NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1507

A Second-Order Artificial Satellite Theory Based on an Intermediate Orbit

Kaare Aksnes

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Foreword

This report is a summary of a dissertation presented by the author to Yale University in candidacy for the degree of Doctor of Philosophy. The work for the thesis was performed with the financial assistance of the Office of Naval Research, the Air Force Office of Scientific Research, and the Royal Norwegian Council for Scientific and Industrial Research. Part of the analysis presented herein was done under the cognizance of the Mission Analysis Division of the Jet Propulsion Laboratory, supported by the National Aeronautics and Space Administration.

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2. Prediction error of the second-order theory for two near-earth satellites: $J_2 = 1.082 \times 10^{-3}$, $J_3 = -2.4 \times 10^{-6}$, $J_4 = 1.7 \times 10^{-6}$; $l_0 = g_0 = h_0 = 0$, $i = 30°$, $a(1 - e) = 6678$ km 32

Abstract

An analytical second-order theory is developed for the motion of a satellite of an oblate planet whose gravitational potential includes the second, third, and fourth zonal harmonics. It is assumed that J_2 is a small quantity of the first order and that J_3 and J_4 are of the second order.

The secular and the periodic perturbations are obtained to the third and to the second order, respectively. The former are contained in the Delaunay variables l'', g'', h'', which are linear functions of the time, while the latter are given as additions to the Hill variables $\dot{r}'', r'', G'', u'', h''$ in the form of trigonometric series with constant coefficients.

The theory is distinguished by a relative simplicity and compactness of the final algorithm achieved by the use of the following special devices and techniques: (1) an intermediate orbit, (2) Hori's perturbation method, and (3) the Hill variables.

A comparison with the results of numerical integration of the equations of motion indicates that the theory is capable of predicting the position of a near-earth satellite to better than one meter over one hundred revolutions.

A Second-Order Artificial Satellite Theory Based on an Intermediate Orbit

I. Introduction

Since *Sputnik I* went aloft in October 1957, a great many papers have been devoted to the mathematical problem posed by the motion of an artificial satellite of an oblate planet.

It is here assumed that the satellite moves in an axially symmetric gravitational field whose potential is of the form

$$V = -\frac{\mu}{r}\left[1 - \sum_{2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k(\sin\theta)\right], \qquad (1)$$

where μ denotes the gravitational constant times the mass of the planet, whose equatorial radius is R; J_k is a constant; and $P_k(\sin\theta)$ is the Legendre polynomial of degree k in $\sin\theta$. Furthermore, the polar coordinates of the satellite are the radius vector r, the declination θ, and the right ascension ϕ. In the present treatment the series (1) will be truncated at $k = 4$, and it is assumed that J_2 is a small quantity of the first order and that J_3 and J_4 are of the second order.

The most comprehensive investigations of this problem were done in the course of a few years, notably by Brouwer, Kozai, Sterne, Garfinkel, Vinti, and others. It is natural to discuss a satellite theory in terms of (1) the type of reference orbit or intermediate orbit, (2) the perturbation method, and (3) the coordinates and orbital elements that are used. Thus Brouwer (Ref. 1) and Kozai (Ref. 2) used an ellipse as a first approximation and obtained the perturbations in the Delaunay variables (Ref. 3) by means of von Zeipel's perturbation method (Ref. 4). On the other hand, the theories of Sterne (Ref. 5), Garfinkel (Refs. 6–8), Vinti (Refs. 9, 10), and the author in earlier work (Refs. 11, 12) are based on four different intermediate orbits, which incorporate at least the first-order secular perturbations due to the oblateness of the primary. A comparison of the first three intermediaries, defined in terms of spherical coordinates, was recently offered in the author's work with Garfinkel (Ref. 13). Vinti's intermediary, which utilizes oblate spheroidal coordinates, apparently has the highest accuracy over a fairly short arc, but Garfinkel's intermediary (Ref. 8) should secularly be the most accurate of them all. On the other hand, with regard to the ease of computing the higher-order perturbations, the author's intermediary seems preferable because of a simplified disturbing function. In this respect, Vinti's intermediary appears to be the least attractive one. Except for Sterne, who did not construct a complete first-order theory, these authors also used the von Zeipel technique to derive perturbations in the Delaunay variables in the manner of Brouwer's example.

Of the aforementioned theories, only the ones due to Kozai (Ref. 2) and Vinti (Ref. 10) include second-order periodic terms arising from J_2, J_3, and J_4, but Vinti's theory does not include all such terms. The explicit expressions for the perturbations in the position coordinates in Kozai's theory are not entirely complete, inasmuch as short-period terms, whose total contribution does not exceed 0.″1 of geocentric arc, have been omitted. However, the rather unwieldy form of the solution and, above all, an essential singularity at zero eccentricity, are more serious objections from a practical point of view.

By comparison, the second-order solution presented here takes a much simpler form. It is not singular at zero eccentricity, and it includes all the terms up to the second order, as well as the secular terms of the third order. The theory makes use of the following concepts:

(1) An intermediate orbit (Ref. 11), which incorporates all first-order secular terms and facilitates the calculation of the higher-order perturbations.

(2) Hori's perturbation method (Ref. 14), which is canonically invariant and avoids the mixing of old and new variables in any one expression.

(3) A modification of the Hill variables (Ref. 15) to calculate perturbations in the coordinates and to preclude the appearance of mixed secular terms.

The following sections contain a detailed discussion of these subjects, each of which is fundamental to the present satellite theory.

II. The Intermediate Orbit

Although an account of this intermediate orbit can be found in Ref. 11, a brief derivation is included here for the sake of completeness.

In terms of spherical coordinates r, θ, ϕ and the conjugate momenta p_1, p_2, p_3, the equations of motion are

$$\left.\begin{aligned} \frac{dp_1}{dt} &= \frac{\partial F}{\partial r}, & \frac{dp_2}{dt} &= \frac{\partial F}{\partial \theta}, & \frac{dp_3}{dt} &= \frac{\partial F}{\partial \phi}, \\ \frac{dr}{dt} &= -\frac{\partial F}{\partial p_1}, & \frac{d\theta}{dt} &= -\frac{\partial F}{\partial p_2}, & \frac{d\phi}{dt} &= -\frac{\partial F}{\partial p_3}, \end{aligned}\right\} \tag{2}$$

where the Hamiltonian F is given by

$$F = -\frac{1}{2}\left(p_1^2 + \frac{p_2^2}{r^2} + \frac{p_3^2}{r^2\cos^2\theta}\right) - V. \tag{3}$$

As a first step, consider a canonical transformation from the variables $r, \theta, \phi, p_1, p_2, p_3$ to new variables l, g, h, L, G, H with the aid of a determining function

$$S = S\,(r, \theta, \phi, L, G, H)$$

depending on the old coordinates and the new momenta. The equations of motion in terms of the new variables are

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dH}{dt} &= \frac{\partial F}{\partial h}, \\ \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}, \end{aligned}\right\} \tag{4}$$

while the relation between the old and the new variables is given by

$$\left.\begin{aligned} p_1 &= \frac{\partial S}{\partial r}, & p_2 &= \frac{\partial S}{\partial \theta}, & p_3 &= \frac{\partial S}{\partial \phi}, \\ l &= \frac{\partial S}{\partial L}, & g &= \frac{\partial S}{\partial G}, & h &= \frac{\partial S}{\partial H}. \end{aligned}\right\} \tag{5}$$

We know that for elliptic motion the proper choice of S is one that separates the corresponding Hamilton–Jacobi partial differential equation in S. Such a solution for S is still possible if, instead of the approximation $V_0 = -\mu/r$, the potential in expression (1) is approximated to

$$V_0 = -\frac{\mu}{r} + \mu c_1 J_2 \frac{R^2}{r^2} P_2(\sin\theta), \tag{6}$$

where c_1 is some suitably chosen constant *which may be a function of the momenta L, G, and H*. The treatment of c_1 as a dynamic variable here is a crucial part of the procedure. By contrast, Sterne (Ref. 5) and Garfinkel (Ref. 6) treat the disposable constants in their potentials as *absolute* constants.

It can easily be verified that a solution for S is

$$S = \int_{r_{\min}}^{r} \left(-\frac{\mu^2}{L^2} + 2\frac{\mu}{r} - \frac{G^2}{r^2}\right)^{1/2} dr + \int_0^{\theta} \left[G^2 - \frac{H^2}{\cos^2\theta} - 2\mu c_1 J_2 R^2 P_2(\sin\theta)\right]^{1/2} d\theta + H\phi. \tag{7}$$

This S-function becomes the generating function for the familiar Delaunay variables if $c_1 = 0$.

With the aid of Eqs. (3), (5), (6), and (7) we find that

$$F = \frac{\mu^2}{2L^2} + V_0 - V, \tag{8}$$

where, according to Eqs. (1) and (6),

$$V_0 - V = \mu J_2 \frac{R^2}{r^2} P_2(\sin\theta)\left(c_1 - \frac{1}{r}\right) + O(J_2^2). \tag{9}$$

It is natural to choose c_1 so as to minimize the time-average, the so-called secular part, of $V_0 - V$. If terms beyond the first order are neglected, use can be made of some well-known Keplerian relations to prove that this secular part vanishes if

$$c_1 = \frac{\mu}{G^2}. \tag{10}$$

The intermediate orbit is now defined as the orbit that results by dropping the perturbation term $V_0 - V$ in Eq. (8). It immediately follows that L, G, H, g, h are constants, while l is a linear function of time,

$$l = \frac{\mu^2}{L^3} t + \sigma, \qquad \sigma = \text{const.} \tag{11}$$

It still remains to express these constants in terms of the original variables.

With the aid of Eqs. (5) and (10), let Eq. (7) be rewritten as

$$S = \int_{r_{\min}}^{r} p_1\, dr + \int_0^{\theta} p_2\, d\theta + \int_0^{\phi} p_3\, d\phi, \tag{12}$$

where

$$\left.\begin{aligned} p_1 &= \left[-\frac{\mu^2}{L^2} + 2\frac{\mu}{r} - \frac{G^2}{r^2}\right]^{1/2}, \\ p_2 &= \left[G^2 - \frac{H^2}{\cos^2\theta} - 2\mu^2 \frac{J_2 R^2}{G^2} P_2(\sin\theta)\right]^{1/2}, \\ p_3 &= H. \end{aligned}\right\} \tag{13}$$

But by Eqs. (2) and (3),

$$p_1 = \dot{r}, \qquad p_2 = r^2\dot{\theta}, \qquad p_3 = r^2 \cos^2\theta\, \dot{\phi}. \tag{14}$$

Elimination of p_1, p_2, p_3 from the last two sets of equations yields, finally,

$$\left.\begin{aligned} H &= r^2 \cos^2\theta\, \dot{\phi}, \\ G^4 - r^4 &(\cos^2\theta\, \dot{\phi}^2 + \dot{\theta}^2) G^2 \\ &- 2\mu^2 J_2 R^2 P_2(\sin\theta) = 0, \\ \frac{\mu^2}{L^2} &= -\dot{r}^2 + 2\frac{\mu}{r} - \frac{G^2}{r^2}. \end{aligned}\right\} \tag{15}$$

The new momenta are thus uniquely given in terms of the initial coordinates and velocities.

As regards the new coordinates, they can with the aid of Eqs. (5), (12), and (13) be written as

$$\left.\begin{aligned} l &= \frac{\mu^2}{L^3}\int_{r_{\min}}^{r} \frac{dr}{p_1}, \\ g &= -G\int_{r_{\min}}^{r} \frac{dr}{r^2 p_1} + G\int_0^{\theta} \frac{1 + 2\gamma P_2(\sin\theta)}{p_2}\, d\theta, \\ h &= \phi - H\int_0^{\theta} \frac{d\theta}{\cos^2\theta\, p_2}, \end{aligned}\right\} \tag{16}$$

where

$$\gamma = \frac{\mu^2 J_2 R^2}{G^4}. \tag{17}$$

The appearance of the dimensionless constant γ in Eqs. (16) is due to the treatment of c_1 as a function of G.

It will be noticed that the two r-integrals in Eqs. (16) do not involve J_2; they are therefore the same as for elliptic motion. In the first of these integrals, introduce the eccentric anomaly E by $r = a(1 - e\cos E)$, and in the latter the true anomaly f by $r = a(1 - e^2)/(1 + e\cos f)$, where a and e are defined by the relations

$$a = \frac{L^2}{\mu}, \qquad 1 - e^2 = \frac{G^2}{L^2}. \tag{18}$$

The integration over E results in the Kepler equation,

$$l = E - e\sin E, \tag{19}$$

while the second integral reduces to the true anomaly,

$$G\int_{r_{\min}}^{r} \frac{dr}{r^2 p_1} = f. \tag{20}$$

The two θ-integrals in Eqs. (16) are elliptic integrals. For practical purposes it is most expedient to expand the integrands in powers of J_2 and integrate term by term.

Let an angle i, which for an elliptic orbit reduces to the inclination, be defined implicitly by

$$H = G \cos i \, [1 + \gamma (3 \cos^2 i - 2)]^{1/2}, \qquad 0 \leqq i \leqq \pi, \tag{21}$$

such that $\cos i$ will be a root of the following quadratic in $\cos^2 \theta$:

$$p_2^2 \cos^2 \theta = G^2 \cos^2 \theta \, [1 + \gamma (3 \cos^2 \theta - 2)] - H^2 = 0.$$

To save writing we now introduce the abbreviations

$$c = \cos i, \qquad s = \sin i. \tag{22}$$

We then have

$$p_2^2 \cos^2 \theta = G^2 (s^2 - \sin^2 \theta) \, [1 + \gamma (4 - 3s^2 - 3 \sin^2 \theta)] = 0,$$

and since by Eqs. (14) p_2 and $\dot{\theta}$ vanish simultaneously, it follows that $s = \sin (\max \theta)$. It is therefore permissible to put

$$\sin \theta = s \sin u, \tag{23}$$

where u corresponds to the argument of latitude in elliptic motion.

By means of the above relations, the first two equations of (16) may be written in the form

$$\left.\begin{aligned} g + f &= \int_0^u [1 + \gamma (3s^2 \sin^2 u - 1)] \, [1 + \gamma (4 - 3s^2 - 3s^2 \sin^2 u)]^{-1/2} \, du, \\ h &= \phi - c \int_0^u [1 + w (1 - s^2 \sin^2 u)]^{-1/2} \, [1 - s^2 \sin^2 u]^{-1} \, du, \end{aligned}\right\} \tag{24}$$

where (Garfinkel, Ref. 8)

$$w = 3\gamma \, [1 + \gamma (1 - 3s^2)]^{-1}.$$

The quadratures in Eqs. (24) can easily be performed if the integrands are first expanded in powers of γ and w. It is sufficient to retain periodic terms to the second order and secular terms to the third order. After rearranging the resulting series for $(g + f)$ and h slightly, the algorithm for the determination of r, θ, ϕ from a, e, i, σ, g, h can be expressed as follows:

$$\left.\begin{aligned} E - e \sin E &= l = n_1 t + \sigma, \\ r &= a (1 - e \cos E), \\ \tan \frac{1}{2} f &= \left(\frac{1 + e}{1 - e}\right)^{1/2} \tan \frac{1}{2} E, \\ \bar{u} &= (1 + g_{21}) (g + f), \\ u &= \bar{u} + \frac{3}{16} \gamma \, [6 - \gamma (4 - 3s^2)] \, s^2 \sin 2\bar{u} + \frac{261}{256} \gamma^2 s^4 \sin 4\bar{u} + O(\gamma^3), \\ \sin \theta &= s \sin u, \\ \phi &= \tan^{-1} (c \tan u) + h + g_{32} \bar{u} - \frac{27}{32} \gamma^2 c s^2 \sin 2\bar{u} + O(\gamma^3), \end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned}
n_1 &= \frac{\mu^2}{L^3} = \left(\frac{\mu}{a^3}\right)^{1/2}, \\
g_{21} &= -\frac{3}{4}\gamma(1-5c^2) - \frac{1}{64}\gamma^2(41+30c^2-135c^4) + \frac{5}{256}\gamma^3(7+159c^2-531c^4+621c^6), \\
g_{32} &= -\frac{3}{16}c\left[8\gamma + \gamma^2(7-33c^2) + \frac{1}{8}\gamma^3(103-534c^2+1143c^4)\right].
\end{aligned}\right\} \tag{26}$$

This result agrees with that obtained in Ref. 13, where ψ was written for u and ϵ for 3γ.

If needed, the three velocities $\dot{r}, \dot{\theta}, \dot{\phi}$ may be computed from the relations

$$\left.\begin{aligned}
r\dot{r} &= e L \sin E, \\
r^2 \cos\theta\,\dot{\theta} &= \\
&\quad G s \cos u\,[1+\gamma(3\cos^2\theta + 3c^2 - 2)]^{1/2}, \\
r^2\cos^2\theta\,\dot{\phi} &= H,
\end{aligned}\right\} \tag{27}$$

which are readily derived from Eqs. (15).

To derive expressions for the mean anomalistic motion n_1, the mean nodal motion n_2, and the mean sidereal motion n_3, we now introduce the action-angle variables $J_1, J_2, J_3, w_1, w_2, w_3$. The temporary notation J_2 and J_3 here must not be confused with J_2 and J_3 in Eq. (1). The action variables are defined by the phase integrals

$$\left.\begin{aligned}
J_1 &= \frac{1}{2\pi}\oint p_1\,dr = L - G, \\
J_2 &= \frac{1}{2\pi}\oint p_2\,d\theta \equiv J_2(G,H), \\
J_3 &= \frac{1}{2\pi}\oint p_3\,d\phi = H,
\end{aligned}\right\} \tag{28}$$

while the angle variables take the form

$$w_j = n_j t + \text{const}, \qquad j = 1,2,3, \tag{29}$$

where

$$n_j = -\frac{\partial F_0(J_1,J_2,J_3)}{\partial J_j}, \qquad j = 1,2,3. \tag{30}$$

The integrals in Eqs. (28), except for the second one whose explicit value is not needed, have been evaluated with the aid of Eqs. (13). Since

$$F_0 = \frac{\mu^2}{2L^2} = \frac{\mu^2}{2(J_1+G)^2},$$

it follows that

$$n_1 = \frac{\mu^2}{L^3}, \qquad n_2 = n_1\left(\frac{\partial J_2}{\partial G}\right)^{-1}, \qquad n_3 = -n_2\frac{\partial J_2}{\partial H}. \tag{31}$$

The derivation of Eqs. (31) is based on the identities

$$1 \equiv \frac{\partial J_2}{\partial G}\frac{\partial G}{\partial J_2}, \qquad 0 \equiv \frac{\partial J_2}{\partial G}\frac{\partial G}{\partial J_3} + \frac{\partial J_2}{\partial H}.$$

With the use of the original definition in Eqs. (5) for g and h, a little reflection shows that $\partial J_2/\partial G$ and $\partial J_2/\partial H$ are obtainable from Eqs. (25) for u and ϕ, viz.

$$n_2 = n_1(1+g_{21}), \qquad n_3 = n_2(1+g_{32}). \tag{32}$$

The motion in the intermediary can be pictured rigorously as a Keplerian elliptic motion taking place in a plane that revolves around the polar axis with the angular velocity $\dot{\Omega}$, while the line of apsides rotates in this plane with the angular velocity $\dot{\omega}$. The mean values of these angular velocities are easily seen to be

$$\left.\begin{aligned}
n_3 - n_2 &= -\frac{3}{2}n_1\gamma c + O(\gamma^2), \\
n_2 - n_1 &= -\frac{3}{4}n_1\gamma(1-5c^2) + O(\gamma^2).
\end{aligned}\right\} \tag{33}$$

These equations do, indeed, predict the correct first-order secular motions of the orbit as can be ascertained by a comparison with other sources.

III. Hori's Perturbation Method

For a full account of Hori's perturbation method, the reader is referred to his original work (Ref. 14). Here only a brief outline of the method will be given.

The method is applicable to a system of canonical equations

$$\frac{dx_j}{dt} = \frac{\partial F}{\partial y_j}, \qquad \frac{dy_j}{dt} = -\frac{\partial F}{\partial x_j}, \qquad j = 1, 2, \cdots n, \tag{34}$$

where the Hamiltonian F is developed in powers of a small and constant parameter ϵ,

$$F = \sum_{k=0}^{\infty} F_k, \qquad F_k = O(\epsilon^k). \tag{35}$$

Consider a Lie transformation $x_j, y_j \rightarrow x'_j, y'_j$ given by

$$\left.\begin{aligned} x_j &= x'_j + \frac{\partial S}{\partial y'_j} + \frac{1}{2}\left\{\frac{\partial S}{\partial y'_j}, S\right\} + \frac{1}{6}\left\{\left\{\frac{\partial S}{\partial y'_j}, S\right\}, S\right\} + O(\epsilon^4), \\ y_j &= y'_j - \frac{\partial S}{\partial x'_j} - \frac{1}{2}\left\{\frac{\partial S}{\partial x'_j}, S\right\} - \frac{1}{6}\left\{\left\{\frac{\partial S}{\partial x'_j}, S\right\}, S\right\} + O(\epsilon^4), \end{aligned}\right\} \tag{36}$$

where braces denote Poisson brackets and S is a first-order function of x'_j and y'_j, defined as

$$S = \sum_{k=1}^{\infty} S_k, \qquad S_k = O(\epsilon^k). \tag{37}$$

Any function $f(x, y)$ of x_j and y_j may then be expressed in terms of the new variables x'_j and y'_j with the aid of the expansion formula

$$f(x, y) = f(x', y') + \{f, S\} + \frac{1}{2}\{\{f, S\}, S\} + \frac{1}{6}\{\{\{f, S\}, S\}, S\} + O(\epsilon^4). \tag{38}$$

By letting $f(x', y')$ equal x'_j and y'_j, we see that the Eqs. (36) are special cases of (38).

In terms of the new variables the equations of motion become

$$\frac{dx'_j}{dt} = \frac{\partial F'}{\partial y'_j}, \qquad \frac{dy'_j}{dt} = -\frac{\partial F'}{\partial x'_j}, \tag{39}$$

where F' is the new Hamiltonian,

$$F' = \sum_{k=0}^{\infty} F'_k, \qquad F'_k = O(\epsilon^k). \tag{40}$$

If we assume that F does not depend explicitly on the time t, we have the energy integral

$$\sum_{k=0}^{\infty} F_k(x, y) = \sum_{k=0}^{\infty} F'_k(x', y'). \tag{41}$$

By applying the expansion formula (38) to the left-hand side of Eq. (41) we obtain, after some manipulation,

$$\left.\begin{aligned}
&F_0 = F'_0,\\
&\{F_0, S_1\} + F_1 = F'_1,\\
&\{F_0, S_2\} + \frac{1}{2}\{F_1 + F'_1, S_1\} + F_2 = F'_2,\\
&\{F_0, S_3\} + \frac{1}{2}\{F_1 + F'_1, S_2\} + \frac{1}{2}\{F_2 + F'_2, S_1\} + \frac{1}{12}\{\{F_1 - F'_1, S_1\}, S_1\} + F_3 = F'_3,\\
&\text{etc.,}
\end{aligned}\right\} \tag{42}$$

where terms have been grouped according to order of magnitude up to order three.

In order to provide a way of averaging Eqs. (42), a pseudo-time t' is introduced by

$$\frac{dx'_j}{dt'} = \frac{\partial F_0}{\partial y'_j}, \qquad \frac{dy'_j}{dt'} = -\frac{\partial F_0}{\partial x'_j}. \tag{43}$$

A necessary condition for the feasibility of the method is that the system (43) be solvable. Then

$$\{F_0, S_k\} = -\frac{dS_k}{dt'}. \tag{44}$$

Let $A(t')$ be a periodic function of t' with period T. The quantities A_s and A_p given by

$$A_s = \frac{1}{T}\int_0^T A(t')\,dt', \qquad A_p = A - A_s \tag{45}$$

then represent the constant part (secular part) and the periodic part of A.

Suppose that F' does not depend on t'. It then follows that

$$0 = \frac{dF'}{dt'} = \{F', F_0\} = -\{F_0, F'\} = -\frac{dF_0}{dt} = -\frac{dF'_0}{dt},$$

and we have a first integral

$$F'_0(x', y') = \text{const} \tag{46}$$

in addition to the energy integral

$$F'(x', y') = \text{const.} \tag{47}$$

The removal of t' from F' is accomplished by applying the averaging technique of Eqs. (45) to Eqs. (42). This determines S_k and F'_k uniquely as follows:

$$\left.\begin{aligned}
&F'_0 = F_0, \qquad F'_1 = F_{1s}, \qquad S_1 = \int F_{1p}\,dt',\\
&F'_2 = F_{2s} + \frac{1}{2}\{F_1 + F'_1, S_1\}_s,\\
&S_2 = \int\left[F_{2p} + \frac{1}{2}\{F_1 + F'_1, S_1\}_p\right]dt',\\
&F'_3 = F_{3s} + \frac{1}{12}\{\{F_{1p}, S_1\}, S_1\}_s + \frac{1}{2}\{F_2 + F'_2, S_1\}_s + \frac{1}{2}\{F_1 + F'_1, S_2\}_s,\\
&\text{etc.}
\end{aligned}\right\} \tag{48}$$

The whole process can be repeated, starting with Eqs. (39) and seeking a transformation to a third set of canonical variables x_j'', y_j''. This time, F_1' plays the role of F_0 of Eqs. (43) in defining a new pseudo-time t'', etc.

The similarity of Hori's perturbation method to von Zeipel's method (Ref. 4) will now have become apparent. The latter method also eliminates the angular variables from the Hamiltonian with the aid of a succession of canonical transformations. But the Hamiltonian is developed in a Taylor series instead of in the Lie series (38). Because of the canonical invariance of the Poisson brackets, the functions S_k and F_k' in Eqs. (48) are also canonical *invariants*, while the corresponding functions in von Zeipel's method will depend on the particular choice of canonical variables x_j, y_j. A change to another set of canonical variables is therefore likely to be a more cumbersome process with von Zeipel's method. Another, perhaps more serious, disadvantage of this method is that the determining function S depends on a *mixed* set of variables, say x_j' and y_j. This means that a number of series inversions will be required to obtain an explicit solution, beyond the first order, in terms of x_j' and y_j'. However, W. A. Mersman (Ref. 16) has overcome this objection to the use of von Zeipel's method by modifying it to render transformations in explicit form.

The second-order satellite theory of Kozai (Ref. 2) reflects these disadvantages of the von Zeipel method. The occurrence of an essential singularity at $e = 0$ in the solution seems to be a combined effect of the use of a Taylor expansion and Delaunay's variables.

Recently Deprit (Ref. 17) has proposed a perturbation method that is also based on Lie transformations. The method is very similar to Hori's method. As a matter of fact, it can easily be shown that the transformation formula (42) and the corresponding transformation used by Deprit coincide through terms of the second order.[1] Deprit's method does not offer a unique determination of S_k and F_k' in Eqs. (42) insofar as no general averaging technique is supplied. The averaging technique specified in Eqs. (45), being problem-independent as well as variable-independent, is an essential part of Hori's perturbation method, although under special circumstances it may be desirable to use other ways of averaging.

[1] J. A. Campbell and W. H. Jefferys of the University of Texas have recently shown that Hori's and Deprit's perturbation methods are equivalent at least to the sixth order.

IV. Hill's Canonical Variables

There are three good reasons for introducing the Hill variables: (1) They allow a very compact representation of the Poisson brackets in Eqs. (48), which simplifies the process of obtaining F_k' and S_k; (2) the final algorithm becomes simpler and is non-singular at zero eccentricity; and (3) an appropriate modification of the Hill variables precludes the appearance of mixed secular terms.

The Hill variables (Ref. 15) were introduced into artificial satellite theory by Izsak (Ref. 18) for the reason stated in (2) above. He demonstrated that one can obtain the short-period first-order perturbations directly in the Hill variables from the partial derivatives of S_1 (L', G', H', l, g) with respect to the Hill variables. This important discovery can now be seen as a consequence of the fact that S_1 in the von Zeipel method has the same form as in the Hori method, and hence is canonically invariant. The same remark applies to S_1^*, from which the long-period perturbations are derived.

With reference to Eqs. (12) and (13) of Section II, consider the determining function

$$S = r\dot{r} + \int_0^\theta \left[G^2 - \frac{H^2}{\cos^2\theta} - 2\frac{\mu^2}{G^2} J_2 R^2 P_2(\sin\theta) \right]^{1/2} d\theta + H\phi, \tag{49}$$

which defines a canonical transformation from r, θ, ϕ, p_1, p_2, p_3 to a new set of variables with the momenta $\dot{r} = p_1$, G, and $H = p_3$. In accordance with theory, the new coordinates will then be given by

$$\frac{\partial S}{\partial \dot{r}} = r, \qquad \frac{\partial S}{\partial G} = g + f, \qquad \frac{\partial S}{\partial H} = h, \tag{50}$$

where the result $g + f$ follows from a comparison with Eqs. (16) and (20). The canonical variables $\dot{r}$, G, H, r, $g + f$, h reduce to the Hill variables if J_2 vanishes in Eq. (49). These variables suffer from the following defect. The perturbed g and h contain mixed secular terms of the second order, as noted in Refs. 7 and 12, respectively. Both authors also proved that the effect of the respective perturbations in g and h cancels out in the position coordinates, at least through the second order. The mixed secular terms arise because the disturbing function is periodic in the arguments f and $\bar{u} = (1 + g_{21})(g + f)$, where g_{21}, as given by Eqs. (26), depends on G and H. If the motion is degenerate, $g_{21} = 0$, and the mixed secular terms also vanish. Since the calculation of perturbations in g and h involves partial derivatives with respect to G and H, there will appear periodic terms factored by $(g_{21})_G (g + f)$ and $(g_{21})_H (g + f)$, i.e., mixed secular terms.

Obviously, the mixed secular terms will not arise if $\bar{u}$ can be used as a canonical variable in place of $g + f$. It will now be proved that this amounts to adopting $\dot{r}$, $\bar{G}$, and $\bar{H}$ as the new momenta, where

$$\left.\begin{aligned} \bar{G} &= G + \Delta G, \\ \Delta G &= \frac{1}{2\pi} \oint p_2\, d\theta + H - G, \\ \bar{H} &= H. \end{aligned}\right\} \tag{51}$$

This means that G in the determining function (49) must be considered a function of $\bar{G}$ and $\bar{H}$, with the following partial derivatives, which are immediately obtainable from Eqs. (28), (31), and (51), as

$$\frac{\partial G}{\partial \bar{G}} = 1 + g_{21}, \qquad \frac{\partial G}{\partial \bar{H}} = (1 + g_{21})\, g_{32}. \tag{52}$$

The new coordinates will now no longer be given by Eqs. (50), but by

$$\left.\begin{aligned} \frac{\partial S}{\partial \dot{r}} &= r, \\ \frac{\partial S}{\partial \bar{G}} &= \frac{\partial S}{\partial G}\frac{\partial G}{\partial \bar{G}} = (1 + g_{21})(g + f) = \bar{u}, \\ \frac{\partial S}{\partial \bar{H}} &= \frac{\partial S}{\partial H} + \frac{\partial S}{\partial G}\frac{\partial G}{\partial \bar{H}} = h + g_{32}\bar{u} = \bar{h}. \end{aligned}\right\} \tag{53}$$

Note how the use of $\bar{h}$ simplifies the seventh equation of (25). The existence of the canonical set $\dot{r}$, $\bar{G}$, $\bar{H}$, r, $\bar{u}$, $\bar{h}$, where none of the perturbed variables contains mixed secular terms, is in itself a proof that such perturbations cannot occur in the position coordinates.

The new equations of motion now become

$$\left.\begin{aligned} \frac{d\dot{r}}{dt} &= \frac{\partial F}{\partial r}, & \frac{d\bar{G}}{dt} &= \frac{\partial F}{\partial \bar{u}}, & \frac{d\bar{H}}{dt} &= \frac{\partial F}{\partial \bar{h}}, \\ \frac{dr}{dt} &= -\frac{\partial F}{\partial \dot{r}}, & \frac{d\bar{u}}{dt} &= -\frac{\partial F}{\partial \bar{G}}, & \frac{d\bar{h}}{dt} &= -\frac{\partial F}{\partial \bar{H}}, \end{aligned}\right\} \tag{54}$$

where, in view of Eqs. (8) and (15),

$$F = -\frac{1}{2}\left(\dot{r}^2 + \frac{G^2}{r^2}\right) + \frac{\mu}{r} + V_0 - V. \tag{55}$$

The Delaunay variables and the Hill variables will be used to calculate the secular and the periodic perturbations, respectively.

V. Removal of Short-Period Terms From F

In this section, J_3 and J_4 will be disregarded so that the expression (8) for the Hamiltonian may be written

$$F = \frac{\mu^2}{2L^2} + \mu J_2 \frac{R^2}{r^2} P_2(\sin\theta)\left(\frac{\mu}{G^2} - \frac{1}{r}\right). \tag{56}$$

To save writing, the notation

$$A = 1 - 3c^2, \qquad B = 3s^2, \tag{57}$$

will be used in the following. Then, with the aid of Eqs. (25),

$$P_2(\sin\theta) = \frac{1}{4}(A - B\cos 2u), \tag{58}$$

where

$$\begin{aligned} \cos 2u = \cos 2\bar{u} &- \frac{3}{8}\gamma\,[6 - \gamma(4 - 3s^2)]\, s^2 \sin^2 2\bar{u} \\ &- \frac{423}{256}\gamma^2 s^4 (\cos 2\bar{u} - \cos 6\bar{u}) + O(\gamma^3). \end{aligned} \tag{59}$$

If we introduce this result into the Hamiltonian

$$F = F_0 + F_1 + \cdots,$$

we shall find

$$
\left.\begin{aligned}
F_0 &= \frac{\mu^2}{2L^2}, \\
F_1 &= -\frac{\gamma G^2}{4r^2}\left[A\, e \cos f - \frac{B}{2} e \cos(2\bar{u} - f) - \frac{B}{2} e \cos(2\bar{u} + f)\right], \\
F_2 &= -\frac{3\gamma^2 B^2 G^2}{32 r^2}\left[e \cos f - \frac{e}{2}\cos(4\bar{u} - f) - \frac{e}{2}\cos(4\bar{u} + f)\right].
\end{aligned}\right\} \tag{60}
$$

Note that in Eq. (56) the factor $\mu/G^2 - 1/r = -\mu e \cos f/G^2$ contains an odd multiple of f, while only even multiples of $\bar{u}$ will appear in the factor $P_2(\sin\theta)$ if expanded according to Eqs. (58) and (59). Therefore, no trigonometric terms with arguments $k(\bar{u} - f)$, $k = 0, 1, 2 \cdots$, can occur in $F - F_0$, which must be purely short-periodic. Thus F_3, F_4, etc., can be neglected in the following, since only the secular and the long-period terms in F need be carried up to the third order.

At this point we introduce a canonical transformation from x_j, y_j to x'_j, y'_j, where x_j, y_j are regarded either as the Delaunay variables L, G, H, l, g, h or as the Hill variables $\dot{r}, \bar{G}, \bar{H}, r, \bar{u}, \bar{h}$. The desired determining function

$$S(x', y') = S_1 + S_2 + \cdots$$

and Hamiltonian

$$F'(x', y') = F'_0 + F'_1 + F'_2 + \cdots$$

may be obtained from Eqs. (48). Since

$$F_{1s} = F_{2s} = F_{3s} = 0,$$

these equations take the form

$$
\left.\begin{aligned}
F'_0 &= \frac{\mu^2}{2L'^2}, \\
F'_1 &= 0, \\
S_1 + \Delta S_2 &= \int F_1\, dt', \\
F'_2 + \Delta F'_3 &= \frac{1}{2}\{F_1, S_1\}_s, \\
S_2 - \Delta S_2 &= \int\left[F_2 + \frac{1}{2}\{F_1, S_1\}_p\right] dt', \\
F'_3 - \Delta F'_3 &= \frac{1}{12}\{\{F_1, S_1\}, S_1\}_s + \frac{1}{2}\{F_2 + F'_2, S_1\}_s + \frac{1}{2}\{F_1, S_2\}_s,
\end{aligned}\right\} \tag{61}
$$

where the sole purpose of ΔS_2 and $\Delta F'_3$ is to transfer "overflow" parts from S_1 into S_2 and from F'_2 into F'_3 without altering the sums $S_1 + S_2$ and $F'_2 + F'_3$. Since these equations involve only the new variables x'_j, y'_j, no confusion will arise if the primes are dropped from these variables in this section.

As t' is the time of the intermediate orbit,

$$dt' = \frac{r^2}{G} df. \tag{62}$$

With the aid of this relation the integrations in Eqs. (61) can be carried out over f instead of over t'. Remembering that $\bar{u} = (1 + g_{21})(g + f)$, where $g_{21} = 0(J_2)$, we find that the integral

$$\int F_1 dt'$$

has a first-order part,

$$S_1 = -\frac{\gamma G}{4}\left[Ae\sin f - \frac{B}{2}e\sin(2\bar{u} - f) - \frac{B}{6}e\sin(2\bar{u} + f)\right], \tag{63}$$

and a second-order part,

$$\Delta S_2 = -\frac{\gamma}{4}BGg_{21}\left[e\sin(2\bar{u} - f) + \frac{1}{9}e\sin(2\bar{u} + f)\right]. \tag{64}$$

To expand the Poisson brackets in Eqs. (61), the following partial differentials with respect to the Hill variables will be found useful:

$$\left.\begin{aligned}
\frac{\partial c^k}{\partial G} &= -\frac{k}{G}c^k[1 + \gamma(4 - 9c^2)] + O(\gamma^2),\\
\frac{\partial c^k}{\partial H} &= \frac{k}{G}c^{k-1}\left[1 + \gamma\left(1 - \frac{9}{2}c^2\right)\right] + O(\gamma^2),
\end{aligned}\right\} \tag{65}$$

$$\left.\begin{aligned}
\frac{\partial e}{\partial \dot{r}} &= \frac{G}{\mu}\sin f,\\
\frac{\partial e}{\partial r} &= -\frac{G^2}{\mu r^2}\cos f,\\
\frac{\partial e}{\partial G} &= \frac{1}{G}\left(\frac{3}{2}e + 2\cos f + \frac{e}{2}\cos 2f\right),
\end{aligned}\right\} \tag{66}$$

$$\left.\begin{aligned}
\frac{\partial f}{\partial \dot{r}} &= \frac{G}{\mu e}\cos f,\\
\frac{\partial f}{\partial r} &= \frac{G^2}{\mu e r^2}\sin f,\\
\frac{\partial f}{\partial G} &= -\frac{1}{eG}\left(2\sin f + \frac{1}{2}e\sin 2f\right),
\end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned}
\frac{\partial}{\partial \dot{r}}[e^k\cos(j\bar{u} \pm kf)] &= \mp ke^{k-1}\frac{G}{\mu}\sin[j\bar{u} \pm (k-1)f],\\
\frac{\partial}{\partial r}[e^k\cos(j\bar{u} \pm kf)] &= -ke^{k-1}\frac{G^2}{\mu r^2}\cos[j\bar{u} \pm (k-1)f],\\
\frac{\partial}{\partial G}[e^k\cos(j\bar{u} \pm kf)] &= ke^{k-1}\frac{1}{G}\left[\frac{3}{2}e\cos(j\bar{u} \pm kf) + 2\cos[j\bar{u} \pm (k-1)f] + \frac{1}{2}e\cos[j\bar{u} \pm (k-2)f]\right],
\end{aligned}\right\} \tag{68}$$

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial \dot{r}}\left[e^k \sin(j\bar{u} \pm kf)\right] = \pm\, k e^{k-1}\frac{G}{\mu}\cos\left[j\bar{u} \pm (k-1) f\right],\\
&\frac{\partial}{\partial r}\left[e^k \sin(ju \pm kf)\right] = -\, k e^{k-1}\frac{G^2}{\mu r^2}\sin\left[ju \pm (k-1) f\right],\\
&\frac{\partial}{\partial G}\left[e^k \sin(j\bar{u} \pm kf)\right] = k e^{k-1}\frac{1}{G}\left[\frac{3}{2} e \sin(j\bar{u} \pm kf) + 2\sin\left[j\bar{u} \pm (k-1) f\right] + \frac{1}{2} e \sin\left[j\bar{u} \pm (k-2) f\right]\right],
\end{aligned}
\right\}
\qquad (69)
$$

where j and k are positive integers. Equations (65) have been obtained from Eq. (21), while the remaining equations all follow from the relations

$$
e\cos f = \frac{G^2}{\mu r} - 1, \qquad e \sin f = \frac{G}{\mu}\dot{r}. \qquad (70)
$$

With the aid of the preceding formulas, the expansion of the Poisson brackets becomes a simple although laborious process.

Expanding in terms of the Hill variables, noting that by Eqs. (60) and (63), $\partial F_1/\partial \dot{r} = \partial F_1/\partial \bar{h} = \partial S_1/\partial \bar{h} = 0$, we get

$$
\{F_1, S_1\} = -\frac{\partial F_1}{\partial r}\frac{\partial S_1}{\partial \dot{r}} + (1 + g_{21})\left(\frac{\partial F_1}{\partial G}\frac{\partial S_1}{\partial \bar{u}} - \frac{\partial F_1}{\partial \bar{u}}\frac{\partial S_1}{\partial G}\right), \qquad (71)
$$

where we have made use of Eqs. (52) to replace the argument $\bar{G}$ by G. The partials of F_1 and S_1 are easily obtained by means of Eqs. (65–69). After a fair amount of algebra we find that $\{F_1, S_1\}$ can be expressed in the form

$$
\{F_1, S_1\} = \frac{\gamma^2}{r^2}\sum_{j,k} C_{j,k}\, s^{|j|}\, e^{|k|}\cos(j\bar{u} + kf), \qquad (72)
$$

where the coefficients $C_{j,k}$ are finite polynomials in c^2 and e^2, and j and k assume the values 0, 2, 4, and 0, ± 1, ± 2, respectively. It is seen that this sum satisfies the so-called d'Alembert characteristic, since the lowest powers of s and e occurring in the coefficient of a trigonometric term are equal to the respective multiples of $\bar{u}$ and f. All variables in celestial mechanics, which are defined at $e = 0$ and $s = 0$, satisfy this characteristic. Variables like g and h fail to do so because they are not defined at $e = 0$ and $s = 0$, respectively. This affords a very helpful check on the manipulation of the trigonometric series. It should be noted that the d'Alembert characteristic is not preserved in the von Zeipel method (Ref. 4). For instance, S_2 in Kozai's satellite theory (Ref. 2) contains a negative power of e.

Now, according to Eqs. (45), (62), and (72),

$$
\{F_1, S_1\}_s = \frac{1}{2\pi}\int_0^{2\pi}\frac{\gamma^2}{r^2} C_{0,0}\, dl = \gamma^2 \frac{n_1}{G} C_{0,0}.
$$

The standard Hori algorithm, which calls for the elimination of t' from F', would here lead to the elimination of both l and g by a *single* canonical transformation (e.g., Garfinkel, Ref. 7). This would require accounting for J_3 and J_4 also in Eqs. (61). However, it appears more convenient to modify the standard algorithm by proceeding in two steps. The prime purpose of the first step here will be to eliminate the *first-order* short-period terms from F'. Furthermore, F' will be allowed to retain some second-order periodic terms, in addition to the secular terms. In the next step, these second-order periodic terms, along with those arising from J_3 and J_4, will be eliminated by a second canonical transformation. Note that in this approach t' is not eliminated from F' entirely, and F'_0 is therefore not a constant as in Eq. (46).

In order to proceed in two steps, it is convenient to alter the above expression for $\{F_1, S_1\}_s$ to read

$$
\{F_1, S_1\}_s = \frac{\gamma^2}{r^2}\left[C_{0,0} + C_{2,-2}\, s^2 e^2 \cos(2\bar{u} - 2f)\right]. \qquad (73)
$$

Equations (61) will still be valid as long as the relation $A = A_p + A_s$ is preserved. The third-order Poisson brackets in (61) are treated in exactly the same manner, and their secular parts, which are the only parts of interest here, take the same form as $\{F_1, S_1\}_s$ in Eq. (73).

With the Poisson brackets in the form of Eqs. (72) and (73), the last three equations of (61) yield immediately

$$F_2' = -\frac{\gamma^2 G^2}{128 r^2}\left[2 + 12c^2 - 30c^4 + (15 - 54c^2 + 15c^4)\, e^2 - 6\,(1 - 15c^2)\, s^2 e^2 \cos(2\bar{u} - 2f)\right], \tag{74}$$

$$\Delta F_3' = \frac{3\gamma^3 G^2}{128 r^2}\left[(3 + 34c^2 - 69c^4)\, s^2 e^2 - 3\,(1 + 18c^2 - 27c^4)\, s^2 e^2 \cos(2\bar{u} - 2f)\right], \tag{75}$$

$$\begin{aligned} S_2 = -\frac{1}{32}\gamma^2 G\Big[&(5 + 6c^2 - 27c^4)\, e \sin f + \frac{3}{8}(5 - 18c^2 + 5c^4)\, e^2 \sin 2f + \left\{1 - 3c^2 + \frac{3}{2}(1 + c^2)\, e^2\right\} s^2 \sin 2\bar{u} \\ &- (8 - 60c^2)\, s^2 e \sin(2\bar{u} - f) + 4c^2 s^2 e \sin(2\bar{u} + f) + \frac{3}{8}(3 - 13c^2)\, s^2 e^2 \sin(2\bar{u} + 2f) + \frac{3}{16}(2 - e^2)\, s^4 \sin 4\bar{u} \\ &- \frac{9}{2} s^4 e \sin(4\bar{u} - f) - \frac{3}{2} s^4 e \sin(4\bar{u} + f) - \frac{15}{16} s^4 e^2 \sin(4\bar{u} - 2f) + \frac{3}{16} s^4 e^2 \sin(4\bar{u} + 2f)\Big], \end{aligned} \tag{76}$$

$$F_3' = \frac{\gamma^3 G^2}{256 r^2}\left[10 + 30c^2 - 186c^4 + 210c^6 + (7 + 633c^2 - 1815c^4 + 1335c^6)\, e^2 + \frac{3}{4}(57 - 194c^2 + 105c^4)\, s^2 e^2 \cos(2\bar{u} - 2f)\right]. \tag{77}$$

Expressions (63) and (74) for S_1 and F_2' agree with the author's earlier result (Ref. 12), obtained with a much greater effort by means of the von Zeipel method in the Delaunay variables.

VI. Removal of Long-Period Terms From F'

The part of the disturbing function $V_0 - V$ that involves J_3 and J_4 can be written as

$$\Delta F = \frac{\gamma^2 \gamma_4}{r^5}\frac{G^8}{\mu^3}\left[A_0 + A_2 \cos 2u + A_4 \cos 4u\right] + \frac{\gamma^2 \gamma_3}{r^4}\frac{G^6}{\mu^2}\left[A_1 \sin u + A_3 \sin 3u\right], \tag{78}$$

where

$$\left.\begin{aligned} A_0 &= -\frac{3}{64}(3 - 30c^2 + 35c^4), \\ A_1 &= -\frac{3}{8}(1 - 5c^2)\, s, \\ A_2 &= \frac{5}{16}(1 - 7c^2)\, s^2, \\ A_3 &= \frac{5}{8} s^3, \\ A_4 &= -\frac{35}{64} s^4, \end{aligned}\right\} \tag{79}$$

and γ_3 and γ_4 are the zeroth-order, dimensionless quantities

$$\gamma_3 = \frac{J_3}{J_2^2}\frac{a(1 - e^2)}{R}, \qquad \gamma_4 = \frac{J_4}{J_2^2}. \tag{80}$$

With the aid of the expansions

$$\left.\begin{aligned} \cos ku &= \cos k\bar{u} + \frac{9}{16} k \gamma s^2 \{\cos(2 + k)\,\bar{u} \\ &\qquad - \cos(2 - k)\,\bar{u}\} + O(\gamma^2), \\ \sin ku &= \sin k\bar{u} + \frac{9}{16} k \gamma s^2 \{\sin(2 + k)\,\bar{u} \\ &\qquad + \sin(2 - k)\,\bar{u}\} + O(\gamma^2), \end{aligned}\right\} \tag{81}$$

ΔF can be split up into a second-order part $\Delta F_2 = (\Delta F)_{u=\bar{u}}$ and a third-order "overflow" part $\Delta F_3 = (\Delta F - \Delta F_2)_s$. To obtain the additions $\Delta F_2'$ and $\Delta F_3'$ of F_2' and F_3' of the preceding section, it will be necessary to express ΔF_2 and ΔF_3 in terms of the primed variables. Obviously, $\Delta F_2' = \Delta F_2$, while according to Eq. (38),

$$\Delta F_3' = \Delta F_3 + \{\Delta F_2, S_1\}_s ,$$

which yields

$$\Delta F_2' = \frac{\gamma^2 G^2}{r^2}\Bigg[\gamma_4 \sum_{j=\pm(0,2,4)} A_{|j|}\left\{\frac{1}{4}(2+3e^2)\cos j\bar{u} + \frac{3}{8}(4+e^2)\,e\cos(j\bar{u}+f) + \frac{3}{4}e^2\cos(ju+2f) + \frac{1}{8}e^3\cos(j\bar{u}+3f)\right\}$$

$$+ \gamma_3 \sum_{j=\pm(1,3)} \operatorname{sgn}(j)\,A_{|j|}\left\{\frac{1}{4}(2+e^2)\sin j\bar{u} + e\sin(j\bar{u}+f) + \frac{1}{4}e^2\sin(j\bar{u}+2f)\right\}\Bigg], \tag{82}$$

$$\Delta F_3' = -\frac{\gamma^3 G^2}{512 r^2}\Bigg[30\gamma_4\left\{1+3c^2-65c^4+77c^6-2(29-75c^2+99c^4-77c^6)e^2-\frac{1}{8}(119-291c^2+201c^4-77c^6)e^4\right\}$$

$$+15\gamma_4\left\{104-216c^2-56c^4+\frac{1}{4}(129-242c^2-63c^4)e^2\right\}s^2e^2\cos(2\bar{u}-2f) - \frac{15}{8}\gamma_4(59+7c^2)s^4e^4\cos(4\bar{u}-4f)$$

$$-12\gamma_3\{54-80c^2+90c^4+(41-36c^2-5c^4)e^2\}\,se\sin(\bar{u}-f) + 2\gamma_3(89+75c^2)s^3e^3\sin(3\bar{u}-3f)\Bigg], \tag{83}$$

where for simplicity the primes have been dropped from the variables.

The periodic part of the Hamiltonian $F'(x', y')$ is now to be removed by a canonical transformation $x_j', y_j' \rightarrow x_j'', y_j''$. The associated determining function $S^* = \Delta S_2 + S_1^* + S_2^* + \cdots$, where ΔS_2 is short-periodic while the remaining part of S^* is purely long-periodic, and the new Hamiltonian $F''(x'', y'') = F_0'' + F_1'' + \cdots$ are given by

$$\left.\begin{aligned}
F_0'' &= \frac{\mu^2}{2L''^2},\\
F_1'' &= 0,\\
F_2'' &= (F_2' + \Delta F_2')_s,\\
S_1^* + \Delta S_2^* + \Delta S_2 &= \int (F_2' + \Delta F_2')_p\, dt'',\\
F_3'' &= \left[F_3' + \Delta F_3' + \frac{1}{2}\{F_2' + \Delta F_2' + F_2'', S_1^*\}\right]_s,\\
S_2^* - \Delta S_2^* &= \int\left[F_3' + \Delta F_3' + \frac{1}{2}\{F_2' + \Delta F_2' + F_2'', S_1^*\}\right]_p dt''.
\end{aligned}\right\} \tag{84}$$

Here ΔS_2^* serves to transfer the overflow part of S_1^* into S_2^*. Note that the order of a long-period term is reduced by 1 when integrated. Since F_0' is not a constant and $F_1' = 0$, the pseudo-time t'', like t', is the time of the intermediate orbit. The averaging must now be done strictly in accordance with Eqs. (45), if t' is replaced by t'' there. Equations (84) depend on the new variables x_j'', y_j'' only, so no confusion will arise by dropping the primes from these variables in what follows.

We have already obtained suitable expressions for the quantities F_2', $\Delta F_2'$, F_3', and $\Delta F_3'$ in Eqs. (84). The Poisson bracket $\{F_2' + \Delta F_2' + F_2'', S_1^*\}$ is expanded most easily in terms of the Delaunay variables. After some simple but tedious algebra we find

$$F_2'' = -\frac{\gamma^2 G n_1}{128}\left[2 + 12c^2 - 30c^4 + (15 - 54c^2 + 15c^4)\,e^2 + 3\gamma_4\,(3 - 30c^2 + 35c^4)\,(2 + 3e^2)\right], \tag{85}$$

$$\begin{aligned} F_3'' = \frac{\gamma^3 G n_1}{256}\Bigg[& 10 + 30c^2 - 186c^4 + 210c^6 + (7 + 633c^2 - 1815c^4 + 1335c^6)\,e^2 \\ & - 15\gamma_4\left\{1 + 3c^2 - 65c^4 + 77c^6 - 2\,(29 - 75c^2 + 99c^4 - 77c^6)\,e^2 - \frac{1}{8}(119 - 291c^2 + 201c^4 - 77c^6)\,e^4\right\} \\ & + 24\gamma_3^2\left\{1 - 6c^2 + 5c^4 + 2\,(1 - 9c^2 + 10c^4)\,e^2\right\} + \\ & + \frac{3}{8}\left\{2\,(D - D^{(1)} c^2 e^2)\,s^2\,(1 - 5c^2) + D\,(3 - 25c^2 + 30c^4)\,e^2\right\} D s^2 e^2 \Bigg], \end{aligned} \tag{86}$$

$$\begin{aligned} \Delta S_2 = {} & (f - l)\frac{F_2''}{n_1} \\ & + \gamma^2 \gamma_4 G \sum_{j=\pm(0,2,4)}{}' A_{|j|}\left\{\left(\frac{1}{2} + \frac{3}{4}e^2\right)\frac{\sin j\bar{u}}{j} + \left(\frac{3}{2} + \frac{3}{8}e^2\right) e\,\frac{\sin(j\bar{u} + f)}{j+1} + \frac{3}{4}e^2\,\frac{\sin(j\bar{u} + 2f)}{j+2} + \frac{1}{8}e^3\,\frac{\sin(j\bar{u} + 3f)}{j+3}\right\} \\ & - \gamma^2 \gamma_3 G \sum_{j=\pm(1,3)}{}' \operatorname{sgn}(j)\, A_{|j|}\left\{\left(\frac{1}{2} + \frac{1}{4}e^2\right)\frac{\cos j\bar{u}}{j} + e\,\frac{\cos(j\bar{u} + f)}{j+1} + \frac{1}{4}e^2\,\frac{\cos(j\bar{u} + 2f)}{j+2}\right\}, \end{aligned} \tag{87}$$

$$S_1^* = -\frac{\gamma G}{32}\left[D s^2 e^2 \sin(2\bar{u} - 2f) + 16\gamma_3\, s e \cos(\bar{u} - f)\right], \tag{88}$$

$$\begin{aligned} S_2^* = -\frac{\gamma^2 G}{32}\Big[& (D_2 + \bar{D}_2 e^2)\, s^2 e^2 \sin(2\bar{u} - 2f) + D_4 s^4 e^4 \sin(4\bar{u} - 4f) \\ & + \gamma_3\,(D_1 + \bar{D}_1 e^2)\, s e \cos(\bar{u} - f) + \gamma_3 D_3 s^3 e^3 \cos(3\bar{u} - 3f)\Big]. \end{aligned} \tag{89}$$

Note that the primes on the summation signs in Eq. (87) have been added to indicate that any term with a zero divisor must be excluded. Here the D-constants are given by[2]

[2]To avoid some apparent singularities at $s = 0$ in later expressions, a definition different from that used in Ref. 19 is here introduced for the D-constants.

$$
\left.\begin{aligned}
&(1-5c^2)\,D = 1 - 15c^2 + 5\gamma_4(1-7c^2),\\
&(1-5c^2)\,D_2 = \frac{1}{16}(57 - 194c^2 + 105c^4) - 5\gamma_4(13 - 27c^2 - 7c^4) - 2\gamma_3^2(2 - 15c^2)\\
&\qquad\qquad - \frac{1}{16}\{21 + 10c^2 - 95c^4 + 10\gamma_4(3 - 36c^2 + 49c^4)\}\,D,\\
&(1-5c^2)\,\bar{D}_2 = -\frac{5}{32}\gamma_4(129 - 242c^2 - 63c^4) - \frac{1}{32}\{25 - 126c^2 + 45c^4 + 45\gamma_4(1 - 14c^2 + 21c^4)\}\,D,\\
&(1-5c^2)\,D_4 = \frac{1}{128}\{5\gamma_4(59 + 7c^2) + (4 - 30c^2)\,D^2\},\\
&(1-5c^2)\,D_1 = -\frac{1}{2}(153 - 35c^2)\,s^2 - \frac{5}{2}\gamma_4(15 - 168c^2 + 217c^4),\\
&(1-5c^2)\,\bar{D}_1 = -\frac{1}{8}(529 - 1414c^2 + 605c^4) - \frac{5}{8}\gamma_4(37 - 566c^2 + 889c^4) - \frac{1}{4}(9 - 50c^2)\,Ds^2,\\
&(1-5c^2)\,D_3 = \frac{1}{72}\{353 + 255c^2 - 15\gamma_4(1 + 7c^2)\} + \frac{1}{4}(3 - 20c^2)\,D.
\end{aligned}\right\} \quad (90)
$$

In Eq. (86) we have used the notation $D^{(1)} = \partial D/\partial c^2$. We shall later need all the following derivatives:

$$
\left.\begin{aligned}
&(1-5c^2)\,D^{(1)} = -15 - 35\gamma_4 + 5D,\\
&(1-5c^2)\,D_2^{(1)} = -\frac{1}{8}(97 - 105c^2) + 5\gamma_4(27 + 14c^2) + 30\gamma_3^2 - \frac{5}{8}\{1 - 19c^2 - \gamma_4(36 - 98c^2)\}\,D\\
&\qquad\qquad + 5D_2 - \frac{1}{16}\{21 + 10c^2 - 95c^4 + 10\gamma_4(3 - 36c^2 + 49c^4)\}\,D^{(1)},\\
&(1-5c^2)\,\bar{D}_2^{(1)} = \frac{5}{16}\gamma_4(121 + 63c^2) + \frac{9}{16}\{7 - 5c^2 + 35\gamma_4(1 - 3c^2)\}\,D\\
&\qquad\qquad - \frac{1}{32}\{25 - 126c^2 + 45c^4 + 45\gamma_4(1 - 14c^2 + 21c^4)\}\,D^{(1)} + 5\bar{D}_2,\\
&(1-5c^2)\,D_4^{(1)} = \frac{1}{128}\{35\gamma_4 - 30D^2 + (8 - 60c^2)\,DD^{(1)}\} + 5D_4,\\
&(1-5c^2)\,D_1^{(1)} = 94 - 35c^2 + 35\gamma_4(12 - 31c^2) + 5D_1,\\
&(1-5c^2)\,\bar{D}_1^{(1)} = \frac{1}{4}\{707 - 605c^2 + 5\gamma_4(283 - 889c^2) + (59 - 100c^2)\,D - (9 - 50c^2)\,D^{(1)}s^2\} + 5\bar{D}_1,\\
&(1-5c^2)\,D_3^{(1)} = \frac{5}{24}(17 - 7\gamma_4) + \frac{1}{4}(3 - 20c^2)\,D^{(1)} - 5D + 5D_3.
\end{aligned}\right\} \quad (91)
$$

This completes the reduction of the Hamiltonian.

It is now necessary to restore the primes on the variables x_j, y_j, i.e., on Delaunay's or Hill's variables. Primes will also be attached to the symbol γ. The appearance of one or more of the primed symbols in an expression will then indicate which of the three canonical sets x_j, y_j, x_j', y_j', or x_j'', y_j'' should be used to calculate any unprimed quantities in that expression.

VII. Calculation of Perturbations

The new Hamiltonian, $F'' = F_0'' + F_1'' + F_2'' + F_3''$, as given by Eqs. (84), (85), and (86), depends on L'', G'', H'' only. From the canonical equations it follows that L'', G'', H'' are constants, while l'', g'', h'' are linear functions of the time:

$$l'' = l_0 + \dot{l}''t, \qquad g'' = g_0 + \dot{g}''t, \qquad h'' = h_0 + \dot{h}''t \tag{92}$$

Here l_0, g_0, h_0 denote the initial values of l'', g'', h'' whose time-derivatives are given by

$$\begin{aligned}
\dot{l}'' = n_1 &+ \frac{3n_1}{128}\gamma''^2(1-e^2)^{1/2}[8(1-6c^2+5c^4) - 5(5-18c^2+5c^4)e^2 - 15\gamma_4(3-30c^2+35c^4)e^2] \\
&+ \frac{n_1\gamma''^3(1-e^2)^{1/2}}{256}\Bigg[8(2-147c^2+384c^4-255c^6) + 5(7+633c^2- 1815c^4+1335c^6)e^2 \\
&- 15\gamma_4\Bigg\{119-291c^2+201c^4-77c^6-\frac{1}{2}(461-1209c^2+1779c^4-1463c^6)e^2 \\
&- \frac{7}{8}(119-291c^2+201c^4-77c^6)e^4\Bigg\} - 24\gamma_3^2\{1-18c^2+25c^4-10(1-9c^2+10c^4)e^2\} \\
&- \frac{3}{8}D^2s^4\{4-20c^2+(2-30c^2)e^2-7(3-20c^2)e^4\} + \frac{3}{4}(D-D^{(1)}s^2)Ds^2(c^2-5c^4)(4e^2-7e^4)\Bigg],
\end{aligned} \tag{93}$$

$$\begin{aligned}
\dot{g}'' = &-\frac{n_1\gamma''^2}{128}[44-300c^4+(75-378c^2+135c^4)e^2+15\gamma_4\ \{4(3-36c^2+49c^4)+9(1-14c^2+21c^4)e^2\}] \\
&+ \frac{n_1\gamma''^3}{256}\Bigg[4(31+366c^2-1257c^4+1020c^6)+3(21+2609c^2-8673c^4+7035c^6)e^2 \\
&+ 15\gamma_4\Bigg\{105-147c^2+491c^4-609c^6+\frac{1}{2}(1163-3015c^2+2709c^4-1673c^6)e^2 \\
&+ \frac{1}{8}(833-2619c^2+2211c^4-1001c^6)e^4\Bigg\} + 24\gamma_3^2\{11-90c^2+95c^4+2(5-63c^2+90c^4)e^2\} \\
&+ \frac{3}{8}D^2s^4\{4-20c^2+10(3-19c^2)e^2+3(7-60c^2)e^4\} - \frac{3}{4}(D-D^{(1)}s^2)Ds^2c^2\{8-40c^2+5(3-19c^2)e^2\}e^2 \\
&+ \frac{3}{2}\{(D-D^{(1)}s^2)^2(1-5c^2)+8DD^{(1)}s^2\}c^4e^4\Bigg],
\end{aligned} \tag{94}$$

$$\dot{h}'' = \frac{3n_1}{32} c\gamma''^2 [2 - 10c^2 - (9 - 5c^2) e^2 - 5\gamma_4 (3 - 7c^2)\ (2 + 3e^2)]$$

$$- \frac{3n_1}{128} c\gamma''^3 \Big[2 - 48c^2 + 30c^4 + (247 - 1392c^2 + 1425c^4) e^2 + 5\gamma_4 \Big\{ 21 - 34c^2 + 21c^4 - (114 - 150c^2 + 84c^4) e^2$$

$$- \frac{1}{8} (291 - 402c^2 + 231c^4) e^4 \Big\} - 16\gamma_3^2 \{3 - 5c^2 + (9 - 20c^2) e^2\} - \frac{5}{4} D^2 s^4 (e^2 + 2e^4)$$

$$- \frac{1}{2} (D - D^{(1)} s^2) Ds^2 \{1 - 5c^2 + (2 - 15c^2) e^2\} e^2 + \frac{1}{4} \{(D - D^{(1)} s^2)^2 (1 - 5c^2) + 8DD^{(1)} s^2\} c^2 e^4 \Big]. \tag{95}$$

The Delaunay elements L'', G'', H'', l'', g'', h'', or any set of elements derivable from these, will hereafter be referred to as *mean elements*.

We now turn to the derivation of the periodic perturbations. The composite transformation from x_j, y_j to x'_j, y'_j to x''_j, y''_j is provided by the formula

$$f(x, y) = f(x'', y'') + \{f, S + S^*\} + \frac{1}{2} \{\{f, S + S^*\}, S + S^*\} + \frac{1}{2} \{f, \{S, S^*\}\} + O(\epsilon^3), \tag{96}$$

which follows by applying the transformation law (38) twice. Here the determining functions $S(x'', y'')$ and $S^*(x'', y'')$ are given by

$$S = S_1 + S_2, \qquad S^* = \Delta S_2 + S_1^* + S_2^*, \tag{97}$$

where S_1, S_2, ΔS_2, S_1^*, S_2^* are given by Eqs. (63), (76), (87), (88), and (89), provided that x_j, y_j are replaced by x''_j, y''_j in these equations.

The periodic perturbations can be expressed very conveniently in terms of the Hill variables $\dot{r}$, $\bar{G}$, $\bar{H}$, r, $\bar{u}$, $\bar{h}$. We now identify x_1, x_2, x_3, y_1, y_2, y_3 with the Hill variables. If we introduce Eqs. (97) into Eq. (96) and put $f = x_1$, x_2, etc., we find after a slight rearrangement of terms, that

$$\left.\begin{aligned} x_j &= x''_j + \frac{\partial}{\partial y''_j} (S_1 + S_2 + \Delta S_2) + \frac{1}{2} \left\{ \frac{\partial S_1}{\partial y''_j}, S_1 \right\} + \frac{\partial}{\partial y''_j} (S_1^* + S_2^*) + \frac{1}{2} \left\{ \frac{\partial S_1^*}{\partial y''_j}, S_1^* \right\} + \left\{ \frac{\partial S_1}{\partial y''_j}, S_1^* \right\} + O(\epsilon^3), \\ y_j &= y''_j - \frac{\partial}{\partial x''_j} (S_1 + S_2 + \Delta S_2) - \frac{1}{2} \left\{ \frac{\partial S_1}{\partial x''_j}, S_1 \right\} - \frac{\partial}{\partial x''_j} (S_1^* + S_2^*) - \frac{1}{2} \left\{ \frac{\partial S_1^*}{\partial x''_j}, S_1^* \right\} - \left\{ \frac{\partial S_1}{\partial x''_j}, S_1^* \right\} + O(\epsilon^3). \end{aligned}\right\} \tag{98}$$

Since the Hill variable $\bar{G}$ always appears implicitly through the Delaunay variable $G = G(\bar{G}, \bar{H})$, it is convenient to replace $x_2 = \bar{G}$ by G in Eqs. (98) by making use of the differential relations (52). Furthermore, the algorithm becomes a little bit cleaner if, instead of first obtaining $\bar{u}$ and $\bar{h}$, we compute u and h directly as

$$\left.\begin{aligned} u &= \bar{u} + \frac{3}{16} \gamma \{6 - \gamma (1 + 3c^2)\} s^2 \sin 2\bar{u} + \frac{261}{256} \gamma^2 s^4 \sin 4\bar{u}, \\ h &= \bar{h} - \frac{27}{32} \gamma^2 c s^2 \sin 2\bar{u}, \end{aligned}\right\} \tag{99}$$

which agree with Eqs. (25) and (53) except that, for the sake of convenience, the definition of h has been altered in such a way that the right ascension is now given by

$$\phi = \tan^{-1} (c \tan u) + h. \tag{100}$$

Before substituting $\bar{u} = y_2$ and $\bar{h} = y_3$ from Eqs. (98) into (99), it is necessary to express the terms factored by γ and γ^2 in Eq. (99) in terms of the mean elements. Through terms of the second order, this simply amounts to a mechanical replacement of x_j, y_j by x_j'', y_j'', provided that the term $9/8\,\{\gamma'' s^2 \sin 2\bar{u}'', S_1 + S_1^*\}$ is added to the resulting expression for u.

Some of the partial derivatives that occur in Eqs. (98) were obtained earlier, and the remaining ones may be derived by applying the relations (65–69). It is also necessary to differentiate the factor $(f - l)$, which appears in the expression for ΔS_2, with respect to $\dot{r}$, r, and G. These derivatives are given in the aforementioned paper by Izsak (Ref. 18).

After having carried out the simple but very tedious algebra described above, we find that the quantities $\dot{r}, r, G, u, h$ may be expressed by trigonometric series whose constant coefficients depend on L'', G'', H'', while the arguments are multiples of $\bar{u}''$ and f. The following expressions include all the first- and second-order periodic perturbations arising from J_2, J_3, and J_4. Note that these expressions, except those for u and h that do not satisfy the d'Alembert characteristic in s although their sum does, satisfy the d'Alembert characteristic both in s and in e:

$$\dot{r} = \dot{r}'' - \frac{\gamma'' G''^3}{2\mu r''^2}\left[s^2 \sin 2\bar{u}'' - \frac{1}{8} D s^2 e \sin(2\bar{u}'' - f) - \gamma_3 s \cos \bar{u}''\right]$$
$$+ \frac{\gamma''^2 G''^3}{32\mu r''^2}\left[\Sigma s^j e^{|k|}\{\dot{r}^s_{j,k} \sin(j\bar{u}'' + kf) + \dot{r}^c_{j,k}\cos(j\bar{u}'' + kf)\} + \alpha_1 r''^2 a^{-2}(1 - e^2)^{-3/2} e \sin f + \alpha_2 (f - l'') e \cos f\right], \tag{101}$$

$$r = r'' + \frac{\gamma'' G''^2}{4\mu}\left[1 - 3c^2 + s^2 \cos 2\bar{u}'' - \frac{1}{4} D s^2 e \cos(2\bar{u}'' - f) + 2\gamma_3 s \sin \bar{u}''\right]$$
$$+ \frac{\gamma''^2 G''^2}{32\mu}\left[\Sigma s^j e^{|k|}\{r^c_{j,k}\cos(j\bar{u}'' + kf) + r^s_{j,k}\sin(j\bar{u}'' + kf)\} - 2\alpha_1 r'' a^{-1}(1 - e^2)^{-1/2} + \alpha_2 (f - l'') e \sin f\right], \tag{102}$$

$$G = G'' + \frac{\gamma'' G''}{4}\left[3s^2 e \cos(2\bar{u}'' - f) + s^2 e \cos(2\bar{u}'' + f) - \frac{1}{4} D s^2 e^2 \cos(2\bar{u}'' - 2f) + 2\gamma_3 s e \sin(\bar{u}'' - f)\right]$$
$$- \frac{\gamma''^2 G''}{32}\Sigma s^j e^{|k|}\{G^c_{j,k}\cos(j\bar{u}'' + kf) + G^s_{j,k}\sin(j\bar{u}'' + kf)\}, \tag{103}$$

$$u = \bar{u}'' - \frac{1}{4}\gamma''\Big[(2 - 12c^2) e \sin f - \frac{1}{8}(4 + De^2) s^2 \sin 2\bar{u}'' - \left(2 - 5c^2 + \frac{1}{2} D s^2\right) e \sin(2\bar{u}'' - f) + c^2 e \sin(2\bar{u}'' + f)$$
$$- \frac{1}{4}(D - D^{(1)} s^2) c^2 e^2 \sin(2\bar{u}'' - 2f) - 4\gamma_3 s \cos \bar{u}'' - \gamma_3 (1 + c^2) s^{-1} e \cos(\bar{u}'' - f) - \gamma_3 s e \cos(\bar{u}'' + f)\Big]$$
$$- \frac{\gamma''^2}{32}\left[\Sigma e^{|k|}\{u^s_{j,k}\sin(j\bar{u}'' + kf) + u^c_{j,k}\cos(j\bar{u}'' + kf)\} + \alpha_3 (f - l'')\right], \tag{104}$$

$$h = \bar{h}'' - \frac{\gamma'' c}{4}\left[6e \sin f - 3e \sin(2\bar{u}'' - f) - e \sin(2\bar{u}'' + f) + \frac{1}{4}(D - D^{(1)} s^2) e^2 \sin(2\bar{u}'' - 2f) + 2\gamma_3 s^{-1} e \cos(\bar{u}'' - f)\right]$$
$$- \frac{\gamma''^2 c}{32}\left[\Sigma e^{|k|}\{h^s_{j,k}\sin(j\bar{u}'' + kf) + h^c_{j,k}\cos(j\bar{u}'' + kf)\} + \alpha_4 (f - l'')\right]. \tag{105}$$

In Eqs. (101–105) the constant coefficients, including the alphas and the only non-zero doubly subscripted constants, are given as follows:

$$
\left.
\begin{aligned}
\alpha_1 &= -\frac{1}{2}(1 + 6c^2 - 15c^4) - \frac{1}{4}(15 - 54c^2 + 15c^4)\,e^2 - \frac{3}{4}\gamma_4(3 - 30c^2 + 35c^4)(2 + 3e^2),\\
\alpha_2 &= \frac{3}{2}(5 - 18c^2 + 5c^4) + \frac{9}{2}\gamma_4(3 - 30c^2 + 35c^4),\\
\alpha_3 &= 11 - 75c^4 + \frac{3}{4}(25 - 126c^2 + 45c^4)\,e^2 + 15\gamma_4(3 - 36c^2 + 49c^4) + \frac{135}{4}\gamma_4(1 - 14c^2 + 21c^4)\,e^2,\\
\alpha_4 &= -6(1 - 5c^2) + 3(9 - 5c^2)\,e^2 + 15\gamma_4(3 - 7c^2)(2 + 3e^2),\\
\dot{r}^{s}_{0,1} &= \frac{1}{4}(39 - 134c^2 + 71c^4) + \frac{9}{4}\gamma_4(3 - 30c^2 + 35c^4) - 2\gamma_3^2(1 + c^2)\\
&\quad + \frac{1}{32}\{(32 + 2D)\,s^2 + (1 - 5c^2)\,De^2 + 4D^{(1)}c^2s^2e^2\}\,Ds^2 + \alpha_1\{1 + (1 - e^2)^{1/2}\}^{-1},\\
\dot{r}^{s}_{0,2} &= \frac{3}{2}\gamma_4(3 - 30c^2 + 35c^4) - (1 - 3c^2)\,Ds^2 - D^{(1)}s^4c^2,\\
\dot{r}^{s}_{2,0} &= -4(1 - 13c^2) - 10\gamma_4(1 - 7c^2)(2 + e^2) - 8\gamma_3^2,\\
\dot{r}^{s}_{2,-1} &= -\frac{1}{2}(1 - 39c^2) - \frac{15}{2}\gamma_4(1 - 7c^2) - \gamma_3^2 + 2D_2 + 3\bar{D}_2e^2,\\
\dot{r}^{s}_{2,1} &= \frac{3}{4}(13 - 51c^2) - \frac{45}{4}\gamma_4(1 - 7c^2) - 3\gamma_3^2,\\
\dot{r}^{s}_{2,2} &= -2\gamma_4(1 - 7c^2),\\
\dot{r}^{s}_{2,-3} &= \bar{D}_2,\\
\dot{r}^{s}_{4,0} &= 4 + 7\gamma_4(2 + e^2) - De^2,\\
\dot{r}^{s}_{4,-1} &= -\frac{3}{16}(11 - 105\gamma_4) - 3D + \frac{3}{64}D^2e^2,\\
\dot{r}^{s}_{4,1} &= \frac{35}{16}(1 + 5\gamma_4),\\
\dot{r}^{s}_{4,-2} &= \frac{35}{4}\gamma_4 - D + \frac{1}{8}D^2 + D^{(1)}c^2,\\
\dot{r}^{s}_{4,2} &= \frac{9}{4}\gamma_4,
\end{aligned}
\right\}
\qquad (106)
$$

$$
\left.
\begin{aligned}
\dot{r}^{s}_{4,-3} &= \frac{1}{64} D^2 + 4D_4, \\
\dot{r}^{c}_{1,0} &= -\frac{1}{2}\gamma_3 \{4(7-19c^2) + Ds^2 - (2D - D^{(1)}s^2)c^2e^2 - 2D_1 - 4\bar{D}_1 e^2\}, \\
\dot{r}^{c}_{1,1} &= -4\gamma_3 (1-5c^2), \\
\dot{r}^{c}_{1,-2} &= -\frac{1}{8}\gamma_3 \{D(1+3c^2) - 2D^{(1)}s^2c^2 - 8\bar{D}_1\}, \\
\dot{r}^{c}_{3,0} &= -\frac{3}{4}\gamma_3 (12 - De^2), \\
\dot{r}^{c}_{3,-1} &= \frac{2}{3}\gamma_3 (2+3D), \\
\dot{r}^{c}_{3,1} &= -\frac{8}{3}\gamma_3, \\
\dot{r}^{c}_{3,-2} &= \frac{1}{8}\gamma_3 \{3D + 2D^{(1)}c^2 + 24D_3\}, \\
r^{c}_{0,0} &= 9 - 2c^2 - 23c^4 + \frac{9}{4}\gamma_4 (3 - 30c^2 + 35c^4)(2+e^2) + 4\gamma_3^2 s^2 - \frac{1}{16}(4-D)Ds^4e^2, \\
r^{c}_{0,1} &= -\frac{1}{4}(9 - 26c^2 + 41c^4) + \frac{9}{4}\gamma_4 (3 - 30c^2 + 35c^4) + 2\gamma_3^2 (1+c^2) \\
&\quad - \frac{1}{32}\{(32+2D)s^2 + D(1-5c^2)e^2 + 4D^{(1)}c^2s^2e^2\}\, Ds^2 - \alpha_1 \{1 + (1-e^2)^{1/2}\}^{-1}, \\
r^{c}_{0,2} &= -\frac{3}{4}\gamma_4 (3 - 30c^2 + 35c^4) + \frac{1}{2}Ds^2(1-3c^2) + \frac{1}{2}D^{(1)}c^2s^4, \\
r^{c}_{2,0} &= 8(1-7c^2) + 5\gamma_4 (1-7c^2)(2+e^2) + 4\gamma_3^2, \\
r^{c}_{2,-1} &= -\frac{1}{2}(5 - 51c^2) - \frac{15}{2}\gamma_4 (1-7c^2) + \gamma_3^2 - 2D_2 - 3\bar{D}_2 e^2, \\
r^{c}_{2,1} &= -\frac{1}{4}(13 - 51c^2) + \frac{15}{4}\gamma_4 (1-7c^2) + \gamma_3^2, \\
r^{c}_{2,-2} &= -\frac{15}{2}\gamma_4 (1-7c^2) + (1-6c^2)D, \\
r^{c}_{2,2} &= \frac{1}{2}\gamma_4 (1-7c^2),
\end{aligned}
\right\} \quad (106\ \text{contd})
$$

$$
\left.\begin{aligned}
r^{c}_{2,-3} &= \bar{D}_2,\\
r^{c}_{4,0} &= -1 - \frac{7}{4}\gamma_4(2+e^2) + \frac{1}{4}De^2,\\
r^{c}_{4,-1} &= \frac{1}{16}(11 - 105\gamma_4) + D - \frac{1}{64}D^2e^2,\\
r^{c}_{4,1} &= -\frac{7}{16}(1+5\gamma_4),\\
r^{c}_{4,-2} &= -\frac{35}{8}\gamma_4 + \frac{1}{2}D - \frac{1}{16}D^2 - \frac{1}{2}D^{(1)}c^2,\\
r^{c}_{4,2} &= -\frac{3}{8}\gamma_4,\\
r^{c}_{4,-3} &= -\frac{1}{64}D^2 - 4D_4,\\
r^{s}_{1,0} &= -\frac{1}{2}\gamma_3\{4 + 44c^2 + Ds^2 - (2D - D^{(1)}s^2)c^2e^2 - 2D_1 - 4\bar{D}_1e^2\},\\
r^{s}_{1,-1} &= \gamma_3(2 - 10c^2 + s^2D),\\
r^{s}_{1,1} &= -\gamma_3(2 - 10c^2),\\
r^{s}_{1,-2} &= \frac{1}{8}\gamma_3\{(1+3c^2)D - 2D^{(1)}c^2s^2 - 8\bar{D}_1\},\\
r^{s}_{3,0} &= -\frac{1}{4}\gamma_3(12 - De^2),\\
r^{s}_{3,-1} &= \frac{1}{3}\gamma_3(2+3D),\\
r^{s}_{3,1} &= -\frac{2}{3}\gamma_3,\\
r^{s}_{3,-2} &= \frac{1}{8}\gamma_3(3D + 2D^{(1)}c^2 + 24D_3),\\
G^{c}_{0,0} &= 4s^4 + \left(14 - 34c^2 + \frac{3}{2}Ds^2\right)s^2e^2 + 4\gamma_3^2(s^2 - c^2e^2) + \frac{1}{16}D^2s^4(2e^2+e^4) + \frac{1}{8}(D^{(1)}s^2 - D)Dc^2s^2e^4,\\
G^{c}_{0,1} &= -\frac{3}{8}\{64s^2 + 4Ds^2 + (3 - 11c^2)De^2 + 4D^{(1)}c^2s^2e^2\}s^2,
\end{aligned}\right\}\text{(106 contd)}
$$

$$
\left.
\begin{aligned}
G^c_{0,2} &= \frac{3}{2}\{4 - 12c^2 + Ds^2\}\, s^2, \\
G^c_{0,3} &= -\frac{1}{8}\{(3 - 11c^2)\, D + 4D^{(1)}c^2s^2\}\, s^2, \\
G^c_{2,0} &= \{3 - 9c^2 - 5\gamma_4 (1 - 7c^2)\}\,(2 + 3e^2), \\
G^c_{2,-1} &= 6\,(1 + 3c^2) - \frac{15}{2}\gamma_4 (1 - 7c^2)\,(4 + e^2), \\
G^c_{2,1} &= 2\,(5 - 17c^2) - \frac{5}{2}\gamma_4 (1 - 7c^2)\,(4 + e^2), \\
G^c_{2,-2} &= -\frac{1}{2}(13 - 111c^2) - \frac{15}{2}\gamma_4 (1 - 7c^2) + 2D_2 + 2\bar{D}_2 e^2, \\
G^c_{2,2} &= \frac{3}{4}(7 - 33c^2) - \frac{15}{4}\gamma_4 (1 - 7c^2), \\
G^c_{2,-3} &= \frac{5}{2}\gamma_4 (1 - 7c^2), \\
G^c_{2,3} &= -\frac{1}{2}\gamma_4 (1 - 7c^2), \\
G^c_{4,0} &= \frac{3}{4}(2 - e^2) + \frac{35}{4}\gamma_4 (2 + 3e^2) - \frac{3}{2} De^2, \\
G^c_{4,-1} &= -18 + \frac{35}{4}\gamma_4 (4 + e^2) - \frac{1}{2} D\,(1 + 2e^2) + \frac{1}{2} D^{(1)} c^2 e^2, \\
G^c_{4,1} &= -6 + \frac{21}{4}\gamma_4 (4 + e^2) - \frac{3}{8} De^2, \\
G^c_{4,-2} &= -\frac{3}{4}(5 - 35\gamma_4) - \frac{3}{2} D, \\
G^c_{4,2} &= \frac{1}{4}(3 + 35\gamma_4), \\
G^c_{4,-3} &= \frac{35}{4}\gamma_4 - \frac{9}{8} D + \frac{3}{2} D^{(1)} c^2, \\
G^c_{4,3} &= \frac{5}{4}\gamma_4, \\
G^c_{4,-4} &= 4D_4, \\
G^s_{1,0} &= -48\gamma_3 c^2,
\end{aligned}
\right\}
\qquad (106\ \text{contd})
$$

$$\left.\begin{aligned}
G^{s}_{1,-1} &= \gamma_3 \left\{24(1-3c^2) + \frac{3}{8}D(4s^2 + e^2 - 5c^2e^2) + \frac{3}{4}D^{(1)}c^2s^2e^2 - D_1 - \bar{D}_1e^2\right\},\\
G^{s}_{1,1} &= 6\gamma_3(3-7c^2),\\
G^{s}_{1,-2} &= 12\gamma_3c^2,\\
G^{s}_{1,2} &= 4\gamma_3c^2,\\
G^{s}_{3,0} &= -16\gamma_3,\\
G^{s}_{3,-1} &= -18\gamma_3,\\
G^{s}_{3,1} &= -3\gamma_3,\\
G^{s}_{3,-3} &= -\frac{1}{8}\gamma_3(D + 2D^{(1)}c^2 + 24D_3),\\
u^{s}_{0,1} &= \frac{1}{2}(89 - 146c^2 - 303c^4) + \frac{45}{2}\gamma_4(3 - 42c^2 + 63c^4) + \frac{15}{4}\gamma_4(3 - 48c^2 + 77c^4)e^2 - 4\gamma_3^2(1-3c^2)\\
&\quad - \frac{1}{8}Ds^2\left\{4 - 16c^2 + Ds^2 + (16 - 57c^2)e^2 + \frac{3}{2}Ds^2e^2\right\}\\
&\quad + \frac{1}{4}(4 - 10c^2 + Ds^2)(D - D^{(1)}s^2)c^2e^2 - 2\alpha_1\{1 + (1-e^2)^{1/2}\}^{-1},\\
u^{s}_{0,2} &= \frac{1}{2}(7 - 57c^2 + 14c^4) + \frac{3}{2}\gamma_4(3 - 75c^2 + 140c^4) + 2\gamma_3^2c^2 - \frac{1}{4}Ds^2\left\{2 + 3c^2 + \frac{1}{8}Ds^2(1+e^2)\right\}\\
&\quad + \frac{1}{4}(D - D^{(1)}s^2)c^2s^2\left(1 + \frac{1}{4}De^2\right) - \frac{1}{2}\alpha_1\{1 + (1-e^2)^{1/2}\}^{-1},\\
u^{s}_{0,3} &= -\frac{1}{4}\gamma_4(3 - 35c^4) + \frac{1}{8}D(9c^2 - 13c^4) + \frac{1}{2}D^{(1)}c^4s^2,\\
u^{s}_{2,0} &= 3 - 8c^2 + 17c^4 + \frac{3}{8}(5 + 128c^2 - 85c^4)e^2 + 5\gamma_4(1 + 8c^2 - 21c^4)\\
&\quad - \frac{5}{8}\gamma_4(13 - 296c^2 + 427c^4)e^2 + 2\gamma_3^2s^2(5 + e^2) + \left\{\frac{1}{4}D(7 - 39c^2) - D_2 - \frac{3}{2}\bar{D}_2e^2\right\}s^2e^2,\\
u^{s}_{2,-1} &= -47 + 600c^2 - 697c^4 - 5\gamma_4(13 - 152c^2 + 175c^4) - 5\gamma_4(2 - 28c^2 + 35c^4)e^2\\
&\quad + 10\gamma_3^2s^2 + Ds^2\{4 - 21c^2 + 2(1 - 9c^2)e^2\} - 4D_2s^2 - 6\bar{D}_2s^2e^2,\\
u^{s}_{2,1} &= \frac{1}{2}(9 - 16c^2 - 25c^4) + \frac{5}{2}\gamma_4(1 + 24c^2 - 49c^4) + \gamma_4(1 + 12c^2 - 28c^4)e^2\\
&\quad + 6\gamma_3^2s^2 + \frac{1}{4}Ds^2(1 - 6c^2)e^2,
\end{aligned}\right\} \quad (106\ \text{contd})$$

$$
\left.
\begin{aligned}
u^{s}_{2,-2} &= \frac{1}{4}(15 - 244c^2 + 117c^4) + \frac{15}{4}\gamma_4(1 - 7c^2)s^2 + 8\gamma_3^2(1 - 5c^2)^{-1}(2 - 5c^2)s^2 + \frac{1}{2}D(2 - 27c^2 + 33c^4) \\
&\quad + \frac{1}{2}D^{(1)}s^2(13c^2 - 21c^4) + 2D_2(2 - 3c^2) + 2D_2^{(1)}c^2s^2 + \bar{D}_2(1 - 3c^2)e^2 \\
&\quad + 2\bar{D}_2^{(1)}c^2s^2e^2 + \gamma_3^2(1 + 4c^2 - c^4)s^{-2}, \\
u^{s}_{2,2} &= \frac{1}{4}(1 - 4c^2 + 81c^4) + \frac{1}{4}\gamma_4(1 + 112c^2 - 203c^4) + \gamma_3^2 s^2, \\
u^{s}_{2,-3} &= 5\gamma_4(1 - 12c^2 + 14c^4) - \frac{1}{4}Ds^2(5 - 54c^2) - 2\bar{D}_2 s^2, \\
u^{s}_{2,3} &= \gamma_4(4c^2 - 7c^4), \\
u^{s}_{2,-4} &= -\frac{1}{2}\bar{D}_2 s^2, \\
u^{s}_{4,0} &= \frac{1}{2}(2 - 5c^2)s^2 + \frac{1}{4}(2 - 37c^2 + 11c^4)e^2 + \frac{7}{8}\gamma_4(3 - 23c^2)s^2 + \frac{7}{4}\gamma_4(2 - 17c^2)s^2e^2 \\
&\quad - \frac{1}{64}\{8(1 - 13c^2) + Ds^2e^2\}Ds^2e^2, \\
u^{s}_{4,-1} &= \frac{9}{8}(1 + 15c^2)s^2 + \frac{7}{8}\gamma_4\{7 - 47c^2 + (1 - 11c^2)e^2\}s^2 \\
&\quad - \frac{1}{32}Ds^2\{16(1 - 2c^2) + 8(1 - 8c^2)e^2 + 3Ds^2e^2\} + \frac{1}{2}(D - D^{(1)}s^2)c^4e^2, \\
u^{s}_{4,1} &= \frac{1}{8}(5 + 43c^2)s^2 + \frac{1}{8}\gamma_4\{21 - 189c^2 + (5 - 47c^2)e^2\}s^2 + \frac{3}{8}Dc^2s^2e^2, \\
u^{s}_{4,-2} &= -\frac{1}{32}(13 + 286c^2 - 11c^4) + \frac{35}{32}\gamma_4(5 - 29c^2)s^2 - \frac{3}{2}Ds^2(1 - 2c^2) - \frac{1}{32}D^2s^4(5 + e^2) \\
&\quad + \frac{1}{4}D^{(1)}c^2s^4 - 2D_4s^4e^2, \\
u^{s}_{4,2} &= \frac{1}{32}(3 - 78c^2 + 43c^4) + \frac{1}{32}\gamma_4(31 - 311c^2)s^2, \\
u^{s}_{4,-3} &= \frac{35}{8}\gamma_4(1 - 3c^2)s^2 - \frac{5}{32}Ds^2(8 - 36c^2 + Ds^2) - \frac{1}{2}(D - D^{(1)}s^2)(2c^2 - 5c^4) - 8D_4s^4, \\
u^{s}_{4,3} &= \frac{1}{8}\gamma_4(1 - 11c^2)s^2,
\end{aligned}
\right\} \quad \text{(106 contd)}
$$

$$u^s_{4,-4} = \frac{1}{32}(D^{(1)}s^2 - D)c^2\{3Ds^2 - 2(D^{(1)}s^2 - D)c^2\} + \frac{1}{2}DD^{(1)}(1-5c^2)^{-1}c^4s^2 + D_4(1-5c^2)s^2 + 2D_4^{(1)}c^2s^4,$$

$$u^c_{1,0} = \gamma_3\{Ds^2 + D(1-3c^2)e^2 + D^{(1)}c^2s^2e^2 - 2D_1 - 4\bar{D}_1e^2\}s - \gamma_3\{4(3-50c^2+59c^4) - 2(6-35c^2+23c^4)e^2\}s^{-1},$$

$$u^c_{1,-1} = -\gamma_3\left\{\frac{1}{4}D(1-3c^2) + D^{(1)}c^2s^2 + \frac{1}{8}D^{(1)}c^2(1-5c^2)^{-1}(9-24c^2+55c^4)e^2 - \frac{7}{2}D_1 - \frac{1}{2}\bar{D}_1e^2 - 2(D_1^{(1)} + \bar{D}_1^{(1)}e^2)c^2\right\}s + \gamma_3\left\{8 - 132c^2 + 156c^4 - \frac{1}{16}D(3-12c^2+17c^4)e^2 - D_1c^2 - \bar{D}_1c^2e^2\right\}s^{-1},$$

$$u^c_{1,1} = \gamma_3\left\{\frac{1}{4}Ds^2 + \frac{1}{8}D(1-5c^2)e^2 + \frac{1}{4}D^{(1)}c^2s^2e^2 - \frac{1}{2}D_1 - \bar{D}_1e^2\right\}s + 6\gamma_3(2+c^2-7c^4)s^{-1},$$

$$u^c_{1,-2} = -\gamma_3\left\{\frac{1}{4}D(3-7c^2) + \frac{1}{2}D^{(1)}c^2s^2 + 2\bar{D}_1\right\}s - \gamma_3(3-38c^2+23c^4)s^{-1},$$

$$u^c_{1,2} = \gamma_3(3-13c^2)s,$$

$$u^c_{1,-3} = -\gamma_3\left\{\frac{1}{16}D(1-5c^2) + \frac{1}{8}D^{(1)}c^2s^2 + \frac{1}{2}\bar{D}_1\right\}s,$$

$$u^c_{3,0} = -\gamma_3\left\{\frac{2}{3}(1+23c^2) + \frac{3}{2}Ds^2e^2\right\}s - \frac{2}{3}\gamma_3(2-13c^2+5c^4)e^2s^{-1},$$

$$u^c_{3,-1} = -\gamma_3\left\{\frac{1}{6}(43+65c^2) + \left(\frac{15}{6}D + \frac{9}{16}De^2 + \frac{1}{8}D^{(1)}c^2e^2 + \frac{3}{2}D_3e^2\right)s^2\right\}s,$$

$$u^c_{3,1} = \gamma_3\left\{\frac{1}{6}(5-23c^2) - \frac{1}{4}Ds^2e^2\right\}s,$$

$$u^c_{3,-2} = -\gamma_3\left\{\frac{11}{4}D + \frac{1}{2}D^{(1)}c^2 + 6D_3\right\}s^3 - \frac{1}{3}\gamma_3(17-88c^2+35c^4)s^{-1},$$

$$u^c_{3,2} = \frac{1}{3}\gamma_3s^3,$$

$$u^c_{3,-3} = -\gamma_3\left\{\frac{1}{16}Ds^2 - \frac{1}{8}D^{(1)}c^2(9-16c^2+15c^4)(1-5c^2)^{-1} - \frac{1}{2}D_3(1-7c^2) - 2D_3^{(1)}c^2s^2\right\}s - \frac{1}{8}\gamma_3D(13c^2-5c^4)s^{-1},$$

$$h^s_{0,1} = 12(1+3c^2) + \frac{45}{2}\gamma_4(3-7c^2)(4+e^2) - \frac{3}{2}Ds^2 - \frac{3}{8}D(9-13c^2)e^2 - \frac{3}{2}D^{(1)}c^2s^2e^2,$$

$$h^s_{0,2} = \frac{3}{2}(9-5c^2) + \frac{45}{2}\gamma_4(3-7c^2) + \frac{3}{2}Ds^2, \qquad (106\ \text{contd})$$

$$
\left.
\begin{aligned}
h^s_{0,3} &= \frac{5}{2}\gamma_4(3-7c^2) - \frac{1}{8}D(7-11c^2) - \frac{1}{2}D^{(1)}c^2s^2,\\
h^s_{2,0} &= -3(7-3c^2) - 6(4-c^2)e^2 - 10\gamma_4(4-7c^2)(2+3e^2) + \frac{9}{2}Ds^2e^2,\\
h^s_{2,-1} &= -144(1-2c^2) - 15\gamma_4(4-7c^2)(4+e^2) + \frac{3}{2}Ds^2(6+5e^2),\\
h^s_{2,1} &= -8(3-5c^2) - 5\gamma_4(4-7c^2)(4+e^2) + \frac{3}{4}Ds^2e^2,\\
h^s_{2,-2} &= 28 + D(5-9c^2) - D^{(1)}s^2(2-6c^2) + 2D_2 - 2D_2^{(1)}s^2 + 2\bar{D}_2e^2 - 2\bar{D}_2^{(1)}s^2e^2 - 4\gamma_3^2s^{-2},\\
h^s_{2,2} &= -\frac{3}{2}\{13c^2 + 5\gamma_4(4-7c^2)\},\\
h^s_{2,-3} &= 5\gamma_4(4-7c^2) - \frac{21}{4}Ds^2,\\
h^s_{2,3} &= -\gamma_4(4-7c^2),\\
h^s_{4,0} &= \frac{3}{2}s^2 + \frac{3}{4}(7+c^2)e^2 + \frac{35}{4}\gamma_4s^2(2+3e^2) - \frac{3}{2}Ds^2e^2,\\
h^s_{4,-1} &= -18s^2 + \frac{35}{4}\gamma_4s^2(4+e^2) - \frac{1}{2}Ds^2 - \frac{1}{2}D(3-2c^2)e^2 + \frac{1}{2}D^{(1)}c^2s^2e^2,\\
h^s_{4,1} &= -6s^2 + \frac{21}{4}\gamma_4s^2(4+e^2) - \frac{3}{8}Ds^2e^2,\\
h^s_{4,-2} &= \frac{3}{4}(7+5c^2) + \frac{105}{4}\gamma_4s^2 - \frac{3}{2}Ds^2,\\
h^s_{4,2} &= \frac{1}{4}(7-3c^2) + \frac{35}{4}\gamma_4s^2,\\
h^s_{4,-3} &= \frac{35}{4}\gamma_4s^2 - \frac{3}{8}D(7-3c^2) + \frac{3}{2}D^{(1)}c^2s^2,\\
h^s_{4,3} &= \frac{5}{4}\gamma_4s^2,\\
h^s_{4,-4} &= -\frac{1}{16}(D^{(1)}s^2-D)\{Ds^2 - (D^{(1)}s^2-D)c^2\} - \frac{1}{2}DD^{(1)}(1-5c^2)^{-1}c^2s^2 + 4D_4s^2 - 2D_4^{(1)}s^4,\\
h^c_{1,0} &= -12\gamma_3(4-8c^2-e^2)s^{-1},
\end{aligned}
\right\}
\quad \text{(106 contd)}
$$

$$
\left.
\begin{aligned}
h^c_{1,-1} &= -\gamma_3\left\{\frac{1}{2}D - D^{(1)}s^2 - \frac{1}{4}D^{(1)}(3-3c^2+20c^4)(1-5c^2)^{-1}e^2 + 2D_1^{(1)} + 2\bar{D}_1^{(1)}e^2\right\}s \\
&\quad + \gamma_3\left\{40 - 72c^2 - \frac{1}{8}D(1-5c^2)e^2 + D_1 + \bar{D}_1e^2\right\}s^{-1}, \\
h^c_{1,1} &= -6\gamma_3(7-11c^2)s^{-1}, \qquad h^c_{1,-2} = -12\gamma_3 s^{-1}, \qquad h^c_{3,0} = 16\gamma_3 s - 4\gamma_3 e^2 s^{-1}, \\
h^c_{3,-1} &= 18\gamma_3 s, \qquad h^c_{3,1} = 3\gamma_3 s, \qquad h^c_{3,-2} = -12\gamma_3 s^{-1}, \\
h^c_{3,-3} &= -\gamma_3\left\{\frac{1}{4}D^{(1)}(3+c^2)(1-5c^2)^{-1} + 2D_3^{(1)}s^2 - 3D_3\right\}s + \frac{1}{8}\gamma_3 D(9-c^2)s^{-1}.
\end{aligned}
\right\} \quad (106\ \text{contd})
$$

It is of considerable practical importance that the preceding coefficients are constants that need be evaluated only once for each orbit. Note that if the short-period terms are separated from the long-period terms in the manner of earlier satellite theories, not only will it be necessary to recompute the coefficients of the former terms for each new instant, but the total number of trigonometric terms is also thereby increased. For low-eccentric orbits, a great many of the terms factored by powers of e will become negligible. As a matter of fact, if $e = 0$ the number of second-order terms in, for instance, u is reduced from 32 to 4.

It is seen that J_3 introduces singularities at $s = 0$ in the expressions for u and h or, more specifically, in the coefficients associated with the arguments $\bar{u}''$, $\bar{u}'' - f$, $\bar{u}'' + f$, $\bar{u}'' - 2f$, $2\bar{u}'' - 2f$, $3\bar{u}''$, $3\bar{u}'' - 2f$, $3\bar{u}'' - 3f$. If s is of the order of γ'', some of these coefficients may reach the order of unity. This means that Eqs. (104) and (105) can be used, without loss of accuracy, provided that $s \geqq \gamma''$. But if the latter condition is not met, the use of the individual expressions for u and h should be avoided by calculating directly the perturbations in the position coordinates θ and ϕ in the following manner.

From Eqs. (25), and by expanding $c = (1 - s^2)^{1/2}$ in powers of s^2 in Eq. (100), we get $\sin\theta = s\sin u$ and $\phi - u - h = -1/4\, s^2 \sin 2u + O(s^4)$. For the computation of θ we also need to add the equation $\sin i \cos u = s\cos u$. The right-hand sides of these equations are next expanded by Taylor's theorem to the second order about the values G'' and $\bar{u}''$ of G and u. By introducing $G - G''$ and $u - \bar{u}''$ from Eqs. (103) and (104) into these expansions, taking advantage of the simplifying condition $s < \gamma''$, we find to the second order in γ'',

$$
\left.
\begin{aligned}
\sin\theta &= s\sin\bar{u}'' + \frac{1}{4}\gamma''\left[2\gamma_3 e\cos f - 8se\sin(\bar{u}''-f) + 4se\sin(\bar{u}''+f) + \frac{1}{4}Dse^2\sin(\bar{u}''-2f)\right] \\
&\quad + \frac{1}{32}\gamma''^2\gamma_3\left[8(6-e^2) - \left(8 - D_1 - \frac{1}{2}De^2 - \bar{D}_1e^2\right)e\cos f + 8e^2\cos 2f\right], \\
\phi &= (u+h) - \frac{1}{4}s^2\sin 2\bar{u}'' - \frac{1}{4}\gamma''\gamma_3 se\cos(\bar{u}''+f) + \frac{1}{16}\gamma''^2\gamma_3^2e^2\sin 2f, \\
\sin i\cos u &= s\cos\bar{u}'' - \frac{1}{2}\gamma''\left[\gamma_3 e\sin f + se\cos(\bar{u}''-f) - 3se\cos(\bar{u}''+f) + \frac{1}{8}Dse^2\cos(\bar{u}''-2f)\right] \\
&\quad + \frac{1}{32}\gamma''^2\gamma_3\left[\left(56 - D_1 - \frac{1}{2}De^2 - \bar{D}_1e^2\right)e\sin f - 8e^2\sin 2f\right],
\end{aligned}
\right\} \quad (107)
$$

where u and h may be computed from Eqs. (104) and (105) after dropping the singular parts (those factored by s^{-1} and s^{-2}) of their coefficients. In order that Eqs. (107) would satisfy the d'Alembert characteristic, it was necessary that many terms should cancel during the construction of these equations, a circumstance that afforded a helpful test on the expressions involved.

VIII. Position and Velocity From Mean Elements, and Vice Versa

Let it be assumed that the initial conditions are given by a set of mean elements, a, e, i, l_0, g_0, h_0. The following constants may then be computed:[3]

$$\left.\begin{aligned}
&\gamma'' = J_2\left[\frac{R}{a(1-e^2)}\right]^2, \qquad \gamma_3 = \frac{J_3}{J_2^2}\frac{a(1-e^2)}{R}, \qquad \gamma_4 = \frac{J_4}{J_2^2},\\
&c = \cos i, \qquad s = \sin i, \qquad 0 \leqq i \leqq \pi\\
&G'' = [\mu a(1-e^2)]^{1/2}, \qquad H'' = G''c[1+\gamma''(3c^2-2)]^{1/2}, \qquad n_1 = \left[\frac{\mu}{a^3}\right]^{1/2},\\
&g_{21} = -\frac{3}{4}\gamma''(1-5c^2) - \frac{1}{64}\gamma''^2(41+30c^2-135c^4) + \frac{5}{256}\gamma''^3(7+159c^2-531c^4+621c^6),\\
&g_{32} = -\frac{3}{16}c\left[8\gamma'' + \gamma''^2(7-33c^2) + \frac{1}{8}\gamma''^3(103-534c^2+1143c^4)\right].
\end{aligned}\right\} \quad (108)$$

Compute also the auxiliary constants given by Eqs. (90), (91), (93), (94), (95), and (106). Next obtain l'', g'', h'' from Eqs. (92).

The following equations will then suffice for the determination of f, r'', $\dot{r}''$, $\bar{u}''$, $\bar{h}''$ ($\dot{r}''$ is needed for velocity determination only):

$$\left.\begin{aligned}
&E - e\sin E = l'', && \dot{r}'' = \frac{\mu e \sin f}{G''},\\
&r''\sin f = a(1-e^2)^{1/2}\sin E, && \bar{u}'' = (1+g_{21})(g''+f),\\
&r''\cos f = a(\cos E - e), && \bar{h}'' = h'' + g_{32}\bar{u}''.
\end{aligned}\right\} \quad (109)$$

At this point $\dot{r}$, r, G, u, h are available from Eqs. (101–105). The perturbed values of γ and i are now given by

$$\left.\begin{aligned}
&\gamma = \gamma''\left(\frac{G''}{G}\right)^4,\\
&\cos i = \frac{H}{G}\left[1 + \frac{1}{2}\gamma\left(2 - 3\frac{H^2}{G^2}\right) + \frac{3}{8}\gamma^2\left(4 - 20\frac{H^2}{G^2} + 21\frac{H^4}{G^4}\right)\right] + O(\gamma^3),\\
&\sin i = +[1-\cos^2 i]^{1/2},
\end{aligned}\right\} \quad (110)$$

[3]A FORTRAN subroutine is available for this purpose and may be obtained from the author.

where the expansion for $\cos i$ has been obtained from the expression for H in Eq. (21). The computation of θ and ϕ from the relations

$$\left.\begin{aligned} \sin\theta &= \sin i \sin u, \\ \phi &= \tan^{-1}(\cos i \tan u) + h, \end{aligned}\right\} \tag{111}$$

or, if $s < \gamma''$, from Eqs. (107), and $\dot{\theta}$ and $\dot{\phi}$ from

$$\left.\begin{aligned} r^2 \cos\theta\, \dot{\theta} &= G \sin i \cos u \left[1 + \gamma(3\cos^2\theta + 3\cos^2 i - 2)\right]^{1/2}, \\ r^2 \cos^2\theta\, \dot{\phi} &= H, \end{aligned}\right\} \tag{112}$$

completes the calculation of the satellite's position and velocity. Alternatively, if x, y, z denote equatorial Cartesian coordinates, then

$$\left.\begin{aligned} x &= r\left[\cos u \cos h - \sin u \sin h \cos i\right], \\ y &= r\left[\cos u \sin h + \sin u \cos h \cos i\right], \\ z &= r\left[\sin u \sin i\right], \end{aligned}\right\} \tag{113}$$

while the velocity components are given by

$$\left.\begin{aligned} r\dot{x} &= x\dot{r} - (x^2 + y^2)^{-1}(Kxz + Hyr), \\ r\dot{y} &= y\dot{r} - (x^2 + y^2)^{-1}(Kyz - Hxr), \\ r\dot{z} &= z\dot{r} + K, \end{aligned}\right\} \tag{114}$$

where

$$K = G \sin i \cos u \left[1 + \gamma\{4 - 3\sin^2 i\,(1 + \sin^2 u)\}\right]^{1/2}, \tag{115}$$

which are readily derived from Eqs. (111) and (112).

We now turn to the inverse problem. Let the initial conditions be specified by the values of r, θ, ϕ, $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$ at some instant t. The instantaneous values of G and H are then available from the first two equations of (15), while u and h follow from Eqs. (110) and (111). The left-hand sides of Eqs. (101–105) are thus known precisely. In order to solve these equations for $\dot{r}''$, r'', G'', $\bar{u}''$, $\bar{h}''$, it is necessary to proceed by iteration and let the instantaneous elements be a first approximation to the mean elements. The procedure is described as follows:

Step 1. With the use of the current estimate of $\dot{r}''$, r'', G'', $\bar{u}''$, $\bar{h}''$, a, e, c, where

$$\left.\begin{aligned} a &= \mu\left(-\dot{r}''^2 + 2\frac{\mu}{r''} - \frac{G''^2}{r''^2}\right)^{-1}, \\ e &= \left(1 - \frac{G''^2}{\mu a}\right)^{1/2}, \\ c &= \frac{H}{G''}\left[1 + \frac{1}{2}\gamma''\left(2 - 3\frac{H^2}{G''^2}\right) + \frac{3}{8}\gamma''^2\left(4 - 20\frac{H^2}{G''^2} + 21\frac{H^4}{G''^4}\right)\right], \end{aligned}\right\} \tag{116}$$

compute f, l'', g'', h'' from Eqs. (109). The values of g'' and h'' need be computed in the last iteration only, after which the computation is advanced to step 2. Otherwise, use the preceding result to obtain improved values for $\dot{r}''$, r'', G'', $\bar{u}''$, $\bar{h}''$ from (101–105), and go back to step 1.

Step 2. With a, e, i, l'', g'', h'' already determined, l_0, g_0, h_0 are obtained from Eqs. (92), for which purpose all the third-order terms may be dropped in Eqs. (93), (94), and (95).

It is noted that the resulting mean elements are accurate to the second order only, since that is the accuracy of the periodic terms on which the determination of the mean elements depends. The third-order secular accuracy of the analytical solution cannot be fully utilized unless the semimajor axis a, or equivalently the mean motion n_1, is known to that order. But third-order secular terms will be important only when considering several revolutions of the satellite. In that case, n_1 can be accurately determined by a least-squares fit to the observations.

The iterative procedure described above can be avoided if Eqs. (101–105) are inverted to yield $\dot{r}''$, r'', G'', $\bar{u}''$, $\bar{h}''$ as explicit functions of the instantaneous elements. The inverse of Eq. (96) is

$$f(x'', y'') = f(x, y) - \{f, S + S^*\} + \frac{1}{2}\{\{f, S + S^*\}, S + S^*\} - \frac{1}{2}\{f, \{S, S^*\}\} + O(\epsilon^3), \qquad (117)$$

where S and S^* must now be expressed in terms of the unprimed variables. Clearly, the inversion will require little additional effort. Even so, it is hardly worthwhile to perform this inversion, since two or three iterations in the first aproach will accomplish the same task.

IX. Analytical and Numerical Checks

It is of interest to compare the present theory with the one advanced by Brouwer and Kozai (Refs. 1, 2). Let us first look at the secular terms as given by the invariant angular frequencies n_1, n_2, n_3 associated with the coordinates r, θ, ϕ. If we put $g_{21} = g_{32} = 0$ in the Brouwer–Kozai theory, then in both theories (Garfinkel, Ref. 8),

$$\left.\begin{aligned} n_1 &= \dot{l}'', \\ n_2 &= (1 + g_{21})(n_1 + \dot{g}''), \\ n_3 &= (1 + g_{32})\, n_2 + \dot{h}''. \end{aligned}\right\} \qquad (118)$$

Equations (15), (18), and (21), which also hold in the Brouwer–Kozai theory if the terms containing J_2 are dropped, provide a basis for the comparison. In the following, a zero superscript will be attached to the symbols of the Brouwer–Kozai theory. Expressions are available for the right-hand sides of Eqs. (118) in terms of the mean elements L'', G'', H'', or $L^{0''}$, $G^{0''}$, $H^{0''}$. Two relations between these two sets of mean elements follow from the invariance of H and F, while a third relation is obtained by averaging the third equation of (15) over the time, as follows:

$$\left.\begin{aligned} H'' &= H^{0''}, \\ \frac{\mu^2}{L''^2} &= \frac{\mu^2}{L^{0''2}}\left[1 - \frac{1}{2}\eta^0\gamma^0(1 - 3c^{02}) + \frac{1}{32}\eta^0\gamma^{02}\{1 + 6\eta^0 - (6 + 36\eta^0)c^{02} + (45 + 54\eta^0)c^{04}\}\right] + O(\gamma^3), \\ \frac{\mu^2}{L''^2} &= \frac{\mu^2}{L^{0''2}} - 2n_1^0\,(G'' - G^{0''}) + O(\gamma^2), \end{aligned}\right\} \qquad (119)$$

where $\eta^0 = (1 - e^{02})^{1/2}$. These equations are sufficiently accurate for proving that $n_1 = n_1^0$, $n_2 = n_2^0$, and $n_3 = n_3^0$ through terms of the second order. It should be possible to extend the comparison to the third order when an extension of the Brouwer theory becomes available in a form that is free from a singularity at $e = 0$ (Kozai, Ref. 2).

No attempt will be made to prove that the periodic terms of the two theories in question are equivalent. These terms assume a much simpler form in the present theory than in Kozai's theory, as can be verified by a direct inspection of the two algorithms. Much of this simplicity is due to the use of an intermediary instead of a Kepler ellipse. This can easily be demonstrated by noting that the functions F_1^0 and S_1^0 of the Brouwer–Kozai theory are related to our F_1 and S_1 by

$$\left.\begin{aligned} F_1^0 &= F_1 - \frac{\gamma G^2}{4r^2}(A - B\cos 2u), \\ F_1^{0\prime} &= -\frac{\gamma}{4} n_1 GA, \\ S_1^0 &= S_1 - \frac{\gamma G}{4}\left[A(f-l) - \frac{1}{2}B\sin 2u\right]. \end{aligned}\right\} \quad (120)$$

In the process of obtaining $S_2^0, F_2^{0\prime}, F_3^{0\prime}$ it will be necessary to evaluate the Poisson brackets $\{F_1^0, S_1^0\}$ and $\{\{F_1^0, S_1^0\}, S_1^0\}$, which are obviously much more complicated than the corresponding brackets for the intermediate orbit. The equation of the center, $(f - l)$, appearing in S_1^0 is a particular nuisance in this evaluation because of the complicated form of the derivatives of $(f - l)$ as given by Izsak (Ref. 18).

An important geometrical difference between the two orbits can be inferred from the fact that, in contrast to F_1^0 and S_1^0, F_1 and S_1 are factored by the eccentricity. This means that to the first order in J_2, the perturbations in L, G, H, h, and therefore the "out of the plane" perturbations, vanish at zero eccentricity. It has already been pointed out that for such an eccentricity, $e = 0$, the entire second-order theory is drastically simplified.

A rather convincing check of the foregoing theory can be obtained by a comparison with the results of numerical integration of the equations of motion. The adopted procedure is as follows: A set of preliminary mean elements a, e, i, l_0, g_0, h_0 is chosen, and an initial position and velocity are computed from the theory. This provides initial conditions for the numerical integration. It should be noted that the effective mean motion of the integrated orbit can be expected to differ from $n_1 = \mu^{1/2} a^{-3/2}$ through terms of the *third* order (second order for a first-order theory). This discrepancy can be removed by adjusting the semimajor axis a, but leaving the other elements unchanged, so as to minimize the sum of the squares of the residuals, i.e., the distances between integrated and theoretical positions.

The result of such comparisons is shown in Figs. 1 and 2 for two near-earth satellites moving in a circular orbit and in an orbit of eccentricity 0.3. According to Fig. 1, for which the third-order secular and the second-order periodic terms have been neglected, the first-order theory has a position error of about 60 m. The second-order theory reduces this error by a factor of 100 or more according to Fig. 2. It is noteworthy that even for the first-order theory there is no secular trend noticeable in the residuals after about six days (9000 min) of motion.

X. Conclusion

An analytical second-order theory has been developed for the motion of a satellite of an oblate planet whose

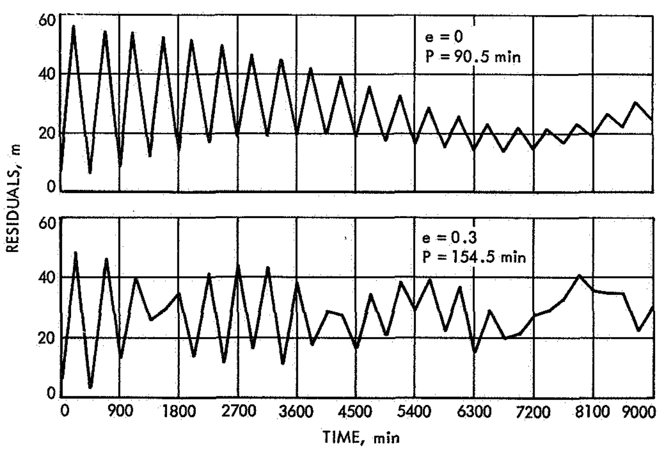

Fig. 1. Prediction error of the first-order theory for two near-earth satellites:
$J_2 = 1.082 \times 10^{-3}$, $J_3 = -2.4 \times 10^{-6}$, $J_4 = 1.7 \times 10^{-6}$; $l_0 = g_0 = h_0 = 0$, $i = 30°$, $a(1 - e) = 6678$ km

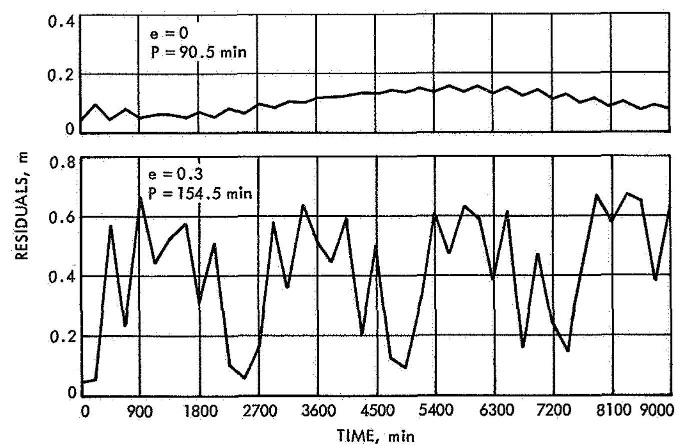

Fig. 2. Prediction error of the second-order theory for two near-earth satellites:
$J_2 = 1.082 \times 10^{-3}$, $J_3 = -2.4 \times 10^{-6}$, $J_4 = 1.7 \times 10^{-6}$; $l_0 = g_0 = h_0 = 0$, $i = 30°$, $a(1 - e) = 6678$ km

gravitational potential includes the J_2, J_3, and J_4 zonal harmonics. Expressions for the secular and the long-period perturbations of the Delaunay variables due to J_5, J_6, J_7, and J_8 have been obtained by Kozai (Ref. 2). A conversion to the Hill variables can readily be made.

The present theory, like all other theories based on series expansions in powers of J_2, breaks down at the critical inclination $i = 63°.4$. The theory does not contain power series in e, and it is valid for circular as well as for equatorial orbits.

Having carefully checked the algebra and the d'Alembert characteristic, and having obtained good agreement with numerical integration of the equations of motion, the author is confident that the theory is free from gross errors. Nevertheless, it is the author's intention to check the preceding hand calculations by the manipulation of trigonometric series in symbolic form on an electronic computer. This approach has already been used by A. Deprit and A. Rom (Ref. 20), and others. It is hoped that such a computer program can also be used to obtain the perturbations due to other sources, such as the higher spherical harmonics, air drag, and the luni-solar attraction.

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16. Abstract

An analytical second-order theory is developed for the motion of a satellite of an oblate planet whose gravitational potential includes the second, third, and fourth zonal harmonics. It is assumed that J_2 is a small quantity of the first order and that J_3 and J_4 are of the second order.

The secular and the periodic perturbations are obtained to the third and to the second order, respectively. The former are contained in the Delaunay variables l", g", h", which are linear functions of the time, while the latter are given as additions to the Hill variables $\dot{r}''$, r", G", u", h" in the form of trigonometric series with constant coefficients.

The theory is distinguished by a relative simplicity and compactness of the final algorithm achieved by the use of the following special devices and techniques: (1) an intermediate orbit, (2) Hori's perturbation method, and (3) the Hill variables.

A comparison with the results of numerical integration of the equations of motion indicates that the theory is capable of predicting the position of a near-earth satellite to better than one meter over one hundred revolutions.

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