REPORT 10909801-F

INVESTIGATION OF STATISTICAL TECHNIQUES TO SELECT OPTIMAL TEST LEVELS FOR SPACECRAFT VIBRATION TESTS

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INVESTIGATION OF STATISTICAL TECHNIQUES TO SELECT OPTIMAL TEST LEVELS FOR SPACECRAFT VIBRATION TESTS

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PREFACE

The studies reported herein were performed by the Digitek Corporation, Marina del Rey, California, for the NASA Goddard Space Flight Center, Greenbelt, Maryland, under Contract NAS 5-21093. The performance period for the studies was 1 November 1969 to 31 October 1970. The principal investigator was Allan G. Piersol. Major contributions were made by John R. Maurer. The NASA Technical Monitor was William F. Bangs who, with Joseph P. Young, provided considerable assistance and valuable guidance throughout the course of studies.

ABSTRACT

Analytical expressions defining "optimal" test levels for various types of vibration testing of spacecraft hardware are derived using a "minimum cost of error" criterion. The developments assume the service environmental loads and hardware strengths are random variables with either lognormal or normal probability density functions. The resulting expressions for "optimal" test levels are functions only of the distribution of environmental loads and cost factors defining the undesirable consequences of potential test and service failures. It is noteworthy that the expressions do not include parameters of the hardware strength, suggesting that the selection of vibration test levels should not be influenced by pretest assessments of the hardware design integrity. The results indicate the principal factors which should influence vibration test levels are the purpose of the test, the "cost" of developing the hardware, and the "cost" of a potential service failure. Specifically, all other things equal, qualification (design verification) tests should be more severe than acceptance (manufacturing quality verification) tests; tests of relatively expensive hardware should be less severe than tests of less expensive hardware; and tests of hardware whose failure in service might produce relatively serious consequences should be more severe than tests of hardware whose failure in service would produce less serious consequences.

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LIST OF SYMBOLS

C_x - cost (or other measure of undesirable consequences) of event x.

d_1 - decision to approve hardware for service use.

d_2 - decision not to approve hardware for service use.

$\exp[x]$ - e^x

$\mathrm{erf}(k)$ - $\frac{1}{\sqrt{2\pi}} \int_0^k \exp[-x^2/2]\, dx$

E - service environmental load.

L - laboratory environmental test level.

L_o - test level which minimizes the expected value of the cost of an error.

M - number of production items of hardware manufactured prior to the qualification test.

N - number of production items of hardware to be manufactured.

p(x) - probability density function of x.

P[x] - probability of x.

$P[x|y]$ - conditional probability of x given y.

P[x, y] - joint probability of x and y.

R_c - ratio of the cost of a test failure to the cost of a service failure.

S - hardware strength.

$\hat{x}$ - natural logarithm of x.

X_o	-	test level L_o expressed in terms of a percentile of p(E).
μ_x	-	mean value of x.
σ_x	-	standard deviation of x.
λ	-	S - L
ν	-	S - E
η	-	S/L
ξ	-	S/E
ω	-	any parameter value of interest.
Ω	-	domain for all possible values of ω.
θ	-	subset of desirable values within Ω
$\bar{\theta}$	-	complement of $\theta = \Omega - \theta$

1.0 INTRODUCTION

The development of spacecraft hardware (the complete spacecraft assembly or any components thereof) is accompanied by considerable vibration testing at various stages of the development and production cycle. The most formal stages of testing are the qualification (design verification) tests and the acceptance (workmanship verification) tests. In broad terms, qualification tests are conducted to verify the adequacy of the hardware design for proper performance in its anticipated service vibration environment. The tests are usually performed on a single representative sample item of the hardware which is not scheduled for service use. Acceptance tests are conducted to verify that the hardware is assembled without workmanship errors or defective materials. The tests are usually performed on all units scheduled for service use. For the case of hardware designs involving only a few or perhaps one unit scheduled for service use, qualification and acceptance testing objectives are often combined into a single qual-acceptance test.

The basic purpose of a qualification test suggests that the test levels and durations must be closely related to the anticipated service environmental levels and durations. For an acceptance test, however, there are two possible approaches to the derivation of test levels and durations. The first approach is to derive testing conditions which simulate the anticipated service environment, as would be done for qualification testing. The argument for this approach is that such a test should reveal manufacturing defects which might cause a service failure. Conversely, if a manufacturing defect is not revealed by the test, it probably would not be detrimental to the service performance. The second approach is to derive testing conditions which are specifically designed to detect workmanship errors and defective materials. For this case, the resulting test conditions need not constitute a simulation of the anticipated service environment. For example, acceptance tests of electronic equipment sometimes include the application of dwell sinusoidal excitation at the resonant frequencies of wire bundles, even though no such excitation exists in the anticipated service environment. This is done solely because the vibration of wire bundles at resonance is an effective way to reveal (quickly fail) poorly soldered electrical connections.

The NASA Goddard Space Flight Center (GSFC) is concerned with both qualification and acceptance tests as well as combined qual-acceptance tests on unmanned spacecraft hardware. In GSFC terminology, qualification tests are called <u>prototype</u> tests, acceptance tests are called <u>flight</u> tests, and combined qual-acceptance tests are called <u>proto-flight</u> tests. For the case of acceptance (flight) tests, GSFC employes the first mentioned approach to the derivation of vibration test levels and durations; that is, acceptance test criteria are based upon predictions for the anticipated service environment. To be specific, acceptance test levels are established at the 97.7 percentile* of the anticipated service vibration levels, and acceptance test durations are selected to be approximately equivalent to the duration of the service vibration exposure. Qualification (prototype) test criteria are then established by setting the qualification test levels to be 1.5 times the acceptance test levels and the qualification test durations to be 2 times the acceptance test durations. Combined qual-acceptance (proto-flight) test criteria are arrived at by using qualification test levels with acceptance test durations.

*equivalent to the $\mu + 2\sigma$ percentile of a normal distribution.

The foregoing GSFC procedures have undoubtedly provided acceptable results, as evidenced by the rare occurrence of confirmed vibration induced service failures in GSFC spacecraft. Nevertheless, certain aspects of the procedures are somewhat arbitrary and, hence, might be improved upon. The study reported herein is concerned with one such aspect of the procedures; namely, the selection of test levels based upon predictions for the anticipated service environmental levels.

Before pursuing the above specific subject, it should be mentioned that the general subject of spacecraft vibration testing raises a number of basic issues which might be debated. Included are the validity and efficiency of direct mechanical vibration tests as opposed to acoustic noise induced vibration tests, input motion controlled tests as opposed to input force and/or response motion controlled tests, full assembly tests as opposed to component tests, and real time environmental simulation tests as opposed to accelerated damage simulation tests. To permit the formulation of a clearly defined problem which can be pursued analytically, such issues are avoided in this study by a series of basic assumptions which are generally compatible with GSFC testing philosophy. These assumptions are as follows:

(a) An acoustic test is simply a special type of vibration test where the input forcing function is a pressure field rather than a mounting point motion.

(b) No matter what parameter (impinging pressure, input motion, input force, or response motion) is used to specify the vibration test levels, the anticipated service environment can be predicted in terms of the probability density function of that parameter.

(c) The basic goals of a vibration test are not influenced by the scope of the hardware to be tested (an acceptance test of a complete spacecraft assembly is performed with the same basic intent as an acceptance test of some component thereof).

(d) Vibration exposure times in service are such that real environmental simulation tests can be performed (no accelerated testing is required).

Further assumptions needed to make the problem tractable are presented later for each testing objective to be considered.

2.0 THEORETICAL BACKGROUND

The selection of vibration test levels for spacecraft hardware ultimately reduces to the determination of an acceptable compromise between the undesirable consequences of overtesting versus undertesting. Some finite risk of overtesting or undertesting always must be accepted since the service environmental load as well as the resistance of the spacecraft hardware to environmental induced failures are not precisely known; i. e., both of these factors are random variables.

To present the problem in more quantitative terms, let the following notation be defined:

S	=	hardware strength; i. e., resistance to environmental induced failures
E	=	service environmental load
L	=	environmental test level
p(S)	=	probability density function of the hardware strength
p(E)	=	probability density function of the service environmental load

Note that S, E, and L may be defined in terms of any load parameter of concern (input acceleration, input force, response acceleration, etc.) so long as it is the same parameter for all three. Further note that S and E are both functions of frequency since the hardware strength as well as the environmental loads are frequency dependent. It follows that an appropriate value for L will also be frequency dependent. However, the notation (f) for the frequency dependence of S, E, and L will be omitted throughout for simplicity.

Given the above definitions, the compromise between overtesting and undertesting can be graphically illustrated as shown in Figure 1. For a test at level L, it is clear from Figure 1 that the probability of the hardware failing the test (related to overtesting) is

$$\beta = P[S \le L] = \int_0^L p(S)\, dS \tag{1}$$

while the probability of the environmental load exceeding the test level (related to undertesting) is

$$\alpha = P[L < E] = \int_L^\infty p(E)\, dE \tag{2}$$

Of greater interest is how the test level L relates to the probability of an environmentally induced service failure. In the absence of any testing, the probability of a service failure is simply the probability that $S \le E$. Specifically, letting $\nu = S - E$, the probability of a service failure is given by

$$P_f = P[S \le E] = P[\nu \le 0] = \int_{-\infty}^0 p(\nu)\, d\nu \tag{3}$$

Now consider the case where the hardware has passed a test at level L. Letting $\lambda = S - L$ and assuming for simplicity that the test causes no damage, the probability of a service failure given the hardware passes a test at level L is

$$P_f(L) = P[S \le E \mid L < S] = \frac{P[S \le E,\ L < S]}{P[L < S]}$$

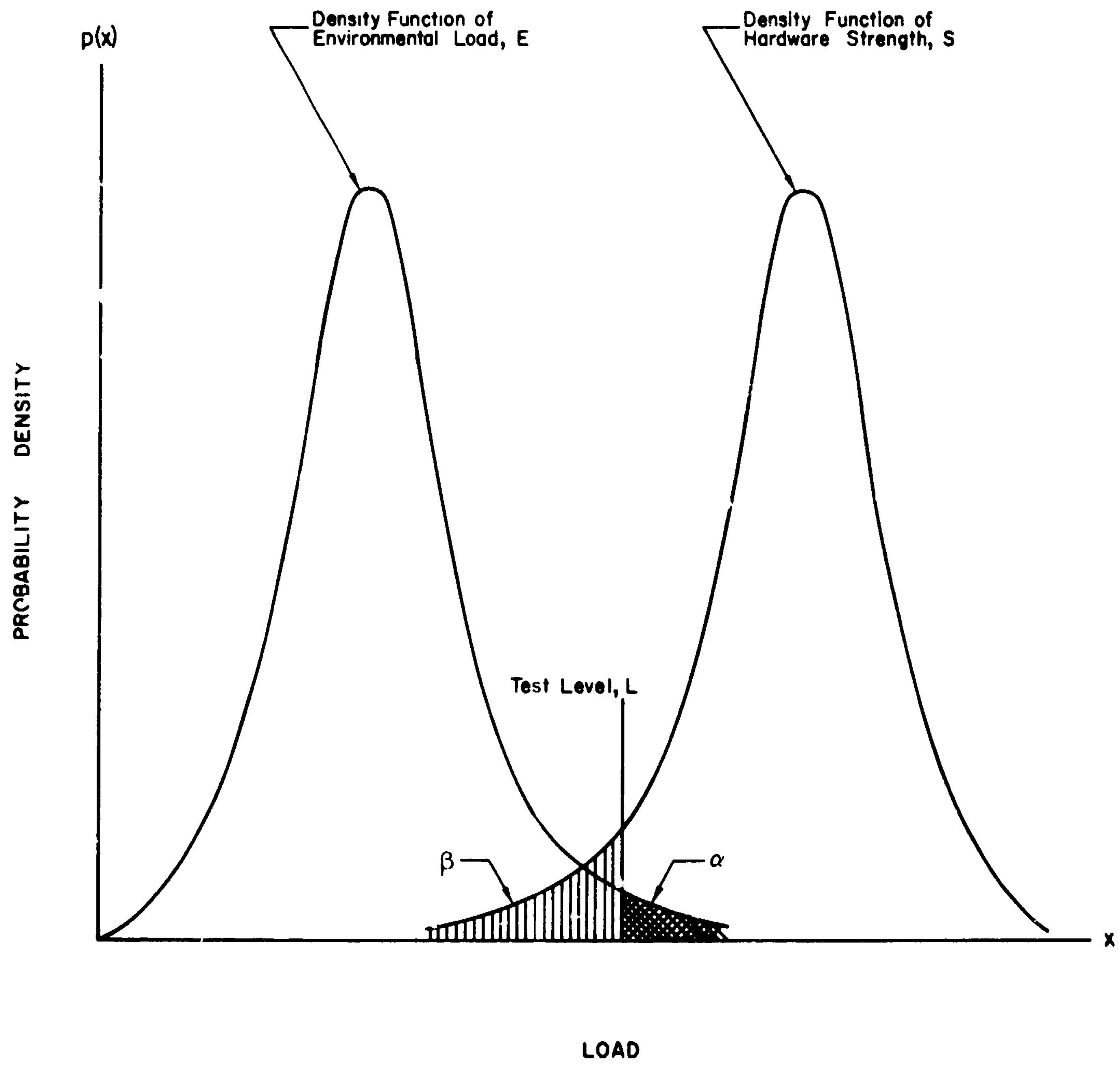

FIGURE I. STATISTICAL DISTRIBUTIONS OF ENVIRONMENTAL LOAD AND HARDWARE STRENGTH

$$= \frac{P\,[\,\nu \le 0,\, \lambda > 0\,]}{P\,[\lambda > 0]} = \frac{\int_{-\infty}^{0} \left\{ \int_{0}^{\infty} p\,(\nu, \lambda)\, d\lambda \right\} d\nu}{\int_{0}^{\infty} p\,(\lambda)\, d\lambda} \tag{4}$$

where $P\,[x \mid y]$ denotes the conditional probability of x given y, and $P\,[x,\ y]$ denotes the joint probability of x and y. For the case of a test at level L = 0 (no test), λ S in Eq. (4). Since S cannot be negative ($0 \le S < \infty$), it follows that $\int_0^{\infty} p\,(\lambda)\, d\lambda = 1$ and $\int_0^{\infty} p\,(\nu,\ \lambda)\, d\lambda = p\,(\nu)$. Hence, for the case of L = 0, Eq. (4) reduces to

$$P_f\,(L = 0) = \int_{-\infty}^{0} p\,(\nu)\, d\nu$$

which is equal to the probability of a service failure defined in Eq. (3), as would be expected.

Although not obvious from Eq. (4), it can be shown that $P_f\,(L) \rightarrow 0$ as $L \rightarrow \infty$. In words, for the idealized case being considered, the probability of a service failure in a successfully tested item of hardware goes down as the test level goes up. Of course the probability of the hardware passing the test, even though it may be satisfactory for the service environment, also goes down as indicated by Eq. (1). Hence the risk of undertesting can be eliminated only by testing at infinite levels which would fail all hardware items, while the risk of overtesting can be eliminated only by testing at nil levels (no testing at all). It is clear that the selection of a test level involves a compromise between these two extremes.

3.0 TEST LEVEL SELECTION CRITERION

The first problem to be resolved is the formulation of a suitable criterion for the required compromise between overtesting and undertesting. A number of possible criteria are available from the general literature on statistical decisions, as summarized for applications to the aerospace vibration testing problem by Rentz [1]*. The most attractive of the possible criteria from the viewpoints of simplicity and applicability to the problem at hand is believed to be a "minimum cost of error" criterion. Such a criterion is applicable to most testing problems involving a simple "yes" or "no" decision where the undesirable consequences of an incorrect decision are finite and subject to quantitative estimation. A simple statement of the approach is presented in [2] with more rigorous developments available from the references to [1].

3.1 REVIEW OF BASIC APPROACH

Let ω be a parameter value of interest, and Ω be the domain for all possible values of ω. Let θ be a set of desirable values within Ω which hopefully includes the parameter value ω, and $\bar{\theta} = \Omega - \theta$ be the set of values within Ω which do not fall within θ (the complement of θ). An experiment is now performed with a result that leads to one of two possible decisions, d_1 or d_2, as follows:

d_1 = the parameter value ω is inside θ ($\omega \in \theta$).

d_2 = the parameter value ω is outside θ ($\omega \in \bar{\theta}$).

It follows that the resulting decision will produce one of four possible situations:

*Numbers in brackets denote references in Section 8.0.

a) decision d_1 is made when in fact $\omega \in \theta$
(a correct decision)

b) decision d_2 is made when in fact $\omega \in \overline{\theta}$
(a correct decision)

c) decision d_1 is made when in fact $\omega \in \overline{\theta}$
(an incorrect decision)

d) decision d_2 is made when in fact $\omega \in \theta$
(an incorrect decision)

The first two situations involve correct decision which are assumed to produce no undesirable consequences. The second two situations involve incorrect decisions which are assumed to produce undesirable consequences that can be quantitatively predicted. Let the incorrect decisions be called errors. It is now asserted that a "good" experiment is one which will produce a decision with the minimum undesirable consequences on the average; that is, the minimum expected value for the cost of an error.

To formulate the criterion, let the following notation be defined.

P_1 = probability of an error in decision d_2; that is, $P\,[d_2 : \omega \in \theta]$

C_1 = cost (or other measure of the undesirable consequences) of an error in decision d_2.

P_2 = probability of an error in decision d_1; that is, $P\,[d_1 : \omega \in \overline{\theta}]$

C_2 = cost (or other measure of the undesirable consequences) of an error in decision d_1.

p = probability that d_1 is correct decision; that is, $P\ [\omega \subset \theta]$.

$1-p$ = probability that d_2 is correct decision; that is, $P\ [\omega \subset \overline{\theta}]$.

Now if $\omega \subset \theta$, decision d_2 would constitute an error; the probability of the error is P_1 and the cost is C_1. Hence, the expected value for the cost of an error in this case is P_1C_1. Similarly, if $\omega \subset \overline{\theta}$, decision d_1 would constitute an error with an expected cost of P_2C_2. It follows that the expected value for the total cost of an error is

$$C_e = p\, P_1C_1 + (1-p)\, P_2C_2 \tag{6}$$

A "good" experiment will then be one where P_1 and P_2 are such that C_e in Eq. (6) is a minimum.

3.2 APPLICATION TO VIBRATION TESTING

Consider an idealized vibration test, as discussed in Section 2.0, where the following assumptions apply.

a) The test is performed to verify that the hardware will function properly in its anticipated service environment.

b) If the hardware passes the test, there is no residual damage caused by the test which will adversely influence the proper performance of the hardware in service (no fatigue damage occurs).

c) If the hardware fails the test, some corrective action (redesign, rework, etc.) is taken and the hardware is retested: that is, the hardware must successfully pass the test without waivers before service use is approved.

The criterion outlined in Section 3.1 may now be applied directly to the design of a "good" experiment (the selection of a "good" test level) for this idealized vibration test as follows:

Let S, E, and L be the hardware strength, the service environmental load, and the test environmental level, respectively, as illustrated in Figure 1. Let d_1 be the decision to approve the hardware for service use which occurs if the hardware passes the test; that is,

$$S > L \longrightarrow d_1$$

and d_2 be the decision not to approve the hardware for service use which occurs if the hardware fails the test; that is,

$$S \leq L \longrightarrow d_2$$

It follows that d_1 will be a correct decision if the hardware in fact is stronger than the service environmental load; that is,

$$\omega \subset \theta = S > E$$

while d_2 will be a correct decision if the hardware in fact is not stronger than the service environmental load; that is,

$$\omega \subset \overline{\theta} = S \leq E$$

Finally, the cost of an error in decision d_1 will be the cost of a service failure, to be denoted by C_f, while the cost of an error in decision d_2 will be the cost of a test failure (redesign, rework, retest, etc.), to be denoted by C_t. Then, referring to Eq. (6), the terms needed to define the expected value of the cost of an error is as follows:

$$P_1 = P[d_2 \mid \omega \subset \theta] = P[S \leq L \mid S > E]$$

$$C_1 = C_t \text{ (cost of a test failure)}$$

$$P_2 = P[d_1 \mid \omega \subset \overline{\theta}] = P[S > L \mid S \leq E]$$

$$C_2 = C_f \text{ (cost of a service failure)}$$

$$p = P[\omega \subset \theta] = P[S > E]$$

$$(1-p) = P[\omega \subset \overline{\theta}] = P[S \leq E]$$

Hence, Eq. (6) becomes

$$\begin{aligned} C_e &= P[S > E]\ P[S \leq L \mid S > E]\, C_t \\ &+ P[S \leq E]\ P[S > L \mid S \leq E]\, C_f \end{aligned} \tag{7}$$

Noting that $P[x/y] = P[x, y] / P[y]$, Eq. (7) reduces to

$$C_e = P[S \leq L,\ S > E]\, C_t + P[S > L,\ S \leq E]\, C_f \tag{8}$$

Based upon the criterion discussed in Section 3.1, a "good" test level L for this idealized example is one which minimizes C_e in Eq. (8). Such a test level will be referred to as an "optimal" test level.

A brief review of Eq. (8) quickly reveals the logic of the criterion as applied to the vibration testing problem. Specifically, Eq. (8) says that the total cost of an error is given by the probability of a test failure due to overtesting ($P[S \leq L,\ S > E]$) times the cost of a test failure due to overtesting (C_t) plus the probability of a service failure due to undertesting ($P[S > L,\ S \leq E]$) times the cost of a service failure due to undertesting (C_f). An optimal test level is one which will minimize this cost on the average.

4.0 FORMULATION OF TEST LEVEL SECTION MODELS

The criterion for selecting an optimal test level, as outlined in Section 3.0, is now applied to the specific types of tests performed by GSFC. Qual-acceptance testing is considered first since it provides the most direct application of the criterion. Acceptance testing is considered next and qualification testing last.

4.1 QUAL-ACCEPTANCE TESTING

Consider a qual-acceptance vibration test, called a protoflight test by GSFC, as discussed in Section 1.0. Let the following assumptions apply.

a) The purpose of the test is to verify that the specific hardware item being tested will function properly in its anticipated service vibration environment; i.e., the design integrity and fabrication quality of the item are adequate.

b) The integrity of the hardware design has not been verified by a prior qualification test on a prototype unit.

c) If the hardware passes the test, there is no residual damage caused by the test which will adversely influence the proper performance of the hardware in its anticipated service environment.

d) If the hardware fails the test, corrective action in the form of redesign and/or rework is taken and the hardware is retested; that is, the hardware must successfully pass a test without waivers before being approved for service use.

Assumptions a) and b) evolve directly from the basic purpose of a qual-acceptance test as defined herein. Assumption c) is inherent in the qual-acceptance test concept and critical to the problem formulation. Unfortunately, in light of assumption b), the validity of assumption c) is questionable in practice. In fact, this assumption points out the primary deficiency in the qual-acceptance test concept. Specifically, in the absence of a prior prototype test, it is difficult to confirm that the qual-acceptance test has not expended a significant portion of the wear-out (fatigue) life of the hardware. Perhaps this confirmation could be obtained from the results of earlier engineering and design evaluation tests, or from analysis of data collected during the qual-acceptance test. In any case, the assumption is necessary and will apply.

Assumption d) requires elaboration. The problem here is defining what constitutes a failure of the hardware. On the one hand, if a structural member were to break due to inadequate design strength, this clearly would be a failure in the context of assumption d). On the other hand, if a wire connection were to break due to faulty soldering, the test might be briefly interrupted to resolder the connection, or simply continued to completion with the corrective soldering accomplished at the conclusion of the test. Either way, this would not be a failure in the context of assumption d) since no major corrective action in the form of extensive analysis and/or retest would be required to make the hardware suitable for service use. In broad terms, a test failure will be defined as any structural failure or performance malfunction which cannot be confidently attributed to an easily repairable workmanship error or material defect.

With the above assumptions, the discussions in Section 3.0 can now be applied to the design of a qual-acceptance test. In fact, the assumptions make the qual-acceptance test equivalent to the idealized vibration test discussed in Section 3.2. Hence, the results in Eq. (8) apply directly; that is, the expected value of the cost of an error in qual-acceptance testing is

$$C_e = P[S \leq L,\ S > E]C_{qt} + P[S > L,\ S \leq E]C_f \qquad (9)$$

where

S = hardware strength

E = service environmental load

L = test environmental load

C_{qt} = cost of qual-acceptance test failure

C_f = cost of service failure

4.2 ACCEPTANCE TESTING

Consider next an acceptance test, called a flight test by GSFC, as discussed in Section 1.0. Let the following assumptions apply.

a) The purpose of the test is to verify that the specific hardware item being tested was manufactured with no workmanship errors or material defects that would impair proper performance in its anticipated service environment.

b) The integrity of the hardware design has been verified by a prior qualification test on a prototype unit.

c) If the hardware passes the test, there is no residual damage caused by the test which will adversely influence the proper performance of the hardware in its anticipated service environment.

d) If the hardware fails the test, corrective action is taken in the form of an investigation to determine if the failure was due to a basic design fault which escaped detection in the qualification test. If so, redesign and/or rework is pursued. If not, that specific item of hardware is rejected as being quality defective beyond repair.

Assumptions a) and b) evolve directly from the basic purpose of an acceptance test as defined herein. Assumption c) is inherent in the acceptance test concept and critical to the problem formulation, as was true for qual-acceptance testing in Section 4. 1. Assumption d) elaborates on the type of action which usually is initiated by an acceptance test failure. However, if an acceptance test failure is defined in the same way as a qual-acceptance test failure in Section 4. 1*, then assumption d) is no different from the corresponding assumption used to formulate the qual-acceptance test case.

*anomalies due to obvious workmanship errors or material defects which are repaired during or after the test and do not initiate extensive analysis and/or retest are not considered to be failures in the context of assumption d).

Hence, in the final analysis, an acceptance test serves the same basic purpose as the qual-acceptance test; i. e., to verify that the specific hardware item being tested will function properly in its anticipated service vibration environment.

The only practical difference between acceptance and qual-acceptance testing is that for the acceptance test case, the design integrity of the hardware in question has already been verified by a prior qualification test on a prototype item. The primary implication of this difference is that assumption c), which is questionable for the qual-acceptance test, is reasonable for the acceptance test assuming the acceptance test is less severe than the prior qualification test. A secondary implication is that the probability distribution of the hardware strength S of concern in the acceptance test may be different from the strength of concern in qual-acceptance testing. Specifically, the expected value of the hardware strength should be somewhat greater for the acceptance test case because of the added confidence provided by the fact that the hardware has successfully passed a prior qualification test.

Based upon the above considerations, the expected value of the cost of an error in acceptance testing is given directly by Eq. (8) as

$$C_e = P[S_a \leq L,\ S_a > E]C_{at} + P[S_a > L,\ S_a \leq E]C_f \qquad (10)$$

which is the same criterion as presented for qual-acceptance testing in Section 4. 1 with two exceptions.

a) The hardware strength is denoted by S_a rather than S to indicate different strength distributions may be involved.

b) The cost of a test error is denoted by C_{at} rather than C_{qt} to indicate different costs for a test failure are involved.

4.3 QUALIFICATION TESTING

Finally consider a qualification test, called a prototype test by GSFC, as discussed in Section 1.0. Let the following assumptions apply.

a) The purpose of the test is to verify the adequacy of the hardware design for proper performance in its anticipated service environment.

b) A decision concerning the adequacy of the hardware design is based upon the results of a single test on one sample item of the hardware. The sample item tested is not itself delivered for service use.

c) If the sample item of hardware passes the test, N items of production hardware will be delivered for service use after appropriate acceptance testing.

d) If the sample item of hardware fails the test, corrective action in the form of redesign is taken and a new reworked sample item of hardware is tested; that is, a sample item of hardware must successfully pass a test without waivers before the hardware design is approved for service use. Furthermore, all items of production hardware which might have been manufactured prior to the test will be refurbished in accordance with the redesign needed to pass the test.

Assumptions a) and b) evolve directly from the basic purpose of a qualification test as defined herein. Assumptions c) and d) are needed to quantitize the potential cost of an error in the decision resulting from the test.

Given the above assumptions, there are several intepretations which can be applied to formulate an expected cost of error for qualification testing. Three possibilities are now considered.

4.3.1 Approach No. 1

The first approach is to select the qualification test level based upon the probability that all production items (rather than the sample item being tested) will function properly in their anticipated service environment. This is done without consideration of the potential results of the acceptance tests to be performed on the production items.

Following the procedure in Section 3, let d_1 (d_2) be the decision to approve (not to approve) the hardware design for service use which occurs if the sample item passes (fails) the test; that is,

$$S_q > L \longrightarrow d_1$$
$$S_q \leq L \longrightarrow d_2$$

where S_q is the strength of the sample item used for the qualification test. Decision d_1 will be considered a correct decision if <u>all production items</u> are stronger than the service environmental loads; that is,

$$S_{a1} > E \text{ and } S_{a2} > E \text{ and} \ldots \text{ and } S_{aN} > E = \bigcap_{i=1}^{N} (S_{ai} > E)$$

where S_{ai} is the strength of the ith production item. Decision d_2 will be considered a correct decision if <u>any production item</u> is not stronger than the service environmental loads; that is,

$$S_{a1} \leq E \text{ or } S_{a2} \leq E \text{ or} \ldots \text{ or } S_{aN} \leq E = \bigcup_{i=1}^{N} (S_{ai} \leq E)$$

The undesirable situations (errors) which might result from the qualification test are then

a) decision d_1 is made when in fact $\bigcup_{i=1}^{N} (S_{ai} \leq E)$.

b) decision d_2 is made when in fact $\bigcap_{i=1}^{N} (S_{ai} > E)$.

It follows that the various terms needed in Eq. (6) become

$$P_1 = P[S_q \leq L \mid \bigcap_{i=1}^{N} (S_{ai} > E)]$$

$$C_1 = C_{qt} \text{ (cost of a qualification test failure)}$$

$$P_2 = P[S_q > L \mid \bigcup_{i=1}^{N} (S_{ai} \leq E)]$$

$$C_2 = C_f \text{ (cost of a service failure)}$$

$$p = P[\bigcap_{i=1}^{N} (S_{ai} > E)]$$

$$(1-p) = P[\bigcup_{i=1}^{N} (S_{ai} \leq E)]$$

Hence, the expected value of the cost of an error is

$$C_e = P[\bigcap_{i=1}^{N}(S_{ai} > E)]\ P[S_q \leq L \mid \bigcap_{i=1}^{N}(S_{ai} > E)]\ C'_{qt}$$

$$+ P[\bigcup_{i=1}^{N}(S_{ai} \leq E)]\ P[S_q > L \mid \bigcup_{i=1}^{N}(S_{ai} \leq E)]\ C_f$$

$$= P[S_q \leq L, \bigcap_{i=1}^{N}(S_{ai} > E)]\ C'_{qt} + P[S_q > L, \bigcup_{i=1}^{N}(S_{ai} \leq E)]\ C_f \quad (11)$$

Note that the cost of a test failure, as denoted by C'_{qt} in Eq. (11), must include the cost of refurbishing all production items which might have been manufactured prior to the qualification test, as well as the hardware redesign action needed to pass the test. If $M \leq N$ production items were manufactured prior to the test and C_r is the cost of refurbishing a single production item, then

$$C'_{qt} = C_{qt} + MC_r \quad (12)$$

4.3.2 Approach No. 2

The second approach is to select the qualification test level based upon the probability that all production items will pass their acceptance tests and function properly in their anticipated service environment.

For this case, the basic decision process is the same as outlined in Section 4.3.1, except the undesirable situations (errors) which might result from the qualification test must be modified to account for the later decisions produced by the acceptance tests on the production items. Specifically, let d_1 be considered a correct decision if all production items will pass the acceptance test and are stronger than the service environmental loads; that is,

$$S_{a1} > L_a,\ S_{a1} > E \text{ and } S_{a2} > L_a,\ S_{a2} > E \text{ and } \ldots$$

$$\text{and } S_{aN} > L_a,\ S_{aN} > E \;=\; \bigcap_{i=1}^{N} (S_{ai} > L_a,\ S_{ai} > E)$$

where S_{ai} is the strength of the ith production item as before, and L_a the acceptance test level. Decision d_2 will be considered a correct decision if <u>any production item</u> is not stronger than the service environmental loads (whether it will pass an acceptance test is not relevant in this case); that is,

$$S_{a1} \leq E \text{ or } S_{a2} \leq E \text{ or } \ldots \text{ or } S_{aN} \leq E \;=\; \bigcup_{i=1}^{N} (S_{ai} \leq E)$$

Evaluating the undesirable situations (errors) and calculating the terms in Eq. (6) leads to

$$\begin{aligned} C_e &= P[S_q \leq L,\ \bigcap_{i=1}^{N}(S_{ai} > L_a,\ S_{ai} > E)]C'_{qt} \\ &\quad + P[S_q > L,\ \bigcup_{i=1}^{N} (S_{ai} \leq E)]C_f \end{aligned} \tag{13}$$

where C'_{qt} is as defined in Eq. (12).

4.3.3 <u>Approach No. 3</u>

The final approach is to select the qualification test level based upon the probability that the sample item being tested will function properly in its anticipated service environment. For this case, the probability that the production items will function properly in their anticipated service environment is accounted for in the "cost of service failure" term.

The decision to approve (not to approve) the hardware design for service use which occurs if the sample item passes (fails) the test is the same as before; that is,

$$S_q > L \longrightarrow d_1$$

$$S_q \leq L \longrightarrow d_2$$

Now, however, decision d_1 (d_2) will be considered a correct decision if the sample item being tested is stronger (not stronger) than the service environmental loads; that is,

$$S_q > E \quad (d_1 \text{ correct})$$

$$S_q \leq E \quad (d_2 \text{ correct})$$

Hence, the undesirable situations (errors) which might result from the qualification test, and their probability of occurrence, are the same as developed in Section 3.2; namely

$$P_1 = P[S_q \leq L \mid S_q > E]$$

$$P_2 = P[S_q > L \mid S_q \leq E]$$

Furthermore, referring to Eq. (6), the pC_1 term is the same as before except the cost of refurbishing $M \leq N$ items of production hardware which might have been manufactured prior to the qualification test must be added. Specifically, using the notation of Eq. (12),

$$pC_1 = P[S_q > E]C'_{qt} = P[S_q > E]\,(C_{qt} + MC_r) \qquad (14)$$

Now consider the $(1-p)C_2$ term in Eq. (6). Let P_{nf} denote the probability that $n \leq N$ production items will fail in service. Assuming all production items have the same strength distribution function, the probability of n service failures is given by the binomial distribution; that is,

$$P_{nf} = \frac{N!}{(N-n)!\,n!} P_f^{\,n} P_s^{\,N-n}$$

where

$$P_f = \text{probability of a single service failure}$$
$$= P[S_a \leq E]$$

$$P_s = \text{probability of a single service success}$$
$$= P[S_a > E] = 1 - P_f$$

Let C_{nf} denote the cost of n service failures. Assuming all service failures are of equal cost, $C_{nf} = n\,C_f$ where C_f is the cost of a single service failure. Since all failure cost possibilities must be considered, it follows that

$$(1-p)C_2 = \sum_{n=1}^{N} nC_f P_{nf} = C_f \sum_{n=1}^{N} \frac{N!}{(N-n)!\,(n-1)!} P_f^{\,n} P_s^{\,N-n}$$

$$= N\,C_f P_f \sum_{n=1}^{N} \frac{(N-1)!}{(N-n)!\,(n-1)!} P_f^{\,n-1} P_s^{\,N-n}$$

Letting u = N-1 and v = n-1, the summation in the above expression reduces to

$$\sum_{v=0}^{u} \frac{u!}{(u-v)!\,v!} P_f^{\,v} P_s^{\,u-v} = \left(P_f + P_s\right)^u$$

Hence,

$$(1-p)C_2 = N C_f P_f [P_f + P_s]^{N-1} = N C_f P_f$$

$$= N C_f P [S_a \leq E] \qquad (15)$$

Finally, assume that the acceptance tests on the production items eliminate from service use all items with deficient strength due to poor workmanship and/or material defects; i.e., assume $S_a \approx S_q$. Then, using Eqs (14) and (15), the expected value of the cost of an error in qualification testing is given by Eq. (6) as

$$C_e = P[S_q > E]\ P[S_q \leq L \mid S_q > E]\ (C_{qt} + M C_r)$$

$$+ P[S_q \leq E]\ P[S_q > L \mid S_q \leq E]\ (N C_f)$$

$$= P[S_q \leq L,\ S_q > E]\ (C_{qt} + M C_r) + P[S_q > L,\ S_q \leq E]\ N C_f$$

(16)

Of the three approaches developed in this Section, the formulation resulting from the third approach, as given by Eq. (16), will be used to derive optimal levels for qualification testing. The primary reason for this choice is that Eq. (16) is the only one of the three formulations which can be minimized and solved in closed form without introducing extreme assumptions. The only debatable assumption involved in the derivation of Eq. (16) is that $S_a \approx S_q$. This is a reasonable assumption if the prototype hardware item used for the qualification test is truly representative of the production items, as it should be, and if $\mu_S >> \sigma_S$, as it usually is.

5.0 SOLUTION OF TEST LEVEL SECTION MODELS

The determination of an optimal test level from the cost of error models presented in Section 4.0 requires three general steps as follows:

a) Specific probability density functions of the hardware strength S and the environmental load E must be assumed.

b) The probability statements in Eqs (9), (10) and (16) must be reduced to analytical expressions involving S, E, and L based upon the probability density functions assumed for S and E.

c) Each of the resulting cost of error expressions must be differentiated with respect to the test level L, and solved for the value of L which makes the derivative equal to zero.

Each of the above steps is now outlined with analytical details presented in the appendices.

5.1 PROBABILITY DENSITY FUNCTIONS

The determination of an optimal test level will be pursued assuming two different forms for the probability density functions of S and E, as follows:

a) lognormal distribution.

b) normal distribution.

For the lognormal distribution assumption, the probability density functions of S and E are given by

$$p[\hat{S}] = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}} \exp\left[-\frac{1}{2}\left(\frac{\hat{S} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \tag{17a}$$

$$p[\hat{E}] = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{E}}} \exp\left[-\frac{1}{2}\left(\frac{\hat{E} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^2\right] \tag{17b}$$

where

$\hat{S}$ = log S

$\hat{E}$ = log E

$\mu_{\hat{S}}$ = mean value of log S

$\mu_{\hat{E}}$ = mean value of log E

$\sigma_{\hat{S}}$ = standard deviation of log S

$\sigma_{\hat{E}}$ = standard deviation of log E

For the normal distribution assumption, the probability density functions of S and E are given by

$$p[S] = \frac{1}{\sqrt{2\pi}\ \sigma_S} \exp\left[-\frac{1}{2}\left(\frac{S - \mu_S}{\sigma_S}\right)^2\right] \tag{18a}$$

$$p[E] = \frac{1}{\sqrt{2\pi}\ \sigma_E} \exp\left[-\frac{1}{2}\left(\frac{E - \mu_E}{\sigma_E}\right)^2\right] \tag{18b}$$

where

μ_S = mean value of S

μ_E = mean value of E

σ_S = standard deviation of S

σ_E = standard deviation of E

Of the above two probability density functions, the lognormal function is the more realistic for the problem being considered.

Although there is no firm theoretical justification for a lognormal assumption, it has been widely used in the past to describe the probabilistic character of both environmental loads and hardware strength. From a theoretical viewpoint, the lognormal distribution rules out the possibility of negative values for S and E, which is consistent with physical reality. From a practical viewpoint, it makes the distributions of the load and strength parameters normal when these parameters are measured in dB, which is convenient. Further discussions of the lognormal distribution are presented in Appendix A.

The normal probability density function theoretically suggests the possible occurrence of negative values for S and E, which cannot happen in practice. Furthermore, experience has not supported the applicability of the normal assumption to environmental load and/or hardware strength distributions. The normal distribution assumption is employed in these studies primarily to provide an indication of how much the underlying probability density function assumption might impact the resulting optimal test levels provided by the models.

5.2 REDUCTION OF PROBABILITY STATEMENTS

The two basic probability statements which appear in Eqs (9), (10), and (16) are the joint probabilities, $P[S \leq L, S > E]$ and $P[S > L, S \leq E]$. These two statements are reduced to analytical expressions assuming both lognormal and normal distributions of E and S in Appendix B. The results for the lognormal distribution assumption are presented in Eqs (B. 8) and (B. 10). The results for an unrestricted normal distribution assumption are given by Eqs (B. 14) and (B. 15), while the results for a truncated normal distribution assumption (restricted to nonnegative values of S and E) are presented in Eqs (B. 19) and (B. 21).

5.3 SOLUTION FOR TEST LEVEL WHICH MINIMIZES COST OF ERROR

By substituting the appropriate expressions from Appendix B into the equations for C_e developed in Section 4, taking the derivative of C_e with respect to L, and setting this derivative equal to zero, the test level L_o which minimizes the expect value of the cost of an eror C_e is determined for each of the three types of testing. These calculations are presented in Appendix C. The results are summarized below.

5.3.1 Solutions for Qual-Acceptance Testing

The test level which will minimize the cost of error in qual-acceptance testing, as given by Eq. (9), is determined in Appendix C to be as follows:

a) For lognormal distributions of S and E,

$$\mathrm{erf}\left(\frac{\hat{L}_o - \mu_{\hat{\epsilon}}}{\sigma_{\hat{\epsilon}}}\right) = \frac{C_f - C_{qt}}{2(C_f + C_{qt})} \tag{19a}$$

b) For normal distributions of S and E,

$$\mathrm{erf}\left(\frac{L_o - \mu_{\epsilon}}{\sigma_{\epsilon}}\right) = \frac{C_f - C_{qt}}{2(C_f + C_{qt})} \tag{19b}$$

In the above equations,

$$\mathrm{erf}(k) = \frac{1}{\sqrt{2\pi}} \int_o^k \exp\left[-\frac{x^2}{2}\right] dx$$

5.3.2 Solutions for Acceptance Testing

The cost of error model for acceptance testing, as given by Eq. (10), is of the same form as the model for qual-acceptance testing given in Eq. (9). They differ only by the terms for cost of a test failure and hardware strength. The latter term does not appear in the solution for the test level which minimizes the cost of error. Hence, the results for acceptance testing are the same as presented in Eq. (19) for qual-acceptance testing except C_{at} replaces C_{qt}; that is,

a) For lognormal distributions of S and E,

$$\operatorname{erf}\left(\frac{\hat{L}_o - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) = \frac{C_f - C_{at}}{2(C_f + C_{at})} \tag{20a}$$

b) For normal distribution of S and E,

$$\operatorname{erf}\left(\frac{L_o - \mu_E}{\sigma_E}\right) = \frac{C_f - C_{at}}{2(C_f + C_{at})} \tag{20b}$$

5.3.3 Solutions for Qualification Testing

Using Eq. (16) as the cost of error model for qualification testing, it is clear that the qualification test case is the same as the qual-acceptance test case except that the cost of a test failure is given by $C'_{qt} = C_{qt} + MC_r$, and the cost of service failures is given by NC_f. Hence, the test level which will minimize the cost of error in qualification testing is as follows:

a) For lognormal distributions of S and E,

$$\operatorname{erf}\left(\frac{\hat{L}_o - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) = \frac{N\,C_f - (C_{qt} + M\,C_r)}{2(N\,C_f + C_{qt} + M\,C_r)} \tag{21a}$$

b) For normal distributions of S and E,

$$\operatorname{erf}\left(\frac{L_o - \mu_E}{\sigma_E}\right) = \frac{N\,C_f - (C_{qt} + M\,C_r)}{2(N\,C_f + C_{qt} + M\,C_r)} \tag{21b}$$

6.0 EVALUATION OF RESULTS

The optimal test levels presented in Section 5.0 for all three types of testing are particularly gratifying in three important respects. First, the hardware strength S does not enter into the results indicating the optimal test levels are independent of the strength of the hardware being tested. This is of great practical significance since the hardware strength is generally an unknown parameter in practice. Second, the results can be reduced to an optimal test level in terms of a percentile of the environmental load distribution as a function of a ratio of the cost of a test failure to the cost of a service failure. This is convenient in terms of practical applications as will be discussed later. Third, the results are fully consistent with a lower bound for optimal vibration test levels previously derived by Choi and Piersol in [3].

6.1 RESULTS FOR QUAL-ACCEPTANCE AND ACCEPTANCE TESTING

Since the optimal test levels for qual-acceptance testing and acceptance testing are similar in form, they will be discussed together. First, let the optimal test level expressions as given by Eqs (19) and (20) be converted to a more convenient form as follows.

Let X_o denote the optimal test level L_o in terms of a percentile of the distribution function of the environmental load E; that is,

$$X_o = 100 \int_{-\infty}^{L_o} p(E)\, dE$$

For either a lognormal or normal distribution of E, it follows that

$$X_o = 100\left[\frac{1}{2} + \int_o^k p(z)dz\right] = 100\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_o^k \exp\left(-\frac{x^2}{2}\right)dx\right]$$

$$= 100\left[\frac{1}{2} + \operatorname{erf}(k)\right] \tag{22}$$

where

$$k = \begin{cases} \left(\dfrac{\hat{L}_o - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) & \text{for a lognormal distribution} \\ \left(\dfrac{L_o - \mu_E}{\sigma_E}\right) & \text{for a normal distribution} \end{cases}$$

Hence, Eqs (19) and (20) may be written as

$$X_o = 100\left[\frac{1}{2} + \frac{(C_f - C_t)}{2(C_f + C_t)}\right] = 100\left[\frac{C_f}{C_f + C_t}\right]$$

$$= 100\left[\frac{1}{1 + R_c}\right] \tag{23}$$

where

$$R_c = \begin{cases} \dfrac{C_{qt}}{C_f} & \text{for qual-acceptance testing} \\ \dfrac{C_{at}}{C_f} & \text{for acceptance testing} \end{cases}$$

The form of Eq. (23) is desirable for two reasons. First, it expresses the optimal test level in terms of a percentile of the assumed environmental distribution, which is usually the way test levels are arrived at in practice. Second, it expresses the cost terms as a ratio of the test failure cost to service failure cost. This is important for the following reason. It might be difficult in practice to individually assess the anticipated cost of a service or test failure in absolute terms. However, it might be quite reasonable to assess the relative importance of the two possibilities. For example, it may be impractical to say that a test failure would cost $\$10^4$ and a service failure would cost $\$10^5$, but completely within reason to say that a service failure would be 10 times as undesirable as an unnecessary test failure. Further discussion of the cost terms is presented in Section 6.3.

It is interesting to compare the optimal test level given by Eq. (20) with a previously determined lower bound on the optimal level for vibration tests derived by Choi and Piersol in [3].* For conditions appropriate to qual-acceptance and acceptance testing, the test level bound given in [3] is

$$X_o > 100 \left[1 - R_c\right] \tag{24}$$

where R_c is the same as defined in Eq. (23). This bound is plotted along with the optimal level cf Eq. (23) in Figure 2.

It is seen in Figure 2 that the optimal test levels developed herein are consistently above the bounding optimal levels derived in [3].

*The basic decision criterion used in [3] is slightly different from the criterion employed here in that [3] seeks to minimize the expected cost of any decision resulting from the test, rather than the expected cost of just the incorrect decisions. This fact, however, should not detract from the validity of [3] as a bound on the results developed herein.

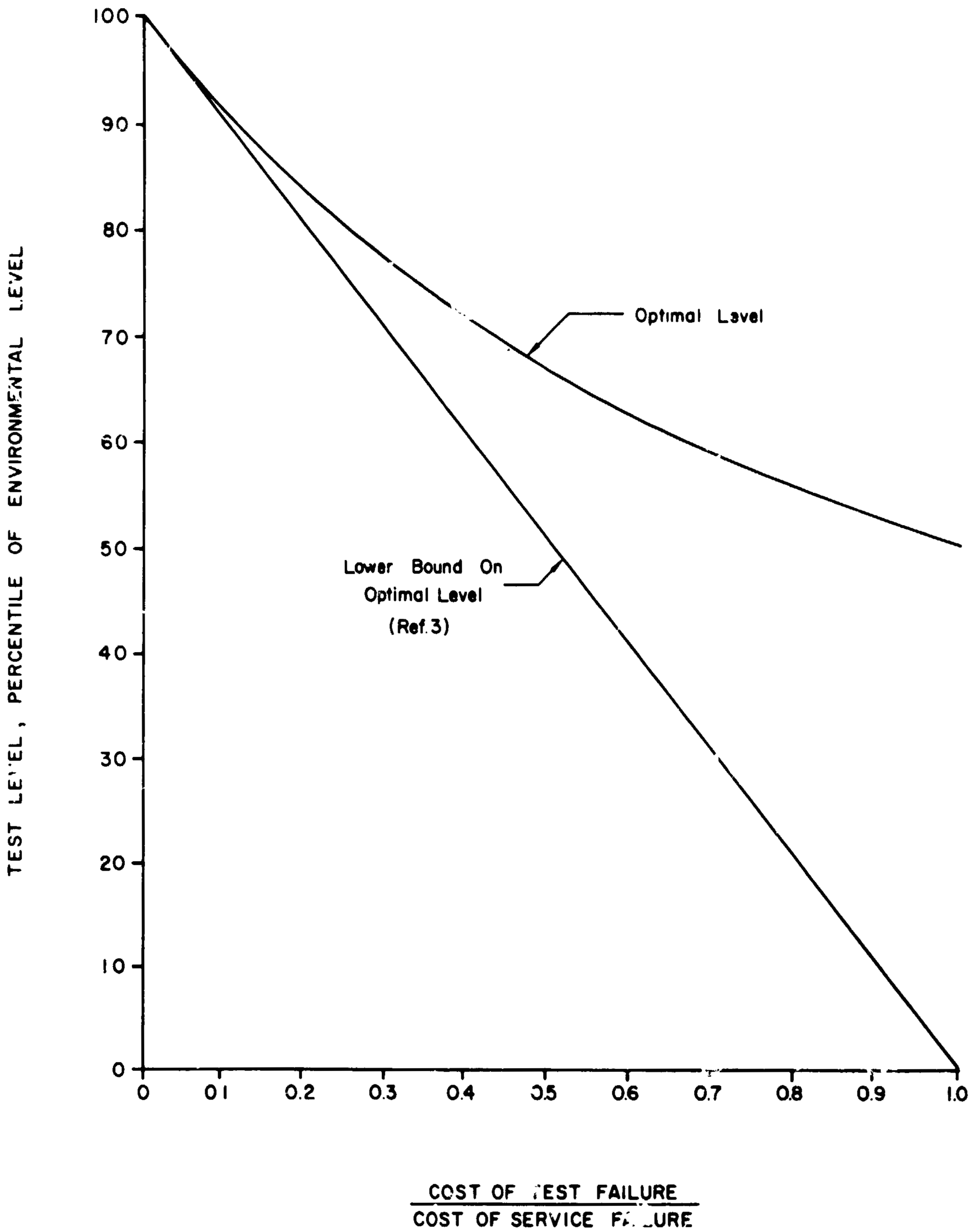

FIGURE 2. OPTIMAL TEST LEVEL VERSUS COST RATIO FOR QUAL-ACCEPTANCE AND ACCEPTANCE TESTS

The results indicate the optimal level approaches the bound of [3] as the test level increases (as the cost ratio decreases). At the one extreme where the cost ratio is zero, both curves converge to the 100th percentile, suggesting an infinite test level in this case. This is logical since an extremely rigorous test would be warranted if the cost of a test failure were negligible compared to the cost of a service failure. At the other extreme where the cost ratio is unity, the bounding value suggests only that the optimal test level is greater than zero. The optimal value determined herein, however, is a test at the 50th percentile of the environment. This means a test with an equal chance of over and under testing, which is logical if the costs of a test failure and a service failure are equal.

6.2 RESULTS FOR QUALIFICATION TESTING

The optimal test level for qualification testing, as given by Eq. (21), can be written in terms of a percentile of the distribution function of the environmental load as follows:

$$
\begin{aligned}
X_o &= 100 \left[\frac{1}{2} + \operatorname{erf}(k)\right] \\
&= 100 \left[\frac{1}{2} + \frac{N\,C_f - (C_t + M\,C_r)}{2(N\,C_f + C_t + M\,C_r)}\right] = \left[\frac{N\,C_f}{N\,C_f + C_t + M\,C_r}\right] \\
&= 100 \left[\frac{N}{N + R_c'}\right] \qquad (25)
\end{aligned}
$$

where

$$
R_c' = \frac{C_t + M\,C_r}{C_f}
$$

Equation (25) is similar in form to Eq. (20), which presents the results for qual-acceptance and acceptance testing. Hence, the form of Eq. (25) is desirable for the same reasons as discussed in Section 6.1. Plots of Eq. (25) for various values of N are presented in Figure 3.

Again referring to [3], a lower bound on the optimal levels for qualification vibration tests was derived to be

$$X_o \leq 100 \left[1 - R_c'\right]^{1/N} \tag{26}$$

where R_c' is the same as defined in Eq. (25). This bound is plotted along with the optimal level of Eq. (25) for the case of N = 8 in Figure 4. Note that the optimal test levels developed herein are again above the bounding optimal levels desired in [3] for the case of N = 8. The fact that the optimal level will always exceed the bound for all values of N is demonstrated in Appendix D.

6.3 DISCUSSION OF COST ITEMS

The selection of optimal test levels in terms of a minimum cost of error criterion reduces to the assessment of the ratio of the undesirable consequences of failing the test due to overtesting versus failing in service due to undertesting. The assessment of this ratio R_c depends on the hardware being tested and the type of test being performed, as is now discussed.

6.3.1 Influence of Hardware

It is clear that the value of R_c which might be appropriate for testing of a single component could be quite different from the value of R_c which would apply to an entire spacecraft assembly. Even among single components, however, an appropriate value for R_c could vary widely.

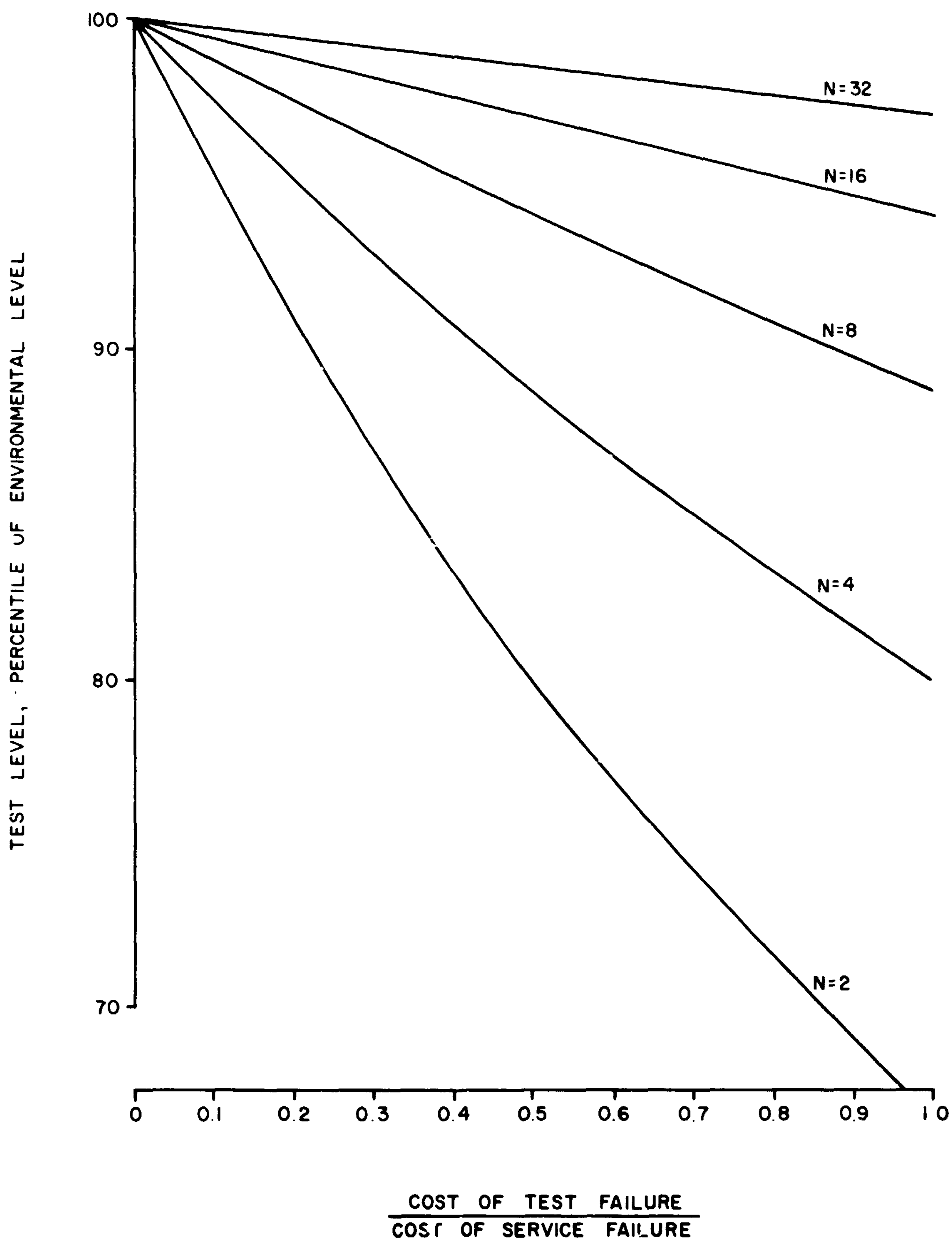

FIGURE 3. OPTIMAL TEST LEVEL VERSUS COST RATIO FOR QUALIFICATION TESTS — VARIOUS PRODUCTION RUN SIZES

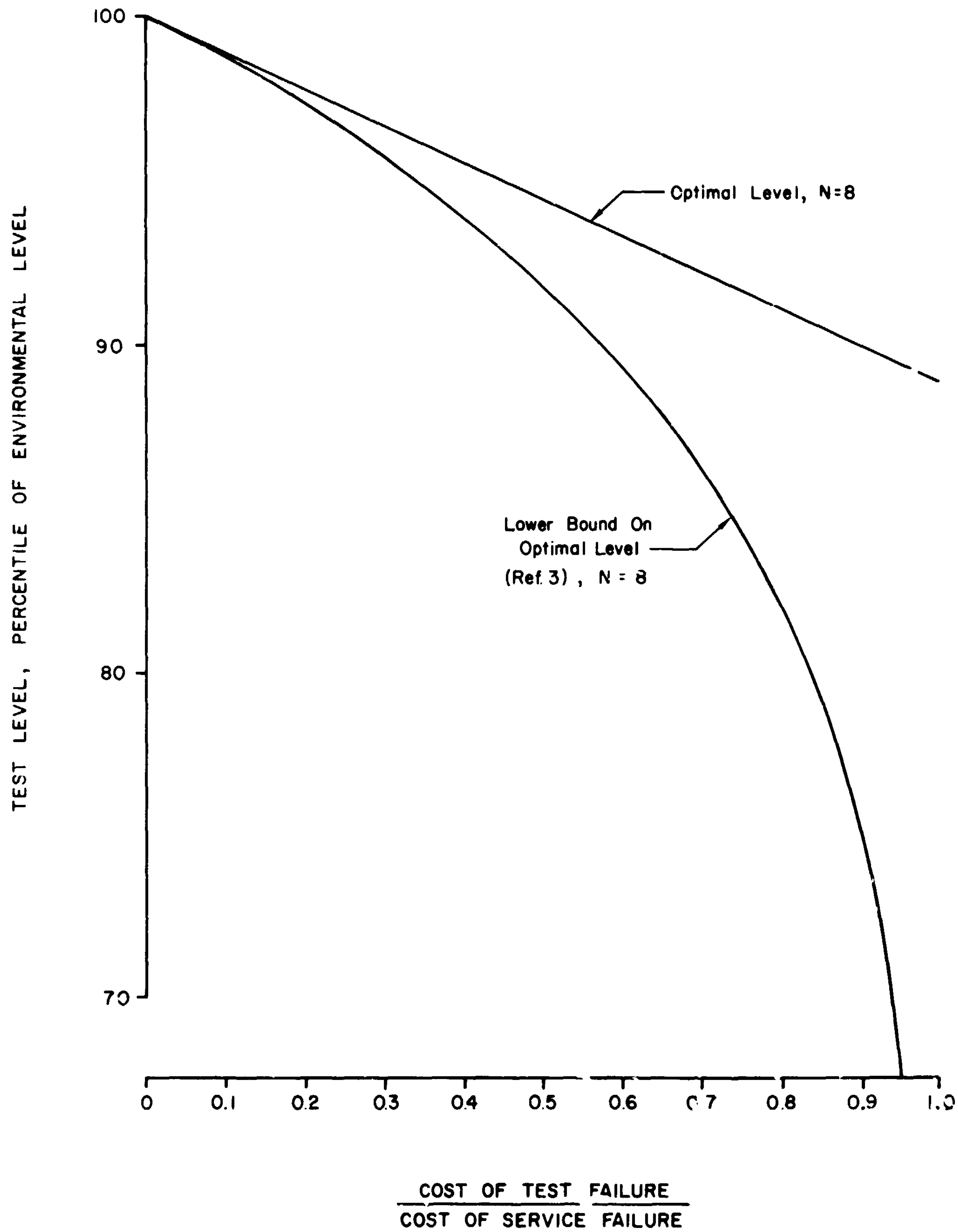

FIGURE 4. OPTIMAL TEST LEVEL VERSUS COST RATIO FOR QUALIFICATION TESTS - PRODUCTION RUN SIZE OF 8

For example, if the component in question is a relatively inexpensive "off the shelf" item which can be procured from several competitive vendors, but its failure in service could mean the loss of the entire spacecraft, then R_c would be a very small number: say, 0.01 or less. On the other hand, if the component in question is an engineered item involving high development costs, but its failure in service would mean the loss of only one of a collection of experiments to be performed by the spacecraft, then R_c would be a somewhat larger number; say, 0.1 or greater. For the case of qual-acceptance or acceptance testing, Eq. (23) suggests that the first hypothetical component should be tested at no less than the 99th percentile of the environmental load distribution while the second component should be tested at no greater than the 91st percentile.

Although not consistent with current GSFC practice, the above results are fully consistent with intuition. On the one hand, if a component is readily available at low cost from several sources so that a test failure would require only a new procurement from a different source, but the component is critical to the overall success of the spacecraft mission, then the risk of undertesting should be minimized even at the expense of a high probability of overtesting; that is, the demand for an unusually rugged component should be high. On the other hand, if the component is the result of an expense development program where a test failure would lead to redevelopment, but the component is critical only to a limited portion of the spacecraft mission, then the risk of overtesting should receive increased concern; that is, the demand for ruggedness should be more moderate.

6.3.2 Influence of Testing Objective

A comparison of Eqs (23) and (25) reveals that for similar cost factors and $N > 1$, the optimal level for a qualification test is always higher than for a qual-acceptance or acceptance test. Of course, for a given item of hardware, the cost ratios would not necessarily be the same for all three testing objectives. Nevertheless, it is unlikely that the variations in R_c would be sufficient to alter the basic conclusion that qualification demands higher test levels. This conclusion is consistent with both current practice and intuition. Simple judgement strongly supports the idea that the most severe test should be the one intended to verify the integrity of the basic hardware design.

The implications of the results to qual-acceptance verus acceptance test levels are less obvious. The same test level selection rule, as given by Eq. (23), applies to both of these cases. However, the cost ratio term R_c may be different for the two cases, and in practice, usually is. Specifically, the cost of a service failure for the two cases would be the same for similar hardware since only one item of hardware is involved. On the other hand, the cost of a test failure generally would be different for similar hardware since an acceptance occurs much later in the production cycle than a qual-acceptance test; that is, the acceptance test occurs after the hardware integrity has been supposedly verified and the design cycle has been closed. It follows that a failure in acceptance testing (as defined in Section 4.2) probably would lead to more involved corrective action in terms of both redesign and refurbishing than a failure in qual-acceptance testing. Referring to Eq. (23), this means that $C_{at} > C_{qc}$, and thus, $R_{ac} > R_{qc}$, all other things equal. Hence, the optimal test levels for acceptance testing would generally be <u>lower</u> than the optimal test levels for qual-acceptance testing. This conclusion is consistent with current GSFC policy, as outlined in Section 1.0.

6.3.3 Determination of Cost Ratios

There are two possible approaches to the determination of appropriate values for the cost ratio R_c in Eqs (23) and (25). The first is to employ quantitative cost figures (in dollars) available from accounting data. The second is to rely solely upon subjective judgment.

The simplest quantitative approach is to use worst case figures for the cost of a test or service failure. The worst case figure for a test failure would be the total cost of developing or purchasing the hardware item being tested (the worst possible failure would be one that required a completely new design). The worst case figure for a service failure would be the total cost of the experiment which that failure might abort. Some interesting data on the cost of spacecraft launches is available from [4].

The subjective approach involves an estimate for R_c based upon qualitative considerations including perhaps a "gut" feeling for the relative importance of a test and service failure. Note that such qualitative considerations have traditionally been involved in vibration testing thoughout the aerospace industry. The procedure of granting a "waiver" for a test failure is nothing more than a qualitative decision that the cost of ordering redesign or rework needed to pass the test is not worth the limited improvement that might be obtained in service reliability. Such "waivers" are based upon an engineer's evaluation of the relevance of the failure (was it due to overtesting) and the cost of correcting it, versus the risk and consequences of a service failure which might result if the problem is not corrected. This same type of judgment could be used to arrive at estimates for R_c in Eqs (23) and (25).

6.4 COMPARISONS WITH CURRENT GSFC POLICY

It is noted in Section 6.3.2 that the optimal test levels suggested by the results herein are generally consistent with current GSFC policy in terms of testing objectives.

Specifically, the results indicate qualification tests should involve the most severe levels while acceptance tests should be the least severe and qual-acceptance tests something in between. Although GSFC performs qual-acceptance (proto-flight) tests with the same test levels as qualification (prototype) tests, the qual-acceptance test duration is only half as long. Hence, the procedure is somewhat compatible with the results.

The specific differences between qualification and acceptance test levels called for in current GSFC policy are not necessarily consistent with the results herein. GSFC specifies the qualification levels in terms of a fixed margin (50%) over the acceptance test levels. The results of Eqs (23) and (25) indicate the margin should be in terms of a difference in the percentile level of the environmental load distribution. Furthermore, the results herein indicate that there might be differences in the optimal test levels for different types of hardware, as discussed in Section 6. 3. 1. Current GSFC policy does not provide for such differences.

To pursue the latter point further, GSFC policy calls for acceptance (flight) test levels at the 97. 7 percentile of the anticipated service environmental loads, regardless of the component being tested. From Eq. (23), this corresponds to a cost ratio of $R_c \approx 0.023$, that is, an assessment that a service failure due to undertesting is about 43 times more undesirable than a test failure due to overtesting. This does not appear to be an unreasonable assessment on the average. Based upon the past experience of this author, cost ratio assessments in the range from $R_c = 0.1$ to $R_c = 0.01$ would probably cover most aerospace hardware in practice (excluding manned spacecraft). These cost ratios correspond to test levels in the range from the 91st to the 99th percentile. Current GSFC test levels fall well within this range.

7. CONCLUSIONS

The principal conclusions of the studies are as follows:

1. optimal test levels for qualification, qual-acceptance, and acceptance vibration tests can be derived by minimizing the expected value of the undesirable consequences (cost) of an error in the decision resulting from the test.

2. the resulting optimal test levels are a function of the distribution of the service environmental loads, the cost of an unnecessary test failure, and the cost of a potential service failure. For qualification tests, the number of production items which might have been manufactured prior to the qualification test, as well as the total number of production items to be manufactured, also influence the optimal test level.

3. the optimal test levels are not an explicit function of the hardware strength. This is an important conclusion which might not be intuitively obvious at first glance. However, by considering the alternative (what impact should a low or high expected value for strength have on the selection of a minimum cost of error test level), the lack of an explicit functional relationship appears reasonable.

4. all other things equal, the optimal level for qualification tests is more severe than for acceptance tests.

5. all other things equal, the optimal level for a test increases as:

   (a) the cost of redesigning the hardware to pass the test decreases,

   (b) the cost of a service failure of the hardware increases,

   (c) the number of production items to be manufactured increases (qualification tests only).

6. the results are generally consistent with intuition and reasonably consistent with current GSFC policies.

7. for the case of acceptance tests, the results indicate that an optimal test level in most cases would be a test at the 91st to 99th percentile of the anticipated service environmental loads. This compares favorably with the current GSFC policy of testing at the 97.7 percentile of the anticipated service environmental loads.

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APPENDIX A

PROPERTIES OF THE LOGNORMAL DISTRIBUTION

Let $\hat{x} = \log x$ be a normally distributed random variable with a mean value of $\mu_{\hat{x}}$ and a variance of $\sigma_{\hat{x}}^2$; that is

$$p(\hat{x}) = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \exp\left[-\frac{1}{2}\left(\frac{\hat{x} - \mu_{\hat{x}}}{\sigma_{\hat{x}}}\right)^2\right] \tag{A-1}$$

Then the random variable $x = \exp[\hat{x}]$ is said to have a <u>lognormal distribution</u>.

A. 1 MEAN VALUE OF LOGNORMAL VARIABLES

Given the lognormal variable x, the mean value of x is defined by

$$\mu_x = E[x] = E\left[\exp(\hat{x})\right] = \int_{-\infty}^{\infty} \exp[\hat{x}]\, p(\hat{x})\, d\hat{x} \tag{A-2}$$

Substituting Eq. (A-1) into (A-2) yields

$$\mu_x = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \int_{-\infty}^{\infty} \exp[\hat{x}] \exp\left[-\frac{1}{2}\left(\frac{\hat{x} - \mu_{\hat{x}}}{\sigma_{\hat{x}}}\right)^2\right] d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \int_{-\infty}^{\infty} \exp\left[\hat{x} - \frac{1}{2\sigma_{\hat{x}}^2}\left(\hat{x}^2 - 2\mu_{\hat{x}}\hat{x} + \mu_{\hat{x}}^2\right)\right] d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \exp\left[\mu_{\hat{x}} + \frac{\sigma_{\hat{x}}^2}{2}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{x} - \left(\mu_{\hat{x}} + \sigma_{\hat{x}}^2\right)}{\sigma_{\hat{x}}^2}\right)^2\right] d\hat{x}$$

Now let

$$y = \frac{\bar{x} - \mu_{\hat{x}} + \sigma_{\hat{x}}^{2}}{\sigma_{\hat{x}}} \qquad dy = \frac{d\hat{x}}{\sigma_{\hat{x}}}$$

It follows that

$$\mu_x = \frac{1}{\sqrt{2\pi}} \exp\left[\mu_{\hat{x}} + \frac{\sigma_{\hat{x}}^{2}}{2}\right] \int_{-\infty}^{\infty} \exp\left[-\, y^2/2\right] dy$$

Hence,

$$\boxed{\mu_x = \exp\left[\mu_{\hat{x}} + \frac{\sigma_{\hat{x}}^{2}}{2}\right]} \tag{A-3}$$

A. 2 VARIANCE OF LOGNORMAL VARIABLES

Given the lognormal variable x, the variance of x is defined by

$$\sigma_x^{2} = E\left[\left(x - \mu_x\right)^2\right] = E\left[x^2\right] - \mu_x^{2} = E\left[\exp\left(2\hat{x}\right)\right] - \mu_x^{2} \tag{A-4}$$

Thus

$$\sigma_x^{2} + \mu_x^{2} = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \int_{-\infty}^{\infty} \exp\left[2\hat{x}\right] \exp\left[-\frac{1}{2}\left(\frac{\hat{x} - \mu_{\hat{x}}}{\sigma_{\hat{x}}}\right)^2\right] d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\,\sigma_{\hat{x}}^{2}}\left(\hat{x}^2 - 2\left(\mu_{\hat{x}} + 2\sigma_{\hat{x}}^{2}\right)\hat{x} + \mu_{\hat{x}}^{2}\right)\right] d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \exp\left[2\left(\mu_{\hat{x}} + \sigma_{\hat{x}}^{2}\right)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{x} - (\mu_{\hat{x}} + 2\sigma_{\hat{x}}^{2})}{\sigma_{\hat{x}}}\right)^2\right] d\hat{x}$$

Now let

$$y = \frac{\bar{x} - \left(\mu_{\hat{x}} + 2\sigma_{\hat{x}}^{2}\right)}{\sigma_{\hat{x}}} \quad \text{and } dy = \frac{d\hat{x}}{\sigma_{\hat{x}}}$$

It follows that

$$\sigma_x^{2} + \mu_x^{2} = \frac{1}{\sqrt{2\pi}} \exp\left[2\left(\mu_{\hat{x}} + \sigma_{\hat{x}}^{2}\right)\right] \int_{-\infty}^{\infty} \exp\left[-y^{2}/2\right] dy$$

Hence,

$$\boxed{\sigma_x^{2} = \exp\left[2\left(\mu_{\hat{x}} + \sigma_{\hat{x}}^{2}\right)\right] - \mu_x^{2}} \qquad \text{(A. 5)}$$

A. 3 INVERSE RELATIONSHIPS

Equations (A. 3) and (A. 5) may be solved for the mean and variance of $\hat{x}$ as a function of the mean and variance of x as follows. From Eq. (A. 3)

$$\mu_{\hat{x}} + \frac{\sigma_{\hat{x}}^{2}}{2} = \log \mu_x \qquad \text{(A. 6)}$$

From Eq. (A. 5),

$$\mu_{\hat{x}} + \sigma_{\hat{x}}^{2} = \frac{1}{2}\log\left(\mu_x^{2} + \sigma_x^{2}\right) = \frac{1}{2}\log \mu_x^{2}\left(1 + \frac{\sigma_x^{2}}{\mu_x^{2}}\right)$$

$$= \log \mu_x + \frac{1}{2}\log\left(1 + \frac{\sigma_x^{2}}{\mu_x^{2}}\right) \qquad \text{(A. 7)}$$

Hence,

$$\boxed{\begin{aligned} \mu_{\hat{x}} &= \log \mu_x - \frac{1}{2} \log\left(1 + \frac{\sigma_x^2}{\mu_x^2}\right) && \text{(A. 8a)} \\ \sigma_{\hat{x}}^2 &= \log\left(1 + \frac{\sigma_x^2}{\mu_x^2}\right) && \text{(A. 8b)} \end{aligned}}$$

APPENDIX B

REDUCTION OF PROBABILITY STATEMENTS

This Appendix outlines the reduction of the joint probability statements, $P[S \leq L, S > E]$ and $P[S > L, S \leq E]$, to analytical expressions assuming both lognormal and normal distributions of S and E.

B.1 SOLUTIONS ASSUMING LOGNORMAL DISTRIBUTIONS OF S AND E

First consider the probability statement, $P[S \leq L, S > E]$. Let the following transformations be defined.

$$\eta = S/L \qquad \hat{\eta} = \log \eta = \log S - \log L = \hat{S} - \hat{L} \qquad \text{(B.1a)}$$

$$\xi = S/E \qquad \hat{\xi} = \log \xi = \log S - \log E = \hat{S} - \hat{E} \qquad \text{(B.1b)}$$

Noting that S and E cannot take on negative values in practice, it follows that

$$\begin{aligned} P[S \leq L, S > E] &= P[0 < \eta \leq 1, \xi > 1] \\ &= P[\hat{\eta} \leq 0, \hat{\xi} > 0] \\ &= \int_{-\infty}^{0} \left[\int_{0}^{\infty} p(\hat{\eta}, \hat{\xi}) d\hat{\xi} \right] d\hat{\eta} \end{aligned} \qquad \text{(B.2)}$$

as illustrated in Figure B.1. Now from [5],

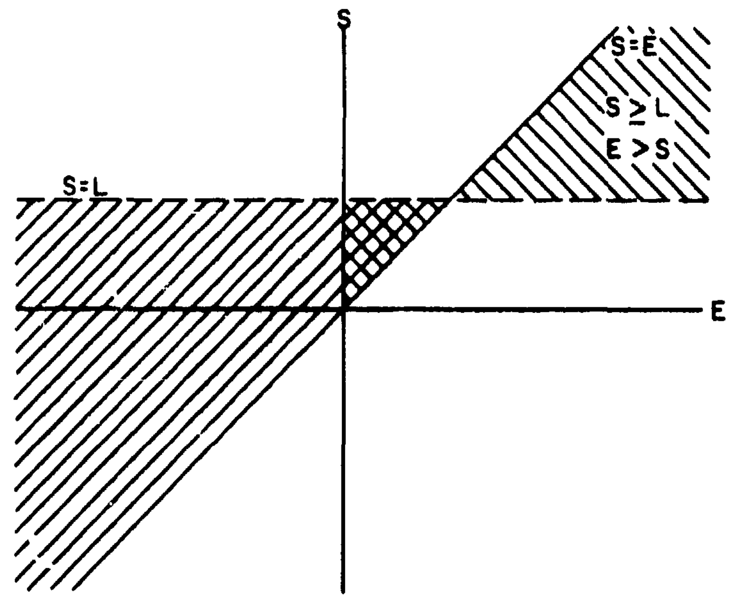
$$p(\hat{\eta}, \hat{\xi}) = \frac{1}{|J(\hat{S}, \hat{E})|} p_{\hat{\xi}, \hat{E}}(S, E) \qquad \text{(B.3)}$$

From Eq. (B.1), $\hat{S} = \hat{\eta} + \hat{L}$ and $\hat{E} = \hat{\eta} + \hat{L} - \hat{\xi}$. Hence, the joint probability density function in Eq. (B.3) is given by

(S,E)

$P(0 \leq S \leq L,\ S > E \geq 0)$

$$= \int_0^L \left[\int_E^L p(S,E)\, dS \right] dE$$

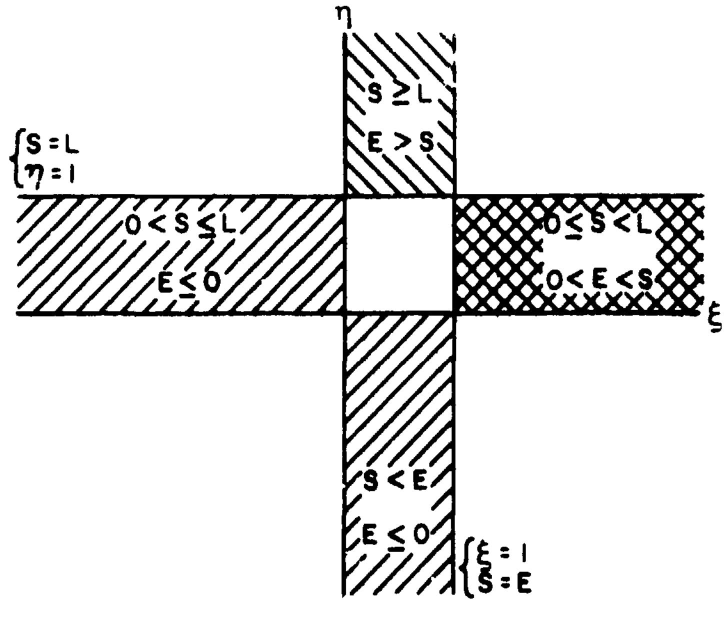


(η, ξ)

$\eta = S/L \quad , \quad \xi = S/E$

$P(0 < \eta \leq 1\ ,\ \xi \geq 1)$

$$= \int_0^1 \left[\int_1^\infty p(\eta,\xi)\, d\xi \right] d\eta$$



$(\hat{\eta}, \hat{\xi})$

$\hat{\eta} = \log \eta \quad , \quad \hat{\xi} = \log \xi$

$P(\hat{\eta} < 0\ ,\ \hat{\xi} \geq 0)$

$$= \int_{-\infty}^0 \left[\int_0^\infty p(\hat{\eta},\hat{\xi})\, d\hat{\xi} \right] d\hat{\eta}$$

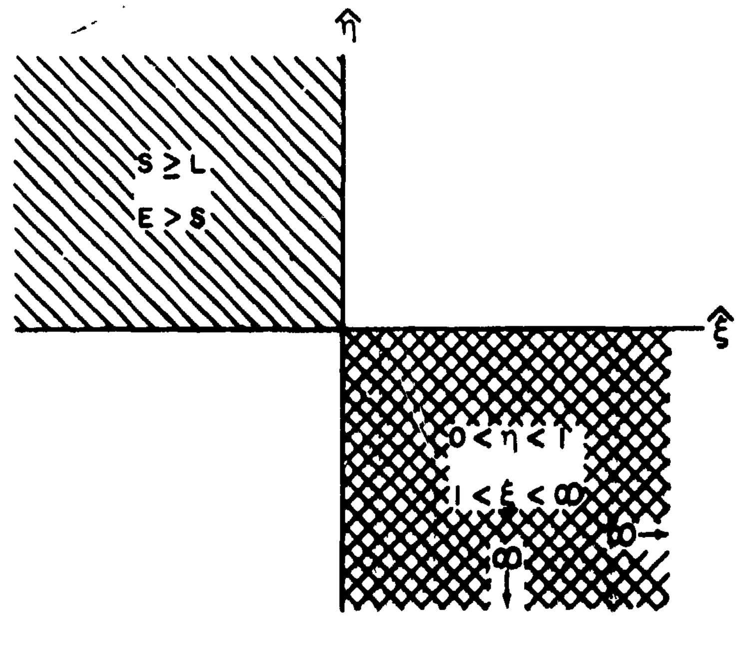


FIGURE B.1. TRANSFORMATION OF VARIABLES — LOG NORMAL CASE

$$p_{\hat{S},\hat{E}}(\hat{S},\ \hat{E}) = p_{\hat{S},\hat{E}}(\hat{\eta}+\hat{L},\ \hat{\eta}+\hat{L}-\hat{\xi})$$

and the Jacobian reduces to

$$J(\hat{S},\ \hat{E}) = \begin{vmatrix} \frac{\partial\hat{\eta}}{\partial\hat{S}} & \frac{\partial\hat{\eta}}{\partial\hat{E}} \\ \frac{\partial\hat{\xi}}{\partial\hat{S}} & \frac{\partial\hat{\xi}}{\partial\hat{E}} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1$$

giving $|J(\hat{S},\ \hat{E})| = 1$. Also, since S and E are independent, it follows that $\hat{S}$ and $\hat{E}$ are independent. Thus, Eq. (B.3) becomes

$$p(\hat{\eta},\ \hat{\xi}) = p_{\hat{S}}(\hat{\eta}+\hat{L})\ p_{\hat{E}}(\hat{\eta}+\hat{L}-\hat{\xi}) \tag{B. 4}$$

and Eq. (B.2) reduces to

$$P\left[S\leq L,\ S>E\right] = \int_{-\infty}^{0} p_{\hat{S}}(\hat{\eta}+\hat{L})\left[\int_{0}^{\infty} p_{\hat{E}}(\hat{\eta}+\hat{L}-\hat{\xi})d\hat{\xi}\right]d\hat{\eta} \tag{B. 5}$$

For the case of a lognormal distribution of S and E, the variables $\hat{S}$ and $\hat{E}$ are normally distributed with mean values of $\mu_{\hat{S}}$ and $\mu_{\hat{E}}$, respectively, and variances of $\sigma_{\hat{S}}^{2}$ and $\sigma_{\hat{E}}^{2}$, respectively. Substituting the appropriate normal density functions into Eq. (B.5) yields

$$P\left[S\leq L,\ S>E\right] = \frac{1}{2\pi\sigma_{\hat{S}}\sigma_{\hat{E}}}\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right]\left\{\int_{0}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\hat{\xi}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^{2}\right]d\hat{\xi}\right\}d\hat{\eta} \tag{B. 6}$$

Now let

$$u = -\frac{\hat{\eta}+\hat{L}-\hat{\xi}-\mu_{\hat{E}}}{\sigma_{\hat{E}}} \quad \text{and} \quad du = \frac{d\hat{\xi}}{\sigma_{\hat{E}}} \tag{B. 7}$$

Eq. (B. 5) reduces to

$$P\left[S\leq L,\ S>E\right] = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\xi}}}\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right)^2\right]\left\{\frac{1}{\sqrt{2\pi}}\int_{-\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\varepsilon}}}{\sigma_{\hat{\varepsilon}}}\right)}^{\infty} \exp\left[-\frac{u^2}{2}\right] du\right\} d\hat{\eta}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\xi}}}\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right)^2\right]\left[\frac{1}{2}+\operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\varepsilon}}}{\sigma_{\hat{\varepsilon}}}\right)\right] d\hat{\eta}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\xi}}}\left\{\frac{1}{2}\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right)^2\right] d\hat{\eta}\right.$$

$$\left.+\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right)^2\right] \operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\varepsilon}}}{\sigma_{\hat{\varepsilon}}}\right) d\hat{\eta}\right\}$$

$$\boxed{\begin{aligned} P\left[S\leq L,\ S>E\right] &= \frac{1}{4}+\frac{1}{2}\operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right) \\ &+ \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\xi}}}\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\xi}}}{\sigma_{\hat{\xi}}}\right)^2\right]\operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{\varepsilon}}}{\sigma_{\hat{\varepsilon}}}\right) d\hat{\eta} \qquad \text{(B. 8)} \\ \text{where } \operatorname{erf}(k) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{k} \exp\left[-x^2/2\right] dx \end{aligned}}$$

Now consider the probability statement, $P\left[S>L,\ S\leq E\right]$. Using the transformations and results of Eqs (B. 1) through (B. 4), it follows that

$$\begin{aligned} P\left[S>L,\ S\leq E\right] &= P\left[\eta>1,\ 0<\xi\leq 1\right] \\ &= P\left[\hat{\eta}>0,\ \hat{\xi}\leq 0\right] \\ &= \int_{0}^{\infty}\left[\int_{-\infty}^{0} p(\hat{\eta},\ \hat{\xi})d\hat{\xi}\right] d\hat{\eta} \\ &= \int_{0}^{\infty} p_{\hat{\xi}}\ (\hat{\eta}+\hat{L})\left[\int_{-\infty}^{0} p_{\hat{\varepsilon}}\ (\hat{\eta}+\hat{L}-\hat{\xi})d\hat{\xi}\right] d\hat{\eta} \qquad \text{(B. 9)} \end{aligned}$$

Again assuming a lognormal distribution for S and E, and using the substitutions of Eq. (B. 7), the result in Eq. (B. 9) reduces to

$$P\left[S > L,\ S \leq E\right] = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{S}}} \int_0^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \left\{ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)} \exp\left[-\frac{u^2}{2}\right] du \right\} d\hat{\eta}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{S}}} \int_0^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \left[\frac{1}{2} - \operatorname{erf}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)\right] d\hat{\eta}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{S}}} \left\{ \frac{1}{2} \int_0^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] d\hat{\eta} \right.$$

$$\left. - \int_0^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \operatorname{erf}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) d\hat{\eta} \right\}$$

$$\boxed{\begin{aligned} P\left[S > L,\ S \leq E\right] &= \frac{1}{4} - \frac{1}{2} \operatorname{erf}\left(\frac{\hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right) \\ &\quad - \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{S}}} \int_0^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \operatorname{erf}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) d\hat{\eta} \end{aligned}} \tag{B. 10}$$

where the erf function is as defined in Eq. (B. 8).

B. 2 SOLUTIONS ASSUMING NORMAL DISTRIBUTIONS OF S AND E

Let the following transformations be defined

$$\lambda = S - L \tag{B. 11a}$$

$$\nu = S - E \tag{B. 11b}$$

If no restrictions are placed on the values of S and E (if negative values are permitted), it follows that

$$P\left[S \leq L,\ S > E\right] = P\left[\lambda \leq 0,\ \nu > 0\right]$$

$$= \int_{-\infty}^{0}\left[\int_{0}^{\infty} p(\lambda,\ \nu)d\nu\right]d\lambda \qquad \text{(B. 12)}$$

as illustrated in Figure B. 2. Similarly,

$$P\left[S > L,\ S \leq E\right] = P\left[\lambda > 0,\ \nu \leq 0\right]$$

$$\int_{0}^{\infty}\left[\int_{-\infty}^{0} p(\lambda,\ \nu)d\nu\right]d\lambda \qquad \text{(B. 13)}$$

Now, a comparison of Eqs (B. 11) through (B. 13) with Eqs (B. 1), (B. 2) and (B. 9) reveals that the formulations here are identical to those developed for the lognormal case where

S	is analogous	to	$\hat{S}$
E	"	"	"	$\hat{E}$
λ	"	"	"	$\hat{\eta}$
ν	"	"	"	$\hat{\xi}$

Hence, for the case where S and E are normally distributed with mean values of μ_s and μ_E, and variances of σ_s^2 and σ_E^2, respectively, it follows that

$$\boxed{P\left[S \leq L,\ S > E\right] = \frac{1}{4} + \frac{1}{2}\,\mathrm{erf}\left(\frac{L-\mu_s}{\sigma_s}\right) + \frac{1}{\sqrt{2\pi}\,\sigma_s}\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_s}{\sigma_s}\right)^2\right]\mathrm{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)d\lambda}$$

(B. 14)

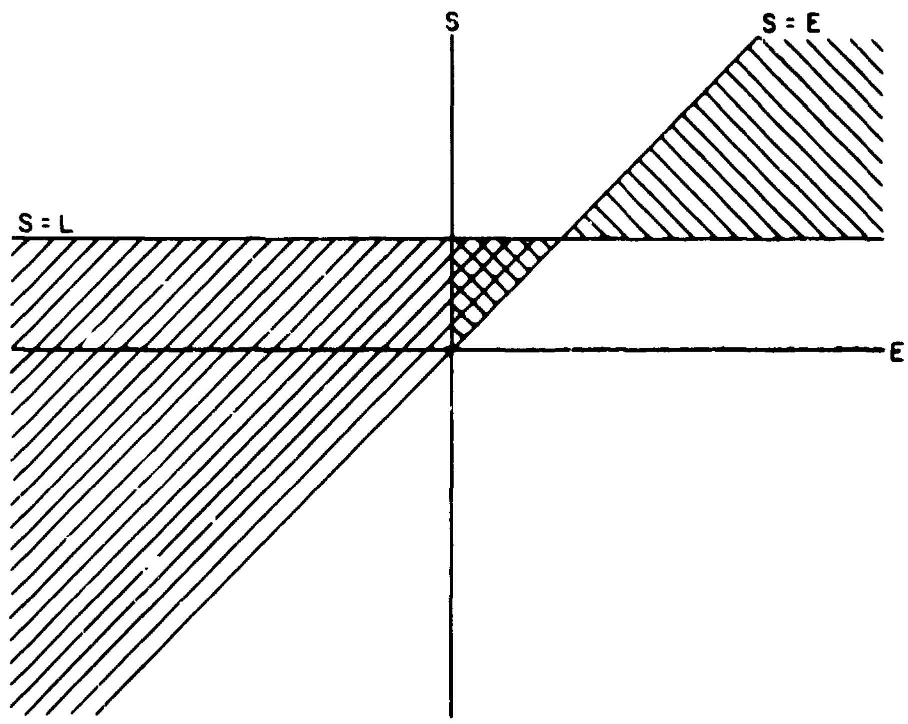
(S, E)

$P(S \leq L, \ S > E)$

$$= \int_{-\infty}^{L} \left[\int_{E}^{L} p(S,E)\, dS \right] dE$$

$P(0 \leq S \leq L, \ S > E \geq 0)$

$$= \int_{0}^{L} \left[\int_{E}^{L} p(S,E)\, dE \right] dS$$



(λ, ν)

$\lambda = S - L, \quad \nu = S - E$

$P(\lambda \leq 0, \ \nu > 0)$

$$= \int_{0}^{\infty} \left[\int_{-\infty}^{0} p(\lambda,\nu)\, d\nu \right] d\lambda$$

$P(-L < \lambda \leq 0, \ 0 < \nu \leq \lambda + L)$

$$= \int_{-L}^{0} \left[\int_{0}^{\lambda + L} p(\lambda,\nu)\, d\nu \right] d\lambda$$

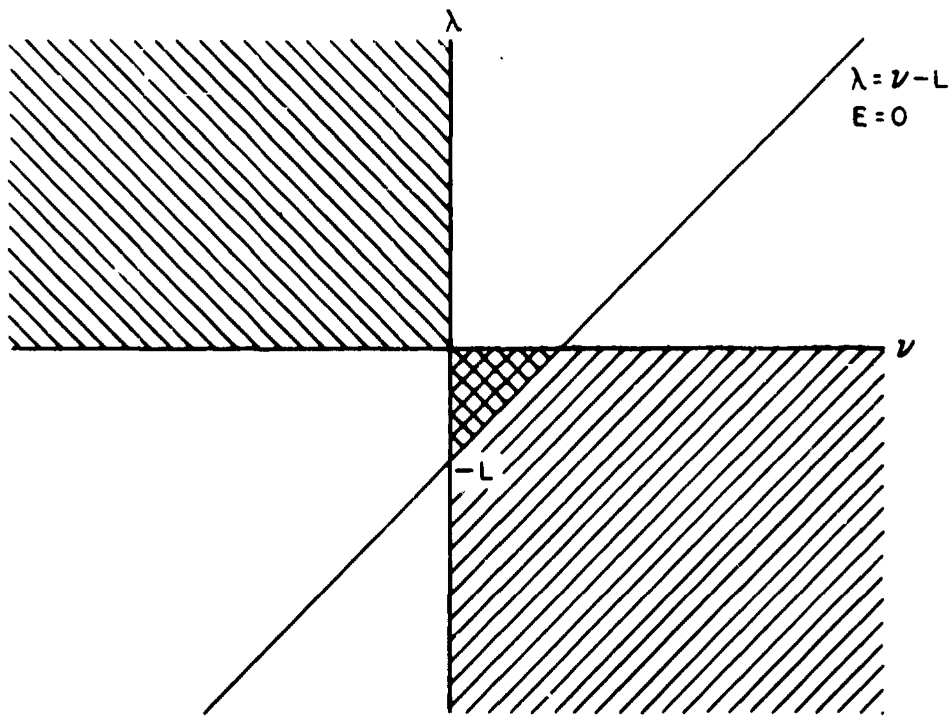


FIGURE B.2 TRANSFORMATION OF VARIABLES — NORMAL CASE

$$
\boxed{P\left[S>L,\ S\leq E\right] = \frac{1}{4} - \frac{1}{2}\,\mathrm{erf}\left(\frac{L-\mu_S}{\sigma_S}\right) - \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_0^{\infty}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]\mathrm{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)d\lambda}
$$

(B. 15)

where the erf function is as defined in Eq. (B. 8). Note that Eqs (B. 14) and (B. 15) are arrived at assuming S and E can take on negative values, which is not possible in practice.

B. 3 SOLUTIONS ASSUMING TRUNCATED NORMAL DISTRIBUTIONS OF S AND E

Finally consider the physically realizable case where S and E are normally distributed, but restricted to nonnegative values. For simplicity assume that

$$
\int_0^{\infty} p(S)ds = \int_{-\infty}^{\infty} p(S)dS = 1
$$

$$
\int_0^{\infty} p(E)dE = \int_{-\infty}^{\infty} p(E)dE = 1
$$

which are reasonable assumptions if $\mu_S > 2\sigma_S$ and $\mu_E > 2\sigma_E$. Using the transformations of Eq. (B. 11), it follows that

$$
P\left[0\leq S\leq L,\ S>E\geq 0\right] = P\left[-L<\lambda\leq 0,\ 0<\nu\leq\lambda+L\right]
$$

$$
= \int_{-L}^{0}\left[\int_0^{\lambda+L} p(\lambda,\ \nu)d\nu\right]d\lambda \qquad \text{(B. 16)}
$$

as illustrated in Figure B. 2. Using the relationship of Eq. (B. 3), the above probability reduces to

$$P\left[0\le S\le L,\ S>E\ge 0\right] = \int_{-L}^{0} p_{S}(\lambda+L)\left[\int_{0}^{\lambda+L} p_{E}(\lambda+L-\nu)d\nu\right] d\lambda$$

$$= \frac{1}{2\pi\sigma_S\,\sigma_E}\int_{-L}^{0}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]$$

$$\text{times}\left\{\int_{0}^{\lambda+L}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\nu-\mu_E}{\sigma_E}\right)^2\right]d\nu\right\}d\lambda$$

(B. 17)

Now let

$$u = -\frac{\lambda+L-\nu-\mu_E}{\sigma_E} \text{ and } du = \frac{d\nu}{\sigma_E} \qquad \text{(B. 18)}$$

Eq. (B. 16) reduces to

$$P\left[0\le S\le L,\ S>E\ge 0\right] = \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_{-L}^{0}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]$$

$$\text{times}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)}^{\left(\frac{\mu_E}{\sigma_E}\right)}\exp\left[-\frac{u^2}{2}\right]du\right\}d\lambda$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_{-L}^{0}\left[\operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right)+\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)\right]$$

$$\text{times}\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]d\lambda$$

$$= \operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right)\left[\operatorname{erf}\left(\frac{\mu}{\sigma}\right) + \operatorname{erf}\left(\frac{L-\mu_S}{\sigma_S}\right)\right]$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_{-L}^{0} \exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]$$

$$\text{times } \operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right) d\lambda \qquad \text{(B. 19)}$$

Note that for the special case where $\mu_S \gg \sigma_S$ and $\mu_E \gg \sigma_E$, $\operatorname{erf}(\mu_S/\sigma_S) \approx \operatorname{erf}(\mu_E/\sigma_E) \approx 1/2$, and Eq. (B. 18) reduces to

$$P\left[0 \le S \le L,\ S > E \ge 0\right] \approx \frac{1}{4} + \frac{1}{2}\operatorname{erf}\left(\frac{L-\mu_S}{\sigma_S}\right)$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right) d\lambda \qquad \text{(B. 20)}$$

which is the same result that was obtained for the unrestricted normal distribution of S and E in Eq. (B. 14). It can be shown by similar developments that the probability statement $P\left[S > L \ge 0,\ 0 \le S \le E\right]$ is given by

$$P\left[S > L \ge 0,\ 0 \le S \le E\right] = \frac{1}{4} - \frac{1}{2}\operatorname{erf}\left(\frac{L-\mu_S}{\sigma_S}\right)$$

$$- \frac{1}{\sqrt{2\pi}\,\sigma_S}\int_{0}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right) d\lambda \qquad \text{(B. 21)}$$

which is the same as the result obtained for the unrestricted normal case in Eq. (B. 15).

APPENDIX C

SOLUTIONS FOR MINIMUM AVERAGE COST OF ERROR

This Appendix outlines the solutions for the test level which minimizes the average cost of error formula developed in Section 3.0 and applied to various GSFC testing objectives in Section 4.0. From Eq. (8), the expected value for the cost of error in vibration testing is given by

$$C_e = P[S \leq L,\ S > E]C_t + P[S > L,\ S \leq E]C_f \tag{C.1}$$

where the two probability statements are defined for both lognormal and normal distributions of S and E in Appendix B. The test level which will minimize the average cost of error in Eq. (C.1) is given by that level L_o which satisfies the expression

$$\frac{dCe}{dL} = C_t \frac{dP[S \leq L,\ S > E]}{dL} + C_f \frac{dP[S > L,\ S \leq E]}{dL} = 0 \tag{C.2}$$

C.1 ASSUMING LOGNORMAL DISTRIBUTIONS OF S AND E

Assuming lognormal distributions of S and E, the first term in Eq. (C.2) may be evaluated as follows. From Eq. (B.8),

$$\frac{dP[S \leq L,\ S > E]}{dL} = \frac{1}{L}\ \frac{dP[S \leq L,\ S > E]}{d\hat{L}} = \frac{1}{2\sqrt{2\pi}\ \sigma_{\hat{s}} L} \exp\left[-\frac{1}{2}\left(\frac{L - \mu_{\hat{s}}}{\sigma_{\hat{s}}}\right)^2\right]$$

$$- \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{s}} L} \int_{-\infty}^{0} \left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{s}}}{\sigma_{\hat{s}}^{\,2}}\right) \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{s}}}{\sigma_{\hat{s}}}\right)^2\right] \operatorname{erf}\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) d\hat{\eta}$$

$$+ \frac{1}{2\pi \sigma_{\hat{s}} \sigma_{\hat{E}} L} \int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{s}}}{\sigma_{\hat{s}}}\right)^2 + \left(\frac{\hat{\eta} + \hat{L} - \mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^2\right)\right] d\hat{\eta} \tag{C.3}$$

where $\hat{S}$, $\hat{E}$, and $\hat{L}$ are the logarithms of S, E, and L, respectively, $\hat{\eta}$ is as defined in Eq. (B.1a), and the erf function is as defined in Eq. (B.8). Consider the first integral in Eq. (C.3); namely,

$$I = \int_{-\infty}^{0} \left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}^{2}}\right) \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right] \operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) d\hat{\eta}$$

Integration by parts with

$$u = \operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) \qquad dv = \left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}^{2}}\right) \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right] d\hat{\eta}$$

$$du = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{E}}} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^{2}\right] d\hat{\eta} \qquad v = -\exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right]$$

leads to the result

$$\begin{aligned}
I &= uv\Big|_{-\infty}^{0} - \int_{-\infty}^{0} v\,du \\
&= -\exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right] \operatorname{erf}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)\Bigg|_{-\infty}^{0} \\
&\quad + \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{E}}} \int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^{2}\right] \exp\left[-\frac{1}{2}\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right] d\hat{\eta} \\
&= -\exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2}\right] \operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) \\
&\quad + \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{E}}} \int_{-\infty}^{0} \exp\left\{-\frac{1}{2}\left[\left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^{2} + \left(\frac{\hat{\eta}+\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)^{2}\right]\right\} d\hat{\eta}
\end{aligned}$$

Substitution of this expression into Eq. (C. 2) gives

$$\frac{dP\left[S \leq L,\ S > E\right]}{dL} = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}\ L}\left\{\exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right]\right\}\left[\frac{1}{2} + \operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)\right] \tag{C. 4}$$

Star.ing with Eq. (B. 10), it can be shown by a similar development that the second term in Eq. (C. 2) reduces to

$$-\frac{dP\left[S > L,\ S \leq E\right]}{dL} = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}\ L}\left\{\exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right]\right\}\left[\operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) - \frac{1}{2}\right] \tag{C. 5}$$

The derivative of the average cost of error is now obtained by substituting Eqs (C. 4) and (C. 5) into Eq. (C. 2), as follows.

$$\frac{dC_e}{dL} = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}\ L}\ \exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right]\left\{C_{qt}\left[\frac{1}{2} + \operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)\right] + C_f\left[\operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) - \frac{1}{2}\right]\right\} \tag{C. 6}$$

Hence, the optimal test level L_o is given by that level which satisfied the relationship

$$C_t\left[\frac{1}{2} + \operatorname{erf}\left(\frac{\hat{L}_o-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right)\right] + C_f\left[\operatorname{erf}\left(\frac{\hat{L}_o-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) - \frac{1}{2}\right] = 0$$

or

$$\boxed{\operatorname{erf}\left(\frac{\hat{L}_o-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) = \frac{C_f - C_t}{2(C_f + C_t)}} \tag{C. 7}$$

C.2 ASSUMING NORMAL DISTRIBUTIONS OF S AND E

Assuming unrestricted normal distributions of S and E, the first term in Eq. (C.2) may be evaluated as follows. From Eq. (B.14),

$$\frac{dP\left[S \leq L,\ S > E\right]}{dL} = \frac{1}{2\sqrt{2\pi}\,\sigma_s} \exp\left[-\frac{1}{2}\left(\frac{L-\mu_s}{\sigma_s}\right)^2\right]$$

$$- \frac{1}{\sqrt{2\pi}\,\sigma_s}\int_{-\infty}^{0}\left(\frac{\lambda+L-\mu_s}{\sigma_s^2}\right)\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_s}{\sigma_s}\right)^2\right]\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)d\lambda$$

$$+ \frac{1}{2\pi\,\sigma_s\,\sigma_E}\int_{-\infty}^{0}\exp\left\{-\frac{1}{2}\left[\left(\frac{\lambda+L-\mu_s}{\sigma_s}\right)^2 + \left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right]\right\}d\lambda \qquad (C.8)$$

where λ is as defined in Eq. (B.11a) and the erf function is as defined in Eq. (B.8). Now, a comparison of Eq. (C.8) with Eq. (C.3) reveals that the formulations here are identical (excluding an L in the denominator) to those developed for the lognormal case where

S is analogous to		$\hat{S}$
E	" "	$\hat{E}$
L	" "	$\hat{L}$
λ	" "	$\hat{\eta}$

Hence, it follows from Eq. (C.4) that

$$\frac{dP\left[S \leq L,\ S > E\right]}{dL} = \frac{1}{\sqrt{2\pi}\,\sigma_s}\left\{\exp\left[-\frac{1}{2}\left(\frac{L-\mu_s}{\sigma_s}\right)^2\right]\right\}\left[\frac{1}{2} + \operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right)\right]$$

(C.9)

By a similar development, it can be shown that

$$\frac{dP\left[S>L,\ S\leq E\right]}{dL} = \frac{1}{\sqrt{2\pi}\ \sigma_S}\left\{\exp\left[-\frac{1}{2}\left(\frac{L-\mu_S}{\sigma_S}\right)^2\right]\right\}\left[\operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right)-\frac{1}{2}\right] \tag{C. 10}$$

Finally, substituting Eqs (C. 9) and (C. 10) into Eq. (C. 2) and solving for the optimal test level L_o yields

$$\boxed{\operatorname{erf}\left(\frac{L_o-\mu_E}{\sigma_E}\right) = \frac{C_f - C_t}{2(C_f + C_t)}} \tag{C. 11}$$

C. 3 ASSUMING TRUNCATED NORMAL DISTRIBUTIONS OF S AND E

Now consider the physically realizable case where S and E are normally distributed, but restricted to nonnegative values. The first term in Eq. (C. 2) may be evaluated as follows. From Eq. (B. 19),

$$\begin{aligned}\frac{dP\left[0\leq S\leq L,\ S>E\geq 0\right]}{dL} &= \frac{1}{\sqrt{2\pi}\ \sigma_S}\operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right)\left\{\exp\left[-\frac{1}{2}\left(\frac{L-\mu_S}{\sigma_S}\right)^2\right]-\exp\left[-\frac{1}{2}\left(\frac{\mu_S}{\sigma_S}\right)^2\right]\right\}\\ &- \frac{1}{\sqrt{2\pi}\ \sigma_S}\int_{-L}^{0}\left(\frac{\lambda+L-\mu_S}{\sigma_S^2}\right)\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)d\lambda\\ &+ \frac{1}{2\pi\ \sigma_S\ \sigma_E}\int_{-L}^{0}\exp\left\{-\frac{1}{2}\left[\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2+\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right]\right\}d\lambda\end{aligned} \tag{C. 12}$$

where λ is as defined in Eq. (B. 11a) and the erf function is as defined in Eq. (B. 8). Consider the first integral in Eq. (C. 12); namely

$$I = \int_{-L}^{0}\left(\frac{\lambda+L-\mu_S}{\sigma_S^2}\right)\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]\operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)d\lambda$$

Integration by parts with

$$u = \operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right) \qquad dv = \left(\frac{\lambda+L-\mu_S}{\sigma_S^2}\right) \exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right] d\lambda$$

$$du = \frac{1}{\sqrt{2\pi}\,\sigma_E} \exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right] d\lambda \qquad v = -\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2\right]$$

leads to the result

$$I = uv\Big|_{-L}^{o} - \int_{-L}^{o} v\,du = -\exp\left[-\frac{1}{2}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right] \operatorname{erf}\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)\Bigg|_{-L}^{o}$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_E} \int_{-L}^{o} \exp\left\{-\frac{1}{2}\left[\left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2 + \left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right]\right\} d\lambda$$

$$= -\exp\left[-\frac{1}{2}\left(\frac{\mu_S}{\sigma_S}\right)^2\right] \operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right) + \exp\left[-\frac{1}{2}\left(\frac{L-\mu_S}{\sigma_S}\right)^2\right] \operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right)$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_E} \int_{-L}^{o} \exp\left\{-\frac{1}{2}\left[\left(\frac{\lambda+L-\mu_S}{\sigma_S}\right)^2 + \left(\frac{\lambda+L-\mu_E}{\sigma_E}\right)^2\right]\right\} d\lambda$$

Substitution of this expression into Eq. (C. 3) gives

$$\frac{P\left[0 \le S \le L,\ S > E \ge 0\right]}{dL} = \frac{1}{\sqrt{2\pi}\,\sigma_S} \left\{\exp\left[-\frac{1}{2}\left(\frac{L-\mu_S}{\sigma_S}\right)^2\right]\right\} \left[\operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right) + \operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right)\right] \qquad \text{(C. 13)}$$

Starting with Eq. (B. 21), it can be shown by a similar development that the second term in Eq. (C. 2) reduces to

$$\frac{P\left[S > L \ge 0,\ 0 \le S \le E\right]}{dL} = \frac{1}{\sqrt{2\pi}\,\sigma_S} \left\{\exp\left[-\frac{1}{2}\left(\frac{L-\mu_S}{\sigma_S}\right)^2\right]\right\} \left[\operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right) - \frac{1}{2}\right] \qquad \text{(C. 14)}$$

Hence, substituting Eqs (C. 13) and (C. 14) into Eq. (C. 2) and solving for the optimal test level L_o yields

$$\operatorname{erf}\left(\frac{L_o - \mu_E}{\sigma_E}\right) = \frac{\frac{1}{2} C_f - \operatorname{erf}(\mu_E/\sigma_E)\, C_t}{C_f + C_t} \tag{C. 15}$$

Note that for the case where $\mu_E >> \sigma_E$, $\operatorname{erf}\left(\frac{\mu_E}{\sigma_E}\right) \approx \frac{1}{2}$ and Eq. (C. 15) reduces to the same result obtained for the unrestricted normal distributions of S and E in Eq. (C. 11).

C. 4 ALTERNATE SOLUTION

The solution for an optimal test level may be obtained more directly if the expression for the average cost of error in Eq. (C. 1) is written in an alternate form. This may be done by using the following relationships.

$$P\left[S \le L,\ S > E\right] = P\left[S > E\right] - P\left[S > L,\ S > E\right] \tag{C. 16}$$

$$P\left[S > L,\ S > E\right] = P\left[S > L\right] - P\left[S > L,\ S \le E\right] \tag{C. 17}$$

Substituting Eq. (C. 17) into Eq. (C. 16) yields

$$P\left[S \le L,\ S > E\right] = P\left[S > E\right] - P\left[S > L\right] + P\left[S > L,\ S \le E\right] \tag{C. 18}$$

Now substituting Eq. (C. 18) into Eq. (C. 1) gives

$$C_e = \left\{ P\left[S > E\right] - P\left[S > L\right] \right\} C_t + P\left[S > L,\ S \le E\right] (C_t + C_f) \tag{C. 19}$$

Hence, the optimal test level is given by that level L_o which satisfies the expression

$$\frac{dC_e}{dL} = -\frac{dP\left[S>L\right]}{dL} C_t + \frac{dP\left[S>L,\ S\le E\right]}{dL} (C_t+C_f) = 0 \qquad (C.20)$$

For the case of lognormal distributions of S and E

$$P\left[S>L\right] = \int_L^\infty p(S)dS = \int_{\hat{L}}^\infty p(\hat{S})d\hat{S}$$

$$= \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}} \int_{\hat{L}}^\infty \exp\left[-\frac{1}{2}\left(\frac{\hat{S}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] d\hat{S}$$

$$= \frac{1}{2} - \operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)$$

Thus,

$$\frac{dP\left[S>L\right]}{dL} = -\frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}\ L} \exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right] \qquad (C.21)$$

The derivative of the joint probability statement in Eq. (C. 20) is given by Eq. (C. 5). Substituting Eqs (C. 5) and (C. 21) into Eq (C. 20) yields

$$\frac{dC_e}{dL} = \frac{1}{\sqrt{2\pi}\ \sigma_{\hat{S}}\ L} \exp\left[-\frac{1}{2}\left(\frac{\hat{L}-\mu_{\hat{S}}}{\sigma_{\hat{S}}}\right)^2\right]\left\{C_t + \left[\operatorname{erf}\left(\frac{\hat{L}-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) - \frac{1}{2}\right]\left[C_t+C_f\right]\right\} = 0$$

Solving for the optimal level L_o again gives

$$\operatorname{erf}\left(\frac{\hat{L}_o-\mu_{\hat{E}}}{\sigma_{\hat{E}}}\right) = \frac{C_f-C_t}{2(C_f+C_t)} \qquad (C.22)$$

which agrees with the previous result of Eq. (C. 7).

For the case of unrestricted normal distributions of S and E,

$$P\left[S>L\right] = \int_L^\infty p(S)dS = \frac{1}{\sqrt{2\pi}\,\sigma_s}\int_L^\infty \exp\left[-\frac{1}{2}\left(\frac{S-\mu_s}{\sigma_s}\right)^2\right] dS$$

$$= \frac{1}{2} - \operatorname{erf}\left(\frac{L-\mu_s}{\sigma_s}\right)$$

Thus,

$$\frac{dP\left[S>L\right]}{dL} = -\frac{1}{\sqrt{2\pi}\,\sigma_s} \exp\left[-\frac{1}{2}\left(\frac{L-\mu_s}{\sigma_s}\right)^2\right] \tag{C. 23}$$

The derivative of the joint probability statement in Eq. (C. 20) is given by Eq. (C. 10). Substituting Eqs (C. 10) and (C. 23) into Eq. (C. 20) yields

$$\frac{dC_e}{dL} = \frac{1}{\sqrt{2\pi}\,\sigma_s} \exp\left[-\frac{1}{2}\left(\frac{L-\mu_s}{\sigma_s}\right)^2\right]\left\{C_t + \left[\operatorname{erf}\left(\frac{L-\mu_E}{\sigma_E}\right) - \frac{1}{2}\right]\left[C_t + C_f\right]\right\} = 0$$

Solving for the optimal level L_o again gives

$$\operatorname{erf}\left(\frac{L_o-\mu_E}{\sigma_E}\right) = \frac{C_f - C_t}{2(C_f + C_t)} \tag{C. 24}$$

which agrees with the previous result of Eq. (C. 11).

APPENDIX D

EVALUATION OF OPTIMAL TEST LEVEL BOUND

In Section 6, an optimal level for qualification vibration tests in terms of a percentile of the environmental load distribution function is shown to be

$$X_o = 100 \left[\frac{N}{N + R_{qc}} \right] \tag{D. 1}$$

where N is the number of items scheduled for service use and R_{qc} is the ratio of the cost of a qualification test failure to a service failure. In [3], a lower bound on this cost was determined to be

$$X_o \geq 100 \left[1 - R_{qc} \right]^{1/N} \tag{D. 2}$$

It will now be shown that Eq (D. 2) is indeed a lower bound on Eq. (D. 1) for all values of the positive integer N and all values of R_{qc} in the applicable range $0 \leq R_{qc} \leq 1$.

From Eqs (D. 1) and (D. 2),

$$\frac{N}{N + R} \geq (1 - R)^{1/N} \tag{D. 3}$$

where the subscript on R is omitted for clarity. It follows that

$$1 \geq \left(1 + \frac{R}{N}\right)^N (1 - R) \tag{D. 4}$$

Now expanding $\left(1+\frac{R}{N}\right)^N$ into a binomial series, Eq. (D.4) reduces to

$$1 \geq \left[\sum_{h=o}^{N} \frac{N!}{(N-h)!\,h!}\left(\frac{R}{N}\right)^h\right](1-R)$$

$$\geq \sum_{h=o}^{N} \frac{N!}{(N-h)!\,h!}\left(\frac{R}{N}\right)^h - \sum_{k=o}^{N} \frac{N!}{(N-k)!\,k!}\left(\frac{R^{k+1}}{N^k}\right) \qquad (D.5)$$

Letting h=j+1 and k=j, Eq. (D.5) becomes

$$1 \geq 1 + \sum_{j=o}^{N-1} \frac{N!}{(N-j-1)!\,(j+1)!}\left(\frac{R^{j+1}}{N^{j+1}}\right) - \sum_{j=o}^{N} \frac{N!}{(N-j)!\,j!}\left(\frac{R^{j+1}}{N^j}\right)$$

$$\geq 1 - \frac{R^{N+1}}{N^N} + \sum_{j=o}^{N-1}\left[\frac{(N-1)!}{(N-j-1)!\,(j+1)!} - \frac{N!}{(N-j)!\,j!}\right]\frac{R^{j+1}}{N^j}$$

(D.6)

Noting that the term in the brackets reduces to

$$\frac{(N-j)(N-1)! - N!\,(j+1)}{(N-j)!\,(j+1)!} = \frac{(N-1)!\,[N-j-N(j+1)]}{(N-j)!\,(j+1)!}$$

$$= -\frac{(N-1)!\,(j)\,(1+N)}{(N-j)!\,(j+1)!}$$

Equation (D.6) may be written as

$$1 \geq 1 - \frac{R^{N+1}}{N} - \sum_{j=o}^{N-1} \frac{(N-1)!\,(j)\,(1+N)}{(N-j)!\,(j+1)!} \left(\frac{R^{j+1}}{N_j} \right) \qquad \text{(D.7)}$$

Since R, N, and j are always positive quantities and N is always greater than j, it is clear that all terms in the series of Eq. (D. 7) will be positive. Hence, the inequality of Eq. (D. 7) and, correspondingly, the inequality of Eq. (D. 3) must always be true for all values of N.