TM-70-2015-6

TECHNICAL MEMORANDUM

MASS DISTRIBUTION OF ASTEROIDS

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AUTHOR(S)- J. S. Dohnanyi

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ABSTRACT

As a result of mutual collisions that are frequent on a geologic time scale the mass distribution of asteroids undergoes constant change. Nonetheless, for an arbitrary range of asteroid masses the distribution can be faithfully represented by the solution of a steady state approximation of the stochastic equation that describes the evolution of the asteroid population. For such a mass range a power law with index $\sim 11/6$, obtained earlier by the writer, is shown to be the only term left of a convergent power series expansion solving the steady state equation, and is therefore unique. The approximate steady state solution fails for the largest asteroids: these are broken up by collisions without being replenished. For asteroids in this mass range an approximate time dependent solution is obtained, which asymptotically approaches the solution valid for the lower mass range. This time dependent solution is sensitive to the comminution law for collisional fragments which is here assumed to be similar to that derived from

laboratory experiments with semi-infinite basalt targets, impacted by high velocity projectiles. The good agreement of the theoretical predictions with observations in the accessible mass range lends confidence in the validity of the stochastic model. The cosmological implication is that the present distribution of asteroids is not likely to reveal much about the original distribution, since the latter has been altered beyond recognition by the frequent occurrence of random inelastic collisions.

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SUBJECT: Mass Distribution of Asteroids
Case 340

DATE: December 1, 1970

FROM: J. S. Dohnanyi

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

This paper is the continuation of a study undertaken by the writer, in an effort to model some of the physical processes that have given rise to the observed size distribution of stray bodies in the asteroidal belt. Interest in this problem is due to a desire to gain insight into the origin of asteroids and to estimate their distribution in size ranges too small to be telescopically observed.

In a previous study (Dohnanyi, 1969), which will be referred to as D-I, a stochastic model of asteroidal collisions was formulated in the form of a differential equation defining the rate at which the number density of asteroids, in a given mass range, changes in time because of inelastic collisions, which break up some objects but create many collisional fragments. D-I is therefore seen to be more detailed than earlier particles-in-the-box calculations (cf Jones, 1968; Piotrowski, 1953), which did not include the fragmentation process. D-I is furthermore complementary to studies of the statistics of asteroidal orbits (see Wetherill, 1969, for review and references) as well as statistical studies based on the astronomy of individual asteroids (Anders, 1965, Hartmann and Hartmann, 1968).

A particular solution to the asteroidal number density was obtained in D-I. This solution is valid for asteroids whose masses are far away from the limiting largest and smallest masses of the distribution, provided that the population of asteroids has reached a steady state distribution under the action of mutual collisions. This solution has the simple form of

$$f(m)dm = Am^{-\alpha}\,dm, \quad \alpha \approx 11/6 \tag{1.1}$$

where $f(m)dm$ is the number of asteroids per unit volume having masses in the range of m to $m + dm$.

In this paper we have made a slight change in the mathematical modeling of the fragmentation process (D-I), which results in an improved physical model and simpler mathematics. We then discuss some of the asymptotic properties for long times in the model, and show that eq. 1.1 is the only steady state solution expandable in a power series in m far away from the limiting masses of the distribution. We consequently obtain an approximate solution for the distribution of large asteroids with masses near the limiting largest mass of the population.

2.0 OBSERVATIONAL EVIDENCE

The observational material on asteroids we shall discuss is presented in Fig. 1. Plotted in this figure are the cumulative number of observed asteroids (solid histogram) as well as the probable true number of asteroids (dashed-line histogram) versus absolute photographic magnitude g, as given by Kuiper et al (1958). The curve of Kuiper et al is complete up to g = 9.5; i.e., the observed number of these objects is believed to equal the true number. Above $g \gtrsim 9.5$ the difference between the true and the observed number of asteroids begins to increase owing to selection effects. The dashed-line histogram is the probable number of asteroids determined by using the "completeness" factors of Kuiper et al (1958). These authors have tabulated the maximum and minimum probable numbers of asteroids, and the dashed-line histogram in Figure 1 is their mean. A mass scale, based on a material density of 3.5×10^3 Kg/m^3 for spherical asteroids with an albedo of .2 (D-I), gives a nominal relation between the mass m (Kg) and absolute photographic magnitude:

$$\log_{10} m = 22.67 - 0.6g. \qquad (2.1)$$

This mass scale is also indicated in Fig. 1.

The solid curve in Fig. 1 is the cumulative number N(M) of asteroids heavier than M,

$$N(M) = \int_M^{M_\infty} f(m)\,dm. \qquad (2.2)$$

as a function of mass M (or g) obtained in D-I. In that paper we took $M_\infty = 1.86 \times 10^{20}$ Kg corresponding to g = 4 and

$$f(m) = 2.59 \times 10^{16} m^{-1.837}. \qquad (2.3)$$

The exponent on m was obtained theoretically and the numerical (normalization) factor is empirical. It can be seen from Figure 1 that there is close agreement between theory and experiment.

The theoretical formula (eq. 2.3) was derived only for masses m

$$m << M_\infty, \text{ or } g >> 4 \tag{2.4}$$

where M_∞ is the largest mass present. Furthermore, the uniqueness of the result of Eq 2.3 was not examined, nor was the effect on the asteroidal population of the depletion of large asteroids by collisions analyzed quantitatively. The purpose of the present study is, therefore, to examine the uniqueness of Eq. 2.3 as well as the applicability of this equation to the distribution of large asteroids.

3.0 COLLISIONAL MODEL

3.1 Collision Equation

The asteroid belt contains over 1,600 catalogued asteroids; it is generally assumed that it contains many more smaller ones that cannot or have not been telescopically detected because they are not bright enough. It has been estimated (See e.g., Piotrowski, 1953) that collisions between asteroids must be frequent, when measured on an astronomical time scale.

Collisions between asteroids must undoubtedly affect their mass distribution. In order to see how this occurs, one may regard them as molecules in a box and then use the methods of kinetic theory of gases insofar as they apply. This has been discussed in D-I, where an equation was derived for the number density of asteroids as their population evolves under the influence of collisions.

Let $f(m,t)dm$ be the number density per unit volume of space of asteroids having a mass in the range m to $m + dm$ at a time t. Assuming uniform spatial distribution within the asteroid belt, it can be shown that the influence of collisions on the number density $f(m,t)$ can be expressed as the sum

$$\partial f(m,t)/\partial t = \left.\frac{\partial f}{\partial t}\right|_{\text{erosion}} + \left.\frac{\partial f}{\partial t}\right|_{\substack{\text{catastrophic}\\ \text{collisions}}} + \left.\frac{\partial f}{\partial t}\right|_{\text{creation}} \tag{3.1}$$

where $\partial f/\partial t$ is the rate at which the number density of the asteroids changes because of collisions.

The first term on the right hand side of this equation is the contribution of the erosive reduction in the particle masses. We define erosive collisions as collisions where the small projectile particle is too small to catastrophically disrupt the larger target object. The result of such a collision is then the removal of some mass from the target object by cratering.

The second term on the right hand side of Eq. 3.1 is the reduction rate due to catastrophic collisions in which, in addition to a crater at the site of impact by a small projectile, a spall is detached from the opposite side of the larger target object. This definition is more detailed than that used in D-I and reflects new information on this process provided by recent experiments (Gault and Wedekind, 1969).

The last term on the right hand side of Eq. 2.1 denotes the contribution of particle creation by the fragmentation of larger objects; it is the number of fragments in a mass range m to m + dm created per unit volume and unit time by both the erosive and catastrophic fragmentation of larger colliding objects.

3.2 Simplified Model for Fragmentation

We shall derive here an explicit expression for the term $\partial f/\partial t_{creation}$ in Eq. 3.1. We shall revise slightly the earlier physical model, employed in D-I, and obtain a simpler mathematical formulation.

Using experimental results by Gault et al (1963), the following crushing law for the mass distribution of fragments produced during impact was assumed in D-I.

$$g(m;M,M_2)dm = C(M,M_2)m^{-\eta}dm \qquad (3.2)$$

where $g(m;M,M_2)dm$ is the number of fragments in the mass range m to m + dm produced when a projectile of mass M impacts a larger target mass M_2. For η = constant < 2 and from mass conservation during collisions it follows that

$$C(M,M_2) = (2-\eta)\ M_e M_b^{\eta-2} \qquad (3.3)$$

where M_e is the total ejected mass and M_b is the limiting mass of the largest fragment.

In D-I we assumed that M_b is proportional to the mass of the projectile particle. We now assume, more correctly, that M_b is proportional in catastrophic collisions to the mass M_2 of the larger colliding object and in erosive collisions to the total ejected mass produced during impact.

$$M_b = M_e/\lambda_e \quad \text{(erosive collisions)} \qquad (3.4)$$

$$M_b = M_2/\lambda \quad \text{(catastrophic collisions)}$$

where λ and λ_e are constants to be determined by experiment.

In terms of the notation of D-I,

$$\lambda_e = \Gamma/\Lambda \qquad (3.4a)$$

where Γ is the total mass of fragments cratered out by a projectile with unit mass and Λ is a constant.

The comminution law, Eq. 3.2, is based on experiments by Gault, et al., (1963) who fired high-velocity projectiles into effectively semi-infinite blocks of basalt. They found a value for η of approximately 1.8 ± 0.1 and therefore the approximate form of $C(M,M_2)$ in Eq. 3.3 for $\eta<2$ is justified.

More recently, Gault and Wedekind (1969) fired high velocity projectiles into "finite" glass spheres. They found that the resulting fragments, including the "core" were distributed according to $\eta \approx 1.7$. While glass has generally different properties from basalt, it is interesting to note that the high velocity impact experiments with effectively semi-infinite glass targets, reported in the same paper, yielded a value of $\eta \approx 1.8$, just like basalt.

These authors also found that at a certain threshold impact energy, a spall fragment was formed and "ejected" in addition to the ordinary, erosive, cratering process. Such a spall forms at the surface of the sphere opposite the point of impact, above the threshold of an observed impact energy of about 10^6 erg/gram for single-piece glass spheres.

We now proceed with the derivation of our fragmentation model. The number of collisions per unit volume of space and unit time, $\delta^2 n$ between spherical particles with masses in the range M_1 to $M_1 + dM_1$ and M_2 to $M_2 + dM_2$ is (cf., Dohnanyi 1969):

$$\delta^2 n = K(M_1^{1/3} + M_2^{1/3})^2 \, f(M_1,t) \, dM_1 f(M_2,t) \, dM_2 \qquad (3.5)$$

where

$$K = (3\pi^{1/2}/4\rho)^{2/3}\overline{V} \qquad (3.6)$$

with $\overline{V}$ the effective encounter velocity. $K(M_1^{1/3} + M_2^{1/3})^2$ is the product of the geometrical collision cross section of two spherical particles with masses M_1 and M_2 and their encounter velocity.

The total ejected mass M_e when two objects with masses M_1 and M_2 collide catastrophically, is

$$M_e = M_1 + M_2. \qquad (3.7)$$

Combining this with Eq. 3.2 - 3.4 and 3.5, we obtain the number of fragments in a mass range m to m + dm created per unit time and volume by catastrophic collisions between masses in the range M_1 to $M_1 + dM_1$ and M_2 to $M_2 + dM_2$ (with $M_2 > M_1$).

This is:

$$m^{-\eta}\, dm(2-\eta)\lambda^{2-\eta} M_2^{\eta-2} (M_1 + M_2)\ K(M_1^{1/3} + M_2^{1/3})^2 \qquad (3.8)$$

$$f(M_1,t)\ dM_1\ f(M_2.t)dM_2$$

and holds for

$$m \leq \frac{M_2}{\lambda} \qquad (3.9)$$

since m cannot exceed the mass of the largest fragment produced by the catastrophic collision of M_1 with M_2. (cf Eq. 2.4).

Integrating expression 3.8 over all permissible masses M_2 and M_1, we obtain the positive contribution of fragmentation to $\partial f/\partial t$:

$$\left.\frac{\partial f^{(m,t)}}{\partial t}\right|_{\text{creation}} = K(2-\eta)\ \lambda^{2-\eta}\ m^{-\eta} \int_{\lambda m}^{M_\infty} dM_2 \int_{M_2/\Gamma'}^{M_2} dM_1 \qquad (3.10)$$

$$M_2^{\eta-2}(M_1 + M_2)(M_1^{1/3} + M_2^{1/3})^2\ f(M_1,t)\ f(M_2,t)$$

Here M_∞ is the limiting mass of the largest asteroid present and, $1/\Gamma'$ is the mass of the smallest projectile capable of catastrophically disrupting a target object of unit mass (cf D-I).

3.3 Explicit Mathematical Formulation

The explicit mathematical form of Eq. 3.1 has been given in D-I. In the previous section of this paper, we have modified the creation by fragmentation term for the catastrophic process. Thus, replacing that term as given in D-I by Eq. 3.10 and replacing Λ by Γ/λ_e (Eq. 3.4a) in D-I's expression for particle creation by erosive collisions, we obtain:

$$dm \frac{\partial f(m,t)}{\partial t} = dm \Bigg\{ -K\, f(m,t) \int_{m/\Gamma'}^{M_\infty} f(M,t) \tag{3.11}$$

$$(m^{1/3} + M^{1/3})^2\, dM + K(2-\eta)\lambda^{2-\eta} m^{-\eta} \int_{\lambda m}^{M_\infty} dM_2 \int_{M_2/\Gamma'}^{M_2} dM_1$$

$$M_2^{\eta-2}\,(M_1 + M_2)\quad (M_1^{1/3} + M_2^{1/3})^2\, f(M_1,t)\, f(M_2,t)$$

$$- \frac{\partial}{\partial m}\,[f(m,t)\ (dm/dt)] + K(2-\eta)\ \Gamma^{\eta-1}\lambda_e^{2-\eta}\ m^{-\eta}$$

$$\int_{\lambda_e m/\Gamma}^{M_\infty/\Gamma'} dM_1 \int_{\Gamma' M_1}^{M_\infty} dM_2\ \ M_1^{\eta-1}\ \ (M_1^{1/3} + M_2^{1/3})^2$$

$$f(M_1,t)\ f(M_2,t) \Bigg\} .$$

The first two terms on the right hand side of this equation are the contributions of catastrophic processes and the last two are the contributions of the erosive collisions. The first term on the right hand side of Eq. 3.11 is the rate per unit volume at which objects in the mass range m to m + dm are destroyed by catastrophic collisions and the second term is the rate per unit volume at which objects in this mass range are created by the fragmentation of colliding larger objects.

The third term, on the right hand side of Eq. 3.11, is the rate per unit volume at which the number of objects in the relevant mass range is changing because their mass changes in time as a result of erosive collisions.

The rate of mass change, dm/dt, of an object because of erosion is (D-I),

$$\frac{dm}{dt} = -\Gamma K \int_{\mu}^{m/\Gamma'} M\, f(M,t)\, (m^{1/3} + M^{1/3})^2\, dM \qquad (3.12)$$

Here Γ is the mass cratered out by an erosive particle of unit mass and Γ' is the largest mass catastrophically fragmented by a projectile of unit mass; μ is the mass of the smallest particle present.

The last term on the right hand side of Eq. 3.11 is the rate at which objects are created in the relevant mass range by erosive collisions.

We now derive a simpler form of Eq. 3.11, valid for large masses. It can readily be shown that as the value of the mass m approaches $(\Gamma/\Gamma')\ (M_\infty/\lambda_e)$, particle creation by erosive collisions ceases. This happens because the largest erosive projectile, M_∞/Γ', craters out of M_∞ a total mass of $\Gamma M_\infty/\Gamma'$, and the largest individual object so formed is $(\Gamma/\Gamma')(M_\infty/\lambda_e)$, according to Eq. 3.4.

Furthermore, when m approaches M_∞/λ, the creation term due to catastrophic fragmentation vanishes: no creation mechanism by catastrophic collisions is possible for masses greater than the largest fragment produced when M_∞ is disrupted catastrophically.

We now assume, for the moment, that

$$\lambda < \frac{\Gamma'}{\Gamma} \lambda_e \quad . \tag{3.13}$$

The significance of this assumption will be considered later and it will be found satisfied in the cases of interest in this study.

Then, we can write, for $m \gtrsim M_\infty/\lambda$,

$$\frac{\partial f(m,t)}{\partial t} = - \frac{\partial}{\partial m} [f(m,t)\ dm/dt] - K\ f(m,t) \int_{m/\Gamma'}^{M_\infty} f(M,t) \quad (m^{1/3} + M^{1/3})^2\ dM. \tag{3.14}$$

This is the collision equation for masses near the limiting value M_∞; it only contains the contribution of erosive mass reduction and of catastrophic collisions.

4.0 STEADY STATE SOLUTION FOR SMALL MASSES

4.1 Steady State Solution

The general solution of the collision equation (Eq. 3.11 and 14) is difficult to obtain. We shall, therefore, solve it for an important special case. We shall show that a steady state solution exists and is unique; this solution is identical with the result of D(I).

In the absence of a source to replenish the largest asteroids (Eq. 3.14) the number of objects in all finite mass ranges will go eventually to zero.

Hence

$$\lim_{t\to\infty} f(m,t) = 0, \; m > 0 \tag{4.1}$$

Also, according to Eq. 3.11 and 14, it follows from Eq. 4.1 that

$$\lim_{t\to\infty} \frac{\partial f(m,t)}{\partial t} = 0 \qquad m > 0, \tag{4.2}$$

Thus, the time rate of change in the particle number density goes smoothly to zero as $t\to\infty$, indicating that no collapses or other discontinuities occur as $t\to\infty$.

We also note that Eq. 4.1 implies 4.2 for all masses but vice versa Eq. 4.2 does not imply Eq. 4.1 for masses smaller than M_∞/λ. Indeed, if the time derivative $\partial f(m,t)/\partial t$ goes to zero much faster than $f(m,t)$ for small masses, the solution of the homogeneous equation (obtained by letting $\partial f/\partial t = 0$ in equation 3.11) gives a first approximation to the solution as $t\to\infty$. The significance of this approximate solution becomes clearer if we note that the condition

$$\lim_{t\to\infty} \frac{\partial f(m,t)}{\partial t} = 0 \qquad m > 0$$

means that a constant τ exists, such that

$$\left|\frac{-\partial f(m,t)}{\partial t}\right| < \frac{f(m,t)}{\tau}, \tag{4.3}$$

for sufficiently large t. Whence if Eq. 4.3 is satisfied, the homogeneous equation obtained from Eq. 3.11 by letting $\partial f(m,t)/\partial t = 0$ is a reasonable first approximation as $t\to\infty$.

4.2 Series Solution for Small Masses

Assuming that Eq. 4.3 is satisfied, we obtain a first approximation for the number density of particles, by solving the homogeneous version of Eq. 3.11 resulting from setting $\partial f(m,t)/\partial t$ equal to zero. It then follows that the various collisional processes balance, i.e., the rate of removing objects by collisions from a given mass range equals the rate of supplying objects into this mass range by the collisional fragmentation of larger objects. It is clear that such a steady state situation can be attained even in an approximate way only for masses much smaller than M_∞/λ, because in the range of larger masses, from M_∞/λ to M_∞, no fresh objects are created by fragmentation and all collisions decrease the number of objects in this mass range. However, for mathematical convenience we shall approximate the statement $m << M_\infty/\lambda$ by the mathematically stronger statement,

$$M_\infty \to \infty \tag{4.4}$$

In order for Eq. 3.11 to be valid, it is also necessary that $m >> \Gamma'\mu$. For $m \lesssim \Gamma'\mu$ all collisions are catastrophic in the framework of the present model and the influence of erosion on our model is expressed by a different mathematical relation. We shall take, therefore, the mathematically stronger statement,

$$\mu \to 0 \tag{4.5}$$

After discussing the solution for the unlimited population characterized by Eq. 4.4 and 4.5, we shall relax Eq. 4.4 and discuss in the next section the influence of a finite limiting mass M_∞ on the distribution.

Since power law functions successfully approximate the distribution of interplanetary particles over various mass ranges, a trial solution of the form,

$$f(m) = Am^{-\alpha} \tag{4.6}$$

was attempted in D-I. It was found that with $\alpha \simeq 11/6$, Eq. 4.6 solves the homogeneous part of Eq. 3.11 for the unlimited distribution, Eq. 4.4 and 4.5.

In an effort to examine the uniqueness of the solution Eq. 4.6 and to find alternate solutions if they exist, we use here a systematic method for solving the homogeneous part of Eq. 3.11 with the conditions Eq. 4.4 and 4.5.

We let,

$$f(m,t) \approx f(m) = \sum_{j=0}^{\infty} a_j m^{-\delta_j} \tag{4.7}$$

where f(m) is positive definite for $0 \leq m \leq \infty$ and where the complex numbers a_j and δ_j, dependent on j, are to be determined from the boundary conditions of the problem. A priori, the series in Eq. 4.7 may be an infinite series or a polynomial, depending on whether the numbers j form an infinite or a finite sequence.

We substitute Eq. 4.7 into the homogeneous part of Eq. 3.11 and, under the boundary conditions Eq. 4.4 and 5 we obtain an explicit equation for the coefficients a_j and exponents δ_j in Eq. 4.7. Detailed derivation of the resulting equation is given in Appendix A; this equation has the form,

$$\sum_j \sum_\ell w_{j,\ell}\, m^{-\delta_j-\delta_\ell+5/3} = 0 \tag{4.8}$$

where

$$w_{j,\ell} = a_j a_\ell [-Kg_\ell + q_{j,\ell} + p_{j,\ell} - v_{j,\ell}] \tag{4.9}$$

and where the quantities inside the square brackets are defined by Eqs. A-3, 6, 10, and 13 of Appendix A.

We let

$$\delta_j = \delta' j + \alpha \tag{4.10}$$

where α is some constant.

Equating the coefficients of like powers of m in Eq. 4.8 to zero gives

$$\sum_{j=0}^{n} w_{j,n-j} = 0 \qquad n = 0,1,2... \tag{4.11}$$

The term defined by n = 0 gives an indicial equation,

$$Ka_0^2 \left\{ -\frac{(\Gamma')^{\alpha-1}}{\alpha-1} + \frac{(2-\eta)\lambda^{-2\alpha+11/3}}{2\alpha-\eta-5/3}\,\frac{(\Gamma')^{\alpha-1}}{\alpha-1} + \frac{(2-\eta)\,\Gamma^{2\alpha-8/3}\,\lambda^{-2\alpha+11/3}}{2\alpha-\eta-5/3}\,\frac{(\Gamma')^{-\alpha+5/3}}{\alpha-5/3} - \frac{\Gamma(2\alpha-8/3)\,(\Gamma')^{\alpha-2}}{2-\alpha} \right\} \approx 0 \tag{4.12}$$

where only the leading terms have been retained, as discussed in Appendix A. The conditions

(A4) $R(\delta_\ell) > 5/3$

(A8) $R(\delta_j + \delta_\ell) > \eta + 5/3$

(A15) $R(\delta_\ell) < 2$

and the fact that Γ' is a large number (of the order of $10^4 - 10^5$) have also been used to obtain Eq. 4.12. The expressions inside the brackets in Eq. 4.12 are the respective contributions of catastrophic collisions, catastrophic creation, erosive creation and erosion. The latter two expressions are small because $\Gamma << \Gamma'$ and have only been included for the sake of completeness.

The remarkable property of Eq. 4.12 is that it has the non-trivial solution,

$$\alpha = 11/6 \tag{4.13}$$

for which the catastrophic and erosive processes balance individually i.e., the sum of the first two terms as well as the sum of the last two terms individually equals zero.

It was shown in D-I that if the neglected terms are included in Eq. 4.12, the result is only a small perturbation on α. The solution to the leading terms of the indicial equation Eq. 4.13, as well as the corresponding solution to the complete indicial equation were shown in D-I to be the one and only solution for real values of α that satisfy the conditions Eq. A-4, 8, and 15.

We shall presently show that the real part of δ'_j in Eq. 4.10 must be zero. Obviously, the expansion of f(m), Eq. 4.9 is either the sum of an infinite or a finite number of terms. If it is an infinite series, then according to Eq. 4.10, the condition Eq. A-15 will be violated for sufficiently large values of j. It therefore follows that the power series, Eq. 4.9 and Eq. 4.10 must be a finite polynomial. If, however, f(m) is a polynomial in $m^{-\delta' j\alpha}$, then for $m \to 0$ or for $m \to \infty$ some term in the polynomial other than $m^{-\alpha}$ will dominate and we have,

$$f(m) \to \text{constant } m^{-\delta' j-\alpha} \tag{4.14}$$

In view, however, of the uniqueness of α (or the corresponding solution of the complete equation) Eq. 4.14 cannot be satisfied for real δ'.

If δ' is complex, say

$$\delta' = x + iy, \tag{4.15}$$

where $i = \sqrt{-1}$, Eq. 4.14 becomes

$$f(m) \to \text{constant } m^{-x_j-\alpha} \{\cos(y_j \ln[m] + + i \sin(y_j \ln[m])\} \tag{4.16}$$

which expression is not positive definite and is therefore not a physically admissible solution. It therefore follows that δ' must be zero and the exponent α must be real.

We conclude that

$$f(m) = a_o m^{-\alpha} \tag{4.17}$$

with real a_o and α is the only physically admissible solution that can be expanded into a Taylor series in m. The parameter α is approximately 11/6 and can be obtained more precisely by numerically solving Eq. 4.11 with n = 0.

5.0 SOLUTION FOR LARGE MASSES

5.1 Asymptotic Form of the Collision Equation for Large Masses

In the preceding sections, we obtained a steady-state solution for the collision equation (Eq. 3.11), for small masses $m << M_\infty$. We shall in this section examine the influence of a finite limiting mass M_∞ on this solution. We do so by obtaining an approximate solution valid for small masses ($m << M_\infty$) as well as large ones ($m \sim M_\infty$).

We note again that the chief difference between the number density of small masses and large ones is that for large enough masses the creation by fragmentation processes is no longer sufficient to replenish the objects removed from a given mass range by collisions and a net change in the particle population results. It is therefore clear that, as $m \to M_\infty$, the time derivative of the number density $\partial f(m,t)/\partial t$ can no longer be assumed neglible in comparison with the collision terms. It thus becomes necessary to solve the complete time dependent equation. That task is, however, difficult and we shall derive an approximate form for the time dependent equation that can readily be solved for the number density function f(m,t).

We first note that a number density of the form

$$f(m) = A\ m^{-\alpha} \tag{5.1}$$

with

$$\alpha \approx 11/6 \tag{5.2}$$

fits remarkably well the observational data of large asteroids (Fig. 1). It has been shown in D-I that for such a distribution, the erosive contribution to the collision equation is small in comparison with the contribution of catastrophic collisions. We shall, therefore, disregard the contribution of erosion to the evolution of the population of large asteroids. It has also been shown in D-I that for a population of the form Eq. 5.1 and 5.2 the dominant contribution to $\partial f(m,t)/\partial t$ arises from collisions with much smaller objects provided that $(\Gamma')^{\alpha-1} >> 1$. Since Γ' is of the order of 10^4 to 10^5 and $\alpha = 11/6$, we see that this inequality is satisfied.

Using these considerations, we obtain the approximate linear equation

$$\frac{\partial f(m,t)}{\partial t} \approx -Km^{2/3}\ f(m,t) \int_{\frac{m}{\Gamma'}}^{M_\infty} A\ M^{-\alpha}\ dM \tag{5.3}$$

$$+ K(2-\eta)\lambda^{2-\eta}\ m^{-\eta} \int_{\lambda m}^{\frac{M_\infty}{\lambda}} dM_2 f(M_2,t)\ M_2^{\eta-1/3} \int_{\frac{M_2}{\Gamma'}}^{M_2}$$

$$dM_1\ AM_1^{-\alpha}$$

for

$$m \leq M_\infty/\lambda^2 . \tag{5.4}$$

and where the contribution of masses in the range M_∞/λ to M_∞ has been neglected from the creation term.

The upper bound, Eq. 5.4, on the range of validity of Eq. 5.3 follows because the largest mass that can be created by the disruption of M_∞/λ is M_∞/λ^2. The fragmentation term in Eq. 5.3 therefore vanishes for masses $m > M_\infty/\lambda^2$.

The first term on the right hand side of Eq. 5.3 is the rate at which $f(m,t)$ changes with time because of collisions with point objects capable of disrupting objects with masses in the range m to $m + dm$. Corrections to Eq. 5.3 because of the finite size of the projectile masses are quite small for the distributions we shall consider; we approximate the number density of projectiles by $Am^{-\alpha}$ with α obtained from a steady state solution of masses far away from M_∞.

The second term on the right hand side of Eq. 5.3 is the rate of change of $f(m,t)$ because of the creation of objects into the mass range m to $m + dm$. The creation by catastrophic fragmentation term is dominated by the rate at which target objects with number density $f(m,t)$ are catastrophically impacted by smaller projectile objects with a number density $AM^{-\alpha}$. Clearly, this creation term approaches 0 as $m \to M_\infty/\lambda$.

We have neglected in Eq. 5.3 the contribution to the particle creation by fragmentation of objects having masses greater than M_∞/λ. Since we shall be mainly interested in solutions of the collision equation for λ close to one, this approximation is reasonable.

It is easily seen that 5.3 can be solved by using the method of the separation of variables. Let, as a mathematical device,

$$f(m,t) = -m^{\alpha-\eta-2/3}\, T(t)\, \frac{dW(m)}{dm} \tag{5.5}$$

Substituting Eq. 5.5 into Eq. 5.3 gives, after integration and rearrangement of terms:

$$\frac{1}{T(t)}\; \frac{\alpha-1}{AK(\Gamma')^{\alpha-1}}\; \frac{dT(t)}{dt} = -C \tag{5.6}$$

and

$$0 = (m^{-\alpha\, +\, 5/3} - C)\,\frac{dW(m)}{dm} - (2-\eta)\,\lambda^{2-\eta} m^{-\alpha\, +\, 2/3}\, \times \tag{5.7}$$

$$\times\ [W(M_\infty/\lambda) - W(\lambda m)]$$

where C is the constant arising from the separation of variables and the approximation $(\Gamma')^{\alpha-1} >> 1$ has been used. The physical significance of C can be readily seen by noting that for $m = M_\infty/\lambda^2$ Eq. 5.7 becomes

$$0 = [(M_\infty/\lambda^2)^{-\alpha\, +\, 5/3} - C]\, \frac{dW(m)}{dm}\Big|_{m\, =\, \frac{M_\infty}{\lambda^2}} \tag{5.8}$$

Since

$$\frac{dW(m)}{dm}\Big|_{m\, =\, M_\infty/\lambda^2} \neq 0 \tag{5.9}$$

it follows that

$$C = (M_\infty/\lambda^2)^{-\alpha\, +\, 5/3} \tag{5.10}$$

is an effective "cut-off mass" raised to the power indicated. The quantity C is a scaling criterion and determines whether a certain mass m is large or small, depending on its numerical value relative to $C^{1/(-\alpha + 5/3)}$; i.e., if

$$m^{-\alpha + 5/3} >> C \tag{5.11}$$

then m is small and, to a good approximation, Eq. 5.7 reduces to 4.9 with the erosion terms disregarded. For sufficiently small masses then, the solution of Eq. 5.7 for f(m,t) approaches asymptotically the solution of Eq. 4.8 (or 4.12).

We now solve Eq. 5.7 for the simple case*

$$\lambda = 1. \tag{5.12}$$

We now let

$$\phi = W(m) - W(M_\infty); \tag{5.13}$$

using Eq. 5.12, 13 we can express Eq. 5.7 in this form

$$(m^{-\alpha + 5/3} - C)\ d\phi/dm + (2-\eta) m^{-\alpha + 2/3} \phi(m) = 0. \tag{5.14}$$

This is readily solved for ϕ; the result is

$$\phi(m) = \phi_o\ (m^{-\alpha + 5/3} - C)^{-(2-\eta)/(-\alpha + 5/3)} \tag{5.15}$$

---

*The general solution with arbitrary λ but infinite C has been discussed elsewhere by Chu (1970). He has proven the interesting result that for $1 \leq \lambda < e^{\frac{.57}{2-\eta}}$ the general solution of Eq. 5.7 with $C = \infty$ is of the form Eq. 4.17 with $\alpha = 11/6$.

and using Eq. 5.5 and 5.13, we get

$$f(m,t) = m^{\alpha-11/3}\,(1-C\,m^{\alpha-5/3})^{\frac{2-\eta}{\alpha-5/3}-1}\,f_o T(t). \tag{5.16}$$

where

$$f_o = (2-\eta)\phi_o$$

To obtain the time dependent function T(t), we note that Eq. 5.16 should approach,

$$f \to A\,m^{-\alpha} \text{ as } m/M_\infty \to o \tag{5.17}$$

and whence, the normalization factor A should have the same time dependence as T(t). From Eqs. 5.3, 16 and 17 we obtain

$$T(t) = A(t),\ f_o = 1 \tag{5.18}$$

and, from Eq. 5.6 we get,

$$A(t) = \frac{A_o}{1 + C'\,A_o t} \tag{5.19}$$

where A_o is the value of A(t) at time t = 0 and,

$$C' = CK\,(\Gamma')^{\alpha-1}/(\alpha-1) \tag{5.20}$$

is constant.

It can be shown (D-I) that

$$C' = \frac{1}{\tau_\infty a} \tag{5.21}$$

where $1/\tau_\infty$ is the dominating term of the probability per unit time for any of the limiting masses M_∞ to undergo a catastrophic collision (cf Eq. 64 of D-I); a is the value of A(t) at the present time. Since T(t) is a monotonically decreasing function of time, it is clear (from Eq. 5.19) that for long times Eq. 4.3 is satisfied and our asymptotic formulae are justified.

Collecting terms, we get,

$$f(m,t) = A(t)\ m^{\alpha-11/3}\ (1 - Cm^{\alpha-5/3})^{\frac{2-\eta}{\alpha-5/3}-1} \tag{5.22}$$

where A(t) is given by Eq. 5.19.

Using Eq. 5.17 and 22 we conclude that

$$- \alpha = \alpha - 11/3 \tag{5.23}$$

and hence

$$\alpha = 11/6 \tag{5.24}$$

is the self-consistent solution of the problem.

6.0 DISCUSSION AND CONCLUSIONS

The main analytical result of this paper is that the number density f(m) of asteroids is

$$f(m) = Am^{-11/6} \left\{1 - (m/M_\infty)^{1/6}\right\}^{6(2-\eta)-1} \tag{6.1}$$

where we have used Eqs. 5.10, 16, 17, and 24. For masses that are small in comparison with M_∞ the quantity within the curly brackets has a value very close to one. Eq. 6.1 is valid only if a long period of time has elapsed since the creation of the asteroidal population, to the extent that the specific form of the initial distribution has been obliterated and the resulting distribution is the asymptotic limit for long times.

For masses that are very much smaller than M_∞ the distribution is a power law type with exponent - 11/6 and is independent of η. For large masses, close to M_∞, the distribution is no longer a simple power law type and depends on the value of η.

Eq. 6.1 has been derived for the condition $\lambda = 1$ (cf Eqs. 3.4, 5.7, and 5.12). This means that the largest possible fragment in catastrophic collision is the total ejected mass M_e. For small projectiles, the projectile mass may be neglected and $\lambda = 1$ means that the largest possible fragment is the mass of the target object itself.

The simplest way to compare Eq. 6.1 with observation is to integrate it and obtain the cumulative number of asteroids N(M) having a mass m or greater.

$$N(m) = \int_{m}^{M_\infty} f(m)\, dm \qquad (6.2)$$

The result is Figure 2. This is a repetition of Figure 1 with plots of Eq. 6.2 (for various values of η) superposed. A value of 1.86×10^{20} Kg(g = 4) has been taken for M_∞ and a value is used for A that brings N(m) into agreement with the observed cumulative number at g = 9.

It can be seen from the figure that a fragmentation index $\eta = 11/6$ provides excellent agreement between theory and observation. The agreement is also good for $\eta = 23/12$ and $\eta = 5/3$; in the former case the observed number of large asteroids is somewhat overestimated and in the latter case it is somewhat underestimated. The curve for $\eta = 3/2$ shows that the agreement with observation begins to deteriorate; for this case the observed number of large asteroids is substantially underestimated.

Experimental values for η have, however, been obtained for objects whose cohesive energy was much greater than that of asteroids. Since the surface to volume ratio decreases with the size of an object, one might expect that the relative number of small fragments is greater when a large object of the order of kilometers is shattered than is the case when an object of the order of centimeters is shattered. It appears then, keeping other variables fixed, that η would tend toward a higher value for large target masses. Consideration of this effect, therefore, gives further support to our results.

Using the particles-in-a-box approach, an empirical power-law-type comminution law with a constant exponent η and scaling parameters Γ, Γ', λ and λ_1, we have derived a stochastic model of asteroidal collisions. It was shown that, after a sufficiently long time, asteroids may reach the unique steady state distribution derived in the text. This distribution is shown to be in good agreement with observation by Kuiper et al (1958). These results imply then that the belt asteroids of Kuiper et al (1958) are in statistical equilibrium with respect to the various collisional rate processes discussed in the text. It does not appear possible, therefore, to estimate the age or the initial distribution of the asteroidal population from the present observational data.

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Attachments
References
Appendix
Figures 1 & 2

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APPENDIX

This appendix discusses the derivation of Eq. 4.9 in the main text by substitution of the series expansion, Eq. 4.7, represented as follows,

$$\boxed{\begin{array}{c}\text{catastrophic}\\ \text{collisions}\end{array}} + \boxed{\begin{array}{l}\text{catastrophic}\\ \text{creation}\end{array}} + \boxed{\begin{array}{l}\text{erosive}\\ \text{creation}\end{array}} + \boxed{\text{erosion}} = \frac{f(m,t)}{\partial t} \approx 0 . \tag{A.1}$$

The explicit form of the separate terms is given in the main text, and we present these terms now in expanded form, subject to the conditions Eqs. 4.4 and 4.5, $M_\infty \to \infty$ and $\mu \to 0$. We thus obtain

$$\boxed{\begin{array}{c}\text{catastrophic}\\ \text{collisions}\end{array}} = -K \sum_j \sum_\ell a_j a_\ell g_\ell m^{-\delta_j-\delta_\ell + 5/3} \tag{A.2}$$

where

$$g_\ell = (\Gamma')^{\delta_\ell-1}/(\delta_\ell-1) + 2(\Gamma')^{\delta_\ell-4/3}/(\delta_\ell-4/3) + + (\Gamma')^{\delta_\ell-5/3}/(\delta_\ell-5/3) \tag{A.3}$$

provided that

$$R(\delta_\ell) > 5/3 \qquad (A.4)$$

where R denotes the real part.

g_ℓ (Eq. A.3) includes the contribution of three processes. The first term on the right hand side of Eq. A.3 is the contribution of projectiles able to produce catastrophic collisions, treated as geometrical points. The second term, on the right hand side of Eq. A.3, is the contribution of grazing collisions defined as collisions in which the target object m would be missed if the projectile were a point particle. These grazing collisions give rise to the second term in Eq. A.3 that depends on the product of the radii of the colliding objects (second term on the right hand side of Eq. A.3). The last term (Eq. A.3) represents catastrophic collisions where finite sized projectiles are impacting point particle targets, m.

If $\delta_\ell = 5/3$, the right hand side of Eq. A.3 can be shown to diverge as $\ln M_\infty$; for $\delta_\ell < 5/3$, the divergence is even stronger ($M_\infty^{-\delta_\ell + 5/3}$). This behavior indicates, that terms with $\delta_\ell \leq 5/3$ are dominated by masses near the limiting mass of the distribution M_∞. Since these large masses do not have a steady state distribution, it follows that if they dominate the collisional processes for smaller objects, the distribution of the latter will also be transient. The condition Eq. A.4 is therefore necessary.

Next, using Eqs. 3.11, 4.4 and 4.7, we obtain

$$\boxed{\begin{array}{c}\text{catastrophic}\\ \text{creation}\end{array}} = \sum_j \sum_\ell a_j q_{j\ell}\, m^{-\delta_\ell - \delta_j + 5/3} \tag{A.5}$$

where

$$g_j = K(2-\eta)\, \frac{\lambda^{-\delta_j - \delta_\ell + 11/3}}{\delta_j + \delta_\ell - \eta - 5/3}\, Q_\ell \tag{A.6}$$

with

$$Q_\ell = \frac{1-(\Gamma')^{\delta_\ell - 1}}{1-\delta_\ell} + 2\,\frac{1-(\Gamma')^{\delta_\ell - 4/3}}{4/3-\delta_\ell} + \frac{1-(\Gamma')^{\delta_\ell - 5/3}}{5/3-\delta_\ell} + \tag{A.7}$$

$$+ \frac{1-(\Gamma')^{\delta_\ell - 2}}{2-\delta_\ell} + 2\,\frac{1-(\Gamma')^{\delta_\ell - 7/3}}{7/3-\delta_\ell} + \frac{1-(\Gamma')^{\delta_\ell - 8/3}}{8/3-\delta_\ell}$$

and provided that

$$R(\delta_j + \delta_\ell) > \eta + 5/3 \tag{A.8}$$

where R denotes the real part.

We note that Q_ℓ is non-singular because each of its terms converges to $\ln(\Gamma')$ as its denominator goes to zero. Since Γ' is of the order of $10^4 - 10^5$ and by Eq. A.4, $\delta_\ell > 5/3$, the first term in Q_ℓ is dominant. The factors in Eq. A.5 multiplied by the first term in Q_ℓ, represent the rate of supply into the mass range m to m + dm by fragmentation of target objects with mass λm or greater, by projectiles of mass $\lambda m/\Gamma'$ or greater, treated as point masses, and not contributing to the total mass of fragments. The other terms in Q_ℓ are smaller corrections due to grazing collisions and finite projectile mass.

The condition Eq. A.8 is necessary because Eq. A.5 diverges for $\delta_j + \delta_\ell \leq \eta + 5/3$ for $M_\infty \to \infty$. If η were greater than permitted by Eq. A.8, an excessive amount of small debris would be created, and in any finite mass range the population could not be replenished at the required rate to sustain a steady state.

The next term in Eq. A.1 is

$$\boxed{\begin{array}{c}\text{erosive}\\ \text{creation}\end{array}} = \sum_j \sum_\ell a_j a_\ell \, p_{j\ell} \, m^{-\delta_j - \delta_\ell + 5/3} \qquad \text{(A.9)}$$

and

$$p_{j\ell} = K(2-\eta) \frac{\Gamma^{\delta_j + \delta_\ell - 8/3} \lambda^{-\delta_j - \delta_\ell + 11/3}}{\delta_j + \delta_\ell - \eta - 5/3} P_\ell \qquad \text{(A.10)}$$

where

$$P_\ell = \frac{(\Gamma')^{-\delta_\ell + 5/3}}{\delta_\ell - 5/3} + 2\,\frac{(\Gamma')^{-\delta_\ell + 4/3}}{\delta_\ell - 4/3} + \frac{(\Gamma')^{-\delta_\ell + 1}}{\delta_\ell - 1} \qquad \text{(A.11)}$$

and where the boundary conditions Eqs. A.4 and A.8 are assumed satisfied.

Because of Eq. A.4, the first term on the right hand side of Eq. A.11 dominates. This is the contribution to the creation rate of objects in the mass range m to m + dm by point particle projectiles having masses $\lambda_e m/\Gamma$ or greater erosively impacting finite sized targets having masses of $\Gamma'\lambda_e m/\Gamma$ or greater. The remaining two terms of P_ℓ (Eq. A.11) are small corrections due to grazing collisions.

Finally,

$$\boxed{\text{erosion}} = -\sum_j \sum_\ell a_j a_\ell \, v_{j\ell} \, m^{-\delta_j-\delta_\ell + 5/3} \qquad (A.12)$$

where

$$v_{j\ell} = \Gamma K(\delta_j + \delta_\ell - 8/3) \, V_\ell \qquad (A.13)$$

and where

$$V_\ell = \frac{(\Gamma')^{\delta_\ell - 2}}{2-\delta_\ell} + 2\,\frac{(\Gamma')^{\delta_\ell - 7/3}}{7/3 - \delta_\ell} + \frac{(\Gamma')^{\delta_\ell - 8/3}}{8/3 - \delta_\ell} \qquad (A.14)$$

and provided that

$$R(\delta_\ell) < 2 \qquad (A.15)$$

where R represents the real part of the succeeding expression.

The dominant term in V_ℓ is the first one on the right hand side of Eq. A.14. This is the contribution of point projectiles with masses m/Γ' or smaller, erosively impacting finite target objects in the mass range m to $m + dm$, thereby reducing the number of objects in this mass range. The other two terms defining V_ℓ (Eq. A.14) are small corrections due to the small but finite size of the small projectiles.

If the condition Eq. A.15 is violated, Eq. A.12 can be shown to diverge when $\mu \to o$ and no series solution for a steady-state distribution far away from the limit μ exists. Physically, this means that erosion by small particles will dominate over other processes giving rise to a transient distribution.

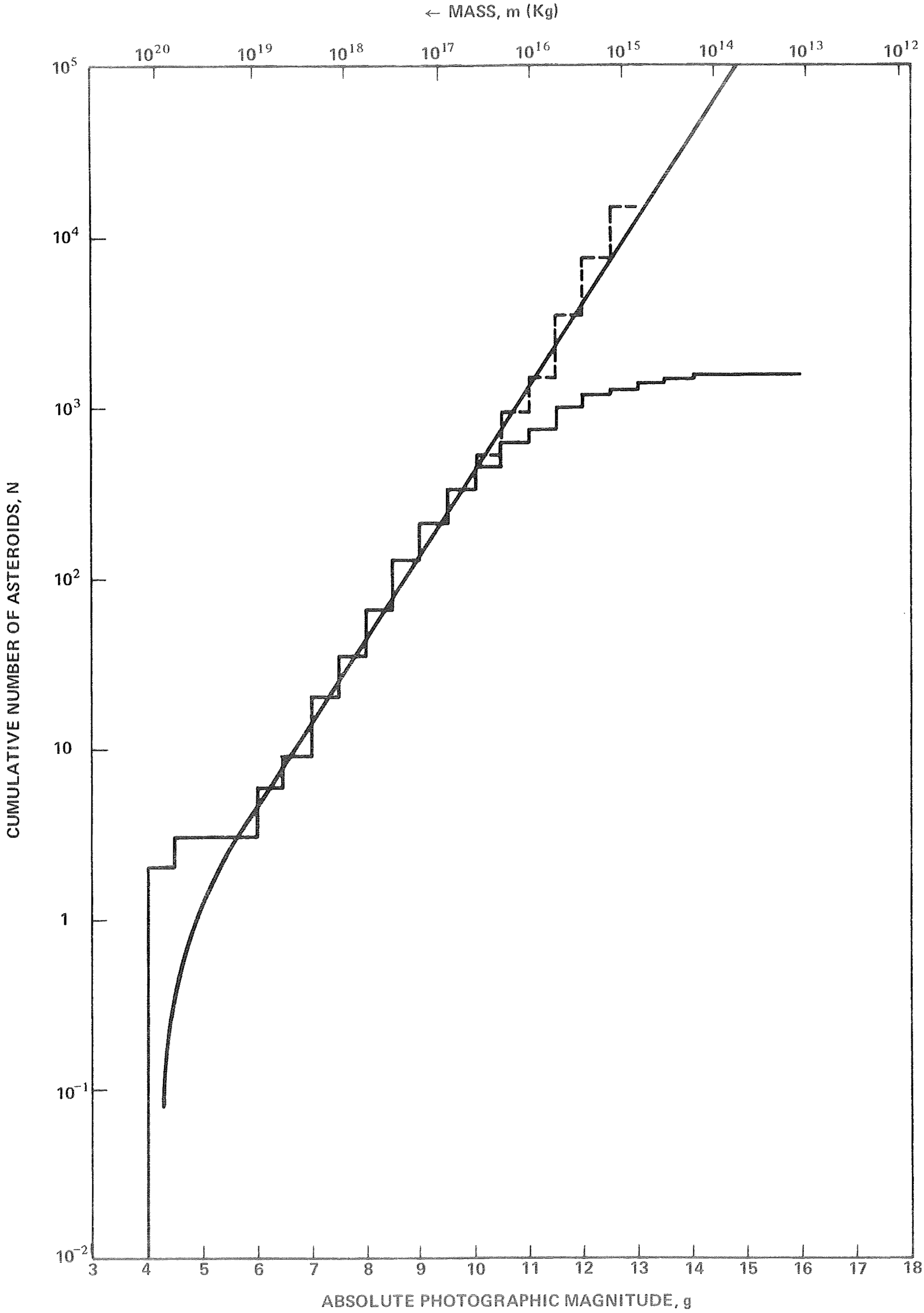

FIGURE 1 - CUMULATIVE NUMBER OF ASTEROIDS HAVING AN ABSOLUTE PHOTOGRAPHIC MAGNITUDE g OR SMALLER (I.E. MASS m OR GREATER).
OBSERVED VALUE = SOLID LINE HISTOGRAM
PROBABLE VALUE = DASHED LINE HISTOGRAM
EARLIER THEORY (DOHNANYI, 1969) = SOLID CURVE

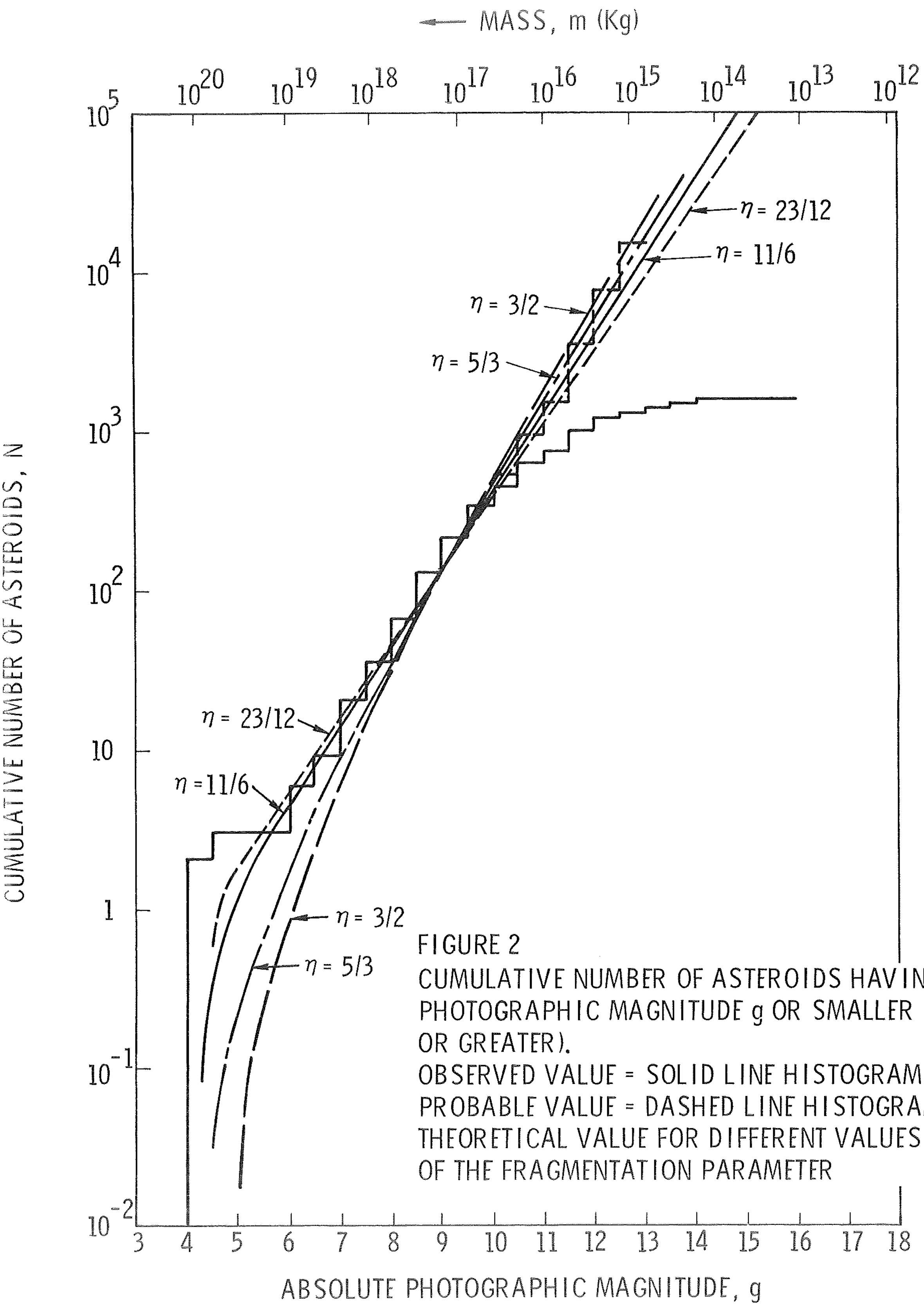

FIGURE 2

CUMULATIVE NUMBER OF ASTEROIDS HAVING AN ABSOLUTE PHOTOGRAPHIC MAGNITUDE g OR SMALLER (i. e. MASS m OR GREATER).

OBSERVED VALUE = SOLID LINE HISTOGRAM

PROBABLE VALUE = DASHED LINE HISTOGRAM

THEORETICAL VALUE FOR DIFFERENT VALUES OF THE FRAGMENTATION PARAMETER } AS INDICATED