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TWO SETS OF LINEARIZED AIRCRAFT EQUATIONS OF MOTION FOR CONTROL SYSTEM ANALYSIS

By Paul S. Rempfer and Lloyd Stevenson

Guidance and Control Research Directorate
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INTRODUCTION

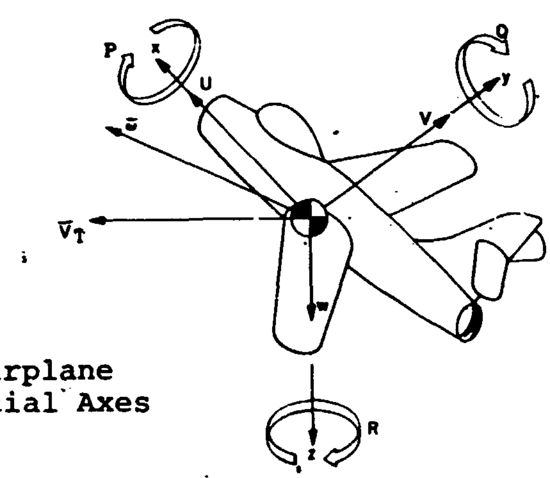
In developing a mathematical model for a tandem rotor helicopter to use in the analysis of an automatic approach and landing system for that helicopter, linearized equations not normally used in aircraft stability analysis were derived. These equations are derived herein along with those normally used (refs. 1,2).

DERIVATION OF THE NON-LINEAR EQUATIONS OF MOTION

Before developing the linearized equations of motion for analysis purposes, the non-linear equations are developed.

Rotary Motion

Consider the motion of an aircraft as shown in this figure.



Motion of An Airplane
Referred to Inertial Axes

Assume that the x-z plane is one of symmetry and define the inertia matrix as

$$I \triangleq \begin{bmatrix} I_{xx} & 0 & J_{xz} \\ 0 & I_{yy} & 0 \\ J_{xz} & 0 & I_{zz} \end{bmatrix}$$

where

$$\left.\begin{aligned} J_{xy} &\triangleq -\int xy\ dm = 0 \\ J_{zy} &\triangleq -\int zy\ dm = 0 \end{aligned}\right\} \text{ due to x-z symmetry}$$

$$J_{xz} \triangleq -\int xz\ dm$$

Note in Figure 1 that

$$\overline{\omega} \triangleq \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

where P, Q, R is the angular velocity resolved along the body axes x, y, and z, respectively. Similarly, let the applied torque be denoted as

$$\overline{T} \triangleq \begin{bmatrix} L \\ M \\ N \end{bmatrix} .$$

With the assumption that the angular momentum of any rotating machinery on-board, such as engines and, in helicopters, the rotor, is negligible, the equations in inertial space are written simply

$$\overline{T} = \frac{d}{dt}(I\overline{\omega}) . \tag{1}$$

On the assumption that the Earth is inertial, then Eq. (1) is written in the body axis system as

$$\overline{T} = I\frac{d'\overline{\omega}}{dt} + \overline{\omega} x I \overline{\omega} \tag{2}$$

where the prime denotes "as seen by an observer fixed in the body axes," and therefore

$$\frac{d'\overline{\omega}}{dt} \triangleq \begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix}$$

where the dot denotes total time derivative. Substituting the appropriate definitions into Eq. (2) results in the equation set

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + J_{xz}\dot{R} \\ I_{yy}\dot{Q} \\ J_{xz}\dot{P} + I_{zz}\dot{R} \end{bmatrix} + \begin{bmatrix} J_{xz}PQ + I_{zz}QR - I_{yy}QR \\ I_{xx}RP + J_{xz}(R^2-P^2) - I_{zz}PR \\ I_{yy}PQ - I_{xx}PQ - J_{xz}RQ \end{bmatrix} \tag{3}$$

These equations define the rotary motion of the aircraft.

Translatory Motion

Assume that the mass of the aircraft is constant and denoted m. Let the force vector be denoted

$$\overline{F} \triangleq \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where X, Y, and Z, lie along the body axes. Note from Figure 1 that the total velocity

$$\overline{V}_T \triangleq \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

is resolved along the body axes. The equation in inertial space may then be written

$$\overline{F} = \frac{d}{dt}(m\overline{V}_T) \tag{4}$$

On the assumption that the Earth is inertial, then Eq. (4) is written in the body axis system as

$$\overline{F} = m\frac{d'\overline{V}_T}{dt} + \overline{\omega}\times m\overline{V}_T. \tag{5}$$

Making appropriate substitutions into Eq. (5) gives the equation set

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} m\dot{U} \\ m\dot{V} \\ m\dot{W} \end{bmatrix} + \begin{bmatrix} mQW - mRV \\ mRU - mPW \\ mPV - mQU \end{bmatrix} \tag{6}$$

These equations define the translatory motion of the aircraft.

Euler Angle Transformations

Consider the aircraft of Figure 1. We wish to describe the attitude of the aircraft with respect to a set of axes fixed in the Earth. To do this we define an Euler angle set denoted Ψ, Θ, Φ. These three angles are the azimuth change, elevation change, and roll required to arrive at the aircraft attitude from the inertial axes. They must be taken in the given order. If a vector is denoted in the inertial coordinates as $\bar{C}_i$ and viewed from a coordinate system which has been slewed through Ψ, we get

$$\bar{C}_1 = T_z(\Psi)\bar{C}_i \tag{7}$$

where

$$T_z(\Psi) \triangleq \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If the observer is then elevated through Θ, he sees

$$\bar{C}_2 = T_y(\Theta)\bar{C}_1 = T_y(\Theta)T_z(\Psi)\bar{C}_i \tag{8}$$

where

$$T_y(\Theta) \triangleq \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}$$

Finally, if the observer is rolled through Φ, he sees from aircraft body axes

$$\overline{C}_b = T_x(\Phi)\overline{C}_2 = T_x(\Phi)T_y(\Theta)T_z(\Psi)\overline{C}_i \ . \tag{9}$$

where

$$T_x(\Phi) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix}$$

Gravity Forces

With Eq. (9) the gravity force may be written in body axes as

$$\overline{W}_b = T_x(\Phi)T_y(\Theta)T_z(\Psi)\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = mg\begin{bmatrix} -\sin\Theta & \\ \sin\Phi & \cos\Theta \\ \cos\Phi & \cos\Theta \end{bmatrix} \tag{10}$$

Euler Angle Rate Equations

The body axis rates may be written as functions of the Euler angle rates:

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = T_x(\Phi)\left[\begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix} + T_y(\Theta)\left[\begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} + T_z(\Psi)\begin{bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{bmatrix}\right]\right]$$

$$= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Theta & \sin\Phi\cos\Theta \\ 0 & -\sin\Theta & \cos\Phi\cos\Theta \end{bmatrix}\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} \tag{11}$$

Inertial Velocity Equations

If the velocity in inertial coordinates is denoted

$$\overline{V}_{T_i} \triangleq \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} ,$$

then we may write the body axis velocities as functions of the inertial velocities

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = T_x(\Phi)T_y(\Theta)T_z(\Psi)\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

$$= \begin{Bmatrix} V_x\cos\Theta\cos\Psi + V_y\cos\Theta\sin\Psi - V_z\sin\Theta \\ V_x(\sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi) \\ V_x(\cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi) \end{Bmatrix}$$

$$+ \begin{Bmatrix} 0 \\ V_y(\sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi) + V_z\sin\Phi\cos\Theta \\ V_y(\cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi) + V_z\cos\Theta\cos\Phi \end{Bmatrix} \quad (12)$$

LINEARIZING OF EQUATIONS USING DELTA PERTURBATIONS

The purpose of this linear model is to analyze an automatic approach and landing system. As such, the system is tied to the Approach Navigation Frame. This is an Earth-fixed coordinate system with the origin at the desired touchdown point, the X axis along the runway, and the Z axis down along the local vertical. The system is considered inertial. The variables to be commanded will be in the ANF such as V_x, V_y, and V_z and Ψ, Θ, Φ. The control system will therefore be feeding back these quantities from an inertial platform. The linearized equations are then desired in terms of these variables. The derivation of these equations follows.

Body Translatory Equations

Rewrite equation set (6) with the forces being divided into aerodynamic forces and the gravity forces of Eq. (10)

$$\dot{U} = RV - QW + \frac{X_A}{m} - g\sin\Theta$$

$$\dot{V} = PW - RU + \frac{Y_A}{m} + g\sin\Phi\cos\Theta \quad (13)$$

$$\dot{W} = QU - PV + \frac{Z_A}{m} + g\cos\Phi\cos\Theta$$

Assume the following perturbations

$$\left.\begin{array}{lll} U = U_0 + \Delta U & P = \Delta P & \Theta = \Theta_0 + \Delta\Theta \\ V = \Delta V & Q = \Delta Q & \Phi = \Delta\Phi \\ W = W_0 + \Delta W & R = \Delta R & \Psi = \Delta\Psi \end{array}\right\} \tag{14}$$

Note that the perturbations are simply added to the Euler angles. If steady flight is assumed, the steady flight equations are obtained from Eq. (13):

$$\begin{aligned} 0 &= \frac{X_{A0}}{m} - g \sin \Theta_0 \\ 0 &= \frac{Y_{A0}}{m} \\ 0 &= \frac{Z_{A0}}{m} + g \cos \Theta_0 \quad . \end{aligned} \tag{15}$$

Next note that

$$\left.\begin{aligned} \sin (\Theta_0 + \Delta\Theta) &= \sin \Theta_0 + (\cos \Theta_0)\, \Delta\Theta \\ \cos (\Theta_0 + \Delta\Theta) &= \cos \Theta_0 - (\sin \Theta_0)\, \Delta\Theta \end{aligned}\right\} . \tag{16}$$

Substituting the definitions of Eq. (14) into Eqs. (13), using Eq. (16) and subtracting off steady flight, Eq. (15), yields the following perturbation equations

$$\begin{aligned} \Delta\dot{U} &= -W_0 \Delta Q - (g \cos \Theta_0)\Delta\Theta + \frac{\Delta X_A}{m} \\ \Delta\dot{V} &= W_0 \Delta P - U_0 \Delta R + (g \cos \Theta_0)\Delta\Phi + \frac{\Delta Y_A}{m} \\ \Delta\dot{W} &= U_0 \Delta Q - (g \sin \Theta_0)\Delta\Theta + \frac{\Delta Z_A}{m} \end{aligned} \tag{17}$$

Body Rate Equations

Substituting the perturbation definitions of Eq. (14) into Eq. (3) and dropping second-order terms gives

$$\Delta\dot{P} = -\frac{J_{xz}}{I_{xx}}\Delta\dot{R} + \frac{\Delta L}{I_{xx}}$$

$$\Delta\dot{Q} = \frac{\Delta M}{I_{yy}} \qquad (18)$$

$$\Delta\dot{R} = -\frac{J_{xz}}{I_{zz}}\Delta\dot{P} + \frac{\Delta N}{I_{zz}}$$

Euler Angle Equations

The relation between Euler angle perturbed rates and body perturbed rates is desired. This is obtained by inverting equation set (11) and applying the perturbation definitions to give

$$\Delta\dot{\Theta} = \Delta Q$$

$$\Delta\dot{\Phi} = \Delta P + (\tan\Theta_0)\ \Delta R \qquad (19)$$

$$\Delta\dot{\Psi} = \frac{\Delta R}{\cos\Theta_0}$$

Equation Set with Aerodynamic Partial Derivatives

The motion of the aircraft is described by Eqs. (17), (18), and (19). The force and moment perturbations are, in general, functions of the motion and can be written to first order as a linear function of the motion variables. Motions in the longitudinal plane are assumed to separate from the lateral-directional to give the total equation set which follows.

$$\Delta\dot{U} = -W_0\Delta Q - (g\cos\Theta_0)\Delta\Theta + \frac{X_u}{m}\Delta U + \frac{X_w}{m}\Delta W + \frac{X_q}{m}\Delta Q + \frac{X_\delta}{m}\delta$$

$$\Delta\dot{V} = W_0\Delta P - U_0\Delta R + (g\cos\Theta_0)\Delta\Phi + \frac{Y_v}{m}\Delta V + \frac{Y_p}{m}\Delta P + \frac{Y_r}{m}\Delta R + \frac{Y_\delta}{m}\delta$$

$$\Delta\dot{W} = U_0\Delta Q - (g \sin \Theta_0)\Delta\Theta + \frac{Z_u}{m}\Delta U + \frac{Z_w}{m}\Delta W + \frac{Z_q}{m}\Delta Q + \frac{Z_\delta}{m}\delta$$

$$\Delta\dot{P} = -\frac{J_{xz}}{I_{xx}}\Delta\dot{R} + \frac{L_v}{I_{xx}}\Delta V + \frac{L_p}{I_{xx}}\Delta P + \frac{L_r}{I_{xx}}\Delta R + \frac{L_\delta}{I_{xx}}\delta$$

$$\Delta\dot{Q} = \frac{M_u}{I_{yy}}\Delta U + \frac{M_w}{I_{yy}}\Delta W + \frac{M_q}{I_{yy}}\Delta Q + \frac{M_\delta}{I_{yy}}\delta$$

$$\Delta\dot{R} = -\frac{J_{xz}}{I_{zz}}\Delta\dot{P} + \frac{N_v}{I_{zz}}\Delta V + \frac{N_p}{I_{zz}}\Delta P + \frac{N_r}{I_{zz}}\Delta R + \frac{N_\delta}{I_{zz}}\delta \qquad (20)$$

$$\Delta\dot{\Theta} = \Delta Q$$

$$\Delta\dot{\Phi} = \Delta P + (\tan \Theta_0)\Delta R$$

$$\Delta\dot{\Psi} = \frac{\Delta R}{\cos \Theta_0}$$

Equation Set with Body Rates Eliminated

The equation set (20) may be reduced in variables by eliminating the body rates with the last three equations. The result is

$$\Delta\dot{U} = \frac{X_u}{m}\Delta U + \frac{X_w}{m}\Delta W - \left(W_0 - \frac{X_q}{m}\right)\Delta\dot{\Theta} - (g \cos \Theta_0)\Delta\Theta + \frac{X_\delta}{m}\delta$$

$$\Delta\dot{V} = \frac{Y_v}{m}\Delta V + \left(W_0 + \frac{Y_p}{m}\right)\Delta\dot{\Phi} + (g \cos \Theta_0)\Delta\Phi - \left(\sin \Theta_0\left(W_0 + \frac{Y_p}{m}\right)\right.$$
$$\left. + \cos \Theta_0\left(U_0 - \frac{Y_r}{m}\right)\right)\Delta\dot{\Psi} + \frac{Y_\delta}{m}\delta$$

$$\Delta\dot{W} = \frac{Z_u}{m}\Delta U + \frac{Z_w}{m}\Delta W + \left(U_0 + \frac{Z_q}{m}\right)\Delta\dot{\Theta} - (g \sin \Theta_0)\Delta\Theta + \frac{Z_\delta}{m}\delta$$

$$
\left.
\begin{aligned}
\Delta\ddot{\Phi} &= \frac{L_v}{I_{xx}}\Delta V + \frac{L_p}{I_{xx}}\Delta\dot{\Phi} - \left(\frac{J_{xz}}{I_{xx}}\cos\,\Theta_0 - \sin\,\Theta_0\right)\Delta\ddot{\Psi} - \left(\frac{L_p}{I_{xx}}\sin\,\Theta_0\right. \\
&\quad \left. - \frac{L_r}{I_{xx}}\cos\,\Theta_0\right)\Delta\dot{\Psi} + \frac{L_\delta}{I_{xx}}\delta \\
\Delta\ddot{\Theta} &= \frac{M_u}{I_{yy}}\Delta U + \frac{M_w}{I_{yy}}\Delta W + \frac{M_q}{I_{yy}}\Delta\dot{\Theta} + \frac{M_\delta}{I_{yy}}\delta \\
\Delta\ddot{\Psi} &= \frac{1}{I_{zz}\cos\,\Theta_0 - J_{xz}\sin\,\Theta_0}\Big(N_v\Delta V - J_{xz}\Delta\ddot{\Phi} + N_p\Delta\dot{\Phi} \\
&\quad - (N_p\sin\,\Theta_0 - N_r\cos\,\Theta_0)\Delta\dot{\Psi} + N_\delta\delta\Big)
\end{aligned}
\right\} \quad (21)
$$

Reduced Equation Set in Vector-Matrix Notation

Putting the equation set (21) in vector-matrix notation results in

<u>Longitudinal Equations</u>

(ANF co-ordinate system)

$$
\begin{bmatrix}
S-\frac{X_u}{m}, & -\frac{X_w}{m}, & \left(W_0-\frac{X_q}{m}\right)S + g\cos\,\Theta_0 \\
-\frac{Z_u}{m}, & S-\frac{Z_w}{m}, & -\left(U_0+\frac{Z_q}{m}\right)S + g\sin\,\Theta_0 \\
-\frac{M_u}{I_{yy}}, & -\frac{M_w}{I_{yy}}, & \left(S-\frac{M_q}{I_{yy}}\right)S
\end{bmatrix}
\begin{bmatrix} \Delta U \\ \Delta W \\ \Delta\Theta \end{bmatrix}
=
\begin{bmatrix} \frac{X_\delta}{m} \\ \frac{Z_\delta}{m} \\ \frac{M_\delta}{I_{yy}} \end{bmatrix}\delta
\qquad (22)
$$

Lateral-Directional Equations

(ANF co-ordinate system)

$$\begin{bmatrix} s-\frac{Y_v}{m} & , -\left(W_0+\frac{Y_p}{m}\right)s - g\cos\theta_0 & , \left(\left(W_0+\frac{Y_p}{m}\right)\sin\theta_0 + \left(U_0-\frac{Y_r}{m}\right)\cos\theta_0\right)s \\ -\frac{L_v}{I_{xx}} & , \left(s-\frac{L_p}{I_{xx}}\right)s & , \left(\left(\frac{J_{xz}}{I_{xx}}\cos\theta_0-\sin\theta_0\right)s + \left(\frac{L_p}{I_{xx}}\sin\theta_0-\frac{L_r}{I_{xx}}\cos\theta_0\right)\right)s \\ -\frac{N_v}{B} & , \left(\frac{J_{xz}}{B}s-\frac{N_p}{B}\right)s & , \left(s + \left(\frac{N_p}{B}\sin\theta_0-\frac{N_r}{B}\cos\theta_0\right)\right)s \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\Phi \\ \Delta\Psi \end{bmatrix} = \begin{bmatrix} \frac{Y_\delta}{m} \\ \frac{L_\delta}{I_{xx}} \\ \frac{N_\delta}{B} \end{bmatrix} \delta \qquad (23)$$

where $B \triangleq I_{zz}\cos\theta_0 - J_{xz}\sin\theta_0$ and $s \triangleq \frac{d}{dt}$ the Laplace differential operator.

Inertial Velocity Equations

If the equation set (12) is linearized, the result is given

$$\left.\begin{aligned} \Delta U &= \Delta V_x\cos\theta_0 - \Delta V_z\sin\theta_0 - W_0\Delta\theta \\ \Delta V &= \Delta V_y + W_0\Delta\Phi - (U_0\cos\theta_0 + W_0\sin\theta_0)\Delta\Psi \\ \Delta W &= \Delta V_x\sin\theta_0 + \Delta V_z\cos\theta_0 + U_0\Delta\theta \end{aligned}\right\} \qquad (24)$$

With these equations, the body axes velocities can be eliminated from the equation set (21) and the result is given

$$\begin{aligned} \Delta\dot{V}_x = {} & \left(\frac{X_u}{m} + \frac{X_w}{m}\tan\theta_0\right)\Delta V_x + (\tan\theta_0)\Delta\dot{V}_z - \left(\frac{X_u}{m}\tan\theta_0 - \frac{X_w}{m}\right)\Delta V_z \\ & + \frac{1}{\cos\theta_0}\left(\frac{X_q}{m}\Delta\dot{\theta} - \left(\frac{X_u}{m}W_0 - \frac{X_w}{m}U_0 + g\cos\theta_0\right)\Delta\theta + \frac{X_\delta}{m}\delta\right) \\ \Delta\dot{V}_y = {} & \frac{Y_v}{m}\Delta V_y + \frac{Y_p}{m}\Delta\dot{\Phi} + \left(\frac{Y_v}{m}W_0 + g\cos\theta_0\right)\Delta\Phi - \left(\frac{Y_p}{m}\sin\theta_0\right. \\ & \left. - \frac{Y_r}{m}\cos\theta_0\right)\Delta\dot{\Psi} - \frac{Y_v}{m}\left(U_0\cos\theta_0 + W_0\sin\theta_0\right)\Delta\Psi + \frac{Y_\delta}{m}\delta \end{aligned}$$

$$
\left.
\begin{aligned}
\Delta\dot{V}_z &= -(\tan\Theta_0)\Delta\dot{V}_x + \left(\frac{Z_u}{m} + \frac{Z_w}{m}\tan\Theta_0\right)\Delta V_x - \left(\frac{Z_u}{m}\tan\Theta_0 - \frac{Z_w}{m}\right)\Delta V_z \\
&\quad + \frac{1}{\cos\Theta_0}\left(\frac{Z_q}{m}\Delta\dot{\Theta} - \left(\frac{Z_u}{m}W_0 - \frac{Z_w}{m}U_0 + g\sin\Theta_0\right)\Delta\Theta + \frac{Z_\delta}{m}\delta\right) \\[2ex]
\Delta\ddot{\Theta} &= \left(\frac{M_u}{I_{yy}}\cos\Theta_0 + \frac{M_w}{I_{yy}}\sin\Theta_0\right)\Delta V_x - \left(\frac{M_u}{I_{yy}}\sin\Theta_0 - \frac{M_w}{I_{yy}}\cos\Theta_0\right)\Delta V_z \\
&\quad + \frac{M_q}{I_{yy}}\Delta\dot{\Theta} - \left(\frac{M_u}{I_{yy}}W_0 - \frac{M_w}{I_{yy}}U_0\right)\Delta\Theta + \frac{M_\delta}{I_{yy}}\delta \\[2ex]
\Delta\ddot{\Phi} &= \frac{L_v}{I_{xx}}\Delta V_y + \frac{L_p}{I_{xx}}\Delta\dot{\Phi} + \frac{L_v}{I_{xx}}W_0\Delta\Phi - \left(\frac{J_{xz}}{I_{xx}}\cos\Theta_0 - \sin\Theta_0\right)\Delta\ddot{\Psi} \\
&\quad - \left(\frac{L_p}{I_{xx}}\sin\Theta_0 - \frac{L_r}{I_{xx}}\cos\Theta_0\right)\Delta\dot{\Psi} - \frac{L_v}{I_{xx}}\Big(U_0\cos\Theta_0 \\
&\quad + W_0\sin\Theta_0\Big)\Delta\Psi + \frac{L_\delta}{I_{xx}}\delta \\[2ex]
\Delta\ddot{\Psi} &= \frac{1}{B}\Big(N_v\Delta V_y - J_{xz}\Delta\ddot{\Phi} + N_p\Delta\dot{\Phi} + N_vW_0\Delta\Phi - (N_p\sin\Theta_0 \\
&\quad - N_r\cos\Theta_0)\Delta\dot{\Psi} - N_v(U_0\cos\Theta_0 + W_0\sin\Theta_0)\Delta\Psi + N_\delta\delta\Big)
\end{aligned}
\right\}\ (25)
$$

where again $B \overset{\Delta}{=} I_{zz}\cos\Theta_0 - J_{xz}\sin\Theta_0$.

Inertial Velocity Equations in Vector-Matrix Notation

The equation set (25) may be put into vector-matrix notation with the result

Longitudinal Equations

(ANF coordinate system)

$$\begin{bmatrix} \left[(\cos\theta_0)s - \left(\frac{X_u}{m}\cos\theta_0 + \frac{X_w}{m}\sin\theta_0\right)\right] & \left[(-\sin\theta_0)s + \left(\frac{X_u}{m}\sin\theta_0 - \frac{X_w}{m}\cos\theta_0\right)\right] & \left[-\frac{X_q}{m}s + \frac{X_u}{m}W_0 - \frac{X_w}{m}U_0 + g\cos\theta_0\right] \\ \left[(\sin\theta_0)s - \left(\frac{Z_u}{m}\cos\theta_0 + \frac{Z_w}{m}\sin\theta_0\right)\right] & \left[(\cos\theta_0)s + \left(\frac{Z_u}{m}\sin\theta_0 - \frac{Z_w}{m}\cos\theta_0\right)\right] & \left[-\frac{Z_q}{m}s + \frac{Z_u}{m}W_0 - \frac{Z_w}{m}U_0 + g\sin\theta_0\right] \\ \left[-\frac{M_u}{I_{yy}}\cos\theta_0 - \frac{M_w}{I_{yy}}\sin\theta_0\right] & \left[\frac{M_u}{I_{yy}}\sin\theta_0 - \frac{M_w}{I_{yy}}\cos\theta_0\right] & \left[s^2 - s\frac{M_q}{I_{yy}} + \frac{M_u}{I_{yy}}W_0 - \frac{M_w}{I_{yy}}U_0\right] \end{bmatrix} \begin{bmatrix} \Delta V_x \\ \Delta V_z \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} \frac{X_\delta}{m} \\ \frac{Z_\delta}{m} \\ \frac{M_\delta}{I_{yy}} \end{bmatrix}\delta \qquad (26)$$

Lateral Equations

(ANF coordinate system)

$$\begin{bmatrix} \left[s - \frac{Y_v}{m}\right] & \left[-\frac{Y_p}{m}s - \left(g\cos\theta_0 + \frac{Y_v}{m}W_0\right)\right] & \left[\left(\frac{Y_p}{m}\sin\theta_0 - \frac{Y_r}{m}\cos\theta_0\right)s + \frac{Y_v}{m}V_{x0}\right] \\ -\frac{L_v}{I_{xx}} & \left[s^2 - \frac{L_p}{I_{xx}}s - \frac{L_v}{I_{xx}}W_0\right] & \left[\left(\frac{J_{xz}}{I_{xx}}\cos\theta_0 - \sin\theta_0\right)s^2 + \left(\frac{L_p}{I_{xx}}\sin\theta_0 - \frac{L_r}{I_{xx}}\cos\theta_0\right)s + \frac{L_v}{I_{xx}}V_{x0}\right] \\ -\frac{N_v}{B} & \left[\frac{J_{xz}}{B}s^2 - \frac{N_p}{B}s - \frac{N_v}{B}W_0\right] & \left[s^2 + \left(\frac{N_p}{B}\sin\theta_0 - \frac{N_r}{B}\cos\theta_0\right)s + \frac{N_v}{B}V_{x0}\right] \end{bmatrix} \begin{bmatrix} \Delta V_y \\ \Delta\phi \\ \Delta\psi \end{bmatrix} = \begin{bmatrix} \frac{Y_\delta}{m} \\ \frac{L_\delta}{I_{xx}} \\ \frac{N_\delta}{B} \end{bmatrix}\delta \qquad (27)$$

where $V_{x0} \triangleq U_0\cos\theta_0 + W_0\sin\theta_0$, $B \triangleq I_{zz}\cos\theta_0 - J_{xz}\sin\theta_0$ and $s \triangleq \frac{d}{dt}$ the Laplace differential operator.

LINEARIZATION OF EQUATIONS USING EULER PERTURBATIONS

In the previous section linearization was done to develop a math model for approach and landing system analysis. As such, it was desired referenced to the ANF. In this section the equations are developed for use in the analysis of a stability augmentation system. Such a system will generally feed back only body-referenced rates P, Q, R and U, V, W. The linearizations then are desired in terms of these variables.

In the previous section linearization was done by representing the perturbed angles as the initial angles plus delta perturbations of the Euler angles so that

$$\left.\begin{aligned} \Theta &= \Theta_0 + \Delta\Theta \\ \Phi &= \Phi_0 + \Delta\Phi \\ \Psi &= \Psi_0 + \Delta\Psi \end{aligned}\right\} \tag{28}$$

In this section the perturbed angles will be arrived at by taking the initial angles and adding to them another set of Euler angles referenced to the initial body axes. These perturbation Euler angles will be denoted ψ, θ, ϕ. The translational velocity and angular velocity perturbations will be as in Eq. (14).

Body Translatory Equations

The perturbed equations will differ only in the gravity force which is described with Ψ, Θ, Φ. This perturbed force is written

$$\Delta\bar{F}_{grav} = \left[T_x(\phi)T_y(\theta)T_z(\psi)\begin{bmatrix} -\sin\Theta_0 \\ 0 \\ \cos\Theta_0 \end{bmatrix} - \begin{bmatrix} -\sin\Theta_0 \\ 0 \\ \cos\Theta_0 \end{bmatrix} \right] mg \tag{29}$$

Carrying out the matrix multiplication we get

$$\Delta\bar{F}_{grav} = \begin{bmatrix} -\theta\cos\Theta_0 \\ \psi\sin\Theta_0 + \phi\cos\Theta_0 \\ -\theta\sin\Theta_0 \end{bmatrix} mg \tag{30}$$

The perturbation equations then become

$$\Delta\dot{U} = -W_0\Delta Q - (g\cos\Theta_0)\theta + \frac{\Delta X_A}{m}$$

$$\Delta\dot{V} = W_0\Delta P - U_0\Delta R + (g\cos\Theta_0)\phi + (g\sin\Theta_0)\psi + \frac{\Delta Y_A}{m} \qquad (31)$$

$$\Delta\dot{W} = U_0\Delta Q - (g\sin\Theta_0)\theta + \frac{\Delta Z_A}{m}$$

Body Rate Equations

Since angles are not involved, these equations are just those of set (18).

Euler Angle Equations

With the defined transformations, the body rate perturbations can be written

$$\begin{bmatrix}\Delta P\\ \Delta Q\\ \Delta R\end{bmatrix} = T_x(\phi)\left[\begin{bmatrix}\dot{\phi}\\ 0\\ 0\end{bmatrix} + T_y(\theta)\left[\begin{bmatrix}0\\ \dot{\theta}\\ 0\end{bmatrix} + T_z(\psi)\begin{bmatrix}0\\ 0\\ \dot{\psi}\end{bmatrix}\right]\right] . \qquad (32)$$

With small angle assumptions on the perturbations and by dropping second-order terms the result is

$$\begin{bmatrix}\Delta P\\ \Delta Q\\ \Delta R\end{bmatrix} = \begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} . \qquad (33)$$

This means that for this linearization, unlike that of the prior section (see Eq. (19)), the angular perturbations are simply the integrals of the body rates. Although, for convenience in the remainder of this section, the angles ψ, θ, ϕ will be used, their simple relations to the body rates should be remembered.

Equation Set with Aerodynamic Partial Derivatives

Using assumptions similar to those of set (20) and Eqs. (33) to eliminate body rates yields the following equations:

$$
\left.\begin{aligned}
\Delta\dot{U} &= -W_0\dot{\theta} - (g\cos\Theta_0)\theta + \frac{X_u}{m}\Delta U + \frac{X_w}{m}\Delta W + \frac{X_q}{m}\Delta Q + \frac{X_\delta}{m}\delta \\
\Delta\dot{V} &= W_0\dot{\phi} - U_0\dot{\psi} + (g\sin\Theta_0)\psi + (g\cos\Theta_0)\phi + \frac{Y_v}{m}\Delta V + \frac{Y_r}{m}\dot{\psi} \\
&\quad + \frac{Y_p}{m}\dot{\phi} + \frac{Y_\delta}{m}\delta \\
\Delta\dot{W} &= U_0\dot{\theta} - (g\sin\Theta_0)\theta + \frac{Z_u}{m}\Delta U + \frac{Z_w}{m}\Delta W + \frac{Z_q}{m}\Delta Q + \frac{Z_\delta}{m}\delta \\
\ddot{\phi} &= -\frac{J_{xz}}{I_{xx}}\ddot{\psi} + \frac{L_p}{I_{xx}}\dot{\phi} + \frac{L_r}{I_{xx}}\dot{\psi} + \frac{L_v}{I_{xx}}\Delta V + \frac{L_\delta}{I_{xx}}\delta \\
\ddot{\theta} &= \frac{M_q}{I_{yy}}\dot{\theta} + \frac{M_u}{I_{yy}}\Delta U + \frac{M_w}{I_{yy}}\Delta W + \frac{M_\delta}{I_{yy}}\delta \\
\ddot{\psi} &= -\frac{J_{xz}}{I_{zz}}\ddot{\phi} + \frac{N_p}{I_{zz}}\dot{\phi} + \frac{N_r}{I_{zz}}\dot{\psi} + \frac{N_v}{I_{zz}}\Delta V + \frac{N_\delta}{I_{zz}}\delta
\end{aligned}\right\} \qquad (34)
$$

Reduced Equation Set in Vector-Matrix Notation

Putting the equation set (34) in vector-matrix notation with Laplace notation results in a set of longitudinal equations identical to set (22) for the delta perturbations.

<u>Longitudinal Equations</u>

(body axis system)

$$
\begin{bmatrix}
S-\frac{X_u}{m}, & -\frac{X_w}{m}, & \left(W_0-\frac{X_q}{m}\right)S + g\cos\Theta_0 \\
-\frac{Z_u}{m}, & S-\frac{Z_w}{m}, & -\left(U_0+\frac{Z_q}{m}\right)S + g\sin\Theta_0 \\
-\frac{M_u}{I_{yy}}, & -\frac{M_w}{I_{yy}}, & \left(S-\frac{M_q}{I_{yy}}\right)S
\end{bmatrix}
\begin{bmatrix} \Delta U \\ \Delta W \\ \theta \end{bmatrix}
=
\begin{bmatrix} \frac{X_\delta}{m} \\ \frac{Z_\delta}{m} \\ \frac{M_\delta}{I_{yy}} \end{bmatrix}\delta \qquad (35)
$$

The lateral equations, however, are different.

Lateral-Directional Equations

(body axis system)

$$\begin{bmatrix} S-\frac{Y_v}{m}\, , & -\left(W_0+\frac{Y_p}{m}\right)S - g\cos\Theta_0\, , & \left(U_0-\frac{Y_r}{m}\right)S - g\sin\Theta_0 \\ -\frac{L_v}{I_{xx}}\, , & \left(S-\frac{L_p}{I_{xx}}\right)S & , \left(\frac{J_{xz}}{I_{xx}}S-\frac{L_r}{I_{xx}}\right)S \\ -\frac{N_v}{I_{zz}}\, , & \left(\frac{J_{xz}}{I_{zz}}S-\frac{N_p}{I_{zz}}\right)S & , \left(S-\frac{N_r}{I_{zz}}\right)S \end{bmatrix} \begin{bmatrix} \Delta V \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \frac{Y_\delta}{m} \\ \frac{L_\delta}{I_{xx}} \\ \frac{N_\delta}{I_{zz}} \end{bmatrix} \delta \quad (36)$$

where $S \triangleq \frac{d}{dt}$, the Laplace differential operator.

CONCLUDING REMARKS

Two sets of linearized equations of motion for an aircraft initially in steady flight have been derived. The first set is written in terms of Euler angles and approach navigation frame velocities for the purpose of analyzing an automatic approach and landing system. The second set is written in terms of body rates and body-referenced velocities for the purpose of analyzing simple stability augmentation systems. The two equation sets are clearly different. The difference depends on the aircraft and its steady flight condition. Any assumption that the second more common set may be used in place of the first is not, therefore, automatically correct for all aircraft and all steady flight conditions but should be verified in each instance.

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