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THEORETICAL INVESTIGATION OF THE REFLECTION OF IONIZING SHOCKS

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ORGANISATION EUROPÉENNE DE RECHERCHES SPATIALES
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THEORETICAL INVESTIGATION OF THE REFLECTION OF IONIZING SHOCKS

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ABSTRACT. The reflection of an ionizing shock from the end wall of a shock tube is studied theoretically following the experimental model of J. Smith. The observed perturbations in the wall pressure history are found to agree with this theory. To describe the first perturbation, a decrease in pressure due to the ionization part in the reflected shock structure, the flow equations are linearized but the rate equations are used in nonlinear form. The second perturbation, an increase in pressure due to the ionization part of the incident shock structure, is studied using Whitham's theory and assuming equilibrium behind the reflected shock.

1. INTRODUCTION

In a series of shock-tube experiments with xenon, Smith¹ found that the end-wall pressure was not constant after the reflection of an ionizing shock. Similar results were found for argon by Camac and Feinberg.² A typical form of the pressure profiles measured by Smith is shown in Figure 1 which also gives a description of the model used to study this problem.

The incident shock wave propagating in the shock tube leaves behind a gas at very high temperature. In the equilibrium state this gas must be ionized. However, as was shown by Petschek and Byron,³ this gas does not reach an equilibrium state immediately after the passage of the shock; there is a region (2 in Figure 1) in which the gas is almost in a frozen state with zero degree of ionization. After that, in region 6, the gas is assumed to have reached an equilibrium degree of ionization. Between regions 2 and 6 there is a thin ionization front.^{1,2,3,4} A similar process takes place behind the reflected shock. First, in region 4 the gas is almost frozen and reaches equilibrium after a certain time in region 5. Because of the higher temperatures behind the reflected shock, the equilibrium state is reached faster than behind the incident shock.

The first drop in the end wall pressure (Figure 1) observed by Smith¹ was associated with the transition from frozen to equilibrium state behind the reflected shock. This problem will be studied in Chapter 2. After the first drop in pressure, Smith¹ observed a region of almost constant pressure followed by an increase. He associated this increase in pressure with a shock wave produced by the interaction of the ionization front (behind the incident shock) with the reflected shock. It will be shown that this interaction sometimes produces a family of weak compression waves rather than a shock. This problem will be studied in Chapter 3.

To study the transition from region 4 to 5, it will be assumed that the equilibrium degree of ionization is so small that the equilibrium flow properties differ only in the first order from the frozen flow properties. This will allow us to linearize the flow equations. However, it will not be possible to linearize the equations governing the production of ions and electrons, since in these equations appear terms such as $\exp(-\theta_i/T)$, where θ_i is the ionization temperature and T the temperature of the gas. Terms like this can only be linearized for such small degrees of ionization that the properties of the flow will not experience any significant change. A study similar to this was made by Spence,⁴ who studied the problem of a

piston moving in a gas with such a velocity that the vibrating states were excited (or dissociation produced); however, he linearized the rate equations.

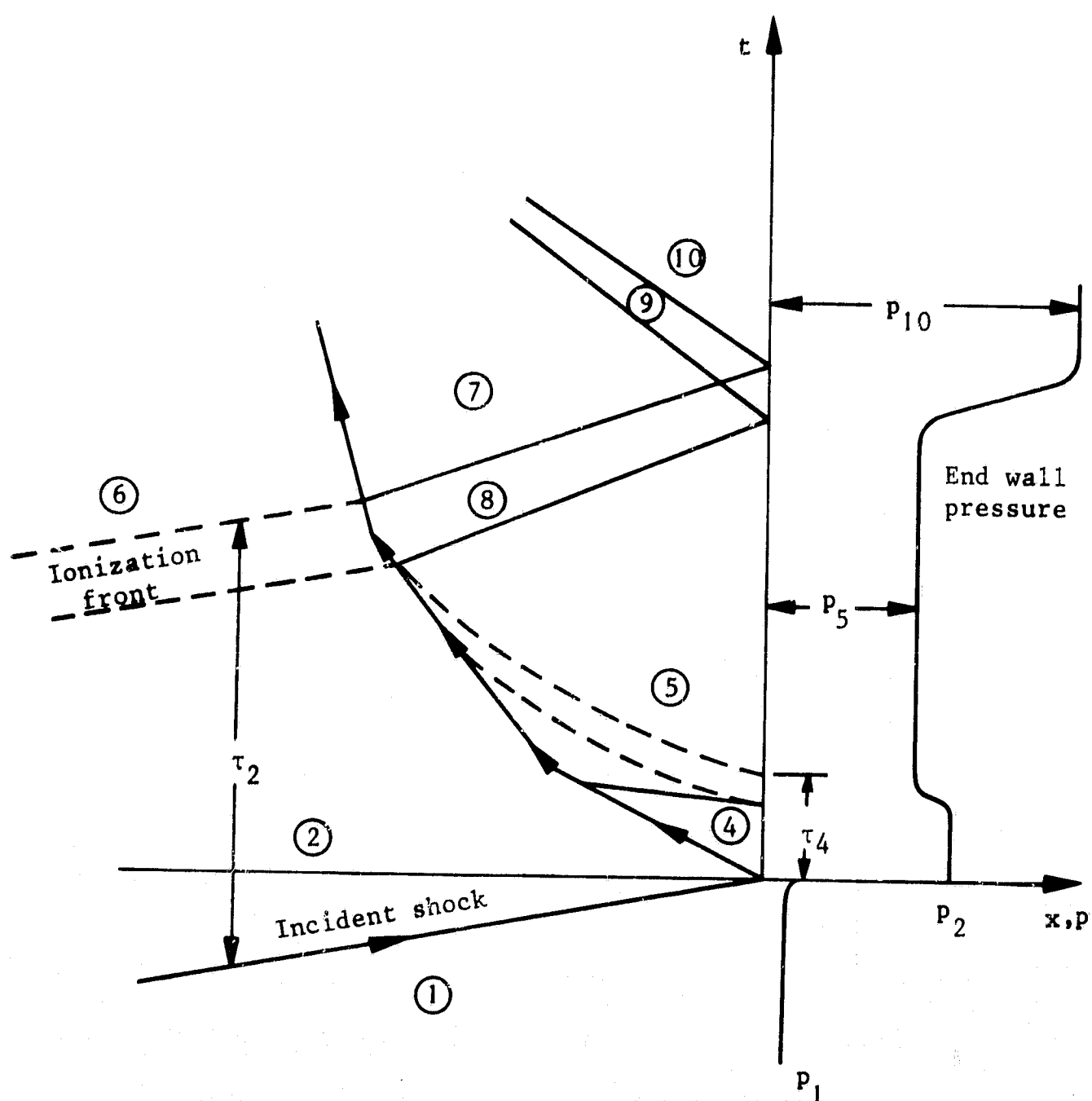


Figure 1.- x-t diagram showing shock trajectories, ionization fronts, and their interactions. On the right is shown a typical form of the end wall pressure profiles measured by Smith.¹

The rate equations used governing the production of electrons and ions will be those given by Morgan and Morrison.⁵ Their model can be described as follows: after the reflection of the shock there are no electrons; the first electrons are produced by atom-atom collision according to the model proposed by Harwell and Jahn;⁶ when enough electrons are produced, the dominating mechanism is electron-atom collision (Petschek and Byron³). It will be shown that the perturbation in pressure at the wall decreases proportionally to the increase in degree of ionization according to the formula:

$$p_4 - p_{\text{wall}}(t) = \frac{2}{3} \frac{(\theta_i/T_4)p_4}{(1 + \sqrt{3})} \alpha_{\text{wall}}(t)$$

where p_4 and T_4 are the frozen pressure and temperature about which we linearized, θ_i is the ionization temperature, and α is the degree of ionization at the wall. This behaviour agrees with Smith's results,¹ which show a decrease in pressure in the region where α increases, and a constant pressure in the equilibrium region where α is constant.

Camac and Feinberg¹⁰ say that this decrease in pressure is due to an expansion wave preceding the ionization front. If that were so the expansion wave would be in a frozen region and could be considered as an isentropic expansion. We do not think that this is a correct description of what really happens. According to our results the decrease in pressure is due to the fact that the process is not isentropic. The ionization acts like a number of heat sinks distributed over the flow field. These heat sinks decrease the entropy of the atom gas. This decrease in entropy combined with the non-stationary character of the process are the factors that produce the decrease in pressure at the wall.

The compression wave between regions 5 and 7 is assumed to be at the equilibrium degree of ionization as given by the Saha equation. Our calculations have been made for degrees of ionization of the order of 30%. Under these conditions it is observed that temperature and degree of ionization

do not experience any significant changes. This seems to be consistent with the fact that, at such degrees of ionization, γ , the ratio of the specific heats, is close to unity. Whitham's rule⁷ is used to calculate the second decrease of the reflected shock speed, during its interaction with the ionization front following the incident shock. An extension of Whitham's rule was made by Lick;⁸ both rules are, in our problem, equivalent to assuming that the curvature of the shock does not produce a significant change in the entropy of the gas behind the shock. This approximation seems to be good. As the reflected shock curves, it sends a family of compression waves to the wall. If they intersect they produce a shock; it will be shown that sometimes this is not the case. The interaction of this family of compression waves with the wall is studied assuming that temperature and degree of ionization remain constant. It will be seen that this interaction produces an increase in pressure on the end wall. Our results are in good agreement with Smith's measurements regarding the total amount of the increase in pressure and the time at which it happens, but in Smith's measurements the characteristic time of the pressure rise is longer.

2. RELAXATION REGION BEHIND THE INCIDENT SHOCK

2.1 INTRODUCTORY REMARKS

In this chapter a study will be made of the transition region from 4 to 5 (Figure 1). Region 4 of frozen flow does not exist in reality. The transition to region 5 will start immediately after the reflection of the shock. However, in the initial stages the ionization is almost negligible, and we can assume a certain frozen region behind the reflected shock. Similar arguments can be used for regions 2 and 6 behind the incident shock: This last transition is assumed to be much longer than that behind the reflected shock. The frozen conditions in region 4 are calculated by the conventional methods of gas dynamics applied to a perfect monatomic gas ($\gamma = 5/3$). We know the conditions in region 1 and the Mach number, M_1 , of the incident

shock; conditions in region 2 will be given by the usual jump conditions across the shock:

$$\frac{p_2}{p_1} = 1 + \frac{2}{\gamma+1} (M_1^2 - 1)$$

$$u_2 = M_1 a_1 \left(1 - \frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2} \right)$$

where p is the pressure, u the flow velocity (relative to the end wall of the shock tube), and a the speed of sound. Similar formulae hold for the other flow quantities in region 2 (see Liepmann and Roshko,⁹ Chapter 2). Once we know the flow quantities in region 2, we apply the same jump conditions to calculate the flow properties in region 4, where the speed of the gas has to be zero. If c_4 is the speed of the reflected shock:

$$\frac{c_4 + u_2}{c_4} = \frac{1 + \frac{\gamma+1}{\gamma-1} \frac{p_4}{p_2}}{\frac{\gamma+1}{\gamma-1} + \frac{p_4}{p_2}}$$

$$c_4 + u_2 = a_2 \left(\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma} \frac{p_4}{p_2} \right)^{\frac{1}{2}}$$

These two equations give us p_4 and c_4 . The rest of the flow properties in region 4 can be easily calculated from the other jump conditions. For large incident Mach numbers the limiting values of the conditions in 4 take a simpler form:

$$M_1 \gg 1$$

$$\frac{p_4}{p_2} = \frac{3\gamma-1}{\gamma-1} = 6 \quad (1a)$$

$$\frac{T_4}{T_2} = \frac{3\gamma-1}{\gamma} = \frac{12}{5} \quad (1b)$$

$$\frac{a_4}{c_4} = \sqrt{\frac{3\gamma-1}{2(\gamma-1)}} = \sqrt{3} \quad (1c)$$

$$\frac{a_2}{c_4} = \sqrt{\frac{\gamma}{2(\gamma-1)}} = \sqrt{\frac{5}{4}} \quad (1d)$$

The movement of the gas in the transition from 4 to 5 will be considered one-dimensional and non-stationary. The x-axis will be taken along the shock tube; the origin will be at the end wall. The origin of time will be the moment at which the incident shock hits the end wall (see Figure 1). The problem will be solved for such small degrees of ionization that the flow properties can be considered a small perturbation of the frozen flow properties.

$$\rho = \rho_4 + \rho'(x,t) \quad \rho'/\rho_4 \ll 1 \quad (2a)$$

$$u = u'(x,t) \quad u'/a_4 \ll 1 \quad (2b)$$

$$T = T_4 + T'(x,t) \quad T'/T_4 \ll 1 \quad (2c)$$

$$p = p_4 + p'(x,t) \quad p'/p_4 \ll 1 \quad (2d)$$

where T is the temperature, and a_4 is the speed of sound of the frozen gas.

The effect of the ionization will be taken into account only in the energy equation. As the monatomic gas becomes ionized, the translational energy of its atoms is transformed into ionization energy. The ionization

effect can be considered as a series of heat sinks distributed over the flow field.*

2.2 FLOW EQUATIONS

The flow equations will be:

$$\frac{\partial \rho'}{\partial t} + \rho_4 \frac{\partial u'}{\partial t} = 0 \quad (3)$$

$$\rho_4 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \quad (4)$$

$$T_4 \frac{\partial S'}{\partial t} = - \frac{E_{ion}}{m_A} \frac{\partial \alpha}{\partial t} \quad (5)$$

where α is the degree of ionization, E_{ion} is the energy required to ionize an atom, m_A the mass of an atom and S' is the perturbed entropy of the gas:

$$S = S_4 + S'$$

Equation (3) is the continuity equation, (4) the momentum equation, and (5) the energy equation.

The ionization temperature θ_i is defined by:

$$\theta_i = \frac{E_{ion}}{m_A R}$$

R is the perfect gas constant:

$$R = c_p - c_v, \quad c_p/c_v = \gamma = 5/3$$

* The electronic excitation of the atoms also acts like another heat sink. This effect could be important in the first stages of the ionization. It has been neglected in this analysis.

Using well-known thermodynamic relations, the energy equation can be written:

$$\frac{\partial \rho'}{\partial t} - \frac{1}{a_4^2} \frac{\partial p'}{\partial t} = \frac{\gamma-1}{\gamma} \rho_4 \frac{\theta_i}{T_4} \frac{\partial \alpha}{\partial t} \quad (6)$$

This equation tells us how small α has to be in order for conditions (1) to be satisfied.

$$(\theta_i/T_4) \alpha \ll 1 \quad (7)$$

For argon θ_i is 183 000 °K, and in our case T_4 is of the order of 10 000 °K. Then (θ_i/T_4) is a large quantity, and α being small is not sufficient for conditions (1) to be satisfied.

If α were known, we would have three equations (3), (4) and (6) with three unknowns u' , p' , ρ' . We need to find more information about the degree of ionization α .

To obtain equation (6) we used the equation of state:

$$\frac{p}{\rho_4} = \frac{\rho}{\rho_4} \frac{T}{T_4}$$

where p , ρ , and T are the properties of the atom gas. It could be argued that this is not true because of the presence of the electron gas. However, in our case this will only introduce a lower order correction. We should remember that:

$$1 > \frac{\theta_i}{T_4} \alpha \gg \alpha$$

A similar argument can be used for the continuity equation. Since θ_i/T_4 is so large, the most important effect of the ionization is to take heat out of the atom gas; any other effect is negligible in comparison with this.

2.3 EQUATIONS DETERMINING IONIZATION RATES

It will be assumed that the mechanisms that determine ionization rates are three: atom-atom collision, electron-atom collision, and recombination. We will give a brief description of these three.

(a) *Atom-atom collisions.* This process is that which produces the first electrons. It has been studied by Harwell and Jahn.⁶ Two atoms collide, and, if the collision energy is large enough, one of the atoms becomes excited and reaches a certain metastable energy level. An atom that has reached this energy level ionizes very easily. The rate of ionization is then determined by the rate at which the atoms reach this excited state. Morgan and Morrison⁵ give an expression for the rate at which electrons are produced by this process:

$$\begin{aligned} \left(\frac{Dn_e}{Dt}\right)_{aa} &= 4S_2 n_a^2 (\pi m_a)^{-\frac{1}{2}} (kT)^{\frac{3}{2}} \left(\frac{E_1}{2kT} + 1\right) \exp\left(-\frac{E_1}{kT}\right) \\ &= n_a^2 R_{aa} \end{aligned} \quad (8)$$

$\frac{D}{Dt}$ is the derivative along the gas flow:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$$

k is the Boltzmann constant, n_e is the number of electrons per unit volume, n_a the number of atoms per unit volume, m_a the atom mass, E_1 the excitation energy of the metastable state, and S_2 a constant determining the inelastic collision cross-section. Harwell and Jahn⁶ found that the atom-atom cross-section for excitation from the ground state is represented by

$$S_2(\epsilon - E_1) \text{ for } \epsilon > E_1; \quad 0 \text{ for } \epsilon < E_1 \quad (9)$$

where ϵ is the energy of the collision in electron volts in the centre of mass system. According to them for argon E_2 is 11.7 eV and S_2 is 7.1×10^{-19} cm²/eV. However, Morgan and Morrison⁵ say that S_2 is perhaps 7.1×10^{-20} cm²/eV. More recent experiments by Kelly¹⁶ give an intermediate value for S_2 of 1.2×10^{-19} cm²/eV.

(b) *Electron-atom collisions.* This mechanism is described in detail by Petschek and Byron.³ Once the first electrons are produced, electron-atom collisions take place; these are more effective than atom-atom collisions. An electron collides inelastically with an atom and the atom reaches a certain metastable state. Then the atom ionizes very easily, as in the case of atom-atom collision. The electron loses energy in the inelastic collision and recovers it by elastic collision with atoms and ions. This requires the electron gas to have a lower temperature than the atom gas. Both ion and atom gas are assumed to have the same temperature, and this temperature is the one that appears in the flow equations (2.2). Morgan and Morrison⁵ give an expression for the rate of production of electrons by electron-atom collision:

$$\begin{aligned} \left(\frac{Dn_e}{Dt}\right) &= 8n_e n_a S_1 (2\pi m_e)^{-\frac{1}{2}} (kT_e)^{\frac{3}{2}} \left(\frac{E_1}{2kT_e} + 1\right) \exp\left(-\frac{E_1}{kT_e}\right) \\ &= n_e n_a R_{ea} \end{aligned} \quad (10)$$

where T_e is the temperature of the electron gas, S_1 has a meaning corresponding to that given in equation (9), and m_e is the mass of the electron. Petschek and Byron³ give a value of 7×10^{-18} cm²/eV for S_1 with argon.

(c) *Recombination.* When the degree of ionization is close to equilibrium, recombination causes the rate of production of electrons to decrease and, as the degree of ionization tends to the equilibrium value, to tend to zero.

In this study we will use the three-body recombination model proposed by Camac and Feinberg.¹⁰ This process is the opposite of ionization by electron-atom collision:



Two electrons collide with an ion and produce an electron and an atom. Two electrons are needed in order to satisfy the momentum conservation. For details see Camac and Feinberg.¹⁰ According to (11) the rate of production of electrons will be:

$$\frac{Dn_e}{Dt} = k_{ion} n_e n_a - k_{rec} n_e^2 n_i \quad (12)$$

where k_{rec} is the recombination rate constant and k_{ion} the ionization rate constant. The value of k_{ion} can be found from (10) and (8). In equilibrium

$$(k_{rec})_{eq} = (k_{ion})_{eq} \frac{(n_a)_{eq}}{(n_e)_{eq} (n_i)_{eq}}$$

Assuming that the electrical forces are large compared with the dynamic forces:

$$n_e \simeq n_i$$

Also from condition (7)

$$n_e \ll n_a$$

$$\alpha = \frac{n_e}{n_e + n_a} \approx \frac{n_e}{n_a}$$

$$(k_{rec})_{eq} = (k_{ion})_{eq} \frac{1}{n_a} (\alpha_{eq} (p_{eq}, T_{eq}))^{-2} \quad (13)$$

α_{eq} can be determined from the Saha equation as a function of the flow properties at equilibrium, $\alpha_{eq}(p_{eq}, T_{eq})$. Camac and Feinberg¹⁰ assume that relation (13) also holds in non-equilibrium situations, with α_{eq} given by the Saha equation, but with the pressure and the electron temperature that the gas really has in the non-equilibrium conditions:

$$k_{rec} = k_{ion} \frac{1}{n_a} (\alpha_{eq}(p, T_e))^{-2} \quad (13a)$$

Because of conditions (2), n_a can be considered a constant. There is some arbitrariness in the fact that we have chosen the electron temperature instead of the atom temperature to determine this fictitious α_{eq} . Camac and Feinberg¹⁰ say that it is the electrons that determine the ionization rate, and consequently the electron temperature is the significant one (see also Smith¹).

Combining the three results obtained previously, (8), (10) and (13a), we get:

$$k_{ion} = \frac{n_a}{n_e} (R_{aa} + \alpha R_{ea})$$

$$\frac{Dn_e}{Dt} = n_a^2 (R_{aa} + \alpha R_{ea}) \left(1 - (\alpha/\alpha_{eq})^2\right) \quad (14)$$

We see that when α approaches its equilibrium value $\frac{Dn_e}{Dt}$ tends to zero, as was expected. The Saha equation gives:

$$\frac{\alpha_{eq}^2}{1 - \alpha_{eq}^2} = (3.35 \times 10^{-2} \frac{\text{newtons}}{\text{m}^2 (\text{°K})^{5/2}}) \frac{2(C_1 + C_2 \exp(-\theta_a/T_e) + \dots)}{(C_3 + C_4 \exp(-\theta_e/T_e) + \dots)} \times \frac{T_e^{5/2}}{p} \exp(-\theta_i/T_e) \quad (15)$$

where C_1 and C_2 are the degeneracies of ground state and first excited state of the ion, and C_3 and C_4 those of the atom. θ_a is the excitation temperature of the first excited state of the ion. The contributions of the other excited states are neglected. For the values of these constants see, for example, Witte.¹¹

We now face the problem of how to linearize equations (14) and (15). We will assume that

$$T_e = T_4 + T'_e \quad T'_e/T_4 \ll 1 \quad (16)$$

This is to be expected because in equilibrium atoms and electrons have the same temperature, and, because of conditions (2), the flow properties are a small perturbation of the equilibrium flow properties. This assumption will be checked later when an equation to determine T_e is found, and when the numerical results of the problem are obtained.

In expressions (8), (10), (14) and (15) appear terms like:

$$\exp\left(-\frac{\theta_i}{T_e}\right) = \exp\left(-\frac{\theta_i}{T_4 + T'_e}\right) \approx \exp\left(-\frac{\theta_i}{T_4}\right) \exp\left(\frac{\theta_i}{T_4} \frac{T'_e}{T_4}\right)$$

and similar terms with T . The linearization of these terms would require:

$$\frac{\theta_i}{T_4} \frac{T'_e}{T_4} \ll 1 \quad \frac{\theta_e}{T_4} \frac{T'_e}{T_4} \ll 1$$

But these are much stronger conditions than (2) and (7). Since the perturbations are expected to be of order $\alpha(\theta_i/T_4)$, the above conditions would require $(\theta_i/T_4)^2 \propto \alpha$ to be small compared with unity. Under these circumstances the perturbed flow quantities would be so small as to render this analysis meaningless. We proceed supposing that:

$$\left(\frac{\theta_i}{T_4}\right)^2 \propto \alpha \quad \text{is not necessarily small} \quad (17)$$

All the terms of equation (14) can be linearized except those containing exponentials of the form indicated. We notice that:

$$\frac{E_1}{kT} = \frac{\theta_e}{T}$$

where θ_e is 135 000 °K for argon, that is, of the same order of magnitude as θ_i .

The semi-linearized equation (14) becomes:

$$\frac{\partial \alpha}{\partial t} = A (e^{-\theta_e/T} + B e^{-\theta_e/T_e} \alpha) (1 - C \alpha^2 e^{\theta_i/T_e}) \quad (18)$$

where A, B, and C are constants

$$A = 4n_4 S_2 (\pi m_a)^{-1/2} (kT_4)^{3/2} \left(\frac{\theta_e}{2T_4} + 1 \right)$$

$$B = 2(S_1/S_2) (m_a/2m_e)^{1/2}$$

$$C = \left[3.35 \times 10^{-2} \frac{\text{newtons}}{m^2 (\text{°K})^{5/2}} \right]^{-1} \left[\frac{C_3}{2(C_1 + C_2 \exp(-\theta_a/T_4))} \right] \frac{P_4}{T_4^{5/2}}$$

θ_a is only 2060 °K for argon, so it is possible to linearize the exponential containing θ_a . The other terms of the partition functions are ignored because they are small.

T_e can be calculated using the equation for the conservation of energy of the electron-gas. The electrons lose their energy by inelastic collisions with the atoms and recover it by elastic collisions with the ions and atoms. This process has been studied in detail by Petschek and Byron³ and by Morgan and Morrison.⁵

The rate of energy transfer by elastic collisions between electrons and ions was calculated by Landau and is given by Morgan and Morrison:⁵

$$Q_i = \frac{n_e^2 e^4}{m_i} \left(\frac{8\pi m_e}{kT_e} \right)^{1/2} \frac{(T_i - T_e)}{T_e} \ln \left(\frac{9(kT_e)^3}{8\pi n_e e^6} \right) \quad (18a)$$

where e is the electron charge in Gaussian units:

$$e = 4.8 \times 10^{-10} \text{ cm} \times \text{dyne}^{\frac{1}{2}}$$

T_i is the ion temperature that, in our case, is equal to T (the atom temperature).

The rate of energy transfer by elastic collisions between atoms and ions was calculated by Petschek and Byron.³ We use the expression given by Morgan and Morrison:⁵

$$Q_a = 4.36 \times 10^{46} \frac{m_e^2}{m_a} \left(\frac{T}{T_e} - 1 \right) n_a n_e 4\pi \left(\frac{m_e}{2\pi k T_e} \right)^{3/2} \times \int_0^\infty S(\epsilon) \epsilon^{5/2} \exp \left(-\frac{11600\epsilon}{T_e} \right) d\epsilon^{\frac{1}{2}} \quad (18b)$$

where all quantities are in the c.g.s. system, except ϵ (translational energy of the electron) which is expressed in eV. $S(\epsilon)$ is the electron-atom momentum transfer cross-section. For the numerical analysis the values of $S(\epsilon)$ will be taken from von Engel.¹²

The energy balance for the electrons will be:⁵

$$Q_i + Q_a = \left(\frac{Dn_e}{Dt} \right)_{ea} \left(E_{ion} + \frac{3}{2} kT_e \right) \quad (19)$$

We now apply conditions (2) and (16) to this equation and assume that in our case the electron temperature is much smaller than the ionization temperature (18a, b, 19):

$$\left[D\alpha \ln \left(\frac{E}{\alpha} \right) + F \right] (T - T_e) = A B e^{-\theta_e/T_e} (1 - C\alpha^2 e^{\theta_i/T_e}) \quad (20)$$

where D , E and F are new constants depending on the unperturbed conditions of region 4. A , B and C are the constants appearing in equation (18).

We now have a system of four differential equations (3), (4), (6) and (18) with six unknowns (p , ρ , u , T , T_e and α). There are two additional algebraic equations, (20) and the equation of state:

$$\frac{p}{p_4} = \frac{\rho}{\rho_4} \frac{T}{T_4} \quad (21)$$

2.4 BOUNDARY CONDITIONS

At the wall the speed of the gas is zero.

$$x = 0, \quad u = 0 \quad (22)$$

Other boundary conditions are given by the Rankine-Hugoniot equations across the reflected shock. Upstream of the reflected shock the conditions are constant, and downstream there is a small perturbation of the constant frozen conditions 4. We expect that the speed of the reflected shock will be a small perturbation of the frozen speed of the shock c_4 . Then, to a first approximation, we can apply the boundary conditions on the frozen reflected shock $x = -c_4 t$.

In region 2 we have some fixed values of the flow quantities: p_2 , ρ_2 , T_2 , u_2 . The Rankine-Hugoniot equations give the conditions behind the reflected shock as a function of these fixed values and the speed of the reflected shock. Calling this speed c we have:

$$\begin{array}{ll} c = c_4 + c' & c'/c_4 \ll 1 \\ p_4 + p' = f_1(c_4 + c') & p_4 = f_1(c_4) \\ \rho_4 + \rho' = f_2(c_4 + c') & \rho_4 = f_2(c_4) \\ T_4 + T' = f_3(c_4 + c') & T_4 = f_3(c_4) \\ u' = f_4(c_4 + c') & 0 = f_4(c_4) \end{array}$$

then

$$p' = \left(\frac{df_1}{dc} \right)_{c=c_4} c' \quad (23a)$$

$$\rho' = \left(\frac{df_2}{dc} \right)_{c=c_4} c' \quad (23b)$$

$$T' = \left(\frac{df_3}{dc} \right)_{c=c_4} c' \quad (23c)$$

$$u' = \left(\frac{df_4}{dc} \right)_{c=c_4} c' \quad (23d)$$

Eliminating c' :

$$p' = \frac{\left(\frac{df_1}{dc} \right)_{c=c_4}}{\left(\frac{df_4}{dc} \right)_{c=c_4}} u' \quad \text{on } x = -c_4 t$$

$$T' = \frac{\left(\frac{df_3}{dc} \right)_{c=c_4}}{\left(\frac{df_1}{dc} \right)_{c=c_4}} p' \quad \text{on } x = -c_4 t$$

We could obtain a further relation between p' and ρ' , but that would be a combination between the last equation and the equation of state. The last two equations can be reduced to:

$$p' = -k_1 a_4 \rho_4 u' \quad \text{on } x = -c_4 t \quad (24)$$

$$T' = \frac{T_4}{p_4} k_2 p' \quad \text{on } x = -c_4 t \quad (25)$$

where

$$k_1 = \frac{[(\gamma+1) + (\gamma-1) (p_4/p_2)]^{\frac{1}{2}}}{[(p_4/p_2) (\gamma+1) + 3\gamma - 1]} \left[\frac{2}{\gamma(p_4/p_2)} \right]^{\frac{1}{2}} \left[(\gamma-1) + (\gamma+1) \frac{p_4}{p_2} \right] \quad (26)$$

$$k_2 = 1 + \frac{p_4/p_2}{[(\gamma+1)/(\gamma-1)] + (p_4/p_2)} - \frac{(p_4/p_2) (\frac{\gamma+1}{\gamma-1})}{1 + (\frac{\gamma+1}{\gamma-1}) (\frac{p_4}{p_2})} \quad (27)$$

For $\gamma = 5/3$

$$k_1 = \left[\frac{4 + (p_4/p_2)}{5p_4/p_2} \right]^{\frac{1}{2}} \left[\frac{1 + (4p_4/p_2)}{3 + (2p_4/p_2)} \right] \quad (28)$$

$$k_2 = 1 + \frac{p_4/p_2}{4 + (p_4/p_2)} - \frac{4p_4/p_2}{1 + (4p_4/p_2)} \quad (29)$$

And for $M_1 \gg 1$ we saw in (i) that p_4/p_2 tends to 6 and

$$k_1 = 0.96 \quad (28a)$$

$$k_2 = 0.64 \quad (29a)$$

The fact that k_1 is so close to unity has a very important physical significance:

$$p' \approx -\rho_4 a_4 u' \quad \text{along the shock}$$

This means that the characteristics intersecting the shock are absorbed by the shock, and they almost do not reflect. The shock almost behaves like a characteristic (cf. Whitham⁷). It should also be pointed out that k_1 remains

close to unity for every value of p_4/p_2 between one and six (see (28)). In particular for $(p_4/p_2) = 1.0$, $k_1 = 1.0$.

Spence^h used in his analysis the strong shock jump conditions, because he was working with an incident shock. His boundary condition cannot be applied in this work because the reflected shock can never be considered a strong shock ($p_4/p_2 = 6$).

Since the degree of ionization is zero in region 2 we have also:

$$\alpha = 0 \quad \text{on} \quad x = -c_4 t \quad (30)$$

A further boundary condition is that initially there is no ionization and all quantities are frozen.

$$\text{At} \quad x = 0, \quad t = 0$$

$$\alpha = p' = \rho' = T' = u = 0 \quad (31)$$

Equations (22), (24), (25), (30) and (31) will be our boundary conditions.

The perturbed speed of the shock can be obtained from (23a):

$$c' = \left(1 / (df_1/dc)_{c=c_4} \right) p' = k_3 \frac{a_4}{p_4} p'$$

where

$$k_3 = (2\gamma)^{\frac{1}{2}} \left(\frac{\gamma+1}{4\gamma} \right) \left(\frac{p_4}{p_2} \right)^{\frac{1}{2}} \left[(\gamma+1) + (\gamma-1) \frac{p_4}{p_2} \right]^{-\frac{1}{2}} \quad (32)$$

and p' is the perturbed pressure behind the shock. For $\gamma = 5/3$

$$k_3 = \frac{2}{\sqrt{5}} \left(\frac{p_4}{p_2} \right)^{\frac{1}{2}} \left[4 + \frac{p_4}{p_2} \right]^{-\frac{1}{2}} \quad (32a)$$

And for $M_1 \gg 1$, p_4/p_2 is 6;

$$k_3 = 0.692 \quad (32b)$$

2.5 NUMERICAL RESULTS

The system of equations (3), (4), (6), (18), (20) and (21) with boundary conditions (22), (24), (25), (30) and (31) was solved numerically. To do this the flow equations were put in characteristic form:

$$\left(\frac{\partial}{\partial t} + a_4 \frac{\partial}{\partial x}\right) (p' + a_4 \rho_4 u') = - (\gamma-1) \frac{\theta_i}{T_4} p_4 g(\alpha, T, T_e) \quad (33a)$$

$$\left(\frac{\partial}{\partial t} - a_4 \frac{\partial}{\partial x}\right) (p' - a_4 \rho_4 u') = - (\gamma-1) \frac{\theta_i}{T_4} p_4 g(\alpha, T, T_e) \quad (33b)$$

$$\frac{\partial}{\partial t} \left(\frac{\gamma}{\gamma-1} \frac{T'}{T_4} - \frac{p'}{p_4} + \frac{\theta_i}{T_4} \alpha \right) = 0 \quad (33d)$$

$$\frac{\partial \alpha}{\partial t} = g(\alpha, T, T_e) \quad (33e)$$

g is given in equation (18). The characteristics are the lines:

$$x - a_4 t = \text{const.}$$

$$x + a_4 t = \text{const.}$$

$$t = \text{const.}$$

In this problem the characteristics and the boundaries are fixed straight lines.

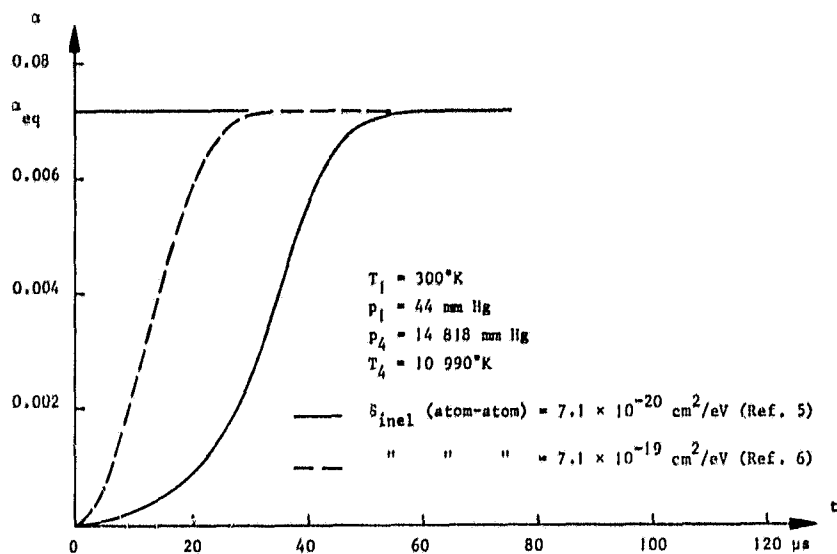


Figure 2.- Degree of ionization at the wall.

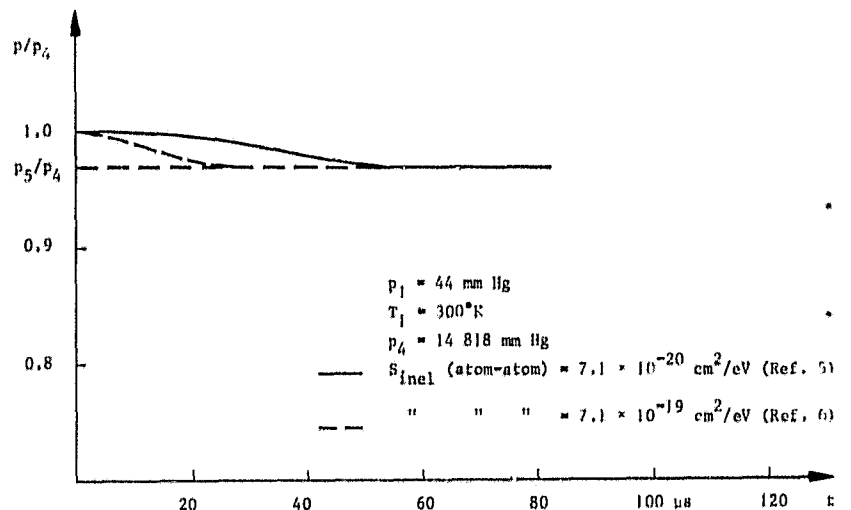


Figure 3.- End wall pressure distribution.

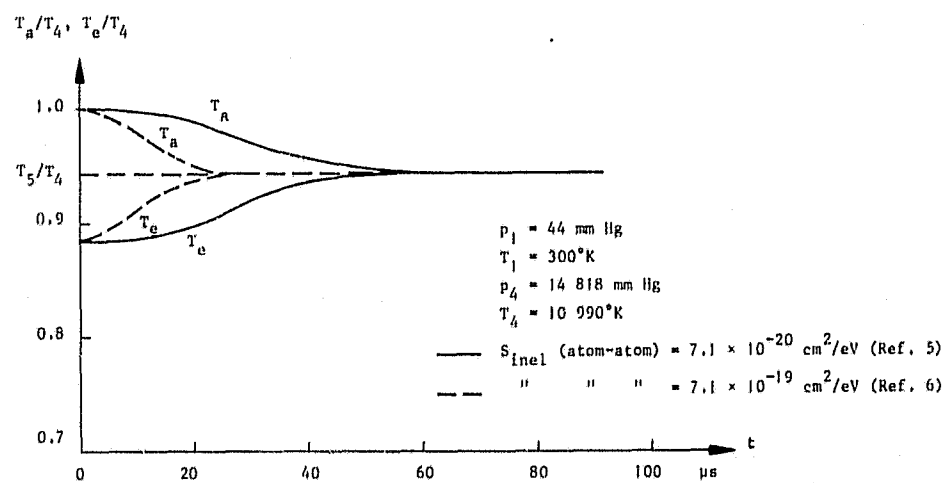


Figure 4.- Electron and atom temperatures at the end wall.

In Figures 2, 3, 4, 5 and 6 are represented some of the results obtained from the numerical analysis. As was pointed out before, the value of S_2 (equation (9)) is uncertain. So the calculations were performed for the two different values of S_2 given by Harwell and Jahn⁶ and Morgan and Morrison.⁵

None of Smith's experiments¹ was done for such low degrees of ionization as would allow linearization. Consequently we cannot compare them quantitatively with our calculations. However, we can see that there is a decrease

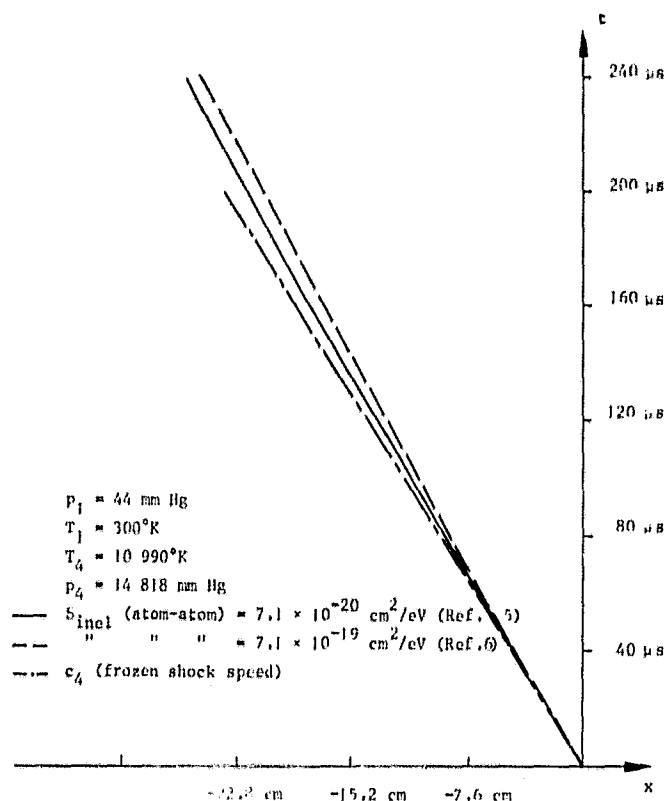


Figure 5.- Reflected shock trajectory.

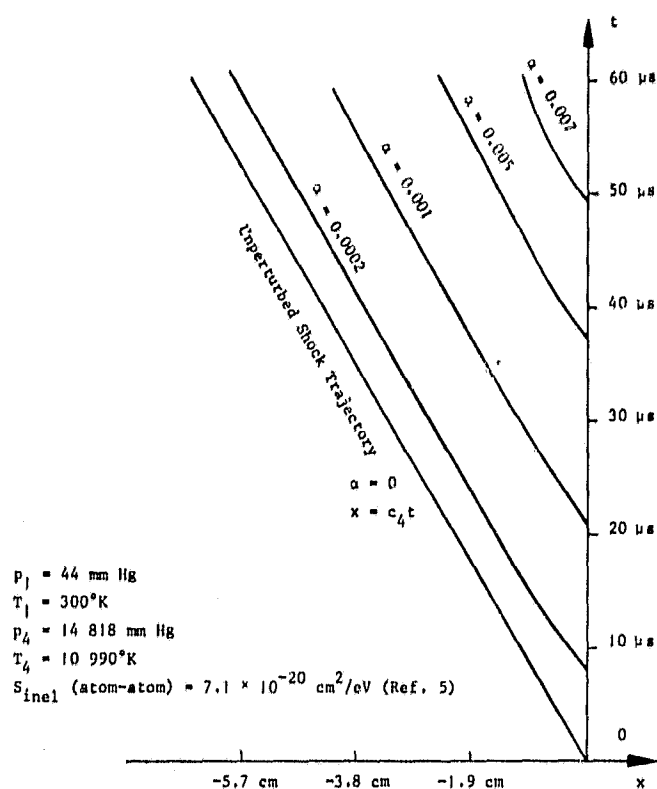


Figure 6.- Distribution of degree of ionization in the x,t plane.

in pressure that corresponds to an increase in the degree of ionization, and that after a certain time the degree of ionization reaches asymptotically its equilibrium value and the pressure also reaches a constant equilibrium value (Figures 2 and 3). When the equilibrium value is reached (Figure 4) electrons and atoms reach the same temperature; this was to be expected because of the energy balance in the electron gas, see equation (20). In Figure 5 is shown the shock trajectory; the ionization tends to decrease the shock speed and after a certain time the shock moves at a constant equilibrium speed.

A very interesting result is that given in Figure 6, where the lines of constant degree of ionization are represented. These lines are very nearly parallel to the unperturbed shock trajectory. The reason for this is not very clear. It could be argued that the exponentials of equation (18) can be linearized; then we have:

$$\frac{\partial \alpha}{\partial t} = \text{function}(\alpha)$$

This equation can be integrated independently of the others, and by using the boundary condition (30) we obtain:

$$\alpha = \text{function } (x + c_4 t)$$

This result is similar to that given in Figure 6. However, it can be shown that by doing this, we obtain values of α that are two or three times larger than the ones obtained without linearizing the exponentials. Linearization of the exponentials is not justified for our problem.

Camac and Feinberg¹⁰ extrapolated the experimental data of Petschek and Byron³ in order to calculate the relaxation time behind the reflected shock τ_4 ; according to them:

$$\tau_4 = \frac{0.156}{p_1} \left(\frac{\rho_2}{\rho_1}\right) \left(\frac{\rho_2}{\rho_4}\right) \exp\left(\frac{87\,000\text{ }^\circ\text{K}}{T_4}\right) \mu\text{s}$$

p_1 is in mm Hg. For the conditions of Figure 2 this formula gives a value of 17 μs for τ_4 . This value seems to be in good agreement with the profile of Figure 2 which corresponds to

$$S_2 = 7.1 \times 10^{-19} \text{ cm}^2/\text{eV} \quad (\text{Harwell and Jahn}^6)$$

In this case our value of τ_4 is about 20 μs . However, in Petschek and Byron's experiments impurities probably played an important role, and they have been ignored in the present analysis.

2.6 APPROXIMATE ANALYSIS OF THE PROBLEM

Although the linearization of the exponentials of equation (18) is not justified in our problem, we can still make an analysis similar to that of Spence⁴ by using for α the form:

$$\alpha = \alpha\left(t + \frac{x}{c_4}\right) \quad (34)$$

By eliminating ρ' from (6) and using (3) we obtain:

$$\frac{1}{a_4^2} \frac{\partial p'}{\partial t} + \rho_4 \frac{\partial u'}{\partial x} = - \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \rho_4 \frac{\partial \alpha}{\partial t} \quad (35)$$

By using (35) and (4) we obtain two separate equations for p' and u'

$$\frac{1}{a_4^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = - \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \rho_4 \frac{\partial^2 \alpha}{\partial t^2} \quad (36a)$$

$$\frac{1}{a_4^2} \frac{\partial^2 u'}{\partial t^2} - \frac{\partial^2 u'}{\partial x^2} = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{\partial^2 \alpha}{\partial t \partial x} \quad (36b)$$

If α were known, the solution of these equations would be:

$$p' = (\gamma-1) \frac{\theta_i}{T_4} \rho_4 \frac{c_4^2}{a_4^2 - c_4^2} \alpha(t + \frac{x}{c_4}) + \rho_4 a_4 f_5(t - \frac{x}{a_4}) - \rho_4 a_4 f_6(t + \frac{x}{a_4}) \quad (37)$$

$$u' = - \frac{(\gamma-1)}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)^2} \alpha(t + \frac{x}{c_4}) + f_5(t - \frac{x}{a_4}) + f_6(t + \frac{x}{a_4}) \quad (38)$$

where we used also the lower order equations (35) and (4) and the functional form (34) for α . f_5 and f_6 are two arbitrary functions to be determined by the boundary conditions. Using (22), (24), and (30), we obtain:

$$f_5(t) + f_6(t) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \alpha(t)$$

$$\begin{aligned} \rho_4 a_4 f_5 \left(\left(1 + \frac{c_4}{a_4}\right) t \right) - \rho_4 a_4 f_6 \left(\left(1 - \frac{c_4}{a_4}\right) t \right) = \\ = -k_1 \rho_4 a_4 f_5 \left(\left(1 - \frac{c_4}{a_4}\right) t \right) - k_1 \rho_4 a_4 f_6 \left(\left(1 - \frac{c_4}{a_4}\right) t \right) \end{aligned}$$

These two equations can be put in the form:

$$\begin{aligned} f_5(z) &= \frac{1-k_1}{1+k_1} f_6 \left(\left(1 - \frac{c_4^2}{a_4^2}\right) z \right) \\ \frac{(1-k_1)}{(1+k_1)} f_6 \left(\left(1 - \frac{c_4^2}{a_4^2}\right) z \right) + f_6(z) &= \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \alpha(z) \end{aligned}$$

The solution of these equations is:

$$f_6(z) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \sum_{n=0}^{\infty} (-1)^n \frac{(1-k_1)^n}{(1+k_1)^n} \alpha \left[\left(1 - \frac{c_4^2}{a_4^2}\right)^n z \right] \quad (39a)$$

$$f_5(z) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1-k_1)^n}{(1+k_1)^n} \alpha \left[\left(1 - \frac{c_4^2}{a_4^2}\right)^n z \right] \quad (39b)$$

These series can be shown to be uniformly convergent for any finite interval of z . The values of p' and u' are given as functions of x and t by (37), (38) and (39) if α is known.

Our case is different from Spence's⁴ because we are considering monatomic gases ($\gamma = 5/3$), and because the reflected shock can never be strong.

This makes k_1 very close to unity (28a) and allows us to make some simplifications. We can consider f_5 zero and take only the first term in the series expansion for f_6 (39). Then:

$$p' = (\gamma-1) \frac{\theta_i}{T_4} \frac{c_4^2 p_4}{(a_4^2 - c_4^2)} \alpha\left(t + \frac{x}{c_4}\right) - (\gamma-1) \frac{\theta_i}{T_4} p_4 \frac{a_4 c_4}{(a_4^2 - c_4^2)} \alpha\left(t + \frac{x}{a_4}\right) \quad (40)$$

$$u' = - \frac{(\gamma-1)}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \alpha\left(t + \frac{x}{c_4}\right) + \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2 - c_4^2)} \alpha\left(t + \frac{x}{a_4}\right) \quad (41)$$

and at the wall where x is zero:

$$p'_{\text{wall}}(t) = - (\gamma-1) \frac{(\theta_i/T_4) p_4}{(1+a_4/c_4)} \alpha(t) \quad (42)$$

$$u' = 0$$

and in the shock:

$$x = -c_4 t$$

$$p'_{\text{shock}}(t) = - (\gamma-1) \frac{\theta_i}{T_4} \frac{p_4 a_4 c_4}{(a_4^2 - c_4^2)} \alpha\left[\left(1 - \frac{c_4}{a_4}\right) t\right]$$

where p'_{shock} is the pressure behind the shock. Using (32) we have for the perturbed speed of the shock:

$$c'(t) = - k_3 (\gamma-1) \frac{\theta_i}{T_4} \frac{c_4}{(1-c_4^2/a_4^2)} \alpha\left[\left(1 - \frac{c_4}{a_4}\right) t\right] \quad (43)$$

Then the speed of the shock is:

$$c = c' + c_4$$

These results will be interpreted in the next section.

2.7 INTERPRETATION OF THE RESULTS

Using for α the functional form (34), equations (33a) and (33b) become:

$$\frac{d}{dt} \left[p' + a_4 \rho_4 u' + (\gamma-1) \frac{\theta_i}{T_4} \frac{p_4}{\left(1 + \frac{a_4}{c_4}\right)} \alpha \right] = 0 \quad \text{at } x = a_4 t + \text{const.} \quad (44)$$

$$\frac{d}{dt} \left[p' - a_4 \rho_4 u' - (\gamma-1) \frac{\theta_i}{T_4} \frac{p_4}{\left(\frac{a_4}{c_4} - 1\right)} \alpha \right] = 0 \quad \text{at } x = -a_4 t + \text{const.} \quad (45)$$

Suppose we want to calculate the conditions at point B of Diagram 1 below. B is at the wall and u_B is zero. Using (44) along CB we get:

$$p'_B = (p'_C + \rho_4 a_4 u'_C) - (\gamma-1) \frac{\theta_i}{T_4} \frac{p_4}{\left(1 + \frac{a_4}{c_4}\right)} \alpha_B \quad (45a)$$

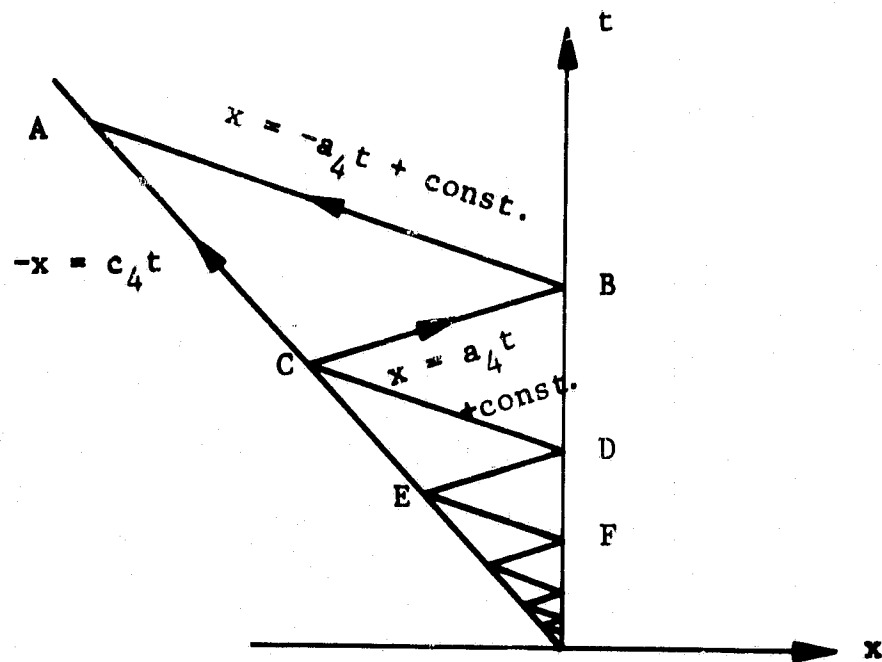


Diagram 1.

since α_C is zero. At C (in the shock) we have a relation between p' and u' (24). Using (45) along CD and (24) at C we get:

$$p'_C = \frac{k_1}{1+k_1} \left[p'_D - (\gamma-1) \frac{\theta_i}{T_4} \frac{p_4}{\frac{a_4}{c_4} - 1} \alpha_D \right]$$

We can continue along the characteristics DE, EF, etc. until we get arbitrarily close to the origin, where α is zero. It is easy to check that the values of p_B or p_C obtained in this way correspond to the series expansions obtained previously (37), (39) and (40). However, the boundary condition (24) tells us that:

$$p'_C + \rho_4 a_4 u'_C \approx 0$$

It is not necessary to calculate the values of the flow quantities at C, D, E, F, etc. because all the information we need is contained in the characteristic CB; the characteristic CD is almost totally absorbed by the shock and does not affect what happens along CB. This simplification corresponds to taking only the first term in the series expansion (39). Equation (45a) gives the same result as that obtained in the previous section (42). This shows that the perturbation in pressure at the wall is proportional to the degree of ionization at that time at the wall, and that the pressure decreases as the ionization increases. These results are in agreement with Smith's experiments.¹

A similar argument can be used to calculate the conditions at A behind the shock. In this case the only relevant characteristics will be AB and BC, because of the above arguments. So the conditions at A can only be affected by the degree of ionization at B; this agrees with equation (43) which can be written:

$$c'_A = -k_3 (\gamma-1) \frac{\theta_i}{T_4} \left(\frac{c_4^2}{1 - \frac{c_4^2}{a_4^2}} \right) \alpha_B$$

We can say that the perturbation in the speed of the shock is proportional to the degree of ionization at the point of the wall that is on the same C^- characteristic as the point in the shock.

In all these equations we could use approximations (1). For example, c_4/a_4 is $1/\sqrt{3}$.

It is also interesting to note that all perturbations of the flow quantities are proportional to the factor $(\theta_i/T_4) \propto$. This confirms the validity of the perturbation method and assumption (7).

We must finally note that the analysis of Sections 2.6 and 2.7 is subject to the condition that $\alpha(x + c_4 t)$ is known. To do this we think the only possible method is the numerical solution indicated in 2.5.

3. INTERACTION OF THE REFLECTED SHOCK WITH THE IONIZATION FRONT FOLLOWING THE INCIDENT SHOCK

3.1 INTRODUCTORY REMARKS

We saw in Chapter 2 that the reflected shock, after a certain time, reaches an equilibrium speed and leaves behind an ionized gas in equilibrium. There is experimental evidence that the ionization time behind the reflected shock is shorter than behind the incident shock. The reason is that the temperature is higher behind the reflected shock. (Smith¹, Camac and Feinberg¹⁰.) To study this problem we will assume that the gas behind the incident shock is relaxing and behind the reflected shock is in equilibrium. In the transition from region 2 to 6 (Figure 1) the gas is relaxing, and in regions 5, 7, 8, 9 and 10 the gas is in equilibrium. This approximation will be exact as τ_1/τ_4 tends to infinity. The experiments of Smith¹ and Camac and Feinberg¹⁰ seem to indicate that this ratio is only of order five. However, as the degree of ionization increases behind the incident shock new electrons are produced, and as the reflected shock advances it meets

regions where there are more electrons. It has been shown in Chapter 2 that the initial and much longer stage of the ionization process is by atom-atom collision, and that this process ceases to be important as soon as there are enough electrons for the electron-atom collision process to take over. Consequently it is to be expected that when the reflected shock meets regions where there are already electrons, the ionization time will be much smaller than τ_4 . Then the assumption of equilibrium conditions behind the reflected shock is greatly improved.

We now want to calculate the structure of the ionization front behind the incident shock. These fronts have been studied theoretically and experimentally by many people (Morgan and Morrison⁵, Smith¹, Petschek and Byron³ and Wong and Bershader¹³). This problem is similar to the one worked out in Chapter 2, but simpler, because by taking coordinates fixed in the shock the flow becomes stationary. Wong and Bershader¹³ measured the ionization profile behind the incident shock. They give $\alpha(\tau)$; τ is the distance in laboratory time behind the incident shock:

$$\tau = t - x/c_1$$

c_1 is the speed of the incident shock (Figure 1). In the numerical analysis we will use the experimental values of α obtained by Wong and Bershader¹³ for argon and the theoretical results obtained by Smith¹ for xenon.*

Since we know the values of α behind the incident shock, we can calculate the rest of the flow properties there. Petschek and Byron³ give these expressions:

$$p_a = p_2 \tag{46}$$

$$T_a = T_2 - \frac{2}{5} \alpha_a \theta_i \tag{47}$$

$$\rho_a = \rho_2 \left(1 - \frac{2}{5} \alpha_a \frac{\theta_i}{T_2} \right)^{-1} \tag{48}$$

$$u_a = M_1 a_1 \left(\frac{3}{4} + \frac{1}{10} \alpha_a \frac{\theta_i}{T_2} \right) \tag{49}$$

* Some of these results were given to us by Smith in a private communication.

These equations represent conservation of momentum (46), energy (47), and mass (49). By using the equation of state we obtain (48). The degree of ionization is assumed to be small, but not the products $\frac{\theta_i}{T_2} \alpha_a$ (compare with equation (2)). The subscript "a" represents any point behind the incident shock, and subscript 2 represents frozen conditions immediately after the passage of the shock. Conditions in region 2 are calculated by the usual Rankine-Hugoniot relations (see section 2.4).

Region 2 does not exist in reality. The ionization degree is never zero, except immediately after the passage of the shock; however, it is close to zero for some time. This allows us to define region 2.

We have all the information about the ionization front following the incident shock and can study its interaction with the reflected shock.

3.2 TRAJECTORY OF THE REFLECTED SHOCK

We have the problem of a shock propagating in a region of known but not constant properties, and we want to calculate the trajectory of the shock and the properties of the gas behind it.

Across the shock we have conservation of mass, momentum and energy.

$$\rho_a(u_a + c) = \rho(u + c) \quad (50)$$

$$p_a + \rho_a(u_a + c)^2 = p + \rho(u + c)^2 \quad (51)$$

$$h_a + \frac{1}{2}(u_a + c)^2 = h + \frac{1}{2}(u + c)^2 \quad (52)$$

p_a , ρ_a , h_a and u_a are known as functions of the position of the shock. In particular (see (46), (47), (48) and (49)):

$$h_a = c_p T_a + R \theta_i \alpha_a = h_2 = \text{const.} \quad (53)$$

ρ , u , p and h are the density, velocity, pressure and enthalpy of the gas behind the reflected shock; c is the speed of the reflected shock. These five quantities are unknown.

The gas behind the reflected shock is in equilibrium, consequently its properties are related by:

$$h = (1 + \alpha) c_p T + \theta_i \alpha R \quad (54)$$

$$p = (1 + \alpha) R \rho T \quad (55)$$

$$\alpha = \alpha(p, T) \quad (56)$$

The last equation is the Saha equation, similar to equation (15). (In our case the gas is in equilibrium and electrons and atoms have the same temperature.) Behind the reflected shock the degree of ionization is higher than behind the incident shock; then α is not necessarily small behind the reflected shock.

For argon the coefficients of equations (54) and (55) are

$$R = \frac{2}{5} c_p = 2.08299 \times 10^2 \frac{\text{m}^2}{\text{sec}^2 \text{ } ^\circ\text{K}}$$

To find a further condition along the shock we shall use Whitham's theory.⁷ Whitham says that behind a shock that is propagating in a non-uniform region, the same relation that holds along the characteristics intersecting the shock also holds along the shock. In our case (see, for example, Courant and Friedrichs,¹⁴ Chapter III):

$$dp - \rho a du = 0 \quad (57)$$

where "a" is the speed of sound of the ionized gas in equilibrium. This relation is not exactly true. Let us consider Figure 7. During the time that the shock meets the frozen gas of region 2, which has constant properties, there is a uniform region 5 behind the shock. As the shock meets

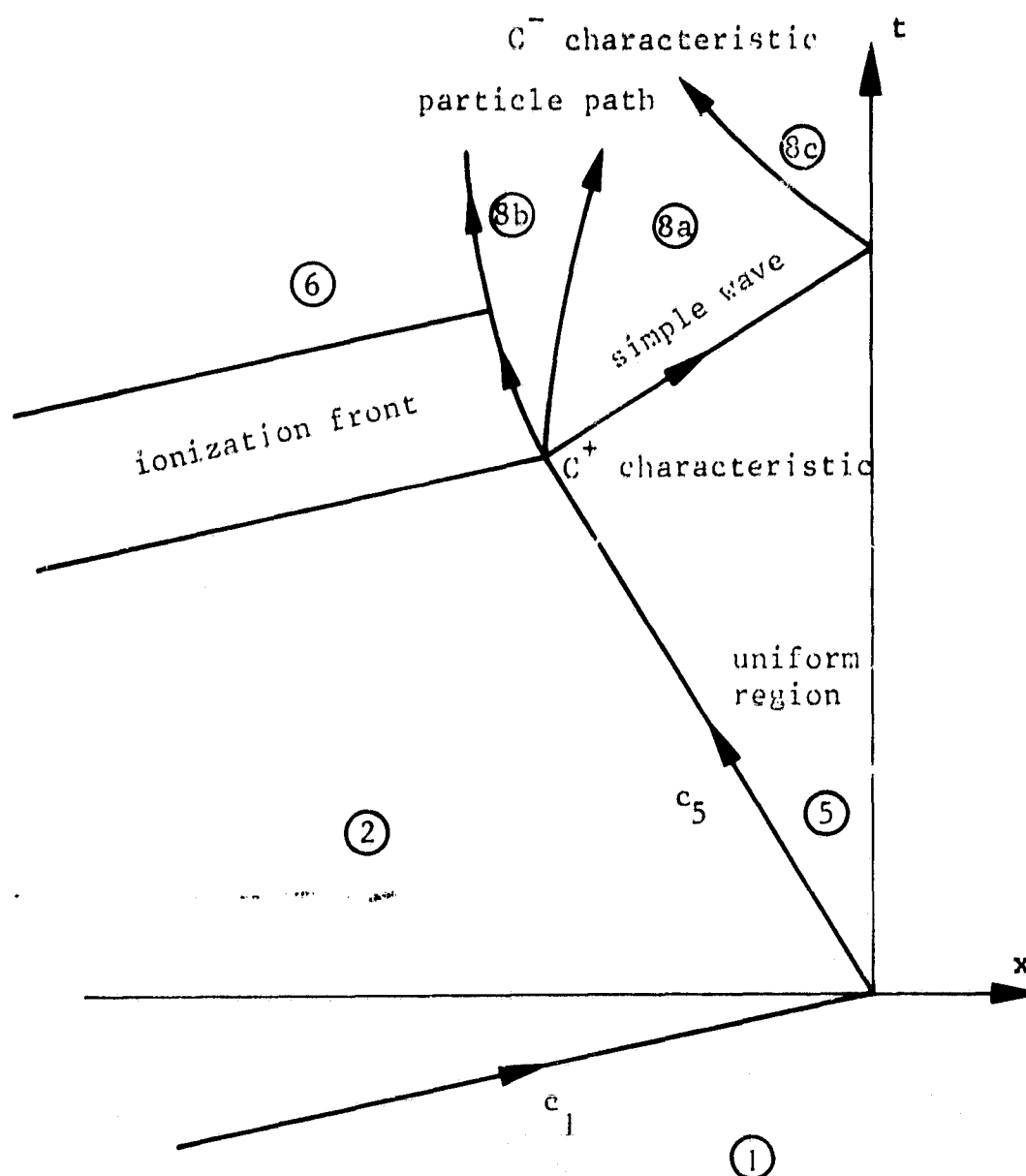


Figure 7.- x-t diagram describing the interaction of the ionization front with the reflected shock.

the transition region behind the incident shock it starts to bend. We then have, as indicated in Figure 7, region 8a which is a simple wave, region 8b which is non-isentropic because of the curvature of the shock, and region 8c which is isentropic but not a simple wave. Along the C^- characteristics relation (57) holds, and since region 8a is isentropic we can integrate equation (57):

$$u - \int_{p_5}^p \frac{dp}{\rho a} = 0 \quad (58)$$

This is satisfied over the whole region 8a. The integral should be calculated at constant entropy. We will explain later how this can be done in the case of ionized gases in equilibrium; at the moment we just notice that for constant entropy ρ and "a" can be expressed as functions only of p . Since equation (58) holds over the whole region 8a, differentiating it we find that (57) also holds over the whole region 8a, because p_5 is constant. However, (57) does not hold over the whole region 8b, but holds only along the C^- characteristics of that region, because 8b is not isentropic and the integral relation (58) does not hold there. Relation (57) proposed by Whitham⁷ is then only an approximation. Lick⁸ extended Whitham's theory and said that relation (57) holds over the whole non-isentropic region behind the shock. Lick's theory is equivalent in our case to assuming that the region adjoining the shock is also a simple wave. The justification of these assumptions is not very clear. For further information the reader is referred to the works of Whitham⁷ and Lick⁸. Our numerical results will also confirm these assumptions.

The speed of sound of an ionized gas in equilibrium is (Witte¹¹):

$$a^2 = \left(\frac{dp}{d\rho} \right)_{S=\text{const}} = \frac{c_p}{\left[c_p \left(\frac{\partial \rho}{\partial p} \right)_T - \frac{T}{\rho^2} \left(\frac{\partial \rho}{\partial T} \right)_p \right]}$$

$$= \frac{5}{3} \frac{p}{\rho} \frac{1 + \frac{5}{4} \left(1 + \frac{2}{5} \frac{\theta_i}{T} \right)^2 \alpha(1-\alpha) + \frac{1}{2} \left(1 + \frac{2}{5} \frac{\theta_i}{T} \right) \alpha(1-\alpha) \psi(T)}{1 + \left[\frac{5}{4} + \frac{\theta_i}{T} + \frac{1}{3} \left(\frac{\theta_i}{T} \right)^2 \right] \alpha(1-\alpha) + \frac{1}{2} \left(1 + \frac{2}{3} \frac{\theta_i}{T} \right) \alpha(1-\alpha) \psi(T)} \quad (59)$$

where

$$\psi(T) = \frac{C_2 \theta_a}{T} \frac{\exp(-\theta_a/T)}{C_1 + C_2 \exp(-\theta_a/T)} - \frac{C_4 \theta_e}{T} \frac{\exp(-\theta_e/T)}{C_3 + C_4 \exp(-\theta_e/T)}$$

The term in $\psi(T)$ does not appear in Witte's paper; it is given by the contributions of the first excited states of the atom and of the ion atom. Equation (59) is obtained by using (55) and (56).

The position of the shock is given by

$$dx/dt = -c \quad (60)$$

In Figure 7 it is assumed that the reflected shock propagates from the beginning with a speed c_5 that leaves behind a gas in equilibrium and at rest. We know that this is not true. As shown in Diagram 2, below, there is a certain time, τ_4 , during which the ionization degree at the wall is zero. The first particles that are ionized are the ones close to the wall, because they are the ones that have been for a longer time in region 4. The shock will propagate with the speed c_4 that leaves behind a gas frozen and at rest until it feels the effect of the ionization. This will happen when the frozen characteristic leaving the wall at the time τ_4 meets the shock.

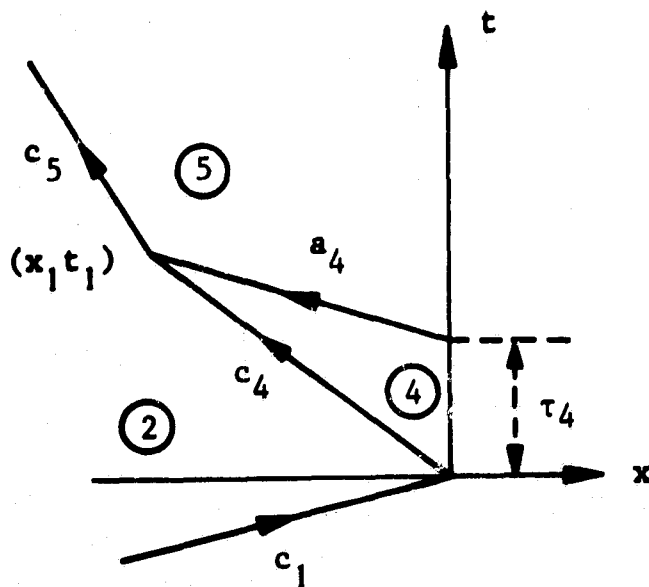


Diagram 2.

Instead of supposing that the reflected shock starts propagating with speed c_5 from the origin, we will suppose that it propagates with that velocity from a point of coordinates (x_1, t_1) :

$$t_1 = \frac{\tau_4}{\left(1 - \frac{1}{\sqrt{3}}\right)} \quad (61a)$$

$$x_1 = \frac{c_4 \tau_4}{\left(1 - \frac{1}{\sqrt{3}}\right)} \quad (61b)$$

$$a_4/c_4 = \sqrt{3} \quad (\text{see 1c})$$

The values of τ_4 were taken from Smith¹ for xenon and Camac and Feinberg² for argon. This approximation cannot be thought to be a rigorous estimation of the effect of ionization behind the reflected shock. However, it should be regarded as an improvement over an approximation in which the shock propagates with equilibrium speed from the origin. The way to calculate c_4 and a_4 was indicated in 2.1.

To calculate the initial speed of the shock c_5 and the initial flow properties ρ_5 , p_5 , h_5 , we solve the algebraic equations (50), (51), (52), (54), (55) and (56), taking the initial velocity u_5 equal to zero and the properties ahead of the shock p_a , ρ_a , h_a , u_a equal to the frozen values p_2 , ρ_2 , h_2 , u_2 . The initial conditions will be:

$$t = t_1, x = x_1, c = c_5, \rho = \rho_5, h = h_5, p = p_5, \alpha = \alpha_5,$$

$$T = T_5, a = a_5, u = 0$$

We can now solve the system of equations (50), (51), (52), (54), (55), (56), (57), (59) and (60) for x , c , ρ , h , p , α , T , a and u as functions of time with the initial conditions given above. This was done numerically. The initial conditions were also calculated numerically.

In Figures 8a, b, and c are represented some of the calculated shock trajectories. In Table I are given the calculated flow properties behind the shock.

An interesting result is that behind the reflected shock the values of the degree of ionization, temperature, enthalpy, and speed of sound remain nearly constant, and consequently the ratio p/ρ also remains constant. This result seems to hold for all the numerical cases solved. The values of the degree of ionization varied between 20% and 40% for the different examples worked out.

Table 1. Flow properties behind the reflected shock as functions of the shock position

x mm	t μ s	u mm/ μ s	p mm Hg	α	a mm/ μ s	T°K
$M_1 = 16.3, p_1 = 3$ mm Hg, argon						
2.75	1.27	0	4644	0.348	2.25	15 509
3.53	2.20	0.06	4782	0.348	2.25	15 539
4.01	2.79	0.10	4895	0.348	2.25	15 562
4.30	3.18	0.20	5152	0.347	2.26	15 610
4.41	3.35	0.40	5704	0.343	2.26	15 696
4.55	3.66	0.67	6538	0.336	2.26	15 795
4.58	3.74	0.79	6942	0.332	2.26	15 834
$M_1 = 15.4, p_1 = 5$ mm Hg, argon						
2.27	1.30	0	6991	0.297	2.21	15 567
2.97	2.11	0.05	7186	0.297	2.21	15 594
3.46	2.69	0.08	7299	0.296	2.22	15 609
3.69	2.98	0.15	7554	0.296	2.22	15 641
3.86	3.22	0.35	8387	0.292	2.22	15 729
3.92	3.32	0.42	8727	0.291	2.22	15 760
3.97	3.41	0.58	9464	0.287	2.22	15 816
4.00	3.49	0.65	9802	0.285	2.23	15 840
$M_1 = 15.1, p_1 = 0.5$ mm Hg, xenon						
7.96	6.13	0	655	0.410	1.08	11 400
8.23	6.85	0.03	679	0.410	1.08	11 428
8.39	7.30	0.10	730	0.409	1.09	11 478
8.46	7.56	0.27	869	0.402	1.09	11 582
8.47	7.64	0.37	967	0.396	1.09	11 636
8.48	7.70	0.41	1004	0.393	1.09	11 654
8.49	8.11	0.46	1060	0.390	1.09	11 679

Along the shock the pressure behind it increased with time. In the different cases considered, the total increase in pressure varied between 20% and 80% of the initial pressure.

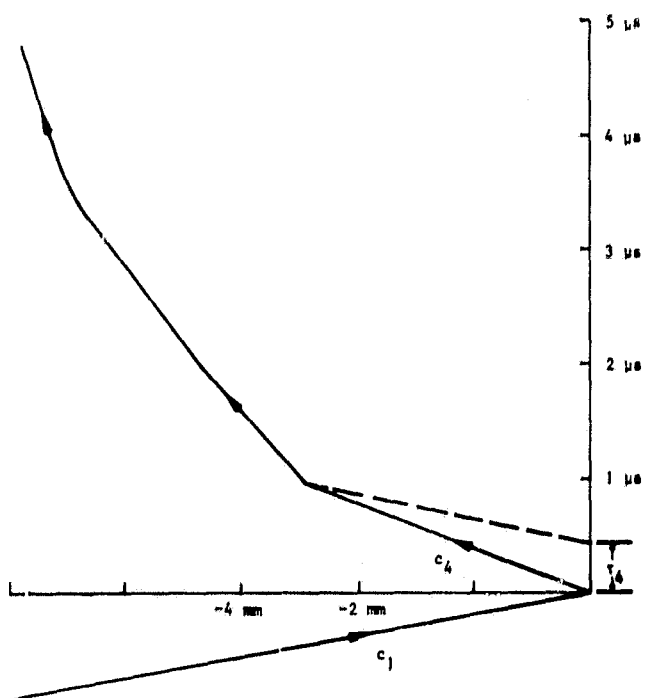


Figure 8a.- Reflected shock trajectory
(α from Wong and Bershader.¹³
 $M_1 = 16.3$, $p_1 = 3$ mm Hg, $T_1 = 300^\circ\text{K}$, argon).

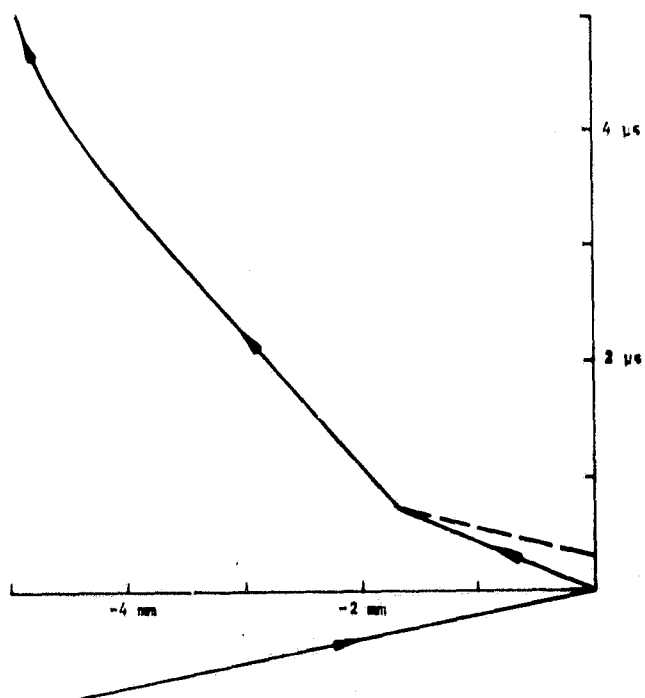


Figure 8b.- Reflected shock trajectory
(α from Wong and Bershader.¹³
 $M_1 = 15.0$, $p_1 = 5$ mm Hg, argon).

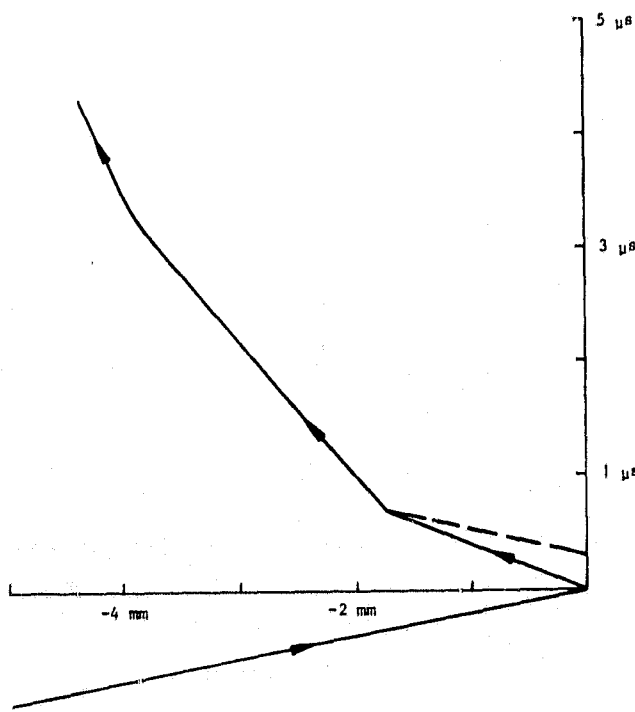


Figure 8c.- Reflected shock trajectory
(α from Wong and Bershader.¹³
 $M_1 = 15.4$, $p_1 = 5$ mm Hg, argon).

The speed of the shock decreases with time. The interaction of the ionization front with the shock bends the shock towards the wall (because the density is increasing ahead of the shock).

The speed of the gas behind the shock, which initially is zero, is towards the wall and increases in magnitude as the shock bends. This means that there is a mechanism to stop the gas, because the velocity at the end wall must be zero. To study this phenomenon we have to consider regions 8a, 8b and 8c (Figure 7); this will be done in the following sections.

3.3 FLOW FIELD BEHIND THE REFLECTED SHOCK

It is well known that an ionized gas in equilibrium tends to behave as if γ , the ratio of the specific heats, were close to unity. This means that, if the gas behaves isentropically, the classical relation:

$$p/\rho^\gamma = \text{const.}$$

will tend to (p/ρ) equal to constant. If the degree of ionization and the temperature of the gas remain constant, it is clear from equation 55 that this relation holds. The entropy of equilibrium ionized argon (Witte¹¹) is:

$$S/R = (1 + \alpha) \ln \frac{T^{5/2}}{p} + \ln \left[\frac{(1+\alpha)^{1+\alpha}}{(1-\alpha)^{1-\alpha}} \alpha^{-2\alpha} \right] - 10.354\alpha \quad (62)$$

We claim that for moderate changes in pressure (80%) and high degrees of ionization, constant entropy means constant temperature and constant degree of ionization. From the expression above we see that the term that gives the largest changes in entropy is 10.354α . So if the entropy is going to be constant, α should also remain almost constant. On the other hand, if we differentiate the Saha equation (15) we obtain:

$$d\alpha = \left(\frac{1-\alpha^2}{2} \right) \left(\frac{\theta_i}{T} + \frac{5}{2} + \psi(T) \right) \frac{dT}{T} - \left(\frac{1-\alpha^2}{2} \right) \alpha \frac{dp}{p}$$

$\psi(T)$ is given in (59). By far the largest changes in this expression will come from the term containing (θ_i/T) . This term corresponds to the exponential of the Saha equation. If α is going to be constant then T also has to be almost constant. Further, if we examine equation (59) for the speed of sound and take limits for (θ_i/T) large, and assume that α is not small, we obtain:

$$a^2 \rightarrow p/\rho$$

But if we assume that at constant entropy α and T are constants we find:

$$\frac{p}{\rho} = R (1+\alpha) T = \text{const.}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \frac{p}{\rho} = a^2$$

This can also be checked by looking at Figure 5 in Witte's paper.¹¹

It is now evident that the theories of Whitham⁷ and Lick⁸ give us a very good approximation. In Section 4.2 we saw that behind the reflected shock the relation:

$$\frac{p}{\rho} = \text{const.}$$

held, and that α , T and " a " remained almost constant. This means that the curvature of the shock does not change very much the entropy behind the shock. As a result regions 8a and 8b can be considered as a single simple-wave region where the C^+ characteristics are straight lines along which the flow properties remain constant. Since the conditions behind the shock are known, we also know all the flow properties in regions 8a and 8b.

The C^+ characteristics are convergent lines because their slope is given by $(a+u)$, and u increases along the shock. However for the moment we will assume that the characteristics do not intersect in 8a or 8b. We will study this case later. This family of C^+ characteristics forms a compression wave going to the wall.

3.4 INTERACTION OF THE COMPRESSION WAVE WITH THE WALL

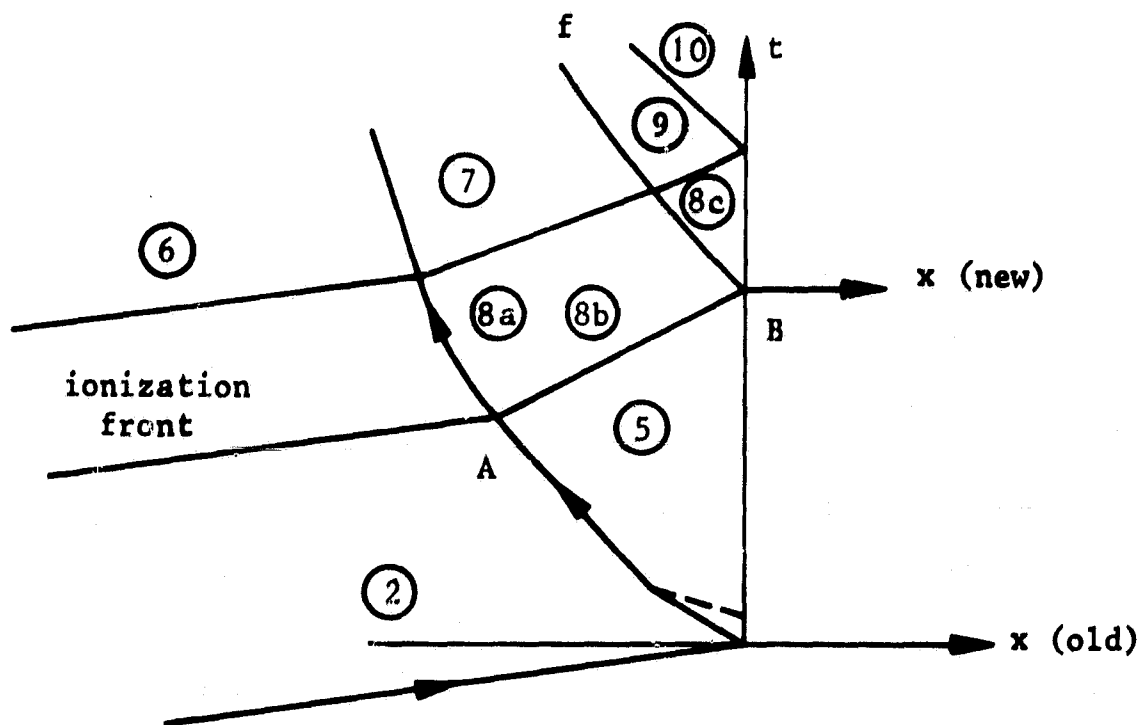


Diagram 3.

Let us suppose that at point A of Diagram 3 the shock starts to bend by a significant amount. The characteristic leaving A hits the wall at a certain point B and then reflects. Let us call f the reflected characteristic. The trajectory of f and the flow properties along it are known from the results of the previous section. Between f and the wall is the region 8c that we want to study.

Let us choose B as our new origin of coordinates. Region 8c is isentropic because all the particle paths crossing this region come from region 5, where the entropy is constant. In this region then:

$$u + \int \frac{dp}{\rho a} = \text{const. along } \frac{dx}{dt} = u + a$$

$$u - \int \frac{dp}{\rho a} = \text{const. along } \frac{dx}{dt} = u - a$$

The integral has to be evaluated at constant entropy (see, for example, Chapter III in Courant and Friedrichs¹⁴). The evaluation of this integral, although straightforward, is extremely complicated, and it does not look as though we would be able to obtain it in closed form. From (55), (56), (59) and (62), eliminating α and T (constant entropy), we could obtain ρ and a in terms of p . We shall, however, use the approximations indicated in Section 3.3, which show that, for an ionized gas in equilibrium with high degrees of ionization and with moderate changes in pressure involved in the process, the constant entropy condition is equivalent to taking:

$$\alpha = \text{const.}, \quad T = \text{const.}, \quad \frac{p}{\rho} = a^2 = \text{const.}$$

Then:

$$\int \frac{dp}{\rho a} \approx a \ln \left(\frac{p}{p_5} \right) + \text{const.}$$

The constant pressure p_5 was added for convenience. Then in region 8c:

$$u + a \ln \frac{p}{p_5} = 2r = \text{const. along } \frac{dx}{dt} = u + a \quad (63)$$

$$u - a \ln \frac{p}{p_5} = -2s = \text{const. along } \frac{dx}{dt} = u - a \quad (64)$$

where r and s are the Riemann invariants. If we now change the independent variables to r and s , equations (63) and (64) become:

$$u = r - s \quad (65)$$

$$a \ln \frac{p}{p_5} = r + s \quad (66)$$

$$x_s = (r - s + a) t_s \quad (67)$$

$$x_r = (r - s - a) t_r \quad (68)$$

The subscripts r , s denote derivatives. Eliminating x between (67) and (68) we obtain:

$$2a t_{sr} + t_s + t_r = 0 \quad (69)$$

This equation combined with (67) and (68) (and the boundary conditions) will give us $x(r,s)$, $t(r,s)$. Inverting these functions we get $s(x,t)$ and $r(x,t)$ and by using (65) and (66) $u(x,t)$ and $p(x,t)$. The inversion of $x(r,s)$ and $t(r,s)$ is possible if the Wronskian of the transformation is different from zero. In simple-wave regions this Wronskian is zero; we will discuss later under what other circumstances this could happen; for the moment we assume that this is not the case in region 8c.

We must have some boundary conditions in order to solve (69). At the wall, from (65) we have:

$$u = 0 \quad x = 0$$

$$r = s \quad x = 0$$

On the line $r = s$, x has to be constant (equal to zero):

$$x_r dr + x_s ds = (x_r + x_s) dr = 0$$

$$x_r = -x_s$$

Using (67) and (68) we get the boundary condition:

$$t_r = t_s \quad \text{on} \quad r = s \quad (70)$$

Another boundary condition is given by the fact that we know the flow properties along the C^- characteristic f (see diagram 3) and its trajectory. Along f :

$$s = \text{const.} = 0$$

because on B , u is zero and p is p_5 . We then have:

$$t = T(r) \quad \text{on} \quad s = 0 \quad (71)$$

where $T(r)$ is known from the calculations of Section 3.3. A typical form of $T(r)$ is given in Figure 9. When the reflected shock has crossed the ionization front following the incident shock, the conditions behind the reflected shock (region 7) are constant. Region 7 is uniform, and region 9 is a simple wave; there our theory is no longer correct. The value of r_7 is:

$$r_7 = \frac{1}{2} (u_7 + a \ln \frac{p_7}{p_5}) \quad (72)$$

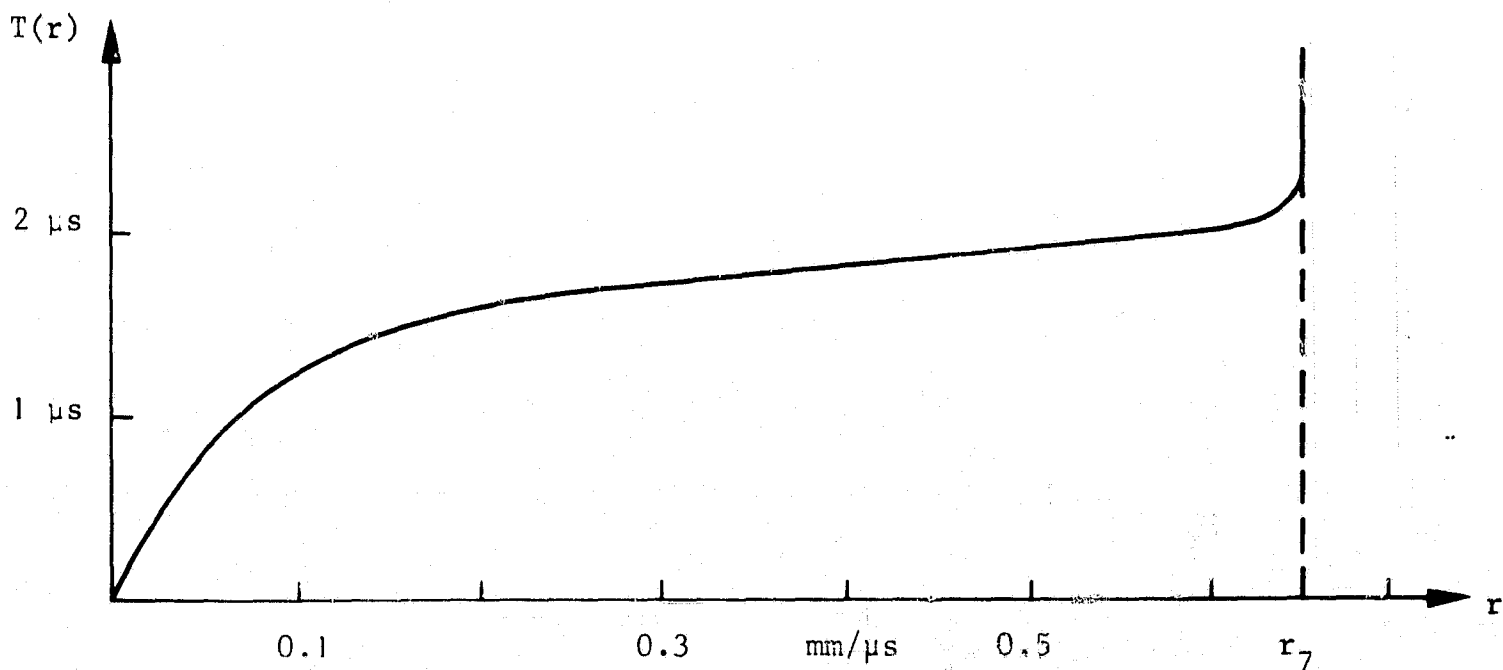


Figure 9.- Values of the Riemann invariant along the leading C^- characteristic.
($M_1 = 15.0$, $p_1 = 5 \text{ mm Hg}$, argon).

Condition (71) should rather be:

$$t = T(r) \quad \text{on} \quad s = 0 \quad \text{for} \quad 0 \leq r \leq r_7 \quad (73)$$

We now make the transformation:

$$t = e^{-\frac{1}{2a}(s+r)} w \quad (74)$$

Equation (69) becomes:

$$4a^2 w_{rs} - w = 0 \quad (75)$$

and the boundary conditions (70) and (73):

$$w_r = w_s \quad \text{on} \quad r = s \quad (76)$$

$$w = T(r) e^{r/2a} \quad \text{on} \quad s = 0 \quad \text{and} \quad 0 \leq r \leq r_7 \quad (77)$$

Since the problem is completely symmetrical we can substitute boundary condition (76) by:

$$w = T(s) e^{s/2a} \quad \text{on} \quad r = 0 \quad \text{and} \quad 0 \leq s \leq r_7 \quad (78)$$

It is evident that equation (75) with boundary conditions (77) and (78) will give us a solution that also satisfies (76).

The Riemann function of equation (75) is (see Chapter V §5 of Courant and Hilbert¹⁵):

$$R(x, y, \xi, \eta) = J_0 \left[\sqrt{-\frac{1}{a^2} (x-\xi) (y-\eta)} \right] = I_0 \left[\frac{1}{a} \sqrt{(\xi-x) (\eta-y)} \right]$$

where I_0 is the modified Bessel function of order zero. The solution of equation (75) with boundary conditions (77) and (78) is then:

$$w(r,s) = \int_0^s I_0 \left(\frac{1}{a} \sqrt{r(s-z)} \right) \beta'(z) dz + \int_0^r I_0 \left(\frac{1}{a} \sqrt{s(r-z)} \right) \beta'(z) dz$$

$$r \geq s \quad \text{and} \quad 0 \leq r \leq r_7 \quad (79)$$

Condition $r \geq s$ means that we are to the left of or at the wall. The value of β is:

$$\beta(z) = T(z) e^{z/2a}$$

It can be checked directly by substitution that (79) is a solution of (75), (76) and (77).

We are interested in calculating the pressure distribution at the wall where u is zero. From (65) and (66)

$$\frac{p_{\text{wall}}}{p_5} = e^{2r/a} \quad (80)$$

and from (79) and (74) we get:

$$t_{\text{wall}}(r) = 2e^{-r/a} \int_0^r I_0 \left(\frac{\sqrt{r(r-z)}}{a} \right) \beta'(z) dz \quad (81)$$

Formulae (80) and (81) give $p(t)$ at the wall in parametric form with r as a parameter. We will give later a simpler interpretation of this result.

In the examples we have worked out the ratio r/a was never larger than 0.4 and the modified Bessel function oscillated between 1 and 1.04. A very good approximation to (81) could be obtained by taking I_0 equal to unity:

$$t_{\text{wall}} = 2 e^{-r/2a} T(r) \quad (82)$$

The reason for this is that the expansion of I_0 is

$$I_0 = 1 + O(r^2/4a^2) \quad (82a)$$

r/a can be regarded as the Mach number of the flow. In Figure 10 we have $p(t)$ at the wall for the different examples calculated by this theory. We did not find any experimental data to compare the results of Figure 10. Later we will try to check Smith's experimental results;¹ we cannot do it with this theory because there may be shock formation in these cases. However we can see that the interaction of the ionization front with the reflected shock produces an increase in pressure at the end wall of the shock tube as the experiments seem to indicate.

As we have already noted, once the shock has crossed the ionization front it leaves behind a uniform region 7; adjoining this region is the simple-wave region 9, and between this and the wall the uniform region 10. After the compression wave has hit the wall, the pressure at the wall remains constant (Figure 10). This pressure can be easily determined. Along a C^+ characteristic going through regions 7, 9 and 10 we have (see diagram 3 and Figure 1)

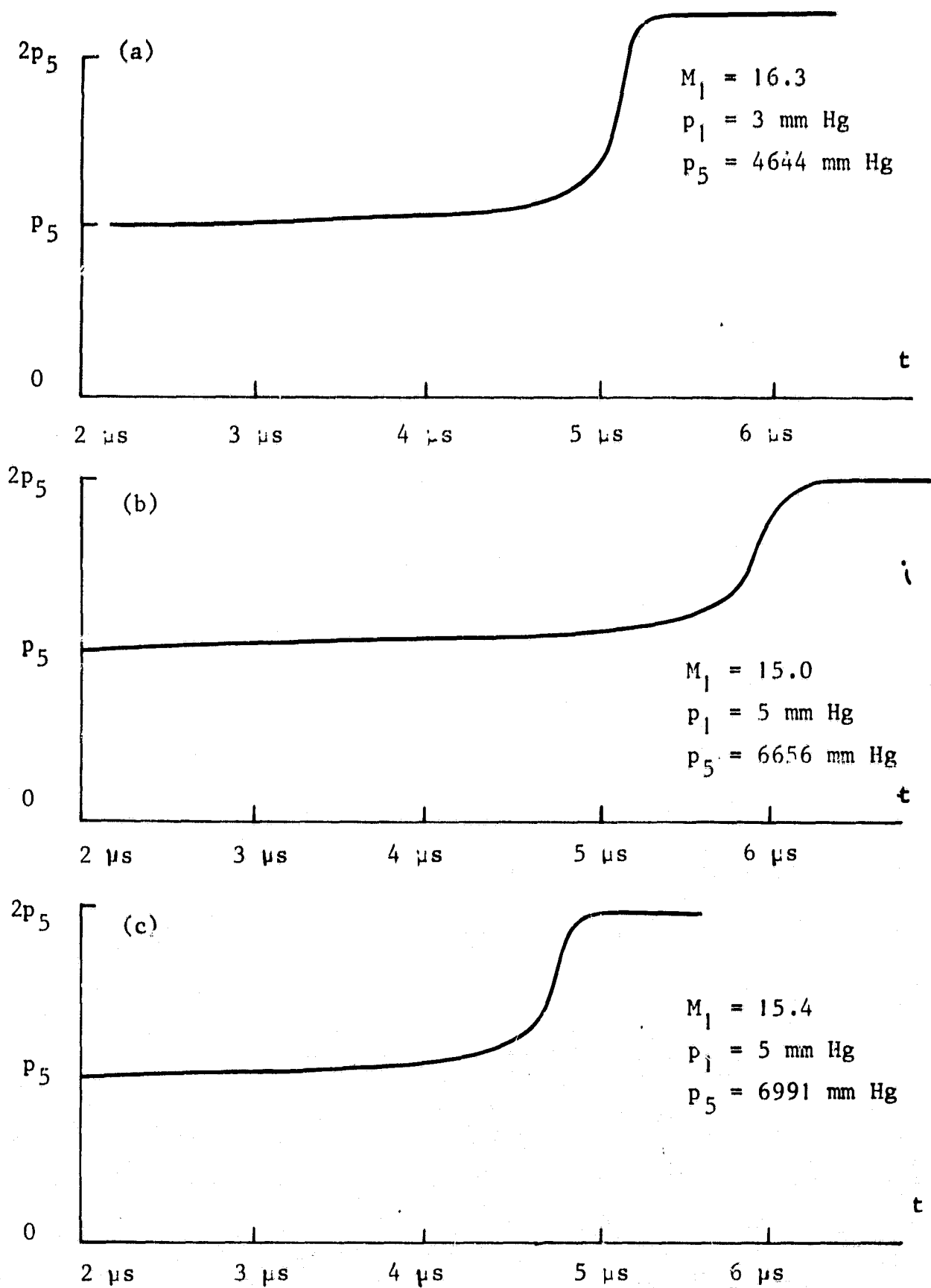


Figure 10.- End wall pressure increase as a result of the interaction of the ionization front with the reflected shock (argon).

$$a \ln \frac{p_{10}}{p_5} = 2r_7 = u_7 + a \ln \frac{p_7}{p_5}$$

and along a C^- characteristic going from region 5 to 7:

$$0 = u_7 - a \ln \frac{p_7}{p_5}$$

These equations give us:

$$p_{10} = p_5 \left(\frac{p_7}{p_5} \right)^2 \quad (83)$$

A result similar to this can be obtained, instead of equation (80), for the region of changing pressure. Using the relations (63) and (64) along a C^+ characteristic going from the shock to the wall and along the shock (that behaves like a C^- characteristic):

$$u_{sh} - a \ln \frac{p_{sh}}{p_5} = 0$$

$$u_{sh} + a \ln \frac{p_{sh}}{p_5} = a \ln \frac{p_{wall}}{p_5}$$

$$p_{wall} = p_5 \left(\frac{p_{sh}}{p_5} \right)^2 \quad (84)$$

p_{sh} is the pressure behind the shock at a point that is on the same C^+ characteristic as the point at the wall. The fact that (p_{sh}/p_5) is raised to the power two means that the pressure at the wall is greater than behind the shock, because behind the shock the gas has a speed towards the wall. Although this is a very simple interpretation, we must not forget that we still need to know the trajectory of the C^+ characteristic and in particular the point at which it hits the wall. This is given by equation (81).

3.5 CASES FOR WHICH A SHOCK WAVE IS FORMED IN THE SIMPLE-WAVE REGION

If the ionization front behind the incident shock is very thin the reflected shock will bend very sharply. It is then possible that the characteristics leaving the reflected shock will intersect. We assume that this intersection occurs somewhere in the simple-wave regions 8a or 8b (Figure 7).

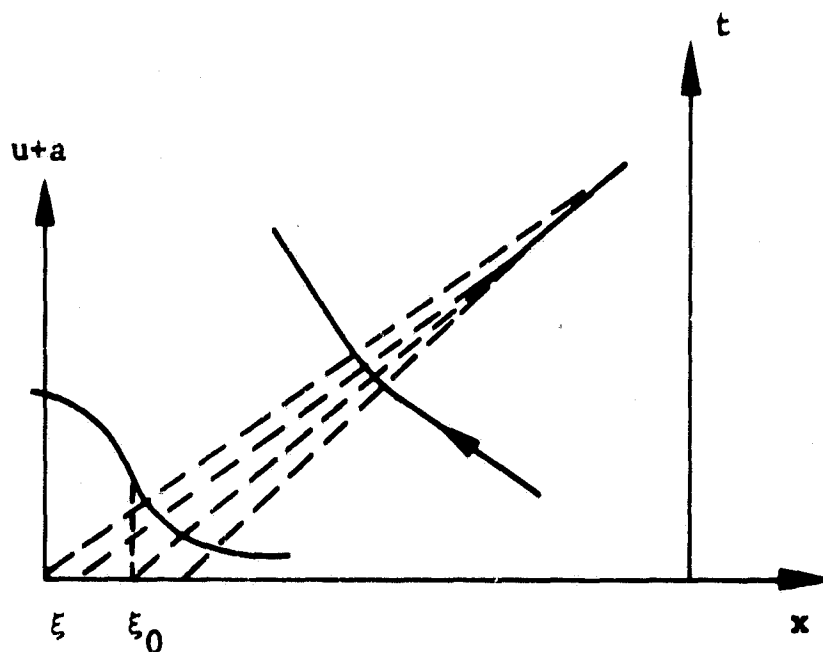


Diagram 4.

Characteristic equation:

$$x = \xi + c(\xi) t$$

envelope of characteristics:

$$\frac{\partial x}{\partial \xi} = 1 + c'(\xi) t = 0$$

$$\xi + c(\xi) t = x$$

cusp of the envelope:

$$\frac{dt}{d\xi} = \frac{c''(\xi)}{c'(\xi)^2} = 0 \quad c''(\xi) = 0$$

To obtain the point at which the shock starts to form, we prolong the characteristics to the left and calculate the coordinates ξ where they intersect the x-axis (as indicated in Diagram 4). Then we represent the speeds of the characteristics, $c = u + a$, as a function of ξ . The characteristic where $c''(\xi_0)$ is zero is the one on which the cusp of the envelope of the characteristics lies. The coordinates of the cusp are:

$$t_c = - \frac{1}{c'(\xi_0)}$$

$$x_c = \xi_0 - \frac{c(\xi_0)}{c'(\xi_0)}$$

Now we have to continue constructing the shock. To do this we examine first the conditions that the gas satisfies on both sides of the shock. From energy conservation we get:

$$c_p T_a (1 + \alpha_a) + R \theta_i \alpha_a + \frac{1}{2} (U - u_a)^2 = c_p T_b (1 + \alpha_b) + R \theta_i \alpha_b + \frac{1}{2} (U - u_b)^2$$

U is the shock speed. Subscripts a and b indicate conditions on both sides of the shock. The velocities are of the same order as $c_p T$. Since θ_i is much larger than T , the dominant term in this expression is $\theta_i \alpha$; then across the shock α is approximately constant. Using the same arguments of Section 3.3 we can conclude that α , T , a , and p/ρ are constants over the whole field, independently of the presence of this shock. The entropy will also be a constant* (Section 3.3). The mechanical conditions across the shock are:

$$\rho_a (U - u_a) = \rho_b (U - u_b) = m \quad (85a)$$

$$p_a + m(U - u_a) = p_b + m(U - u_b) \quad (85b)$$

* There should be an increase in entropy across the shock, but it is small and does not affect the flow field.

and from the above arguments:

$$\frac{p_a}{\rho_a} = \frac{p_b}{\rho_b} = a^2 = \text{const.} \quad (85c)$$

From these three equations we obtain:

$$U = \frac{u_a + u_b + \sqrt{(u_a - u_b)^2 + 4a^2}}{2} = \frac{(u_a + a) + (u_b + a)}{2} + \frac{1}{8} \frac{(u_a - u_b)^2}{a} + \dots \quad (86)$$

where the expansion is made for small values of the ratio $(u_a - u_b)^2/4a^2$ (see equation 82a). The first term of the expansion is the classical result that the speed of the shock is the average of the speeds of the characteristics on both sides of the shock. The first two terms of the expansion (85) are the same as those obtained by Courant and Friedrichs (paragraph 72)¹⁴ for a perfect gas with constant specific heats. However, their method cannot be applied for γ equal to one. Using equations (85) and (86) it is easy to show that the s Riemann invariant also remains constant across the shock (to a first approximation):

$$p_b - p_a = m(u_b - u_a)$$

$$\frac{p_b}{p_a} = 1 + \frac{m}{p_a} (u_b - u_a) = 1 + \frac{1}{a} (u_b - u_a) + O\left(\left(\frac{u_a}{a}\right)^2\right)$$

$$\frac{p_b}{p_a} = \exp\left(\frac{u_b - u_a}{a}\right) + O\left(\left(\frac{u_a}{a}\right)^2\right)$$

$$\ln \frac{p_b}{p_a} = \frac{u_b - u_a}{a} + O\left(\left(\frac{u_a}{a}\right)^2\right)$$

$$u_b - a \ln \frac{p_b}{p_a} = u_a - a \ln \frac{p_a}{p_a} + O\left(\left(\frac{u_a}{a}\right)^2\right) \quad (87)$$

This means that the region behind the shock is also a simple-wave with straight C^+ characteristics along which the flow properties are constants. With this latter result and equation (86), we can continue constructing the shock and calculate all the flow field (see Figure 11). However, the method fails as soon as the effect of the wall is felt. This happens when the non-simple wave region 8c is reached.

We could study region 8c by using an analysis similar to the one of Section 3.4 and applying to the shock the conditions (86) and (87), but in the examples that we are going to deal with we can apply a further simplification. We assume that by the time the shock arrives at region 8c the conditions behind and after the shock are almost uniform; they are a small perturbation of uniform conditions 5 upstream of the shock and of conditions 7 downstream. All the characteristics, across which the velocity and pressure suffer large changes, are absorbed by the shock before it arrives at the non-simple wave region adjoining the wall. We can apply the linearized theory of sound in this region. Upstream of the shock the characteristics propagate with speeds $\pm a$ (u_5 is zero) and downstream with $u_7 \pm a$. The shock moves with velocity:

$$U = a + \frac{u_7}{2}$$

This shock reflects at the wall leaving behind a region 10 (Figure 11) that, to a first approximation, is at rest; the characteristics move with velocities $\pm a$ in that region. Across this reflected shock the Riemann invariant r of the C^+ characteristics remains constant. The speed of this shock is given by the average of the speeds of the C^- characteristics on 7 and 10.

$$|U| = a - \frac{u_7}{2}$$

Since we know the location of all the characteristics we can apply equation (84) (see Section 3.4):

$$p_{\text{wall}} = \left(\frac{p_{\text{sh}}}{p_5} \right)^2 p_5$$

p_{sh} is the pressure behind the reflected shock (that interacts with the ionization front) at a point that is on the same C^+ characteristic as the point at the wall. Since we also know the location of the secondary shocks, we can calculate the pressure distribution at the wall.

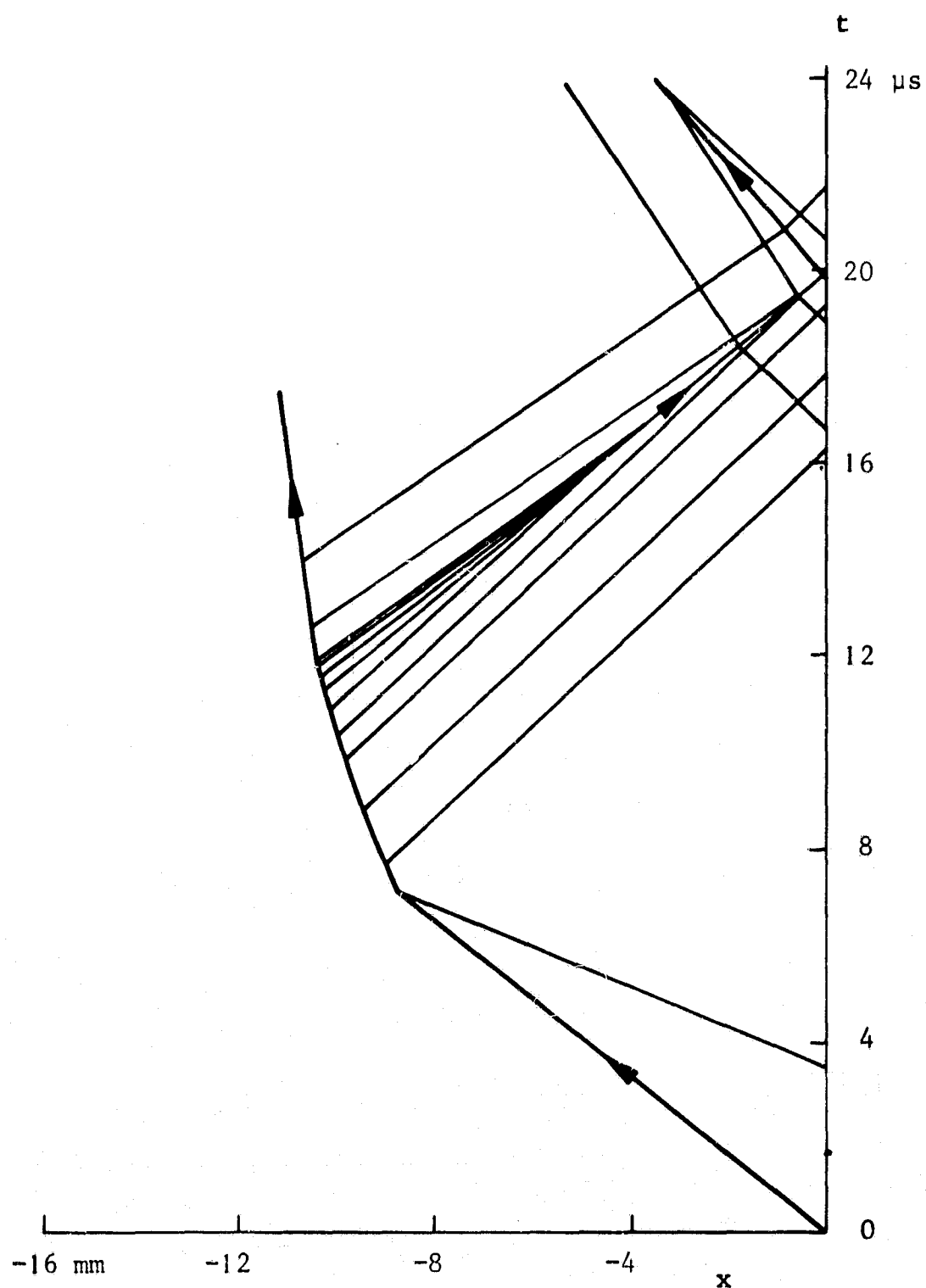


Figure 11.- x-t diagram showing the formation of the shock and the reflected shock trajectory ($M_1 = 14.0$, $p_1 = 0.5$ mm Hg, xenon).

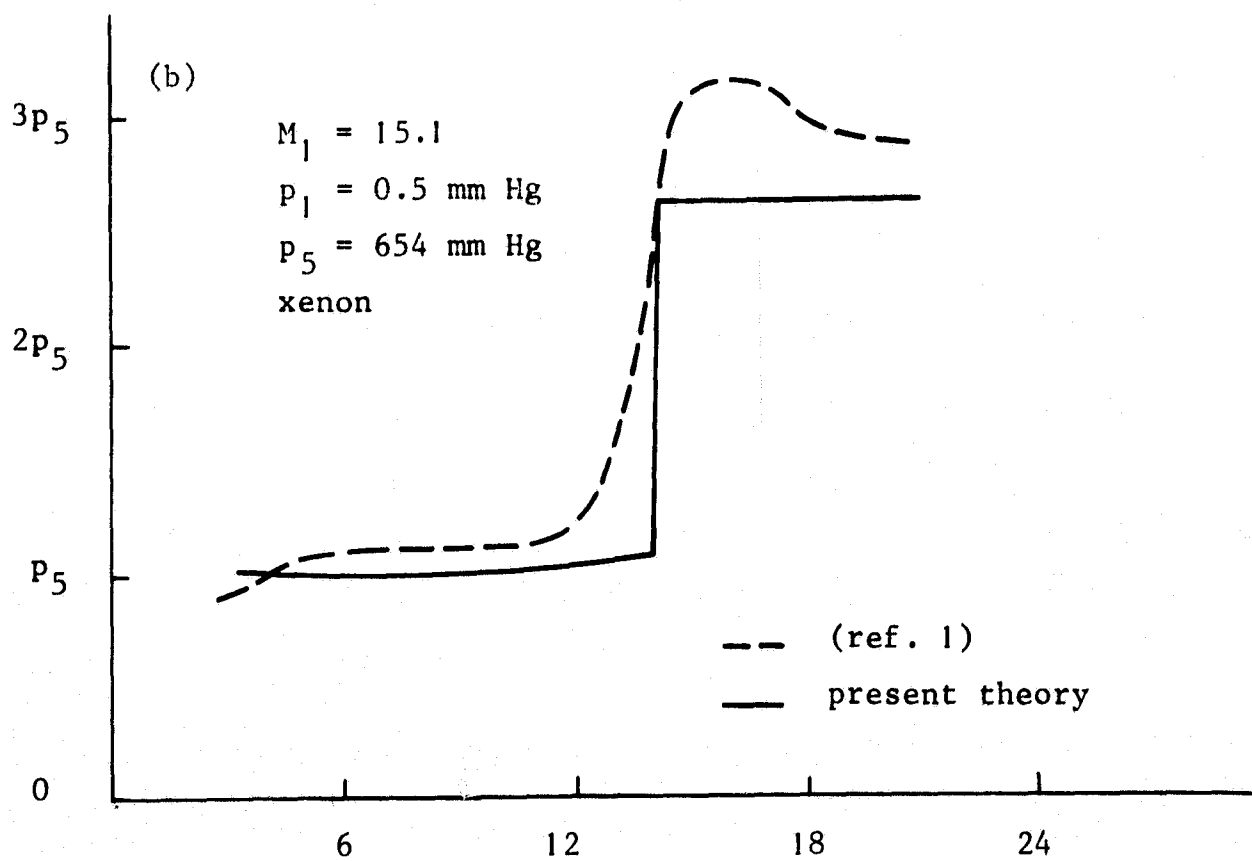
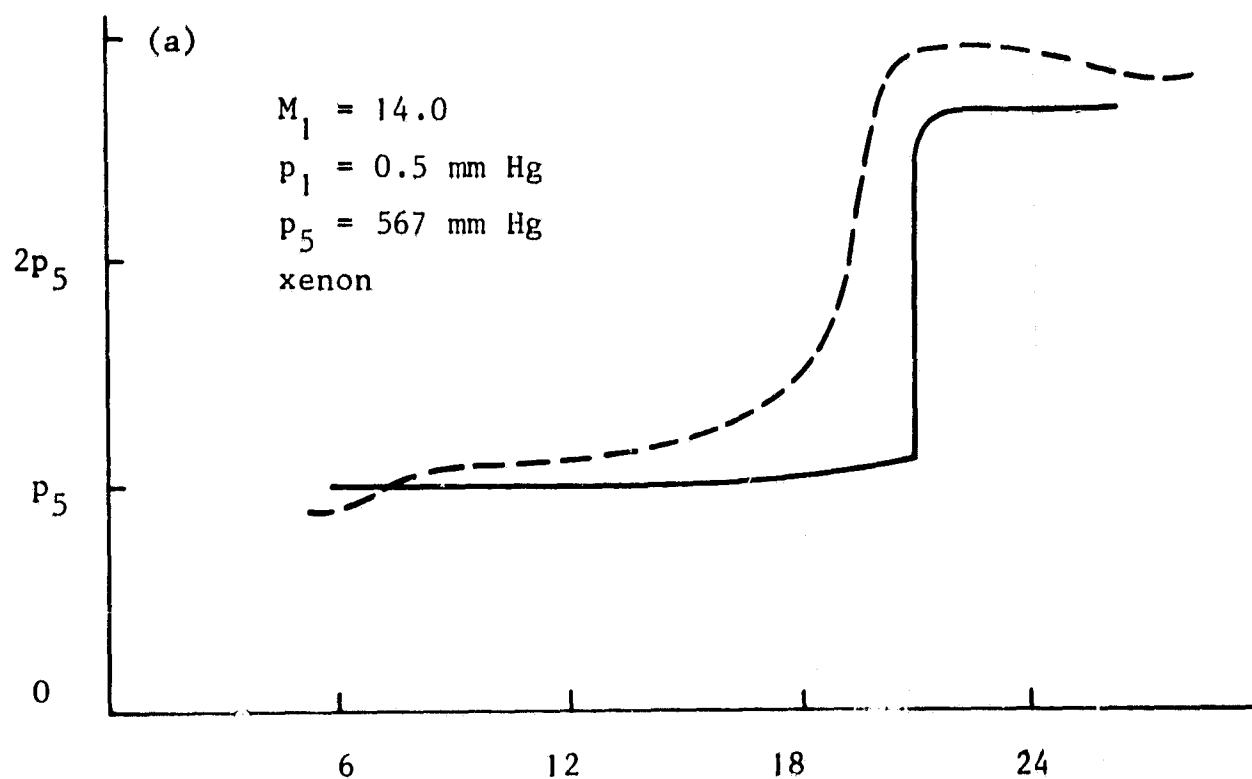


Figure 12.- End wall pressure increase as a result of the interaction of the ionization front with the reflected shock (xenon).

In Figure 12 are represented the calculated pressure distributions at the wall; Smith's measurements¹ are also shown. These seem to be much more spread out than our theoretical results, and their form seems to be more like the examples presented in Figure 10 in which there was no shock formation. Perhaps impurities played an important role in the process of ionization behind the incident shock, and the actual ionization front was thicker than the one resulting from the theoretical model proposed by Smith.¹ It could also be that our assumption of equilibrium degree of ionization is not a good one, and there are relaxation and end wall boundary layer effects that produce this spreading.

The agreement with Smith's experiments¹ regarding the location and the amount of the jump in pressure seems to be good.

3.6 FORMATION OF THE SHOCK INSIDE THE REGION ADJOINING THE WALL

We shall now consider the possibility of shock formation in region 8c. In 3.4 we saw that the coordinates x and t could be expressed in the forms $x(r,s)$ and $t(r,s)$, where r and s are the Riemann invariants. r can be expressed as $r(t,s)$, and the family of C^- characteristics can be put in the parametric form:

$$x = x(r(t,s), s) = x(t, s)$$

If these C^- characteristics intersect they will form an envelope of equations:

$$x = x(t, s)$$

$$\left(\frac{\partial x}{\partial s} \right)_t = 0$$

By a simple manipulation we get;

$$\left(\frac{\partial x}{\partial r}\right)_s \left(\frac{\partial r}{\partial s}\right)_t + \left(\frac{\partial x}{\partial s}\right)_r = 0$$

$$\left(\frac{\partial r}{\partial s}\right)_t = - \frac{(\partial t / \partial s)_r}{(\partial t / \partial r)_s}$$

$$\left(\frac{\partial x}{\partial r}\right)_s \frac{(\partial t / \partial s)_r}{(\partial t / \partial r)_s} - \left(\frac{\partial x}{\partial s}\right)_r = 0$$

and by using (67) and (68) this last equation reduces to:

$$(r - s - a)t_s - (r - s + a)t_s = 0$$

$$(\partial t / \partial s)_r = 0 \quad (88)$$

In the same way the envelope of the C^+ characteristics is given by

$$(\partial t / \partial r)_s = 0 \quad (89)$$

The procedure used in 3.4 was correct as long as the Wronskian was different from zero:

$$\left(\frac{\partial x}{\partial r}\right)_s \left(\frac{\partial t}{\partial s}\right)_r - \left(\frac{\partial x}{\partial s}\right)_r \left(\frac{\partial t}{\partial r}\right)_s \neq 0$$

If we use equations (67) and (68) we see that for the Wronskian to be zero we need:

$$x_r t_s - x_s t_r = 0$$

$$(r - s + a)t_r t_s - (r - s + a)t_s t_r = -2at_r t_s = 0$$

For this to be true either (88) or (89) has to be satisfied. If the Wronskian vanishes in the non-simple wave region 8c there is shock formation in that region.

Using (79), (74) and the approximation (82a):

$$t = e^{-r/2a} T(s) + e^{-s/2a} T(r) \quad r \geq s$$

Taking this expression into (88) and (89) we get:

$$T(r) = 2a e^{\frac{s-r}{2a}} T'(s) \quad r \geq s \quad (90)$$

$$T(s) = 2a e^{\frac{r-s}{2a}} T'(r) \quad r \geq s \quad (91)$$

Equations (90) and (91) together with the functions $x(r,s)$ and $t(r,s)$ give us the envelopes of the C^- and C^+ characteristics respectively. T is a known function.

The values of r and s that satisfy one of the equations (90) or (91) and give a minimum value of t correspond to the cusp of the envelope. If we consider that (90) gives us $r(s)$, then the cusp will satisfy:

$$t_r \frac{dr}{ds} + t_s = 0$$

and since t_s is zero on the envelope of the C^- characteristics:

$$\frac{dr}{ds} = 0$$

$$T'(s) + 2aT''(s) = 0 \quad (92)$$

The values of r and s that correspond to the cusp of the envelope of the C^- characteristics will be given by equations (90) and (92) and those corresponding to the cusp of the envelope of the C^+ characteristics by (91) and:

$$T'(r) + 2a T''(r) = 0 \quad (93)$$

We should first check if the C^+ characteristics have an envelope. If that is so, on the cusp of the envelope a shock wave will start to form. This shock will propagate towards the wall and will reflect from it. If the C^+ characteristics do not collapse, we look for the envelope of the C^- characteristics.

In none of the cases studied in 3.4 were (90) or (91) satisfied at any point of 8c.

It must be remembered that since the C^- characteristics are convergent lines, they must collapse somewhere in the simple-wave region 9 (Figure 1) if they did not do so before. However, in our examples this shock will be of moderate strength and will not affect the previous results.

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