N71-12381
NASw-1952
CR-111687

R-538-NASA

October 1970

ACQUISITION BEHAVIOR OF SOME ADAPTIVE ARRAYS

W. Doyle

A Report prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Rand
SANTA MONICA, CA. 90406

This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASw-1952. This report does not necessarily represent the views of Rand or of the National Aeronautics and Space Administration.

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Published by The RAND Corporation

PREFACE

This report is part of a Rand study for NASA on the technology of communications satellites. It specifically concerns adaptive array antennas that appear to offer attractive solutions to the problems of operating earth receiving antennas which must share spectrum use with terrestrial services producing strong interfering signals. The earth receiving antennas continually adjust or adapt their patterns to provide the best discrimination in favor of the wanted signal from the satellites. If the interfering signals are changing, or if the earth antenna is changing in orientation or location, it is important for the adaptive system to have an appropriate response--or behavior with time, which is the subject of this report.

SUMMARY

The performance of two adaptive array schemes is analyzed. In scheme A the desired signal is specified by a beam-forming vector, in scheme W by a known pilot signal. In a stationary interference environment, if loop gains are high enough and the effective time constants are long enough, the weights in both systems will converge to values for which the output signal-to-interference ratio is maximum. One result of the analysis is a criterion for choosing loop gain and time constants which should be useful in designing an AGC system, a practical necessity for both schemes. The A scheme converges somewhat more rapidly than the W scheme when parameters are adjusted for the same asymptotic performance. For the A scheme, the output signal-to-interference enhancement history during adaptation is found under the assumption that smoothing is sufficient to make weight fluctuations small and slow. For the W scheme, settling time varies inversely with pilot-to-interference power ratio. Digital simulation experiments confirm the analysis. The results presented in this report are for the narrow-band case.

ACKNOWLEDGMENT

I am indebted to John Mallett for extensive and helpful discussions of the problems considered here.

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I. PROBLEM BACKGROUND

There are c channels, each delivering an output composed of a desired signal, a random channel noise, and interference from undesired signals. A scheme is required which automatically forms that weighted average of the channel signals for which the output signal-to-interference ratio is a maximum. The required optimum weights are easily derived,[1] but some notation and definitions are needed to express the results.

All signals are represented as complex numbers. These may be viewed as in-phase and quadrature components of band-pass signal relative to some nominal carrier frequency. Let the relative magnitudes and phases of the components of the desired signal in the c channels be embodied in a row vector $r = (r_1, \ldots, r_c)$. These relative amplitudes and phases are assumed to be constant. If the channels are the outputs of c identical antenna elements, then all r_i have the same magnitude, which may be made 1 by using this common level to establish the unit of power. In this case each $r_j = \exp(i\psi_j)$, where the phase angles ψ_j are computable from array and signal geometry and cable transmission delays.

Let $w = (w_1, \ldots, w_c)$ be a row vector of complex components and let $u = (u_1, \ldots, u_c)$ be the vector of channel outputs with the desired signal absent. The system output is defined by

$$\beta = uw^* , \tag{1}$$

where * applied to any matrix denotes its conjugate transpose. For a fictitious, and absent, desired signal of unit envelope power the corresponding signal output power is

$$S = |rw^*|^2 , \tag{2}$$

while the output interference† power is

†It is really noise plus interference. The abbreviation seems less clumsy and should cause no confusion.

$$I = E\{\beta^*\beta\} = wE\{u^*u\}w^* = wMw^* \quad , \tag{3}$$

which defines the interference covariance matrix M.

From Ref. 1, the optimum weights are given by

$$Mw^* = r^* \quad , \tag{4}$$

and the resulting optimum output S/I is

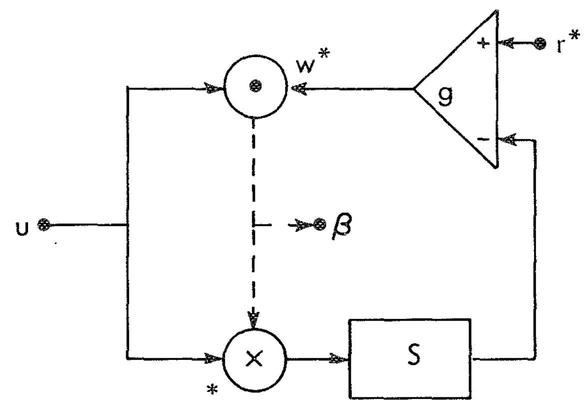
$$\frac{S}{I} = \frac{|rw^*|^2}{wMw^*} = rM^{-1}r^* \quad . \tag{5}$$

From the same reference (Chap. X, Sect. 7) it follows that Eq. (5) is the unique maximum of S/I.

In this report two analog schemes for solving Eq. (4) continuously are examined: the A scheme[2] and the W scheme.[3] Main emphasis falls on A and the problem is to investigate the action during the process of adaptation. A particular goal is to predict the time needed to get close to equilibrium for any given array and interference configuration.

II. THE A SCHEME AND ITS EQUATIONS

Consider this block diagram:



Solid links denote c channels. The dashed link is a single channel containing the output, β. The $\odot$ is intended to suggest that a sum of products is formed, namely $\beta = uw^*$. In $\otimes$ each input channel quantity, u^*, is multiplied by β. The * indicates that u^* is to be used. The box S contains c identical smoothing filters.

From inspection of the diagram, the weight vector w satisfies

$$w^* = g[r^* - S(u^* u w^*)]. \tag{6}$$

It is no loss of generality to assume that for any constant vector, r^*,

$$Sr^* = r^* = S^{-1} r^* .$$

This says only that the smoothing filters pass constant inputs (the carrier frequency) unchanged and with unity gain. A set of practical time-invariant filters is representable by

$$S^{-1} = 1 + a_1 d/dt + \dots a_k d^k/dt^k, \tag{7}$$

with constant coefficients, a_i; for the single time constant filter

$$S^{-1} = 1 + \tau d/dt. \tag{8}$$

It follows that Eq. (6) is equivalent to

$$g^{-1}S^{-1}w^{*} = r^{*} - u^{*}uw^{*}. \tag{9}$$

The weight vector at any instant has an expected value which will be written $\bar{w}^{*}$. Throughout this report it is assumed that the smoothing is sufficient to reduce the fluctuations of w to values small relative to $\bar{w}$ itself. Under this assumption a good approximation for the result of taking expected values of both sides of Eq. (9) is

$$g^{-1}S^{-1}\bar{w}^{*} = r^{*} - M\bar{w}^{*}. \tag{10}$$

Equation (10) describes the mean drift of the weights and, if the weight fluctuations are small, it closely describes the trajectory of the weights during the adaptation process. If M remains fixed, Eq. (10) may be used to deduce the asymptotic equilibrium value for $\bar{w}$. Let $t \to \infty$. We expect all derivatives of $\bar{w}$ to tend to zero; hence the eventual equilibrium w satisfies ($t = \infty$):

$$g^{-1}\bar{w}^{*} = r^{*} - M\bar{w}^{*}. \tag{11}$$

Clearly, for sufficiently high g, Eq. (11) provides a good approximation to Eq. (4), so that the A scheme is capable of getting as near to ideal performance as desired. There is no obvious direct way to discover what g is required for any given M (i.e., for any given array and interference environment); however, g can be chosen, Eq. (11) solved for $\bar{w}$, and then the resulting $\bar{w}$ evaluated by substituting in Eqs. (2) and (3) to find S/I.

III. WEIGHT DRIFT AND ACTIVITY IN NORMAL CHANNELS: SCHEME A

From its form in an expression for real power, M is Hermitian. Furthermore, since channel noise never vanishes, M is positive definite. A unitary transformation P and a set of positive characteristic numbers λ_i therefore exist for which

$$P^* MP = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_c\}.$$

The transformation P permits the introduction of normal coordinates. The constant linear transformation, P, clearly commutes with S^{-1}; hence, introducing

$$q = rP \qquad y = wP \tag{12}$$

reduces Eq. (10) to

$$g^{-1}S^{-1}\bar{y}^* = q^* - \Lambda\bar{y}^* \quad , \tag{13}$$

which is a set of c independent equations. For the single time constant filter Eq. (8), the solution of Eq. (13) is trivial once the initial conditions are chosen. Suppose the interference environment is established and then the scheme is turned on with filters initially empty. Then $\bar{y}(o) = gq$, from the block diagram, and the solution for each component of Eq. (13) is

$$\bar{y}_i(t) = \frac{gq_i}{1+g\lambda_i}\left[1 + g\lambda_i e^{\frac{-(1+g\lambda_i)t}{\tau}}\right]. \tag{14}$$

The asymptotic value of $\bar{y}_i$ is $\bar{y}_i(\infty) = gq_i/(1+g\lambda_i)$.

Let the input vector, u, also be transformed into v = uP, along with the transformations (12); then the basic equation (9) becomes

$$g^{-1}S^{-1}y^* = q^* - v^*vy^* \tag{15}$$

The transformed input processes, v, now have the diagonal covariance matrix, Λ, and are uncorrelated. The envelope power stimulating the ith normal channel is

$$E\{v_i^* v_i\} = \lambda_i$$

Two practical interference regimes may be distinguished. In the first there are fewer independent interferers, say n, than channels and only the presence of a small channel noise component keeps M non-singular. The total input envelope powers in the two coordinate systems, namely trace (M) and trace $(\Lambda) = \Sigma\, \lambda_i$, are the same. If the channel noises were set equal to zero, the ranks of M and Λ would become $n < c$ and the total powers would represent only the interference. For small channel noise powers tending to zero, n of the λ_i differ negligibly from their values for zero channel noise, while the remaining $c - n$ values of λ_i tend to zero.

From Eq. (14), the effective time constant in the ith channel is $\tau/(1+g\lambda_i)$; thus one might expect the settling time of the system to be determined by the smallest λ_i. For the case under consideration, $n < c$, this would be an error. To see this, compare the values of $\bar{y}_i(o)$ and $\bar{y}_i(\infty)$. For λ_i near zero they are nearly the same; thus, even though the effective time constant is only the relatively big τ of the smoothing filters, the associated normal weights, y_i, undergo negligible average change during adaptation. In the normal coordinate system these will be referred to as the "noise" weights, noise characteristic values, noise channels, etc. The noise weights vary relatively slowly but by a negligible amount; hence they ultimately have no practical effect on the settling time at all, at least so far as concerns the mean weight drift. To finish this argument it is only necessary to demonstrate that the noise weights also have a negligibly small fluctuation about their mean drift. A detailed argument is given later; however, the conclusion is obvious if one recalls that $\lambda_i = E\{v_i^* v_i\}$ is the envelope power in the ith normal channel. The noise channels thus have relatively negligible input power and their activity will be correspondingly negligible.

The other practical regime is characterized by a covariance matrix that would have full rank c even if there were no channel noise. This is true if there are at least as many independent* interferers as there are adaptive channels. For this case there are no tiny noise characteristic numbers, although particular interference configurations can lead to a wide range of λ_i. Even in this case, however, the arguments above suggest that the settling time of the system is not mainly determined by the smallest λ_i but by some (admittedly undemocratic) function of all of them. In particular, since a small λ_i corresponds to a normal channel with less input power, its influence should be correspondingly less. The relative contributions are evaluated in the next section.

The results just obtained are not peculiar to the single resonator smoother. For higher order filters the same arguments apply: activity in the ith channel varies with the input power, λ_i, to the channel. The total drift of the mean weight, $\bar{y}_i(t)$, between its initial and asymptotic values will be small, even though slow, for one of the noise channels in the first regime.

For higher order filters (Eq. (7)), Eq. (13) becomes

$$\frac{a_k}{(1+g\lambda_i)}\frac{d^k\bar{y}_i}{dt^k} + \ldots + \frac{a_1}{(1+g\lambda_i)}\frac{d\bar{y}_i}{dt} + \bar{y}_i = \frac{gq_i}{1+g\lambda_i} \qquad (16)$$

If the filters are regarded as initially empty, the trajectories of the mean weights, $\bar{y}_i$, after turn-on can be found in various tables of Laplace transforms. Input (right-hand side) is a step of magnitude $gq_i/(1+g\lambda_i)$ and one wants the inverse transform of $s^{-1}[1 + a_1 s/(1+g\lambda_i) + \ldots + a_k s^k/(1+g\lambda_i)]$.

---

*N sources strung along a ray from the origin and emitting statistically independent signals are equivalent to a single such source of the same total power.

IV. WEIGHT FLUCTUATIONS AND S/I HISTORY DURING ADAPTATION: SCHEME A

Define the fluctuation component, x_i, of the normal weight, y_i, by

$$y_i = \bar{y}_i + x_i.$$

The ensemble average output interference component at any instant is

$$N(t) = \bar{y} \Lambda \bar{y}^* + \sum_{i=1}^{c} \lambda_i E\{|x_i|^2\}. \tag{17}$$

The history of the first term of N(t) during adaptation follows from Eq. (14) or a solution to Eq. (13) in the case of higher order filters.

A derivation for the second term of Eq. (17) begins with the conjugate of the difference of Eqs. (15) and (13), namely

$$g^{-1}S^{-1}x_i = \lambda_i \bar{y}_i - v_i \sum_{k=1}^{c} v_k^* y_k$$

or

$$(g^{-1}S^{-1}+\lambda_i)x_i = -v_i \sum_{k\neq i}^{c} v_k^* y_k - s_i y_i , \tag{18}$$

where

$$s_i = v_i v_i^* - \lambda_i .$$

Equation (18) exhibits x_i as the output of a filter whose input (right-hand side) has a much more rapid fluctuation, due to the factors v and s, than the weights, $\bar{y}_i$, or their fluctuations, x_i. The factors v and s fluctuate rapidly about zero, while the y_j have in practical application a comparatively slow and small fluctuation about their means, $\bar{y}_j$. The right-hand side of Eq. (18) may therefore be approximated by replacing y_j by $\bar{y}_j$. The resulting approximation

$$(g^{-1}S^{-1}+\lambda_i)x_i = -v_i \sum_{k\neq i} v_k^* \bar{y}_k - s_i \bar{y}_i \tag{19}$$

is valid as long as

$$E\{|x_i|^2\} << |\bar{y}_i|^2 . \tag{20}$$

Now assume that the input processes v_i are all Gaussian. The fact that the v_i are uncorrelated then implies that they are independent and hence that the terms on the right-hand side of Eq. (19) are uncorrelated. For example, $E\{v_i v_k^* s_i\} = 0$ because $v_i s_i$ and v_k^* are independent and have mean zero. Therefore the input power is the sum of the powers of the separate terms. The power of $v_i v_k^* \bar{y}_k$ is

$$|\bar{y}_k|^2 \; E\{|v_i|^2 \quad |v_k|^2\} = \lambda_i \lambda_k |\bar{y}_k|^2 .$$

For a Gaussian v_i

$$E\{|s_i|^2\} = \lambda_i^2 ;$$

hence the total power on the right-hand side of Eq. (19) is

$$P_{in} = \lambda_i \sum_{k=1}^{c} \lambda_k |\bar{y}_k|^2 . \tag{21}$$

For single time constant filters, $S^{-1} = 1 + \tau d/dt$, and the operator on the left side of Eq. (19) is equivalent to

$$\frac{1 + g\lambda_i}{g} \left(1 + \frac{\tau}{1 + g\lambda_i} \frac{d}{dt}\right) ,$$

which is a combination of a gain factor and smoothing with effective time constant

$$\tau_i = \frac{\tau}{1 + g\lambda_i} .$$

If a series of independent pulses with mean zero and power P_{in} is smoothed with time constant τ_i (measured in pulses), then the output power is $P_{out} = P_{in}/(2\tau_i)$. If we assume, for the adaptive scheme, that τ represents the filter time constants in units of independent input pulse times, then we may apply this result to Eq. (19). In an analog realization, the effective number of samples averaged, replacing $2\tau_i$, is the ratio of input bandwidth to smoothing filter bandwidth. In a digital realization, the effective number of samples averaged is the actual number of samples averaged, as long as the sampling frequency does not exceed twice the input bandwidth. If the sampling frequency is higher, in a digital realization the effective number of samples averaged is the actual number averaged times twice the input bandwidth divided by the sampling frequency. Taking the gain factor into account as well, we have

$$E\{|x_i|^2\} = \frac{g^2\lambda_i}{2\tau(1 + g\lambda_i)} \sum_{k=1}^{c} \lambda_k|\bar{y}_k|^2 \quad . \tag{22}$$

If Eqs. (22) and (17) are now combined, the result for single time constant smoothing is

$$N(t) = \bar{y}\Lambda\bar{y}^* [1 + E] \quad , \tag{23}$$

where

$$\bar{y}\Lambda\bar{y}^* = g^2 \sum_{i=1}^{c} \frac{\lambda_i|q_i|^2}{(1 + g\lambda_i)^2} \left[1 + g\lambda_i\, e^{-\frac{(1 + g\lambda_i)t}{\tau}}\right]^2 \tag{24}$$

is the interference output power that would be produced during adaptation if the weights simply followed their mean drift exactly, and

$$E = \sum_{i=1}^{c} \frac{g^2\lambda_i^2}{2\tau(1 + g\lambda_i)} \tag{25}$$

is the average fractional excess output interference power that can be expected due to the slow fluctuations in the weights.

The signal output power is $S = A^2 = |wr^*|^2 = |yq^*|^2$. Its expected variation during adaptation follows from Eq. (14) and is

$$S(t) = g^2 \left| \sum_{i=1}^{c} |q_i|^2 \frac{1+g\lambda_i \exp[-t(1+g\lambda_i)/\tau]}{1 + g\lambda_i} \right|^2 . \tag{26}$$

Two important special points on the trajectory Eq. (26) are

$$S(o) = g^2 \left| \sum_{i=1}^{c} |q_i|^2 \right|^2 = g^2 c^2$$

and

$$S(\infty) = g^2 \left| \sum_{i=1}^{c} \frac{|q_i|^2}{1+g\lambda_i} \right|^2 = A^2(\infty) .$$

Equation (23) follows from the facts that $rr^* = c$ and P is unitary.

Equations (26) and (24) provide the expected behavior of signal and interference outputs during adaptation, while Eqs. (23) and (25) indicate how much the output interference may fluctuate due to the finite smoothing times.

It is interesting to note that the fractional excess, E, is independent of time and involves only g, τ, and the λ_i. Also note that $E\{|\lambda_i|^2\}$, and hence E, have been computed on the assumption (Eq. (20)) that the weight fluctuations are small compared to the weights themselves. In an application it will presumably also be wished that the slow fluctuations in output interference power stay small compared to the asymptotic average output interference power. This will be the case if E is kept reasonably below 1. In Section VI E is used to determine suitable bounds on g and τ.

These results also complete the verification of the observation, in the preceding section, that the effects of the "noise" channels are negligible during adaptation.

V. ILLUSTRATIVE CASES: SCHEME A

A digital simulation of several cases confirms the preceding results. In this simulation all sources emit complex random numbers distributed over a square centered at the origin and parallel to the axes. The sizes of the squares are set so each source produces the specified envelope power. Such sources are far from Gaussian; however, since the scheme depends only on sums and products of signals, it should suffice to match source powers to those in the theoretical calculations. The experiments bear this out.

For all of the following cases the interference comes from five sources at azimuths 0, 72, 144, 216, and 288 deg and angles down from zenith of 91.25, 87.5, 93.75, 85, and 90 deg, respectively. The desired source is at the zenith. Each interfering source has unit power. For cases EG5A, EF5A, and ED5A the noise power is 10^{-5} per channel, the gain is 1000, and the filter time constants are 150,000. For the case DD4 the values are 10^{-6}, 10,000, and 1,500,000.

CASE EG5A

There are six elements equally spaced on a circle of radius 0.5 wavelength plus a seventh element at the center of the circle. The parameters for the normal channels are summarized in Table 1.

Table 1

NORMAL PARAMETERS FOR CASE EG5A

i	λ_i	τ_i	$\lvert q_i \rvert^2$	$N(o)/g^2$	$N(\infty)/g^2$	$A(\infty)/g$
1	14.1	11	0.000008	0.000116	$< 10^{-6}$	0.000000
2	8.4	18	0.202	1.69	$< 10^{-6}$	0.000024
3	7.7	19	0.223	1.71	$< 10^{-6}$	0.000029
4	2.69	56	0.000002	0.000006	$< 10^{-6}$	0.000000
5	2.06	73	0.000003	0.000006	$< 10^{-6}$	0.000000
6	0.000011	148,000	3.42	0.000036	0.000036	3.39
7	0.000010	148,000	3.15	0.000032	0.000031	3.12

Table 1 illustrates and confirms the assertions about noise channels for the case when there are more channels than independent sources of interference. The heading τ_i denotes the effective time constants

$$\tau_i = \frac{\tau}{1 + g\lambda_i} . \tag{27}$$

The initial contributions to output noise (interference) come mainly from normal channels 2 and 3, while the ultimate output noise comes mainly from the noise channels 6 and 7. Notice that, as expected, the contributions from the noise channels remain virtually unchanged by the adaptation. For these channels no significant change is necessary and none occurs; thus the very big effective time constants (148,000) are harmless. Notice also that the normal beam-forming vector, q_i, has most of its weight in the noise channels and that they contribute most of the ultimate output signal, $A(\infty)$.

This case was simulated for 1000 independent sample vectors from all sources. Table 2 lists theoretical and experimental output S/I histories. The experimental S/I is obtained by taking the weight vector, w, at any instant (sample) in the course of the simulation and evaluating it with the formula

$$\frac{S}{I} = \frac{|wr^*|^2}{wMw^*} . \tag{28}$$

Table 2

S/I HISTORIES FOR CASE EG5A

t	Theory	Simulation
0	11.6	11.6
50	34.3	30.2
100	54.5	48.6
150	57.9	53.1
200	58.0	55.9
250	58.0	57.2

The weight vector, w, fluctuates slowly. A snapshot of w, after the system has settled, yields a random vector whose expected value is $\bar{w}$. It is therefore appropriate to compare Eq. (28) with the theoretical S/I, omitting the effects of weight fluctuations. To verify the slight extra degradation caused by the weight fluctuations, the values of Eq. (28) for a series of samples should be used to estimate the additional noise component.* This has not been done.

The theoretical output S/I is 58.3 dB for this case if the weights are fixed at their ultimate average values. Weight fluctuation for this case introduces a theoretical degradation of 0.5 dB, giving the estimate of 57.8 dB for performance after settling. Between samples 250 and 1000, snapshots were taken every 50 samples. The range of S/I was from 57.1 to 58.1 dB.

CASE EF5A

The array consists of six elements. Three are equally spaced around a circle of radius 0.6 wavelength, with a fourth at the center. The remaining two lie at 0.5 wavelength intervals above the center. The two tables which follow summarize computed and measured performance.

Table 3

NORMAL PARAMETERS FOR CASE EF5A

i	λ_i	τ_i	$\lvert q_i\rvert^2$	$N(o)/g^2$	$N(\infty)/g^2$	$A(\infty)/g$
1	18.9	8	0.175	3.3	↑	0.000009
2	6.9	22	1.83	12.6	$< 10^{-6}$	0.000264
3	3.6	42	0.045	0.16	↓	0.000013
4	0.61	246	1.22	0.74	0.000002	0.0020
5	0.0144	9767	0.066	0.00095	0.000004	0.0043
6	0.000010	149,000	2.67	0.000026	0.000026	2.64

*It should be possible to estimate this from the set of S/I achieved with the last n snapshots taken. See your statistician.

Table 4

S/I HISTORIES FOR CASE EF5A

t	Theory	Simulation
0	3.3	3.3
50	13.8	13.2
100	15.8	15.6
200	18.6	18.4
400	24.7	24.6
800	35.8	36.0

Optimum, fixed weights for this case yield output S/I of 54.4 dB. If the weights were fixed at the theoretical mean to which they converge for g = 1000, output S/I would be 53.4 dB. The fluctuations of the weights theoretically cause about another 0.4 dB loss. This case was not run long enough to approach limiting performance.

CASE DD4

This case differs from EF5A only in that noise power per channel is 10^{-6}, gain is 10,000, and filter time constant is 1,500,000. The simulation was run for 25,000 steps, which allows the lengthy comparison of Table 5. The characteristic values and other normal parameters are similar to those for case EF5A, except for the last, or noise, channel, for which τ_6 = 1,500,000.

Table 5

S/I HISTORIES FOR CASE DD4

t	Theory	Simulation
0	3.3	3.3
20	9.0	8.0
40	12.8	12.8
60	14.5	13.8
80	15.2	14.5
100	15.8	14.4

Table 5 (Cont'd)

t	Theory	Simulation
3000		40.6
3200	41.5	
6000		43.0
6400	44.1	
12,800	49.3	
15,000		50.0
25,000		59.0
25,600	59.1	
51,200	68.3	

For this case the optimum performance possible is around 69 dB. Allowing for the finite gain, the expected value for the weight vector would give S/I = 68.4 dB.

CASE ED5A

In the preceding three examples there have been more adaptive channels than independent interfering sources. The present case retains the interference configuration of the other three; however, the array is reduced to four elements. Three elements are equally spaced on a circle of radius 0.5 wavelength, with the fourth at the center.

The parameters in normal coordinates are given in Table 6.

Table 6

NORMAL PARAMETERS FOR CASE ED5A

i	λ_i	τ_i	$\|q_i\|^2$	$N(o)/g^2$	$N(\infty)/g^2$	$A(\infty)/g$
1	9.9	15	0.52	5.2		0.000053
2	5.1	29	0.000062	0.00032	$< 10^{-6}$	0.000000
3	4.5	33	0.78	3.5		0.000173
4	0.53	285	2.7	1.42	0.0000054	0.005138

Theoretical optimum S/I is 7.3 dB. Computed and simulated S/I histories are given in Table 7.

Table 7

S/I HISTORIES FOR CASE ED5A

t	Theory	Simulation
0	2.0	2.0
25	6.0	
50	7.1	6.8
100	7.2	7.1

From Table 6 it is apparent that significant interference reduction occurs in all four normalized channels. The poor performance possible in this case is achieved early, however, since it occurs by the time the activity in channels 2 and 3 has begun to settle down.

The tables of normal parameters can be used to make rough estimates of settling times. Consider the columns $N(o)/g^2$ and $N(\infty)/g^2$, which give the initial and ultimate interference-plus-noise contributions for each normal channel. If one determines which channels experience a significant reduction in N, then their effective time constants, τ_i, are the only ones influencing the overall settling time. In case EG5A only time constants up to 73 matter; hence settling should occur in a few multiples of 73 steps. The simulation had about 56 dB of a possible 58.3 in 200 steps. In case EF5A the N for channel 5 has to be reduced by a factor of about 250 to approach the theoretical performance. Thus action is required in a channel whose effective time constant is near 10,000. The DD4 case, which had similar normal parameters, took about 20,000 samples to get to the performance limit (DD4 had $N = 10^{-6}$, so its ultimate performance was better than that of the EF5A case) for the EF5A case.

VI. CHOOSING τ AND g: SCHEME A

Equation (25) for the extra output noise due to slow fluctuations in the weights may be used to choose either g or τ to keep that excess small. An examination of two extreme cases leads to a useful rule which may be applied directly in the original coordinates. The two extreme cases occur when (1) the power is equally divided among the normal channels and (2) when one channel has all the power and the remainder none.

CASE 1: ALL λ_i EQUAL

Let p denote the total power from all interfering sources and channel noise and let c be the number of channels; then $\sum_i \lambda_i = cp = c\lambda_i$ implies each $\lambda_i = p$ and the fractional excess noise is

$$E = \frac{g^2p^2c}{2\tau(1+gp)} \approx \frac{gcp}{2\tau}. \tag{29}$$

The last approximation requires $gp >> 1$, which will nearly always be the case.

CASE 2: ONE BIG λ

Put $\lambda_1 = \Sigma\, \lambda_i = cp$ and $\lambda_2 = \ldots = 0$; then

$$E = \frac{g^2c^2p^2}{2\tau(1+gcp)} \approx \frac{gcp}{2\tau} \tag{29}$$

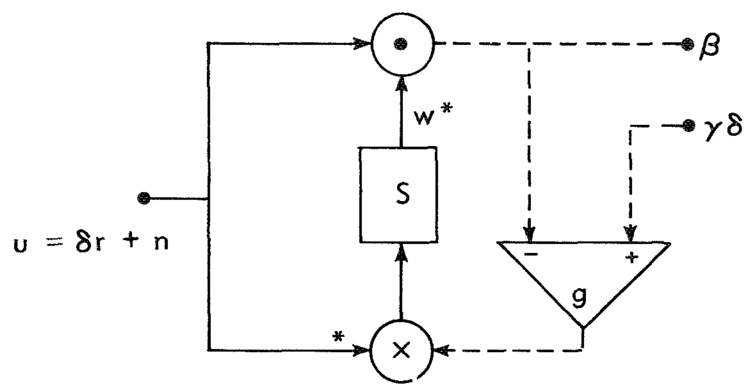
just as for case 1. This coincidence suggests that Eq. (29) should be a good general guide for choosing g and τ. If, for example, E is to be less than 0.1, then Eq. (29) requires that

$$\frac{\tau}{g} > 5\ pc\ . \tag{30}$$

Equation (30) can form the basis of a first attempt at an AGC scheme. If the filter time constants have been fixed, Eq. (30) provides an upper limit on the allowable g. Values of g greatly above this limit will lead to excessively short effective time constants and excessive fluctuations in the weights. The total channel power p can be estimated by putting a square law envelope detector on one channel and smoothing its output.

VII. THE W SCHEME: ITS EQUATIONS AND WEIGHT DRIFT DURING ADAPTATION

Widrow, *et al.*[3] discuss a number of variations of an adaptive array scheme. The one discussed in this section is a simple variation that is similar, in some respects, to the A scheme. Consider this block diagram:



Symbols and notation are the same as for the A scheme. However, it is now supposed that the array contains a pilot contribution, δr, and that a magnified and phase-shifted replica, $\gamma\delta$, is known. The pilot modulation (including the case of no modulation) is represented by δ, while $r = (r_1, \ldots, r_c)$, as before, accounts for the relative magnitudes and phases of the pilot contributions in the c channels. n denotes the rest of the channel activity, interference plus noise. As for the A scheme, it is assumed that no information signal from the desired source is present in the adaptive loops.

From inspection of the sketch, the weight vector w^* satisfies

$$S^{-1}w^* = u^*g(\gamma\delta - uw^*) \ . \tag{31}$$

Following the same line of argument as for the A scheme leads to similar results for the mean weight drift during adaptation and for the equation determining the ultimate equilibrium weights. For the mean weight drift,

$$S^{-1}\bar{w}^* = g(\gamma p r^* - C\bar{w}^*) \ , \tag{32}$$

where $p = E\{|\delta|^2\}$ is the pilot power and

$$C = E\{u^* u\} = p r^* r + M \tag{33}$$

is the channel covariance matrix. Note that C is the interference covariance matrix plus a contribution due to the pilot signal.

Since C is Hermitian and positive definite also (channel noise, for example, never vanishes), a unitary transformation, Q, and a set of characteristic numbers μ_i exist for which $C = Q \operatorname{diag} \{\mu_i\} Q^*$. If the quantities involved are transformed to normal coordinates,

$$uQ = v \qquad rQ = q \qquad wQ = y \ ,$$

then Eq. (32) reduces to the c independent equations

$$S^{-1}\bar{y}_i = g(\gamma p q_i - \mu_i \bar{y}_i) \ ,$$

whose solutions for single time constant smoothing are

$$\bar{y}_i = \frac{g\gamma p q_i}{1+g\mu_i}\left(1 - e^{\frac{-(1+g\mu_i)t}{\tau}}\right). \tag{34}$$

The filters are assumed to be initially empty; hence $\bar{y}(o) = 0$. This accounts for the negative sign in Eq. (34). It is interesting to compare the forms of Eqs. (14) and (34). Notice, however, that the characteristic numbers, μ_i, in Eq. (34) are different from the characteristic numbers, λ_i, in Eq. (14). If the interference environments are the same for the A and W schemes, one would expect all the μ_i to be slightly greater than the corresponding λ_i because of the added pilot contribution, and if the number of channels exceeds the number of independent sources, then one would expect there to be one less "noise" channel for the matrix C in the W scheme than for the matrix M.

In order for the weights to behave anything like Eq. (34), it is necessary that the smoothing be sufficiently great so that the effective time constants, $(1+g\mu_i)/\tau$, are greater than the time necessary to obtain a good estimate of $E\{u^* \delta\} = pr^*$. Otherwise, instead of the term γpr^* in Eq. (32) there should be a term of the form $f(t)r^*$, representing the gradual build-up of this reference. This topic is discussed further after some examples.

For the ultimate equilibrium weights, Eq. (32) implies

$$\bar{w}^* = g(\gamma pr^* - pr^* r\bar{w}^* - M\bar{w}^*)$$

or

$$(g^{-1}I + M)\bar{w}^* = pr^*(\gamma - r\bar{w}^*) \quad . \tag{35}$$

Since the parenthesis on the right-hand side is a complex scalar, Eq. (35) implies that as $g \to \infty$ the solution $\bar{w}^*$ tends to a value proportional to the ideal weights given by Eq. (4). Thus, if the smoothing and gain are great enough the W scheme approaches ideal performance.

VIII. COMPARISON OF A AND W SCHEMES

Some idea of what to expect may be had by comparing the basic equations for the two schemes. It is clear that both

$$\text{A:} \qquad g^{-1}w^* = r^* - S(u^*uw^*) \qquad (6)$$

and

$$\text{W:} \qquad g^{-1}w^* = \gamma S(u^*\delta) - S(u^*uw^*) \qquad (31)$$

schemes are continuously estimating the channel covariance matrix, since for adequate smoothing $S(u^*uw^*) \approx S(u^*u)\bar{w}^*$. For the A scheme, $S(u^*u)$ approximates M, the channel covariance matrix for interference only. For the W scheme, $S(u^*u)$ approximates $pr^*r + M$, which is a covariance matrix for one more source. If M has rank less than c and the desired signal is not on a ray to one of the interferers, then $pr^*r + M$ has rank c + 1 and it would appear that the W scheme has a more difficult task.

A more important difference, however, is in the first term. Since $S(u^*\delta)$ approximates pr^*, the W scheme is also using the known pilot to estimate r^*, a vector supplied cleanly in the A scheme. If the interference is very heavy, the fluctuations $n^*\delta$ in $u^*\delta = \delta^*\delta r^* + n^*\delta$ will be very great and much more smoothing will be required to produce a clean enough estimate of r^*.

It is interesting to note that one method of operation proposed in Ref. 3 for a W scheme is to connect the input channels alternately to the real world and to a synthesized, clean, beam-forming signal, ar^*, while applying a and zero alternately to the pilot input terminal (where $\gamma\delta$ enters in the sketch on p. 20). The above comparison suggests that if the extra effort of generating the beam-forming vector r^* is to be expended, it might be simpler and better to use the A scheme directly. The main advantage of the W scheme is its ability to estimate r^* when provided only with a pilot signal.

The following example provides one illustration of the convergence of both the A and W schemes under similar conditions. The behavior of the W scheme is given for several pilot levels. For comparison, the A scheme was run at several values of τ, the idea being that for a fair comparison one should use a τ that leads to about the same ultimate fluctuation in S/I values about their equilibrium value. All the W cases settled down to S/I of about 46.7 dB while the A cases settled somewhat higher. The times given below are those to cross the S/I = 43.7 dB level (3 dB from W scheme's asymptote). Both examples are for the source and element configuration of case EG5A on p. 12. The channel noise was 10^{-4} in both cases. Other pertinent parameters are as follows.

W scheme: τ = 5M, g = 50K, local pilot always inserted at unity power level. Incoming pilot signal at levels 10, 1, 0.1, and 0.01 relative to power in each of five interferers.

A scheme: g = 10K, τ = 0.5M, 1M, and 2M for three A cases. The τ that produce the same eventual S/I fluctuation as the W cases is somewhere between 1M and 2M.

Table 8

COMPARISON OF A AND W SCHEMES

Item	W Scheme Pilot Power				A Scheme τ		
	10	1	0.1	0.01	0.5M	1M	2M
Time to reach 43.7 dB, sample units	155	175	570	4200	56	82	160
Ultimate S/I fluctuation, dB	1.5	1.4	0.8	1.6	2.2	1.1	0.4

The results in Table 8 demonstrate the importance of pilot power in determining the settling time of the W scheme. Some light is shed on this by considering the ratio of the useful reference power to the fluctuation power in the term $u^*\delta = \delta^*\delta r^* + n^*\delta$. If i is the total interference power, then the pilot-to-interference ratio is p/i. In order for the W scheme to settle down, $u^*\delta$ needs sufficient smoothing to produce $E\{u^*\delta\} \approx pr^*$ with an acceptable level of residual fluctuation. The number of independent samples to be smoothed in order to reach a given output p/i varies inversely with the pilot power, p. This relationship seems to be roughly followed for the lower levels of p in Table 8.

The fluctuations about eventual equilibrium for the W cases are all in the 1 to 1.5 dB range, so a fair comparison A case would be one with τ about 0.8M, for which the time to reach 43.7 dB would be about 78. In this example the A scheme is two or three times as fast as the W scheme, depending on what pilot-to-interference ratio one considers practical.

REFERENCES

1. Gantmacher, F. R., *Matrix Theory*, Vol. 1, Chelsea Publishers, New York, 1959.

2. Applebaum, S. P., *Adaptive Arrays*, Syracuse University Research Corporation, Report SPL709, June 1966.

3. Widrow, B., P. E. Mantey, L. J. Griffiths, and B. B. Goode, "Adaptive Antenna Systems," *Proc. IEEE*, December 1967, p. 2143.