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DEFINITIONS IN COMMUNICATION TECHNOLOGY

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The purpose of this memorandum is to acquaint personnel who have limited background in communication theory, but deal with certain aspects of communications work, with some of the more common vernacular in the area of communication technology. The concentration here will be on signal-to-noise ratios, bit errors, quantization, and the like.

Probably the best place to start is with a definition of signal-to-noise ratio. As defined by Shannon in his classical theorem on channel capacity, it is the ratio of the variance of the signal to the variance of the noise; that is, if the signal variable is x and the noise variable y, with $z = x + y$, then

$$\left(\frac{S}{N}\right)_1 = \frac{\left(E\left[z^2\right] - E^2[z]\right)\Big|_{y=0}}{\left(E\left[z^2\right] - E^2[z]\right)\Big|_{x=0}}$$

where we have denoted an ensemble average by $E[\cdot]$. This definition also has the advantage of not being affected by any large dc biases which might add to the signal power but not to the signal content.

Another definition which has common use is

$$\left(\frac{S}{N}\right)_2 = \frac{\left(E^2[z]\right)\Big|_{y=0}}{E^2[z] - E^2[z]} \quad ,$$

which stems from the Tchebycheff inequality,

$$P\left\{\,\left|z - E(z)\right| \geq \epsilon\right\} \leq \frac{\left[E\left(z^2\right) - E^2(z)\right]}{\epsilon^2} \quad .$$

When we set

$$\epsilon = \rho E[z] \quad ,$$

we have

$$P\left\{\,\left|\frac{z}{E[z]} - 1\right| \geq \rho\right\} \leq \frac{1}{\rho^2\left(\frac{S}{N}\right)} \quad ,$$

If the noise has a constant and known mean, $E[z] = E[z]\big|_{y=0} + C$. This definition is useful when the sample mean is the primary statistic of interest. This holds true in the areas of detection theory, where the signal-to-noise ratio at the output of a matched filter is defined in this manner, and in the field of photo detection devices where the signal-to-noise ratio of the current flow is defined in this manner. This definition is not too meaningful when the sample mean is zero.

Another definition useful in spectrum analysis is in essence a mean squared error criterion:

$$\left(\frac{S}{N}\right)_3 = \frac{\int d\omega \left[\text{signal power spectrum}\right]}{\int d\omega \left[\text{noise power spectrum}\right]}$$

which resulted from Wiener's classic work in Harmonic Analysis.

When used in their proper context, these definitions are closely related and can usually be intermixed. Notice that these definitions are power definitions as opposed to voltage definitions commonly used in the sciences. To avoid any confusion, therefore, it is recommended that one use the engineering units of decibel when conversing. Thus

$$\begin{aligned}\left(\frac{S}{N}\right) &= 10 \log_{10}\left[\text{power signal-to-noise ratio}\right]\\ &= 20 \log_{10}\left[\text{voltage signal-to-noise ratio}\right]\end{aligned}$$

and each can use his own numbers and still communicate.

The next important topic is how the signal-to-noise ratio is related to fidelity. Clearly, when speaking of continuous time functions, the signal-to-noise ratio is measuring the deviation of a sample $\hat{x}$ from the true function x based on a quadratic cost $(x - \hat{x})^2$ referenced to the signal level. In other words, if the signal is 1 then $1/(x - \hat{x})^2$ is related to the number of gray levels we can distinguish. When this number is large, the average variation in gray level is small and becomes unnoticeable. These are the considerations that one takes into account when quantizing a signal. Suppose, for example, we wish to quantize a signal. We would like to know how the signal-to-noise ratio is related to the number M of distinguishable quantization levels. For the sake of argument, suppose that the signal is uniformly distributed between 0 and 1. We divide the interval (0, 1) into M levels. If a signal falls anywhere within the interval, we select the center of that interval as the sample. At the receiver we always receive the center of the interval independent of where the signal actually was. This introduces what is called quantization noise.

Using the first definition for the signal-to-noise ratio we see that

$$\begin{aligned}\left[E\left[z^2\right] - E^2[z]\right]_{y=0} &= \int_0^1 (1) \cdot x^2\,dx - \left[\int_0^1 (1) \cdot x\,dx\right]^2\\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12},\end{aligned}$$

while the quantization noise σ_q^2 is:

$$\sigma_q^2 = \left[E\left[z^2\right] - E^2[z] \right] = \int_{i/M}^{\frac{i+1}{M}} My^2\, dy - \left[\int_{i/M}^{\frac{i+1}{M}} My\, dy \right]^2$$

$$= \frac{1}{12M^2} \quad .$$

Thus:

$$\left(\frac{S}{N}\right)_1 = M^2$$

is the quantization signal-to-noise ratio. This would be the first step in determining how many levels to choose. We can also view this another way. Suppose the signal has a bandwidth B requiring a minimum of 2B samples per second. Each sample can be quantized into M levels with $\log_2 M$ bits. Therefore, a minimum of $2B \log_2 M$ bits per second is required.

Assuming we can apply Shannon's capacity formula, we see that the minimum signal-to-noise ratio required can be obtained by equating

$$2B \log_2 M = B \log_2 \left[1 + \frac{S}{N}\right]$$

or

$$1 + \frac{S}{N} = M^2.$$

Stated another way, we have

$$\frac{100}{M} = \frac{100\%}{\sqrt{1 + \left(\frac{S}{N}\right)}}$$

resolution for a signal-to-noise ratio S/N.

The major reason for quantizing signals is that high-speed data processing is done in this form. However, it is also true in communications that an advantage can be gained using binary (PCM) forms of modulation. To illustrate this point, we will compare an amplitude modulation system to the same system quantized.

Consider a signal X uniformly distributed between 0 and 1 with gaussian noise superimposed upon it. This simulates a communications channel. Consider Figure 1.

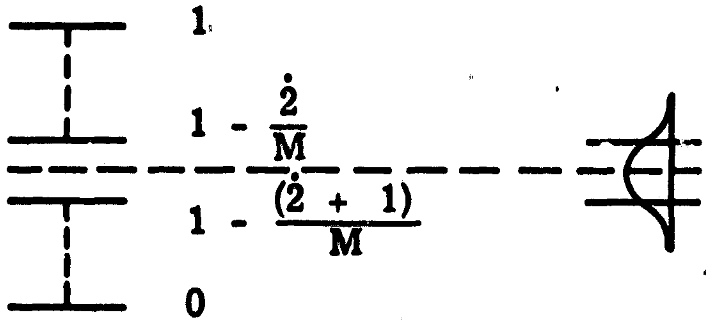

Figure 1

Whether the signal is sampled or not, the noise will have the effect of relocating the desired level. If we have the requirement of a resolution of $1/M$ with a certain confidence, this means that with probability

$$\int_{-\ell\sigma}^{\ell\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-x^2}{2\sigma^2}} dx$$

the signal will be in the prescribed interval. Thus $2\ell\sigma = 1/M$, with ℓ specified by the percentage of time this will be true. For a 95 percent confidence interval (about the minimum for good quality TV), $\ell = 2$. Thus:

$$\sigma^2 = \frac{1}{(2\ell M)^2} = \frac{1}{16M^2}$$

and, since the signal variance is $1/12$:

$$\left(\frac{S}{N}\right) = \frac{4}{3} M^2 .$$

Now if the same signal is quantized into M levels, this requires $\log_2 M$ binary digits. In addition, the signal will have to be sampled at least twice per cycle. This means that the bandwidth of the binary system will be $2 \log_2 M$ times larger than the original bandwidth. Thus if we have S/N for the unquantized system, we will have a signal-to-noise ratio of

$$\frac{S_B}{N'} = \frac{S_B}{N\, 2 \log_2 M} = \frac{S_B}{S} \cdot \frac{S}{N} \cdot \frac{1}{2 \log_2 M} = \frac{2M^2}{3 \log_2 M} \cdot \frac{S_B}{S}$$

for the binary system, where S_B is the signal power required for the binary transmission. The relationship between signal-to-noise ratio and bit error p for a coherent rf binary system is given in the following table:

$p = \int_{\sqrt{2\frac{S_B}{N'}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx$	$\left(\frac{S_B}{N'}\right)$
10^{-3}	5
10^{-4}	7.5
10^{-5}	9.5
10^{-6}	11.5

Thus, for example if $p = 10^{-5}$, the fractional power required for the binary transmission is S_B/S, which is equal to

$$\frac{S_B}{S} \cong \frac{15 \log_2 M}{M^2} \; .$$

Hence, for more than four levels of quantization $\left(\frac{S}{N} > 12\text{dB}\right)$ the binary transmission would be more efficient.

The final question remaining is: How does the bit error enter into the calculation of a signal-to-noise ratio? It has been shown that for the uniformly distributed signal, the noise variance due to bit errors alone is:

$$\sigma_B^2 = \frac{1}{3} p \left[1 - \frac{1}{M^2} \right] .$$

Therefore, if we again choose to reconstruct the signal as the center of the interval selected, the total variance of the reconstructed signal after the binary transmission will be the sum of the quantization variance and the bit error variance or

$$\sigma_T^2 = \frac{1}{3} p \left[1 - \frac{1}{M^2} \right] + \frac{1}{12M^2}$$

with.

$$\left(\frac{S}{N}\right)_T = \frac{1}{4p\left[1 - \frac{1}{M^2}\right] + \frac{1}{M^2}} \quad .$$

This is plotted in Figure 2. Notice that

$$p = \frac{1}{4M^2} \simeq \frac{1}{3(S/N)}$$

is the crossover point between being quantization-noise-limited and bit-error limited. This point, in fact, would be the most efficient place to fix the design, since any increase in signal power would not improve the processed output (quantization-noise-limited) whereas a decrease in power would degrade the output below the desired level. Attainable outputs lie to the right of the solid line and below the dashed line. Notice, for example, that it is incompatible to require an output S/N ratio of 40 dB with less than 2^7 quantization levels (7 bit word) or more than a bit error of 10^{-5} (uncoded PCM). One can also see that the major contribution to bit error noise is an error in the most significant digit, hence the insensitivity to M.

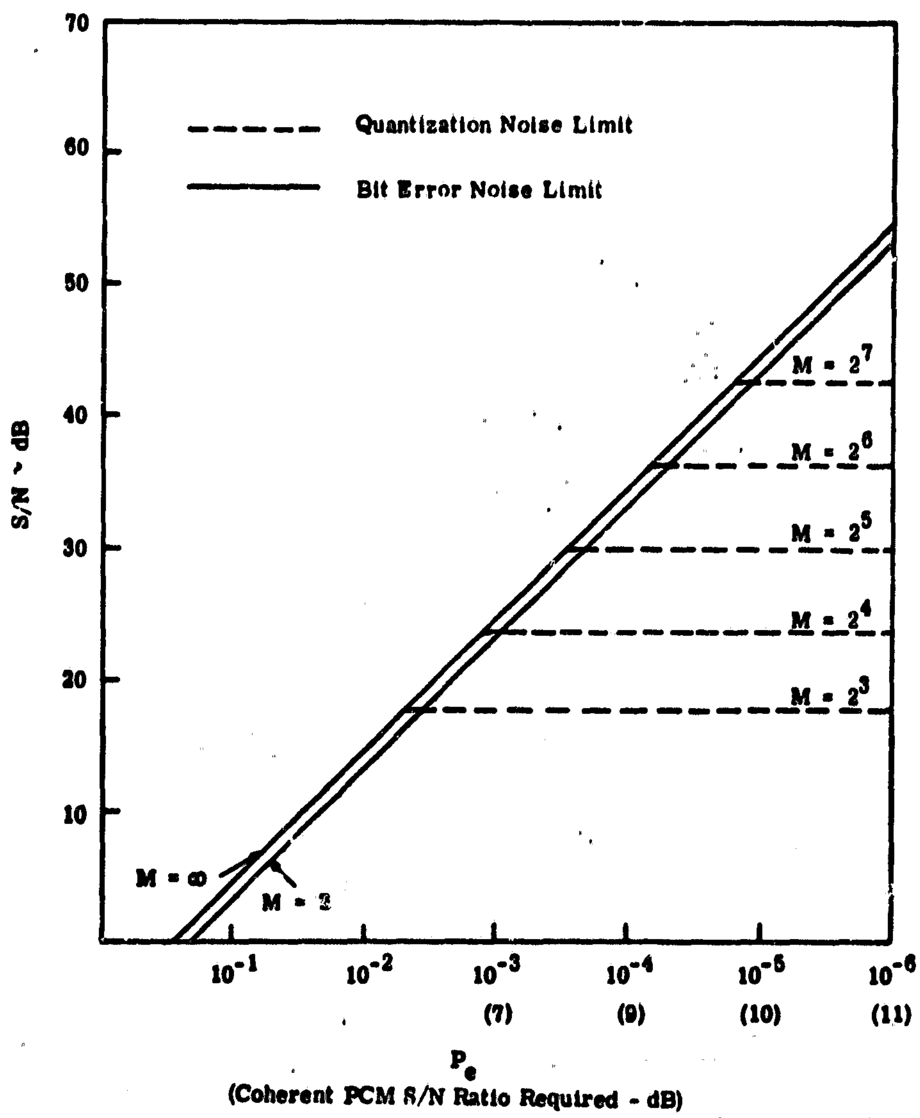

Figure 2