partial \phi}{\partial x}(x - v\lambda) dx \tag{D22}$$

We can further specify the tensor, T , as

$$T = -\underline{ee}\Omega''(v\lambda) - (\mathbb{I} - \underline{ee}) \frac{\Omega'(v\lambda)}{v\lambda} \tag{D23}$$

where the overlap integral, Ω , is given by

$$\Omega(v\lambda) = \int dx \phi(x) \phi(x - v\lambda) \tag{D24}$$

We notice the following fact

$$\begin{aligned}
\int_0^\infty \lambda d\lambda \Omega''(v\lambda) &= \lambda \Omega'(v\lambda) \Big|_0^\infty - \int_0^\infty d\lambda \Omega'(v\lambda) \\
&= 0 - \Omega(\infty + \Omega(0)) \\
&= \Omega(0)
\end{aligned} \tag{D25}$$

and furthermore

$$\int_0^\infty \lambda d\lambda \frac{\Omega'(v\lambda)}{v\lambda} = -\Omega(0) \tag{D26}$$

We then obtain a completely explicit representation for the operator that represents the asymptotic value of the transient,

$$\begin{aligned}
c_2' &= \Omega(0) \frac{\partial}{\partial v} \left\{ -\underline{ee} + \Pi - \underline{ee} \right\} \frac{\partial}{\partial v} \\
&= \Omega(0) \frac{\partial}{\partial v} \left\{ \Pi - 2\underline{ee} \right\} \frac{\partial}{\partial v}
\end{aligned} \tag{D27}$$

This result coincides with the one obtained previously by Ramanathan and Sandri. The main purpose of the theorem proves that f^2 is well behaved, in particular in a bounded function. This means that the decomposition carried out for large τ_0 is a legitimate one; it is, therefore, correct to split the three terms that appear in the second-order equation as done in Eq. (D17). Compatibility at this stage is rather trivial. It simply refers to the behavior with respect to τ_0 . We now have the problem of decomposing properly Eq. (D16). This is the mathematical point which will allow us, if done properly, to carry out a full decomposition of the third-order theory. We have to consider

$$\frac{\partial f^0}{\partial \tau_2} + \frac{\partial f^1}{\partial \tau_1} = c_2 f^0 \tag{D28}$$

We shall now analyze this equation with regard to τ_1 , in analogy to the analysis carried out for Eq. (D13) with regard to τ_0 . For large τ_1 , we conclude that

$$\frac{\partial f^1}{\partial \tau_1} = 0 \tag{D29}$$

so that we can strengthen the first-order result on the f^1 function to

$$f^1 = f_1(\vec{\tau}_2) \tag{D30}$$

Subtracting (D28) from (D27), we obtain

$$\frac{\partial f^0}{\partial \tau_2} = c_2 f^0 \tag{D31}$$

whose solution is, in general, given as

$$f^0(\vec{\tau}_2) = e^{C_2 \tau_2} f^0(\vec{\tau}_3) \quad (D32)$$

C_2 we know is negative definite so that the solutions of (D31) are fully bounded.

Third-order theory. We now calculate the third-order results in the same spirit used above for second-order theory. In particular, we shall investigate the equations successively for large values of each of the increasing scales and demand that the results be well behaved so that the decomposition be meaningful. In fact, we shall see that the commutativity or compatibility of the equations and the uniformity criteria become very intimately related.

The third-order equation can be written as

$$\begin{aligned} \frac{\partial f^0}{\partial \tau_3} + \frac{\partial f^1}{\partial \tau_2} + \frac{\partial f^2}{\partial \tau_1} + \frac{\partial f^3}{\partial \tau_0} = \int dx I \int_0^{\tau_0} d\lambda I(x-v\lambda) f^1 + \\ + \int dx I \int_0^{\tau_0} d\lambda e^{-k(\tau_0-\lambda)} I \int_0^\lambda d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial f^0}{\partial v} \end{aligned} \quad (D33)$$

The right-hand side of this equation can be written as

$$\tau_0 - \lambda = z \quad (D34)$$

where we have used the change of variable

$$\begin{aligned} R = \int_0^{\tau_0} d\lambda K_2(v, \lambda) f^1(\vec{\tau}_2) + \\ + \int dx I \int_0^{\tau_0} d\lambda e^{-Kz} I \int_0^{\tau_0-z} d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial f^0}{\partial v} \end{aligned} \quad (D35)$$

We can rewrite the right-hand side once more as

$$\begin{aligned}
R = & \int_0^\infty d\lambda K_2(v, \lambda) f^1(\vec{\tau}_2) - \int_{\tau_0}^\infty d\lambda K_2(v, \lambda) f^1(\vec{\tau}_2) + \\
& + \int dx I \int_0^{\tau_0} d\lambda e^{-K\lambda} I \int_0^{\tau_0 - \lambda} d\mu \frac{\partial \phi}{\partial x}(x - v\mu) \cdot \frac{\partial f^0}{\partial v} \quad (D36)
\end{aligned}$$

A final explicit representation can be given as

$$\begin{aligned}
R = & c_2 f^1(\vec{\tau}_2) - \int_{\tau_0}^\infty d\lambda K_2(\vec{v}, \lambda) f^1(\vec{\tau}_2) + \\
& + \int dx I \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x - v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_0^{\tau_0 - \lambda} d\mu \frac{\partial \phi}{\partial x}(x - v[\mu + \lambda]) \cdot \frac{\partial f^0}{\partial v} + \\
& - \int dx I \int_{\tau_0}^\infty d\lambda \frac{\partial \phi}{\partial x}(x - v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_\lambda^{\tau_0} dz \frac{\partial \phi}{\partial x}(x - vz) \cdot \frac{\partial f^0}{\partial v} \quad (D37)
\end{aligned}$$

where we have used

$$\mu + \lambda = z \quad (D38)$$

Making full use of the change of variable given in Eq. (D38), we can write

$$\begin{aligned}
R = & c_2 f^1(\vec{\tau}_2) - \int_{\tau_0}^\infty d\lambda K_2(\vec{v}, \lambda) f^1(\vec{\tau}_2) + \\
& + \int dx I \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x - v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_\lambda^{\tau_0} d\mu \frac{\partial \phi}{\partial x}(x - v\mu) \cdot \frac{\partial f^0}{\partial v} + \\
& - \int dx I \int_{\tau_0}^\infty d\lambda \frac{\partial \phi}{\partial x}(x - v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_x^{\tau_0} d\mu \frac{\partial \phi}{\partial x}(x - v\mu) \cdot \frac{\partial f^0}{\partial v} \quad (D39)
\end{aligned}$$

In order to separate the asymptotic part, we can further rewrite this quantity as

$$\begin{aligned}
R = & C_2 f^1(\tau_2) + \int dx I \xi^* I \xi^* I f^0 - \int_{\tau_0}^{\infty} d\lambda K_2(\vec{v}, \lambda) f^1(\tau_2) + \\
& - \int dx I \int_0^{\infty} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_{\tau_0}^{\infty} d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial f^0}{\partial v} + \\
& - \int dx I \int_{\tau_0}^{\infty} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_{\lambda}^{\tau_0} d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial f^0}{\partial v} \quad (D40)
\end{aligned}$$

The asymptotic value of the operator that acts on f^0 is

$$\Gamma_3 = \int dx I \xi^* I \xi^* I \quad (D41)$$

This is a third-order operator with regards to velocity derivatives. The operator C_3 should be the second-order part of Γ_3 in order to commute with C_2 . A more precise proof of this statement will be given later. We thus write

$$\Gamma_3 = \alpha_3 C_2 + \Delta_3 \quad (D42)$$

A full display of the operator Γ_3 is given as

$$\begin{aligned}
\Gamma_3 = & \int dx I \xi^* I \xi^* I f^0 \\
= & \int dx \frac{\partial \phi}{\partial x}(x) \cdot \frac{\partial}{\partial v} \int_0^{\infty} d\lambda e^{-K\lambda} \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^{\infty} d\mu e^{-K\mu} \frac{\partial \phi}{\partial x} \cdot \frac{\partial f^0}{\partial v} \\
= & \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_0^{\infty} d\mu \frac{\partial \phi}{\partial x}(x-v\lambda-v\mu) \cdot \frac{\partial f^0}{\partial v} \\
= & \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^{\infty} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_0^{\infty} dz \frac{\partial \phi}{\partial x}(x-v\lambda-v\mu) \cdot \frac{\partial f^0}{\partial v} \quad (D43)
\end{aligned}$$

The decomposition that is achieved then for large values of the argument can be displayed as follows. For the kinetic condition, we have

$$\frac{\partial f^0}{\partial \tau_3} = \alpha_3 c_2 f^0 \quad (D44)$$

$$\frac{\partial f^1}{\partial \tau_2} = c_2 f^1 + \Delta_3 f^0 \quad (D45)$$

In addition, we have the following two simple statements

$$\frac{\partial f^{2*}}{\partial \tau_1} = \frac{\partial f^2(0, \dot{\tau}_1)}{\partial \tau_1} + 0 \quad (D46)$$

$$\frac{\partial f^2(\dot{\tau}_1)}{\partial \tau_1} = 0 \quad (D47)$$

Finally, the transient is given by

$$\begin{aligned} \frac{\partial f^3}{\partial \tau_0} = & - \int_{\tau_0}^{\infty} d\lambda K_2(\dot{v}, \lambda) f^1(\tau_2) \\ & - L \xi^* I \int_{\tau_0}^{\infty} d\mu \frac{\partial \phi(x - v\mu)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \\ & - L \int_{\tau_0}^{\infty} d\lambda \frac{\partial \phi(x - v\lambda)}{\partial x} \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_{\lambda}^{\tau_0} d\mu \frac{\partial \phi(x - v\mu)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (D48) \end{aligned}$$

The form (D44) would guarantee compatibility and an entropy theorem. This form can, in fact, be demonstrated clearly by an explicit representation of the operators involved. This will be given later.

APPENDIX E. EXPLICIT REPRESENTATIONS

In the table below, we summarize the quantities that have to be understood. We have to deal with essentially eleven unknowns, governed by nine equations, plus asymptotic requirements. In addition, we have a set of operators whose structure must be understood clearly if we are to carry out the calculations successfully. In this table, we put in a solid box those quantities that enter the Bogolubov calculation or the Bogolubov-Chou-Uhlenbeck approach. Quantities outside the box do not enter in their calculations, even though several of them can be treated adiabatically.

~~f^0~~	~~g^0~~	~~h^0~~			
~~f^1~~	~~g^1~~				
~~f^2~~	g^2				
f^3			c_2	c'_2	
f^4			c_3	c'_3	
			c_4	c'_4	
					U

Table 1

We first discuss some general properties of the perturbation equations.

General integration theorem. The perturbation equations have the form of a continuity equation for incompressible flow with a source. One can then think roughly of the solutions in terms of a flow of water which is generated by a definite source. The basic equation is given by

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) A(\vec{x}, t) = \sigma(\vec{x}, t) \quad (\text{E1})$$

The general solution of this equation can be written in two different forms

$$\begin{aligned} A(\vec{x}, t) &= \alpha(\vec{x} - \vec{v}[t - \tau]) + \int_{\tau}^t d\lambda \sigma(\vec{x} - \vec{v}[t - \lambda], \lambda) \\ &= \alpha(\vec{x} - \vec{v}[t - \tau]) + \int_0^{t-\tau} d\mu \sigma(\vec{x} - \vec{v}\mu, t - \mu) \end{aligned} \quad (E2)$$

We shall call the first form given in Eq. (E2) "first representation" and the second term "second representation." The function α that appears in Eq. (E2) is given by

$$A(\vec{x}, \tau) = \alpha(\vec{x}) \quad (E3)$$

The parameter that enters the solution given in (E2) is an arbitrary parameter and can be chosen to be any real number. A special case of the solution is obtained by setting

$$\tau = 0 \quad (E4)$$

We then obtain the special form of the solution

$$\begin{aligned} A(\vec{x}, t) &= \alpha(\vec{x} - \vec{v}t) + \int_0^t d\lambda \sigma(\vec{x} - \vec{v}[t - \lambda], \lambda) = \\ &= \alpha(\vec{x} - \vec{v}t) + \int_0^t d\mu \sigma(\vec{x} - \vec{v}\mu, t - \mu) \end{aligned} \quad (E5)$$

with

$$A(\vec{x}, 0) = \alpha(\vec{x}) \quad (E6)$$

Proof. We introduce the following notation

$$K = \vec{v} \cdot \vec{\nabla} \quad (E7)$$

Therefore, we can write (E1) in the form

$$\left(\frac{\partial}{\partial t} + K \right) A(t) = \sigma(t) \quad (E8)$$

Using an integrating factor, we can write

$$\begin{aligned}\frac{\partial}{\partial t}(e^{Kt} A(t)) &= e^{Kt} \left(\frac{\partial}{\partial t} + K \right) A(t) \\ &= e^{Kt} \sigma(t)\end{aligned}\tag{E9}$$

Integration of (E9) from the arbitrary point τ up to the point t yields

$$\int_{\tau}^t d\lambda \frac{\partial}{\partial \lambda} (e^{K\lambda} A(\lambda)) = \int_{\tau}^t d\lambda e^{K\lambda} \sigma(\lambda)\tag{E10}$$

Carrying out the integration explicitly, we obtain

$$e^{Kt} A(t) - e^{K\tau} A(\tau) = \int_{\tau}^t d\lambda e^{K\lambda} \sigma(\lambda)\tag{E11}$$

Therefore

$$A(t) = e^{-K(t-\tau)} A(\tau) + \int_{\tau}^t d\lambda e^{-K(t-\lambda)} \sigma(\lambda)\tag{E12}$$

To obtain the second representation, we let

$$t - \lambda = \mu, \quad \text{Q.E.D.} \tag{E13}$$

We have demonstrated the general theorem and will now exploit it to study some asymptotic properties.

Asymptotic theorem for the adiabatic theory. We note that the adiabatic theory requires the asymptotic theorem before phase mixing; the nonadiabatic theory uses the nonasymptotic theorem after phase mixing. The two theories are therefore closely related. We will consider the following cases: the source, σ , can have either

(i) short range, namely,

$$\sigma(\vec{x}, t) = 0 \quad , \quad |\vec{x}| > a \sim r_0 \quad (\text{E14})$$

and

(ii) saturates rapidly to a time-asymptotic value, namely,

$$\sigma(\vec{x}, t) = \sigma^*(\vec{x}) \quad , \quad t > \tau_0 \sim r_0/v \quad (\text{E15})$$

The case of slow variation as required by the nonadiabatic theory is studied separately. As an example of the source satisfying the requirements for (E14) and (E15), we have the first-order theory, namely, the quantity

$$\sigma(\vec{x}, t) = \frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} \int_0^t d\mu \frac{\partial \phi(\mathbf{x} - \mathbf{v}\mu)}{\partial \mathbf{x}} \cdot \frac{\partial f^0}{\partial \mathbf{v}} \quad (\text{E16})$$

The asymptotic theorem of the adiabatic theory states that for the simple initial value problem, or at least in the case where the initial correlations are short, we have

$$A \approx \int_0^\infty \sigma^*(\vec{x} - \vec{v}\lambda) d\lambda \equiv A^* \quad (\text{E17})$$

It can be seen from the derivation of this result that, in addition, a travelling wave which reflects the initial conditions will move in the direction $\vec{v}$ and travel out to infinity. The result (E17) is true, therefore, for fixed $\mathbf{x}$ and $\mathbf{v}$ and is not a uniform result if we allow it to move over the entire phase space; namely, it neglects the travelling wave. A travelling wave is also important for the nonadiabatic theory. In order to prove (E17), we take the range of the following time variables

$$t > \tau \quad (\text{E18})$$

and the parameter τ is chosen so that

$$\tau > \tau_0 \quad (\text{E19})$$

We study the first representation of A and write

$$A(\vec{x}, t) = A(\vec{x} - \vec{v}[t - \tau], \tau) + \int_{\tau}^t d\lambda \sigma(\vec{x} - \vec{v}[t - \lambda], \lambda) \quad (\text{E20})$$

Since we have the following condition

$$\lambda > \tau > \tau_0 \quad (\text{E21})$$

we can write

$$\begin{aligned} \int_{\tau}^t d\lambda \sigma(\vec{x} - \vec{v}[t - \lambda], \lambda) &= \int_{\tau}^t d\lambda \sigma^*(\vec{x} - \vec{v}[t - \lambda]) \\ &= \int_0^{t - \tau} d\mu \sigma^*(\vec{x} - \vec{v}\mu) \end{aligned} \quad (\text{E22})$$

where we have made use of

$$\mu = t - \lambda \quad (\text{E23})$$

Now we can write

$$A \underset{t \uparrow}{\sim} A(\vec{x} - \vec{v}[t^* - \tau], \tau) + \int_0^{\infty} d\mu \sigma^*(\vec{x} - \vec{v}\mu) \quad (\text{E24})$$

We note that

$$\tau \sim \tau_0 \sim r_0/v \quad (\text{E25})$$

Therefore, the velocity, τ , is very small for velocities of the order of the thermal speed. We shall use t^* to denote large values of t . We shall now relate the first term in (E24) to the initial A . Since we have

$$A(\vec{x}, t) = A(\vec{x} - \vec{v}t, 0) + \int_0^t d\lambda \sigma(\vec{x} - \vec{v}[t - \lambda], \lambda) \quad (\text{E26})$$

we can write, replacing t by τ ,

$$A(\vec{x}, \tau) = A(\vec{x} - \vec{v}\tau, 0) + \int_0^\tau d\lambda \sigma(\vec{x} - \vec{v}[\tau - \lambda], \lambda) \quad (\text{E27})$$

We are interested in a special case of the first argument of this function, namely

$$\begin{aligned} A(\vec{x} - \vec{v}[t^* - \tau], \tau) &= A(\vec{x} - \vec{v}t^* + \vec{v}\tau - \vec{v}\tau, 0) + \\ &+ \int_0^\tau d\lambda \sigma(\vec{x} - \vec{v}t^* + \vec{v}\tau - \vec{v}\tau + \vec{v}\lambda, \lambda) \\ &= A(\vec{x} - \vec{v}t^*, 0) + \int_0^\tau d\lambda \sigma(\vec{x} - \vec{v}t^* + \vec{v}\lambda, \lambda) \quad (\text{E28}) \end{aligned}$$

We now see the following. Since

$$\lambda < \tau < r_0/v \quad (\text{E29})$$

we have

$$\vec{x} + \vec{v}\lambda \sim \vec{x} \quad (\text{E30})$$

where the approximate equality is valid in the entire range of integration. If we thus consider

$$t^* \gg r_0/v \quad (\text{E31})$$

we shall have

$$|\vec{x} + \vec{v}\lambda - \vec{v}t^*| > r_0 \quad (\text{E32})$$

It is clear therefore that for fixed x and v we can conclude

$$\int_0^\tau d\lambda \sigma(\vec{x} + \vec{v}\lambda - \vec{v}t^*, \lambda) = 0 \quad (\text{E33})$$

This should leave us with the free-streamed initial correlation as indicated by (E28). It is worthwhile, however, to keep in mind that (E33) is not uniform in phase space and, in fact, (E33)

represents the wave that is travelling away to infinity. The term (E33) will vanish at a given location in space for fixed velocity, but it can be followed as a wave travelling in the direction, v , towards infinity.

The result can be rechecked formally as follows.

$$\frac{\partial A}{\partial t} + v \cdot \nabla A = \sigma(x, t) \quad (\text{E34})$$

We now introduce B as any solution of

$$\frac{\partial B}{\partial t} + v \cdot \nabla B = \sigma^*(x) \quad (\text{E35})$$

We notice that the quantity A^* is the asymptotic value of a B function. Now, let

$$\Delta = A - B \quad (\text{E36})$$

The function Δ must then satisfy

$$\frac{\partial \Delta}{\partial t} + v \cdot \nabla \Delta = \sigma(x, t) - \sigma^*(x) \quad (\text{E37})$$

Clearly, we have

$$\frac{\partial \Delta}{\partial t} + v \cdot \nabla \Delta = 0, \quad t \gg \tau \quad (\text{E38})$$

We thus conclude that the difference between A and A^* is a free-streaming function.

Now we want to generate explicit representations for the functions of interest.

Zero-order correlation. We write the zero-order correlation as

$$\frac{\partial g^0}{\partial t} + v \cdot \nabla g^0 = \frac{\partial \phi}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (\text{E39})$$

The integration can be written as

$$g^0 = \int_0^t d\mu \frac{\partial \phi(x-v\mu)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (\text{E40})$$

The asymptotic value of this expression is given by

$$g^{0*} = \int_0^\infty d\mu \frac{\partial \phi(x-v\mu)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (\text{E41})$$

Commutator lemma. It is very important to keep in mind

$$e^{-K\lambda} \frac{\partial}{\partial v} e^{K\lambda} = \frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \quad (\text{E42})$$

Proof. We simply differentiate the exponential operator, obtaining

$$\begin{aligned} \frac{\partial}{\partial v} e^{v \cdot \nabla \lambda} &= \lambda \nabla e^{v \cdot \nabla \lambda} + e^{v \cdot \nabla \lambda} \frac{\partial}{\partial v} \\ &= e^{K\lambda} \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \end{aligned} \quad (\text{E43})$$

Upon multiplication by the inverse operator, we recover (E42).

First-order correlation. Exploiting the lemma, we can now write the first-order correlation as

$$g^{1*} = \int_0^\infty d\lambda \frac{\partial \phi(x-v\lambda)}{\partial x} \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_0^\infty d\mu \frac{\partial \phi(x-v\mu-v\lambda)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (\text{E44})$$

Making use of the change of variables,

$$\mu + \lambda = z \quad (\text{E45})$$

we have the asymptotic value

$$g^{1*} = \int_0^\infty d\lambda \frac{\partial \phi(x-v\lambda)}{\partial x} \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_\lambda^\infty dz \frac{\partial \phi(x-vz)}{\partial x} \cdot \frac{\partial f^0}{\partial v} \quad (\text{E46})$$

It is important to note that the first-order correlation is made down at the origin by the interaction function as it generates a

second-order correlation. Similarly, the zero-order correlation is confined to short distances in the source of the first-order correlation. The nonadiabatic effects are therefore not so important here; they are of greater importance in the phase mixing of the three-body correlations that generate two-body correlations in second order.

Leading-order three-body correlation. The three-body correlation satisfies the equation

$$\begin{aligned}
 \frac{\partial h_{12}^0}{\partial t} + \mathbf{v} \cdot \nabla h_{12}^0 &= I_1 g_2^0 + I_2 g_1^0 \\
 &= \frac{\partial \phi(\xi_1)}{\partial \mathbf{x}} \cdot \frac{\partial g_2^0}{\partial \mathbf{v}} + \frac{\partial \phi(\xi_2)}{\partial \mathbf{x}} \cdot \frac{\partial g_1^0}{\partial \mathbf{v}} \\
 &= \sigma(\xi_1 \xi_2 t)
 \end{aligned} \tag{E47}$$

with

$$g_1^0 = \int_0^t d\mu \frac{\partial \phi(\xi_1 - \mathbf{v}\mu)}{\partial \mathbf{x}} \cdot \frac{\partial f^0}{\partial \mathbf{v}} \tag{E48}$$

The full nonadiabatic expression can be written more compactly by using

$$g_1^0 = \int_0^{\tau_0} d\lambda e^{-K\lambda} I_1 f_0 \tag{E49}$$

We then have

$$\sigma = \left(I_1 \int_0^{\tau_0} d\lambda e^{-K\lambda} I_2 + I_2 \int_0^{\tau_0} d\lambda e^{-K\lambda} I_1 \right) f_0 \tag{E50}$$

so that

$$h_{12} = \int_0^{\tau_0} d\lambda e^{-K(\tau_0 - \lambda)} \left\{ I_1 \int_0^\lambda d\mu e^{-K\mu} I_2 + I_2 \int_0^\lambda d\mu e^{-K\mu} I_1 \right\} f_0 \tag{E51}$$

We give a direct proof of the result to eliminate all doubts with regard to the streaming operators. Use of an integrating factor gives:

Proof.

$$\begin{aligned} \frac{\partial}{\partial \tau_0} \left(e^{K\tau_0} h_{12} \right) &= e^{K\tau_0} \left(\frac{\partial h_{12}}{\partial \tau_0} + K h_{12} \right) \\ &= e^{K\tau_0} \left(I_1 \int_0^{\tau_0} d\lambda e^{-K\lambda} I_2 f_0 + 1 \longleftrightarrow 2 \right) \quad (\text{E52}) \end{aligned}$$

Integration then gives

$$e^{K\tau_0} h_{12}(\tau_0) = \int_0^{\tau_0} d\lambda e^{K\lambda} \left(I_1 \int_0^{\lambda} d\mu e^{-K\mu} I_2 f_0 + 1 \longleftrightarrow 2 \right) \quad (\text{E53})$$

We can therefore write the electron two-ion correlation as

$$\begin{aligned} h_{12} &= e^{-K\tau_0} \int_0^{\tau_0} d\lambda e^{K\lambda} \left(I_1 \int_0^{\lambda} d\mu e^{-K\mu} I_2 f_0 + 1 \longleftrightarrow 2 \right) \\ &= \int_0^{\tau_0} d\lambda e^{-K(\tau_0 - \lambda)} \left(I_1 \int_0^{\lambda} d\mu e^{-K\mu} I_2 f_0 + 1 \longleftrightarrow 2 \right) \quad (\text{E54}) \\ &\quad \text{Q.E.D.} \end{aligned}$$

We can use an alternative representation which is introduced by using

$$\tau_0 - \lambda = z, \quad \lambda = \tau_0 - z \quad (\text{E55})$$

We then obtain the three-particle correlation as

$$h_{12} = \int_0^{\tau_0} dz e^{-Kz} \left(I_1 \int_0^{\tau_0 - z} d\mu e^{-K\mu} I_2 f_0 + 1 \longleftrightarrow 2 \right) \quad (\text{E56})$$

We give the explicit representations for some of the operators that are of importance.

Collision operators. The second-order collision operator can be given a number of representations. For the time being, we shall omit the Fourier representations. We can write C_2 in any of the following six forms:

$$C_2 = L\zeta^* I \quad (E57)$$

$$= \frac{\alpha}{v^3} \hat{\nabla}_v^2 \quad (E58)$$

$$= \int_0^\infty d\lambda K(\lambda) \quad (E59)$$

$$= \alpha \frac{\partial}{\partial v_i} \left(\delta_{ij} - e_i e_j \right) \frac{\partial}{\partial v_j} \quad (E60)$$

$$= \int dx I(x) \zeta^* I(x) \quad (E61)$$

$$= \int dx I(x) \int_0^\infty d\lambda I(x-v\lambda) \quad (E62)$$

where we have used the auxiliary notation

$$K(\lambda) = \int dx I(x) I(x-v\lambda) \quad (E63)$$

$$K(\lambda) = \frac{\partial}{\partial v} \cdot T \cdot \frac{\partial}{\partial v} \quad (E64)$$

$$T = \int \nabla \phi(x) \partial \phi(x-v\lambda) dx \quad (E65)$$

We note that the phase mixing should be done as soon as possible. We also note that the operator C_2 has very different representations when acting on a function of speed only, as contrasted to a function of speed and position.

We now consider the third-order operator, Γ^3 , given by

$$\Gamma^3 \equiv L(\zeta^* I)^2 = L\zeta^* I \zeta^* I \quad (E66)$$

Making the phase mixing explicit, we have

$$\Gamma^3 = \int dx I \zeta^* I \zeta^* I \quad (E67)$$

Making the streaming operators explicit, we have

$$\begin{aligned} \Gamma^3 &= \int dx \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda e^{-K\lambda} \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \times \int_0^\infty d\mu e^{-K\mu} \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \\ &= \int dx \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda e^{-K\lambda} \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \times \int_0^\infty d\mu \frac{\partial \phi(x-v\mu)}{\partial x} \cdot \frac{\partial}{\partial v} \quad (E68) \end{aligned}$$

Making use of the commutation rule, we then find

$$\Gamma_3 = \int dx \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi(x-v\lambda)}{\partial x} \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \times \int_0^\infty d\mu \frac{\partial \phi(x-v\lambda-v\mu)}{\partial x} \cdot \frac{\partial}{\partial v} \quad (E69)$$

We can now prove some properties of this operator. It is energy-conserving and satisfies the equilibrium property. In fact, for the energy conservation, we have

$$\begin{aligned} \Gamma_3 \frac{v^2}{2} &= L \zeta^* I \int_0^\infty d\mu e^{-K\mu} \frac{\partial \phi(x)}{\partial x} \cdot \frac{\partial}{\partial v} \frac{v^2}{2} \\ &= L \zeta^* I \int_0^\infty d\mu e^{-K\mu} K \phi(x) \\ &= -L \zeta^* I \int_0^\infty d\mu \frac{\partial}{\partial \mu} (e^{-K\mu}) \phi(x) \\ &= L \zeta^* I (\phi(x) - e^{-K\infty} \phi(x)) \\ &= L \zeta^* I \phi(x) \\ &= L \zeta^* \frac{\partial \phi}{\partial x}(x) \cdot \left(\frac{\partial}{\partial v} \phi(x) \right) = 0 \quad (E70) \end{aligned}$$

The proof of the equilibrium property is given as

$$\begin{aligned}
\Gamma_3 f(|\vec{v}|) &= L\zeta^* I \int_0^\infty d\mu e^{-K\mu} \frac{\partial \phi}{\partial x} \cdot \vec{v} \left(\frac{f'}{v} \right) \\
&= -L\zeta^* I \int_0^\infty d\mu \frac{\partial}{\partial \mu} (e^{-K\mu}) \phi(x) \cdot \frac{f'}{v} \\
&= L\zeta^* I \phi(x) \frac{f'}{v} \\
&= L\zeta^* \frac{\partial \phi}{\partial x}(x) \cdot \frac{\partial}{\partial v} \phi(x) \frac{f'}{v} \\
&= L\zeta^* \vec{v} \cdot \frac{\partial}{\partial x} \left(\frac{\phi^2}{2} \right) \frac{1}{v} \left(\frac{f'}{v} \right) \tag{E71}
\end{aligned}$$

We have

$$\zeta^* K = 1 \tag{E72}$$

for finite range functions. Therefore, we can write

$$\begin{aligned}
\Gamma_3 f(|\vec{v}|) &= \int dx \frac{\partial \phi(x)}{\partial x} \frac{\partial}{\partial v} \left(\frac{\phi^2}{2} \right) \frac{1}{v} \left(\frac{f'}{v} \right) \\
&= \left(\int dx \frac{\partial \phi(x)}{\partial x} \frac{\phi^2}{2} \right) \cdot \frac{\partial}{\partial v} \left(\frac{1}{v} \left(\frac{f'}{v} \right) \right) \\
&= 0 \tag{E73}
\end{aligned}$$

proving the equilibrium property for the entire operator, Γ_3 . This property is somewhat misleading because only the C_3 part of Γ_3 is of relevance for the kinetic theory. In fact, when we intend to prove the positive definiteness of Γ_3 , we have

$$\begin{aligned}
(\alpha, \Gamma_3 \alpha) &= \int dv \alpha L\zeta^* I \zeta^* I \alpha \\
&= \iint dv dx \alpha \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \zeta^* \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \zeta^* \frac{\partial \phi}{\partial x} \cdot \frac{\partial \alpha}{\partial v} \\
&= \iint dv dx \left(\frac{\partial \phi}{\partial x} \cdot \frac{\partial \alpha}{\partial v} \right) \zeta^* \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \zeta^* \left(\frac{\partial \phi}{\partial x} \cdot \frac{\partial \alpha}{\partial v} \right) \tag{E74}
\end{aligned}$$

We introduce

$$A(xv) = \frac{\partial \phi}{\partial x} \cdot \frac{\partial \alpha}{\partial v} = -A(-x, v) \quad (\text{E75})$$

Therefore, we can write

$$(\alpha, \Gamma_3 \alpha) = - \iint dv dx A \left(\xi^* \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \xi^* \right) A \quad (\text{E76})$$

We notice that we have the operator

$$O = \xi^* \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \xi^* = O^+ \quad (\text{E77})$$

Because A is odd in little x , we will only have terms which appear; in particular, we have the contributions

$$\mathcal{P}\delta, \quad \delta\mathcal{P} \quad (\text{E78})$$

Clearly, the resulting Γ_3 is not positive definite. This is the crux of the failure of the old theory, namely, the absence of the entropy theorem. Γ_3 , in order to obtain a correct kinetic theory, must be decomposed into a true collision operator which commutes with C_2 and a rest which is contributing to the modifications of the perturbation f^1 .

Second-order transients. We can make the second-order transient quite explicit by using the following successive steps:

$$\begin{aligned}
\frac{\partial f^2}{\partial \tau_0} &= L(\xi - \xi^*) I f^0 \\
&= \int dx I(x) (\xi - \xi^*) I f^0 \\
&= \int dx I(x) \left\{ \int_0^{\tau_0} d\lambda e^{-K\lambda} - \int_0^{\infty} d\lambda e^{-K\lambda} \right\} I(x) f^0 \\
&= \int_0^{\tau_0} d\lambda \int dx I(x) e^{-K\lambda} I(x) f^0 + \\
&\quad - \int_0^{\infty} d\lambda \int dx I(x) e^{-K\lambda} I(x) f^0 \\
&= - \int_{\tau_0}^{\infty} d\lambda \Omega(v, \lambda) f^0
\end{aligned} \tag{E79}$$

where we have used

$$\Omega(v, \lambda) = \frac{\partial}{\partial v} \cdot T(\vec{v}, \lambda) \cdot \frac{\partial}{\partial v} \tag{E80}$$

Therefore, we can write the transient in second-order theory as

$$f^2 = f^2(0, \vec{\tau}_1) - \int_0^{\tau_0} d\mu \int_{\mu}^{\infty} d\lambda \Omega(\lambda) f^0 \tag{E81}$$

Using the integration by parts lemma, we can rewrite it as

$$f^2 = f^2(0, \vec{\tau}_1) - \tau_0 \int_{\tau_0}^{\infty} d\lambda \Omega(\lambda) f^0 - \int_0^{\tau_0} \lambda d\lambda \Omega(\lambda) f^0 \tag{E82}$$

This formula is needed to guarantee that we have obtained a uniform approximation, namely, that the transient itself is a well behaved quantity.

A similar manipulation can be carried out to construct the transient in third order. The third-order transient can be represented by

$$\begin{aligned}
\frac{\partial f^3}{\partial \tau_0} &= L(\xi_I \xi - \xi^*_I \xi^*) I f^0 = c'_3 f^0 \\
&= \int dx I(x) [\xi_I \xi - \xi^*_I \xi^*] I f^0 \\
&= \int dx I(x) \left[\int_0^{\tau_0} d\lambda e^{-K\lambda} I(x) \int_0^{\tau_0} d\mu e^{-K\mu} - \int_0^{\infty} e^{-K\lambda} d\lambda I(x) \int_0^{\infty} e^{-K\mu} d\mu \right] I f^0
\end{aligned} \tag{E83}$$

We can now decompose the integral in a variety of fashions; in particular, we can write

$$\begin{aligned}
c'_3 &= L \xi_I \xi_I - L \xi^*_I \xi^*_I \\
&= -L \int_{\tau_0}^{\infty} d\lambda e^{-K\lambda} I \int_0^{\infty} d\mu e^{-K\mu} I \\
&\quad -L \int_0^{\tau_0} d\lambda e^{-K\lambda} I \int_{\tau_0}^{\infty} d\mu e^{-K\mu} I \\
&= -\int_{\tau_0}^{\infty} d\lambda \int dx I(x) e^{-K\lambda} I \int_0^{\infty} d\mu e^{-K\mu} I \\
&\quad -\int_0^{\tau_0} d\lambda \int dx I(x) e^{-K\lambda} I \int_{\tau_0}^{\infty} d\mu e^{-K\mu} I
\end{aligned} \tag{E84}$$

A full analysis of this operator is still lacking; however, it looks like the boundedness properties of second order can be translated to third order. It is worth remarking that also the third-order transient should be shown to be bounded to have a complete theory as indicated in Table 1.

APPENDIX F. SPLITTING OF SECOND- AND THIRD-ORDER OPERATORS

We shall now investigate in more detail the splitting of the relevant operators by introducing a notation which is suitable for the asymptotic analysis.

We introduce

$$\begin{aligned}
 a^2 &= L\xi^*_I \\
 &= \int_0^\infty d\lambda \int dx \frac{\partial\phi}{\partial x}(x) \cdot \frac{\partial}{\partial v} \frac{\partial\phi}{\partial x}(x-v\lambda) \cdot \frac{\partial}{\partial v} \\
 &= \int_0^\infty d\lambda \int dx \frac{\partial\phi}{\partial x_1}(x) \left[\frac{\partial}{\partial v_1} \frac{\partial\phi}{\partial x_j}(x-v\lambda) \right] \frac{\partial}{\partial v_j} + \\
 &\quad + \int_0^\infty d\lambda \int dx \frac{\partial\phi}{\partial x_1}(x) \frac{\partial\phi}{\partial x_j}(x-v\lambda) \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_j}
 \end{aligned} \tag{F1}$$

which we write as

$$a^2 = a^2_I + a^2_{II} \tag{F2}$$

The subscript indicates the order of the operator. We have

$$\begin{aligned}
 a^2_I &= \int_0^\infty d\lambda \int dx \frac{\partial\phi}{\partial x_1}(x) \left[\lambda \frac{\partial}{\partial \lambda v_1} \frac{\partial\phi}{\partial x_j}(x-v\lambda) \right] \frac{\partial}{\partial v_j} \\
 &= - \int_0^\infty \lambda d\lambda \int dx \frac{\partial\phi}{\partial x_1}(x) \frac{\partial^2\phi}{\partial x_1 \partial x_j}(x-v\lambda) \frac{\partial}{\partial v_j} \\
 &= - \int \lambda d\lambda \vec{A}(\vec{v}\lambda) \cdot \vec{\nabla}_v
 \end{aligned} \tag{F3}$$

We introduce the vector

$$\begin{aligned}
A_j(\vec{v}\lambda) &= \int dx \frac{\partial \phi}{\partial x_1}(x) \frac{\partial^2 \phi}{\partial x_1 \partial x_j}(x-v\lambda) \\
&= - \int dx \phi(x) \frac{\partial^3}{\partial x_1 \partial x_1 \partial x_j} \phi(x-v\lambda)
\end{aligned} \tag{F4}$$

We notice the following identity

$$\frac{\partial \phi}{\partial x_1}(x-v\lambda) = - \frac{1}{\lambda} \left[\frac{\partial}{\partial v_1} \phi(x-v\lambda) \right] \tag{F5}$$

In terms of the overlap integral given by

$$\Omega(v\lambda) = \int dx \phi(x) \phi(\vec{x}-\vec{v}\lambda) \tag{F6}$$

the vector function, A_j , can be written as

$$\begin{aligned}
A_j(\vec{v}\lambda) &= - \int dx \phi(x) \frac{\partial^3}{\partial (v_1\lambda) \partial (v_1\lambda) \partial (v_j\lambda)} \phi(x-\vec{v}\lambda) \\
&= - \frac{\partial^3}{\partial (v_1\lambda) \partial (v_1\lambda) \partial (v_j\lambda)} \Omega(v\lambda)
\end{aligned} \tag{F7}$$

We can further write

$$\begin{aligned}
A_j(\vec{v}\lambda) &= - \frac{\partial^2}{\partial (v_1\lambda) \partial (v_1\lambda)} \left[\frac{\partial}{\partial v_j\lambda} \Omega(v\lambda) \right] \\
&= - \frac{\partial^2}{\partial (v_1\lambda) \partial (v_1\lambda)} \left[\frac{v_j}{v} \Omega'(v\lambda) \right]
\end{aligned} \tag{F8}$$

We shall use the two lemmas

$$\frac{\partial e_1}{\partial v_j} = \frac{1}{v} (\delta_{1j} - e_1 e_j) \quad , \quad \frac{\partial e_1}{\partial v_1} = \frac{2}{v} \tag{F9}$$

We can then obtain, with the notation

$$\begin{aligned}\vec{\mu} &= \vec{v}\lambda \\ \mu &= v\lambda\end{aligned}\tag{F10}$$

the following expression for the vector A

$$\begin{aligned}A_j &= -\frac{\partial}{\partial v_1 \lambda} \left(\frac{\partial}{\partial \mu_1} [e_j \Omega'(\mu)] \right) \\ &= -\frac{\partial}{\partial \mu_1} \left\{ \frac{1}{v\lambda} (\delta_{1j} - e_1 e_j) \Omega' + e_j e_1 \Omega'' \right\} \\ &= -\left\{ \frac{1}{v\lambda} \left(-\frac{\partial e_1}{\partial \mu_1} e_j - e_1 \frac{\partial e_j}{\partial \mu_1} \right) \Omega' + \frac{\partial e_j}{\partial \mu_1} e_1 \Omega'' + e_j \frac{\partial e_1}{\partial \mu_1} \Omega'' \right\} \\ &= \frac{1}{v\lambda} \left(\frac{\partial e_1}{\partial \mu_1} e_j + e_1 \frac{\partial e_j}{\partial \mu_1} \right) \Omega' - \frac{\partial e_j}{\partial \mu_1} e_1 \Omega'' - e_j \frac{\partial e_1}{\partial \mu_1} \Omega'' \\ &= \frac{1}{v\lambda} \frac{2}{v\lambda} e_j \Omega' - e_j \frac{2}{v\lambda} \Omega'' \\ &= \left(\frac{\Omega'}{v\lambda} - \Omega'' \right) \frac{2}{v\lambda} e_j\end{aligned}\tag{F11}$$

The final expression is given by

$$\vec{A}(\vec{\mu}) = \vec{e} \frac{2}{\mu} \left(\frac{\Omega'(\mu)}{\mu} - \Omega'' \right)\tag{F12}$$

We can now complete the representation of the first-order part of $\mathcal{A}$. We write

$$\begin{aligned}A_I^2 &= -\int_0^\infty \lambda d\lambda \vec{A}(\vec{v}\lambda) \cdot \vec{\nabla}_v \\ &= -\int_0^\infty \lambda d\lambda \frac{2}{v\lambda} \left(\frac{\Omega'(v\lambda)}{v\lambda} - \Omega'' \right) \vec{e} \cdot \vec{\nabla}_v\end{aligned}\tag{F13}$$

with

$$\mu = v\lambda \quad (\text{F14})$$

Then we have

$$\begin{aligned} a_{\text{I}}^2 &= -\frac{1}{v^2} \int_0^\infty \mu d\mu \left(\frac{\Omega'}{\mu} - \Omega'' \right) \vec{e} \cdot \vec{\nabla}_v \\ &= -\frac{2}{v^2} \left\{ \int_0^\infty d\mu \left(\frac{\Omega'}{\mu} - \Omega'' \right) \right\} \vec{e} \cdot \vec{\nabla}_v \end{aligned} \quad (\text{F15})$$

We notice that

$$\int_0^\infty \Omega'' d\mu = \Omega'(\infty) - \Omega'(0) = 0 \quad (\text{F16})$$

Therefore, we conclude .

$$a_{\text{I}}^2 = -\frac{2\alpha}{v^2} \vec{e} \cdot \vec{\nabla}_v \quad (\text{F17})$$

with

$$\alpha = \int_0^\infty d\mu \frac{\Omega'}{\mu} \quad (\text{F18})$$

We now consider the second-order part as

$$\begin{aligned} a_{\text{II}}^2 &= \int_0^\infty d\lambda \int dx \frac{\partial \phi}{\partial x_1}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_j} \\ &= \int_0^\infty d\lambda T_{1j}(\vec{v}, \lambda) \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_j} \end{aligned} \quad (\text{F19})$$

The tensor T can be written as

$$\begin{aligned}
T_{1j} &= \int dx \frac{\partial \phi}{\partial x_1}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) = - \int dx \phi(x) \frac{\partial^2 \phi}{\partial x_1 \partial x_j}(x-v\lambda) \\
&= - \frac{\partial^2}{\partial(v_1 \lambda) \partial(v_j \lambda)} \Omega(v\lambda) = - \frac{\partial}{\partial v_1 \lambda} (e_j \Omega') \\
&= - \frac{1}{v\lambda} (\delta_{1j} - e_1 e_j) \Omega' - e_j e_1 \Omega'' \quad (F20)
\end{aligned}$$

We therefore have

$$\begin{aligned}
A_{II}^2 &= - \int_0^\infty d\lambda \left[\frac{\Omega'}{\mu} (\delta_{1j} - e_1 e_j) + e_1 e_j \Omega'' \right] \frac{\partial^2}{\partial v_1 \partial v_j} \\
&= - \frac{1}{v} \left\{ \int_0^\infty d\mu \frac{\Omega'}{\mu} \right\} (\delta_{1j} - e_1 e_j) \frac{\partial^2}{\partial v_1 \partial v_j} \quad (F21)
\end{aligned}$$

Using the number α defined in (F18), we have

$$A_{II}^2 = - \frac{\alpha}{v} (\delta_{1j} - e_1 e_j) \frac{\partial^2}{\partial v_1 \partial v_j} \quad (F22)$$

Using the identity

$$e_1 e_j \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_j} = (e \cdot \nabla)^2 \quad (F23)$$

we obtain the final representation

$$A_{II}^2 = - \frac{\alpha}{v} \nabla_v^2 + \frac{\alpha}{v} (e \cdot \nabla_v)^2 \quad (F24)$$

This completes the structure analysis of the second-order operator.

Third-order operator. We write the third-order operator as

$$a^3 = \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \left(\frac{\partial}{\partial v} + \lambda \frac{\partial}{\partial x} \right) \int_0^\infty d\mu \frac{\partial \phi}{\partial x}(x-v\lambda-v\mu) \cdot \frac{\partial}{\partial v} \quad (F25)$$

Making use of the change of variable

$$\lambda + \mu = z \quad (F26)$$

we obtain

$$\begin{aligned} a^3 &= \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \frac{\partial}{\partial v} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial}{\partial v} \\ &+ \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \lambda \frac{\partial}{\partial x} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial}{\partial v} \quad (F27) \end{aligned}$$

We now push the velocity derivatives to the right to obtain

$$\begin{aligned} a^3 &= \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v} \frac{\partial \phi}{\partial x}(x-v\mu) \right] \cdot \frac{\partial}{\partial v} \\ &+ \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-v\mu) : \frac{\partial}{\partial v} \frac{\partial}{\partial v} \\ &+ \int dx \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial v} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \cdot \lambda \frac{\partial}{\partial x} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-v\mu) \cdot \frac{\partial}{\partial v} \quad (F28) \end{aligned}$$

Using the last derivative, we obtain $a^3 = \sum_{i=1}^9 a_i^3$ where

$$a_1^3 = \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \right] \frac{\partial}{\partial v_i} \quad (F29)$$

$$a_2^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v_k} \frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \right] \cdot \frac{\partial}{\partial v_i} \quad (F30)$$

$$a_3^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \right] : \frac{\partial}{\partial v_k} \frac{\partial}{\partial v_i} \quad (F31)$$

$$a_4^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x_1}(x-v\mu) : \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_1} \quad (\text{F32})$$

$$a_5^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_1}(x-v\mu) \right] : \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_1} \quad (\text{F33})$$

$$a_6^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x_1}(x-v\mu) : \frac{\partial}{\partial v_k} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_1} \quad (\text{F34})$$

$$a_7^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \cdot \lambda \frac{\partial}{\partial x_j} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x_1}(x-v\mu) \cdot \frac{\partial}{\partial v_1} \quad (\text{F35})$$

$$a_8^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \cdot \lambda \frac{\partial}{\partial x_j} \int_\lambda^\infty d\mu \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_1}(x-v\mu) \right] \cdot \frac{\partial}{\partial v_1} \quad (\text{F36})$$

$$a_9^3 = + \int dx \frac{\partial \phi}{\partial x_k}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \cdot \lambda \frac{\partial}{\partial x_j} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x_1}(x-v\mu) : \frac{\partial}{\partial v_k} \frac{\partial}{\partial v_1} \quad (\text{F37})$$

The sixth term is written as

$$\begin{aligned} a_6^3 &= \int dx \frac{\partial \phi}{\partial x_1}(x) \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-v\lambda) \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x_k}(x-v\mu) \frac{\partial^3}{\partial v_1 \partial v_j \partial v_k} \\ &= \int_0^\infty d\lambda \int_\lambda^\infty d\mu \int dx \frac{\partial \phi}{\partial x_1}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial \phi}{\partial x_k}(x-v\mu) \nabla_{1jk} \\ &= \int_0^\infty d\lambda \int_\lambda^\infty d\mu T_{1jk}^{(3,6)} \nabla_{1jk} \end{aligned} \quad (\text{F38})$$

The third-rank tensor is given by the triple overlap

$$\begin{aligned}
T_{ijk} &= \int dx \frac{\partial \phi}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial \phi}{\partial x_k}(x-v\mu) \\
&= - \int dx \phi(x) \left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x-v\lambda) \frac{\partial \phi}{\partial x_k}(x-v\mu) + \right. \\
&\quad \left. + \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial^2 \phi}{\partial x_i \partial x_k}(x-v\mu) \right] \quad (F39)
\end{aligned}$$

Very important combinations are the first- and second-order parts of this operator; in particular, the first-order part can be written

$$a_{\text{I}}^3 = \int_0^\infty d\lambda \int_\lambda^\infty d\mu \vec{A}^{1,2,7,8} \cdot \vec{\nabla}_v \quad (F40)$$

where the vector A is written as

$$\begin{aligned}
\vec{A}_1^{1,2,7,8} &= \int dx \frac{\partial \phi}{\partial x_k}(x) \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \left[\frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \right] \\
&\quad + \int dx \frac{\partial \phi}{\partial x_k}(x) \left(\frac{\partial}{\partial x_j} \phi(x-v\lambda) \right) \left[\frac{\partial}{\partial v_k} \frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \right] \\
&\quad + \lambda \int dx \frac{\partial \phi}{\partial x_k}(x) \left[\frac{\partial}{\partial v_k} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \phi(x-v\mu) \right) \\
&\quad + \lambda \int dx \frac{\partial \phi}{\partial x_k}(x) \left(\frac{\partial \phi}{\partial x_j}(x-v\lambda) \right) \left[\frac{\partial}{\partial x_j} \frac{\partial}{\partial v_k} \frac{\partial}{\partial x_i} \phi(x-v\mu) \right] \quad (F41)
\end{aligned}$$

Similarly, the second-order operator can be written

$$a_{\text{II}}^3 = \int_0^\infty d\lambda \int_\lambda^\infty d\mu T_{ki}^{3,4,5,9} \frac{\partial^2}{\partial v_k \partial v_i} \quad (F42)$$

where the tensor kernel is written

$$\begin{aligned}
T_{ki}^{3,4,5,9} = & \int dx \frac{\partial \phi}{\partial x_k}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial}{\partial v_j} \frac{\partial \phi}{\partial x_i}(x-v\mu) \\
& + \int dx \frac{\partial \phi}{\partial x_j}(x) \left[\frac{\partial}{\partial v_j} \frac{\partial}{\partial x_k} \phi(x-v\lambda) \right] \frac{\partial}{\partial x_i} \phi(x-v\mu) \\
& + \int dx \frac{\partial \phi}{\partial x_j}(x) \frac{\partial \phi}{\partial x_k}(x-v\lambda) \frac{\partial}{\partial v_j} \frac{\partial}{\partial x_i} \phi(x-v\mu) \\
& + \lambda \int dx \frac{\partial \phi}{\partial x_k}(x) \frac{\partial \phi}{\partial x_j}(x-v\lambda) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \phi(x-v\mu) \quad (F43)
\end{aligned}$$

Boundedness. The problems associated with the boundedness of these quantities can be resolved by analysis of the space defined by the temporal parameters of integration. In particular, consider

$$\int_0^\infty d\lambda \int_\lambda^\infty d\mu \int dx \phi(x) \psi(x-v\lambda) \chi(x-v\mu) \quad (F44)$$

Introducing the phase-mixed expression as

$$F(v\lambda, v\mu) = \int dx \phi(x) \psi(x-v\lambda) \chi(x-v\mu) \quad (F45)$$

we can see that the finite range will hold if we notice the two equivalent diagrams of the parameter space, which follow.

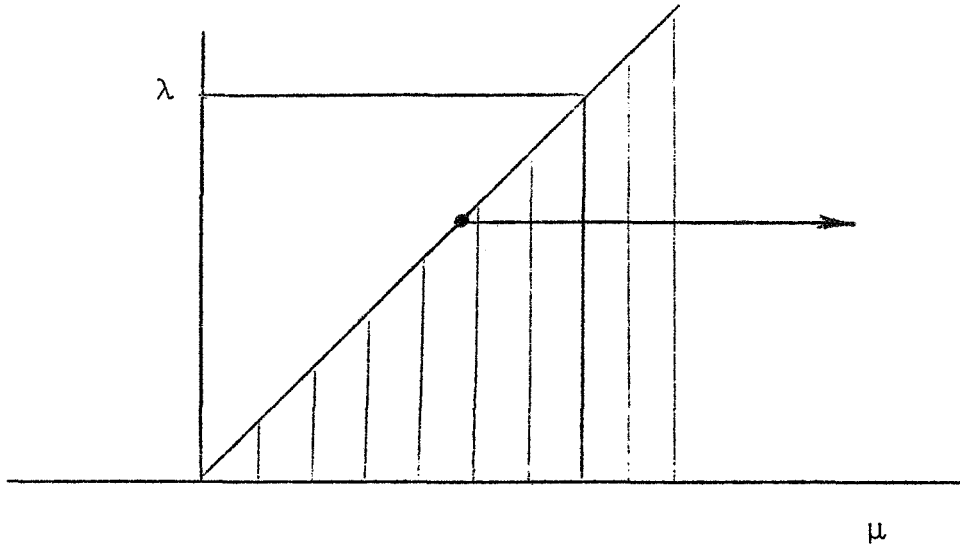


Figure 1

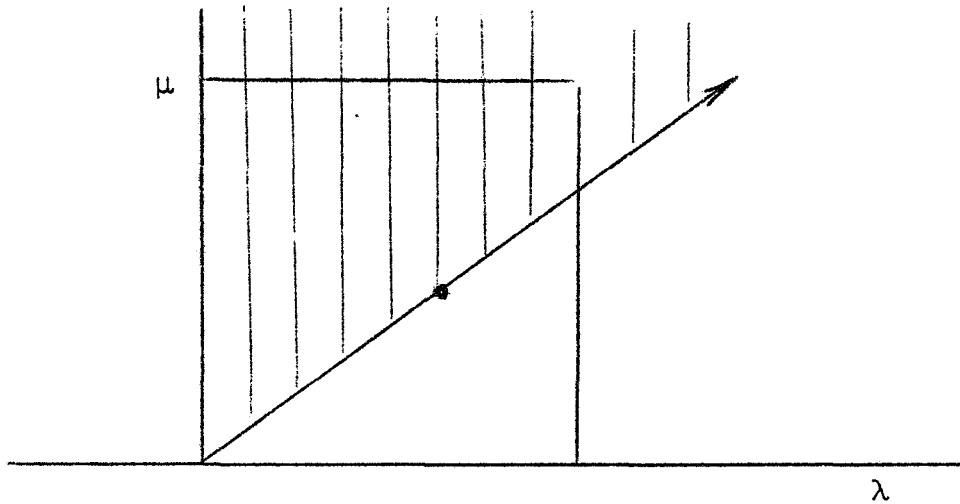


Figure 2

We see that if the quantities ϕ , ψ , χ have finite range, then in the parameter space a rectangular domain of integration will be carved; but since the integration space is the triangular region shown, the boundedness of the ϕ , ψ , χ will imply the boundedness of the entire expression (F44). Similarly, in fourth order, we can consider the expression

$$\int dx \phi(x) \int_0^\infty d\lambda \psi(x-v\lambda) \int_0^\infty d\mu \chi(x-v\lambda-v\mu) \int_0^\infty dv \omega(x-v\lambda-v\mu-v\nu) \quad (\text{F46})$$

A change of variable

$$\lambda + \mu = z \quad (\text{F47})$$

replaces the last two integrals by

$$\int_{\lambda}^{\infty} dz \chi(x-vz) \int_0^{\infty} dv \omega(x-vz-vv) \quad (\text{F48})$$

We now change variables

$$w = z + v \quad (\text{F49})$$

The last integral is given by

$$\int_z^{\infty} dw \omega(x-vw) \quad (\text{F50})$$

The expression then can be written

$$\int dx \phi(x) \psi(x-v\lambda) \chi(x-v\mu) \omega(x-vv) \quad (\text{F51})$$

where the ranges of λ , μ , v are given, respectively, by

$$(0, \infty) \quad , \quad (\lambda, \infty) \quad , \quad (\mu, \infty) \quad (\text{F52})$$

We thus have the three-dimensional analogues of the previous discussion. Thus, the boundedness of the quantities denoted by Greek letters, and their finite range, will guarantee the boundedness of the total expression.

APPENDIX G. THREE-BODY DAMPING OF THE TWO-PARTICLE CORRELATION
IN THE δ -FUNCTION MODEL

We study an exactly soluble model of the BBGKY hierarchy. The Lorentz gas equations are modeled by replacing the interaction function, I , with a δ -function. The first three equations of the hierarchy with a simple choice of ordering are given by

$$\frac{\partial f}{\partial t} = -\epsilon \int_{-\infty}^{+\infty} d\xi I(\xi) g(\xi, t) \quad (G1)$$

$$\frac{\partial g}{\partial t}(\xi_1) + \frac{\partial g}{\partial \xi_1} = I(\xi_1) f - \alpha \epsilon \int_{-\infty}^{+\infty} d\xi_2 I(\xi_2) h_{12}(\xi_1, \xi_2) \quad (G2)$$

$$\frac{\partial h_{12}}{\partial t} + \left(\frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) h_{12} = I(\xi_1) g(\xi_2) + I(\xi_2) g(\xi_1) \quad (G3)$$

where we have used the notation

$$\xi_i = x - x_i \quad (G4)$$

to denote the separation between the electron and the i 'th ion. The equations are written one-dimensionally and the electron is always moving to the right with unit velocity. The model extended in an obvious way to the entire hierarchy is exactly soluble and the solutions show a damping of all the correlations, as well as amplitude amplification due to the direct interaction terms that have been omitted here for simplicity. In order to obtain an asymptotic approximation to the exact solution, we introduce both time and space scales in the following fashion

$$\tau_n = \epsilon^n t \quad (G5)$$

$$\sigma_n^1 = \epsilon^n \xi_1 \quad (G6)$$

$$\sigma_n^2 = \epsilon^n \xi_2 \quad (G7)$$

We now extend the equations by introducing a suitable hyperplane in the phase mixing integrations and a reaction function in the source of correlations. The purpose of the reaction function, R , is that of allowing for the feedback of the rate of change of f on g (for times on the kinetic scale). The feedback then gives a spatial damping to the correlation on the mean free path scale. The equations, extended in this manner, are given by

$$\frac{Df}{Dt} = - \epsilon \iint d\sigma_0 d\sigma_1 \delta(\sigma_1 - \epsilon\sigma_0) I(\sigma_0) g(\sigma_0, \sigma_1) \quad (G8)$$

$$\begin{aligned} \frac{Dg}{Dt}(\sigma^1) &= I(\sigma_0^1) R(\sigma_1^1 - \epsilon\sigma_0^1) f + \\ &- \epsilon \alpha \iint d\sigma_0^2 d\sigma_1^2 \delta(\sigma_1^2 - \epsilon\sigma_0^2) I(\sigma_0^2) h(\sigma_0^1, \sigma_1^1) \sigma_0^2, \sigma_1^2 \end{aligned} \quad (G9)$$

$$\frac{\partial h_{12}}{\partial \tau_0} + \left(\frac{\partial}{\partial \sigma_0^1} + \frac{\partial}{\partial \sigma_0^2} \right) h_{12} = I(g_0^1) g^0(\sigma_0^2, \sigma_1^2 | \tau_0) + I(\sigma_0^2) g^0(\sigma_0^1, \sigma_1^1 | \tau_0) \quad (G10)$$

where we have the important requirement

$$R(0) = 1 \quad (G11)$$

In order to attain the approximation, we expand all quantities in powers of ϵ , requiring the expansions to be uniform. We can use the Taylor expansions

$$\delta(\sigma_1 - \epsilon\sigma_0) = \delta(\sigma_1) - \epsilon\sigma_0 \delta^1(\sigma_1) \quad (G12)$$

$$R(\sigma_1 - \epsilon\sigma_0) = R(\sigma_1) - \epsilon\sigma_0 R^1(\sigma_1) \quad (G13)$$

because the interaction function has a sharp cutoff and therefore neutralizes the apparent growths in (G12) and (G13). Expanding in powers of ϵ , we obtain for the velocity distribution function

$$\frac{\partial f^0}{\partial \tau_0} = 0 \quad (\text{G14})$$

$$\frac{\partial f^0}{\partial \tau_1} + \frac{\partial f^1}{\partial \tau_0} = - \int d\sigma_0 I(\sigma_0) g^0(\sigma_0, 0) \quad (\text{G15})$$

$$\frac{\partial f^0}{\partial \tau_2} + \frac{\partial f^1}{\partial \tau_1} + \frac{\partial f^2}{\partial \tau_0} = - \int d\sigma_0 I(\sigma_0) g^1(\sigma_0, 0) \quad (\text{G16})$$

We notice that the correlation function contributes under the phase mixing only on the short space scale, as required by physical reasons (e.g., Grad's derivation of the kinetic equation). The leading-order correlation satisfies

$$\frac{\partial g^0}{\partial \tau_0} + \frac{\partial g^0}{\partial \sigma_1} = I(\sigma_0^1) R(\sigma_1^1) f^0 \quad (\text{G17})$$

For the δ -function model, this equation yields

$$g^0 = \theta(\sigma_0) \theta(\tau_0 - \sigma_0) R(\sigma_1) f^0(\tau_1) \quad (\text{G18})$$

After some calculation, we obtain for g^1

$$\frac{\partial g^1}{\partial \tau_0} + \frac{\partial g^1}{\partial \sigma_0} = I(\sigma_0) R(\sigma_1) f^1 + \theta(\sigma_0) \theta(\tau_0 - \sigma_0) \left\{ R(\sigma_1) (1 - \alpha) - R'(\sigma_1) \right\} f^0(\tau_1) \quad (\text{G19})$$

We notice that while the first term on the right-hand side is short-ranged in space, the second one is not. It is thus

necessary to put to zero the quantity in the round bracket. If this were not done, the quantity in the second term in (G19) would give rise to a growing contribution in g^1 . This is precisely what is done in Bogolubov's theory. Setting the round bracket to zero, we have

$$R' = (1-\alpha)R \quad (G20)$$

whose solution is given by

$$R = e^{\sigma_1(1-\alpha)} \quad (G21)$$

utilizing the boundary conditions given by (G11). Inserting this result into (G18) for the classical calculation, and returning to the original time and space variables using (G5), (G6), and (G7), we obtain

$$\begin{aligned} g^0 &= \theta(t)\theta(t-x)e^{\epsilon x(1-\alpha)} e^{-\epsilon t} \\ &= \theta(t)\left\{\theta(t-x)e^{-\epsilon(t-x)}\right\}e^{-\alpha\epsilon x} \end{aligned} \quad (G22)$$

The quantity in the round brackets represents a wave travelling to the right, carrying the information that some electrons have interacted with the ion at the time of the disturbance. The damping factor ($0 < \alpha < 1$) then insures that the two-particle correlation has a peak that is slowly damped on the mean free path scale as the separation from the ion increases. Thus, we see that the two-particle correlation has the shape of a teardrop, detaching itself slowly from the ion along the electron velocity and the damping factor of ions makes the tear dissolve slowly in the background due to the three-body interactions.

Within the model, $\alpha > 1$ is possible, giving a slow increase in the peak value of the two-particle correlation function whose range, however, remains finite, of the order of the mean free path.

This is suggestive of an attractive potential since, in this case, three-body collisions may generate bound pairs, thus strengthening the value of the two-body correlation rather than damping it. In the next appendix, we see how this effect is arising in the Lorentz gas.

APPENDIX H. DAMPING OF THE TWO-BODY CORRELATION BY THREE-BODY COLLISIONS IN THE LORENTZ GAS

We discuss the effect of three-body interactions on the two-body correlation and show that the result is slow damping of the peak value of the two-body correlation. This effect has been discussed in the previous appendix for a model. In the Lorentz gas, we obtain, for the three-particle correlation function in leading order,

$$h_{12}^0 = \xi[I_1 g_2^0 + I_2 g_1^0] \quad (H1)$$

The phase-mixing contribution to the second-order two-particle correlation function is given by

$$L_2 \xi I_1 g_2^0 + L_2 \xi I_2 g_1^0 \quad (H2)$$

The nature of these two terms is greatly different if one considers that the location of the second ion is the one that is undergoing integration, while the location of the first ion is the free variable of the expression. Thus, the first term in Eq. (H2) is localized sharply by the presence of the interaction, I_1 , which has the usual finite range; while the second term in Eq. (H2) does not have a finite range since the free variable of the expression is associated with the two-particle correlation which is essentially a wave travelling in the direction of the electron's velocity. Therefore, we must not, as in the Choh-Uhlenbeck theory, include both terms as three-particle contributions to the one-body velocity distribution function. Rather,

the second term in (H2) contributes to the damping of the two-particle correlation function along the lines suggested by Frieman and Goldman. In the Lorentz case, the calculations can be carried out explicitly. We then find, from the second-order equation for the correlation function, the following condition for the removal of these secular terms in the correlation function:

$$\frac{\partial g^0}{\partial \tau_2} + \frac{\partial g^0}{\partial \sigma_2} = c_2 g^0 \quad (\text{H3})$$

Here, we have used the notation

$$c_2 = L\zeta^* I \quad (\text{H4})$$

The expression for the leading-order correlation function can be written as

$$\begin{aligned} g^0 &= \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial \sigma_0} (\sigma_0 - v\lambda) \cdot \frac{\partial}{\partial v} R(\sigma_2) f^0(\tau_2) \\ &= \Omega R f^0 \end{aligned} \quad (\text{H5})$$

By using this expression and exploiting the kinetic condition, we can then obtain from Eq. (H3) the equation that determines the reaction function, R . The kinetic condition is written as

$$\frac{\partial f^0}{\partial \tau_2} = c_2 f^0 = L\zeta^* I f^0 \quad (\text{H6})$$

Substitution of (H6) into (H5) and carrying out the differentiations in (H3), we obtain

$$\Omega R c_2 f^0 + \Omega R' f^0 = c_2 \Omega R f^0 \quad (\text{H7})$$

This equation is valid when applied to an arbitrary distribution

function, f^0 . Therefore, Eq. (H7) can be rewritten in the Heisenberg form

$$(\Omega R)' = [C_2, \Omega R] \quad (H8)$$

where the square bracket represents the commutator. The Heisenberg equation is solved by

$$\Omega R = e^{\sigma_2 C_2} \Omega e^{-\sigma_2 C_2} \quad (H9)$$

since we have the boundary condition

$$R(0) = 1 \quad (H10)$$

Inserting (H9) into (H5), for the two-particle correlation we have

$$\begin{aligned} g^0 &= e^{\sigma_2 C_2} \Omega e^{-\sigma_2 C_2} f(\vec{\tau}_2) \\ &= e^{\sigma_2 C_2} \Omega e^{+(\tau_2 - \sigma_2) C_2} f(\vec{\tau}_3) \end{aligned} \quad (H11)$$

The quantity to the right of the operator, Ω , represents the finite spatial range of the two-particle correlation function. The damping due to the three-body effects comes from the first term on the left of the operator, Ω . In order to see the effect of this operator, Ω , we rewrite it in the following form

$$\begin{aligned} C_2 &= \int d\mathbf{x}_2 I(\xi_1) \xi^* (i\mathbf{v} \cdot \nabla) I(\xi_2) \\ &= \int d\mathbf{x}_2 I(\xi_2) \left\{ \frac{\mathcal{P}}{\mathbf{v} \cdot \nabla} - \pi \delta(\mathbf{v} \cdot \nabla) \right\} I(\xi_2) \end{aligned} \quad (H12)$$

We can then express the two-particle correlation as

$$g^0 = e^{\epsilon^2} \times C_2 \left\{ \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-v\lambda) \right\} \cdot \frac{\partial}{\partial v} e^{\epsilon^2(t-x)C_2[f(\frac{1}{\tau_3})]} \quad (H13)$$

It is clear that in order to understand the effect of C_2 in this relation, we must understand its effect on the vector

$$\vec{\gamma} = \int_0^t d\lambda \frac{\partial \phi}{\partial \vec{x}}(x-v\lambda) \quad (H14)$$

An explicit representation of this vector can be obtained by suitable manipulations with the variables. After some calculations, we find

$$\begin{aligned} \vec{\gamma} = & \vec{x}_\perp \frac{1}{v} \int_{x_\parallel - \tau_0}^{x_\parallel} \frac{\phi'(\sqrt{x_\perp^2 + z^2})}{\sqrt{x_\perp^2 + z^2}} dz + \\ & + \frac{1}{v} \left[\phi(x) - \phi \sqrt{x_\perp^2 - (x_\parallel - v\tau_0)^2} \right] \end{aligned} \quad (H15)$$

By inspection of (H12), we now see that the only contributions can occur when the directional derivative of $\vec{x}$ in the direction of the velocity is very small. We see that the first term of γ therefore can never contribute since it depends strongly on $x_\parallel$. All the contributions must come from the second term which, by the way, is the term contributing in thermodynamic equilibrium. C_2 is an isotropizing operator and this result is quite reasonable. The contributions that are obtained are those that come from the δ -function which, since it carries a minus sign and is a positive operator, allows us to determine the effect of C_2 on g . The principal value cannot contribute because ϕ is symmetric. We now see that if the potential is repulsive and, therefore, C_2 is positive, the effect of C_2 on g is damping. It is very interesting to notice that if, on the other hand, the potential were attractive, the effect of C_2 on g would be an enhancement. This means that as time goes on the formation of bound pairs can increase the correlation between particle pairs.

APPENDIX I. COMPATIBLE SECULARITY CONDITIONS

In this appendix we discuss the method for the extraction of secularities in a compatible way in space-time scale equations. For this purpose, we write the three-particle correlation function as

$$h_{12}^{0*} = \zeta^* I_1 g_2^{0*} + \zeta^* I_2 g_1^{0*} = \zeta^* (I_1 \zeta^* I_2 + I_1 \zeta^* I_1) f^0 \quad (I1)$$

Similarly, the two-particle correlation is written as

$$g_1^{0*} = \zeta^* I_1 f^0 \quad (I2)$$

$$g_1^{1*} = \zeta^* I_1 f^1 + (\zeta^* I_1) g_1^{0*} \quad (I3)$$

$$g_1^{2*} = \zeta^* I_1 f^2 + (\zeta^* I_1) g_1^{1*} + \int dx_2 I_2 \zeta^* I_1 g_2^{0*} \quad (I4)$$

Successive substitution of (I2) into (I3) and (I3) into (I4), with the use of (I1), gives the following expression

$$g_1^{1*} = (\zeta^* I_1) f^1 + (\zeta^* I_1)^2 f^0 \quad (I5)$$

$$g_1^{2*} = (\zeta^* I_1) f^2 + (\zeta^* I_1)^2 f^1 + (\zeta^* I_1)^3 f^0 + \int dx_2 I_2 (\zeta^* I_1)^2 f^0 \quad (I6)$$

The important quantities are those associated with the phase mixing which can then be written as

$$L g^{0*} = L \zeta^* I f^0 = c_2 f^0 = (\Gamma^2) f^0 \quad (I7)$$

$$L g^{1*} = c_2 f^1 + \Gamma_3 f^0 = c_2 f^1 + c_3 f^0 + \Delta_3 f^0 \quad (I8)$$

$$Lg^{2*} = c_2 f^2 + \Gamma_3 f^1 + \Gamma_4 f^0 = c_2 f^2 + c_3 f^1 + \Delta_3 f^1 + c_4 f^0 + \Delta_4 f^0 \quad (I9)$$

The point of these expressions is to decompose all quantities relative to the order of the velocity distribution function. If only we had written

$$\Gamma_3 = c_3 + \Delta_3 \quad (I10)$$

$$\Gamma_4 = \Gamma_{4(2)} + \Gamma_{4(3)} = c_4 + \Delta'_4 + \Delta_4 \quad (I11)$$

The determination of these operators is tedious but straightforward. We have the kinetic conditions

$$\frac{\partial f^0}{\partial \tau_2} = c_2 f^0 \quad (I12)$$

$$\frac{\partial f^0}{\partial \tau_3} = c_3 f^0 \quad (I13)$$

$$\frac{\partial f^0}{\partial \tau_4} = c_4 f^0 \quad (I14)$$

We now have to obtain the equations for the first-order distribution. They are of the following form

$$\frac{\partial f^1}{\partial \tau_2} = c_2 f^1 + \Delta_3 f^0 \quad (I15)$$

$$\frac{\partial f^1}{\partial \tau_3} = c_3 f^1 + \Delta_4 f^0 \quad (I16)$$

Compatibility determines the precise form of Δ_4 . We have, by cross differentiating,

$$\frac{\partial^2 f^1}{\partial \tau_3 \partial \tau_2} = c_2 \frac{\partial f^1}{\partial \tau_3} + \Delta_3 \frac{\partial f^0}{\partial \tau_3} = c_2 c_3 f^1 + c_2 \Delta_4 f^0 + \Delta_3 c_3 f^0 \quad (I17)$$

$$\frac{\partial^2 f^1}{\partial \tau_2 \partial \tau_3} = c_3 \frac{\partial f^1}{\partial \tau_2} + \Delta_4 \frac{\partial f^0}{\partial \tau_2} = c_3 c_2 f^1 + c_3 \Delta_3 f^0 + \Delta_4 c_2 f^0 \quad (I18)$$

If these two equations are to be compatible, one readily shows that

$$[\Delta_3, c_3] = [\Delta_4, c_2] \quad (I19)$$

Since the kinetic condition in fourth order is given by

$$\frac{\partial f^2}{\partial \tau_2} + \frac{\partial f^1}{\partial \tau_3} + \frac{\partial f^0}{\partial \tau_4} = Lg^{2*} = c_2 f^2 + c_3 f^1 + \Delta_3 f^1 + c_4 f^0 + \Delta_4' f^0 + \Delta_4^1 f^0 \quad (I20)$$

we see that we must adjust Δ_4' so as to satisfy (I19) and then we are left with the kinetic condition for the second-order correction given by

$$\frac{\partial f^2}{\partial \tau_2} = c_2 f^2 + \Delta_3 f^1 + \Delta_4^1 f^0 \quad (I21)$$

The operators C and Δ are expressible in terms of the potential function quite generally. The expressions are too long to be given here.

Since the compatibility conditions are expressed by the commutativity of the C and Δ operators at each order, one must calculate the diagonal part of an operator relative to the lower order. Klimas and Sandri have obtained a precise construction. As an example, their formula gives in third order

$$c_3 = D_2(\Gamma_3) \\ = \sum_n |n\rangle \langle n| \Gamma_3 |n\rangle \langle n|$$

where $|n\rangle$ is a complete set of eigenfunctions for C_2 .

APPENDIX J. CONSTRUCTION OF THE COLLISION INTEGRAL

We give the method that yields the collision integrals to any order in the Lorentz gas. We illustrate the method with the third-order case, and avoid giving formulae for the higher-orders since they become unwieldy. We need the diagonal part of the quantity

$$L(\xi^* I)^2 \quad (J1)$$

where by the diagonal part we mean that part that commutes with the second-order collision integral, C_2 . The construction of the diagonal part of Eq. (J1) is greatly simplified if we use the following notation. We introduce a vector

$$\psi_i = \frac{\partial \phi}{\partial x_i} \quad (J2)$$

We then form a table in which the displacements are represented symbolically by Faltung symbols. Thus,

$$\begin{array}{ccccccc} \psi_i & * & \partial_{v_i} \psi_j & * & \psi_k & \times & \partial_{v_j}^2 v_k \\ \psi_i & * & \psi_j & * & \partial_{v_i} \psi_k & \times & \partial_{v_j}^2 v_k \\ \psi_j & * & \psi_\ell & * & \partial_{v_\ell} \psi_k & \times & \partial_{v_j}^2 v_k \\ \psi_j & * & \psi_\ell & * & \lambda \partial_{v_\ell} \psi_k & \times & \partial_{v_j}^2 v_k \end{array}$$

Table 2

The notation becomes immediately clear if one now considers the explicit representation of the result; namely, we write

$$T_{jk} = \int_0^\infty d\lambda \int_0^\infty d\mu \int d\vec{x} A_{jk} \quad (J3)$$

where the tensor A is given by

$$\begin{aligned} A_{jk} = & \left[\frac{\partial \phi}{\partial x_i}(x) \right] \left[\frac{\partial}{\partial v_i} \frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \left[\frac{\partial \phi}{\partial x_k}(x-v\lambda-v\mu) \right] + \\ & + \left[\frac{\partial \phi}{\partial x_i}(x) \right] \left[\frac{\partial \phi}{\partial x_j}(x-v\lambda) \right] \left[\frac{\partial}{\partial v_i} \frac{\partial \phi}{\partial x_k}(x-v\lambda-v\mu) \right] + \\ & + \left[\frac{\partial \phi}{\partial x_j}(x) \right] \left[\frac{\partial \phi}{\partial x_\ell}(x-v\lambda) \right] \left[\frac{\partial}{\partial v_\ell} \frac{\partial \phi}{\partial x_k}(x-v\lambda-v\mu) \right] + \\ & + \left[\frac{\partial \phi}{\partial x_j}(x) \right] \left[\frac{\partial \phi}{\partial x_\ell}(x-v\lambda) \right] \left[\lambda \frac{\partial}{\partial x_\ell} \frac{\partial \phi}{\partial x_k}(x-v\lambda-v\mu) \right] \end{aligned} \quad (J4)$$

Clearly, the contribution to (J1) which is of second order in velocity derivatives is given by

$$\left\{ L(\xi^* I)^2 \right\}_2 = T_{jk} \frac{\partial^2}{\partial v_j \partial v_k} \quad (J5)$$

Since the tensor T is a second-rank symmetric tensor, depending on the single vector, $\vec{v}$, we have

$$T_{jk} = T_{jk}(\vec{v}) = \delta_{jk} \alpha(v) + e_i e_j \beta(v) \quad (J6)$$

Therefore, the second-order contribution to the phase mixing is given by

$$\begin{aligned}
\left\{L(\xi^*I)^2\right\}_2 &= \alpha(v)\nabla^2 + \beta(v)(e\cdot\nabla)^2 \\
&= \alpha(v)\hat{\nabla}^2 + \left\{(\beta-\alpha)(e\cdot\nabla)^2 - \frac{2\alpha}{v} e\cdot\nabla\right\} \quad (J7)
\end{aligned}$$

It can be proved that the only part of (J7) that commutes with the angular part of the Laplacian is the part proportional to $\hat{\nabla}^2$. We can extract this coefficient by noticing the following two properties:

$$\sum_i T_{ii} = 3\alpha + \beta \quad (J8)$$

$$e_i T_{ij} e_j = \alpha + \beta \quad (J9)$$

By subtracting these two equations, we immediately obtain

$$\alpha = \frac{1}{2}(\text{Trace } T - e\cdot T\cdot e) \quad (J10)$$

This quantity is readily constructed from the expressions (J3) and (J4). The third-order kinetic equation is then given by

$$\frac{\partial f^0}{\partial \tau_3} = c_3 f^0 \quad (J11)$$

where the collision integral is

$$c_3 = \alpha(v)c_2 = \frac{\alpha(1)}{v^2} c_2 = \frac{\alpha(1)}{v^3} \hat{\nabla}^2 \quad (J12)$$

The compatibility condition of this equation with the second-order kinetic equation which represents the Fokker-Planck theory is clear since the Fokker-Planck equation is given by

$$\frac{\partial f^0}{\partial \tau_2} = c_2 f^0 \quad (J13)$$

It is now clear the entropy theorem is satisfied because the operator C_2 has only negative eigenvalues and f^0 can be written

$$f^0 = e^{C_2 t \left(\frac{a}{v} + \epsilon \frac{\alpha(1)}{v^3} \right)} \quad (J14)$$

It is clear that in view of the negative-definite property of C_2 , we need not worry about the numerical coefficient α being positive. This is in sharp contrast with the Choh-Uhlenbeck theory where the entropy theorem could not be proved.