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AN EXTENSION TO LOWER PRESSURES OF THE NAVIER-STOKES THEORY FOR THE VISCOUS FLOW OF GASES BETWEEN RELATIVELY ROTATING COAXIAL CYLINDERS

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16. Abstract

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SUMMARY

An extension of the Grad 13-moment approximation is applied to the velocity-linearized Boltzmann kinetic equation for the problem of viscous gases flowing between two relatively rotating coaxial cylinders. Precise boundary conditions are employed in deriving a velocity distribution function which is exact through quadratic terms in the Knudsen number and which thus provides reliable generalizations of previous work to lower pressures; in particular, the Navier-Stokes expression for the torque experienced by the inner cylinder is shown to suffice for Knudsen numbers extending well into the transition flow regime if the separation between the cylinders is small compared with their radii. Low-pressure effects for larger ratios of these distances differ significantly, however, from the calculations of Lin and Street based on the Burnett equations. At more moderate pressures corresponding to the retention of terms through the square of the Knudsen number in the torque and through the first power of the Knudsen number in the velocity distribution function, the simple Navier-Stokes theory is always valid and consistent in the problem described.

Results of the present research are compared also with those of Wang Chang and Uhlenbeck, who use the obsolete Maxwellian separation of reflected particles into specular and diffuse groups, and with previous formulations based on Chapman-Enskog expansions and the Grad 13-moment approximation. The more important conclusions are summarized as follows: (1) the velocity jumps at the cylinder surfaces generally follow Maxwell's original predictions rather than Wang Chang and Uhlenbeck's later modifications, (2) the basic character of the velocity profile in the midst of the gas does not change as the surfaces are approached, and (3) 13 moments are insufficient to describe low-pressure systems.

INTRODUCTION

The problem of viscous gases flowing between two relatively rotating coaxial cylinders has received much attention both on its own merit and because such systems are especially suited for the measurement of velocity slip at solid surfaces. (See ref. 1, pp. 291-300, for the earliest work.) Despite the interest, however, several important questions remain unanswered concerning the precise kinetic behavior of a gas near a boundary and the range of validity of the Navier-Stokes expression (ref. 1) for the torque experienced by the stationary inner cylinder. In particular, the assumption that the velocity distribution function calculated in the midst of the gas extends uniformly to the cylinder surfaces needs to be investigated, as does the appearance of terms proportional to the square of the mean free path in macroscopic relations derived from the second Chapman-Enskog approximation (ref. 2). More recent research on similar subjects in the case of plane Couette flow is summarized in reference 3.

A prime objective of the present research is to resolve these difficulties by exactly solving the corresponding velocity-linearized Boltzmann kinetic equation for Maxwellian particles through quadratic terms in the Knudsen number and imposing the conservation of particles, momentum, and energy at the boundaries. These conservation conditions replace the obsolete Maxwellian hypothesis (refs. 4 and 5) that separates the reflected particles into specular and diffuse groups and adds contributions to the velocity distribution function which are nonanalytic in the mean free path (ref. 6). As explained in reference 3, the Maxwellian hypothesis is nonphysical because it imposes the same accommodation or absorption coefficient for every macroscopic moment of the gas. The restriction to gases of Maxwellian particles (i.e., particles which attract or repel each other with forces varying as the inverse fifth power of their separation) is not expected to alter the basic conclusions.

SYMBOLS

$\vec{c}$	particle velocity
d	intercylinder separation distance
D	drag force per unit area in limit of infinite cylinder radii
f	velocity distribution function
f_o	Maxwellian distribution function relative to local flow velocity
$f_o(r_1)$	Maxwellian distribution function relative to stationary inner cylinder
$f_o(r_2)$	Maxwellian distribution function relative to rotating outer cylinder

g, g_1	contributions to perturbation function
G	arbitrary velocity function
h_1	perturbation function relevant to gas at surface of inner cylinder
h_2	perturbation function relevant to gas at surface of outer cylinder
k	Boltzmann's constant
l	mean free path
L	torque per unit length of stationary inner cylinder
m	particle mass
n	particle number density
p	scalar gas pressure
$\overleftrightarrow{P}$	pressure tensor
$\overset{\circ}{\overleftrightarrow{P}}$	traceless pressure tensor
$\overleftrightarrow{Q}$	heat-flux tensor
r, θ, z	cylindrical coordinates (r measured from axis of coaxial cylinders); also signify vector and tensor components when used as subscripts
r_1, r_2	radii of inner and outer cylinders, respectively
$\hat{r}, \hat{\theta}$	unit vectors associated with cylindrical coordinate system
t	time
T	temperature
$\vec{u}$	particle velocity relative to frame of reference moving with local gas flow, nondimensionalized, $\left(\frac{m}{2kT}\right)^{1/2}(\vec{c} - \vec{v})$

$\overleftrightarrow{U}$ unit tensor

$\vec{v}$ local gas flow velocity

v_1 magnitude of gas flow velocity at surface of stationary inner cylinder; also called slip velocity or velocity jump

v_2 magnitude of gas flow velocity at surface of outer cylinder

v_w rotational speed of outer cylinder

$\bar{v}$ mean equlibrium particle speed

$\vec{X}$ body force per unit mass

α_1 accommodation coefficient (inner cylinder) for energy absorption

$\vec{\gamma}$ particle velocity relative to stationary inner cylinder, nondimensionalized, $\left(\frac{m}{2kT}\right)^{1/2}\vec{c}$

ζ slip distance

ζ_1,ζ_2 slip distances relative to inner and outer cylinders, respectively

η viscosity

$\vec{\mu}$ particle velocity relative to rotating outer cylinder, nondimensionalized, $\left(\frac{m}{2kT}\right)^{1/2}\left(\vec{c} - \hat{\theta}v_w\right)$

ρ mass density

σ_1,σ_2 accommodation coefficients (inner and outer cylinders, respectively) for parallel momentum absorption

σ_{N1} accommodation coefficient (inner cylinder) for perpendicular momentum absorption

σ_1' accommodation coefficient (inner cylinder) for parallel heat-flux absorption (see eq. (34))

ϕ first-order perturbation function

ω angular frequency of outer cylinder

Special notation:

$\left(\frac{\partial}{\partial t}\right)_c$ collisional time derivative

$\langle G \rangle$ velocity average of G referred to complete distribution function

$\langle G|h_1\rangle, \langle G|h_2\rangle$ velocity averages of G referred to $f_o(r_1)h_1$ and $f_o(r_2)h_2$, respectively

Vector and tensor quantities without arrows signify magnitudes or components.

GENERAL CONSIDERATIONS

The torque exerted on unit length of a stationary inner cylinder (radius r_1, infinite length) by the shearing action of a viscous gas flowing between it and a rotating outer coaxial cylinder (radius r_2, infinite length) is given by

$$L = -2\pi r_1^2 P_{r\theta}(r_1) \tag{1}$$

where $P_{r\theta}$ is the $r\theta$-component of the pressure tensor

$$\overleftrightarrow{P} = nm\left\langle\left(\vec{c} - \vec{v}\right)\left(\vec{c} - \vec{v}\right)\right\rangle \tag{2}$$

Hence, the determination of $P_{r\theta}$ from kinetic theory is the essential task and comprises the bulk of the present research.

Calculations of such quantities usually begin with the general time-independent kinetic equation (ref. 2)

$$\vec{c} \cdot \nabla f + \vec{X} \cdot \frac{\partial f}{\partial \vec{c}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{3}$$

the $m\vec{c}$-moment of which yields the macroscopic equation of motion

$$\nabla \cdot \left(\overset{\leftrightarrow}{P}^{\,o} + p\overset{\leftrightarrow}{U} + nm\vec{v}\vec{v}\right) = 0 \tag{4}$$

in the absence of body forces. The traceless pressure tensor $\overset{\leftrightarrow}{P}^{\,o}$ is defined by

$$\overset{\leftrightarrow}{P}^{\,o} = \overset{\leftrightarrow}{P} - p\overset{\leftrightarrow}{U} \tag{5}$$

If no temperature gradients are applied between the cylinders, the scalar pressure p depends on the radial distance r only through effects induced by the friction-generated heat. Since such effects are proportional to the square of the Mach number, as are the diagonal elements of $\overset{\leftrightarrow}{P}^{\,o}$ in the midst of the gas, the direct expansion of equation (4) gives

$$\nabla \cdot \overset{\leftrightarrow}{P}^{\,o} = \nabla \cdot \left[\left(\hat{r}\hat{\theta} + \hat{\theta}\hat{r}\right)P_{r\theta}(r)\right] = \hat{\theta}\left(\frac{dP_{r\theta}}{dr} + \frac{2P_{r\theta}}{r}\right) = 0 \tag{6}$$

through first-order (i.e., linear) terms in the flow velocity. A reasonable tentative assumption (to be confirmed or rejected by the solution of the kinetic equation subject to boundary conditions) is that equation (6) applies also in the immediate proximity of the cylinder surfaces.

Although equation (6) is an important differential relation, its utilization in the present problem is more indirect than direct because the integration constant $P_{r\theta}(r_1)$, rather than the variation of $P_{r\theta}$ with r, is the significant parameter in equation (1). What equation (6) does provide is a differential relation for the flow velocity $\vec{v}$ if $P_{r\theta}$ can be expressed in terms of $\vec{v}$ through the solution of the kinetic equation. The integration constants in the solution of the differential equation for $\vec{v}$ must be related to boundary conditions at the surfaces and also to the relative rotation rate of the two cylinders; hence, the procedure for writing L in equation (1) as a function of measurable quantities is outlined by this discussion.

Unfortunately, however, the difficulties inherent in cylindrical geometries preclude exact closed-form first-order velocity distribution functions of the type found for simple Couette flow between parallel plates (ref. 3); as a result, additional precautions are necessary to ensure complete first-order descriptions of phenomena relevant to cylindrical problems. Other difficulties arise from the nonlinear character of the velocity profile, the spatial variations of which are a priori unknown and must be derived according to the procedure in the preceding paragraph.

KINETIC THEORY

A first-order solution of equation (3) can be written in the convenient form

$$f = f_o(1 + \phi) = f_o\left[1 + \frac{2}{p}P_{r\theta}u_r u_\theta + g\left(r, \vec{u}\right)\right] \tag{7}$$

where g is an unknown function, f_o is the Maxwellian distribution function

$$f_o = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-u^2} \tag{8}$$

and $\vec{u}$ is a dimensionless particle velocity defined by

$$\vec{u} = \left(\frac{m}{2kT}\right)^{1/2}\left(\vec{c} - \vec{v}\right) \tag{9}$$

All thermodynamic quantities are constant through first order in the absence of applied gradients, and the flow velocity satisfies

$$\vec{v} = \hat{\theta}v(r) \tag{10}$$

in the present problem.

Since

$$\vec{c} \cdot \nabla f_o \approx 2f_o\left(\frac{dv}{dr} - \frac{v}{r}\right)u_r u_\theta \tag{11}$$

and

$$\vec{c} \cdot \nabla\left(P_{r\theta}u_r u_\theta\right) \approx -\left(\frac{2kT}{m}\right)^{1/2}\left[\left(\frac{P_{r\theta}}{r} - \frac{dP_{r\theta}}{dr}\right)u_r^{\,2} - \frac{P_{r\theta}}{r}u_\theta^{\,2}\right]u_\theta \tag{12}$$

the substitution into equation (3) of equations (7), (11), and (12) yields

$$\begin{aligned}
\vec{c} \cdot \nabla f &\approx \vec{c} \cdot \nabla f_o + f_o\vec{c} \cdot \nabla\phi \\
&\approx 2f_o\left(\frac{dv}{dr} - \frac{v}{r}\right)u_r u_\theta - \frac{6f_o P_{r\theta}}{pr}\left(\frac{2kT}{m}\right)^{1/2}\left(u_r^{\,2} - \frac{1}{3}u_\theta^{\,2}\right)u_\theta + f_o\left(\frac{2kT}{m}\right)^{1/2}\vec{u} \cdot \nabla g \\
&= \left(\frac{\partial f}{\partial t}\right)_c = -\frac{2f_o}{\eta}P_{r\theta}u_r u_\theta + \left[\frac{\partial\left(f_o g\right)}{\partial t}\right]_c
\end{aligned} \tag{13}$$

with the aid of equation (6) and the well-known properties of collision integrals for Maxwellian particles (ref. 2). The viscosity η correctly represents the detailed collision dynamics.

Equation (13) is called the velocity-linearized Boltzmann equation because quadratic and higher powers of v are neglected in its derivation. The complexities introduced by the cylindrical geometry are obvious: As r approaches infinity, corresponding to flat plates, the exact first-order solution is equation (7) with g set to zero. In the present problem, however, g is more complicated and must contain, at the very least, some seldom-used elements of the general heat-flux tensor

$$\vec{\vec{Q}} = p\left(\frac{2kT}{m}\right)^{1/2}\langle\vec{u}\vec{u}\vec{u}\rangle \qquad (14)$$

More specifically, g can be expressed as

$$g = \frac{4Q_{rr\theta}}{p}\left(\frac{m}{2kT}\right)^{1/2}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)u_\theta + g_1(r,\vec{u}) \qquad (15)$$

in order to balance the $\left(u_r^2 - \frac{1}{3}u_\theta^2\right)u_\theta$ term in equation (13).

The insertion of equation (15) into the collision integral of equation (13) gives the expression

$$\left(\frac{dv}{dr} - \frac{v}{r}\right)u_r u_\theta - \frac{3P_{r\theta}}{pr}\left(\frac{2kT}{m}\right)^{1/2}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)u_\theta + \frac{1}{2}\left(\frac{2kT}{m}\right)^{1/2}\vec{u}\cdot\nabla g$$

$$= -\frac{1}{\eta}\left[P_{r\theta}u_r + 3Q_{rr\theta}\left(\frac{m}{2kT}\right)^{1/2}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)\right]u_\theta + \frac{1}{2f_o}\left[\frac{\partial(f_o g_1)}{\partial t}\right]_c \qquad (16)$$

Accordingly, the $P_{r\theta}$ term in ϕ is proportional to

$$\frac{l}{\bar{v}}\left(\frac{dv}{dr} - \frac{v}{r}\right)$$

and the $Q_{rr\theta}$ term is proportional to

$$\frac{l^2}{r\bar{v}}\left(\frac{dv}{dr} - \frac{v}{r}\right)$$

if the mean free path l and the mean equilibrium particle speed $\bar{v}$ are defined by

$$l = \frac{2\eta}{\rho \bar{v}} \tag{17}$$

and

$$\bar{v} = 2\left(\frac{2kT}{\pi m}\right)^{1/2} \tag{18}$$

This analysis permits a definite statement about the physical consequences of dropping the $\vec{u} \cdot \nabla g$ contribution to equation (16) and the collisional time derivative of $f_o g_1$ to obtain

$$\begin{aligned}\left(\frac{dv}{dr} - \frac{v}{r}\right)u_r u_\theta - \frac{3P_{r\theta}}{pr}\left(\frac{2kT}{m}\right)^{1/2}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)u_\theta \\ \approx -\frac{1}{\eta}\left[P_{r\theta}u_r + 3Q_{rr\theta}\left(\frac{m}{2kT}\right)^{1/2}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)\right]u_\theta\end{aligned} \tag{19}$$

The expansion is twofold in the sense that terms proportional to

$$\frac{l}{r_2 - r_1}\left(\frac{v_w}{\bar{v}}\right)^2 \qquad \text{or} \qquad \frac{l^3}{r_1^2(r_2 - r_1)}\frac{v_w}{\bar{v}}$$

are neglected, as are higher powers of l and the Mach number $v_w/\bar{v}$ of the outer cylinder at $r = r_2$. (Note that this definition of the Mach number differs by a constant factor from the one used in aerodynamics.) From this point forward the inner cylinder is considered to be stationary.

Equations (7), (15), and (17) to (19) combine to give the following first-order results, which are complete through terms containing the square of the mean free path:

$$P_{r\theta} = -\eta\left(\frac{dv}{dr} - \frac{v}{r}\right) = -\frac{4pl}{\pi\bar{v}}\left(\frac{dv}{dr} - \frac{v}{r}\right) \tag{20}$$

$$Q_{rr\theta} = \frac{2\eta}{\rho r}P_{r\theta} = \frac{2l}{r}\left(\frac{2kT}{\pi m}\right)^{1/2}P_{r\theta} \tag{21}$$

and

$$\phi = \frac{2P_{r\theta}}{p}\left[u_r + \frac{4l}{\pi^{1/2}r}\left(u_r^2 - \frac{1}{3}u_\theta^2\right)\right]u_\theta \tag{22}$$

These expressions, together with the differential relation

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0 \tag{23}$$

obtained from equations (6) and (20), complete the kinetic-theory description of the gas between two relatively rotating coaxial cylinders. They extend the Navier-Stokes representation to larger Knudsen numbers through the appearance of l^2 terms in the velocity distribution function; in addition, they are derived from a kinetic equation incorporating the divergence of the traceless pressure tensor on the left-hand side, which is not normal procedure in the second Chapman-Enskog approximation (ref. 2).

In line with the discussion following equation (6), the assumption is now made that equations (20) to (23) are valid in all regions occupied by the gas, including the close proximity to the cylinder surfaces. The only physical criterion for this assumption is whether the boundary conditions can be satisfied: If so, the description is accurate through first order in the flow velocity and through squares of the Knudsen number; if not, equation (22) is inadequate and higher moments must be considered.

BOUNDARY CONDITIONS

A convenient formulation of equations (7) and (22) for the application of boundary conditions at the surface $\left(r = r_1\right)$ of the stationary inner cylinder is obtained from the following manipulation of equation (8):

$$\begin{aligned} f_o &= n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}\left(c^2 - 2vc_\theta + v^2\right)\right] \\ &\approx n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\gamma^2}\left[1 + 2\left(\frac{m}{2kT}\right)^{1/2} v\gamma_\theta\right] \equiv f_o\left(r_1\right)\left(1 + \frac{4v}{\pi^{1/2}\bar{v}}\gamma_\theta\right) \end{aligned} \tag{24}$$

where

$$\vec{\gamma} = \left(\frac{m}{2kT}\right)^{1/2}\vec{c} \tag{25}$$

Accordingly, the first-order velocity distribution function becomes

$$f = f_o\left(r_1\right)\left(1 + h_1\right) \tag{26}$$

with

$$h_1 = \frac{4v}{\pi^{1/2}\bar{v}} \gamma_\theta + \frac{2P_{r\theta}}{p}\left[\gamma_r + \frac{4l}{\pi^{1/2}r}\left(\gamma_r{}^2 - \frac{1}{3}\gamma_\theta{}^2\right)\right]\gamma_\theta \tag{27}$$

The introduction of h_1 permits the conservation laws for particles, the r-, θ-, and z-components of momentum, and energy to be expressed at $r = r_1$ by the relations

$$0 = \left\langle \gamma_r \middle| h_1 \right\rangle \tag{28}$$

$$\sigma_{N1}\left\langle \gamma_r{}^2 \middle| h_1 \right\rangle_{\gamma_r<0} = \left\langle \gamma_r{}^2 \middle| h_1 \right\rangle_{\gamma_r<0} - \left\langle \gamma_r{}^2 \middle| h_1 \right\rangle_{\gamma_r>0} \tag{29}$$

$$\sigma_1\left\langle \gamma_r\gamma_\theta \middle| h_1 \right\rangle_{\gamma_r<0} = \left\langle \gamma_r\gamma_\theta \middle| h_1 \right\rangle \tag{30}$$

$$\sigma_1\left\langle \gamma_r\gamma_z \middle| h_1 \right\rangle_{\gamma_r<0} = \left\langle \gamma_r\gamma_z \middle| h_1 \right\rangle \tag{31}$$

and

$$\alpha_1\left\langle \gamma^2\gamma_r \middle| h_1 \right\rangle_{\gamma_r<0} = \left\langle \gamma^2\gamma_r \middle| h_1 \right\rangle \tag{32}$$

respectively, where 0, σ_{N1}, σ_1, σ_1, and α_1 are accommodation coefficients and the "bra-ket" notation refers to velocity averages symbolized by the definition

$$\left\langle G\left(r_1,\vec{\gamma}\right) \middle| h_1 \right\rangle \equiv \frac{1}{n}\int f_o\left(r_1\right) h_1 G\left(r_1,\vec{\gamma}\right) d\vec{c} \tag{33}$$

Each of equations (28) to (32) is interpreted in the same manner, the left-hand sides representing the average amounts of the corresponding macroscopic properties absorbed by unit surface area in 1 second and the right-hand sides representing the net fluxes toward the surface of the same properties. As explained in reference 3, these conditions replace the obsolete Maxwellian hypothesis that the reflected particles can be separated into diffuse and specular groups relating to a single accommodation coefficient for all moments. The coefficients are generally different for different moments.

Also mentioned in reference 3 is the fact that additional steady-state conservation laws are superfluous in the sense that they do not constrain the distribution function beyond what is implicit in equations (28) to (32). For example, the result of imposing the heat-flux condition

$$\sigma_1' \left\langle \gamma^2 \gamma_r \gamma_\theta \middle| h_1 \right\rangle_{\gamma_r<0} = \left\langle \gamma^2 \gamma_r \gamma_\theta \middle| h_1 \right\rangle \tag{34}$$

on equation (27) is merely to write σ_1' in terms of σ_1, which does not affect the solution of the kinetic equation in any way because equations (28) to (32) completely define (on the average) the gas-surface interactions. Yet a restriction similar to equation (34), but with σ_1' incorrectly equated to σ_1 according to Maxwell's hypothesis, was imposed by Wang Chang and Uhlenbeck (ref. 6) and led to complicated nonanalytic (in the mean free path) contributions to both the distribution function and the calculated macroscopic properties. Such contributions are not implied by the more general and more correct boundary conditions employed in the present research.

Of the five constraints listed in equations (28) to (32), only equation (30) is satisfied nontrivially by the perturbation function in equation (27). This condition provides a relation between the flow velocity v_1 at $r = r_1$ and the accommodation coefficient σ_1 according to the following development:

$$\left\langle \gamma_r \gamma_\theta \middle| h_1 \right\rangle_{\gamma_r<0} = -\frac{1}{\pi}\left[\frac{v_1}{\bar{v}} - \frac{\pi P_{r\theta}(r_1)}{4p}\left(1 - \frac{4l}{\pi r_1}\right)\right] \tag{35}$$

$$\left\langle \gamma_r \gamma_\theta \middle| h_1 \right\rangle = \frac{P_{r\theta}(r_1)}{2p} \tag{36}$$

and

$$\frac{v_1}{\bar{v}} = -\frac{\pi \zeta_1 P_{r\theta}(r_1)}{4pl}\left(1 + \frac{4l^2}{\pi \zeta_1 r_1}\right) \tag{37}$$

where ζ_1 satisfies

$$\zeta_1 = \frac{(2 - \sigma_1)l}{\sigma_1} \tag{38}$$

Note that to the slip velocity v_1 in equation (37) is added a new contribution (the second term) which originates from the heat-flux tensor element $Q_{rr\theta}$ (instead of the vector heat flux which vanishes according to the $u^2\vec{u}$-moment of eq. (22)) and thus is not predicted by the Grad 13-moment velocity distribution function. Because of the explicit cancellation of the slip distance ζ_1 and except for the implicit dependence of $P_{r\theta}(r_1)$ on the same parameter, this term enhances the slip velocity independently of the value of the accommodation coefficient σ_1; in particular, the slip velocity is nonvanishing even when the average momentum component parallel to the cylinder surface is exactly reversed $(\sigma_1 = 2)$ by the reflection. Although the precise physical explanation of this phenomenon is not clear, it can be called a geometric effect with some justification because of the r_1 in the denominator. As r_1 approaches infinity, corresponding to a transition from cylindrical to rectilinear gas flow, the effect becomes negligibly small.

The only previous derivations of the geometric effect, which was not singled out or discussed as such, were those of Schamberg (ref. 7) and Lin and Street (ref. 8) on the basis of the third Chapman-Enskog approximation corresponding to the Burnett equations. Both references give slip velocities which differ from equation (37) by the substitution of the factor 10/3 for the factor 4 in the second term within the parentheses. Of more importance, however, is the fact that references 7 and 8 predict the form of the first-order expression for the velocity slip to be independent of the interparticle interaction potential; hence, except for the numerical value of the viscosity coefficient, which is somewhat irrelevant to the present research, strong support is provided for the assertion that the formal results of this paper are not restricted by the limitation of the detailed development to Maxwellian molecules.

A similar study relating to the surface $(r = r_2)$ of the outer cylinder begins with the first-order distribution function given by the expressions

$$f = f_o(r_2)(1 + h_2) \tag{39}$$

$$f_o(r_2) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\mu^2} \tag{40}$$

and

$$h_2 = \frac{4(v - v_w)}{\pi^{1/2}\bar{v}}\mu_\theta + \frac{2P_{r\theta}}{p}\left[\mu_r + \frac{4l}{\pi^{1/2}r}\left(\mu_r^2 - \frac{1}{3}\mu_\theta^2\right)\right]\mu_\theta \tag{41}$$

where v_w is the rotational speed of the surface and $\vec{\mu}$ satisfies

$$\vec{\mu} = \left(\frac{m}{2kT}\right)^{1/2}\left(\vec{c} - \hat{\theta}v_w\right) \tag{42}$$

and continues with the application of the conservation law

$$\sigma_2 \left\langle \mu_r \mu_\theta \middle| h_2 \right\rangle_{\mu_r > 0} = \left\langle \mu_r \mu_\theta \middle| h_2 \right\rangle \tag{43}$$

evaluated at r_2.

The relations analogous to equations (35) to (37) are

$$\left\langle \mu_r \mu_\theta \middle| h_2 \right\rangle_{\mu_r > 0} = \frac{1}{\pi}\left[\frac{v_2 - v_w}{\bar{v}} + \frac{\pi P_{r\theta}(r_2)}{4p}\left(1 + \frac{4l}{\pi r_2}\right)\right] \tag{44}$$

$$\left\langle \mu_r \mu_\theta \middle| h_2 \right\rangle = \frac{P_{r\theta}(r_2)}{2p} \tag{45}$$

and

$$\frac{v_w - v_2}{\bar{v}} = -\frac{\pi \zeta_2 P_{r\theta}(r_2)}{4pl}\left(1 - \frac{4l^2}{\pi \zeta_2 r_2}\right) \tag{46}$$

where v_2 is the flow velocity at r_2 and

$$\zeta_2 = \frac{(2 - \sigma_2)l}{\sigma_2} \tag{47}$$

Equations (37) and (46) provide the integration constants for equation (23), the solution of which can be written

$$v = \frac{1}{\left(r_2^2 - r_1^2\right)r}\left[r_1\left(r_2^2 - r^2\right)v_1 + r_2\left(r^2 - r_1^2\right)v_2\right] \tag{48}$$

A velocity profile thus exists which conforms both to the simple first-order distribution function of equation (22) and to the complete set of boundary conditions listed in equations (28) to (32) at $r = r_1$ and a similar set at $r = r_2$; hence, the present description is valid throughout the gas. In particular, the assumption is justified that microscopic and macroscopic relations derived in the midst of the gas apply also in the immediate proximity of the cylinder surfaces.

The remainder of this report is concerned with the computation of the torque experienced by the inner cylinder.

TORQUE ON THE INNER CYLINDER

As is evident from equation (1), the element $P_{r\theta}$ of the pressure tensor is the essential parameter for the computation of the torque exerted on the inner cylinder by the shearing action of the rotating gas. The results of the preceding section are sufficient to determine this parameter through linear terms in the flow velocity and cubic terms in the mean free path; in particular, equations (20), (37), (46), and (48) combine to yield

$$P_{r\theta} = -\frac{8pl r_1 r_2}{\pi\bar{v}\left(r_2^2 - r_1^2\right)r^2}\left(r_1 v_2 - r_2 v_1\right)$$

$$= -\frac{2r_1 r_2}{\left(r_2^2 - r_1^2\right)r^2}\left[\zeta_1 r_2 P_{r\theta}\left(r_1\right)\left(1 + \frac{4l^2}{\pi\zeta_1 r_1}\right) + \zeta_2 r_1 P_{r\theta}\left(r_2\right)\left(1 - \frac{4l^2}{\pi\zeta_2 r_2}\right) + \frac{4pl r_1 v_w}{\pi\bar{v}}\right] \quad (49)$$

Accordingly, $P_{r\theta}\left(r_1\right)$ and $P_{r\theta}\left(r_2\right)$ satisfy the simultaneous expressions

$$\left[1 + \frac{2\zeta_1 r_2^2}{\left(r_2^2 - r_1^2\right)r_1}\left(1 + \frac{4l^2}{\pi\zeta_1 r_1}\right)\right]P_{r\theta}\left(r_1\right) + \frac{2\zeta_2 r_2}{r_2^2 - r_1^2}\left(1 - \frac{4l^2}{\pi\zeta_2 r_2}\right)P_{r\theta}\left(r_2\right) = -\frac{8pl r_2 v_w}{\pi\bar{v}\left(r_2^2 - r_1^2\right)} \quad (50)$$

and

$$\frac{2\zeta_1 r_1}{r_2^2 - r_1^2}\left(1 + \frac{4l^2}{\pi\zeta_1 r_1}\right)P_{r\theta}\left(r_1\right) + \left[1 + \frac{2\zeta_2 r_1^2}{\left(r_2^2 - r_1^2\right)r_2}\left(1 - \frac{4l^2}{\pi\zeta_2 r_2}\right)\right]P_{r\theta}\left(r_2\right) = -\frac{8pl r_1^2 v_w}{\pi\bar{v}\left(r_2^2 - r_1^2\right)r_2} \quad (51)$$

Finally, the combination of equations (1) and (17) with the angular frequency

$$\omega = \frac{v_w}{r_2} \quad (52)$$

and the solution of equations (50) and (51) for $P_{r\theta}\left(r_1\right)$ gives

$$L = \frac{4\pi\eta\omega r_1^2 r_2^2}{r_2^2 - r_1^2}\left\{1 - \frac{2\left(\zeta_1 r_2^3 + \zeta_2 r_1^3\right)}{r_1 r_2\left(r_2^2 - r_1^2\right)} + \frac{4\left(\zeta_1 r_2^3 + \zeta_2 r_1^3\right)^2}{r_1^2 r_2^2\left(r_2^2 - r_1^2\right)^2}\left[1 - \frac{2l^2\left(r_1^2 + r_2^2\right)\left(r_2^2 - r_1^2\right)^2}{\pi\left(\zeta_1 r_2^3 + \zeta_2 r_1^3\right)^2}\right]\right\} \quad (53)$$

The justification for retaining l^3 terms follows from the first equality in equation (49) and the observation that v_1 and v_2 are correct through l^2 since the leading contribution to $P_{r\theta}$ appearing in equations (37) and (46) is proportional to l.

Besides correcting in a general manner the existing expansion (see ref. 1, p. 298)

$$L = \frac{4\pi\eta\omega {r_1}^2 {r_2}^2}{{r_2}^2 - {r_1}^2}\left[1 + \frac{2\left(\zeta_1 {r_2}^3 + \zeta_2 {r_1}^3\right)}{r_1 r_2\left({r_2}^2 - {r_1}^2\right)}\right]^{-1}$$
$$\approx \frac{4\pi\eta\omega {r_1}^2 {r_2}^2}{{r_2}^2 - {r_1}^2}\left[1 - \frac{2\left(\zeta_1 {r_2}^3 + \zeta_2 {r_1}^3\right)}{r_1 r_2\left({r_2}^2 - {r_1}^2\right)} + \frac{4\left(\zeta_1 {r_2}^3 + \zeta_2 {r_1}^3\right)^2}{{r_1}^2 {r_2}^2\left({r_2}^2 - {r_1}^2\right)^2}\right] \tag{54}$$

through cubic terms in the Knudsen number $l/(r_2 - r_1)$, equation (53) is applicable for even greater rarefication if the cylinder separation $d = r_2 - r_1$ is small compared with r_1. In particular, the neglect of quadratic and higher powers of d/r_1 in equation (53) yields

$$L \approx \frac{2\pi\eta\omega {r_1}^3}{d}\left(1 + \frac{3d}{2r_1}\right)\left\{1 + \frac{\zeta_1 + \zeta_2}{d}\left[1 + \frac{3d}{2r_1}\left(\frac{\zeta_1 - \zeta_2}{\zeta_1 + \zeta_2}\right)\right]\right\}^{-1} \tag{55}$$

which corresponds to the drag force per unit area

$$D = \frac{\eta v_w}{d}\left(1 + \frac{\zeta_1 + \zeta_2}{d}\right)^{-1} \tag{56}$$

in plane Couette flow (ref. 3) as r_1 approaches infinity.

Since the approach of r_1 to infinity, or more precisely the neglect of l/r_1 compared with unity, makes equation (22) an exact solution of the velocity-linearized kinetic equation, the following reduction of equation (55) is limited only by negligible values of d/r_1 and small values of $(l/d)\left(v_w/\bar{v}\right)$:

$$L = \frac{2\pi\eta\omega {r_1}^3}{d}\left(1 + \frac{\zeta_1 + \zeta_2}{d}\right)^{-1} \tag{57}$$

No specific restriction is placed on l/d by itself; hence, for sufficiently small Mach numbers of the outer cylinder, equation (57) is valid for Knudsen numbers extending well into the transition flow regime. Actually, the limitation on d/r_1 need not be quite as strict as stated because the contribution to L from the term in equation (22) which does not survive the approach of r_1 to infinity also does not survive the elimination of quadratic and higher powers of d/r_1 in equation (53). Thus equation (55) is no more restricted than equation (57) as far as the Knudsen number is concerned and permits larger values of d/r_1.

If ζ_1 and ζ_2 are each equated to ζ, equation (57) becomes identical with the very early (1913) expression deduced by Timiriazeff (ref. 1, p. 298). The preceding discussion explains why this formula was found to hold closely even down to pressures where it was expected to fail on the basis of less rigorous kinetic theory. Validity criteria available at that time were based on the Knudsen number alone instead of the quantity $(l/d)\left(v_w/\bar{v}\right)$ suggested by equations (20) and (22).

Another consequence of equation (53) is that low-pressure reductions in the torque are enhanced by the same geometric effect discussed in the paragraph following equations (37) and (38). For example,

$$L \approx \frac{4\pi\eta\omega r_1^2 r_2^2}{r_2^2 - r_1^2}\left[1 - \frac{8l^2\left(r_1^2 + r_2^2\right)}{\pi r_1^2 r_2^2}\right] \tag{58}$$

in the absence of the conventional contribution to the velocity slip, that is, with σ_1 and σ_2 equated to 2 or ζ_1 and ζ_2 equated to zero. The physical explanation of this effect, which is not predicted by the Grad 13-moment approximation because it derives from seldom-used elements of the complete heat-flux tensor, is again not apparent.

Nevertheless, the geometric effect may be very important at low pressures and for the intercylinder separation a sizable fraction of r_1. The significance is illustrated by the following sample calculations for $r_2 = 2r_1$ and the slip distances ζ_1 and ζ_2 equal to l:

$$L = \frac{16\pi\eta\omega r_1^2}{3}\left(1 - \frac{3l}{r_1} + \frac{9l^2}{r_1^2}\right) \tag{59}$$

from equation (54) and

$$L = \frac{16\pi\eta\omega r_1^2}{3}\left(1 - \frac{3l}{r_1} + \frac{5.82l^2}{r_1^2}\right) \tag{60}$$

from equation (53). A decrease of 35 percent occurs in the last term in parentheses.

CONCLUDING REMARKS

The velocity-linearized Boltzmann equation is solved in the present research for the problem of viscous gases flowing between two relatively rotating coaxial cylinders. Boundary conditions corresponding to the conservation of particles, momentum, and energy are used in place of the obsolete Maxwellian hypothesis that the reflected particles can be divided into diffuse and specular groups. Unlike the results of Wang Chang and Uhlenbeck, who employed the Maxwellian hypothesis and thus restricted the accommodation coefficients to a single value for all moments, no nonanalytic (in the mean free path) contributions to the velocity distribution function are found. Hence, the use of the mean free path as a perturbation expansion parameter is valid, at least for the terms retained in this report which is the complete specification through linear terms in the Mach number and quadratic terms in the Knudsen number.

Calculated results include the generalization to cubic terms in the Knudsen number of the Navier-Stokes expression for the torque exerted on the inner cylinder by the shearing action of the flowing gas. Grad's 13-moment velocity distribution function is shown to be insufficient for this purpose; in particular, seldom-used elements of the complete heat-flux tensor are required unless the separation between the cylinders is small compared with their radii. If this last condition is satisfied and the relative Mach number of the rotating cylinders is sufficiently small, a major restriction on the mean free path is removed and the present formulas are valid for Knudsen numbers extending well into the transition flow regime.

Besides defining and comparing the limitations on various expressions for the torque, the present research introduces contributions to the velocity slip and the torque which, except for a shear factor in the slip, act independently of the gas-surface accommodation coefficients for parallel momentum absorption. Although the physical interpretation of these contributions is not clear, their effects can be very substantial at low pressures and for intercylinder separations which are sizable fractions of the radii. They are not predicted at all by the 13-moment distribution function and are predicted incorrectly by previous applications of the Burnett equations.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., October 9, 1970.

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