NASA TECHNICAL NOTE

NASA TN D-6064

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NASA TN D-6064

INTEGRATING-MATRIX METHOD FOR DETERMINING THE NATURAL VIBRATION CHARACTERISTICS OF PROPELLER BLADES

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1970

1. Report No.
NASA TN D-6064

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle
INTEGRATING-MATRIX METHOD FOR DETERMINING THE NATURAL VIBRATION CHARACTERISTICS OF PROPELLER BLADES

5. Report Date
December 1970

6. Performing Organization Code

7. Author(s)
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8. Performing Organization Report No.
L-5539

9. Performing Organization Name and Address
NASA Langley Research Center
Hampton, Va. 23365

10. Work Unit No.
721-01-12-01

11. Contract or Grant No.

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Technical Note

14. Sponsoring Agency Code

15. Supplementary Notes Most of the information contained herein was included in a thesis entitled "Integrating Matrix Method for Determining the Natural Vibrations of a Rotating, Unsymmetrical Beam With Application to Twisted Propeller Blades" submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering Mechanics, Virginia Polytechnic Institute, Blacksburg, Virginia, June 1967.

16. Abstract

A numerical method is presented for determining the natural lateral vibration characteristics of a rotating twisted propeller blade having a nonuniform, unsymmetrical cross section and cantilever boundary conditions. Two coupled fourth-order differential equations of motion are derived which govern the motion of such a beam having displacements in two directions. A development of the integrating matrix, which is the basis of the method of solution, is given. By expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the differential equations are integrated and formulated into an eigenvalue problem whose solutions may be determined by various methods. Numerical examples are presented and the computed results are compared with experimental data and exact solutions.

17. Key Words (Suggested by Author(s))
Integrating matrix
Numerical methods
Propeller blades
Natural vibrations
Rotating beams
Eigenvalue problem

18. Distribution Statement
Unclassified – Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	71	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

INTEGRATING-MATRIX METHOD FOR DETERMINING THE NATURAL VIBRATION CHARACTERISTICS OF PROPELLER BLADES*

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SUMMARY

A numerical method is presented for determining the natural lateral vibration characteristics of a rotating twisted propeller blade having a nonuniform, unsymmetrical cross section and cantilever boundary conditions. Two coupled fourth-order differential equations of motion are derived which govern the motion of such a beam having displacements in two directions. A development of the integrating matrix, which is the basis of the method of solution, is given. By expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the differential equations are integrated and formulated into an eigenvalue problem whose solutions may be determined by various methods. Numerical examples are presented, and the computed results are compared with experimental data and exact solutions.

INTRODUCTION

The determination of the natural vibration characteristics is of fundamental importance in the design of propeller blades. Propellers often have serious resonant vibration problems with the excitation frequencies usually being equal to the rotational speed or some multiple of it. To insure that conditions susceptible to resonance do not exist within the range of operating speeds, it is necessary that the natural frequencies be determined accurately as a function of the rotational speed. Also, the natural modes, because of their orthogonality relationships, are very suitable for use in "series" solutions of response problems.

This paper formulates a numerical solution for the natural vibration frequencies and the modal functions of twisted propeller blades. In addition to presenting a means

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for determining accurately the vibration characteristics of propeller blades, this paper presents a numerical method for solving linear ordinary differential equations.

The twisted propeller is idealized in this analysis as a rotating cantilevered beam which has a nonuniform and unsymmetrical cross section. The term "twist" is used to define a variable orientation along the length of the beam of the principal axes relative to the plane of rotation and is not to be confused with torsion. Either twist or an unsymmetrical cross section causes the beam to have transverse displacements in two directions. Two coupled differential equations, which describe the behavior of such a beam having motions in two directions, and the associated orthogonality relationships are derived in the appendixes.

A number of studies have been published concerning the determination of the natural vibration characteristics of propeller blades by energy methods, by adaptations of the Holzer method, and by the Stodola method. Earlier analyses neglected the twist of the blades and also assumed the cross sections to be symmetrical. More recent studies (refs. 1 to 4) have treated the twisted propeller which has lateral deflections in two directions. In reference 1 this problem was analyzed through the use of energy principles. Extensions of the Holzer method were used in references 2 and 3. Reference 4 gives a solution based upon the Stodola method.

In this paper the integrating matrix is developed as the basis for the method of solution. The integrating matrix is a means of numerically integrating a function that is expressed in terms of the values of the function at equal increments of the independent variable. It is derived by expressing the integrand as a polynomial in the form of Newton's forward-difference interpolation formula. Integrating matrices based upon polynomials of degrees one to seven are given.

The solution to the governing differential equations of motion is developed entirely in matrix notation. First, the differential equations, which are fourth-order linear homogeneous equations having variable coefficients, are expressed in a matrix equation. This matrix differential equation is then integrated repeatedly by using the integrating matrix as an operator. Next, the constants of integration are evaluated by applying the boundary conditions. Finally, the resulting matrix equation is expressed in the familiar concise form of the eigenvalue problem. The solutions to this eigenvalue problem may be obtained by any one of several methods. Two methods which have been utilized for this problem are described. The solutions define the natural frequencies and the modal displacements. The dominant eigenvalues correspond to the lower frequencies.

The continuous problem is treated instead of a discrete or lumped mass system. Integral equations as such and influence coefficients, which are typical of other integration methods for boundary-value problems, are not used. In developing the solution, all functions are in effect represented by high-degree polynomials at the boundaries as well as elsewhere within the beam. Since the polynomials approximate the continuous functions very accurately, the integration of these polynomial representations yield extremely small errors. In addition to treating the continuous problem and yielding accurate results, the method is appealing because the numerical solution may be formulated quickly from the differential equations and may be easily programed for computations by a digital computer. Also, the inputs are generated very simply since they are merely the coefficients appearing in the differential equations.

Numerical examples are presented to show that the solution is applicable to propeller blades and to give an indication of the accuracy of the method. The computed natural vibration frequencies of a rotating propeller blade are compared with experimental data. Also, the numerical solutions for the natural vibration characteristics of a nonrotating cantilevered beam, which has a uniform and symmetric cross section, are compared with the exact solutions.

SYMBOLS

A — cross-sectional area, in^2 (m^2)

$\vec{a}$ — acceleration vector

a_i — polynomial coefficient where $i = 0, 1, 2, \ldots, r$

$a_{j,k}$ — element in the jth row and kth column of the matrix $[A]$ where $j,k = 1, 2, \ldots, n$

C_1 — constant of integration

c_j — scaling constant

D_i — determinant of the matrix $[A]$ with the ith and (n+i)th rows and the ith and (n+i)th columns removed

E — shifting operator; modulus of elasticity, lb/in^2 (N/m^2)

f — arbitrary function of x

g — function of p

h — increment of the independent variable x; length of beam interval, in. (m)

I	moments and product of inertia, in^4 (m^4)
i	station number
$\vec{i}, \vec{j}, \vec{k}$	unit vectors along the X-, Y-, and Z-axes, respectively
j	index denoting jth natural mode of vibration
k_i, k_{j1}, k_{j2}	constants of integration
l	length of beam, in. (m)
M	bending moment, in-lb (m-N)
$M_{j,k}$	minor of the element $a_{j,k}$
m	mass per unit length, $lb\text{-}sec^2/in^2$ ($N\text{-}sec^2/m^2$)
n	number of intervals
p	nondimensionalized independent variable
R	radius of propeller blade, in. (m)
r	degree of polynomial approximation; index denoting the rth natural mode of vibration
$\vec{r}$	position vector
s	index denoting the sth natural mode of vibration
T	axial tensile force, lb (N)
t	time, sec
u	elastic displacement of the centroidal axis in the X-direction, in. (m)
$\hat{u}$	total elastic displacement in the X-direction of a point not on the centroidal axis, in. (m)

V	shear force, lb (N)
v,w	components of the elastic displacement of the centroidal axis in the directions parallel and normal, respectively, to the plane of rotation, in. (m)
$\bar{v},\bar{w}$	vibration amplitude for v and w, respectively, in. (m)
$\hat{w}$	variable in numerical example
X,Y,Z	axes of rotating orthogonal reference frame
x	independent variable; longitudinal body-axis coordinate, in. (m)
y,z	cross-sectional body-axis coordinates which are parallel and normal, respectively, to the plane of rotation when the propeller is in the unstrained condition, in. (m)
y',z'	inclined cross-sectional body-axis coordinates, in. (m)
$\alpha_{j,k}$	cofactor of the element $a_{j,k}$
β	orientation of the y',z' reference with respect to the y,z reference, deg or rad
$\Delta^k f_0$	kth forward difference
ϵ	longitudinal strain, in./in. (m/m)
ζ	arbitrary function
λ	eigenvalue, sec^2
Ω	angular velocity of the propeller blade, rad/sec or rpm
ω	natural vibration frequency, rad/sec or Hz
$\vec{\omega}$	angular velocity vector

Subscripts:

i	denotes function evaluated at $x = x_i$ where $i = 0, 1, 2, \ldots, n$
j	denotes jth natural vibration mode
r	denotes degree of polynomial approximation
Y,Z	denote directions of moments and shear forces
y,z,y',z'	denote axes about which moments and products of inertia are taken

Matrix notation:

$[\;\;]$	square matrix
$\lceil\;\;\rfloor$	diagonal matrix
$\{\;\;\}$	column matrix
$\lfloor\;\;\rfloor$	row matrix
$[\;\;]^{-1}$	inverted matrix
$[\;\;]^{T}$	transposed matrix
$[I],[\tilde{I}]$	integrating matrices
$\lceil 1 \rfloor$	unit or identity matrix
$\{1\}$	column matrix with all elements unity
$\{0\}$	column matrix with all elements zero

Matrices and the equations by which they are defined are as follows:

Matrix	Equation number
$[A]$	(C7)
$[A_r]$	(A1) to (A7)
$[B]$	(10c)
$\lfloor B_0 \rfloor$	(F9a)
$\lfloor B_n \rfloor$	(F9b)
$[\diagdown C \diagdown]$	(22e)
$[D_0]$	(34a)
$[D_n]$	(34b)
$[F]$	(41)
$[\bar{F}]$	(F13b)
$\{f\}$	(8c)
$[G]$	(45c)
$[H]$	(45d)
$[I]$	(12)
$[\tilde{I}]$	(24a)
$\{K_j\}$	(27), (35), (36), (37), (38), (40)
$[L]$	(C4c)
$[\diagdown m \diagdown]$	(22b)
$[\diagdown \bar{m} \diagdown]$	(F3d)

Matrix	Equation number
$[N]$	(46b)
$[N(s)]$	(E8)
$\lceil P \rfloor$	(43b)
$\lfloor Q \rfloor$	(E1b)
$[S]$	(22d)
$[\bar{S}]$	(F3c)
$[S']$	(C4b)
$\lceil T \rfloor$	(22c)
$\{T\}$	(39b)
$\{w\}$	(F3b)
$\lceil x \rfloor$	(28b)
$\{\Phi\}$	(E2)
$\{\varphi\}$	(22a)
$[\Psi(s)]$	(E6a)
$\left\{\int_{x_{i-1}}^{x_i} f(x)dx\right\}$	(8b)
$\left\{\int_{x_0}^{x_i} f(x)dx\right\}$	(10b)

ANALYSIS

In the analysis, a development of the integrating matrix is presented. A statement of the governing differential equations and boundary conditions is given. The differential equations are expressed in matrix notation and are then integrated numerically by using the integrating matrix as an operator. After the evaluation of the constants of integration, an eigenvalue problem results. Also, procedures for solving the eigenvalue problem are described.

Development of the Integrating Matrix

The integrating matrix is a means by which a continuous function may be integrated with the use of a finite-difference approach. This numerical method is based upon the assumption that the function $f(x)$ may be represented by a polynomial of degree r as given in the following equation:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r \tag{1}$$

Let f_i denote the value of the function $f(x)$ at the station $x = x_i$ where $i = 0, 1, 2, \ldots, r$. If the stations are assumed to be equally spaced, then $x_i = x_0 + ih$ where h is the spacing interval. Equation (1) may be expressed in the form of Newton's forward-difference interpolation formula (ref. 5) as

$$\begin{aligned} f(x) &= g(p) \\ &= f_0 + \frac{1}{1!} p \, \Delta f_0 + \frac{1}{2!} p(p - 1)\Delta^2 f_0 + \frac{1}{3!} p(p - 1)(p - 2)\Delta^3 f_0 \\ &\quad + \ldots + \frac{1}{r!} p(p - 1) \ldots (p - r + 1)\Delta^r f_0 \end{aligned} \tag{2}$$

where p is the dimensionless quantity

$$p = \frac{x - x_0}{h} \tag{3}$$

The forward differences appearing in equation (2) are given by

$$\Delta^k f_0 = (E - 1)^k f_0 \tag{4}$$

The shifting operator E is defined by $E^j f_i = f_{i+j}$ where $j = 1, 2, \ldots, r$ and $i = 0$ for the interpolation formula in equation (2).

Since $f(x) \equiv g(p)$ and since from equation (3) $dx = h\,dp$, the integration of $f(x)$ from x_i to x_j is given by

$$\int_{x_i}^{x_j} f(x)dx = h \int_i^j g(p)dp \tag{5}$$

In proceeding with the development of the integrating matrix, it is convenient to choose a specific polynomial. If the function is assumed to be approximated by a third-degree ($r = 3$) polynomial, then $g(p)$ from equation (2) may be substituted into equation (5) and integrated over each of the three intervals between x_0 and x_3 to give

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{24}\left(9f_0 + 19f_1 - 5f_2 + f_3\right) \tag{6a}$$

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{24}\left(-f_0 + 13f_1 + 13f_2 - f_3\right) \tag{6b}$$

$$\int_{x_2}^{x_3} f(x)dx = \frac{h}{24}\left(f_0 - 5f_1 + 19f_2 + 9f_3\right) \tag{6c}$$

The function $f(x)$ may be integrated over a large number of equal-length intervals by repeated use of equation (6b). Equation (6b) is chosen since it is applicable to the center interval and since it has the smallest associated error of the three equations. Thus, from equations (6), the numerical integration of $f(x)$ over each of the n intervals from x_0 to x_n is given by

$$\left.\begin{aligned}
\int_{x_0}^{x_1} f(x)dx &= \frac{h}{24}\left(9f_0 + 19f_1 - 5f_2 + f_3\right) \\
\int_{x_1}^{x_2} f(x)dx &= \frac{h}{24}\left(-f_0 + 13f_1 + 13f_2 - f_3\right) \\
\int_{x_2}^{x_3} f(x)dx &= \frac{h}{24}\left(-f_1 + 13f_2 + 13f_3 - f_4\right) \\
&\;\;\vdots \\
\int_{x_{n-2}}^{x_{n-1}} f(x)dx &= \frac{h}{24}\left(-f_{n-3} + 13f_{n-2} + 13f_{n-1} - f_n\right) \\
\int_{x_{n-1}}^{x_n} f(x)dx &= \frac{h}{24}\left(f_{n-3} - 5f_{n-2} + 19f_{n-1} + 9f_n\right)
\end{aligned}\right\} \tag{7}$$

The relations in equations (7) may be expressed in matrix notation by

$$\left\{\int_{x_{i-1}}^{x_i} f(x)dx\right\} = \left[A_3\right]\left\{f\right\} \tag{8a}$$

where

$$\left\{\int_{x_{i-1}}^{x_i} f(x)dx\right\} = \left\{\begin{matrix} 0 \\ \int_{x_0}^{x_1} f(x)dx \\ \int_{x_1}^{x_2} f(x)dx \\ \cdot \\ \cdot \\ \cdot \\ \int_{x_{n-1}}^{x_n} f(x)dx \end{matrix}\right\} \tag{8b}$$

$$\left\{f\right\} = \left\{\begin{matrix} f_0 \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{matrix}\right\} \tag{8c}$$

$$\left[A_3\right] = \frac{h}{24}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ 9 & 19 & -5 & 1 & 0 & 0 & & & & & \vdots \\ -1 & 13 & 13 & -1 & 0 & 0 & & & & & \vdots \\ 0 & -1 & 13 & 13 & -1 & 0 & \ddots & & & & \vdots \\ 0 & 0 & -1 & 13 & 13 & -1 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & 0 & -1 & 13 & 13 & -1 \\ 0 & \cdots & & & & & 0 & 1 & -5 & 19 & 9 \end{bmatrix} \tag{8d}$$

The matrix $[A_3]$ is an (n + 1) square matrix. As noted earlier, this matrix is based upon a third-degree polynomial. In a similar manner, matrices such as this may be obtained by using polynomials of other degrees. In general, such matrices are identified as $[A_r]$ where r denotes the degree of the polynomial upon which it is based.

The integral of $f(x)$ from x_0 to x_i may be obtained by merely summing the integrals given for each interval from x_0 to x_i. That is,

$$\int_{x_0}^{x_i} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{i-1}}^{x_i} f(x)dx \tag{9}$$

This equation is given in matrix notation for $i = 0, 1, 2, \ldots, n$ by

$$\left\{\int_{x_0}^{x_i} f(x)dx\right\} = [B]\left\{\int_{x_{i-1}}^{x_i} f(x)dx\right\} \tag{10a}$$

where

$$\left\{\int_{x_0}^{x_i} f(x)dx\right\} = \begin{Bmatrix} 0 \\ \int_{x_0}^{x_1} f(x)dx \\ \int_{x_0}^{x_2} f(x)dx \\ \cdot \\ \cdot \\ \cdot \\ \int_{x_0}^{x_n} f(x)dx \end{Bmatrix} \tag{10b}$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & 0 & & & \vdots \\ 1 & 1 & 1 & 0 & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & 1 & 0 \\ 1 & \cdots & & & & 1 & 1 \end{bmatrix} \tag{10c}$$

The matrix $\lceil B \rfloor$ is an (n + 1) square matrix having a lower triangular array of unit elements.

Substituting equation (8a) into equation (10a) and replacing the matrix $\lceil A_3 \rfloor$ by the general matrix $\lceil A_r \rfloor$ gives

$$\left\{\int_{x_0}^{x_i} f(x)dx\right\} = \left[B\right]\left[A_r\right]\left\{f\right\} \tag{11}$$

The integrating matrix is now defined as

$$\left[I\right] = \left[B\right]\left[A_r\right] \tag{12}$$

From this definition, equation (11) becomes

$$\left\{\int_{x_0}^{x_i} f(x)dx\right\} = \left[I\right]\left\{f\right\} \tag{13}$$

which is the desired matrix expression. Thus, the premultiplication by the integrating matrix of a column matrix defining the function $f(x)$ yields the numerical integration of $f(x)$ from x_0 to x_i where $i = 0, 1, 2, \ldots, n$. Note that the number of intervals n is equal to or greater than the degree r of the polynomial representation.

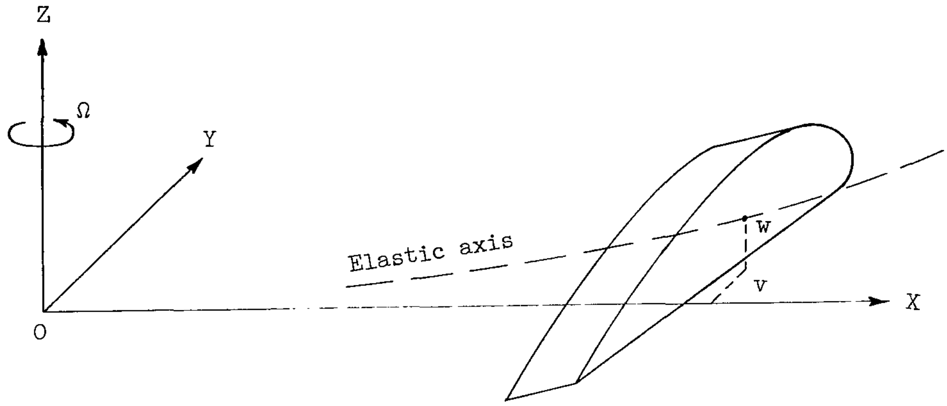
As stated in the Introduction, integrating matrices based upon polynomials of seventh and lower degree have been determined. Appendix A gives the matrix $\lceil A_r \rfloor$ for $r = 1, 2, \ldots, 7$.

In reference 6 an integrating matrix is developed by a different approach. The integrand is approximated over two intervals by passing a parabola through three equally spaced points. Expressions for the coefficients of the equation of the parabola are determined by solving a set of three simultaneous equations. Then the integrating matrix is derived through a series of matrix operations. The integrating matrix obtained is the same as that given by the product $\lceil B \rfloor \lceil A_r \rfloor$ of equation (12) for the specific case of $r = 2$ and $n = 10$. In reference 6 the integrating matrix is used to evaluate the integrals of an "integral series" solution, which is similar to the matrizant method, instead of as an operator by which the differential equations may be directly integrated.

By using Newton's forward-difference interpolation formula, the integrand may be represented conveniently by polynomials of any degree. The interpolation formula avoids the solving of a set of simultaneous equations since it implicitly contains the polynomial coefficients in terms of the values of the integrand at the equally spaced stations. Also, when the integrating matrix presented herein is applied, any number of stations may be chosen (so long as $n \geqq r$) since the integrating matrix was developed in general for any value of n.

Differential Equations and Boundary Conditions

The derivation of the differential equations of motion, along with the statement and discussion of the assumptions made, is given in appendix B. A segment of displaced



Sketch a

blade is shown in sketch a. The equations of motion for a freely vibrating propeller blade having a rotational velocity of Ω are

$$\frac{d^2}{dx^2}\left(EI_{yy}\frac{d^2w}{dx^2} + EI_{yz}\frac{d^2v}{dx^2}\right) - \frac{d}{dx}\left(T\frac{dw}{dx}\right) - \omega^2 mw = 0 \tag{14}$$

$$\frac{d^2}{dx^2}\left(EI_{yz}\frac{d^2w}{dx^2} + EI_{zz}\frac{d^2v}{dx^2}\right) - \frac{d}{dx}\left(T\frac{dv}{dx}\right) - \left(\omega^2 + \Omega^2\right)mv = 0 \tag{15}$$

where the variation of the axial tensile force T with respect to the blade longitudinal coordinate x is given by

$$\frac{dT}{dx} + \Omega^2 mx = 0 \tag{16}$$

Also, in equations (14) and (15) v and w are the in-plane (of rotation) and out-of-plane displacements, respectively; m is the distributed mass; ω is the natural vibration frequency; E is the elastic modulus; I_{yy} and I_{zz} are the moments of inertia of area about cross-sectional body axes which are parallel and perpendicular, respectively, to the plane of rotation when the blade is in the undeformed position; and I_{yz} is the product of inertia. These relations are applicable to twisted propeller blades having a nonuniform and unsymmetrical cross section.

The propeller blade is assumed to have cantilever boundary conditions. Also, the axial force at the tip of the propeller blade is zero. The boundary conditions are listed in equations (17) to (21), where the subscripts 0 and n indicate that the associated quantity is evaluated at the fixed end $x = x_0$ or at the free end $x = x_n$.

$$\left.\begin{aligned} w_0 &= 0 \\ v_0 &= 0 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} \left(\frac{dw}{dx}\right)_0 &= 0 \\ \left(\frac{dv}{dx}\right)_0 &= 0 \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} \left(M_Y\right)_n &= \left(-EI_{yy}\frac{d^2w}{dx^2} - EI_{yz}\frac{d^2v}{dx^2}\right)_n = 0 \\ \left(M_Z\right)_n &= \left(EI_{yz}\frac{d^2w}{dx^2} + EI_{zz}\frac{d^2v}{dx^2}\right)_n = 0 \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} \left(\frac{dM_Y}{dx}\right)_n &= 0 \\ \left(\frac{dM_Z}{dx}\right)_n &= 0 \end{aligned}\right\} \tag{20}$$

$$T_n = 0 \tag{21}$$

Matrix Integration of the Differential Equations

Since the differential equations are applicable for all values of x, they may be written for each of a chosen set of equally spaced stations defined by $x_i = x_0 + ih$ where $i = 0, 1, 2, \ldots, n$. Thus, each of equations (14) and (15) determines a set of $(n + 1)$ equations. These two sets of equations may be combined into a single matrix equation by defining

$$\{\varphi\} = \left\{ \begin{array}{c} w_0 \\ w_1 \\ \vdots \\ w_n \\ \hline v_0 \\ v_1 \\ \vdots \\ v_n \end{array} \right\} \tag{22a}$$

$$\left\lceil m \right\rfloor = \left[\begin{array}{ccc|ccc} m_0 & & & & & \\ & m_1 & & & & \\ & & \ddots \; m_n & & & \\ \hline & & & m_0 & & \\ & & & & m_1 & \\ & & & & & \ddots \; m_n \end{array} \right] \tag{22b}$$

$$\left\lceil T \right\rfloor = \left[\begin{array}{ccc|ccc} T_0 & & & & & \\ & T_1 & & & & \\ & & \ddots \; T_n & & & \\ \hline & & & T_0 & & \\ & & & & T_1 & \\ & & & & & \ddots \; T_n \end{array} \right] \tag{22c}$$

$$
\left[S\right] = \left[\begin{array}{cccc|cccc}
(EI_{yy})_0 & & & & (EI_{yz})_0 & & & \\
& (EI_{yy})_1 & & & & (EI_{yz})_1 & & \\
& & \ddots & & & & \ddots & \\
& & & (EI_{yy})_n & & & & (EI_{yz})_n \\
\hline
(EI_{yz})_0 & & & & (EI_{zz})_0 & & & \\
& (EI_{yz})_1 & & & & (EI_{zz})_1 & & \\
& & \ddots & & & & \ddots & \\
& & & (EI_{yz})_n & & & & (EI_{zz})_n
\end{array}\right] \tag{22d}
$$

$$
\left[C\right] = \left[\begin{array}{ccc|ccc}
0 & & & & & \\
& 0 & & & & \\
& & \ddots \; 0 & & & \\
\hline
& & & 1 & & \\
& & & & 1 & \\
& & & & & \ddots \; 1
\end{array}\right] \tag{22e}
$$

where all off-diagonal elements not shown are zero. Also, each partition of $\left[S\right]$ is a diagonal matrix. By use of the partitioned matrices defined in equations (22), equations (14) and (15) may be expressed in matrix notation as

$$
\frac{d}{dx}\left\{\frac{d}{dx}\left\{\left[S\right]\frac{d^2}{dx^2}\left\{\varphi\right\}\right\} - \left[T\right]\frac{d}{dx}\left\{\varphi\right\}\right\} - \left[\omega^2\left[1\right] + \Omega^2\left[C\right]\right]\left[m\right]\left\{\varphi\right\} = \left\{0\right\} \tag{23}
$$

where $\left[1\right]$ is the unit matrix and $\left\{0\right\}$ is a null column vector.

Since the coupled displacements of the propeller blade lead to a partitioned-matrix equation, the matrix operator necessary to integrate this matrix differential equation is given in terms of the previously developed integrating matrix by

$$\left[\tilde{I}\right] = \left[\begin{array}{c:c} [I] & 0 \\ \hdashline 0 & [I] \end{array}\right] \tag{24a}$$

for the reason that

$$\left\{\begin{array}{c} \int_{x_0}^{x_i} f(x)dx \\ \hdashline \int_{x_0}^{x_i} g(x)dx \end{array}\right\} = \left\{\begin{array}{c} [I]\left\{f(x_i)\right\} \\ \hdashline [I]\left\{g(x_i)\right\} \end{array}\right\} = \left[\begin{array}{c:c} [I] & 0 \\ \hdashline 0 & [I] \end{array}\right] \left\{\begin{array}{c} f(x_i) \\ \hdashline g(x_i) \end{array}\right\} \tag{24b}$$

Because the integration of the differential equations is to be performed in matrix notation, a clearer insight into the process may be gained by considering first the simple equation $d\zeta/dx = f(x,\zeta)$. Integrating this equation from x_0 to x_i gives

$$\zeta_i = \int_{x_0}^{x_i} f(x,\zeta)dx + C_1 \tag{25a}$$

where the constant of integration C_1 is equal to $\zeta(x_0)$ and may be determined from a boundary condition. The subscript i indicates that the function ζ is evaluated at $x = x_i$ where $i = 0, 1, 2, \ldots, n$. In equation (25a) the integral, whose integrand is a function of the unknown dependent variable ζ, may be expressed by some numerical method in terms of the integrand's unknown values corresponding to $x = x_j$ where $j = 0, 1, 2, \ldots, n$. Thus, after the numerical integration is performed, equation (25a) may be rewritten as

$$\zeta_i = \zeta_i(x_j,\zeta_j) \tag{25b}$$

which gives ζ_i as a function of the x_j and ζ_j values. Since $i = 0, 1, 2, \ldots, n$, equation (25b) represents a set of (n + 1) simultaneous linear algebraic equations which may be solved for the ζ_i values to give the numerical solution of the differential equation.

The numerical solution just described is essentially that which is effected when the integrating matrix is applied to a first-order differential equation. However, when the integrating matrix is used, the set of (n + 1) equations is expressed in a matrix equation and all the numerical integrations are performed by a single matrix operation. The integrating-matrix approach also allows the numerical solution of higher order differential equations to be developed in a compact and orderly fashion. For the propeller blade problem a matrix equation, which is analogous to equation (25b) and which relates the displacements of stations along the propeller blade, is obtained.

The second term of equation (23) may be integrated numerically by premultiplying the term by the integrating matrix $\left[\tilde{I}\right]$. Thus, the integration of the equation yields

$$\frac{d}{dx}\left\{\left[S\right]\frac{d^2}{dx^2}\left\{\varphi\right\}\right\} - \left[T\right]\frac{d}{dx}\left\{\varphi\right\} - \left[\tilde{I}\right]\left[\omega^2\left[1\right] + \Omega^2\left[C\right]\right]\left[m\right]\left\{\varphi\right\} + \left\{K_1\right\} = \left\{0\right\} \tag{26}$$

The constant-of-integration matrix is given in general by the 2(n + 1) column matrix:

$$\left\{K_j\right\} = \begin{Bmatrix} k_{j1} \\ k_{j1} \\ \cdot \\ \cdot \\ \cdot \\ k_{j1} \\ \hline k_{j2} \\ k_{j2} \\ \cdot \\ \cdot \\ \cdot \\ k_{j2} \end{Bmatrix} \qquad (j = 1, 2, 3, 4, 5) \tag{27}$$

Equation (16), which is the force equilibrium relation for the X-direction, may be written in matrix notation as

$$\frac{d}{dx}\left[T\right] = -\Omega^2\left[m\right]\left[x\right] \tag{28a}$$

where

$$\left[x\right] = \left[\begin{array}{cccc|cccc} x_0 & & & & & & & \\ & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_n & & & & \\ \hline & & & & x_0 & & & \\ & & & & & x_1 & & \\ & & & & & & \ddots & \\ & & & & & & & x_n \end{array}\right] \tag{28b}$$

Also, the second term of equation (26) may be expressed as

$$\lceil T \rfloor \frac{d}{dx}\{\varphi\} = \frac{d}{dx}\left\{\lceil T \rfloor \{\varphi\}\right\} - \left[\frac{d}{dx}\lceil T \rfloor\right]\{\varphi\} \tag{29}$$

Substituting equation (28a) into equation (29) and then substituting the result into equation (26) leads to

$$\frac{d}{dx}\left\{[S]\frac{d^2}{dx^2}\{\varphi\} - \lceil T \rfloor \{\varphi\}\right\} - \Omega^2 \lceil m \rfloor \lceil x \rfloor \{\varphi\}$$

$$- [\tilde{I}]\left\lceil \omega^2 \lceil 1 \rfloor + \Omega^2 \lceil C \rfloor \right\rfloor \lceil m \rfloor \{\varphi\} + \{K_1\} = \{0\} \tag{30}$$

Integrating this matrix differential equation by again operating with the integrating matrix gives

$$[S]\frac{d^2}{dx^2}\{\varphi\} - \lceil T \rfloor \{\varphi\} - \Omega^2 [\tilde{I}] \lceil m \rfloor \lceil x \rfloor \{\varphi\}$$

$$- [\tilde{I}]^2 \left\lceil \omega^2 \lceil 1 \rfloor + \Omega^2 \lceil C \rfloor \right\rfloor \lceil m \rfloor \{\varphi\} + [\tilde{I}]\{K_1\} + \{K_2\} = \{0\} \tag{31}$$

Equation (31) may be premultiplied by the inverse of the stiffness matrix $[S]$ and then integrated twice. The result after each of the successive integrations is given in the following equations:

$$\frac{d}{dx}\{\varphi\} - [\tilde{I}][S]^{-1}\lceil T \rfloor \{\varphi\} - \Omega^2 [\tilde{I}][S]^{-1}[\tilde{I}]\lceil m \rfloor \lceil x \rfloor \{\varphi\}$$

$$- [\tilde{I}][S]^{-1}[\tilde{I}]^2 \left\lceil \omega^2 \lceil 1 \rfloor + \Omega^2 \lceil C \rfloor \right\rfloor \lceil m \rfloor \{\varphi\} + [\tilde{I}][S]^{-1}[\tilde{I}]\{K_1\}$$

$$+ [\tilde{I}][S]^{-1}\{K_2\} + \{K_3\} = 0 \tag{32}$$

$$\{\varphi\} - [\tilde{I}]^2[S]^{-1}\lceil T \rfloor \{\varphi\} - \Omega^2 [\tilde{I}]^2[S]^{-1}[\tilde{I}]\lceil m \rfloor \lceil x \rfloor \{\varphi\}$$

$$- [\tilde{I}]^2[S]^{-1}[\tilde{I}]^2 \left\lceil \omega^2 \lceil 1 \rfloor + \Omega^2 \lceil C \rfloor \right\rfloor \lceil m \rfloor \{\varphi\} + [\tilde{I}]^2[S]^{-1}[\tilde{I}]\{K_1\}$$

$$+ [\tilde{I}]^2[S]^{-1}\{K_2\} + [\tilde{I}]\{K_3\} + \{K_4\} = 0 \tag{33}$$

The operator matrices $\left[D_0\right]$ and $\left[D_n\right]$, which are used in evaluating the constant-of-integration matrices, are defined as

$$\left[D_0\right] = \left[\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & 0 & 0 & & & & & \\
1 & 0 & \cdots & \cdots & 0 & & \text{All zero} & & & \\
\vdots & \vdots & & & \vdots & & \text{elements} & & & \\
1 & 0 & \cdots & \cdots & 0 & & & & & \\
\hline
 & & & & & 1 & 0 & \cdots & 0 & 0 \\
 & & \text{All zero} & & & 1 & 0 & \cdots & \cdots & 0 \\
 & & \text{elements} & & & \vdots & \vdots & & & \vdots \\
 & & & & & 1 & 0 & \cdots & \cdots & 0
\end{array}\right] \tag{34a}$$

$$\left[D_n\right] = \left[\begin{array}{ccccc|ccccc}
0 & 0 & \cdots & 0 & 1 & & & & & \\
0 & \cdots & \cdots & 0 & 1 & & \text{All zero} & & & \\
\vdots & & & \vdots & \vdots & & \text{elements} & & & \\
0 & \cdots & \cdots & 0 & 1 & & & & & \\
\hline
 & & & & & 0 & 0 & \cdots & 0 & 1 \\
 & & \text{All zero} & & & 0 & \cdots & \cdots & 0 & 1 \\
 & & \text{elements} & & & \vdots & & & \vdots & \vdots \\
 & & & & & 0 & \cdots & \cdots & 0 & 1
\end{array}\right] \tag{34b}$$

In these matrices, each partition is an (n + 1) square matrix.

The constant-of-integration matrix $\left\{K_1\right\}$ is evaluated by premultiplying equation (26) by $\left[D_n\right]$ and applying the boundary conditions given by equations (20) and (21). Since $\left[D_n\right]\left\{K_1\right\} = \left\{K_1\right\}$, the premultiplication leads to

$$\left\{K_1\right\} = \left[D_n\right]\left[\tilde{I}\right]\left[\omega^2\left[1\right] + \Omega^2\left[C\right]\right]\left[m\right]\left\{\varphi\right\} \tag{35}$$

Substituting this expression for $\left\{K_1\right\}$ into equation (31), premultiplying the resulting equation by $\left[D_n\right]$, and applying the boundary conditions of equations (19) and (21) yields

$$\begin{aligned}\left\{K_2\right\} = {} & \Omega^2\left[D_n\right]\left[\tilde{I}\right]\left[m\right]\left[x\right]\left\{\varphi\right\} + \left[D_n\right]\left[\tilde{I}\right]^2\left[\omega^2\left[1\right] + \Omega^2\left[C\right]\right]\left[m\right]\left\{\varphi\right\} \\ & - \left[D_n\right]\left[\tilde{I}\right]\left[D_n\right]\left[\tilde{I}\right]\left[\omega^2\left[1\right] + \Omega^2\left[C\right]\right]\left[m\right]\left\{\varphi\right\}\end{aligned} \tag{36}$$

Premultiplying equation (32) by $\left[D_0\right]$, applying the boundary conditions given in equations (18), and noting that the first row of the integrating matrix $\left[I\right]$ consists only of zero elements gives

$$\left\{K_3\right\} = \left\{0\right\} \tag{37}$$

Similarly, the premultiplication of equation (33) by $\left[D_0\right]$ and the application of the boundary conditions of equations (17) gives

$$\left\{K_4\right\} = \left\{0\right\} \tag{38}$$

A relation for the normal axial force T may be obtained from equation (16). Integrating this equation numerically gives

$$\left\{T\right\} + \Omega^2\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil x \right\rfloor\left\{1\right\} + \left\{K_5\right\} = \left\{0\right\} \tag{39a}$$

where

$$\left\{T\right\} = \left\{\begin{array}{c} T_0 \\ T_1 \\ \cdot \\ \cdot \\ \cdot \\ T_n \\ --- \\ T_0 \\ T_1 \\ \cdot \\ \cdot \\ \cdot \\ T_n \end{array}\right\} \tag{39b}$$

Premultiplying equation (39a) by $\left[D_n\right]$ and applying the boundary condition of equation (21) gives

$$\left\{K_5\right\} = -\Omega^2\left[D_n\right]\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil x \right\rfloor\left\{1\right\} \tag{40}$$

If the matrix $\left[F\right]$ is defined as

$$\left[F\right] = \left[\left[D_n\right] - \left\lceil 1 \right\rfloor\right]\left[\tilde{I}\right] \tag{41}$$

then equation (39a) may be expressed by

$$\left\{T\right\} = \Omega^2\left[F\right]\left\lceil m \right\rfloor\left\lceil x \right\rfloor\left\{1\right\} \tag{42}$$

Equation (33) requires that the matrix representing the normal axial force T be a diagonal matrix; however, the column matrix of equation (42) results since the integrating matrix is used to obtain T. To transfer the elements T_i of the column matrix to a diagonal matrix, let $\lceil P \rfloor$ be a diagonal matrix whose diagonal elements are the same as the elements of the corresponding rows of the column matrix $[F] \lceil m \rfloor \lceil x \rfloor \{1\}$. Therefore,

$$\lceil T \rfloor = \Omega^2 \lceil P \rfloor \tag{43a}$$

where

$$\lceil P \rfloor = \text{diag} [F] \lceil m \rfloor \lceil x \rfloor \{1\} \tag{43b}$$

The substitution of equations (35), (36), (37), (38), and (43a) into equation (33) yields

$$\Big[\lceil 1 \rfloor + \Omega^2 [\tilde{I}]^2 [S]^{-1} \big[-\lceil P \rfloor + [F] \lceil m \rfloor \lceil x \rfloor - [F]^2 \lceil C \rfloor \lceil m \rfloor \big]$$

$$- \omega^2 [\tilde{I}]^2 [S]^{-1} [F]^2 \lceil m \rfloor \Big] \{\varphi\} = \{0\} \tag{44}$$

Dividing equation (44) by ω^2 and rearranging gives

$$[G] \{\varphi\} = \lambda [H] \{\varphi\} \tag{45a}$$

where

$$\lambda = \frac{1}{\omega^2} \tag{45b}$$

$$[G] = [\tilde{I}]^2 [S]^{-1} [F]^2 \lceil m \rfloor \tag{45c}$$

$$[H] = \lceil 1 \rfloor + \Omega^2 [\tilde{I}]^2 [S]^{-1} \big[-\lceil P \rfloor + [F] \lceil m \rfloor \lceil x \rfloor - [F]^2 \lceil C \rfloor \lceil m \rfloor \big] \tag{45d}$$

Equation (45a) is the desired eigenvalue problem which results from the matrix integration of the differential equations. The solutions of the eigenvalue problem define the natural vibration frequencies and the associated modal functions.

Appendix C gives an expression for the stiffness matrix $[S]$ in terms of the moments and product of inertia about a pair of general orthogonal cross-sectional axes. Also, by utilizing the unique properties of the stiffness matrix, a relation is developed in this appendix which expresses explicitly the inverse of $[S]$ and thus avoids the inversion by usual computational methods.

Solutions of the Eigenvalue Problem

The eigenvalue problem defined by equations (45) may be rewritten as

$$\left[N\right]\left\{\varphi(s)\right\} = \lambda(s)\left\{\varphi(s)\right\} \tag{46a}$$

where

$$\left[N\right] = \left[H\right]^{-1}\left[G\right] \tag{46b}$$

and (s) is used to denote the solution corresponding to the sth mode of vibration. Note that the matrix $\left[H\right]$ is nonsingular and that the matrix $\left[N\right]$ is unsymmetrical.

A close examination of the product $\left[H\right]^{-1}\left[G\right]$ shows that the matrix $\left[N\right]$ is of rank 2n since all the elements of the first and (n + 2)th rows are zero. Since the order of $\left[N\right]$ is (2n + 2), there are at least two zero eigenvalues ($\lambda = 0$). This may be proved by the expansion of the characteristic equation $\left|\left[N\right] - \lambda\left[1\right]\right| = 0$. The modes corresponding to the zero eigenvalues have no physical significance.

A discussion of the many various methods of solving eigenvalue problems such as that of equations (46) is beyond the intent of this paper. However, two methods which have been used to obtain the solutions of equations (46) are explained briefly in the following sections.

Solution using the QR transformation.- This method of solution is presently employed in the most general eigenvalue subroutine used by the computer center at the Langley Research Center. First, the matrix $\left[N\right]$ of equations (46) is reduced to upper Hessenberg form by elementary similarity transformations (refs. 7 and 8). Next, the similarity transformations, known as QR transformations (refs. 7, 9, and 10), of Francis are used iteratively to reduce further the matrix to an upper triangular matrix whose diagonal elements are the eigenvalues. The eigenvectors corresponding to the real eigenvalues are computed by using the inverse iteration method of Wielandt (as discussed in ref. 10). Since $\left[N\right]$ is unsymmetrical, there is the possibility of complex eigenvalues; however, complex eigenvalues have not been encountered in the use of this solution method which does determine any imaginary components.

The eigenvectors of the check cases on this subroutine were accurate to 10 significant figures. Thus, it is assumed that the errors associated with the numerical examples of this paper arise from the numerical integration approximations rather than from the solution of the eigenvalue problem.

Solution using sweeping and iteration.- The sweeping and iteration procedure is a well-known and often-used method of solving eigenvalue problems. Iteration is a process by which an eigenvector is obtained by repeated premultiplications of a trial vector by a coefficient matrix. Applying the iteration process to equations (46) gives the first mode of vibration. The higher modes are obtained by using a sweeping matrix to remove the lower mode components from the trial vector. The sweeping matrix is based upon the orthogonality relationship, which is derived in appendix D, between the modes. The development of the sweeping matrix for this problem is given in appendix E. Also, a proof of the convergence of the iteration procedure is given.

NUMERICAL EXAMPLES

The results of two natural vibration problems are presented to give an indication of the accuracy of the analysis and to show the effects of the variation of certain parameters. The numerical examples considered are a typical propeller blade and a nonrotating, cantilevered, uniform beam having lateral displacements in only one direction. To verify the applicability of the differential equations of motion to a practical problem, the natural vibration frequencies of the propeller blade were determined numerically and compared with experimental data. The uniform-beam problem is analyzed to determine how the accuracy is affected by the choice of the integrating matrix and by the number of stations, as well as to substantiate further the accuracy of the presented method of solution.

In appendix F, the differential equation of motion for the nonrotating beam having a symmetric cross section is integrated and formulated into an eigenvalue problem. This eigenvalue problem, along with the one of the propeller blade, was programed for solution by digital computer. For the propeller blade, the inverse of the stiffness matrix as given by equations (C6) and (C13) in appendix C was programed. The solutions to the eigenvalue problems were obtained by using the previously described computer subroutine that utilizes the QR transformation.

Propeller Blade

The propeller blade selected for analysis is the WADC S-5 scale model of reference 11. This blade was chosen since this reference gives a structural description sufficient for the numerical solution as well as experimental data for the natural vibration frequencies. The propeller blade is in effect cantilevered at 6 inches (0.1524 m) from the center of rotation and the tip of the blade is at a radius of 24 inches (0.6096 m). The ratio of the blade thickness to chord length varies from a value of 0.064 at the cantilever radius to a value of 0.021 at the tip. The stiffness characteristics are expressed in terms of the moments of inertia about the principal axes whose orientation with respect to the plane of rotation is given by the twist angle β (see sketch f in appendix C). The variation of β from the root to the tip of the blade is about 40°. In the experimental program, tests were made for various angles of pitch. The pitch settings were defined by the values of β as measured as $x = 0.75R$ where R is the radius from the center of rotation to the tip of the blade. With the exception of the rotational speed, all the input data necessary for numerical solution are given in table 1 for the pitch setting of $\beta = 0^{\circ}$ at 0.75R. Of course, for a different setting, the only change in the input data is a change in the values of β by an amount equal to the setting.

In order to compare numerical results with test data, solutions were computed for cases corresponding to the pitch settings and rotational speeds of the experimental investigation. The first- and second-mode natural frequencies, but not mode shapes, were determined experimentally in an evacuated chamber for various rotational speeds at pitch settings of 0°, 20°, and 40°. The numerical solutions were obtained by using 10 stations, which correspond to nine 2-inch (0.0508-m) intervals, to describe the cross-sectional properties of the propeller blade. Also, a fifth-degree integrating matrix was used in computing the solutions. The terminology "fifth degree" denotes the integrating matrix based upon a fifth-degree polynomial ($r = 5$).

The experimentally and analytically determined natural vibration frequencies are given in table 2 for the various cases. This table also gives the percent error which is the difference between the experimental and computed values expressed as a percentage of the experimental value. These results are presented in the manner typical of propeller-blade analyses in figure 1, which shows graphically the natural frequencies as a function of the rotational speed. The dashed lines (labeled 1P, 2P, and 3P) of the figure relate the natural frequencies as dimensionless multiples of the rotational speed. The intersections of the frequency curves with these dashed lines define the critical rotational speeds (i.e., the speeds at which excitation is most likely to occur). It is observed that the existence of a crossing of the 1P line is dependent upon the pitch of the propeller blade. It is also noted that the pitch has very little effect on vibration frequencies of modes above the fundamental.

A comparison of the modal displacements is not possible since the mode shapes were not determined in the experimental investigation. However, for the sake of presenting examples of typical propeller modes and showing how the mode shapes are affected by rotational speed, figure 2 illustrates the three lowest computed modal functions at rotational speeds of 0 and 6016 rpm with the blade having a pitch angle of $\beta = 20^{\circ}$ at 0.75R.

Uniform Beam

Numerical solutions for the natural vibration characteristics of a cantilevered uniform beam were computed by using various integrating matrices and various numbers of stations. In order to simplify the comparison of the numerically determined natural frequencies with the frequencies of the exact solutions as given by reference 12, the inputs m and h were chosen to be equal to unity and EI was set equal to n^4. Thus, the quantity EI/ml^4 is always equal to unity since the length l is equal to nh. For such a beam, the exact natural vibration frequencies for the first eight modes are as follows:

Mode	Vibration frequency, ω, rad/sec
1	3.5160154
2	22.034492
3	61.697214
4	120.90192
5	199.85953
6	298.55553
7	416.99079
8	555.16525

Figures 3 to 9 present the percent errors, as defined by $\dfrac{(\omega_{computed} - \omega_{exact})10(}{\omega_{exact}}$ of the natural frequencies computed with the use of integrating matrices of degrees one to seven, respectively. Each figure gives the error percentage of the frequencies for tł first eight modes as a function of the number of stations used in the computations. The dashed curves in the figures denote percent errors having a negative sign.

A small portion of the generated modal data is presented in order to give an indication of the accuracy of the various computed mode shapes and to show how the accura is affected by the degree of the integrating matrix and by the number of intervals. Tables 3 to 6 give a comparison of the modal displacements computed by using the vari ous integrating matrices with those of the exact solution. For the computation of the di placements, both 25 intervals (n = 25) and 10 intervals (n = 10) were used, and the displacement at the free end of the beam was taken as two units. However, the data are presented only at x/l increments of 0.2, which is adequate for comparison purposes and for indicating the accuracy of the computed modal functions.

DISCUSSION

Discussion of Accuracy

The results presented in the previous section show that the governing differential equations are indeed applicable to propeller blades. Table 2 gives a comparison of computed natural frequencies with experimentally determined frequencies. The fact that the errors remain relatively small over the wide range of rotational speed and pitch setting indicates that the differential equations adequately describe the motion of the typical propeller blade. The error percentage averages less than 2 percent with a maximum value of 3.6 percent. All the computed natural frequencies are greater than the experimental frequencies except for one case. This indicates that slight inaccuracies may possibly exist in the input data for stiffness and/or distributed mass. In addition to establishing the applicability of the governing equations, these results indicate that the presented integration method is an accurate numerical method for solving differential equations.

An examination of figures 3 to 9, which give the percent error of the computed natural frequencies for a uniform beam, reveals the significance of the degree of the polynomial representation used in developing the integrating matrix. For a given number of intervals, it is seen that the use of the first-degree integrating matrix, which is based upon trapezoidal integration, results in errors much greater than those given by matrices based upon second- and higher degree polynomials. The errors of the frequencies computed with the second- and third-degree integrating matrices are of the same magnitude, with the third-degree matrix being slightly more accurate. The fourth-degree integrating matrix yields accuracy which is significantly better than that of the third-degree matrix. For an error limit in the frequencies of about 1 percent, which is satisfactory for most engineering applications, the figures show that the fourth-, fifth-, sixth-, and seventh-degree integrating matrices each require about the same number of intervals. When these integrating matrices are used, it is seen that an accuracy of 1 percent will result if the number of intervals is slightly greater than twice the number of the highest mode to be determined. However, to obtain very small errors in the frequencies, the number of intervals required appears to decrease as the degree of the integrating matrices increases.

The seventh-degree integrating matrix (fig. 9) yields errors of 0.001 percent when the number of intervals is approximately four times the number of the mode. The results for this example indicate that the solution method would be very useful in applications where extreme accuracy is desired. For example, the first-mode frequency had an error of only 0.000014 percent when the seventh-degree integrating matrix with seven intervals was used.

The curves of these figures show that the solution is apparently stable as the number of intervals increases. There is an oscillation (change in sign) in the value of the percent error for a few of the curves (see figs. 5 to 9); however, in all cases the numerical solution appears to converge, as expected, to the exact solution as the number of intervals increases.

Table 3 shows that all the computed first-mode displacements are very accurate. In fact, a maximum error of 0.00001 in the modal displacements is given by all the integrating matrices except the first-degree matrix when 25 intervals are used and the first- and second-degree matrices when 10 intervals are used. The higher modes (tables 4 to 6) show that the accuracy increases with the degree of the integrating matrix. When the seventh-degree matrix and 25 intervals are used, the maximum error in the computed modal displacements continues to be 0.00001 for the higher modes. As expected, the accuracy is significantly reduced for the higher modes that are computed with 10 intervals. However, with only 10 intervals the seventh-degree integrating matrix yields results for the fourth mode which are satisfactory for most engineering problems. The computed fourth-mode characteristics have a maximum error in the modal displacements of less than 0.01 and an error in the natural vibration frequency of 0.52 percent.

The uniform-beam vibration problem demonstrates that in general, the accuracy of the computed vibration characteristics increases with the degree of the integrating matrix and with the number of stations. Also, it shows that for a given accuracy and a given mode the number of stations required is small relative to the number needed for other numerical methods. Obviously, the higher degree integrating matrices are to be preferred since the same accuracy can be obtained with a fewer number of stations, which means less computational time. Integrating matrices based upon polynomials of degree greater than seven may be developed, but the presented results indicate that the improvement in the accuracy becomes less with each increase in the degree of the polynomial.

The accuracy of the uniform-beam solutions is thought to be typical of the accuracy which may be expected for the propeller-blade problem and for other beam vibration problems which use the integrating matrix to obtain solutions. The uniformity of the example beam is not thought to be a factor that enhances significantly the accuracy of the method since it is the modal function which is first integrated numerically.

The integrating matrix yields exact integrals when the integrand is of degree equal to or less than the degree of the polynomial upon which the integrating matrix is based. However, when the degree of the integrand is greater than the degree of the polynomial, the polynomial representation approximates the integrand, and thus, the integrals given by expressions such as equations (6) become numerical approximations. The error associated with each integral approximation may be determined by expressing the integral as a Taylor's series and then substituting finite-difference approximations for the derivatives. It can be shown that the errors of the approximations used in developing the integrating matrices are $O(h^{r+2})$ where r is the degree of the assumed polynomial. For example, the error of each of equations (6) is of order h^5.

The QR transformation method, which obtains all the eigenvalues at once and does not depend on the eigenvectors, is much preferred over the sweeping and iteration technique. The iteration process yields accurate eigenvalues, but the slight errors which occur in the eigenvectors usually prevent the determination of solutions above the third or fourth mode. A possible reason for this behavior is the fact that the orthogonality relationship is derived from the differential equations rather than from the matrix equation.

Other Applications of Solution Method

The integrating matrix may be used to solve various other boundary-value problems in addition to determining the lateral vibration characteristics of propeller blades and beams. For example, the author has used the integrating matrix to develop solutions for the longitudinal vibration characteristics of beams, the vibration characteristics of beams having coupled lateral and torsional motions, the buckling characteristics of columns, and the lateral displacements of beams under applied loading. When the integrating matrix is used to solve for the natural vibration characteristics of beams, an eigenvalue problem results in which the dominant eigenvalues correspond to the lower frequencies without the need for determining the inverse of the coefficient matrix. Also, for the nonhomogeneous-beam problem, the solution for the displacements is given directly without having to solve a set of simultaneous equations. For coupled differential equations which are of different order, such as those of a beam having bending and torsional displacements, the solution may be formulated by integrating each equation separately and then combining the resulting matrix equations into a single partitioned-matrix equation.

In addition to boundary-value problems, the integrating matrix may be applied to initial-value problems with the solution being developed in the same manner. Although the usefulness of the integrating-matrix method for solving typical engineering initial-value problems has not been demonstrated, simple examples, whose exact solutions are known, show that the method yields results which are extremely accurate relative to those of other methods (e.g., the Euler and center-slope methods) for the same spacing interval. As an example, the solution to the first-order equation $\frac{d\hat{w}}{dx} + \hat{w} = 0$ with the initial condition of $\hat{w}(x=0) = \hat{w}_0$ is found very simply to be

$$\left\{\hat{w}\right\} = \hat{w}_0\left[\left[\,^{\text{\textbackslash}}1_{\text{\textbackslash}}\right] + \left[\,I\,\right]\right]^{-1}\left\{1\right\}$$

where $\left\{\hat{w}\right\}$ is of the same form as $\left\{w\right\}$ which is defined by equation (F3b). After the inverse is evaluated, each $\hat{w}_i$ is related to $\hat{w}_0$ by a constant. To obtain the solution $\hat{w}_i$ for $i > n$, $\hat{w}_n$ becomes the initial condition and replaces $\hat{w}_0$ in the solution. This process may be repeated indefinitely to obtain $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n, \ldots, \hat{w}_{2n}, \ldots, \hat{w}_{3n},$ and so forth. It is noted that the inverse need only be computed once. Also, the solution may be obtained by calculating only the values $\hat{w}_n, \hat{w}_{2n}, \hat{w}_{3n}, \ldots,$ if the intermediate values are not desired. For this case, the spacing interval in effect becomes nh. By choosing the integrating matrix corresponding to $r = n = 2$ with $h = 0.1$ and letting

$\hat{w}_0 = 1.0$, the solution given in the following table is computed and compared with the exact solution:

x	$\hat{w}$	
	Exact	Computed
0	1.0	1.0
.2	.818731	.818731
.4	.670320	.670321
.6	.548812	.548812
.8	.449329	.449330
1.0	.367879	.367880

Thus, it is seen that a very accurate solution results from using only a second-degree integrating matrix. Upon first inverting the 3×3 matrix of the solution formulation, each $\hat{w}_i$ was obtained simply by one multiplication.

The integrating matrix may also be applied to initial-value problems which are nonhomogeneous and higher order. Furthermore, it is not necessary that the spacing interval be constant as was the case in the preceding example. For the nonhomogeneous equation $\frac{d\hat{w}}{dx} + \hat{w} = f(x)$, the solution at each station is found to be given by

$$\{\hat{w}\} = \left[\left\lceil 1 \right\rfloor + \left[I \right]\right]^{-1} \left\{\hat{w}_0\{1\} + \left[I \right]\{f\}\right\}$$

where $\hat{w}_0$ may be replaced as before by $\hat{w}_n$, $\hat{w}_{2n}$, and so forth. Examination of the last row of this matrix equation shows that the solution at every nth station is given by

$$\hat{w}_{(j+1)n} = a\hat{w}_{jn} + \sum_{i=0}^{n} b_i f_{jn+i}$$

where $j = 0, 1, 2, \ldots$, and a and b_i are constants resulting from the matrix multiplications.

Discussion of Method

The integrating-matrix method of solving differential equations is a straightforward approach (e.g., see appendix F). That is, the differential equations are integrated very simply by using the integrating matrix as an operator, and then the constants of integration are evaluated from the boundary conditions. In the presented propeller-blade problem, the development of the solution is complicated considerably by having coupled differential equations and by the centrifugal-force effects. For boundary-value problems, such as those mentioned in the previous section, the solution is formulated very quickly by this method. The solutions are expressed concisely and are readily programed by using standard subroutines in most instances. In obtaining the numerical solutions, the inputs to the computational programs are given simply by the local values of the cross-sectional properties; the generation of stiffness or influence coefficients is avoided. The computed solutions are accurate because high-degree polynomials are used to approximate the functions which are integrated.

As noted in the Introduction, the method is applicable to continuous problems rather than to discrete or lumped mass systems. Since the polynomial representations are of finite degree, the method is best applied to problems in which all functions are smoothly varying. However, with discretion the method may be applied to, for example, the problem of a beam which has an abrupt change in its cross-sectional properties such as the stiffness or the distributed mass. Problems having abrupt changes require a smaller interval size. For such problems it is prudent to obtain a second solution by decreasing the interval size and/or by increasing the degree of the integrating matrix. If either of these changes alters the solution appreciably, then it may be concluded that the interval size was too large in the first solution. The varying of the interval size without changing the solution indicates that the solution is valid.

CONCLUDING REMARKS

A numerical method for determining the natural vibration characteristics of rotating beams, such as propeller blades, is presented. By applying the integrating matrix as an operator, the derived differential equations are integrated and result in a matrix eigenvalue problem. Two methods of solving the eigenvalue problem are described, one of which utilizes the QR transformation and inverse iteration. The other method is a sweeping and iteration procedure that requires a special sweeping matrix based upon the derived orthogonality relationship. The stiffness properties may be transformed to any pair of orthogonal cross-sectional axes, and an explicit expression for the inverse of the stiffness matrix is derived.

The computed natural frequencies for a typical propeller blade were compared with measured frequencies. In addition, an error analysis of the numerical method was made by comparing the computed vibration characteristics of a uniform beam with exact solutions. These two numerical examples showed that the method of solution yields very accurate results. A discussion of the possible application of the solution method to other problems is included.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., September 18, 1970.

APPENDIX A

INTEGRATING MATRICES

The integrating matrix $\left[I \right]$ was defined by equation (12) as

$$\left[I \right] = \left[B \right] \left[A_r \right]$$

where the subscript r denotes the degree of the assumed polynomial upon which the integrating matrix is based. The matrices $\left[A_r \right]$ are as follows for $r = 1, 2, \ldots, 7$:

$$\left[A_1 \right] = \frac{h}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & \ddots & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & & 0 & 1 & 1 \end{bmatrix} \tag{A1}$$

$$\left[A_2 \right] = \frac{h}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & & 0 \\ 5 & 8 & -1 & 0 & 0 & & & \vdots \\ 0 & 5 & 8 & -1 & 0 & \ddots & & \vdots \\ 0 & 0 & 5 & 8 & -1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & 0 & 5 & 8 & -1 \\ 0 & \cdots & & & 0 & -1 & 8 & 5 \end{bmatrix} \tag{A2}$$

$$\left[A_3 \right] = \frac{h}{24} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\ 9 & 19 & -5 & 1 & 0 & 0 & & & & \vdots \\ -1 & 13 & 13 & -1 & 0 & 0 & & & & \vdots \\ 0 & -1 & 13 & 13 & -1 & 0 & \ddots & & & \vdots \\ 0 & 0 & -1 & 13 & 13 & -1 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & -1 & 13 & 13 & -1 & 0 \\ \vdots & & & & & 0 & -1 & 13 & 13 & -1 \\ 0 & \cdots & & & & 0 & 1 & -5 & 19 & 9 \end{bmatrix} \tag{A3}$$

$$
\left[A_4\right] = \frac{h}{720}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & 0 \\
251 & 646 & -264 & 106 & -19 & 0 & 0 & & & & & & & \vdots \\
-19 & 346 & 456 & -74 & 11 & 0 & 0 & & & & & & & \vdots \\
0 & -19 & 346 & 456 & -74 & 11 & 0 & \ddots & & & & & & \vdots \\
0 & 0 & -19 & 346 & 456 & -74 & 11 & \ddots & & & & & & \vdots \\
\vdots & & \ddots & & \ddots & & \ddots & & & & & & & 0 \\
\vdots & & & & & & & & 0 & -19 & 346 & 456 & -74 & 11 \\
\vdots & & & & & & & & 0 & 11 & -74 & 456 & 346 & -19 \\
0 & \cdots & & & & & & & 0 & -19 & 106 & -264 & 646 & 251
\end{bmatrix} \tag{A4}
$$

$$
\left[A_5\right] = \frac{h}{1440}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & 0 \\
475 & 1427 & -798 & 482 & -173 & 27 & 0 & 0 & & & & & & & & \vdots \\
-27 & 637 & 1022 & -258 & 77 & -11 & 0 & 0 & & & & & & & & \vdots \\
11 & -93 & 802 & 802 & -93 & 11 & 0 & 0 & & & & & & & & \vdots \\
0 & 11 & -93 & 802 & 802 & -93 & 11 & 0 & \ddots & & & & & & & \vdots \\
0 & 0 & 11 & -93 & 802 & 802 & -93 & 11 & \ddots & & & & & & & \vdots \\
\vdots & & \ddots & & \ddots & & \ddots & & & & & & & & & 0 \\
\vdots & & & & & & & & & 0 & 11 & -93 & 802 & 802 & -93 & 11 \\
\vdots & & & & & & & & & 0 & -11 & 77 & -258 & 1022 & 637 & -27 \\
0 & \cdots & & & & & & & & 0 & 27 & -173 & 482 & -798 & 1427 & 475
\end{bmatrix} \tag{A5}
$$

$$
\left[A_6\right] = \frac{h}{60\,480}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&\cdots&0\\
19\,087&65\,112&-46\,461&37\,504&-20\,211&6\,312&-863&0&0&&\vdots\\
-863&25\,128&46\,989&-16\,256&7\,299&-2\,088&271&0&0&&\vdots\\
271&-2\,760&30\,819&37\,504&-6\,771&1\,608&-191&0&0&&\vdots\\
0&271&-2\,760&30\,819&37\,504&-6\,771&1\,608&-191&0&&\vdots\\
\vdots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\vdots\\
\vdots&&0&271&-2\,760&30\,819&37\,504&-6\,771&1\,608&-191&0\\
\vdots&&0&0&271&-2\,760&30\,819&37\,504&-6\,771&1\,608&-191\\
\vdots&&0&0&-191&1\,608&-6\,771&37\,504&30\,819&-2\,760&271\\
\vdots&&0&0&271&-2\,088&7\,299&-16\,256&46\,989&25\,128&-863\\
0&\cdots&0&0&-863&6\,312&-20\,211&37\,504&-46\,461&65\,112&19\,087
\end{bmatrix} \tag{A6}
$$

$$
\left[A_7\right] = \frac{h}{120\,960}\begin{bmatrix}
36\,799&139\,849&-121\,797&123\,133&-88\,547&41\,499&-11\,351&1\,375&0&0&\cdots&0\\
-1\,375&47\,799&101\,349&-44\,797&26\,883&-11\,547&2\,999&-351&0&0&&\vdots\\
351&-4\,183&57\,627&81\,693&-20\,227&7\,227&-1\,719&191&0&0&&\vdots\\
-191&1\,879&-9\,531&68\,323&68\,323&-9\,531&1\,879&-191&0&0&&\vdots\\
0&-191&1\,879&-9\,531&68\,323&68\,323&-9\,531&1\,879&-191&0&&\vdots\\
\vdots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\vdots\\
\vdots&&0&-191&1\,879&-9\,531&68\,323&68\,323&-9\,531&1\,879&-191&0\\
\vdots&&0&0&-191&1\,879&-9\,531&68\,323&68\,323&-9\,531&1\,879&-191\\
\vdots&&0&0&191&-1\,719&7\,227&-20\,227&81\,693&57\,627&-4\,183&351\\
\vdots&&0&0&-351&2\,999&-11\,547&26\,883&-44\,797&101\,349&47\,799&-1\,375\\
0&\cdots&0&0&1\,375&-11\,351&41\,499&-88\,547&123\,133&-121\,797&139\,849&36\,799
\end{bmatrix} \tag{A7}
$$

APPENDIX B

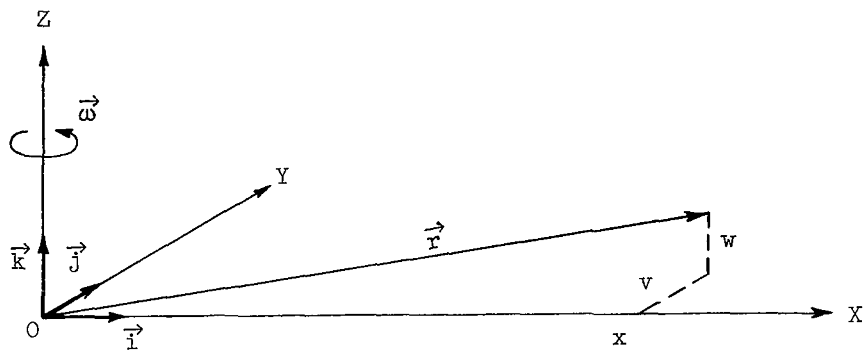
DERIVATION OF THE DIFFERENTIAL EQUATIONS OF MOTION

In this derivation of the equations of motion for a freely vibrating propeller blade, the propeller blade is idealized as a rotating beam having a nonuniform and unsymmetrical cross section. This beam is depicted in sketch b in the unstrained position. The

Sketch b

centroidal axis, which is assumed to be straight, lies along the X-axis of the rotating XYZ orthogonal coordinate system, which has its origin fixed and is rotating about the Z-axis with an angular velocity of Ω. The elastic axis of the beam is assumed to be coincident with the centroidal axis; in other words, the bending and torsional motions are assumed to be uncoupled. This is a sound assumption for the lower vibration frequencies of propeller blades because the proximity of the elastic and centroidal axes renders small coupling. The comparison of experimental vibration data of typical propeller blades with the numerical results of this and other analyses shows that the errors induced by neglecting torsional motion are very small for the lower bending modes. Also, the secondary effects of shear deformation and rotary inertia are considered negligible for propeller blades since the cross-sectional dimensions are small compared with the length. In addition, the cross-sectional properties of the beam are assumed to be continuous functions with no abrupt changes, and the lateral displacements are assumed to be small.

The inertial forces acting on a differential segment of the rotating beam are determined from the acceleration vector of the centroid of the segment. The effect of the longitudinal displacement of the centroid on the inertial forces is small and is assumed



Sketch c

negligible. The position vector (see sketch c) to the centroid of a general differential segment of the strained beam is given by

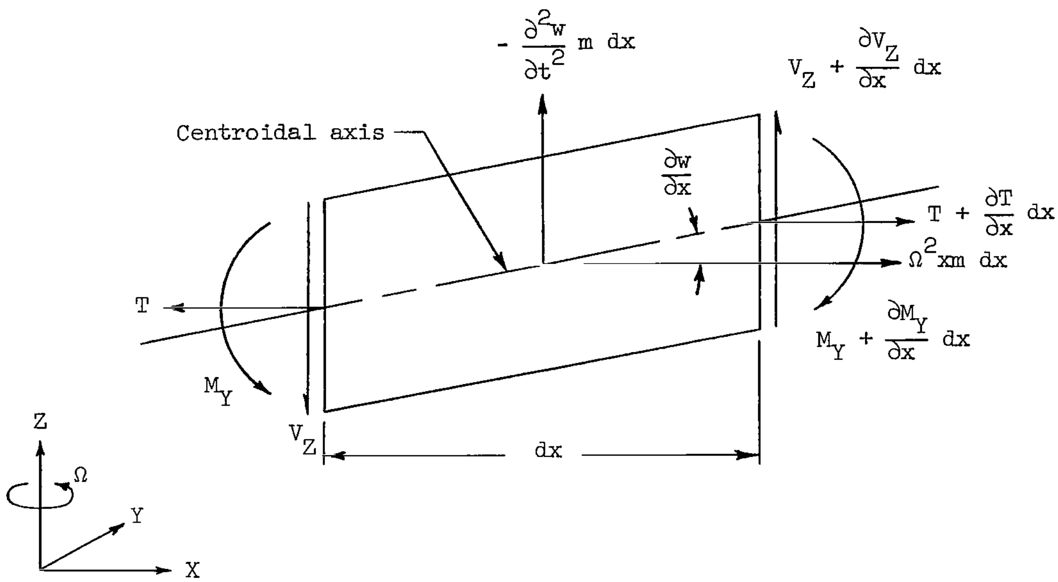
$$\vec{r} = x\vec{i} + v\vec{j} + w\vec{k} \tag{B1}$$

where x is the body-axis coordinate in the X-direction for the unstrained beam. Also, v and w are functions of x and represent the elastic displacements of the centroidal axis in the Y- and Z-directions, respectively. The absolute angular velocity of the rotating system is assumed to be constant and is given by $\vec{\omega} = \Omega\vec{k}$. Differentiating equation (B1) twice with respect to time gives the acceleration vector as

$$\vec{a} = \left(-\Omega^2 x - 2\Omega \frac{\partial v}{\partial t}\right)\vec{i} + \left(\frac{\partial^2 v}{\partial t^2} - \Omega^2 v\right)\vec{j} + \frac{\partial^2 w}{\partial t^2}\vec{k} \tag{B2}$$

It can be shown that the Coriolis acceleration term may be neglected. The natural frequencies which are of particular interest to propeller-blade analyses have the same order of magnitude as the rotational speed Ω. If harmonic motion with the frequency being the same order as the rotational speed is assumed, the Coriolis acceleration term is obviously negligible when compared with $\Omega^2 x$ since the amplitude of v is much less than the magnitude of x for small displacements.

Since the acceleration components are known, the reversed effective inertial forces acting in the X-, Y-, and Z-directions on a differential segment of length dx are $\Omega^2 xm\ dx$, $\left(\Omega^2 v - \frac{\partial^2 v}{\partial t^2}\right)m\ dx$, and $-\frac{\partial^2 w}{\partial t^2}m\ dx$, respectively, where m is the distributed mass of the beam.



Sketch d

The inertial forces, internal forces, and internal moments lying the the XZ-plane are shown acting on a differential beam segment in sketch d where T is axial tensile force, V is shear force, and M is bending moment. The summation of forces in the X-direction and in the Z-direction gives the two following equations:

$$\frac{\partial T}{\partial x} + \Omega^2 mx = 0 \tag{B3}$$

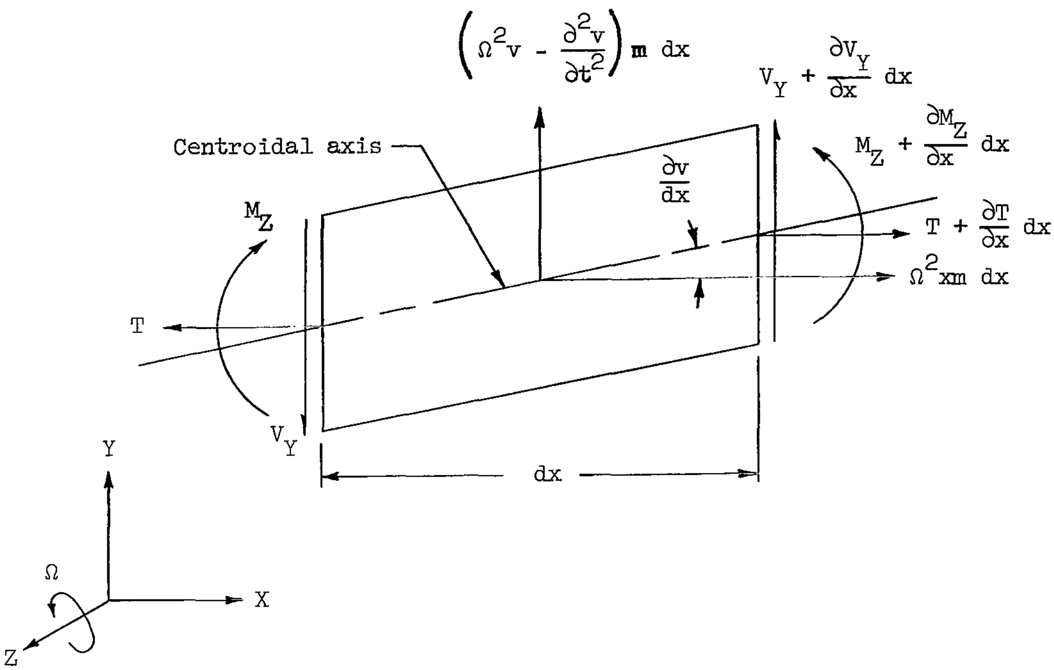
$$\frac{\partial V_Z}{\partial x} - m \frac{\partial^2 w}{\partial t^2} = 0 \tag{B4}$$

After higher order terms are dropped, the summation of moments in the Y-direction yields

$$V_Z = \frac{\partial M_Y}{\partial x} + T \frac{\partial w}{\partial x} \tag{B5}$$

Substituting equation (B5) into equation (B4) gives

$$\frac{\partial^2 M_Y}{\partial x^2} + \frac{\partial}{\partial x}\left(T \frac{\partial w}{\partial x}\right) - m \frac{\partial^2 w}{\partial t^2} = 0 \tag{B6}$$



Sketch e

Sketch e shows the forces and moments acting on the segment in the XY-plane. By proceeding in the same manner as before, the summation of forces in the Y-direction and the summation of moments in the Z-direction yields

$$\frac{\partial V_Y}{\partial x} + \left(\Omega^2 v - \frac{\partial^2 v}{\partial t^2}\right) m = 0 \tag{B7}$$

$$V_Y = -\frac{\partial M_Z}{\partial x} + T\frac{\partial v}{\partial x} \tag{B8}$$

Upon substitution of the expression for V_Y, equation (B7) becomes

$$\frac{\partial^2 M_Z}{\partial x^2} - \frac{\partial}{\partial x}\left(T\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 v}{\partial t^2} - \Omega^2 v\right) m = 0 \tag{B9}$$

Relations for the bending moments M_Y and M_Z in terms of the displacements v and w may be determined by integrating over the cross-sectional area the products of the longitudinal stress and each of the body-axis coordinates of which the stress is a function. It is necessary to determine first an expression for the longitudinal strain over the cross section. The cross-sectional body-axis coordinates are y and z which are parallel to the Y- and Z-axes, respectively, when the beam is in the unstrained condition.

Assuming that plane sections remain plane and making small-angle assumptions for $\partial v/\partial x$ and $\partial w/\partial x$, which define the slope of the centroidal axis, gives the longitudinal displacement of a point defined by y and z within a cross section as

$$\hat{u} = u - y \frac{\partial v}{\partial x} - z \frac{\partial w}{\partial x} \tag{B10}$$

where u is the elastic displacement of the centroidal axis in the X-direction. Since the longitudinal strain is defined by $\epsilon = \frac{\partial \hat{u}}{\partial x}$, equation (B10) leads to

$$\epsilon = \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2} \tag{B11}$$

From Hooke's law, the longitudinal stress is merely the product of the strain and the modulus of elasticity since the normal stresses are zero. If the elastic modulus E is assumed to be constant over the cross section, the bending moments are given by

$$\left.\begin{aligned} M_Y &= E \int_A \epsilon z \, dA \\ M_Z &= -E \int_A \epsilon y \, dA \end{aligned}\right\} \tag{B12}$$

Substituting equation (B11) into equations (B12) and integrating over the cross-sectional area yields

$$M_Y = -EI_{yy} \frac{\partial^2 w}{\partial x^2} - EI_{yz} \frac{\partial^2 v}{\partial x^2} \tag{B13}$$

$$M_Z = EI_{yz} \frac{\partial^2 w}{\partial x^2} + EI_{zz} \frac{\partial^2 v}{\partial x^2} \tag{B14}$$

where the moments and product of inertia are defined by

$$\left.\begin{aligned} I_{yy} &= \int_A z^2 dA \\ I_{zz} &= \int_A y^2 dA \\ I_{yz} &= \int_A yz \, dA \end{aligned}\right\} \tag{B15}$$

The substitution of equations (B13) and (B14) into equations (B6) and (B9), respectively, gives

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(EI_{yy}\frac{\partial^2 w}{\partial x^2} + EI_{yz}\frac{\partial^2 v}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial w}{\partial x}\right) + m\frac{\partial^2 w}{\partial t^2} = 0 \\
&\frac{\partial^2}{\partial x^2}\left(EI_{yz}\frac{\partial^2 w}{\partial x^2} + EI_{zz}\frac{\partial^2 v}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial v}{\partial x}\right) + m\left(\frac{\partial^2 v}{\partial t^2} - \Omega^2 v\right) = 0
\end{aligned}\right\} \tag{B16}$$

An examination of equations (B16) shows that the two displacements v and w are coupled by the product of inertia I_{yz} of the unsymmetrical cross section. Therefore, for $I_{yz} \neq 0$, any deflection in the Y-direction has associated with it a deflection in the Z-direction. The static coupling between the lateral displacements may be verified by using the bending-moment relations of equations (B13) and (B14) or by using directly the strain expression of equation (B11).

By assuming harmonic motion such that $v = \bar{v}e^{i\omega t}$ and $w = \bar{w}e^{i\omega t}$, the equations of motion become

$$\frac{d^2}{dx^2}\left(EI_{yy}\frac{d^2 w}{dx^2} + EI_{yz}\frac{d^2 v}{dx^2}\right) - \frac{d}{dx}\left(T\frac{dw}{dx}\right) - \omega^2 mw = 0 \tag{B17}$$

$$\frac{d^2}{dx^2}\left(EI_{yz}\frac{d^2 w}{dx^2} + EI_{zz}\frac{d^2 v}{dx^2}\right) - \frac{d}{dx}\left(T\frac{dv}{dx}\right) - \left(\omega^2 + \Omega^2\right)mv = 0 \tag{B18}$$

where ω is the natural vibration frequency. These two equations in conjunction with the auxiliary relation of equation (B3), which may be rewritten as

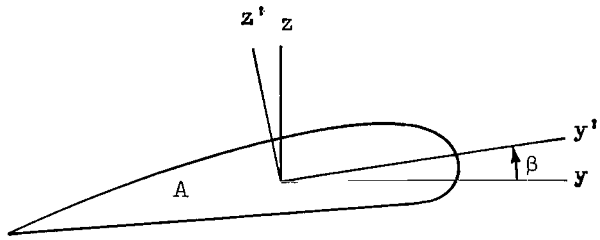
$$\frac{dT}{dx} + \Omega^2 mx = 0 \tag{B19}$$

are the governing differential equations of motion.

APPENDIX C

TRANSFORMATION AND INVERSE OF THE STIFFNESS MATRIX

It may often be desirable to express the stiffness matrix $\left[S\right]$ in terms of the moments and product of inertia about a pair of orthogonal cross-sectional axes other than the axes associated with the y,z body coordinates. As illustrated in sketch f, y'



Sketch f

and z' are defined to be the coordinates of a reference system which is obtained by the rotation of the yz reference through an angle β. The moments and product of inertia in the primed system are defined by

$$\left.\begin{aligned} I_{y'y'} &= \int_A (z')^2 dA' \\ I_{z'z'} &= \int_A (y')^2 dA' \\ I_{y'z'} &= \int_A y'z'\, dA' \end{aligned}\right\} \tag{C1}$$

Also, the coordinate transformation relations are

$$\left.\begin{aligned} y' &= y \cos \beta + z \sin \beta \\ z' &= -y \sin \beta + z \cos \beta \end{aligned}\right\} \tag{C2}$$

By substituting equations (C2) into equations (C1), noting that the Jacobian relating the differential areas in the primed and unprimed systems is equal to unity, and using the definitions of equations (B15), the moments and products of inertia of the two systems are found to be related by

$$\left.\begin{aligned}
I_{y'y'} &= I_{yy} \cos^2\beta + I_{zz} \sin^2\beta - 2I_{yz} \sin\beta \cos\beta \\
I_{z'z'} &= I_{yy} \sin^2\beta + I_{zz} \cos^2\beta + 2I_{yz} \sin\beta \cos\beta \\
I_{y'z'} &= \left(I_{yy} - I_{zz}\right)\sin\beta \cos\beta + I_{yz}\left(\cos^2\beta - \sin^2\beta\right)
\end{aligned}\right\} \tag{C3}$$

From equations (C3), the following matrix identity may be written:

$$\left[S'\right] = \left[L\right]^T\left[S\right]\left[L\right] \tag{C4a}$$

where

$$\left[S'\right] = \left[\begin{array}{cccc|cccc}
\left(EI_{y'y'}\right)_0 & & & & \left(EI_{y'z'}\right)_0 & & & \\
& \left(EI_{y'y'}\right)_1 & & & & \left(EI_{y'z'}\right)_1 & & \\
& & \ddots & & & & \ddots & \\
& & & \left(EI_{y'y'}\right)_n & & & & \left(EI_{y'z'}\right)_n \\
\hline
\left(EI_{y'z'}\right)_0 & & & & \left(EI_{z'z'}\right)_0 & & & \\
& \left(EI_{y'z'}\right)_1 & & & & \left(EI_{z'z'}\right)_1 & & \\
& & \ddots & & & & \ddots & \\
& & & \left(EI_{y'z'}\right)_n & & & & \left(EI_{z'z'}\right)_n
\end{array}\right] \tag{C4b}$$

$$\left[L\right] = \left[\begin{array}{cccc|cccc}
(\cos\beta)_0 & & & & (\sin\beta)_0 & & & \\
& (\cos\beta)_1 & & & & (\sin\beta)_1 & & \\
& & \ddots & & & & \ddots & \\
& & & (\cos\beta)_n & & & & (\sin\beta)_n \\
\hline
-(\sin\beta)_0 & & & & (\cos\beta)_0 & & & \\
& -(\sin\beta)_1 & & & & (\cos\beta)_1 & & \\
& & \ddots & & & & \ddots & \\
& & & -(\sin\beta)_n & & & & (\cos\beta)_n
\end{array}\right] \tag{C4c}$$

and $\left[S\right]$ is defined by equation (22d). Since the transformation matrix $\left[L\right]$ is orthogonal, its inverse is equal to its transpose; thus, from equation (C4a) the stiffness matrix in the yz reference may be expressed in terms of the stiffness values in the y'z' reference by

$$\left[S\right] = \left[L\right]\left[S'\right]\left[L\right]^T \tag{C5}$$

Also, taking the inverse of both sides of equation (C5) gives the transformation of the inverse of the stiffness matrix as

$$\left[S\right]^{-1} = \left[L\right]\left[S'\right]^{-1}\left[L\right]^T \tag{C6}$$

Since it is desirable to avoid the inversion of a matrix whenever possible, an explicit expression will be derived for the inverse of the stiffness matrix which appears in the eigenvalue problem of equations (45). Consider the matrix $\left[A\right]$ given in equation (C7) which is of the same form as the matrices $\left[S\right]$ and $\left[S'\right]$ defined, respectively, by equations (22d) and (C4b). However, in the development which follows it is not necessary for $\left[A\right]$ to be symmetric as are $\left[S\right]$ and $\left[S'\right]$.

$$\left[A\right] = \left[\begin{array}{cccc|cccc}
a_{1,1} & & & & a_{1,n+1} & & & \\
& a_{2,2} & & & & a_{2,n+2} & & \\
& & a_{i,i} & & & & a_{i,n+i} & \\
& & & a_{n,n} & & & & a_{n,2n} \\
\hline
a_{n+1,1} & & & & a_{n+1,n+1} & & & \\
& a_{n+2,2} & & & & a_{n+2,n+2} & & \\
& & a_{n+i,i} & & & & a_{n+i,n+i} & \\
& & & a_{2n,n} & & & & a_{2n,2n}
\end{array}\right] \tag{C7}$$

In the above matrix $\left[A\right]$ all off-diagonal elements in each of the four n-square partition matrices are zero and $i = 1, 2, \ldots, n$.

Notice that the cofactor of each zero element of $\left[A\right]$ is zero since the minor of each such element contains two rows, as well as two columns, which are proportional. Let $\alpha_{j,k}$ represent the cofactor of the element $a_{j,k}$, where $j,k = 1, 2, \ldots, 2n$. Then the inverse of the matrix $\left[A\right]$ is given by

$$[A]^{-1} = \frac{1}{|[A]|}\left[\begin{array}{cccc:cccc}
\alpha_{1,1} & & & & \alpha_{n+1,1} & & & \\
 & \alpha_{2,2} & & & & a_{n+2,2} & & \\
 & & \ddots\ \alpha_{i,i} & & & & \ddots\ \alpha_{n+i,i} & \\
 & & & \ddots\ \alpha_{n,n} & & & & \ddots\ \alpha_{2n,n} \\ \hdashline
\alpha_{1,n+1} & & & & \alpha_{n+1,n+1} & & & \\
 & \alpha_{2,n+2} & & & & \alpha_{n+2,n+2} & & \\
 & & \ddots\ \alpha_{i,n+i} & & & & \ddots\ \alpha_{n+i,n+i} & \\
 & & & \ddots\ \alpha_{n,2n} & & & & \ddots\ \alpha_{2n,2n}
\end{array}\right] \tag{C8}$$

The cofactor $\alpha_{i,i}$ in the upper-left partition of $[A]^{-1}$ is equal to the minor of the element $a_{i,i}$ with a sign. Since the minor of $a_{i,i}$ is equal to the sum of the products of the elements of the (n + i - 1)th row (or column) of the minor and their respective cofactors, the cofactor $\alpha_{i,i}$ may be expressed by

$$\alpha_{i,i} = (-1)^{2i} M_{i,i} = a_{n+i,n+i}(-1)^{2(n+i-1)} D_i = a_{n+i,n+i} D_i \tag{C9a}$$

where $M_{i,i}$ is the minor of the element $a_{i,i}$ and D_i is the determinant of the matrix $[A]$ with the ith and (n + i)th rows and the ith and (n + i)th columns removed.

In a similar manner, the cofactors of the other partitions are found to be

$$\alpha_{i,n+i} = (-1)^{n+2i} M_{i,n+i} = (-1)^n a_{n+i,i} (-1)^{n+2i-1} D_i = -a_{n+i,i} D_i \tag{C9b}$$

$$\alpha_{n+i,i} = (-1)^{n+2i} M_{n+i,i} = (-1)^n a_{i,n+i} (-1)^{n+2i-1} D_i = -a_{i,n+i} D_i \tag{C9c}$$

$$\alpha_{n+i,n+i} = (-1)^{2(n+i)} M_{n+i,n+i} = a_{i,i} (-1)^{2i} D_i = a_{i,i} D_i \tag{C9d}$$

The determinant of $[A]$ may be expressed as the sum of the products of the elements of the ith row and their respective cofactors. Thus,

$$\left|\left[A\right]\right| = a_{i,i}\alpha_{i,i} + a_{i,n+i}\alpha_{i,n+i} = \left(a_{i,i}a_{n+i,n+i} - a_{i,n+i}a_{n+i,i}\right)D_i \tag{C10}$$

From equations (C9) and (C10), it is seen that the elements in $\left[A\right]^{-1}$ of equation (C8) are given by

$$\frac{\alpha_{i,i}}{\left|\left[A\right]\right|} = \frac{-a_{n+i,n+i}}{a_{i,i}a_{n+i,n+i} - a_{i,n+i}a_{n+i,i}} \tag{C11a}$$

$$\frac{\alpha_{i,n+i}}{\left|\left[A\right]\right|} = \frac{-a_{n+i,i}}{a_{i,i}a_{n+i,n+i} - a_{i,n+i}a_{n+i,i}} \tag{C11b}$$

$$\frac{\alpha_{n+i,i}}{\left|\left[A\right]\right|} = \frac{-a_{i,n+i}}{a_{i,i}a_{n+i,n+i} - a_{i,n+i}a_{n+i,i}} \tag{C11c}$$

$$\frac{\alpha_{n+i,n+i}}{\left|\left[A\right]\right|} = \frac{a_{i,i}}{a_{i,i}a_{n+i,n+i} - a_{i,n+i}a_{n+i,i}} \tag{C11d}$$

Substitution of the corresponding elements of equation (C4b) into equations (C11) and then substitution of the results into equation (C8) gives the inverse of $\left[S'\right]$ as

$$\left[S'\right]^{-1} = \left[\begin{array}{cc|cc}
\left(\frac{EI_{z'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_0 & & \left(\frac{-EI_{y'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_0 & \\
 & \ddots & & \ddots \\
 & \left(\frac{EI_{z'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_n & & \left(\frac{-EI_{y'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_n \\
\hline
\left(\frac{-EI_{y'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_0 & & \left(\frac{EI_{y'y'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_0 & \\
 & \ddots & & \ddots \\
 & \left(\frac{-EI_{y'z'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_n & & \left(\frac{EI_{y'y'}}{EI_{y'y'}EI_{z'z'} - EI_{y'z'}^2}\right)_n
\end{array}\right] \tag{C12}$$

It is obvious that a similar expression may be written for $\left[S\right]^{-1}$.

If the angle β represents the orientation of a principal axis of inertia, then since $I_{y'z'} = 0$, $\left[S'\right]^{-1}$ reduces to the following diagonal matrix:

$$\left[S'\right]^{-1} = \left[\begin{array}{ccc|ccc} \left(\frac{1}{EI_{y'y'}}\right)_0 & & & & & \\ & \ddots & & & & \\ & & \left(\frac{1}{EI_{y'y'}}\right)_n & & & \\ \hline & & & \left(\frac{1}{EI_{z'z'}}\right)_0 & & \\ & & & & \ddots & \\ & & & & & \left(\frac{1}{EI_{z'z'}}\right)_n \end{array}\right] \tag{C13}$$

APPENDIX D

DERIVATION OF THE ORTHOGONALITY RELATION

A knowledge of the orthogonality relation between the natural modes is important to the analysis of many vibration problems. The orthogonality relation is often used to obtain the solutions of eigenvalue problems by iteration methods. Also, the relation is a very useful tool in modal series solutions of response problems. Since the displacements w and v are coupled, the conventional orthogonality relation for beams in bending is not valid.

The differential equations of motion for the rth mode are

$$\frac{d^2}{dx^2}\left[EI_{yy}\frac{d^2w(r)}{dx^2} + EI_{yz}\frac{d^2v(r)}{dx^2}\right] - \frac{d}{dx}\left[T\frac{dw(r)}{dx}\right] - \omega^2(r)mw(r) = 0 \qquad \text{(D1)}$$

$$\frac{d^2}{dx^2}\left[EI_{yz}\frac{d^2w(r)}{dx^2} + EI_{zz}\frac{d^2v(r)}{dx^2}\right] - \frac{d}{dx}\left[T\frac{dv(r)}{dx}\right] - \left[\omega^2(r) + \Omega^2\right]mv(r) = 0 \qquad \text{(D2)}$$

Multiplying equation (D1) by $w(s)dx$ and equation (D2) by $v(s)dx$, where $w(s)$ and $v(s)$ are the displacements of the sth mode, and then integrating both equations from x_0 to x_n gives

$$\int_{x_0}^{x_n} w(s)\frac{d^2}{dx^2}\left[EI_{yy}\frac{d^2w(r)}{dx^2} + EI_{yz}\frac{d^2v(r)}{dx^2}\right]dx$$

$$- \int_{x_0}^{x_n} w(s)\frac{d}{dx}\left[T\frac{dw(r)}{dx}\right]dx - \omega^2(r)\int_{x_0}^{x_n} mw(r)w(s)dx = 0 \qquad \text{(D3)}$$

and

$$\int_{x_0}^{x_n} v(s)\frac{d^2}{dx^2}\left[EI_{yz}\frac{d^2w(r)}{dx^2} + EI_{zz}\frac{d^2v(r)}{dx^2}\right]dx$$

$$- \int_{x_0}^{x_n} v(s)\frac{d}{dx}\left[T\frac{dv(r)}{dx}\right]dx - \left[\omega^2(r) + \Omega^2\right]\int_{x_0}^{x_n} mv(r)v(s)dx = 0 \qquad \text{(D4)}$$

Performing the integrations in equations (D3) and (D4) by parts and applying the boundary conditions yields

$$\int_{x_0}^{x_n}\left[EI_{yy}\frac{d^2w(r)}{dx^2} + EI_{yz}\frac{d^2v(r)}{dx^2}\right]\frac{d^2w(s)}{dx^2}\,dx$$

$$+\int_{x_0}^{x_n} T\,\frac{dw(r)}{dx}\,\frac{dw(s)}{dx}\,dx - \omega^2(r)\int_{x_0}^{x_n} mw(r)w(s)dx = 0 \tag{D5}$$

and

$$\int_{x_0}^{x_n}\left[EI_{yz}\frac{d^2w(r)}{dx^2} + EI_{zz}\frac{d^2v(r)}{dx^2}\right]\frac{d^2v(s)}{dx^2}\,dx$$

$$+\int_{x_0}^{x_n} T\,\frac{dv(r)}{dx}\,\frac{dv(s)}{dx}\,dx - \left[\omega^2(r) + \Omega^2\right]\int_{x_0}^{x_n} mv(r)v(s)dx = 0 \tag{D6}$$

In like manner, two additional relations similar to equations (D5) and (D6) may be obtained. First the differential equations are written for the sth mode as

$$\frac{d^2}{dx^2}\left[EI_{yy}\frac{d^2w(s)}{dx^2} + EI_{yz}\frac{d^2v(s)}{dx^2}\right] - \frac{d}{dx}\left[T\,\frac{dw(s)}{dx^2}\right] - \omega^2(s)mw(s) = 0 \tag{D7}$$

$$\frac{d^2}{dx^2}\left[EI_{yz}\frac{d^2w(s)}{dx^2} + EI_{zz}\frac{d^2v(s)}{dx^2}\right] - \frac{d}{dx}\left[T\,\frac{dv(s)}{dx}\right] - \left[\omega^2(s) + \Omega^2\right]mv(s) = 0 \tag{D8}$$

Equations (D7) and (D8) may be multiplied by $w(r)dx$ and $v(r)dx$, respectively, and then integrated from x_0 to x_n. Making the multiplications, performing the integration by parts, and applying the boundary conditions yields

$$\int_{x_0}^{x_n}\left[EI_{yy}\frac{d^2w(s)}{dx^2} + EI_{yz}\frac{d^2v(s)}{dx^2}\right]\frac{d^2w(r)}{dx^2}\,dx$$

$$+\int_{x_0}^{x_n} T\,\frac{dw(r)}{dx}\,\frac{dw(s)}{dx}\,dx - \omega^2(s)\int_{x_0}^{x_n} mw(r)w(s)dx = 0 \tag{D9}$$

and

$$\int_{x_0}^{x_n}\left[EI_{yz}\frac{d^2w(s)}{dx^2}+EI_{zz}\frac{d^2v(s)}{dx^2}\right]\frac{d^2v(r)}{dx^2}\,dx$$

$$+\int_{x_0}^{x_n}T\frac{dv(r)}{dx}\frac{dv(s)}{dx}\,dx-\left[\omega^2(s)+\Omega^2\right]\int_{x_0}^{x_n}mv(r)v(s)dx=0 \tag{D10}$$

Subtracting the sum of equations (D5) and (D6) from the sum of equations (D9) and (D10) gives

$$\left[\omega^2(r)-\omega^2(s)\right]\int_{x_0}^{x_n}m\left[w(r)w(s)+v(r)v(s)\right]dx=0 \tag{D11}$$

If the natural vibration frequencies are assumed to be different for the two modes, then for $r \neq s$,

$$\int_{x_0}^{x_n}m\left[w(r)w(s)+v(r)v(s)\right]dx=0 \tag{D12}$$

which is the desired orthogonality relation.

APPENDIX E

SWEEPING AND ITERATION PROCEDURE

The orthogonality relation derived in appendix D may by use of the integrating matrix be expressed in matrix notation as

$$\lfloor Q \rfloor \left[\tilde{I}\right] \left[m \right] \left[\varphi(r) \right] \left\{ \varphi(s) \right\} = 0 \qquad (r \neq s) \qquad \text{(E1a)}$$

where the row matrix $\lfloor Q \rfloor$, which consists of two partition matrices with (n + 1) elements in each, is given by

$$\lfloor Q \rfloor = \lfloor 0 \; . \; . \; . \; 0 \;\; 1 \;\; \vdots \;\; 0 \; . \; . \; . \; 0 \;\; 1 \rfloor \qquad \text{(E1b)}$$

An arbitrary vector $\{\Phi\}$ may be expressed as a linear combination of the independent eigenvectors by

$$\{\Phi\} = c_1 \{\varphi(1)\} + c_2 \{\varphi(2)\} + \; . \; . \; . + c_{2n+2} \{\varphi(2n + 2)\} \qquad \text{(E2)}$$

In equation (E2) the normalized eigenvectors $\{\varphi(j)\}$ correspond to the eigenvalues $\lambda(j)$ which decrease in magnitude as j varies from 1 to (2n + 2). An expression for the scaling constants c_j may be obtained by premultiplying equation (E2) by $\lfloor Q \rfloor \left[\tilde{I}\right] \left[m \right] \left[\varphi(j) \right]$ and applying the orthogonality relation given by equation (E1). Hence,

$$c_j = \frac{\lfloor Q \rfloor \left[\tilde{I}\right] \left[m \right] \left[\varphi(j) \right] \{\Phi\}}{\lfloor Q \rfloor \left[\tilde{I}\right] \left[m \right] \left[\varphi(j) \right] \{\varphi(j)\}} \qquad \text{(E3)}$$

If the first (s - 1) modes are assumed to be known, the sth mode is obtained by first rewriting equation (E2) as

$$\{\Phi\} - \{\varphi(1)\} c_1 - \{\varphi(2)\} c_2 - \; . \; . \; . \; . - \{\varphi(s - 1)\} c_{s-1}$$

$$= c_s \{\varphi(s)\} + c_{s+1} \{\varphi(s + 1)\} + \; . \; . \; . + c_{2n+2} \{\varphi(2n + 2)\} \qquad \text{(E4)}$$

Upon substituting equation (E3) for $c_1, c_2, \; . \; . \; ., c_{s-1}$, equation (E4) becomes

$$\left[\Psi(s) \right] \{\Phi\} = c_s \{\varphi(s)\} + c_{s+1} \{\varphi(s + 1)\} + \; . \; . \; . + c_{2n+2} \{\varphi(2n + 2)\} \qquad \text{(E5)}$$

where

$$\left[\Psi(s)\right] = \left\lceil 1 \right\rfloor - \frac{\left\{\varphi(1)\right\}\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(1) \right\rfloor}{\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(1) \right\rfloor\left\{\varphi(1)\right\}} - \frac{\left\{\varphi(2)\right\}\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(2) \right\rfloor}{\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(s) \right\rfloor\left\{\varphi(2)\right\}} - \cdots$$

$$- \frac{\left\{\varphi(s-1)\right\}\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(s-1) \right\rfloor}{\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(s-1) \right\rfloor\left\{\varphi(s-1)\right\}} \tag{E6a}$$

The square matrix $\left[\Psi(s)\right]$ is the desired "sweeping" matrix. For computational purposes, it is expressed conveniently by

$$\left[\Psi(s)\right] = \left[\Psi(s-1)\right] - \frac{\left\{\varphi(s-1)\right\}\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(s-1) \right\rfloor}{\left\lfloor Q \right\rfloor\left[\tilde{I}\right]\left\lceil m \right\rfloor\left\lceil \varphi(s-1) \right\rfloor\left\{\varphi(s-1)\right\}} \tag{E6b}$$

where $\left[\Psi(1)\right]$ is the identity matrix.

Premultiplying equation (E5) by $\left[N\right]$ and substituting equation (46a) into the terms on the right-hand side of the resulting equation yields

$$\begin{aligned}\left[N\right]\left[\Psi(s)\right]\left\{\Phi\right\} &= c_s\lambda(s)\left\{\varphi(s)\right\} + c_{s+1}\lambda(s+1)\left\{\varphi(s+1)\right\} \\ &\quad + \cdots + c_{2n+2}\lambda(2n+2)\left\{\varphi(2n+2)\right\}\end{aligned} \tag{E7}$$

Defining $\left[N(s)\right]$ as

$$\left[N(s)\right] = \left[N\right]\left[\Psi(s)\right] \tag{E8}$$

and premultiplying equation (E7) by $\left[N(s)\right]$ gives

$$\begin{aligned}\left[N(s)\right]^2\left\{\Phi\right\} &= c_s\lambda(s)\left[N\right]\left[\Psi(s)\right]\left\{\varphi(s)\right\} + c_{s+1}\lambda(s+1)\left[N\right]\left[\Psi(s)\right]\left\{\varphi(s+1)\right\} \\ &\quad + \cdots + c_{2n+2}\lambda(2n+2)\left[N\right]\left[\Psi(s)\right]\left\{\varphi(2n+2)\right\}\end{aligned} \tag{E9}$$

When the orthogonality relationship of equation (E1) is applied and equation (46a) is substituted into the resulting terms on the right-hand side, equation (E9) becomes

$$\left[N(s)\right]^2\left\{\Phi\right\} = c_s\lambda^2(s)\left\{\varphi(s)\right\} + c_{s+1}\lambda^2(s + 1)\left\{\varphi(s + 1)\right\} + \ldots + c_{2n+2}\lambda^2(2n + 2)\left\{\varphi(2n + 2)\right\} \tag{E10}$$

The repeated premultiplication of equation (E7) by $\left[N(s)\right]$ gives

$$\left[N(s)\right]^k\left\{\Phi\right\} = c_s\lambda^k(s)\left\{\varphi(s)\right\} + c_{s+1}\lambda^k(s + 1)\left\{\varphi(s + 1)\right\} + \ldots + c_{2n+2}\lambda^k(2n + 2)\left\{\varphi(2n + 2)\right\} \tag{E11}$$

For k sufficiently large, equation (E11) reduces to

$$\left[N(s)\right]^k\left\{\Phi\right\} \approx c_s\lambda^k(s)\left\{\varphi(s)\right\} \tag{E12}$$

since $\lambda(s) > \lambda(s + 1) > \ldots > \lambda(2n + 2)$. It is seen that the product obtained by repeated premultiplication of an arbitrary vector by $\left[N(s)\right]$ converges, as k increases, on the normalized eigenvector $\left\{\varphi(s)\right\}$ times a scalar. Hence, the normalized vector of $\left[N(s)\right]^k\left\{\Phi\right\}$ is an eigenvector of the problem since it has been shown to equal $\left\{\varphi(s)\right\}$, provided that k is sufficiently large. By applying the orthogonality relationship and using equation (46a), the premultiplication by $\left[N(s)\right]$ of the normalized eigenvector $\left\{\varphi(s)\right\}$ obtained by the procedure just described yields

$$\left[N(s)\right]\left\{\varphi(s)\right\} = \lambda(s)\left\{\varphi(s)\right\} \tag{E13}$$

and thus, the corresponding eigenvalue is obtained.

After each premultiplication of an arbitrary vector by $\left[N(s)\right]$, the resulting vector may be normalized without affecting the convergence of the process. In the computational solution of the problem, the normalization of the vector after each premultiplication is an expedient method in that it permits a check on the convergence after each premultiplication. From equation (E13) it is known that a solution is obtained when the premultiplication of the normalized vector by $\left[N(s)\right]$ yields the same normalized vector within certain specified computational accuracy limits.

The iteration procedure described for obtaining the sth mode may be used to determine the natural vibration characteristics of each mode. Thus, the solutions of the eigenvalue problem are given by the iteration of equation (E13) since it is proved that such iteration results in convergence to the sth mode.

APPENDIX F

VIBRATIONS OF A NONROTATING CANTILEVERED BEAM

The differential equation of motion for a nonrotating beam vibrating freely in one plane is

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) - \omega^2 mw = 0 \tag{F1}$$

If the beam is assumed to be cantilevered, the boundary conditions are

$$w\Big|_{x=x_0} = 0 \tag{F2a}$$

$$\frac{dw}{dx}\Big|_{x=x_0} = 0 \tag{F2b}$$

$$\left(EI\,\frac{d^2w}{dx^2}\right)\Bigg|_{x=x_n} = 0 \tag{F2c}$$

$$\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right)\Bigg|_{x=x_n} = 0 \tag{F2d}$$

The differential equation may be written for each of a chosen set of equally spaced stations defined by $x_i = x_0 + ih$ where h is the spacing interval and $i = 0, 1, 2, \ldots, n$. Thus equation (F1) determines a set of $(n + 1)$ equations which may be combined into a matrix equation as

$$\frac{d^2}{dx^2}\left\{\left\lceil \bar{S} \right\rfloor \frac{d^2}{dx^2}\{w\}\right\} - \omega^2\left\lceil \overline{m} \right\rfloor\{w\} = \{0\} \tag{F3a}$$

where

$$\{w\} = \begin{Bmatrix} w_0 \\ w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{Bmatrix} \tag{F3b}$$

$$\left\lceil \bar{S} \right\rfloor = \begin{bmatrix} (EI)_0 & & & \\ & (EI)_1 & & \\ & & \ddots & \\ & & & (EI)_n \end{bmatrix} \tag{F3c}$$

$$\left\lceil \bar{m} \right\rfloor = \begin{bmatrix} m_0 & & & \\ & m_1 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix} \tag{F3d}$$

Equation (F3a) may now be integrated four times by using the integrating matrix as an operator. However, after the second integration it is necessary to premultiply the equation by the inverse of the stiffness matrix. The result after each of the successive integrations is given in the following equations:

$$\frac{d}{dx}\left\{\left\lceil \bar{S} \right\rfloor \frac{d^2}{dx^2}\{w\}\right\} - \omega^2 [I] \left\lceil \bar{m} \right\rfloor \{w\} + k_1\{1\} = \{0\} \tag{F4}$$

$$\left\lceil \bar{S} \right\rfloor \frac{d^2}{dx^2}\{w\} - \omega^2 [I]^2 \left\lceil \bar{m} \right\rfloor \{w\} + k_1 [I]\{1\} + k_2\{1\} = \{0\} \tag{F5}$$

$$\frac{d}{dx}\{w\} - \omega^2 [I] \left\lceil \bar{S} \right\rfloor^{-1} [I]^2 \left\lceil \bar{m} \right\rfloor \{w\} + k_1 [I] \left\lceil \bar{S} \right\rfloor^{-1} [I]\{1\}$$

$$+ k_2 [I] \left\lceil \bar{S} \right\rfloor^{-1} \{1\} + k_3\{1\} = \{0\} \tag{F6}$$

$$\{w\} - \omega^2 [I]^2 \left\lceil \bar{S} \right\rfloor^{-1} [I]^2 \left\lceil \bar{m} \right\rfloor \{w\} + k_1 [I]^2 \left\lceil \bar{S} \right\rfloor^{-1} [I]\{1\}$$

$$+ k_2 [I]^2 \left\lceil \bar{S} \right\rfloor^{-1} \{1\} + k_3 [I]\{1\} + k_4\{1\} = \{0\} \tag{F7}$$

where k_i are constants of integration. Also, $\{1\}$ and $\{0\}$ are (n + 1) element column matrices with all elements 1 and 0, respectively. The inverse of the stiffness matrix is given by

$$\lceil \bar{S} \rfloor^{-1} = \begin{bmatrix} \left(\frac{1}{EI}\right)_0 & & & \\ & \left(\frac{1}{EI}\right)_1 & & \\ & & \ddots & \\ & & & \left(\frac{1}{EI}\right)_n \end{bmatrix} \tag{F8}$$

For the purpose of evaluating the constants of integration k_i, the following matrix operators which contain (n + 1) elements are defined.

$$\lfloor B_0 \rfloor = \lfloor 1 \quad 0 \quad 0 \quad . \; . \; . \quad 0 \rfloor \tag{F9a}$$

$$\lfloor B_n \rfloor = \lfloor 0 \quad . \; . \; . \quad 0 \quad 0 \quad 1 \rfloor \tag{F9b}$$

Multiplying equation (F4) by $\lfloor B_n \rfloor$ and applying the boundary condition given by equation (F2d) yields

$$k_1 = \omega^2 \lfloor B_n \rfloor \left[I \right] \lceil \bar{m} \rfloor \{w\} \tag{F10}$$

Multiplying equation (F5) by $\lfloor B_n \rfloor$, applying the boundary condition of equation (F2c), and substituting equation (F10) gives k_2 as

$$k_2 = \omega^2 \lfloor B_n \rfloor \left[I \right]^2 \lceil \bar{m} \rfloor \{w\} - \omega^2 \lfloor B_n \rfloor \left[I \right] \{1\} \lfloor B_n \rfloor \left[I \right] \lceil \bar{m} \rfloor \{w\} \tag{F11}$$

By multiplying equations (F6) and (F7) by $\lfloor B_0 \rfloor$ and applying the remaining boundary conditions, it is found that

$$k_3 = k_4 = 0 \tag{F12}$$

The following eigenvalue problem results by substituting equations (F10), (F11), and (F12) into equation (F7), simplifying, and dividing by ω^2:

$$\frac{1}{\omega^2} \{w\} = \left[I \right]^2 \lceil \bar{S} \rfloor^{-1} \left[\bar{F} \right]^2 \lceil \bar{m} \rfloor \{w\} \tag{F13a}$$

where

$$\left[\bar{F} \right] = \left[\{1\} \lfloor B_n \rfloor - \lceil 1 \rfloor \right] \left[I \right] \tag{F13b}$$

REFERENCES

1. Davidson, Anthony: Calculation of "1P" Propeller Resonance and First Mode Vibration Frequency. Memo. Rep. WCNB-580-19-2, Wright Air Develop. Center, U.S. Air Force, Oct. 15, 1951.

2. Jarrett, G. W.; and Warner, P. C.: The Vibration of Rotating, Tapered-Twisted Beams. J. Appl. Mech., vol. 20, no. 3, Sept. 1953, pp. 381-389.

3. Targoff, Walter P.: The Bending Vibrations of a Twisted Rotating Beam. WADC Tech. Rep. 56-27, DDC AD 97 176, U.S. Air Force, Aug. 1956.

4. Fettis, Henry E.: Simplified Treatment of the Transverse Vibrations of Twisted Cantilever Beams. Contract AF 33(038)-616-1, Wright Air Develop. Center, U.S. Air Force, [1953].

5. Sokolnikoff, I. S.; and Redheffer, R. M.: Mathematics of Physics and Modern Engineering. McGraw-Hill Book Co., Inc., 1958, pp. 691-697.

6. Spector, Joseph: Integral Series Solution for Uncoupled Vibrations of Nonuniform Bars. Master Appl. Mech. Thesis, Univ. of Virginia, May 1952, pp. 77-80.

7. Wilkinson, J. H.: Stability of the Reduction of a Matrix to Almost Triangular and Triangular Forms by Elementary Similarity Transformations. J. Assoc. Comput. Mach., vol. 6, 1959, pp. 336-359.

8. Fox, L.: An Introduction to Numerical Linear Algebra. Oxford Univ. Press, 1965.

9. Francis, J. G. F.: The QR Transformation – A Unitary Analogue to the LR Transformation. Comput. J., vol. 4.
   Pt. 1 – Oct. 1961, pp. 265-271.
   Pt. 2 – Jan. 1962, pp. 332-345.

10. Wilkinson, J. H.: The Algebraic Eigenvalue Problem. Clarendon Press (Oxford), 1965.

11. Carpenter, James E.; and Sullivan, Edward M.: Structural and Vibrational Characteristics of WADC S-5 Model Propeller Blades. WADC Tech. Rep. 56-298, DDC AD 130 787, U.S. Air Force, June 1957.

12. Young, Dana; and Felgar, Robert P., Jr.: Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam. Publ. No. 4913, Eng. Res. Ser. No. 44, Bur. Eng. Res., Univ. of Texas, July 1, 1949.

TABLE 1.- PHYSICAL PROPERTIES OF PROPELLER BLADE

(a) U.S. Customary Units

$[x_0 = 6\text{ in.};\quad h = 2\text{ in.}]$

Station number, i	m, lb-sec^2/in^2	$EI_{y'y'}$, lb-in^2	$EI_{z'z'}$, lb-in^2	β, deg
0	1.026×10^{-3}	0.200×10^6	63×10^6	30.5
1	.696	.110	49	25.2
2	.660	.083	46	20.0
3	.608	.058	44	14.8
4	.564	.042	43	9.6
5	.535	.031	43	4.7
6	.520	.027	44	0
7	.506	.026	47	-4.2
8	.498	.025	51	-7.6
9	.498	.024	56	-10.0

(b) SI Units

$[x_0 = 0.1524\text{ m};\quad h = 0.0508\text{ m}]$

Station number, i	m, N-sec^2/m^2	$EI_{y'y'}$, N-m^2	$EI_{z'z'}$, N-m^2	β, deg
0	7.07	0.574×10^3	181×10^3	30.5
1	4.80	.316	141	25.2
2	4.55	.238	132	20.0
3	4.19	.166	126	14.8
4	3.89	.121	123	9.6
5	3.69	.089	123	4.7
6	3.59	.077	126	0
7	3.49	.075	135	-4.2
8	3.43	.072	146	-7.6
9	3.43	.069	161	-10.0

TABLE 2.- NATURAL VIBRATION FREQUENCIES OF PROPELLER BLADE

β at 0.75R, deg	Mode	Ω, rpm	ω, Hz		Percent error
			Experimental	Computed	
0	1	1567	40.08	40.77	1.7
		2609	58.73	59.85	1.9
		3583	76.52	78.08	2.0
		4486	93.07	95.03	2.1
		5884	117.50	121.24	3.2
	2	1589	107.53	109.05	1.4
		2614	137.02	139.03	1.5
		3585	170.60	171.41	.5
		4537	202.53	204.40	.9
20	1	1568	37.66	38.21	1.5
		2585	53.44	54.32	1.6
		4513	83.74	85.37	1.9
		6016	106.66	109.46	2.6
		6983	122.08	124.86	2.3
	2	1576	107.42	108.62	1.1
		2561	135.45	137.26	1.3
40	1	1506	33.10	34.30	3.6
		2530	46.95	47.65	1.5
		4503	72.28	73.98	2.4
		6007	92.93	94.01	1.2
	2	1484	107.58	106.05	-1.4
		2532	133.93	135.71	1.3
		3583	167.69	170.48	1.7
---	1	0	-----	21.85	---
---	2	0	-----	86.32	---

TABLE 3.- COMPARISON OF COMPUTED FIRST-MODE DISPLACEMENTS FOR $n = 10, 25$ AND $r = 1, 2, \ldots, 7$ WITH EXACT-SOLUTION VALUES

x/l	Exact solution	Number of intervals	Degree of integrating matrix						
			Seventh	Sixth	Fifth	Fourth	Third	Second	First
0	0	25, 10	0	0	0	0	0	0	0
0.2	0.12774	25	0.12774	0.12774	0.12774	0.12774	0.12774	0.12774	0.12754
		10	.12774	.12774	.12774	.12774	.12775	.12778	.12649
0.4	0.45977	25	0.45977	0.45977	0.45977	0.45977	0.45977	0.45978	0.45946
		10	.45977	.45977	.45977	.45977	.45977	.45986	.45782
0.6	0.92227	25	0.92227	0.92227	0.92227	0.92227	0.92227	0.92228	0.92193
		10	.92227	.92227	.92227	.92227	.92227	.92237	.92013
0.8	1.45096	25	1.45096	1.45096	1.45096	1.45096	1.45096	1.45096	1.45070
		10	1.45096	1.45096	1.45096	1.45096	1.45095	1.45099	1.44934
1.0	2.00000	25, 10	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000

TABLE 4.- COMPARISON OF COMPUTED SECOND-MODE DISPLACEMENTS FOR $n = 10, 25$ AND $r = 1, 2, \ldots, 7$ WITH EXACT-SOLUTION VALUES

x/l	Exact solution	Number of intervals	Degree of integrating matrix						
			Seventh	Sixth	Fifth	Fourth	Third	Second	First
0	0	25, 10	0	0	0	0	0	0	0
0.2	0.60211	25	0.60211	0.60211	0.60211	0.60211	0.60212	0.60267	0.60004
		10	.60211	.60210	.60211	.60255	.60262	.60926	.58872
0.4	1.36694	25	1.36694	1.36694	1.36694	1.36695	1.36694	1.36786	1.36583
		10	1.36693	1.36694	1.36699	1.36757	1.36667	1.37777	1.35990
0.6	1.17895	25	1.17895	1.17895	1.17895	1.17896	1.17895	1.17944	1.17933
		10	1.17894	1.17895	1.17901	1.17935	1.17873	1.18344	1.18158
0.8	-0.14007	25	-0.14007	-0.14007	-0.14007	-0.14007	-0.14006	-0.14010	-0.13855
		10	-.14007	-.14008	-.14004	-.13990	-.13925	-.14047	-.13003
1.0	2.00000	25, 10	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000

TABLE 5.- COMPARISON OF COMPUTED THIRD-MODE DISPLACEMENTS FOR $n = 10, 25$ AND $r = 1, 2, \ldots, 7$ WITH EXACT-SOLUTION VALUES

x/l	Exact solution	Number of intervals	Degree of integrating matrix						
			Seventh	Sixth	Fifth	Fourth	Third	Second	First
0	0	25, 10	0	0	0	0	0	0	0
0.2	1.20901	25	1.20901	1.20901	1.20901	1.20919	1.20905	1.21780	1.20348
		10	1.20819	1.21004	1.21164	1.22304	1.20864	1.32008	1.17216
0.4	1.05185	25	1.05185	1.05185	1.05185	1.05196	1.05180	1.05735	1.04898
		10	1.05142	1.05307	1.05309	1.05744	1.04461	1.11491	1.03436
0.6	-0.94753	25	-0.94753	-0.94753	-0.94753	-0.94760	-0.94748	-0.95061	-0.95048
		10	-.94717	-.94842	-.94905	-.94985	-.94014	-.97675	-.96540
0.8	-0.78975	25	-0.78975	-0.78975	-0.78975	-0.78979	-0.78981	-0.79066	-0.79543
		10	-.78907	-.79012	-.79264	-.79192	-.79079	-.78789	-.82736
1.0	2.00000	25, 10	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000

TABLE 6.- COMPARISON OF COMPUTED FOURTH-MODE DISPLACEMENTS FOR $n = 10, 25$ AND $r = 1, 2, \ldots, 7$ WITH EXACT-SOLUTION VALUES

x/l	Exact solution	Number of intervals	Degree of integrating matrix						
			Seventh	Sixth	Fifth	Fourth	Third	Second	First
0	0	25, 10	0	0	0	0	0	0	0
0.2	1.50758	25	1.50757	1.50762	1.50761	1.50914	1.50737	1.54968	1.49965
		10	1.51359	1.54398	1.52492	1.58845	1.46467	2.04705	1.45443
0.4	-0.63112	25	-0.63112	-0.63113	-0.63114	-0.63160	-0.63116	-0.64391	-0.63254
		10	-.63215	-.64456	-.64410	-.65788	-.61936	-.77565	-.63891
0.6	-0.65299	25	-0.65299	-0.65299	-0.65301	-0.65331	-0.65303	-0.66091	-0.65156
		10	-.65380	-.66362	-.66570	-.66231	-.64129	-.72731	-.64517
0.8	1.28608	25	1.28607	1.28607	1.28612	1.28641	1.28592	1.29194	1.29399
		10	1.29344	1.30838	1.30332	1.27247	1.24597	1.28186	1.33918
1.0	2.00000	25, 10	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000

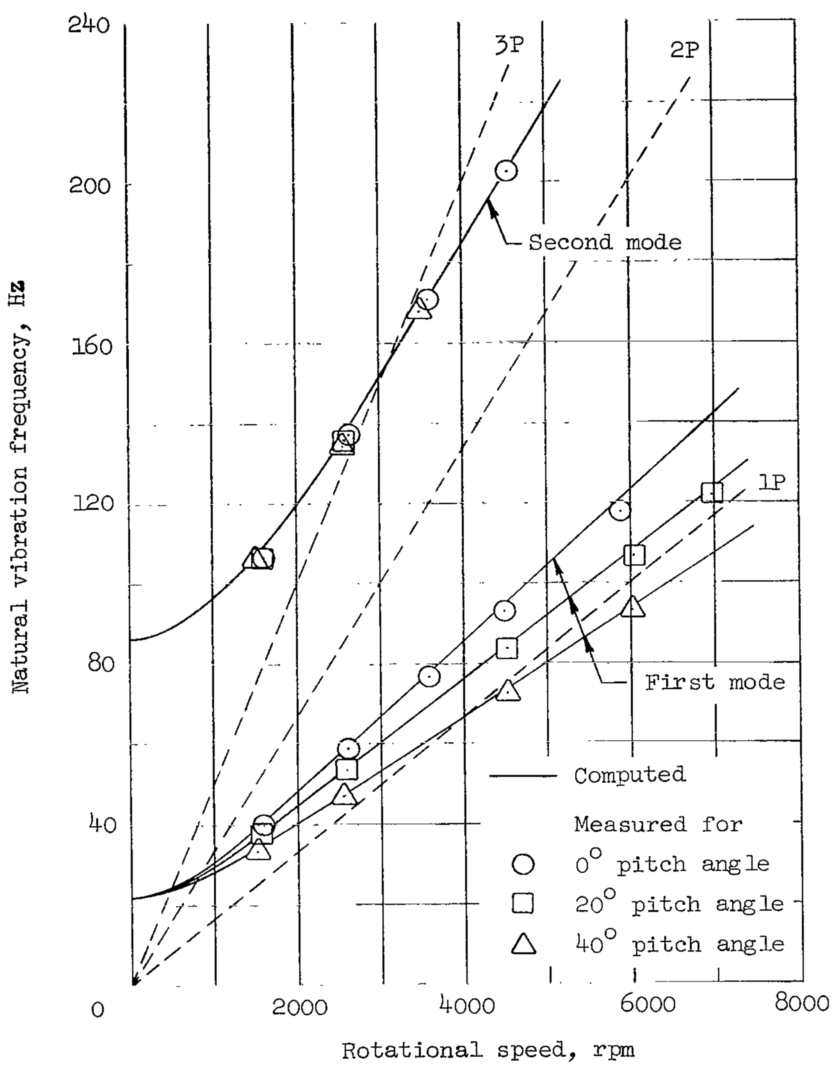
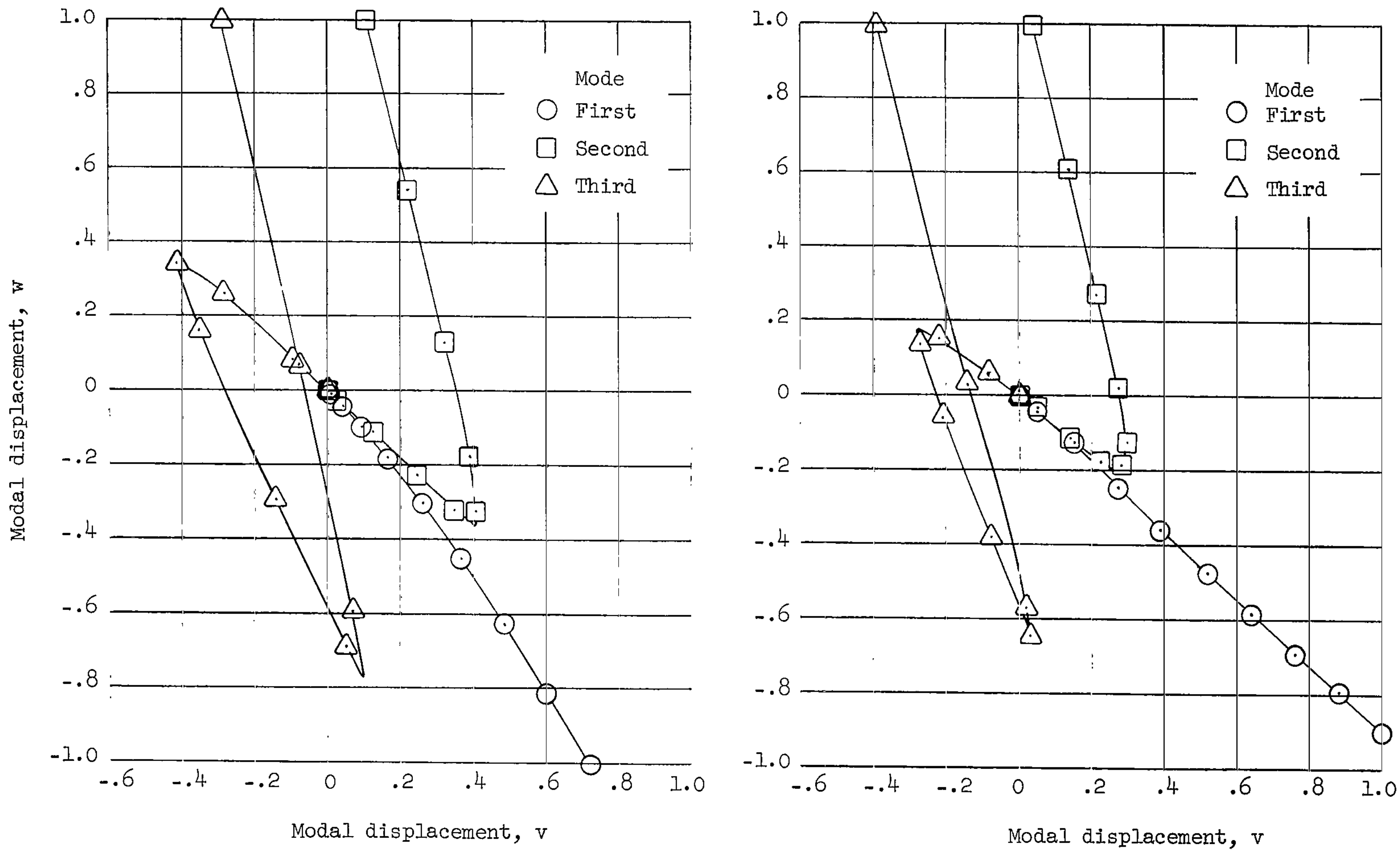


Figure 1.- Computed and measured natural vibration frequencies as a function of rotational speed.



(a) 0 rpm.

(b) 6016 rpm.

Figure 2.- Modal displacements at the equally spaced stations for the propeller blade having a pitch angle β of 20° at $x = 0.75R$.

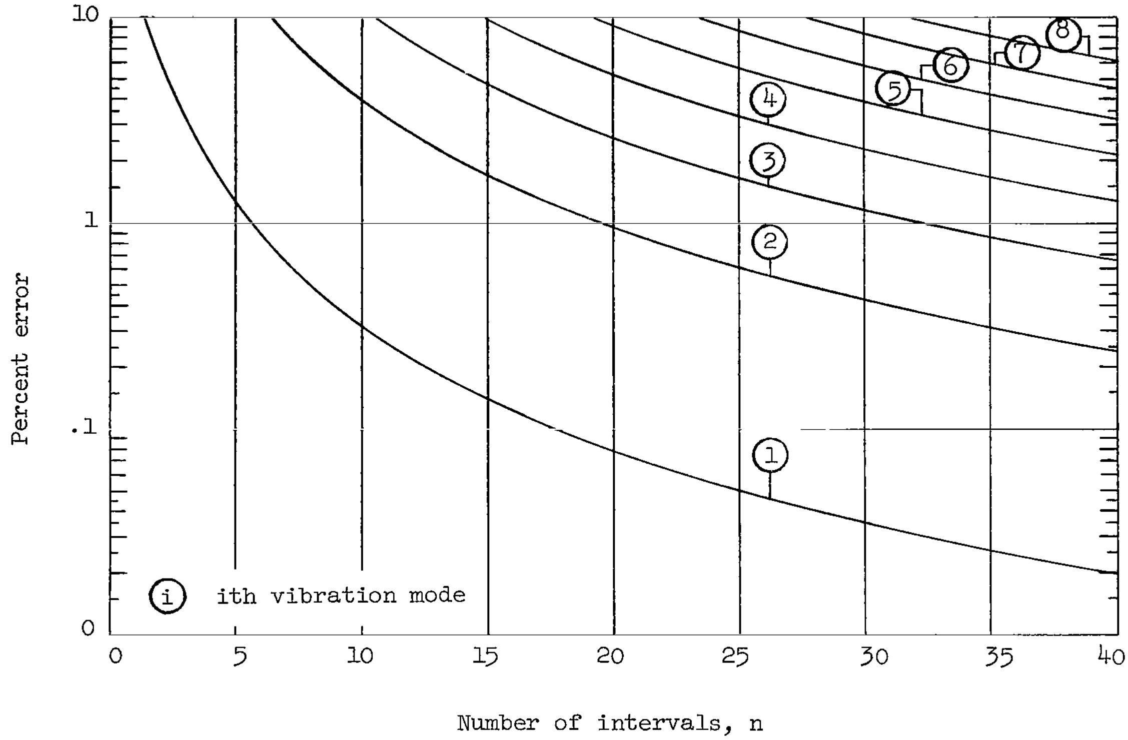


Figure 3.- Percent error of computed natural frequencies of uniform beam as a function of n for the first-degree integrating matrix.

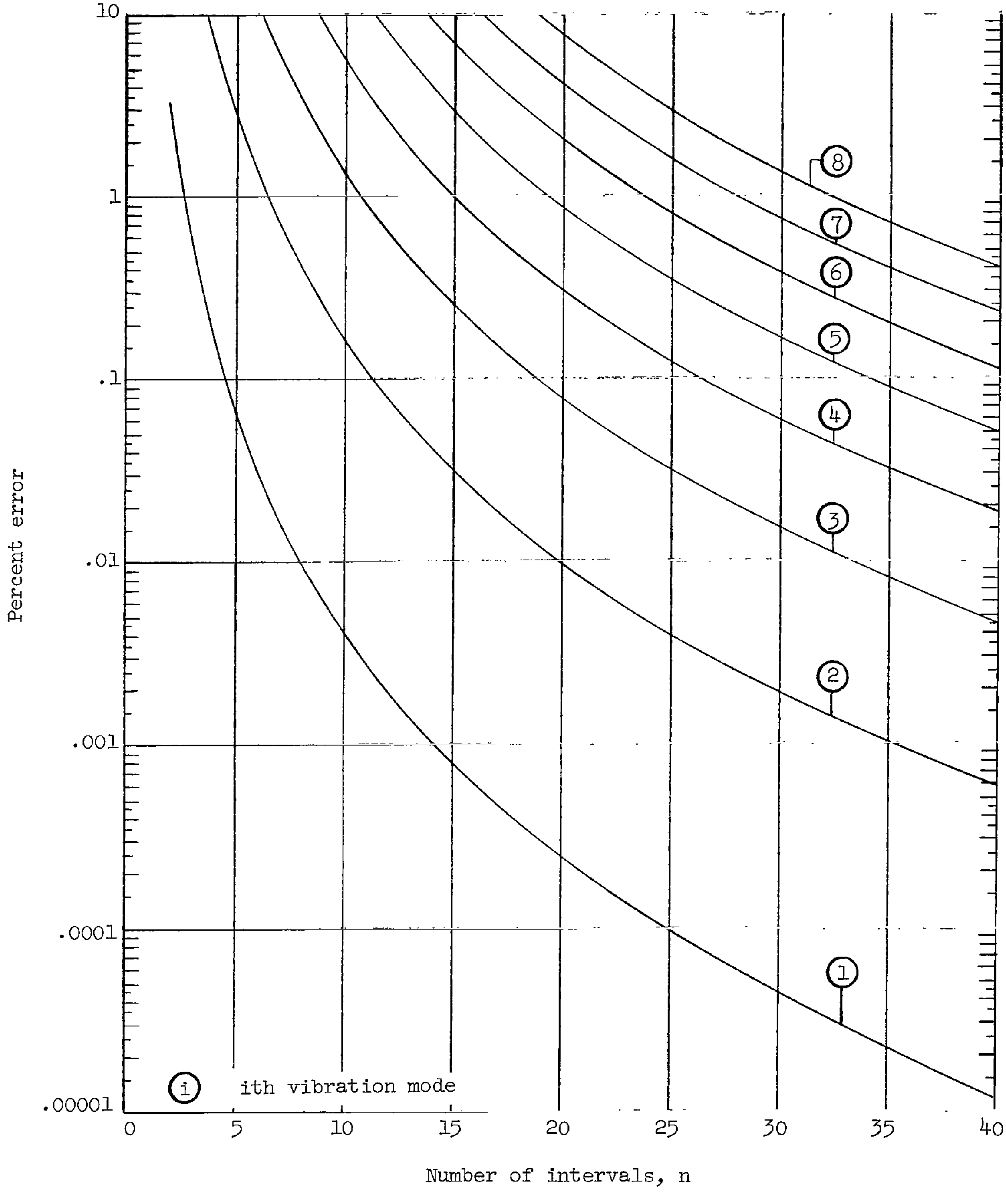


Figure 4.- Percent error of computed natural frequencies of uniform beam as a function of n for the second-degree integrating matrix.

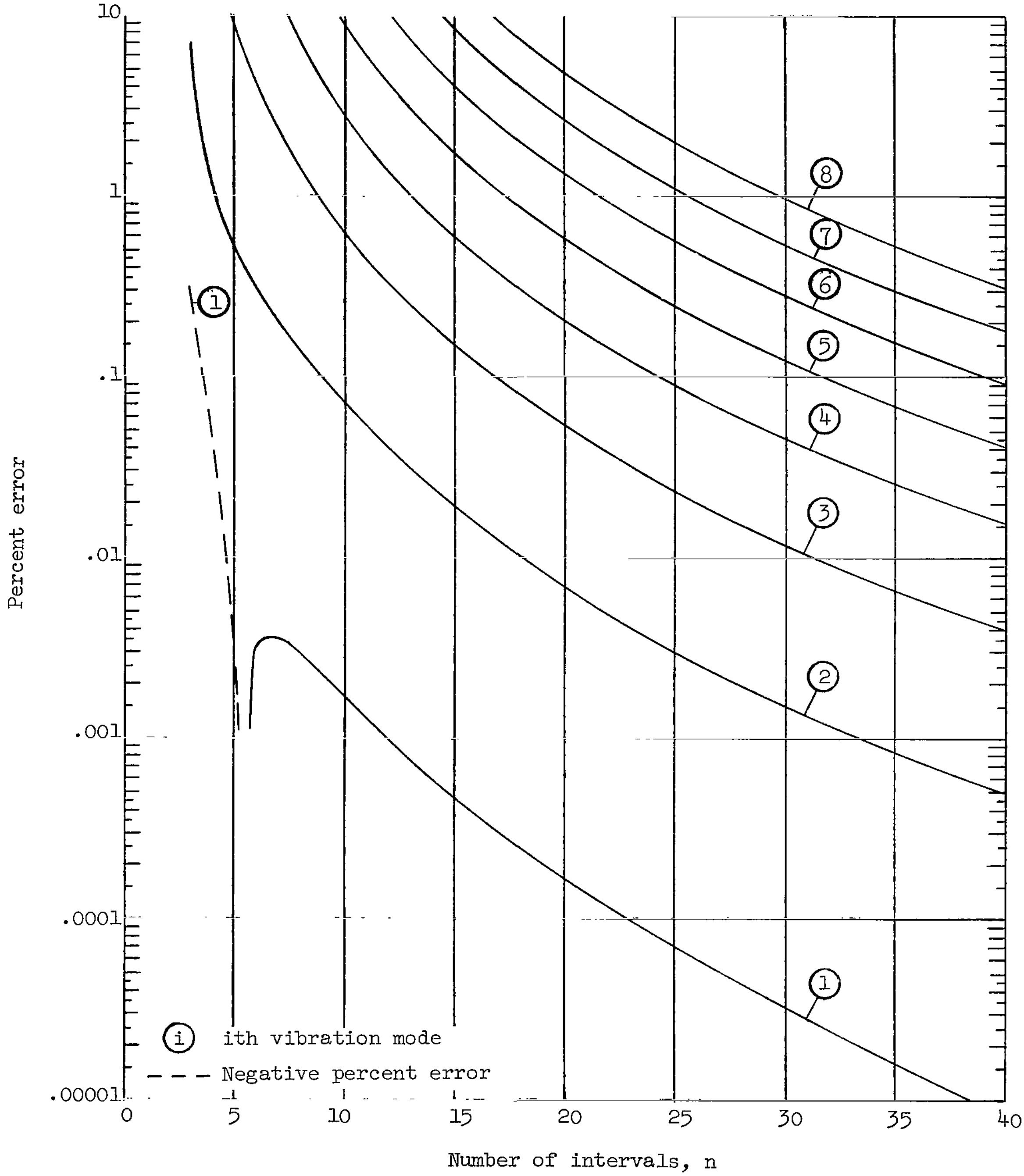


Figure 5.- Percent error of computed natural frequencies of uniform beam as a function of n for the third-degree integrating matrix.

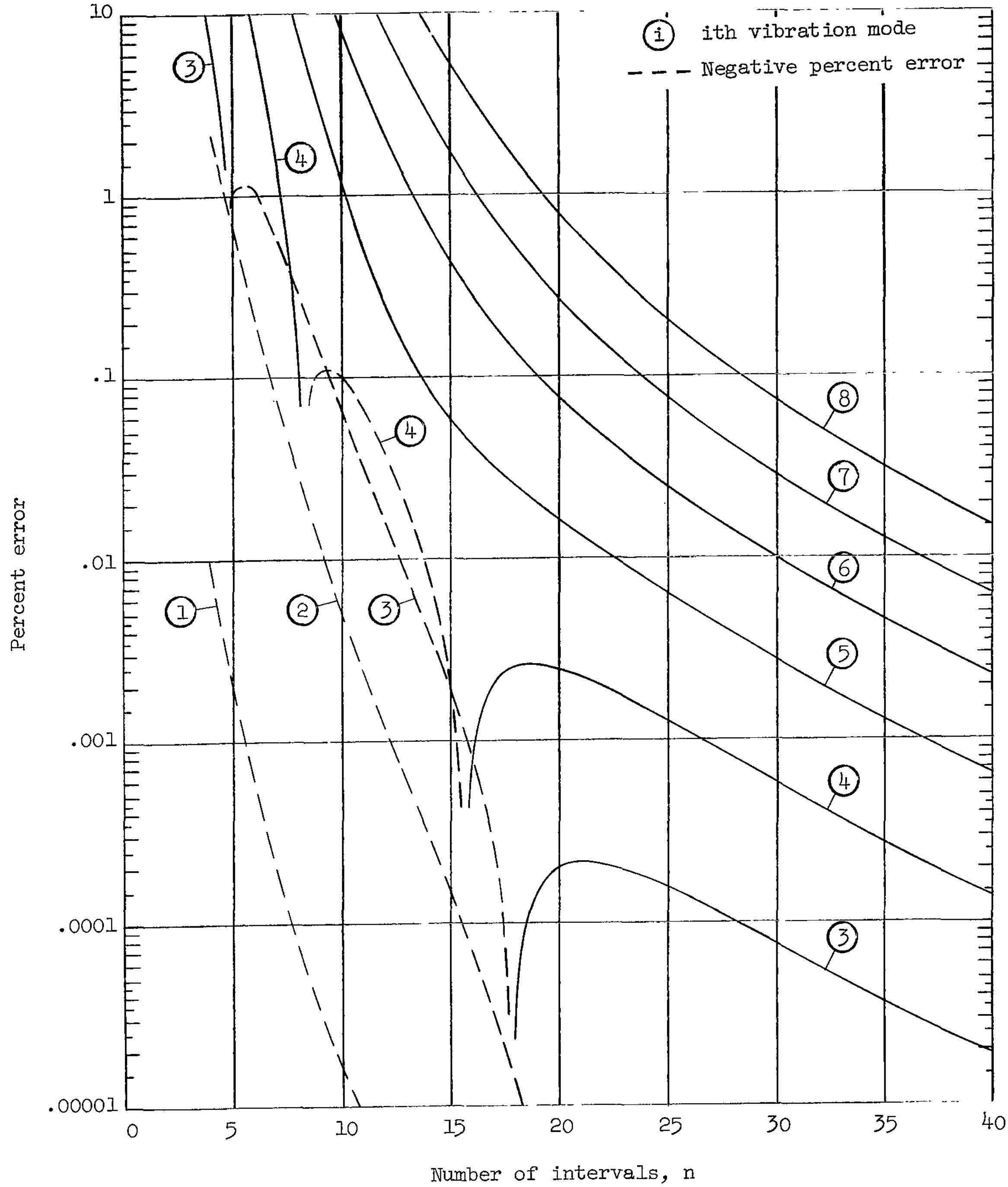


Figure 6.- Percent error of computed natural frequencies of uniform beam as a function of n for the fourth-degree integrating matrix.

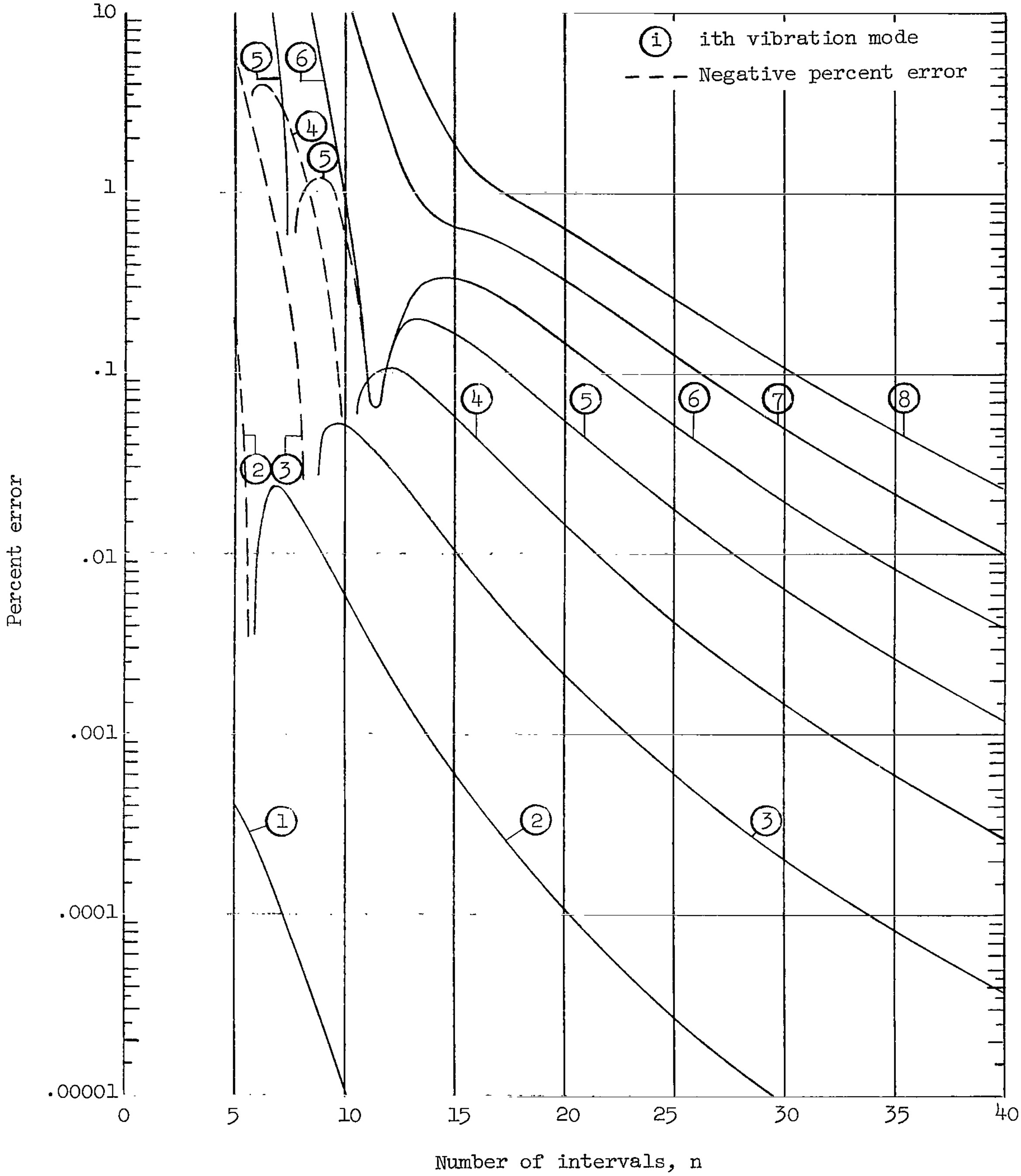


Figure 7.- Percent error of computed natural frequencies of uniform beam as a function of n for the fifth-degree integrating matrix.

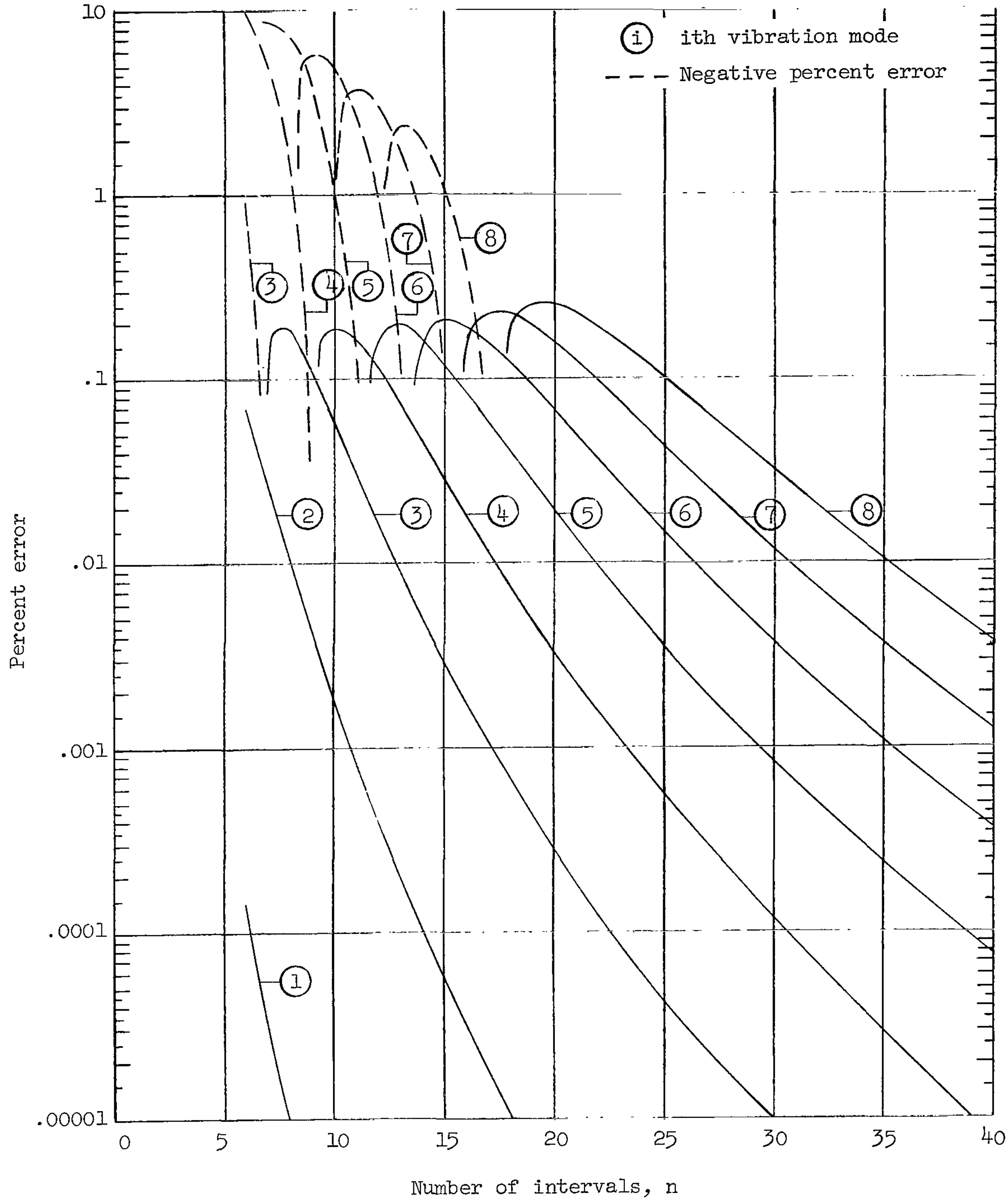


Figure 8.- Percent error of computed natural frequencies of uniform beam as a function of n for the sixth-degree integrating matrix.

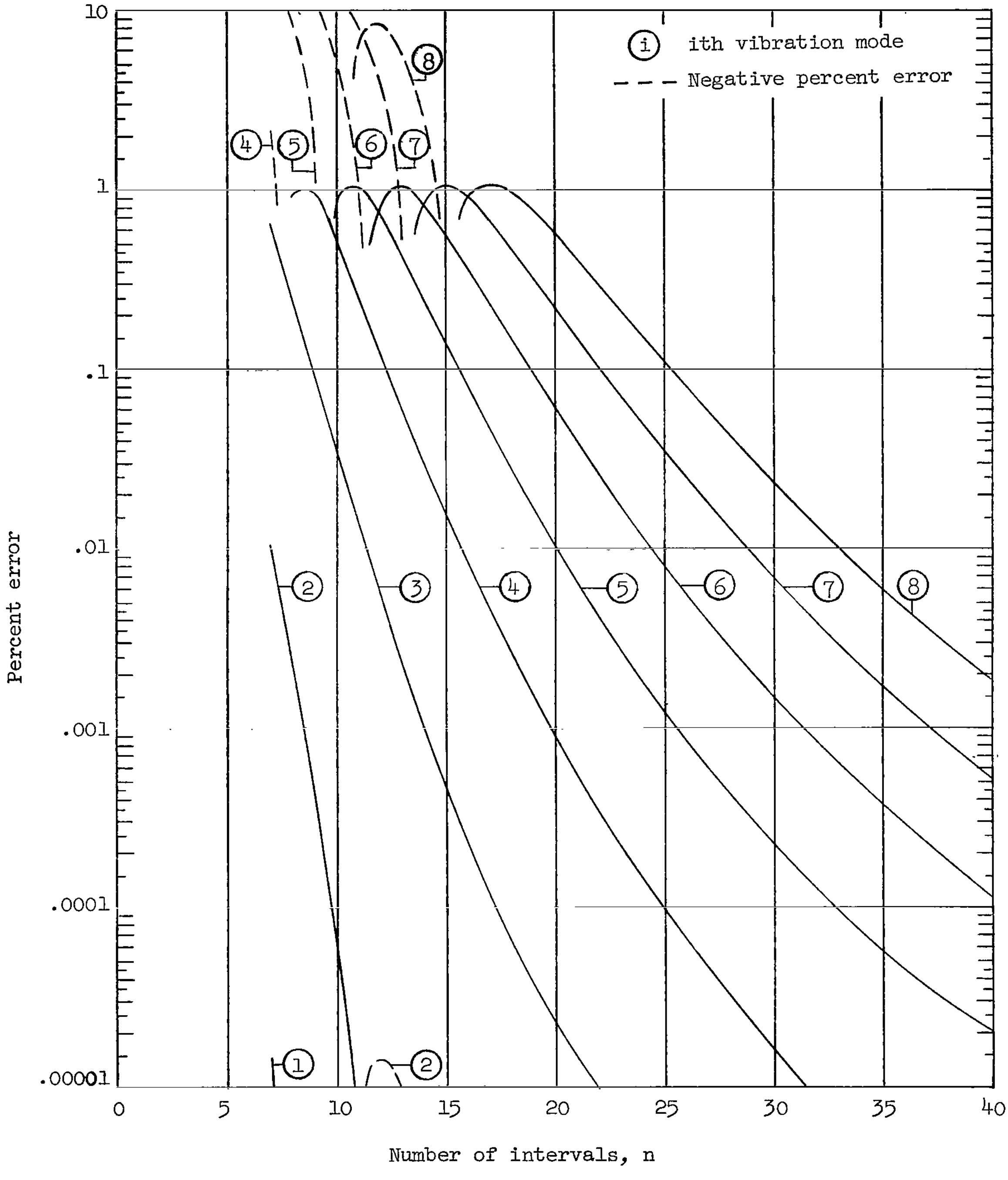


Figure 9.- Percent error of computed natural frequencies of uniform beam as a function of n for the seventh-degree integrating matrix.

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