Particle Flux Associated With Stochastic Processes

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Abstract

The Lagrange expansion, which may be used to derive the Fokker-Planck equation, is here used to derive the corresponding expression for the flux of particles subject to a stochastic scattering process. The coefficients which occur in this expression are, in general, not the same as the coefficients which occur in the Fokker-Planck equation itself. In the special case that the particle distribution involves only one independent variable, the particle flux is determined by the familiar Fokker-Planck coefficients. Evaluation of particle flux is of special interest in the study of stochastic acceleration.

The behavior of particles moving under the influence of random force fields is of interest in astrophysics. One example is the heating and diffusion of charged particles in the solar wind (Barnes, 1967; Jokipii 1966), and another is the acceleration of charged particles by the Fermi-type process of stochastic acceleration (Hall and Sturrock, 1967; Melrose, 1969; Tsytovich, 1966). Under conditions which are normally accepted, such a process may be represented by a Fokker-Planck equation, which gives the formula for the time derivative of the particle distribution function. However, in interpreting various solutions of this equation, it is helpful to know the flux of particles--in real space, momentum space, or in energy, as is appropriate to the problem. The formula for the particle flux is simple, but appears not to be widely known. This paper therefore gives a brief derivation of this formula.

In an earlier publication (Sturrock, 1960; hereafter called paper I), it was shown that the Fokker-Planck equation may conveniently be derived from a generalization of the Lagrange expansion. This publication also presented a formula for particle flux, so it offers a convenient starting point for this paper.

We consider the distribution of particles in a space enumerated by variables x_r, $r = 1, \ldots, n$. These may be spatial coordinates, components of velocity or momentum, or any combination thereof. The distribution function $f(x,t)$ has the property that $f(x,t)d^n x$ is the number of particles in the volume $d^n x$ centered on the position x at time t. If, with the requirement that dx be small compared with the scale determined by the gradient of f, $f\, d^n x$ is not a large number, it is necessary to interpret this number as an "expectation" value.

If, when t changes to $t + \Delta t$, the particle which was at position x moves to position $x + \Delta x$, the distribution function at time $t + \Delta t$ is related to that at time t by the Lagrange expansion. An appropriate change of notation of equation (3.6) of paper I leads to

$$f(x,t+\Delta t) = f(x,t) - \frac{\partial}{\partial x_r}\left(f(x,t)\ \Delta x_r\right) + 1/2\ \frac{\partial^2}{\partial x_r \partial x_s}\left(f(x,t)\ \Delta x_r \Delta x_s\right) - \dots \tag{1}$$

The Fokker-Planck equation is derived from equation (1) by assuming that one may choose the time interval Δt such that (a) the distribution function f changes only slightly in this time; (b) the associated values of Δx_r are small compared with the scale determined by the gradient of f; and (c) that a large number of random processes occur in this time with the result that

$$\langle \Delta x_r \rangle = O(\Delta t),\ \langle \Delta x_r \Delta x_s \rangle = O(\Delta t),\ \langle \Delta x_r \Delta x_s \Delta x_t \rangle = o(\Delta t),\ \text{etc.} \tag{2}$$

where angular brackets denote the expectation value of a quantity. When the time interval Δt is chosen in this way, equation (1) takes the usual form of the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x_r}\left(f\left\langle\frac{\Delta x_r}{\Delta t}\right\rangle\right) + 1/2\ \frac{\partial^2}{\partial x_r \partial x_s}\left(f\left\langle\frac{\Delta x_r \Delta x_s}{\Delta t}\right\rangle\right)\ : \tag{3}$$

Note that the term $\Delta x_r/\Delta t$ may represent the effect of a small steady force field as well as that of a randomly fluctuating force field.

We now wish to obtain an expression for the flux F_r, the derivation of which is to be analogous to the derivation of equation (3).

The same change of notation in equation (3.9) of paper I gives the following expression for the averaged flux:

$$F_r = f\left\langle\frac{\Delta x_r}{\Delta t}\right\rangle - \frac{\partial}{\partial x_s}\left(f\left\langle\frac{d\ \Delta x_r}{dt}\ \Delta x_s\right\rangle\right) \tag{4}$$

We see that, although the first term on the right-hand side of equation (4) also appears in the Fokker-Planck equation (equation 3), the second term does not. Nevertheless, we may confirm that the continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial F_r}{\partial x_r} = 0 \tag{5}$$

is equivalent to equation (3).

For this purpose, it is convenient to rewrite the flux as

$$F_r = f\left\langle\frac{\Delta x_r}{\Delta t}\right\rangle - \frac{\partial}{\partial x_s}\left(f\ \Gamma_{rs}\right) \tag{6}$$

where

$$\Gamma_{rs} = \left\langle\frac{d\Delta x_r}{dt}\ \Delta x_s\right\rangle . \tag{7}$$

This may be expressed as

$$\Gamma_{rs} = \Gamma_{\underline{rs}} + \Gamma_{\underset{\vee}{rs}} \tag{8}$$

where $\Gamma_{\underline{rs}}$ and $\Gamma_{\underset{\vee}{rs}}$ are symmetric and antisymmetric, respectively, in their suffixes. The symmetric component is expressible as a total derivative and may therefore be related to the second term of the Fokker-Planck equation:

$$\Gamma_{\underline{rs}} = 1/2\left\langle\frac{d\Delta x_r}{dt}\ \Delta x_s + \Delta x_r\ \frac{d\Delta x_s}{dt}\right\rangle = 1/2\left\langle\frac{d}{dt}\ \left(\Delta x_r \Delta x_s\right)\right\rangle = 1/2\left\langle\frac{\Delta x_r \Delta x_s}{\Delta t}\right\rangle\ . \tag{9}$$

The antisymmetric term, given by

$$\Gamma_{\underset{\vee}{rs}} = 1/2\left\langle\frac{d\Delta x_r}{dt}\ \Delta x_s - \Delta x_r\ \frac{d\Delta x_s}{dt}\right\rangle \quad , \tag{10}$$

does not appear in the Fokker-Planck equation because the term $\frac{\partial}{\partial x_s}\left(f\ \Gamma_{\underset{\smile}{rs}}\right)$ is divergence free. Substitution of formula (4) into equation (5) therefore yields equation (3).

We see that, in general, the particle flux cannot be expressed in terms of the coefficients of the Fokker-Planck equation alone but in addition requires coefficients of the form $\left\langle \frac{d\ \Delta x_r}{dt}\ \Delta x_s \right\rangle$, which are readily calculated in the same manner as the usual Fokker-Planck coefficients. An important exception to this rule is the case that the system has only one independent coordinate. If, for instance, we are concerned only with the energy E of particles, the Fokker-Planck equation has the form

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial E}\left(f\left\langle\frac{\Delta E}{\Delta t}\right\rangle\right) + 1/2\ \frac{\partial^2}{\partial E^2}\left(f\left\langle\frac{(\Delta E)^2}{\Delta t}\right\rangle\right) \tag{11}$$

and the flux (in energy) is given by

$$F_E = f\left\langle\frac{\Delta E}{\Delta t}\right\rangle - 1/2\ \frac{\partial}{\partial E}\left(f\left\langle\frac{(\Delta E)^2}{\Delta t}\right\rangle\right) \quad , \tag{12}$$

with no additional terms.

This formula is of special interest for discussion of stochastic acceleration, since it is important to know whether a particular solution of the Fokker-Planck equation represents transfer of particles from low energy to high energy (acceleration) or from high energy to low energy (deceleration). This consideration has special relevance to a recent article by Melrose (1969) as will be discussed in more detail in a separate article (Tademaru, Newman and Jones, 1970).

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