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ON THE LIBRATION OF A GRAVITY GRADIENT STABILIZED SPACECRAFT IN AN ECCENTRIC ORBIT

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16. Abstract

General analytical methods for predicting planar and three dimensional attitude motion of a gravity gradient spacecraft in an elliptical orbit are developed. Regions of stability from the linearized planar equation of Mathieu form are plotted versus inertia parameters and orbit eccentricity. Similar plots showing the effects of damping are presented. The planar equation for larger eccentricities is examined with asymptotic expansion methods (KBM) for increased accuracy. An analysis based on the Hamiltonian and using cannonical transformations and the method of averaging is applied to the three degree of freedom equations. Thirty-two combinations of resonant eigenfrequencies, indicating both nonlinear and parametric resonances, and including six previously shown by Breakwell and Pringle, are found. Nineteen of these combinations lead to unbounded increases in the amplitude of oscillation of one or more of the spacecraft modes. Eight lead to significant interchanges of energy between modes. The remaining five have little effect.

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SUMMARY

General analytical methods for the prediction of the attitude motion of a gravity gradient stabilized spacecraft are developed for an elliptic orbit. Both planar and three-dimensional motion are treated.

The linearized small eccentricity equation describing the single degree of freedom in-plane motion (pitch motion) is a Mathieu equation. Combinations of spacecraft inertia parameters and orbit eccentricities for which the spacecraft pitch motion is unbounded are plotted from tables of the Mathieu function. Three regions of unstable motion are found.

The linearized planar equation for larger eccentricities is treated by means of the asymptotic expansion methods (in powers of the eccentricity) of Krylov, Bogoliubov, and Mitropolsky. A more accurate determination of the three regions of unstable motion found from Mathieu's equation is obtained.

Finally an analysis based on the Hamiltonian formulation of the equations of motion demonstrates the effect of both nonlinear and parametric resonances on the three degree of freedom motion of the spacecraft. Canonical transformations and the method of averaging are used to determine combinations of eigenfrequencies of the normal modes which lead to various types of resonances. This is an extension of an earlier work by Breakwell and Pringle in which six resonant frequency combinations were found. Twenty-six additional resonance combinations are found of which sixteen lead to unbounded motion in one or more of the modes and six lead to significant interchanges of energy between various modes.

It is concluded first that the attitude motion of a spacecraft in an eccentric orbit has characteristics significantly different from those of the corresponding motion in a circular orbit and second that theoretical methods are adequate to predict, and hence avoid, the occurrence of unstable modes during the design of gravity gradient spacecraft.

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CONTENTS

Symbol List

a	semi major axis
a	parameter in Mathieu's equation (Chapter 2)
a	dependent variable in asymptotic expansions (Chapter 3)
$b(t)$	forced pitch velocity
$c(t)$	forced pitch momentum
d_{ij}	elements of $[D]$
e	eccentricity
i	$\sqrt{-1}$
i, j, k	integers
k	$3\left(1 + \frac{3e^2}{2} + \frac{15e^4}{8}\right)$
n	mean angular motion
p, q	integers which are relatively prime (Chapter 3)
p_i	generalized momenta
q_i	generalized coordinate
q	parameter in Mathieu's equation (Chapter 2)
r	radius to spacecraft
r_1, r_2	dimensionless inertia ratios; $r_1 = I_1/I_3$, $r_2 = I_2/I_3$
t	time
u_i	terms in asymptotic solution (Chapter 3)
x	general dependent variable
y	general dependent variable
z	general independent variable
v	true anomaly
$s(t)$	series expansion of $T_2(t)$

A_i amplitude correction term (Chapter 3)

B_i phase correction term (Chapter 3)

C_i eigenvectors (Chapter 4)

C_i constants of integration (Chapter 5)

H Hamiltonian

I_1, I_2, I_3 spacecraft moments of inertia relative to principal body axes $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ respectively

L Lagrangian

M mean anomaly

S Hamilton's principal function

$T_1(t)$ difference between spacecraft angular velocity and mean motion

$T_2(t)$ time varying part of $(a/r)^3$

T kinetic energy

V potential energy

Matrices

[A] Direction cosine matrix

[D] Contact transformation matrix

[K] Contact transformation matrix

[M] Contact transformation matrix

[S] Matrix form of H_2

[Z] Symplectic test matrix

Coordinate Frames

$[\vec{e}]$ principal body

$[\vec{E}]$ inertial

$[\vec{\zeta}]$ local vertical

$[\vec{\zeta}']$ mean vertical

α_i generalized momenta in transformed Hamiltonian

β_i generalized coordinates in transformed Hamiltonian

γ constant

δ constant

ϵ a small parameter

η general angle variable

θ roll angle

κ damping constant

λ characteristic root

μ_e gravitational constant times mass of earth

μ rate of exponential growth or decay

π 3.14159

ρ constant

σ constant

τ period of $P(z)$

ϕ pitch angle

ψ yaw angle

ω_i frequency of i^{th} normal mode

Φ transformed pitch angle

Ψ phase variable

Θ phase variable

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CHAPTER 1

INTRODUCTION

Historical Background

Analytic investigations of the effect of an inverse-square gravitational field on a triaxial satellite have been published over a span of nearly 300 years. In 1686, Newton (Reference 1) wrote: "... the figure of the moon would be a spheroid, whose greatest diameter produced would pass through the center of the earth.... Hence it is that the same face of the moon always is turned toward the earth; nor can the body of the moon possibly rest in any other position, but would return always by a libratory motion to this situation...." According to Routh (Reference 2), during the next 200 years the attention of D'Alembert, Lagrange, Laplace, and Poisson was directed toward the gravity-gradient effect on the motion of the moon. By the beginning of the 20th century, correct equations for this torque were routinely published as in Plummer (Reference 3). One of the goals of these early investigators was "selenodesy," i.e., making inferences about the structure of the moon from observation of its motion. Present emphasis, apparently starting with a proposal for a hinged satellite by Breakwell and Roberson in 1954, centers on the other half of this problem, predicting spacecraft motion from a knowledge of its structure.

The particular problem that forms the subject of this investigation—the motion of a gravity-gradient-stabilized spacecraft in an eccentric orbit—was first discussed by Baker (Reference 4). He commented that motion about the orbit normal (i.e. pitch) could be approximately represented by a linearized equation that was similar to a standard Mathieu equation with a forcing term added. Schrello (Reference 5) in his investigations of the gravity-gradient disturbance of aerodynamically stable spacecraft formulated the pitch motion problem in terms of Hill's equation. DeBra (Reference 6) published a numerical investigation of the effect of orbit eccentricity on attitude stability. A number of Russian authors, particularly Beletskii (References 6, 8, and 9) and Chernous'ko (Reference 10), made an analysis in which they applied asymptotic methods to the problem. Kane (Reference 11) applied Floquet theory to a somewhat restricted three-dimensional analysis and discovered

*In its original form, this paper was submitted to the Catholic University of America in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

apparent instabilities in the nominally stable range of spacecraft parameters. Breakwell and Pringle (Reference 12) explained this phenomenon in terms of a nonlinear resonance between the pitch and yaw frequencies.

This dissertation contains several analytical approaches to the problem of determining the librational motion of rigid spacecraft in an eccentric orbit about an ideal spherical earth. The analysis is primarily concerned with the effect of orbit eccentricity on spacecraft motion; no other perturbations are considered. The orbital motion (motion of the spacecraft's center of mass) is considered known *a priori*.

Objectives

The exact equations of motion for a gravity-gradient-stabilized spacecraft in an eccentric orbit are coupled, time-varying, nonlinear, and insoluble in closed form. Thus analytic efforts have been concentrated on a number of special cases such as the linearized equations for three-dimensional motion in a circular orbit, the nonlinear equation for pitch motion in a circular orbit, the linear equation for pitch motion in an eccentric orbit, etc. Since a complete analytical solution is unavailable, parameter selection for any particular spacecraft is based on a combination of linear analysis and numerical simulation. Yet linear analysis is unable to reveal many significant features of the equations of motion; furthermore, the basic phenomena underlying the results obtained from a numerical simulation are seldom explained.

The major goal of this dissertation is to provide an explicit analytical basis for the effect of orbit eccentricity on spacecraft attitude. Two separate approaches to the problem are made. First, the asymptotic methods of Krylov, Bogoliubov, and Mitropolsky are applied to the planar-pitch problem to obtain stability regions and, in the stable regions, to determine the forced responses. Then the nonlinear, three-dimensional problem is discussed, using canonical transformation and the method of averaging to obtain a locus of spacecraft inertia parameters that result in resonant motion.

Other objectives are to provide increased understanding of one of the mechanisms by which nonlinear terms can produce unexpected results in problems with many degrees of freedom, and a basis for application of the two approaches to particular mechanizations of the basic gravity-gradient-stabilization concept (e.g., to consider specific damping concepts).

Outline

Chapter 2 discusses the linearized equation of spacecraft motion about the pitch axis (i.e. orbit normal), with the expansions of orbital parameters restricted to first order in eccentricity, e. This is the simplest form of the problem. There are several possible variations involving choices for reference axis and independent variable. Each of the choices leads to a Mathieu-type equation with a forcing term, but the final equations are not identical. Regions of stable motion are plotted as functions of orbit eccentricity and spacecraft inertia ratio for each of the formulations. The stability plots are similar if e is very small but diverge rapidly as e increases.

Chapter 3 develops an asymptotic solution for the in-plane motion with a more accurate expansion for the orbital elements. When the frequency of oscillation about the pitch axis is 1/2, 1, or 3/2 cycles per orbit, there is a parametric excitation of the pitch motion, as in Chapter 2. The basic asymptotic solution for this problem determines the magnitude of the induced oscillation about the nominal equilibrium axis (i.e., local vertical). However, this form of asymptotic solution fails in the vicinity of each parametric resonance. A second type of solution is developed for each of the resonant frequencies. These solutions are used to obtain boundaries for stable motion similar to those developed in Chapter 2. This completes the analytical treatment of the pitch motion case with one degree of freedom. The next two chapters are concerned with the nonlinear equations for the full three degrees of freedom.

A different approach, first applied to this problem by Breakwell and Pringle (Reference 12) is used in the discussion of spacecraft motion with three degrees of freedom. A linearized description of the motion in terms of normal modes leads to three uncoupled simple-harmonic oscillators in normal coordinates. Nonlinear terms and time-varying terms due to orbital eccentricity tend to cause changes in the fundamental uncoupled oscillations. These changes occur slowly in comparison with the time for a single cycle of any of the normal modes, and only become significant when there is a resonant relationship between the frequency of the forcing terms and the appropriate normal mode(s).

Chapter 4 develops the Hamiltonian for the system to the third order in generalized coordinates and momenta and with orbital motion represented in expansions to fourth order in e; also, a series of cannonical transformations that first decouple the roll-yaw equations and then introduce cyclic coordinates. The resulting equations of motion represent a set of three harmonic oscillators perturbed by second-order terms and forcing functions.

When the method of averaging is applied to the Hamiltonian in Chapter 5, the effect of the nonlinear coupling terms and the forcing terms becomes apparent. The amplitude and phase of the fundamental oscillations vary only slightly from their initial values except when there are resonances between the perturbing terms and the fundamental modes. Approximately 30 resonant relations exist (when the Hamiltonian expansion is limited as noted above); the locus of each of the resonances is plotted in this chapter. It is possible to define regions about each of the resonances in which the motion would change appreciably. The conditions which define these regions are given for each of the possible resonances.

The appendices contain a discussion of the coupling between orbital motion and attitude motion and the specific extensions of the asymptotic formulas used in Chapter 3 for the third-order solutions.

Comments on Linearization and Stability

Linear differential equations are so convenient, and dynamics problems starting with the pendulum are so often linearized, that linearization is frequently resorted to without adequate consideration of its validity. Speaking somewhat broadly, it is valid to linearize equations if the resulting solution does not differ appreciably from the solution to the exact equations. The validity is frequently determined by intuition, or comparison with other similar equations; however, a more correct justification can be obtained by comparing the analytic results for the linear equation with those obtained from integration of the exact equations performed, for instance, on an analog or digital computer. There are rigorous mathematical conditions under which linearization is valid and these are discussed by Struble (Reference 13) and Cesari (Reference 14) among others. Unfortunately, these are only sufficient conditions for linearization to succeed or fail, not necessary ones. In general, if the equilibrium solution to the linear portion of the equation is asymptotically stable and the nonlinear portion meets certain conditions, the equilibrium solution of the complete equation is also asymptotically stable. When the solution to the linear equation is stable but not asymptotically so (as in this problem), no information concerning the adequacy of the linear solution is obtained. In fact, in the last chapter of this dissertation a number of unstable regions are found for the nonlinear equations in regions that satisfy all the stability criteria for the linearized equations.

According to Stoker (Reference 15), stability is a concept that means different things to different people. In this dissertation a linear system will be defined as asymptotically stable if all the characteristic numbers of the Floquet form of solution (Struble, Reference 13) have negative real parts, stable if no root has a positive real part, and unstable if any root has a positive real part. This is contrary to the standard servomechanism practice, in that an undamped bounded oscillator would be stable under the above definition. In discussing stability for nonlinear systems the concept of Liapunov stability (Reference 14) is usually satisfactory. Note that a conclusion of instability based on approximate equations guarantees only that the motion will increase until higher-order terms not included in the analysis become important.

In formulating approximate equations there is no clear-cut rule as to the number or order of terms that should be retained. Similarly with asymptotic solutions there is a question as to the order of approximation to which the solution should be continued. The inclusion of additional terms in the equations or solutions will usually have one of two effects. Either the extension will introduce new phenomena into the solution, (e.g., including the nonlinear term in van-der Pol's equation), or it will increase the precision of the solution of a particular problem (e.g., continuing the expansion of $\sin x$ for a simple pendulum). In this dissertation the intent is to include any reasonable extension that effects a basic change in the solution, but not to add terms solely to increase the precision of a result.

Notation and Definitions

Four right-handed orthogonal coordinate frames shown in Figure 2-2, are used in this dissertation as follows:

$[\vec{E}] = \vec{E}_1, \vec{E}_2, \vec{E}_3$ An inertial system with origin at the earth's center with:

$\vec{E}_1$, a unit vector towards perigee of the spacecraft orbit,

$\vec{E}_3$, a unit vector parallel to the spacecraft-orbit angular-momentum vector, and

$\vec{E}_2 = \vec{E}_3 \times \vec{E}_1$.

$[\vec{\zeta}] = \vec{\zeta}_1, \vec{\zeta}_2, \vec{\zeta}_3$ A local vertical system with origin at the spacecraft center of mass, with:

$\vec{\zeta}_1$, a unit vector parallel to a vector from the earth center of mass to the spacecraft center of mass,

$\vec{\zeta}_3$, a unit vector parallel to $\vec{E}_3$, and

$\vec{\zeta}_2 = \vec{\zeta}_3 \times \vec{\zeta}_1$.

$[\vec{\zeta}'] = \vec{\zeta}_1' \; \vec{\zeta}_2' \; \vec{\zeta}_3'$ A mean vertical system with origin at the spacecraft center of mass, with:

$\vec{\zeta}_1'$, a unit vector parallel to a vector from the earth center of mass to that of an imaginary spacecraft in a circular orbit with the same period and the same phase as the subject spacecraft,

$\vec{\zeta}_3'$, a unit vector parallel to $\vec{E}_3$, and

$\vec{\zeta}_2' = \vec{\zeta}_3' \times \vec{\zeta}_1'$.

$(\vec{e}) = \vec{e}_1, \vec{e}_2, \vec{e}_3$ A principal body-axis system, with:

$\vec{e}_1$, a unit vector along the axis of least inertia,

$\vec{e}_2$, a unit vector along the intermediate axis, and

$\vec{e}_3$, a unit vector along the axis of maximum inertia.

Notation relating to coordinate transformations:

$[A]$ = direction cosine matrix

$A^i(\eta)$ = direction cosine matrix for a rotation through an angle η about the i^{th} axis.

The coordinate frames are located by the following rotation matrices:

Local vertical—inertial

$$[\vec{\zeta}] = A^3(v)\,[\vec{E}]$$

where v is the true anomaly and

$$A^3(v) = \begin{bmatrix} \cos v & \sin v & 0 \\ -\sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mean vertical—inertial

$$[\vec{\zeta}'] = A^3(M)\vec{E}$$

where M is the mean anomaly.

Local vertical—mean vertical

$$[\vec{\zeta}] = A^3 (v - M) [\vec{\zeta}']$$

Body—local vertical

$$[\vec{e}] = A^1 (\psi) A^2 (\theta) A^3 (\phi) [\vec{\zeta}] = [A] [\vec{\zeta}]$$

where ϕ, θ, and ψ are Euler angles designated as pitch, roll, and yaw respectively and

$$[A] = \begin{bmatrix} \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\psi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\theta\cos\psi \end{bmatrix} .$$

Body—mean vertical

$$[\vec{e}] = A^1 (\psi) A^2 (\theta) A^3 (\phi') [\vec{\zeta}]$$

where $\phi = \phi' + v - M$.

Conventions and Units

The unit of time is chosen so that the mean angular motion, n, is one, thus $M = t$.

Units of mass and length are chosen such that $I_3 = 1$; and dimensionless parameters r_1 and r_2 are defined as

$$r_1 \equiv \frac{I_1}{I_3} , \qquad r_2 \equiv \frac{I_2}{I_3} .$$

A superscript dot will indicate a derivative taken with respect to the independent variable which may be either t or v.

A subscript zero indicates an initial value.

CHAPTER 2

PLANAR PITCH LIBRATION AS A MATHIEU FUNCTION

Introduction

A complete formulation of the equations of motion for a rigid body in an inverse-square force field yields six coupled, nonlinear, second-order differential equations. The motion of a gravity-gradient-stabilized spacecraft cannot be determined analytically when presented in this exact form. However, there are possible approximations that lead to significant results. This chapter introduces a series of assumptions that reduce the problem to consideration of the small-angle pitch librations of a spacecraft moving in a known orbit of small eccentricity, and from there to a form of Mathieu's equation. Even this highly restrictive formulation yields interesting results not obtained from circular-orbit analysis.

Beletskii (Reference 8) and Baker (Reference 4) noted the connection between an approximate form of the in-plane pitch-libration problem defined below and a Mathieu equation. Beletskii discussed an equation formulated with true anomaly as the independent variable, and Baker expressed the problem with time as the independent variable. This chapter demonstrates the differences that result from these two choices. A set of graphs are developed herein which present the stability boundaries associated with each of the approaches as a function of the inertia ratios of the spacecraft and the orbital eccentricity. Finally the effect of velocity-dependent damping on spacecraft stability is shown.

The first assumption for this and all the following chapters is that the motion of the spacecraft's center of mass (orbital motion) is not affected by motion of the spacecraft about its center of mass (attitude motion). With this widely accepted simplification, the equations representing orbital motion are identical with those for a point mass in an inverse-square force field.

The converse of this first assumption is not true, and it is necessary to include the orbital parameters as known functions of time when solving for the spacecraft attitude motion. In essence, this is equivalent to assuming that energy can be transferred from orbital motion to attitude motion without affecting the orbital motion. Actually, as shown by Beletskii (Reference 9), the relative magnitudes of the appropriate terms make this a reasonable approximation.

The next assumption is that the equations can be linearized when applied to small-angle motion. The linearized equations for roll and yaw do not involve terms in the pitch motion, and vice versa; pitch motion may be considered independent of roll-yaw motion.

The coupled roll-yaw equations are homogeneous for both circular and eccentric orbits, and there is a solution with $\theta = \dot{\theta} = \psi = \dot{\psi} = 0$ for either case. This chapter and Chapter 3 are concerned with the equation describing pitch motion under the assumption that the roll and yaw variables are identically zero. Thus the exact problem is reduced to consideration of the single linear second-order differential equation for pitch motion, with the orbital parameters known functions of time.

In this chapter the equation for pitch libration will be formulated in several different ways. The variations originate from the different choices of reference frame and independent variable, even when the problem is restricted, as discussed above. Each of the formulations is limited to exclude terms higher than first order in eccentricity, and the resulting equations are transformed into the canonical form of Mathieu's equation. The nature of solutions to Mathieu's equation is well known and regions of bounded and unbounded motion for the pitch libration are determined from these known solutions.

Chapter 3 presents a somewhat more general approach to the same pitch libration problem. The equation is limited to exclude terms higher than third order in eccentricity, and the asymptotic methods of Krylov, Bogoliubov, and Mitropolsky are used to obtain regions of bounded and unbounded motion.

Equations of Motion

The linearized equations of motion for a rigid, gravity-stabilized spacecraft moving in an eccentric orbit are well known. In the notation of this dissertation they are:

$$r_1 \ddot{\psi} + (1 - r_2)\dot{v}^2 \psi + (1 - r_2 - r_1)\dot{v}\dot{\theta} = 0 ,$$

$$r_2 \ddot{\theta} + (1 - r_1)\left(\dot{v}^2 + \frac{\mu_e}{r^3}\right)\theta - (1 - r_2 - r_1)\dot{v}\dot{\psi} = 0 ,$$

and

$$\ddot{\phi} + (r_2 - r_1)\frac{\mu_e}{r^3}\phi = -\ddot{v} .$$

When a circular orbit is assumed, these equations become:

$$r_1 \ddot{\psi} + (1 - r_2)\psi + (1 - r_2 - r_1)\dot{\theta} = 0 ,$$

$$r_2 \ddot{\theta} + 4(1 - r_1)\theta - (1 - r_2 - r_1)\dot{\psi} = 0 ,$$

and

$$\ddot{\phi} + 3(r_2 - r_1)\phi = 0 .$$

One solution of this set of equations is $\phi = \theta = \psi = 0$, and this solution can be shown to be a stable one for any values of r_1 and r_2 in the "Lagrange" or "Delp" regions as shown in Figure 2-1 (after

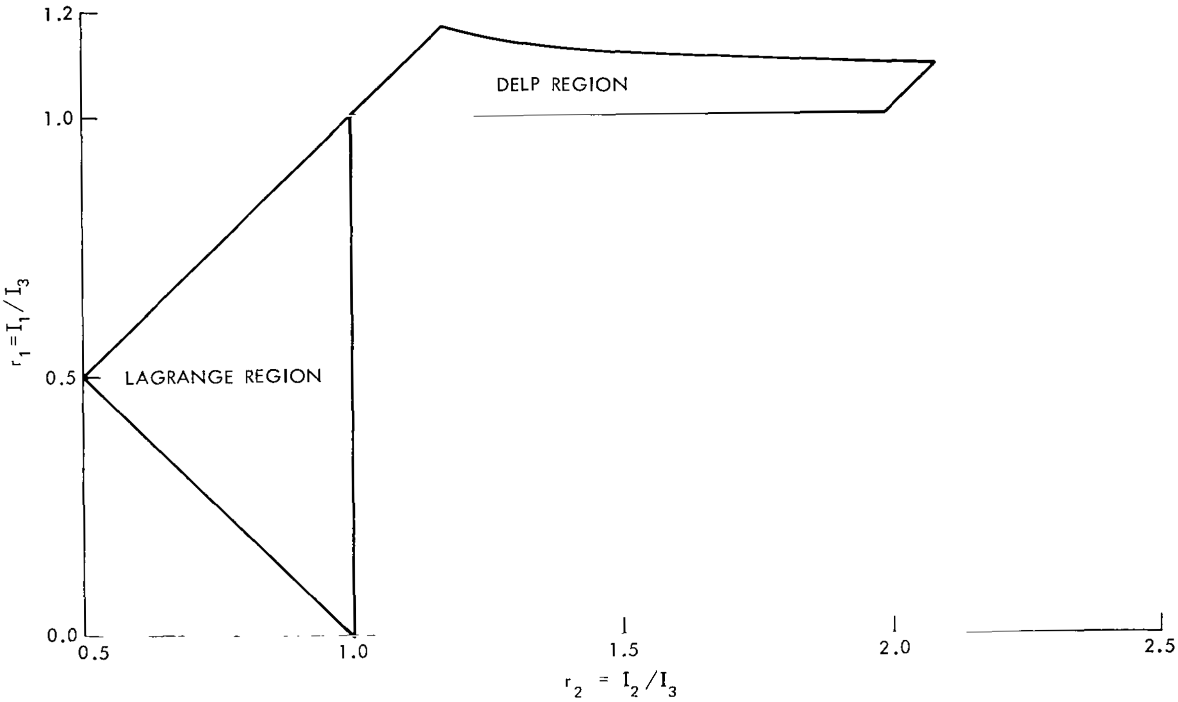

Figure 2-1—Stability regions for a rigid body in a circular orbit.

Debra). Pringle (Reference 16) has demonstrated, by application of Liapunov's second method, that spacecraft in the Lagrange region are stable only if there is any form of pervasive energy dissipation in the system. Therefore there are no practical spacecraft whose parameters fall in the Delp region, and this region will not be considered in the remainder of the dissertation.

In much the same manner, $\theta = \psi = 0$ is a solution of the two linearized roll-yaw equations for motion in an elliptic orbit. It is true that this solution cannot be proven stable throughout the entire Lagrange region. Nevertheless, a number of authors (e.g., Baker (Reference 4), Schrello (Reference 5), Beletskii (Reference 8), Moran (Reference 17), Schecter (Reference 18), etc.) have considered variations of the problem in which roll and yaw are assumed zero and only pitch motion is considered. This class of motion is commonly referred to as the "in-plane (or planar)" pitch-motion problem.

Chapter 5 shows that for some portions of the Lagrange regions it is incorrect to assume: (1) that the equations can be linearized for small-angle motion, or (2) that $\theta = \psi = 0$ is a stable solution to the roll-yaw equations. However, a number of interesting points can be observed by investigation of pitch subject to these restrictions; both this and the next chapter are devoted to such an analysis.

The previous equations are obtained in Appendix A by starting with an exact formulation of the problem and introducing simplifications into the equations. A different approach to the same result would be to write exact equations for a simplified problem; this would help in visualizing the effect of a change in reference systems from the local-vertical to the mean-vertical system.

Consider a spacecraft moving in an eccentric orbit (Figure 2-2) and subject to a torque τ given by

$$\tau = \frac{3\mu_e}{r^3}\left(r_1 - r_2\right)\phi ,$$

or, since

$$\phi' = \phi + v - M ,$$

therefore

$$\tau = \frac{3\mu_e}{r^3}\left(r_1 - r_2\right)\left[\phi' - (v - M)\right] .$$

The angular momentum of the spacecraft about its center of mass is either $I_3\,(\phi + \dot{v})$ or $I_3\,(\phi' + \dot{M})$ depending on the choice of reference frame. Thus when the rate of change of angular momentum is

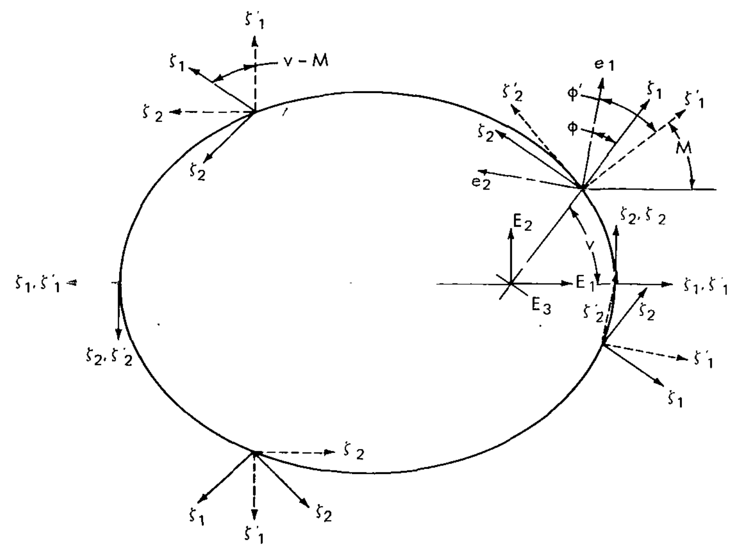

Figure 2-2—Local-vertical and mean-vertical coordinate systems.

equated to the applied torque for motion referenced to the local-vertical frame, the result is

$$\ddot{\phi} + \frac{3\mu_e}{r^3}\left(r_2 - r_1\right)\phi = -\ddot{v} . \tag{2-1}$$

Similarly, when the motion is referred to the mean-vertical frame the result is

$$\ddot{\phi}' + \frac{3\mu_e}{r^3}\left(r_2 - r_1\right)\left[\phi' - (v - M)\right] = 0 ; \tag{2-2}$$

this is the equation studied by Baker (Reference 4).

Both Equations 2-1 and 2-2 are written with time as the independent variable. An alternative formulation in which the independent variable is changed to the true anomaly is of interest. The change of independent variables is accomplished by noting that

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dv^2}\left(\frac{dv}{dt}\right)^2 + \frac{d\phi}{dv}\frac{d}{dv}\left(\frac{dv}{dt}\right)\frac{dv}{dt} ,$$

and from celestial mechanics (see Appendix A)

$$\left(\frac{dv}{dt}\right)^2 = \frac{\mu_e}{r^3}(1 + e \cos v)$$

and

$$\frac{d}{dv}\left(\frac{dv}{dt}\right)\frac{dv}{dt} = -\frac{2\mu_e}{r^3} e \sin v .$$

Thus Equation 2-1 becomes, after some simplification,

$$\frac{d^2\phi}{dv^2} - \frac{2e \sin v}{1 + e \cos v}\frac{d\phi}{dv} + \frac{3\left(r_2 - r_1\right)}{1 + e \cos v}\phi = \frac{2e \sin v}{1 + e \cos v} . \tag{2-3}$$

There is a standard transformation that removes the first derivative term in an equation of this form. Let

$$\Phi(v) = \phi(v)(1 + e \cos v) .$$

Then

$$\frac{d\phi}{dv} = \frac{d\Phi}{dv}(1 + e\cos v)^{-1} + \Phi e \sin v\,(1 + e\cos v)^{-2} ,$$

$$\frac{d^2\phi}{dv^2} = \frac{d^2\Phi}{dv^2}(1 + e\cos v)^{-1} + 2\frac{d\Phi}{dv} e \sin v\,(1 + e\cos v)^{-2}$$

$$+ \Phi e\cos v\,(1 + e\cos v)^{-2} + \Phi\, 2e^2 \sin^2 v\,(1 + e\cos v)^{-3} ,$$

and Equation 2-3 becomes

$$\frac{d^2\Phi}{dv^2} + \frac{3(r_2 - r_1) + e\cos v}{1 + e\cos v}\Phi = 2e\sin v . \tag{2-4}$$

This is the form of the equation that has been studied by Beletskii (References 7, 8, and 9).

At this point in the analysis, Equations 2-1, 2-2, and 2-4 represent three different formulations of a single problem, and each is a second-order differential equation with non-constant coefficients. The radius vector and true anomaly of a particle in an elliptic orbit in an inverse-square force field can be expressed as a function of the mean anomaly. Moulton (Reference 19) gives these functions as

$$r = a\left[1 - e\cos M - \frac{e^2}{2}(\cos 2M - 1) - \cdots\right]$$

and

$$v = M + 2e\sin M + \frac{5}{4}e^2 \sin 2M ,$$

and, for small e,

$$\mu_e r^{-3} = \mu_e a^{-3}\left[1 - e\cos M - \frac{e^2}{2}(\cos 2M - 1) - \cdots\right]^{-3}$$

$$\simeq \mu_e a^{-3}\,[1 + 3e\cos M + \cdots]$$

$$= 1 + 3e\cos t$$

and

$$\ddot{v} = -2e\sin t ,$$

since $M = nt$ and $n = \mu_e a^{-3} = 1$.

Thus, to first order in e, Equations 2-1 and 2-2 become

$$\frac{d^2 \phi}{dt^2} + 3(r_2 - r_1)(1 + 3e \cos t)\phi = 2e \sin t \tag{2-5}$$

and

$$\frac{d^2 \phi'}{dt^2} + 3(r_2 - r_1)(1 + 3e \cos t)\phi' = 6e(r_2 - r_1) \sin t \ . \tag{2-6}$$

Similarly, to first order in e,

$$\begin{aligned} \frac{3(r_2 - r_1) + e \cos v}{1 + e \cos v} &= \left[3(r_2 - r_1) + e \cos v\right]\left[1 - e \cos v + e^2 \cos^2 v - \cdots\right] \\ &\simeq 3(r_2 - r_1) + \left[1 - 3(r_2 - r_1)\right] e \cos v \ , \end{aligned}$$

and Equation 2-4 becomes

$$\frac{d^2 \Phi}{dv^2} + \left\{3(r_2 - r_1) + \left[1 - 3(r_2 - r_1)\right] e \cos v\right\}\Phi = 2e \sin v \ . \tag{2-7}$$

Reduction to Canonical Mathieu Form

Each of the three formulations of the preceding section leads to a nonhomogeneous, linear, differential equation with periodic coefficients. Although in general there is no analytic method for obtaining solutions to this broad class of equations, a number of particular cases have been exhaustively studied. One of the better known examples is Mathieu's equation. It is possible to transform each of Equations 2-5, 2-6, and 2-7 into the canonical form of Mathieu's equation by a suitable change of variables. The solution to this equation is well known and gives the solution to the pitch-motion problem.

Historically, the equation was first studied by Mathieu (Reference 20) in conjunction with the vibration of elliptical membranes. McLachlan (Reference 21) has reviewed the general theory of Mathieu's equation, and its application to the analysis of loud speakers, frequency modulation, propagation in wave guides, oscillatory systems with periodic disturbances, etc. A point of interest is that the linear variational equation used in evaluating the infinitesimal stability of solutions to nonlinear differential equations frequently has the form of Mathieu's equation; see Stoker (Reference 22) and Cesari (Reference 14).

McLachlan gives the canonical form of Mathieu's equation as

$$\frac{d^2 y}{dz^2} + (a - 2q \cos 2z) y = 0 . \tag{2-8}$$

The substitutions

$$2z = t ,$$

$$a = 12(r_2 - r_1) ,$$

and

$$q = -18(r_2 - r_1) e$$

transform Equations 2-5 and 2-6 to

$$\frac{d^2 \phi}{dz^2} + (a - 2q \cos 2z) \phi = 8e \sin 2z \tag{2-9}$$

and

$$\frac{d^2 \phi'}{dz^2} + (a - 2q \cos 2z) \phi' = 24e(r_2 - r_1) \sin 2z . \tag{2-10}$$

The substitutions

$$2z = v ,$$

$$a = 12(r_2 - r_1) ,$$

and

$$2q = (a - 4) e$$

transform Equation 2-7 to

$$\frac{d^2 \Phi}{dz^2} + (a - 2q \cos 2z) \Phi = 8e \sin 2z . \tag{2-11}$$

The effects of the choices of independent variable and reference system could be determined even before these transformations were completed. A comparison of Equations 2-5 and 2-7 indicates that the choice of independent variable affects the periodic part of the homogeneous equation but not the forcing function. The homogeneous equations differ for two reasons: first, the dependent variable is expressed as a function of v instead of t, and second, the dependent variable is changed from ϕ to Φ by a time-varying transformation. Similarly, comparison of Equations 2-5 and 2-6 indicates that the choice of reference frames affects the forcing term but not the homogeneous equation. In Equation 2-5 the forcing term is a result of the acceleration of the reference system, while in 2-6 it is due to the torque which would exist on a perfectly oriented ($\phi' = 0$) spacecraft since the mean vertical is not a zero-torque axis.

When the eccentricity, e, is zero, the true and mean anomalies are identical and there is no difference in the formulas—Equations 2-9, 2-10, and 2-11—resulting from the three approaches. There is an interesting difference when $r_2 - r_1 = 1/3$, i.e., $a = 4$, for any non-zero value of e. The homogeneous part of Equations 2-9 and 2-10 have an unstable solution in the neighborhood of the line $a = 4$, as shown in the next section. However, Equation 2-11 is reduced to a constant coefficient harmonic oscillator when $a = 4$, since $q = 0$ in that case. The difference is caused not by the change of independent variables from t to v, but by the change of dependent variable from ϕ to Φ. The latter transformation removes, to the first order in e, the resonant term corresponding to $1 + e \cos v$ that causes Equations 2-9 and 2-10 to be unstable.

Stability

The complete solution of a general second-order, linear, nonhomogeneous differential equation can be written as the sum of two linearly independent solutions of the homogeneous form of the equation and any particular solution of the nonhomogeneous equation. The homogeneous part of Equations 2-9, 2-10, and 2-11 is identical with the canonical form of Mathieu's equations; thus the nature of the complementary solutions is well known.

There is a general result from Floquet theory (Struble, Reference 13) which says that any solution, y, of a linear differential equation with continuous periodic coefficients of period τ can be represented as

$$y(z) = P(z) \exp(\mu z)$$

where $P(z)$ is periodic with period τ. If $y(z)$ is a solution of Equation 2-8, so is $y(-z)$; thus the complete solution can be expressed as

$$y(z) = c_1 P(z) \exp(\mu z) + c_2 P(-z) \exp(-\mu z) . \tag{2-12}$$

It is apparent from Equation 2-12 that the solution to Equation 2-8 is stable if and only if μ is a pure imaginary number. The form of μ, real, imaginary, or complex, is a function of the parameters a and q. It is possible to plot the regions in which μ is real or imaginary as a function of these parameters; such a plot indicates regions in which the solutions to Equation 2-8 are bounded and those in which the solution is unbounded.

The stability of the complete solution of any of Equations 2-9, 2-10, or 2-11 requires that *both* the complementary solution and the particular solution of the equation are bounded. That is, for stable motion it is necessary but not sufficient that the parameters a and q yield a bounded solution to Equation 2-8.

The range of the parameters a and q for this application is restricted by the physical nature of the problem. Only spacecraft whose inertia range is in the Lagrange region will be considered; thus, $0 < r_1 < r_2 < 1$. The orbital eccentricity must lie in the range $0 \le e < 1$ for a closed orbit; however, it would be wrong to use these bounds in determining the range for q. All three formulations contain series expansions in which terms of second order and higher in e are dropped; this implies $e \ll 1$. If $e \ll 1$ is taken as $e < 0.1$, the range of q can be limited to $-1.8 < q \le 0$ for the first two formulations and $-0.2 < q < 0.4$ for the last one. A second limit exists, since when e exceeds $0.6627 \cdots$ the series developments for r and v used in formulating Equations 2-9 and 2-10 are not convergent, although the expansion for Equation 2-11 is not so restricted. Thus a range of $0 \le a \le 12$ for all three equations, $-12 < q \le 0$ for Equations 2-9 and 2-10, and $-2 < q < 4$ for Equation 2-11 completely covers the admissible range of parameters.

Figure 2-3 shows the regions of bounded and unbounded solutions, mapped from the numerical data in Reference 23.* The regions are symmetric about the a axis; thus it is sufficient to plot only the absolute value of q to represent the range determined above. In using this figure to evaluate the stability of the solutions of Equations 2-9, 2-10, or 2-11 it is necessary to convert the parameters $r_2 - r_1$ and e to the parameters a and q by the appropriate transformation. The conversion from $(r_2 - r_1, e)$ to (q, a) is not the same for Equation 2-11 as it is for Equations 2-9 and 2-10, and neither of the sets of conversions is linear in the variables $r_2 - r_1$ and e. These two facts make it difficult to "see" the effects of changes in the physical parameters $r_2 - r_1$ and e, and obscure the differences resulting from the choice of independent variables. To avoid these difficulties the regions of bounded and unbounded motion are also demarcated with respect to the physical parameters $r_2 - r_1$ and e in Figures 2-4 and 2-5. Figure 2-4 is applicable to both formulations that retain time as the independent variable, (i.e., Equations 2-9 and 2-10); Figure 2-5 corresponds to the choice of true anomaly for independent variable (i.e., Equation 2-11).

These figures differ in two respects: first, there is a region of unbounded motion beginning at $r_2 - r_1 = 1/3$, $e = 0$ in Figure 2-4 but not in Figure 2-5; second, the boundaries of the unstable regions beginning at $r_2 - r_1 = 1/12$, $e = 0$ and $r_2 - r_1 = 3/4$, $e = 0$ differ markedly between the two figures as e increases. In the previous section, the first of the two differences was attributed to the transformation $\Phi(v) = \phi(v)(1 + e \cos v)$ which removes, to the first order in e, the resonance term that produces this region of instability. The instability is, of course, still present, but it cannot be shown with a solution to the first order in e such as is determined herein. The second

*The lines on the figure should be thought of as lines of demarcation, for regions where a given value (q, a) is unstable.

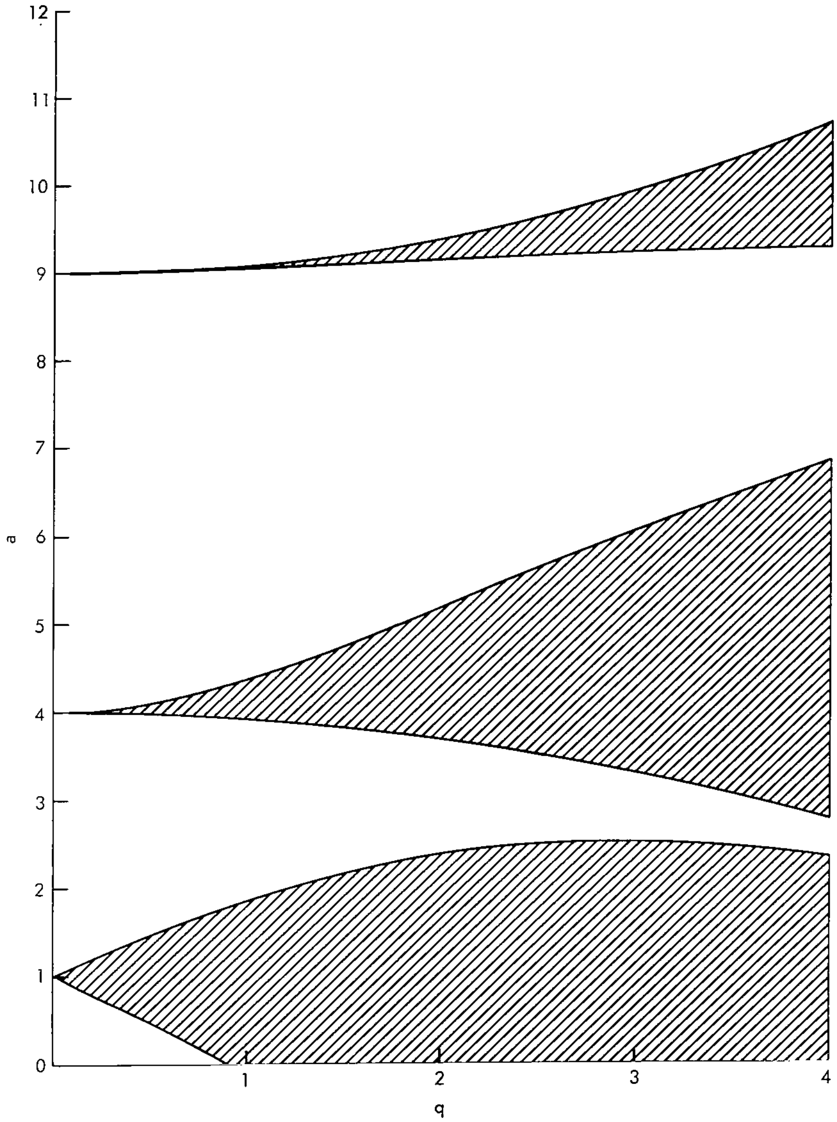

Figure 2-3—Regions of instability for Mathieu's equation.

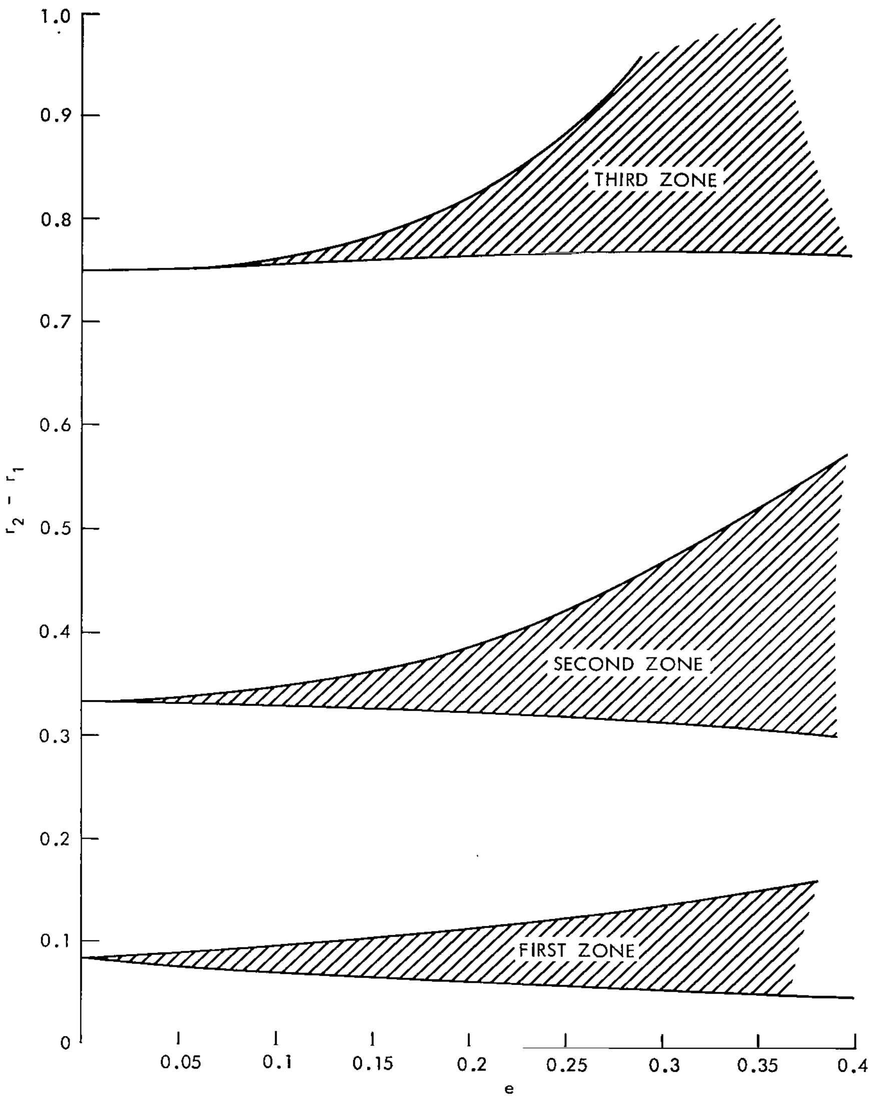

Figure 2-4—Regions of instability for Equations 2-9 and 2-10.

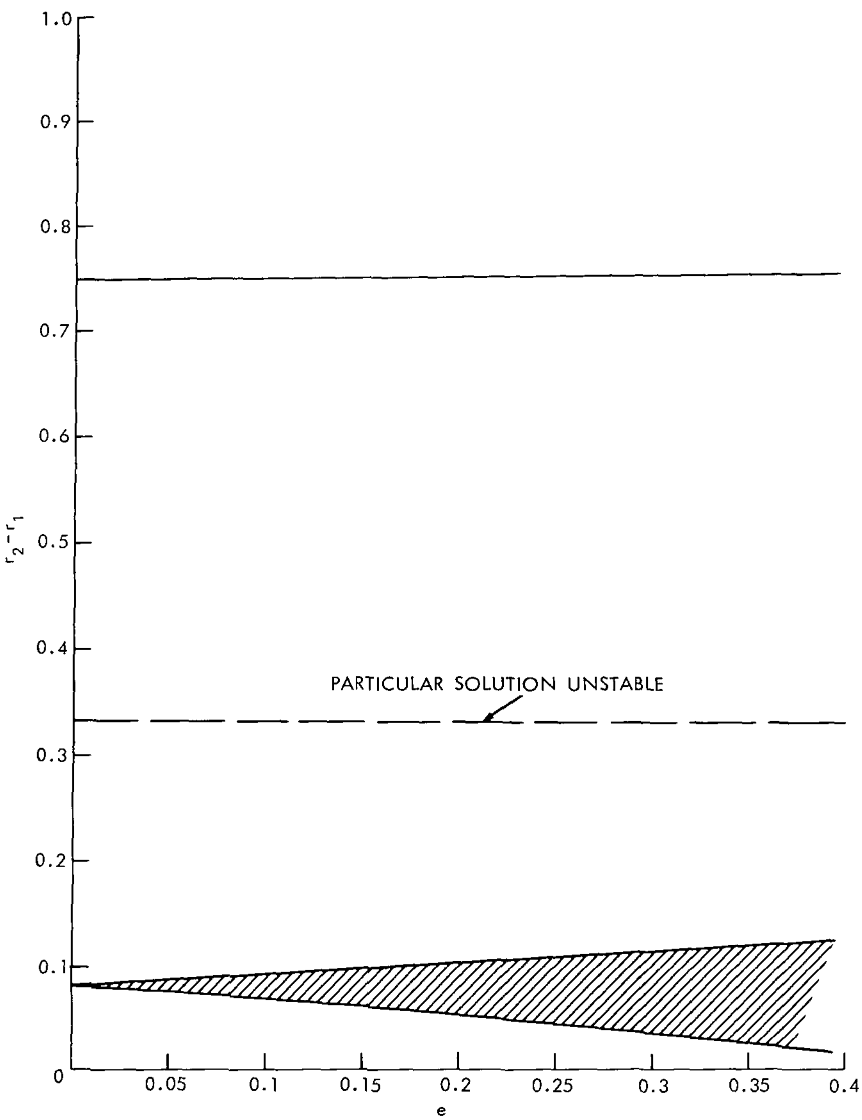

Figure 2-5—Regions of instability for Equation 2-11.

difference is a more obvious result of discarding higher-order terms. When the boundaries are compared for $e \ll 1$ they are quite similar.

It has already been noted that it is necessary to consider also the particular solutions of Equations 2-9 through 2-11 before deciding that a given combination of parameters is stable. The importance of this can be seen by considering the case $r_2 - r_1 = 1/3$, $0 < e < 1$, which is frequently called the "linear pitch resonance case." With these parameters, Equation 2-11 becomes

$$\frac{d^2\Phi}{dz^2} + 4\Phi = 8e \sin 2z \ ,$$

which is the equation of a linear oscillator forced at its resonant frequency. Thus the solution of Equation 2-11 is clearly unstable when $r_2 - r_1 = 1/3$ for any non-zero value of eccentricity. This is indicated by a dashed line in Figure 2-5. Particular solutions of Equations 2-9 and 2-10 may also be unstable in this region but, since the complementary solution is already known to be unbounded, it is not necessary to pursue this aspect of the problem.

Figures 2-4 and 2-5 and the above discussion show that there are three regions of unstable pitch oscillation for any non-zero eccentricity, however small. These regions reduce to values of $r_2 - r_1$ equal to 1/12, 1/3, and 3/4 as e approaches zero. When these values of $r_2 - r_1$ are substituted in the linearized equation for pitch motion in a circular orbit, it can be seen that $r_2 - r_1 = 1/12$ corresponds to a pitch libration with a natural frequency of one-half cycle per orbit. Similarly, $r_2 - r_1 = 1/3$ and $r_2 - r_1 = 3/4$ correspond to natural frequencies of one and three-halves cycles per orbit, respectively.

Damping

The question of stability can be determined for a rigid spacecraft by knowing whether the value of μ corresponding to a given set of parameters (q, a) is real or imaginary. A specific set of parameters actually determines the numerical value of μ and thus the rate of divergence of the pitch motion, when the point (q, a) is in an unstable zone. This rate of divergence becomes important when the effects of energy dissipation (i.e., damping) are considered.

A rigid spacecraft moving in an inverse-square gravitational force field is not subject to damping. In order to model damping correctly it would be necessary either to consider a multiple-body spacecraft or to include other sources of torque such as reaction with the magnetic field. This is beyond the scope of this dissertation; however an idealized velocity-dependent damping term can be inserted in Equation 2-1, without physical justification and solely to obtain a "feeling" for the effect of damping. Thus, we consider the equation

$$\frac{d^2\phi}{dt^2} + 2\kappa \frac{d\phi}{dt} + \frac{3\mu_e}{r^3}\left(r_2 - r_1\right)\phi = -\ddot{v} \ . \qquad (2\text{-}13)$$

When the expansions and simplifications of the second and third sections of this report are applied to Equation 2-13, the final result (which is analogous to Equation 2-9) becomes

$$\frac{d^2\phi}{dz^2} + 4\kappa\frac{d\phi}{dz} + (a - 2q\cos 2z)\phi = 8e\sin 2z \ , \tag{2-14}$$

where, as before,

$$\phi = \text{pitch angle} \ ,$$

$$z = 2t \ ,$$

$$a = 12\left(r_2 - r_1\right) \ ,$$

and

$$q = -18\left(r_2 - r_1\right)e \ .$$

The first derivative term can be removed from Equation 2-14 with the substitution

$$y(z) = \phi(z)\exp(2\kappa z) \tag{2-15}$$

which converts Equation 2-14 to

$$\frac{d^2 y}{dz^2} + \left(a - 4\kappa^2 - 2q\cos 2z\right)y = 8e\sin 2z \ . \tag{2-16}$$

In this case the stability of the complementary solution to Equation 2-16 can be ascertained from the nature of the region surrounding the point $\left(q, \ a - 4\kappa^2\right)$ in Figure 2-3. Once the solution $y(z)$ is known, the pitch motion is determined from the inverse of Equation 2-15 as

$$\phi(z) = y(z)\exp(-2\kappa z) \ .$$

Thus, if $y(z)$ is in a stable region, then the complementary pitch solution is asymptotically stable. However, when the complementary solution to Equation 2-16 is divergent, it does not necessarily follow that the pitch motion is also divergent. In this case the solution to Equation 2-15 has the form

$$\begin{aligned}\phi(z) &= \exp(-2\kappa z)\left[c_1 P(z)\exp(\mu z) + c_1 P(z)\exp(-\mu z)\right] \\ &= c_1 P(z)\exp\left[(\mu - 2\kappa)z\right] + c_2 P(-z)\exp\left[-(\mu + 2\kappa)z\right]\end{aligned}$$

The pitch motion is therefore asymptotically stable, with damping, whenever $|\mu| < 2\kappa$. Values of μ are mapped with regard to e and $\left[3(r_2 - r_1) - \kappa^2\right]^{1/2}$ in Figures 2-6, 2-7, and 2-8, which are based on Reference 24. These charts differ from Figure 2-4 in two respects; first, each chart is applicable only to a portion of the parameter space; second, the vertical coordinate represents the frequency of oscillation of the solution to Equation 2-13 for a circular orbit.

The process of determining stability for the complementary solution to Equation 2-14 can now be reduced to an automatic process. First, given the parameters r_1, r_2, e, and κ, find $\left[3(r_2 - r_1) - \kappa^2\right]^{1/2}$; then, determine the value of μ from the appropriate figure for the point $\left(e, \left[3(r_2 - r_1) - \kappa^2\right]^{1/2}\right)$. The complementary solution is asymptotically stable if $\mu < 2\kappa$, periodic if $\mu = 2\kappa$, divergent if $\mu > 2\kappa$. Thus the complementary solution to the in-plane pitch-motion equation can be asymptotically stable even for those combinations of e and $r_2 - r_1$ that are divergent according to Figure 2-4. Figures 2-6 through 2-8 show that μ increases directly with increasing e, and thus the amount of damping needed for stable motion also increases with increasing orbital eccentricity.

The complete solution to Equation 2-14 consists of the sum of the complementary solution and any particular solution of the nonhomogeneous equation. Although the above analysis indicates that all of the regions of instability previously found for this equation can be eliminated by damping, it does not guarantee that the particular solution introduces no instabilities. However, the particular solution to this equation is analogous to the forced motion of a damped harmonic oscillator, and, as is shown in the next chapter, the forced response (or particular solution) is a bounded, slightly perturbed, sinusoidal oscillation, except when $r_2 - r_1 = 1/3$.

Conclusions

The motion of a gravity-gradient-stabilized spacecraft in an elliptic orbit differs in kind from the motion of the same spacecraft in a circular orbit. It is not correct to assume that the only important effect of eccentricity is to add a forcing term to the constant-coefficient equations for a circular orbit. Actually, it is also necessary to include the periodic variation in the gravity-gradient restoring torque caused by the change in orbit radius with position for an elliptic orbit.

A rigid gravity-gradient spacecraft moving in a circular orbit has a stable position of equilibrium, when the principal axis of least inertia is aligned with the local vertical, and the principal axis of maximum inertia is aligned with the orbit normal. This can be expressed in the notation of this dissertation by a requirement that $0 < r_1 < r_2 < 1$; the portion of parameter space satisfying this requirement is called the Lagrange region. When an elliptic orbit is considered, this stability criterion is no longer sufficient. In fact, three areas of parameter space in the Lagrange region are unstable in pitch even for very small eccentricities. As eccentricity approaches zero, these three regions correspond to a spacecraft with a libration frequency for a circular orbit of one-half, one, or three-halves cycles per orbit, respectively.

The equation describing in-plane pitch motion can be developed in a number of ways. The formulations of Beletskii (Reference 8) and Baker (Reference 4) are compared with a third formulation that has time for an independent variable, as did Baker, but relates the motion to the local

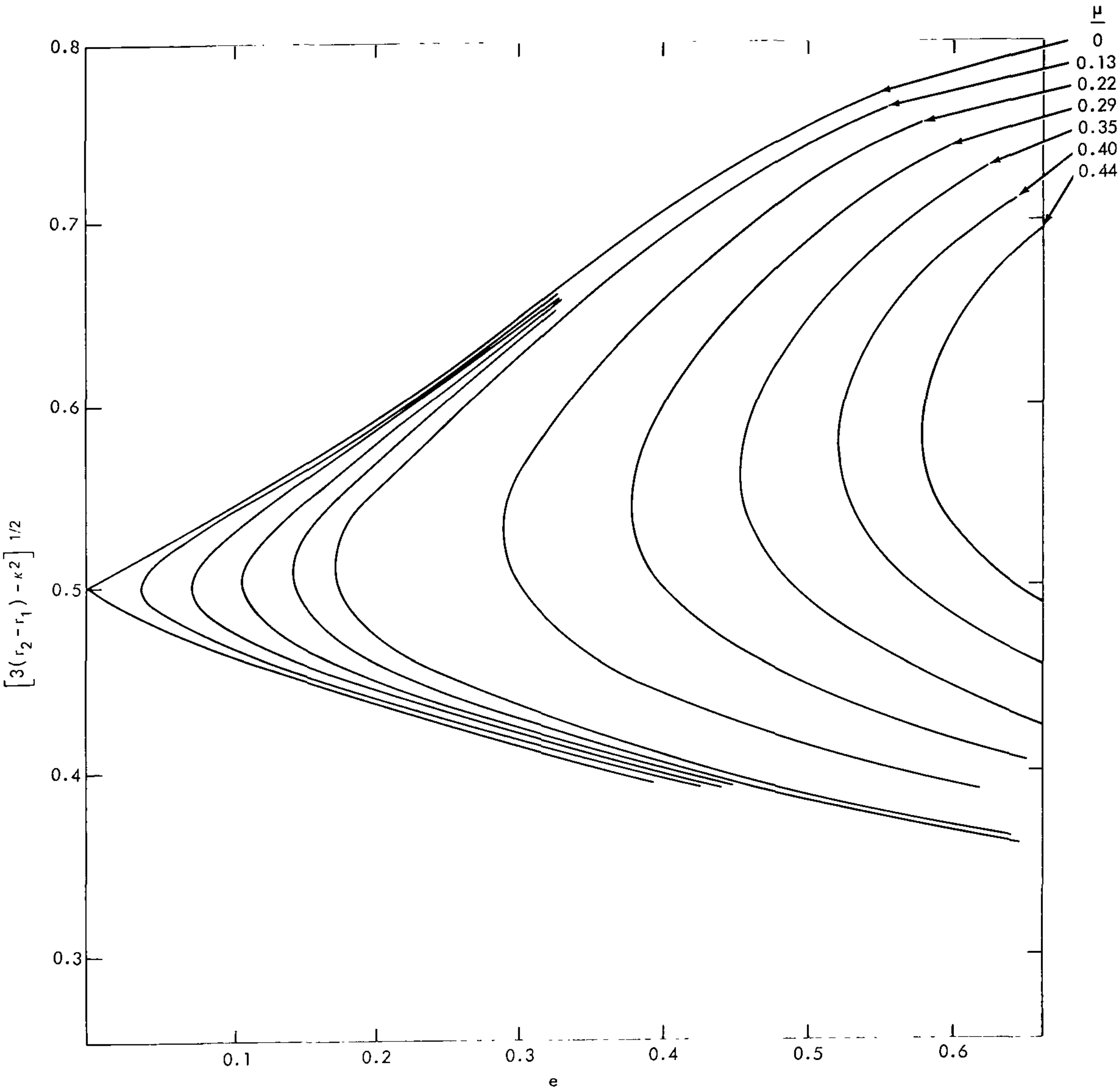

Figure 2-6—Rate of divergence for first unstable zone.

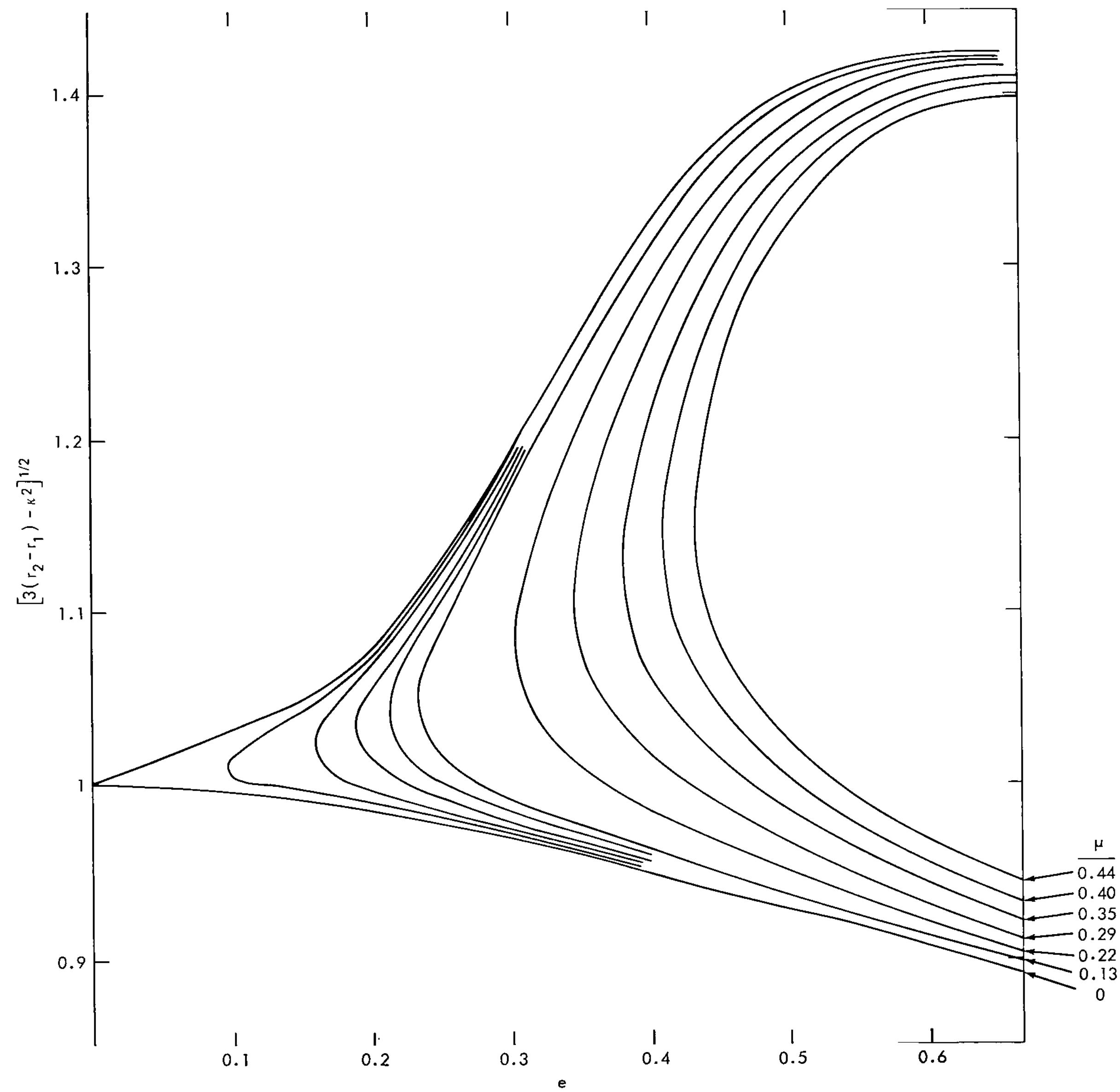

Figure 2-7—Rate of divergence for second unstable zone.

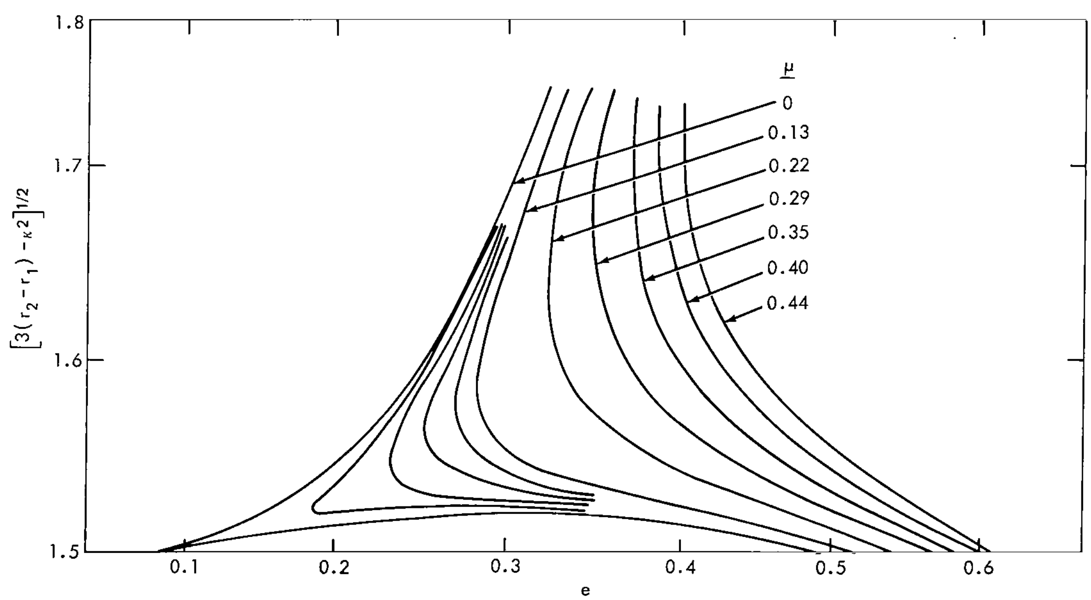

Figure 3-1—Regions of instability for asymptotic expansions.

vertical, as did Beletskii. The regions of unbounded pitch motion are mapped with regard to the inertia parameter, $r_2 - r_1$, and orbital eccentricity, e, for each of the formulations. All three approaches lead to equations that are unstable when $r_2 - r_1 = 1/12, 1/3$, and $3/4$, for all non-zero values of e. The unstable regions expand in size with increasing eccentricity; Figures 2-4 and 2-5 map the regions of unstable motion with regard to eccentricity, e, and the inertia difference, $r_2 - r_1$.

When the spacecraft parameters and orbital eccentricity are in one of the unstable regions, the pitch motion consists of exponentially increasing quasi-periodic oscillations. The rate of exponential growth, μ, is determined by the parameters $r_2 - r_1$ and e; Figures 2-6 through 2-8 plot lines of constant μ with regard to these parameters. Energy dissipation in the form of velocity-dependent damping of sufficient magnitude can suffice to eliminate all the instabilities discussed above. The magnitude of the damping coefficient, κ, required to stabilize the motion of a particular spacecraft increases directly with increasing values of e.

CHAPTER 3

PLANAR PITCH LIBRATION–ASYMPTOTIC EXPANSION THEORY

Introduction

There are several approaches to analyzing a motion when the equations describing it are not easily solved. The most common approach is to consider an equation which is "nearly the same as" the equation of interest but whose solution is known. The suitability of this approach is determined by the nature of the approximations required to reduce the original equation to the form of an equation with a known solution. In Chapter 2 it was necessary to discard terms of order e^2 and higher in the expansions of r and v as functions of t, and for the most part only the solution to the homogeneous portion of the equation was discussed.

A second approach is to obtain an approximate solution to the specific equation being studied. This chapter presents a solution to the in-plane pitch-libration problem in terms of asymptotic expansions in powers of a small parameter (e). This method permits inclusion of terms to any desired order in e (although it will be shown that inclusion of terms of order e^4 and higher does not introduce new phenomena), and leads to the development of a particular solution of the nonhomogeneous equation as well as inequalities defining regions where the solution of the homogeneous equation is unbounded.

An asymptotic solution to the in-plane pitch problem with true anomaly as the independent variable was developed through the second approximation by Beletskii (Reference 8). He obtained a particular solution of the nonhomogeneous equation, similar to the first terms of the solution found herein (subject to the difference in variables). He also noted the existence of the first resonant zone, and the linear resonance for the particular solution derived herein. However, in his formulation it is necessary to consider the third-order solution (which has not been done) to find the second and third resonant zones developed in this chapter.

Equation of Motion

This chapter, also, is concerned with the linearized equation representing in-plane pitch motion of a rigid gravity-gradient spacecraft moving in an elliptic orbit. The equation is

$$\frac{d^2 \phi}{dt^2} + 3\left(r_2 - r_1\right) \frac{\mu_e}{r^3} \phi = \frac{d^2 v}{dt^2} \qquad (3\text{-}1)$$

where

$$r = r(t)$$

and

$$v = v(t) .$$

Expansions of $(a/r)^3$ and v-M in series involving powers of e and the mean anomaly are given by Cayley (Reference 25) through terms of order e^7. The first terms in these expansions are:

$$\left(\frac{a}{r}\right)^3 = 1 + \frac{3e^2}{2} + \frac{15e^4}{8} + 3e \cos t + \frac{9e^2}{2} \cos 2t$$

$$+ e^3 \left[\frac{27}{8} \cos t + \frac{53}{8} \cos 3t\right] + e^4 \left[\frac{7}{2} \cos 2t + \frac{77}{8} \cos 4t\right]$$

and

$$v = M + 2e \sin t + \frac{5}{4} e^2 \sin 2t + e^3 \left[- \frac{1}{4} \sin t + \frac{13}{12} \sin 3t\right]$$

$$+ e^4 \left[- \frac{11}{24} \sin 2t + \frac{103}{96} \sin 4t\right] .$$

With the aid of these two expansions, and with the substitutions

$$\tilde{\alpha}^2 = 3\left(r_2 - r_1\right) ,$$

$$\omega_3^2 = 3\left(r_2 - r_1\right) \left[1 + \frac{3e^2}{2} + \frac{15e^4}{8}\right] ,$$

Equation 3-1 can be written as

$$\frac{d^2 \phi}{dt^2} + \left\{\omega_3^2 + \tilde{\omega}_3^2 \left[3e \cos t + \frac{9e^2}{2} \cos 2t + e^3 \left(\frac{27}{8} \cos t + \frac{53}{8} \cos 3t\right)\right.\right.$$

$$\left.\left. + e^4 \left(\frac{7}{2} \cos 2t + \frac{77}{8} \cos 4t\right)\right]\right\} \phi = 2e \sin t + 5e^2 \sin 2t$$

$$- \frac{e^3}{4} \sin t + \frac{39e^3}{4} \sin 3t - \frac{11e^4}{6} \sin 2t + \frac{103e^4}{6} \sin 4t \quad . \quad \textbf{(3-2)}$$

As in Chapter 2, this is a nonhomogeneous linear second-order differential equation with periodic coefficients. Its total solution is composed of any particular solution to the nonhomogeneous equation plus the complementary solution to the associated homogeneous equation.

Asymptotic Solutions

Although Poincaré wrote the initial discussion of asymptotic expansions in powers of a small parameter, the method used herein was developed by Krylov and Bogoliubov (Reference 26) and in more detail by Bogoliubov and Mitropolsky (Reference 27). Appendix C presents a particular set of formulas that specialize and extend the approach of Reference 28 to include third- and fourth-order terms for an equation of the form

$$\frac{d^2\phi}{dt^2} + \omega_3^2\phi = e\,f_1(\phi, t) + e^2 f_2(\phi, t) + e^3 f_3(\phi, t) + e^4 f_4(\phi, t) \ . \qquad (3\text{-}3)$$

The development is routine; readers interested in the underlying theory or a more detailed discussion should refer to Bogoliubov and Mitropolsky.

In essence the method reduces the solution of Equation 3-3 to a set of simpler problems. Specifically it seeks to find functions

$$u_i(a, \Psi, t)\ , \qquad A_i(a)\ , \qquad B_i(a)\ , \qquad i = 1, \cdots, n$$

such that

$$\phi = a\cos\Psi + e\,u_1(a, \Psi, t) + \cdots + e^m u_m(a, \Psi, t)\ , \qquad (3\text{-}4)$$

where

$$\frac{da}{dt} = e\,A_1(a) + \cdots + e^m A_m(a)$$

and

$$\frac{d\Psi}{dt} = \omega + e\,B_1(a) + \cdots + e^m B_m(a)$$

satisfy Equation 3-3 to order e^{m+1}. When there is a resonance between the perturbing expressions and the natural frequency ω of the unperturbed system, the problem is restated to find functions

$$u_i(a, \Theta, t)\ , \qquad A_i(a, \Theta)\ , \qquad B_i(a, \Theta)\ , \qquad i = 1, \cdots, m\ ,$$

such that

$$\phi = a\cos\left(\frac{pt}{q} + \Theta\right) + e\,u_1(a, \Theta, t) + \cdots + e^m u_m(a, \Theta, t), \tag{3-5}$$

where

$$\frac{da}{dt} = e\,A_1(a, \Theta) + \cdots + e^m A_m(a, \Theta)$$

and

$$\frac{d\Theta}{dt} = e\,B_1(a, \Theta) + \cdots + e^m B_m(a, \Theta)$$

satisfy Equation 3-2 to order e^{m+1} for parameters in which $\omega^2 \approx (p/q)^2$. These two formulations are referred to as the "nonresonance case" and the "resonance case," respectively.

The following section develops a solution to Equation 3-2 for the nonresonant case; it will serve as a particular solution to the nonhomogeneous equation. Examination of this solution will also reveal the values of ω at which resonance occurs. This is followed by a section that develops inequalities that determine when the resonant solution to the homogeneous equation associated with Equation 3-2 is unbounded. These inequalities are plotted in a form similar to that of Chapter 2.

A Particular Solution to the Pitch-Libration Problem

When Equation 3-2 is expressed in the form used in the method of asymptotic solutions, the result is

$$\frac{d^2\phi}{dt^2} + \omega_3^2\phi = e\,f_1(\phi, t) + e^2 f_2(\phi, t) + e^3 f_3(\phi, t) + e^4 f_4(\phi, t), \tag{3-6}$$

where

$$f_1(\phi, t) = 2\sin t - 3\tilde{\omega}^2(\cos t)\phi,$$

$$f_2(\phi, t) = 5\sin 2t - \frac{9\tilde{\omega}^2}{2}(\cos 2t)\phi,$$

$$f_3(\phi, t) = -\frac{1}{4}\sin t + \frac{39}{4}\sin 3t - \left(\frac{27\tilde{\omega}^2}{8}\cos t + \frac{53\tilde{\omega}^2}{8}\cos 3t\right)\phi,$$

and

$$f_4(\phi, t) = -\frac{11}{6}\sin 2t + \frac{103}{6}\sin 4t - \left(\frac{7\tilde{\omega}^2}{2}\cos 2t + \frac{77\tilde{\omega}^2}{8}\cos 4t\right)\phi .$$

The formulas which are used in obtaining asymptotic solutions to equations of this type are developed in Appendix C. In essence Equation 3-4 is substituted in both sides of Equation 3-6 and terms of the same magnitude in ϵ are collected. The first order terms give

$$\omega^2 \frac{\partial^2 u_1}{\partial \Psi^2} + 2\omega \frac{\partial^2 u_1}{\partial \Psi \partial t} + \frac{\partial^2 u_1}{\partial t^2} + \omega^2 u_1 = F_1 + 2\omega A_1 \sin\Psi + 2a\omega B_1 \cos\Psi$$

where

$$F_1 = f_1(a\cos\Psi, t)$$

and as explained in the appendix this equation can be solved for A_1, B_1, and u_1. These results can then be used to expand similar equations for F_2, A_2, B_2 and u_2 and the process can be continued to any desired order. The computations are rather tedious and only a few of the intermediate results are included in the remainder of this section.

In the first approximation,

$$F_1 = 2\sin t - \frac{3a\tilde{\omega}^2}{2}\left[\cos(t+\Psi) + \cos(t-\Psi)\right] ,$$

$$A_1(a) = 0 ,$$

$$B_1(a) = 0 ,$$

and

$$u_1(a, \Psi, t) = \frac{2\sin t}{\omega_3^2 - 1} + \frac{3a\tilde{\omega}^2}{2}\left[\frac{\cos(t+\Psi)}{1+2\omega_3} + \frac{\cos(t-\Psi)}{1-2\omega_3}\right] .$$

The first-order solution to Equation 3-6 is

$$\phi = a\cos\Psi , \tag{3-7}$$

where

$$\frac{da}{dt} = 0 ,$$

$$\frac{d\Psi}{dt} = \omega_3 .$$

In the second approximation

$$F_2 = \left[5 - \frac{3\tilde{\alpha}^2}{\omega_3^2 - 1}\right] \sin 2t - \frac{9a\tilde{\alpha}^4}{2\left(1 - 4\omega_3^2\right)} \cos\Psi - \frac{9a\tilde{\alpha}^2}{4}\left[\left(1 + \frac{\tilde{\omega}^2}{1 + 2\omega_3}\right)\cos(2t + \Psi) + \left(1 + \frac{\tilde{\alpha}^2}{1 - 2\omega_3}\right)\cos(2t - \Psi)\right] ,$$

$$A_2 = 0 ,$$

$$B_2 = \frac{9\tilde{\alpha}^4}{4\omega_3\left(1 - 4\omega_3^2\right)}$$

and

$$u_2 = \left[5 - \frac{3\tilde{\alpha}^2}{\omega_3^2 - 4}\right]\frac{\sin 2t}{\omega_3^2 - 4} + \frac{9a\tilde{\alpha}^2}{16}\left[\left(1 + \frac{\tilde{\alpha}^2}{1 + 2\omega_3}\right)\frac{\cos(2t - \Psi)}{1 + \omega_3} + \left(1 + \frac{\tilde{\omega}^2}{1 - 2\omega_3}\right)\frac{\cos(2t - \Psi)}{1 - \omega_3}\right] .$$

The second-order solution to Equation 3-6 is thus

$$\phi = a\cos\Psi + \frac{3ae\tilde{\alpha}^2}{2}\left[\frac{\cos(t + \Psi)}{1 + 2\omega_3} + \frac{\cos(t - \Psi)}{1 - 2\omega_3}\right] + \frac{2e}{\omega_3^2 - 1}\sin t , \tag{3-8}$$

where

$$\frac{da}{dt} = 0$$

and

$$\frac{d\Psi}{dt} = \omega_3 + \frac{9e^2\tilde{\alpha}^4}{4\omega_3\left(1 - 4\omega_3^2\right)} .$$

In the third approximation

$$F_3 = g_1 \sin t + g_2 \sin 3t - a\left[g_3 \cos(t+\Psi) + g_4 \cos(t-\Psi) + g_5 \cos(3t+\Psi) + g_6 \cos(3t-\Psi)\right],$$

$$A_3 = 0,$$

$$B_3 = 0,$$

and

$$u_3 = \frac{g_1 \sin t}{\omega_3^2 - 1} + \frac{g_2 \sin 3t}{\omega_3^2 - 9} + a\left[\frac{g_3 \cos(t+\Psi)}{1+2\omega_3} + \frac{g_4 \cos(t-\Psi)}{1-2\omega_3} + \frac{g_5 \cos(3t+\Psi)}{3(3+2\omega_3)} + \frac{g_6 \cos(3t-\Psi)}{3(3-2\omega_3)}\right],$$

where

$$g_1 = -\frac{1}{4} - \frac{9\tilde{\omega}^2}{2(\omega_3^2-1)} - \frac{3\tilde{\omega}^2}{2(\omega_3^2-4)}\left[5 - \frac{3\tilde{\omega}^2}{\omega_3^2-1}\right],$$

$$g_2 = \frac{39}{4} - \frac{9\tilde{\omega}^2}{2(\omega_3^2-1)} - \frac{3\tilde{\omega}^2}{2(\omega_3^2-4)}\left[5 - \frac{3\tilde{\omega}^2}{\omega_3^2-1}\right],$$

$$g_3 = \frac{27\tilde{\omega}^2}{16} + \frac{27\tilde{\omega}^4}{32(1+\omega_3)}\left(1 + \frac{\tilde{\omega}^2}{1+2\omega_3}\right) + \frac{27\tilde{\omega}^4}{8(1-2\omega_3)} - \frac{27\tilde{\omega}^6}{4\omega_3(1-4\omega_3^2)}\left(\frac{\omega_3+1}{1+2\omega_3}\right),$$

$$g_4 = \frac{27\tilde{\omega}^2}{16} + \frac{27\tilde{\omega}^4}{32(1-\omega_3)}\left(1 + \frac{\tilde{\omega}^2}{1-2\omega_3}\right) + \frac{27\tilde{\omega}^4}{8(1+2\omega_3)} - \frac{27\tilde{\omega}^6}{4\omega_3(1-4\omega_3^2)}\left(\frac{\omega_3-1}{1-2\omega_3}\right),$$

$$g_5 = \frac{53\tilde{\omega}^2}{16} + \frac{27\tilde{\omega}^4}{32(1+\omega_3)}\left(1 + \frac{\tilde{\omega}^2}{1+2\omega_3}\right) + \frac{27\tilde{\omega}^4}{8(1+2\omega_3)},$$

$$g_6 = \frac{53\tilde{\omega}^2}{16} + \frac{27\tilde{\omega}^4}{32(1-\omega_3)}\left(1 + \frac{\tilde{\omega}^2}{1-2\omega_3}\right) + \frac{27\tilde{\omega}^4}{8(1-2\omega_3)}.$$

The third-order solution to Equation 3-6 becomes

$$\phi = a\cos\Psi + \frac{3ae\tilde{\omega}^2}{2}\left[\frac{\cos(t+\Psi)}{1+2\omega_3} + \frac{\cos(t-\Psi)}{1-2\omega_3}\right]$$

$$+ \frac{9ae^2\tilde{\omega}^2}{16}\left[\left(1 + \frac{\tilde{\omega}^2}{1+2\omega_3}\right)\frac{\cos(2t+\Psi)}{1+\omega_3} + \left(1 + \frac{\tilde{\omega}^2}{1-2\omega_3}\right)\frac{\cos(2t-\Psi)}{(1-\omega_3)} + \frac{2e}{\omega_3^2-1}\sin t + \frac{e^2}{\omega_3^2-4}\left(5 - \frac{3\tilde{\omega}^2}{\omega_3^2-1}\right)\sin 2t\right], \quad (3\text{-}9)$$

where

$$\frac{da}{dt} = 0$$

and

$$\frac{d\Psi}{dt} = \omega_3 + \frac{9e^2 \tilde{\alpha}^4}{4\omega_3 \left(1 - 4\omega_3^2\right)} .$$

As the order increases it becomes harder to find a solution. However, examination of each of the three preceeding solutions—Equations 3-7, 3-8, and 3-9—indicates that each new approximation introduces a significant new result. In particular, u_1 has an unbounded term when $\omega_3 = 1/2$, u_2 when $\omega_3 = 1$, and u_3 when $\omega_3 = 3/2$. However, further approximations do not produce further meaningful discrepancies. In particular, the fourth approximation introduces new terms of the form

$$u_4 = ae^4 \left[\frac{h_1 \cos(4t + \Psi)}{8(2 + \omega_3)} + \frac{h_2 \cos(4t - \Psi)}{8(2 - \omega_3)}\right] + \frac{e^4 h_3}{\omega_3^2 - 16} \sin 4t$$

+ fourth-order corrections to terms found before,

and

$$A_4 = 0 ,$$

$$B_4 = e^4 h_4 ,$$

where h_i $(i = 1, \cdots, 4)$ are constants that could be evaluated with Appendix B formulas. However, since $0 < \omega_3^2 < 3$ for small e, the terms in u_4 are bounded and there is no need to consider them further.

Instead, return to the third-order approximations. Since $da/dt = 0$ for all ω, a particular solution to Equation 3-6 is

$$\phi = \frac{2e}{\omega_3^2 - 1} \sin t + \frac{e^2}{\omega_3^2 - 4} \left(5 - \frac{3\tilde{\omega}^2}{\omega_3^2 - 1}\right) \sin 2t + e^3 \left(\frac{g_1}{\omega_3^2 - 1} \sin t + \frac{g_2}{\omega_3^2 - 9} \sin 3t\right) . \qquad (3\text{-}10)$$

This solution represents the motion of a spacecraft for the particular case in which $a_0 = 0$. It also approximately represents the forced motion of a rigid spacecraft in an eccentric orbit when "light damping" is present. Equation 3-10 is a particular solution to the nonhomogeneous Equation 3-6. In the next section the solution to the homogeneous equation associated with Equation 3-6 will be found for each of the regions $\omega_3 \approx 1/2$, $\omega_3 \approx 1$, and $\omega_3 \approx 3/2$. The combination of these complementary solutions and the particular solution, Equation 3-10 is sufficient to determine completely the nature of the solution to Equation 3-6 in the region of interest.

Regions of Resonant Pitch Motion

An asymptotic solution to the in-plane pitch-libration problem was developed for the non-resonance case in the previous section. This solution represented unstable motion for $\omega_3 = 1/2$, 1, or 3/2, which indicates that a resonance-case type of solution such as Equation 3-5 should be used in these regions. The process is quite similar to that of the preceding section, in that solutions are built up with terms of increasing order of approximation.

It was shown in the previous section that the first three approximations introduce all the resonant phenomena of interest for this problem. Fourth-order terms are therefore ignored in this dissertation.

The homogeneous part of Equation 3-2 can be written to the third order in e as

$$\frac{d^2 \phi}{dt^2} + \omega_3^2 \phi = e f_1(\phi, t) + e^2 f_2(\phi, t) + e^3 f_3(\phi, t) , \tag{3-11}$$

where

$$f_1 = -3\tilde{\omega}^2 (\cos t) \phi ,$$

$$f_2 = -\frac{9\tilde{\omega}^2}{2} (\cos 2t) \phi ,$$

and

$$f_3 = -\frac{27\tilde{\omega}^2}{8} (\cos t) \phi - \frac{53\tilde{\omega}^2}{8} (\cos 3t) \phi .$$

Solutions of Equation 3-11 will be developed for $\omega_3 \approx 1/2$, 1, and 3/2; as before, most of the algebra will be omitted. The first series of solutions will assume $\omega_3 \approx 1/2$, and define $\omega_3^2 - 1/4 = e\Delta$. In the first approximation,

$$F_1 = \frac{-3a\tilde{\omega}^2}{2} \left[\cos(t + \Psi) + \cos(t - \Psi)\right] ,$$

$$A_1 = \frac{3\tilde{\omega}^2}{2} a \sin 2\Theta ,$$

$$B_1 = \frac{3\tilde{\omega}^2}{2} \cos 2\Theta + \Delta ,$$

and

$$u_1 = \frac{3a\tilde{\omega}^2}{4} \cos (t + \Psi) .$$

Then, in the first approximation,

$$\phi = a \cos\left(\frac{t}{2} + \Theta\right) , \qquad (3\text{-}12)$$

where

$$\frac{da}{dt} = \frac{3e\tilde{\omega}^2}{2} a \sin 2\Theta , \qquad (3\text{-}13)$$

$$\frac{d\Theta}{dt} = \frac{3e\tilde{\omega}^2}{2} \cos 2\Theta + \omega_3^2 - \frac{1}{4} . \qquad (3\text{-}14)$$

and a is not a constant. When new variables $u = a\cos\Theta$, $v = a\sin\Theta$ are introduced into Equation 3-13 and 3-14, the system

$$\frac{du}{dt} = \left[\frac{3e\tilde{\omega}^2}{2} - \left(\omega_3^2 - \frac{1}{4}\right)\right] v$$

$$\frac{dv}{dt} = \left[\frac{3e\tilde{\omega}^2}{2} + \left(\omega_3^2 - \frac{1}{4}\right)\right] u$$

results. This can be reduced to either

$$\frac{d^2 u}{dt^2} - \left[\left(\frac{3e\tilde{\omega}^2}{2}\right)^2 - \left(\omega_3^2 - \frac{1}{4}\right)^2\right] u = 0 ,$$

or

$$\frac{d^2 v}{dt^2} - \left[\left(\frac{3e\tilde{\omega}^2}{2}\right)^2 - \left(\omega_3^2 - \frac{1}{4}\right)^2\right] v = 0 .$$

Both u and v are represented by unbounded oscillations when

$$\left|\omega_3^2 - \frac{1}{4}\right| < \left|\frac{3e\tilde{\alpha}^2}{2}\right| , \qquad (3\text{-}15)$$

and, since $a = \left[u^2 + v^2\right]^{1/2}$, inequality 3-15 is also a condition for unbounded oscillation of ϕ.

A more exact boundary can be found by considering the second approximation, in which

$$F_2 = -\frac{9a\tilde{\omega}^4}{8} \cos \Psi + \frac{3a\Delta\tilde{\omega}^2}{2} \cos (t + \Psi) - \frac{9a\tilde{\omega}^2}{4} \left[\left(1 + \frac{\tilde{\omega}^2}{2}\right) \cos (2t + \Psi) + \left(1 - \frac{3\tilde{\omega}^2}{2}\right) \cos (2t + \Psi)\right] ,$$

$$A_2 = 0 ,$$

$$B_2 = -\Delta^2 + \frac{27\tilde{\alpha}^4}{8} ,$$

and

$$\phi = a \cos\left(\frac{t}{2} + \Theta\right) + \frac{3ea\tilde{\alpha}^2}{4} \cos \left(\frac{3t}{2} + \Theta\right) , \qquad (3\text{-}16)$$

where

$$\frac{da}{dt} = \frac{3e\tilde{\omega}^2}{2} a \sin 2\Theta ,$$

$$\frac{d\Theta}{dt} = \frac{3e\tilde{\alpha}^2}{2} \cos 2\Theta + \left(\alpha_3^2 - \frac{1}{4}\right) - \left(\alpha_3^2 - \frac{1}{4}\right)^2 + \frac{27e^2\tilde{\alpha}^4}{8} .$$

The same process that was used in the first approximation indicates that Equation 3-16 represents unbounded motion whenever

$$\left|\omega_3^2 - \frac{1}{4} - \left(\omega_3^2 - \frac{1}{4}\right)^2 + \frac{27e^2\tilde{\omega}^4}{8}\right| < \left|\frac{3e\tilde{\omega}^2}{2}\right| . \qquad (3\text{-}17)$$

It is possible to refine the boundary of the unstable region near $\omega_3 = 1/2$ still further by considering higher-order approximations, but the gain is small.

The unbounded motion for $\omega_3 \approx 1$ appeared in the second-order approximation (Equation 3-8) to Equation 3-2 and will likewise only be apparent in the second-order resonant solution. In this case

ω_3 is nearly one; i.e., $e\Delta = \omega_3^2 - 1$, and, in the first approximation.

$$F_1 = -\frac{3a\tilde{\omega}^2}{2}\left[\cos(t+\Psi) + \cos(t-\Psi)\right] .$$

$$A_1 = 0 ,$$

$$B_1 = \frac{\Delta}{2} ,$$

$$u_1 = \frac{a\tilde{\omega}^2}{2}\cos(t+\Psi) - \frac{3a\tilde{\omega}^2}{2}\cos(t-\Psi) ,$$

and

$$\phi = a\cos(t+\Theta) ,$$

where

$$\frac{da}{dt} = 0 ,$$

$$\frac{d\Theta}{dt} = \frac{\omega_3^2 - 1}{2} .$$

Thus, in the first approximation the amplitude of the oscillations remains constant. This is not the case when the second approximation is considered, for

$$F_2 = \frac{3a\tilde{\omega}^4}{2}\cos\Psi + \frac{a\Delta\tilde{\omega}^2}{2}\cos(t+\Psi) + \frac{3a\Delta\tilde{\omega}^2}{2}\cos(t-\Psi)$$

$$- \frac{9a\tilde{\omega}^2}{4}\left(1 + \frac{\tilde{\omega}^2}{3}\right)\cos(2t+\Psi) - \frac{9a\tilde{\omega}^2}{4}\left(1 - \tilde{\omega}^2\right)\cos(2t-\Psi) ,$$

$$A_2 = \frac{9\tilde{\omega}^2}{8}\left(1 - \tilde{\omega}^2\right) a \sin 2\Theta ,$$

$$B_2 = \frac{9\tilde{\omega}^2}{8}\left(1 - \tilde{\omega}^2\right)\cos 2\Theta - \frac{\Delta^2}{8} + \frac{3\tilde{\omega}^4}{4} ,$$

$$u_2 = -\frac{a\Delta\tilde{\omega}^2}{6}\cos(t+\Psi) + \frac{3a\Delta\tilde{\omega}^2}{2}\cos(t-\Psi) + \frac{9a\tilde{\omega}^2}{32}\left(1 + \frac{\tilde{\omega}^2}{3}\right)\cos(2t+\Psi) ,$$

and

$$\phi = a \cos (t + \Theta) + \frac{a e \tilde{\omega}^2}{2} \cos (2t + \Theta) , \tag{3-18}$$

where

$$\frac{da}{dt} = \frac{9e^2 \tilde{\omega}^2}{8} \left(1 - \tilde{\omega}^2\right) a \sin 2\Theta ,$$

$$\frac{d\Theta}{dt} = \frac{\omega_3^2 - 1}{2} - \frac{\left(\omega_3^2 - 1\right)^2}{8} - \frac{3e^2 \tilde{\omega}^4}{4} + \frac{9e^2 \tilde{\omega}^2}{8} \left(1 - \tilde{\omega}^2\right) \cos 2\Theta .$$

This set of equations represents unbounded motion if

$$\left| \frac{\omega_3^2 - 1}{2} - \frac{\left(\omega_3^2 - 1\right)^2}{8} - \frac{3e^2 \tilde{\omega}^4}{4} \right| < \left| \frac{9e^2 \tilde{\omega}^2}{8} \left(1 - \tilde{\omega}^2\right) \right| . \tag{3-19}$$

This boundary can also be more precisely defined by considering the third approximation, in which

$$\begin{aligned} F_3 = & - 2a\Delta\tilde{\omega}^4 \cos \Psi - \left[\frac{27a\tilde{\omega}^2}{16} \left(1 + 2\tilde{\omega}^2\right) + \frac{9a\tilde{\omega}^4}{64} \left(3 + \tilde{\omega}^2\right) + \frac{7a\Delta^2 \tilde{\omega}^2}{24} + \frac{3a\tilde{\omega}^6}{2}\right] \cos (t + \Psi) \\ & - \left[\frac{27a\tilde{\omega}^2}{16} + \frac{9a\tilde{\omega}^4}{8} + \frac{15a\Delta^2 \tilde{\omega}^2}{8}\right] \cos (t - \Psi) + \left[\frac{a\Delta\tilde{\omega}^4}{4} + \frac{3a\Delta\tilde{\omega}^2}{16} \left(3 + \tilde{\omega}^2\right)\right] \cos (2t + \Psi) \\ & - \frac{9a\Delta\tilde{\omega}^4}{4} \cos (2t - \Psi) , \end{aligned}$$

$$A_3 = \frac{9\Delta\tilde{\omega}^4}{8} a \sin 2\Theta ,$$

$$B_3 = \frac{\Delta^3}{16} + \frac{11\Delta\tilde{\omega}^4}{8} + \frac{9\Delta\tilde{\omega}^4}{8} \cos 2\Theta ,$$

and

$$\phi = a \cos (t + \Theta) + \frac{3ae\tilde{\omega}^2}{2} \left(\omega_3^2 - 1\right) \cos \Theta + \frac{ae\tilde{\omega}^2}{6} \left(4 - \omega_3^2\right) \cos (2t + \Theta) + \frac{9ae^2 \tilde{\omega}^2}{32} \left(1 + \frac{\tilde{\omega}^2}{3}\right) \cos (3t + \Theta), \tag{3-20}$$

where

$$\frac{da}{dt} = \frac{9e^2\tilde{\omega}^2}{8}\left[1 - \tilde{\omega}^2 + \tilde{\omega}^2\left(\omega_3^2 - 1\right)\right] a \sin 2\Theta ,$$

$$\frac{d\Theta}{dt} = \frac{9e^2\tilde{\omega}^2}{8}\left[1 - \tilde{\omega}^2 + \tilde{\omega}^2\left(\omega_3^2 - 1\right)\right] \cos 2\Theta + \frac{\omega_3^2 - 1}{2} - \frac{\left(\omega_3^2 - 1\right)^2}{8} + \frac{\left(\omega_3^2 - 1\right)^3}{16} - \frac{3e^2\tilde{\omega}^4}{4} + \frac{11e^2\tilde{\omega}^4}{8}\left(\omega_3^2 - 1\right) .$$

The motion represented by Equation 3-20 in the region $\omega_3 \approx 1$ is unbounded when

$$\left|\frac{\omega_3^2 - 1}{2} - \frac{\left(\omega_3^2 - 1\right)^2}{8} + \frac{\left(\omega_3^2 - 1\right)^3}{16} - \frac{3e^2\tilde{\omega}^4}{4} + \frac{11e^2\tilde{\omega}^4}{8}\left(\omega_3^2 - 1\right)\right|$$

$$< \left|\frac{9e^2\tilde{\omega}^2}{8}\left[1 - \tilde{\omega}^2 + \tilde{\omega}^2\left(\omega_3^2 - 1\right)\right]\right| . \qquad (3\text{-}21)$$

The last region to be considered is in the vicinity $\omega_3 \approx 3/2$. The existence of this unbounded motion was indicated only in the third-order approximation solution, Equation 3-9, and that is also true when the resonant solution is developed. In this case $e\Delta \equiv \omega_3^2 - 9/4$ and, in the first approximation,

$$F_1 = -\frac{3a\tilde{\omega}^2}{2}\left[\cos\left(t + \Psi\right) + \cos\left(t - \Psi\right)\right] ,$$

$$A_1 = 0 ,$$

$$B_1 = \frac{\Delta}{3} ,$$

$$u_1 = \frac{3a\tilde{\omega}^2}{8} \cos\left(t + \Psi\right) - \frac{3a\tilde{\omega}^2}{4} \cos\left(t - \Psi\right) ,$$

and

$$\phi = a \cos\left(\frac{3t}{2} + \Theta\right) ,$$

where

$$\frac{da}{dt} = 0 ,$$

$$\frac{d\Theta}{dt} = \frac{\omega_3^2 - \frac{9}{4}}{3} .$$

In the second approximation,

$$F_2 = \frac{9a\tilde{\omega}^4}{16}\cos\Psi + \frac{a\Delta\tilde{\omega}^2}{4}\cos(t+\Psi) + \frac{a\Delta\tilde{\omega}^2}{2}\cos(t-\Psi)$$

$$- \frac{9a\tilde{\omega}^4}{4}\left(1+\frac{\tilde{\omega}^2}{4}\right)\cos(2t+\Psi) - \frac{9a\tilde{\omega}^4}{4}\left(1-\frac{\tilde{\omega}^2}{2}\right)\cos(2t+\Psi) .$$

$$A_2 = 0 ,$$

$$B_2 = -\frac{\Delta^2}{27} - \frac{3\tilde{\omega}^4}{16} ,$$

$$u_2 = -\frac{a\Delta\tilde{\omega}^2}{16}\cos(t+\Psi) + \frac{a\Delta\tilde{\omega}^2}{4}\cos(t-\Psi) + \frac{9a\tilde{\omega}^2}{40}\left(1+\frac{\tilde{\omega}^2}{4}\right)\cos(2t+\Psi)$$

$$- \frac{9a\tilde{\omega}^2}{8}\left(1-\frac{\tilde{\omega}^2}{2}\right)\cos(2t-\Psi) .$$

and

$$x = a\cos\left(\frac{3t}{2}+\Theta\right) - \frac{3ae\tilde{\omega}^2}{8}\left[2\cos\left(\frac{t}{2}+\Theta\right) - \cos\left(\frac{5t}{2}+\Theta\right)\right] ,$$

where

$$\frac{da}{dt} = 0$$

and

$$\frac{d\Theta}{dt} = \frac{\omega_3^2 - \frac{9}{4}}{3} - \frac{\left(\omega_3^2 - \frac{9}{4}\right)^2}{27} - \frac{3e^2\tilde{\omega}^4}{16} .$$

Thus even in the second approximation there is no sign of the resonance. However, in the third approximation,

$$
\begin{aligned}
F_3 = & -\frac{9a\Delta\tilde{\omega}^4}{32}\cos\Psi - \left[\frac{27a\tilde{\omega}^2}{16}\left(1-\tilde{\omega}^2\right) + \frac{27a\tilde{\omega}^4}{80}\left(1+\frac{\tilde{\omega}^2}{4}\right) + \frac{5a\Delta^2\tilde{\omega}^2}{72}\right. \\
& \left. + \frac{45a\tilde{\omega}^6}{128}\right]\cos(t+\Psi) - \left[\frac{27a\tilde{\omega}^2}{16}\left(1+\frac{\tilde{\omega}^2}{2}\right) - \frac{27a\tilde{\omega}^4}{16}\left(1-\frac{\tilde{\omega}^2}{2}\right) - \frac{2a\Delta^2\tilde{\omega}^2}{9}\right. \\
& \left. - \frac{9a\tilde{\omega}^4}{64}\right]\cos(t-\Psi) + \left[\frac{3a\Delta\tilde{\omega}^2}{10}\left(1+\frac{\tilde{\omega}^2}{4}\right) + \frac{3a\Delta\tilde{\omega}^4}{32}\right]\cos(2t+\Psi) \\
& - \frac{3a\Delta\tilde{\omega}^2}{4}\left(1+\tilde{\omega}^2\right)\cos(2t-\Psi) - \left[\frac{53a\tilde{\omega}^2}{16} + \frac{27a\tilde{\omega}^4}{80}\left(\frac{7}{2}+\frac{\tilde{\omega}^2}{4}\right)\right]\cos(3t+\Psi) \\
& - \left[\frac{53a\tilde{\omega}^2}{16} - \frac{27a\tilde{\omega}^4}{8}\left(1-\frac{\tilde{\omega}^2}{4}\right)\right]\cos(3t-\Psi) ,
\end{aligned}
$$

$$
A_3 = \left[\frac{53\tilde{\omega}^2}{48} - \frac{9\tilde{\omega}^4}{8}\left(1-\frac{\tilde{\omega}^2}{4}\right)\right] a \sin 2\Theta ,
$$

$$
B_3 = \frac{2\Delta^3}{243} + \frac{13\Delta\tilde{\omega}^4}{96} + \left[\frac{53\tilde{\omega}^2}{48} - \frac{9\tilde{\omega}^4}{8}\left(1-\frac{\tilde{\omega}^2}{4}\right)\right]\cos 2\Theta ,
$$

and

$$
\begin{aligned}
\phi = & \; a\cos\left(\frac{3t}{2}+\Theta\right) - \frac{ae\tilde{\omega}^2}{4}\left[3-\left(\omega_3^2-\frac{9}{4}\right)\right]\cos\left(\frac{t}{2}+\Theta\right) + \frac{ae\tilde{\omega}^2}{16}\left[6-\left(\omega_3^2-\frac{9}{4}\right)\right] \\
& \cos\left(\frac{5t}{2}+\Theta\right) + \frac{9ae^2\tilde{\omega}^2}{40}\left(1+\frac{\tilde{\omega}^2}{4}\right)\cos\left(\frac{7t}{2}+\Theta\right) \\
& - \frac{9ae^2\tilde{\omega}^2}{8}\left(1-\frac{\tilde{\omega}^2}{2}\right)\cos\left(\frac{t}{2}-\Theta\right) , \qquad \textbf{(3-22)}
\end{aligned}
$$

where

$$\frac{da}{dt} = \left[\frac{53e^3\tilde{\omega}^2}{48} - \frac{9e^3\tilde{\omega}^4}{8}\left(1 - \frac{\tilde{\omega}^2}{4}\right)\right] a \sin 2\Theta ,$$

$$\frac{d\Theta}{dt} = \left[\frac{53e^3\tilde{\omega}^2}{48} - \frac{9e^3\tilde{\omega}^4}{8}\left(1 - \frac{\tilde{\omega}^2}{4}\right)\right] \cos 2\Theta + \frac{\omega_3^2 - \frac{9}{4}}{3} - \frac{\left(\omega_3^2 - \frac{9}{4}\right)^2}{27} + \frac{2\left(\omega_3^2 - \frac{9}{4}\right)^3}{243}$$

$$- \frac{3e^2\tilde{\omega}^4}{16} + \frac{13e^2\tilde{\omega}^4}{96}\left(\omega_3^2 - \frac{9}{4}\right) .$$

This system of equations leads to unbounded oscillations whenever

$$\left| \frac{\omega_3^2 - \frac{9}{4}}{3} - \frac{\left(\omega_3^2 - \frac{9}{4}\right)^2}{27} + \frac{2\left(\omega_3^2 - \frac{9}{4}\right)^3}{243} - \frac{3e^2\tilde{\omega}^4}{16} + \frac{13e^2\tilde{\omega}^4}{96}\left(\omega_3^2 - \frac{9}{4}\right)\right|$$

$$< \left|\frac{53e^3\tilde{\omega}^2}{48} - \frac{9e^3\tilde{\omega}^4}{8}\left(1 - \frac{\tilde{\omega}^2}{4}\right)\right| . \qquad (3\text{-}23)$$

It would be necessary to consider the fourth-order approximation if a more precise boundary of this region of unstable motion was desired.

The above analysis has shown that there are three regions in which the solution to the homogeneous equation related to the linear pitch libration problem represents unstable motion. Figure 3-1 demarcates these regions in which this occurs, with respect to $r_2 - r_1$ and e.

Both this and the previous chapter have been concerned with the linear, in-plane, pitch motion of a rigid gravity-gradient spacecraft in an eccentric orbit. The most important result of both chapters is the demonstration that stability criteria developed for circular orbits cannot be extrapolated to even slightly eccentric elliptic orbits.

It is known that the linear equation describing the attitude motion of a gravity-gradient-stabilized spacecraft has a stable equilibrium in the circular-orbit case when the inertia parameters are in the Lagrange region, (i.e., when $0 < r_1 < r_2 < 1$). Actually this region produces motion asymptotically stable at the equilibrium when there is "pervasive damping," no matter how small the damping is (Pringle, Reference 16). Neither of these conclusions is valid for a non-circular orbit, no matter how small the eccentricity. In fact, there may be divergent oscillations for some inertia combinations even with light but pervasive damping.

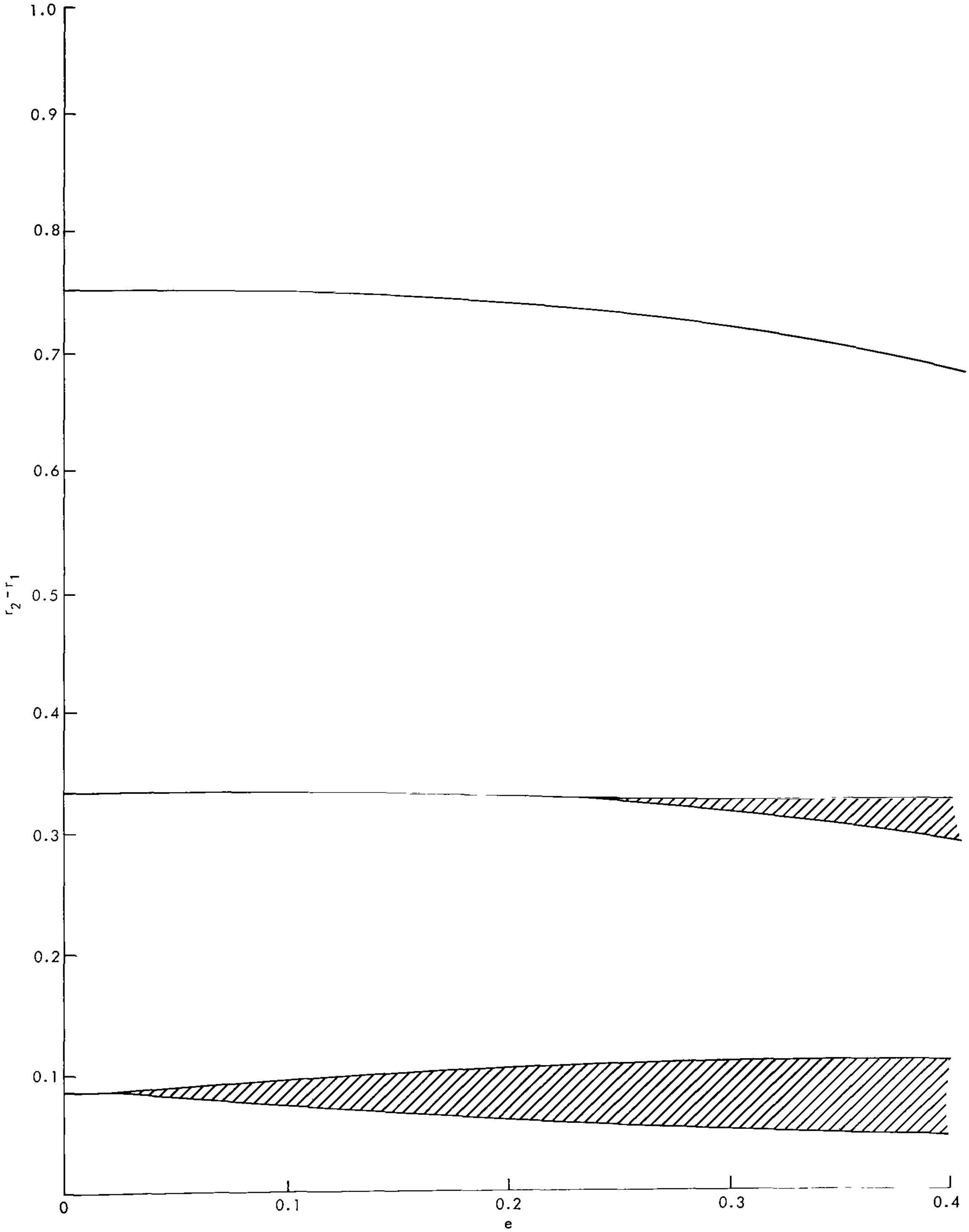

Figure 3-1—Regions of instability for asymptotic expansions.

These instabilities can be attributed to two causes. First, there is the forced response due to the non-uniform orbital rate. This enters the equations as a forcing term (Equation 3-1) and causes the equations of motion to be nonhomogeneous. Equation 3-10 shows the nature of the forced response; evidently the only important instability occurs when $\omega_3 = 1$ (i.e., when $r_2 - r_1 = 1/3$). This same equation also indicates the steady-state behavior of a lightly damped spacecraft, except in the regions of parametric resonance.

The second effect of orbital eccentricity is that the radius vector to the spacecraft, and thus the restoring torque, varies with time. This causes the equations of motion to have periodic coefficients and leads to a phenomenon called "parametric resonance." When $\omega_3 \approx 1/2$, 1, or 3/2 (i.e., when $r_2 - r_1 \approx 1/12$, 1/3, or 3/4), the amplitude of the pitch oscillation may increase without bound, to the order of approximation of the equations, even for arbitrarily small values of e. Figure 3-1 is plotted for a series of inequalities that define regions of unbounded oscillation; see Equations 3-17, 3-21, and 3-23.

The preceeding two chapters have shown that it is not valid to extrapolate results from the linearized equations for a circular orbit to the linearized equations for an elliptic orbit. The remainder of the dissertation will demonstrate that there are inertia parameters in the Lagrange Region for which the linearized equations are not adequate for either circular or eccentric orbits.

CHAPTER 4

GENERAL HAMILTONIAN EQUATIONS OF MOTION

Introduction

A complete description of the motion of a rigid gravity-gradient spacecraft in an eccentric orbit leads to a complex system of equations. Chapter 2 discusses some assumptions frequently used to simplify these equations. One of the most common is that it is valid to linearize the equations and then treat the in-plane or pitch motion as independent of the coupled roll-yaw equations. A second assumption, shown to be false in Chapters 2 and 3, is that a slightly eccentric orbit does not affect stability except for the linear resonance case when the pitch period and orbital period are identical. In fact, the situation is more complex than the results of the previous chapters indicate, because linearization also leads to incorrect results even for small angle motion.

The coupling between the pitch and roll-yaw equations has a significant effect on stability during large amplitude motion, as DeBra (Reference 6) showed by numerical integration. Kane (Reference 11), using a Floquet analysis of an only partially linearized set of equations, showed that this coupling was important even for small-amplitude (e.g. 1 degree) motions. Breakwell and Pringle (Reference 12) confirmed this conclusion in a paper that treated the problem of nonlinear coupling with canonical transformations and the method of averaging. This very significant paper was the first to apply the methods of analytical mechanics to the three-degree-of-freedom problem, although Liu (Reference 28) had previously applied canonical transformations to the in-plane libration of a spacecraft on an elliptic orbit.

This chapter and Chapter 5 extend the method of Breakwell and Pringle to higher-order effects of orbital eccentricity on the motion of a rigid gravity-gradient-stabilized spacecraft in an eccentric orbit. The Hamiltonian for the motion is first developed to third order in terms of the pitch, roll, and yaw angles and their associated momentum variables. Then a series of canonical transformations to new variables (for which the second-order portions of the Hamiltonian represent three uncoupled linear oscillators) is developed. The Hamilton-Jacobi equation is solved to obtain a third canonical transformation, which introduces cyclic coordinates in the second-order Hamiltonian. When the third-order Hamiltonian is expressed in the new system, the resulting Hamiltonian equations represent a set of perturbation equations that show the effects of nonlinear resonance. In Chapter 5 these perturbation equations are averaged and the inertia parameters that lead to various types of resonance are plotted.

Hamiltonian for Three Degree of Freedom Motion

A development of the Hamiltonian for this problem follows in a straightforward manner from the kinetic and potential energies of the system. The kinetic energy, T, associated with the attitude motion of the spacecraft is

$$T = \frac{1}{2}\left(r_1 \Omega_1^2 + r_2 \Omega_2^2 + \Omega_3^2\right)$$

where

$$\Omega_1 = \dot{\psi} - (\dot{\phi} + \dot{v}) \sin\theta ,$$

$$\Omega_2 = \dot{\theta} \cos\psi + (\dot{\phi} + \dot{v}) \cos\theta \sin\psi ,$$

$$\Omega_3 = - \dot{\theta} \sin\psi + (\dot{\phi} + \dot{v}) \cos\theta \cos\psi .$$

The potential energy V associated with attitude motion is

$$V = -\frac{3\mu_e}{2r^3}\left[\left(r_2 - r_1\right) \cos^2\phi \cos^2\theta + \left(r_2 - 1\right)\left(\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi\right)^2\right] ,$$

and the Lagrangian L is

$$L = T - V .$$

The standard procedure at this point would be to introduce generalized momenta p_i, defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_1} ,$$

where

$$q_1 = \psi ,$$

$$q_2 = \theta ,$$

and

$$q_3 = \phi .$$

Unfortunately this leads to complications when terms of similar orders of magnitude are collected. The general nature of the motion of a successfully stabilized spacecraft is well known. There will be small oscillations of ψ and θ about zero, and a small oscillation of ϕ about the local vertical which rotates at a rate of 2π radians/orbit. Thus the angular momentum about the "three" axis, p_3, would have a steady component and a component representing small oscillations. However, this can be eliminated by defining

$$\frac{\partial L}{\partial \dot{q}_3} = p_3 + I_3 n = p_3 + 1 ,$$

with the happy result that all the q_i and p_i are small as compared with 1. The "order" of a given term should be understood as meaning the order of the products of the q_i's and p_i's in the term, (e.g. $q_2^2 p_3$ is 3rd order in coordinates and momenta).

$$p_1 \equiv \frac{\partial L}{\partial \dot{\psi}} = r_1 \Omega_1 ,$$

$$p_2 \equiv \frac{\partial L}{\partial \dot{\theta}} = r_2 \Omega_2 \cos\psi - \Omega_3 \sin\psi ,$$

and

$$p_3 + 1 \equiv \frac{\partial L}{\partial \dot{\phi}} = - r_1 \Omega_1 \sin\theta + r_2 \Omega_2 \cos\theta \sin\psi + \Omega_3 \cos\theta \cos\psi .$$

The Hamiltonian, H, is defined as

$$H = \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L(q, \dot{q}, t) ;$$

thus

$$H = \frac{1}{2} \left[r_1 \Omega_1^2 + r_2 \Omega_2^2 + \Omega_3^2 \right] + \left[r_1 \Omega_1 \sin\theta - r_2 \Omega_2 \cos\theta \sin\psi - \Omega_3 \cos\theta \cos\psi \right] \dot{v} + V ,$$

or

$$H = \frac{1}{2r_1} \left(r_1 \Omega_1 \right)^2 + \frac{1}{2r_2} \left(r_2 \Omega_2 \right)^2 + \frac{1}{2} \Omega_3^2 - \left(p_3 + 1 \right) \dot{v} + V .$$

At this point the series expansions

$$\sin\eta = \eta - \frac{\eta^3}{6} + \cdots , \qquad \cos\eta = 1 - \frac{\eta^2}{2}$$

are introduced into the Hamiltonian. When terms of equal order in the generalized coordinates and momenta are collected and the substitutions $q_1 = \psi$, $q_2 = \theta$ and $q_3 = \phi$ are introduced, the result becomes

$$H = \sum_{i=1}^{n} H_i ,$$

where

$$H_0 = -\dot{v} ,$$

$$H_1 = p_3 (1 - \dot{v}) ,$$

$$H_2 = \frac{1}{2r_1} p_1^2 + \frac{1}{2r_2} p_2^2 + \frac{1}{2} p_3^2 + \frac{3}{2} \frac{\mu}{a^3} \left(\frac{a}{r}\right)^3 \left[(r_2 - r_1 \; q_3^2 + (1 - r_1) q_2^2 \right]$$

$$+ \frac{1}{2} \frac{1 - r_2}{r_2} q_1^2 + \frac{1}{2} q_2^2 + \frac{1 - r_2}{r_2} q_1 p_2 + q_2 p_1 ,$$

$$H_3 = \frac{1 - r_2}{r_2} q_1 p_3 (q_1 + p_2) + q_2 p_3 (q_2 + p_1) + \frac{3}{2} \frac{\mu_e}{a^3} \left(\frac{a}{r}\right)^3 (1 - r_2) q_1 q_2 q_3 ,$$

and fourth-order and higher-order terms will be ignored. The series developments for r and v introduced in Chapter 3 will be used in the form

$$\dot{v} = 1 - T_1 (t) ,$$

where

$$T_1 (t) = -2e \cos t - \frac{5e^2}{2} \cos 2t - \frac{e^3}{4} \left[-\cos t + 13 \cos 3t \right]$$

$$- \frac{e^4}{24} \left[-22 \cos 2t + 103 \cos 4t \right] , \qquad \textbf{(4-1)}$$

and

$$\left(\frac{a}{r}\right)^3 = \frac{k}{3} + T_2(t),$$

where

$$k = 3\left(1 + \frac{3e^2}{2} + \frac{15e^4}{8}\right),$$

and

$$T_2 = \sum_{i=1}^{4} S_i \cos(it) \tag{4-2}$$

where

$$S_1 = 3e + \frac{27e^3}{8},$$

$$S_2 = \frac{9e^2}{2} + \frac{7e^4}{2},$$

$$S_3 = \frac{53e^3}{8},$$

and

$$S_4 = \frac{77e^4}{8}.$$

With these substitutions and with terms $T_2 q_2^2$ and $T_2 q_3^2$ assigned to H_3, H can be written

$$H = \sum_{i=0}^{3} H_i, \tag{4-3}$$

where

$$H_0 = T_1(t) - 1,$$

$$H_1 = p_3 T_1(t),$$

$$H_2 = \frac{1}{2}\left[\frac{1-r_2}{r_2} q_1^2 + \left(1+k-k\,r_1\right) q_2^2 + k\left(r_2 - r_1\right) q_3^2 + \frac{1}{r_1} p_1^2 + \frac{1}{r_2} p_2^2 + p_3^2\right] + \frac{1-r_2}{r_2} q_1 p_2 + q_2 p_1,$$

and

$$H_3 = \frac{1-r_2}{r_2}^2 q_1 p_3 \left(q_1 + p_2\right) + q_2 p_3 \left(q_2 + p_1\right) + \frac{3}{2}\left[\frac{k}{3} + T_2(t)\right]\left(1-r_2\right) q_1 q_2 q_3 + \frac{3T_2(t)}{2}\left[\left(r_2 - r_1\right) q_3^2 + \left(1-r_1\right) q_2^2\right].$$

Diagonalization of Second Order Hamiltonian

When Hamilton's equations are formed from Equation (4-3), the result is six equations that are nonlinear and coupled (i.e., terms q_j, p_j $j \neq i$ appear in the equations for q_i or p_i). The Hamilton equations have the form

$$\dot{q}_i = \frac{\partial H_1}{\partial p_i} + \frac{\partial H_2}{\partial p_i} + \frac{\partial H_3}{\partial p_i},$$

and

$$-\dot{p}_i = \frac{\partial H_1}{\partial q_i} + \frac{\partial H_2}{\partial q_i} + \frac{\partial H_3}{\partial q_i}.$$

Since H_1 contains only first-order terms, it cannot contribute any coupling terms to these two equations; however, both H_2 and H_3 may do so. If the H_3 term is ignored (i.e., if the equations are linearized) it is possible to find a canonical transformation to new coordinates for which the second-order Hamiltonian equations are uncoupled. It is then possible to use the solution to these equations to obtain what amounts to a set of six perturbation equations that determine the effect of including the H_3 terms in the equations of motion.

The first step is to find the transformation that uncouples H_2. This can be done by regarding as a matrix,

$$H_2 = \frac{1}{2} \vec{q}^T [S] \vec{q},$$

where

$$\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{Bmatrix}, \text{ a column vector,}$$

and

$$[S] = \begin{bmatrix} \frac{1-r_2}{r_2} & 0 & 0 & 0 & \frac{1-r_2}{r_2} & 0 \\ 0 & 1+k-kr_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & k(r_2-r_1) & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{r_1} & 0 & 0 \\ -\frac{1-r_2}{r_0} & 0 & 0 & 0 & \frac{1}{r_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and finding a constant-coefficient matrix [D] such that, for q'', a column vector in the new coordinates,

$$\vec{q} = [D]\,\vec{q}\,''$$

represents a canonical transformation to a system $H^{**}(q'', p'', t)$, for which H_2^{**} is a diagonal matrix. This is done in two steps, first a transformation matrix [K] to a system $H^{*}(q', p', t)$ in which

$$H_2^{*} = \sum_{i=1}^{3} \lambda_i\, q_i'\, p_i' ,$$

and then a transformation [M] to a system $H^{**}(q'', p'', t)$ in which

$$H_2^{**} = \frac{1}{2}\sum_{i=1}^{3} (p_i'')^2 + \omega_i^2 (q_i'')^2 .$$

Then the desired matrix [D] is

$$[D] = [K]\,[M] .$$

The procedures used are quite general and could be applied to any similar problem. In the particular case under consideration, the terms p_3 and q_3 in $H_2(q, p, t)$ are already in the desired form. Thus a matrix $[D']$ which diagonalizes the matrix $[S']$ will be developed where

$$[S'] = \begin{bmatrix} \frac{1-r_2}{r_2} & 0 & 0 & \frac{1-r_2}{r_2} \\ 0 & 1+k-kr_1 & 1 & 0 \\ 0 & 1 & \frac{1}{r_1} & 0 \\ \frac{1-r_2}{r_2} & 0 & 0 & \frac{1}{r_2} \end{bmatrix} .$$

The prime notation will be dropped in the remainder of this section; however the meaning is obvious.

A formal procedure for obtaining the first canonical transformation is developed by Pars (Reference 29); only the essential steps are outlined herein. This part of the problem can be restated as: Given a real symmetric matrix $[S]$, find a matrix $[K]$ such that

$$[K]^T [S] [K] = \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix} ,$$

where $[L]$ is a diagonal matrix

$$[L] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} ,$$

and such that

$$[K]^T [Z] [K] = [Z] ,$$

where

$$[Z] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .$$

Then the matrix $[K]$ will define a contact transform

$$\vec{q} = [K] \vec{q}\,'$$

which accomplishes the desired result. The first step in obtaining the matrix [K] is to find the eigenvalues of the matrix [Z] [S] (i.e. find the roots λ of $|S + \lambda Z| = 0$).

This is equivalent to finding the values of λ for which the determinant equation

$$\begin{vmatrix} \frac{1-r_2}{r_2} & 0 & -\lambda & \frac{1-r_2}{r_2} \\ 0 & 1+k-kr_1 & 1 & -\lambda \\ \lambda & 1 & \frac{1}{r_1} & 0 \\ \frac{1-r_2}{r_2} & \lambda & 0 & \frac{1}{r_2} \end{vmatrix} = 0 \tag{4-4}$$

is satisfied. This equation can be expanded to give

$$\lambda^4 + \lambda^2 \left[\frac{1-r_2}{r_1 r_2} + \frac{1+k-kr_1}{r_2} - 2\,\frac{(1-r_2)}{r_2}\right] + \frac{1+k}{r_1 r_2}(1-r_1)(1-r_2) = 0\,. \tag{4-5}$$

The roots of this biquadratic equation are the normal frequencies of the two decoupled normal modes of the linear problem associated with H_2. When $e = 0$, $k = 3$ and Equation 4-5 can be reduced to

$$r_1 r_2 \lambda^4 + \lambda^2 \left[-3r_1^2 + 2r_1 r_2 + 2r_1 - r_2 + 1\right] + 4(1-r_1)(1-r_2) = 0\,,$$

which (with the substitution $\lambda = i\omega$) is the same as Equation 4-4 of Breakwell and Pringle. The roots of this equation occur in complex conjugate pairs; when $0 < r_1 < r_2 < 1$, both the roots of 4-5 and the roots $\pm\lambda_3$ associated with the characteristic equation for the already decoupled pitch motion

$$\lambda^2 + k(r_2 - r_1) = 0 \tag{4-6}$$

are pure imaginary numbers. When λ has any of the four values $+i\omega_1$, $+i\omega_2$, $-i\omega_1$, $-i\omega_2$ (where $\omega_2 > \omega_1$) obtained from Equation 4-5, there is a nontrivial solution to the matrix equation

$$[S]\left\{\vec{C}_i\right\} = -\lambda\left\{\vec{C}_i\right\}, \qquad i = 1, \cdots, 4$$

where the $\vec{C}_i$ are the eigenvectors of [Z] [S]. A set of eigenvectors for the matrix can be obtained from the minors of any column of the associated determinant; see Wiley (Reference 30). One set of eigenvectors obtained from the second column of Equation 4-4 is based on

$$\lambda(1 - r_1 - r_2)$$

$$-\lambda^2 r_1 - 1 + r_2$$

$$\left(1+\lambda^2\right)\left(r_1 - r_1 r_2\right)$$

$$\lambda\left[-\lambda^2 r_1 r_2 - \left(1-r_2\right)\left(1-r_1\right)\right]$$

and in particular is given by

$$C_1 = \rho \begin{Bmatrix} i\omega_1\left(1-r_1-r_2\right) \\ \omega_1^2 r_1 - 1 + r_2 \\ \left(1-\omega_1^2\right)\left(r_1 - r_1 r_2\right) \\ i\omega_1\left[\omega_1^2 r_1 r_2 - \left(1-r_2\right)\left(1-r_1\right)\right] \end{Bmatrix}, \qquad C_2 = \sigma \begin{Bmatrix} i\omega_2\left(1-r_1-r_2\right) \\ \omega_2^2 r_1 - 1 + r_2 \\ \left(1-\omega_2^2\right)\left(r_1 - r_1 r_2\right) \\ i\omega_2\left[\omega_2^2 r_1 r_2 - \left(1-r_2\right)\left(1-r_1\right)\right] \end{Bmatrix},$$

$$C_3 = \gamma \begin{Bmatrix} -i\omega_1\left(1-r_1-r_2\right) \\ \omega_1^2 r_1 - 1 + r_2 \\ \left(1-\omega_1^2\right)\left(r_1 - r_1 r_2\right) \\ -i\omega_1\left[\omega_1^2 r_1 r_2 - \left(1-r_2\right)\left(1-r_1\right)\right] \end{Bmatrix}, \qquad C_4 = \delta \begin{Bmatrix} -i\omega_2\left(1-r_1-r_2\right) \\ \omega_2^2 r_1 - 1 + r_2 \\ \left(1-\omega_2^2\right)\left(r_1 - r_1 r_2\right) \\ -i\omega_2\left[\omega_1^2 r_1 r_2 - \left(1-r_2\right)\left(1-r_1\right)\right] \end{Bmatrix},$$

where ρ, σ, γ, and δ can be any arbitrary (real or complex) constants. Now define K as the matrix formed from the columns C_1, C_2, C_3, C_4, in that order. Then

$$[K]^T [Z] [K] = [F] ,$$

where

$$[F] = \begin{bmatrix} 0 & 0 & f_1 & 0 \\ 0 & 0 & 0 & f_2 \\ -f_1 & 0 & 0 & 0 \\ 0 & -f_2 & 0 & 0 \end{bmatrix},$$

$$f_1 = \left\{\vec{C}_1\right\}^T [Z] \left\{\vec{C}_1\right\} ,$$

and

$$f_2 = \left\{\vec{C}_2\right\}^T [Z] \left\{\vec{C}_2\right\} .$$

Now, if the values of ρ, σ, γ, and δ are chosen such that $f_1 = f_2 = 1$, then, in

$$[K]^T [Z] [K] = [Z] ,$$

the matrix $[K]$ is symplectic and the transformation

$$\vec{q} = [K] \vec{q}'$$

is a contact transformation. In addition,

$$[K]^T [S] [K] = [E] ,$$

where

$$[E] = \begin{bmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \end{bmatrix} ,$$

$$\lambda_1 = i\omega_1 ,$$

$$\lambda_2 = i\omega_2 ,$$

and thus

$$\begin{aligned} H_2 &= \frac{1}{2} \{\vec{q}\}^T [S] \{\vec{q}\} \\ &= \frac{1}{2} \{\vec{q}'\}^T [K]^T [S] [K] \{\vec{q}'\} \\ &= \frac{1}{2} \{\vec{q}'\} [E] \{\vec{q}'\} \\ &= \lambda_1 q_1' p_1' + \lambda_2 q_2' p_2' \end{aligned}$$

Thus the matrix $[K]$ defines a contact transformation which transforms H_2 to H_2^* and satisfies Equation 4-6. The requirement that $f_1 = f_2 = 1$ provides two relationships for determining ρ, σ, γ, and δ:

$$2\delta i\omega_2 \left\{(1 - r_1 - r_2)(1 - \omega_1^2)(r_1 - r_1 r_2) + (1 - r_2 - r_1 \omega_1^2)\left[\omega_1^2 r_1 r_2 - (1 - r_2)(1 - r_1)\right]\right\} = 1 , \qquad (4\text{-}6)$$

and

$$2\sigma\delta i\omega_2 \left\{(1 - r_1 - r_2)\,(1 - \omega_2^2)\,(r_1 - r_1 r_2) + (1 - r_2 - r_1 \omega_2^2)\left[\omega_2^2 r_1 r_2 - (1 - r_2)\,(1 - r_1)\right]\right\} = 1 . \quad (4\text{-}7)$$

Any values of ρ, σ, γ, and δ that satisfy the above equations are satisfactory in defining $[K]$. Thus it would be possible to make two arbitrary choices, such as $\rho = 1$, $\sigma = 1$, and find values of γ and δ. However, it is more convenient to obtain the second transformation matrix $[M]$ and then find two additional relationships that simplify the final transformation matrix $[D]$.

The next step in the process is to find a canonical transformation of the form

$$\{\vec{q}'\} = [M]\,\{\vec{q}''\}$$

that transforms H_2^*, (q_i', p_i') into H_2^{**}, (q_i'', p_i''), where

$$H_2^*(q', p') = \sum_{i=1}^{2} \lambda_i q_i' p_i' ,$$

and

$$H_2^{**}(q'', p'') = \sum_{i=1}^{2} p_i''^2 + \omega_i^2 q_i''^2 .$$

There is a generating function W, given by Whittaker (Reference 31), which can be used to find the needed transformation. Consider

$$W = \sum_{i=1}^{2} p_i'' q_i' - \frac{1}{2}\frac{p_i''^2}{\lambda_i} - \frac{1}{4}\lambda_i q_i'$$

with

$$q_i'' = \frac{\partial W}{\partial p_i''} , \qquad i = 1, 2 ,$$

and

$$p_i' = \frac{\partial W}{\partial q_i'} , \qquad i = 1, 2 .$$

The above leads directly to the relations

$$q_i' = q_i'' + \frac{p_i''}{\lambda_i} ,$$

$$p_i' = -\frac{\lambda_i q_i''}{2} + \frac{p_2''}{2} ,$$

which can be expressed in matrix form as

$$\{\vec{q}'\} = [M] \{\vec{q}''\} ,$$

where

$$[M] = \begin{bmatrix} 1 & 0 & \frac{1}{\lambda_1} & 0 \\ 0 & 1 & 0 & \frac{1}{\lambda_2} \\ -\frac{\lambda_1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{\lambda_1}{2} & 0 & \frac{1}{2} \end{bmatrix} .$$

The first transformation matrix [K] is of the form

$$[K] = \begin{bmatrix} \rho C_{11} & \sigma C_{21} & -\gamma C_{11} & -\delta C_{21} \\ \rho C_{12} & \sigma C_{22} & \gamma C_{12} & \delta C_{22} \\ \rho C_{13} & \sigma C_{23} & \gamma C_{13} & \delta C_{23} \\ \rho C_{14} & \sigma C_{24} & -\gamma C_{14} & -\delta C_{24} \end{bmatrix} ,$$

and thus the matrix [D] would have the form

$$[D] = \begin{bmatrix} \left(\rho + \frac{\lambda_1 \gamma}{2}\right) C_{11} & \left(\sigma + \frac{\lambda_2 \delta}{2}\right) C_{21} & \left(\frac{\rho}{\lambda_1} - \frac{\gamma}{2}\right) C_{11} & \left(\frac{\sigma}{\lambda_2} - \frac{\delta}{2}\right) C_{21} \\ \left(\rho - \frac{\lambda_1 \gamma}{2}\right) C_{12} & \left(\sigma - \frac{\lambda_2 \delta}{2}\right) C_{22} & \left(\frac{\rho}{\lambda_1} + \frac{\gamma}{2}\right) C_{12} & \left(\frac{\sigma}{\lambda_2} + \frac{\delta}{2}\right) C_{22} \\ \left(\rho - \frac{\lambda_1 \gamma}{2}\right) C_{13} & \left(\sigma - \frac{\lambda_2 \delta}{2}\right) C_{23} & \left(\frac{\rho}{\lambda_1} + \frac{\gamma}{2}\right) C_{13} & \left(\frac{\sigma}{\lambda_2} + \frac{\delta}{2}\right) C_{23} \\ \left(\rho + \frac{\lambda_1 \gamma}{2}\right) C_{14} & \left(\sigma + \frac{\lambda_2 \delta}{2}\right) C_{24} & \left(\frac{\rho}{\lambda_1} - \frac{\gamma}{2}\right) C_{14} & \left(\frac{\sigma}{\lambda_2} - \frac{\delta}{2}\right) C_{24} \end{bmatrix} .$$

The freedom which exists in choosing ρ, σ, γ, and δ permits the choice

$$\gamma = \frac{2\rho}{\lambda_1} ,$$

$$\delta = \frac{2\sigma}{\lambda_2} ,$$

which simplifies [D] to the form

$$[D] = \begin{bmatrix} 2\rho C_{11} & 2\sigma C_{21} & 0 & 0 \\ 0 & 0 & \frac{2\rho C_{12}}{\lambda_1} & \frac{2\sigma C_{22}}{\lambda_2} \\ 0 & 0 & \frac{2\rho C_{13}}{\lambda_1} & \frac{2\sigma C_{23}}{\lambda_2} \\ 2\rho C_{14} & 2\sigma C_{24} & 0 & 0 \end{bmatrix} ,$$

and Equations 4-6 and 4-7 become

$$4\rho^2\left\{\left(1 - r_1 - r_2\right)\left(1 - \omega_1^2\right)\left(r_1 - r_1 r_2\right) + \left(1 - r_2 - r_1\omega_1^2\right)\left[\omega_1^2 r_1 r_2 - \left(1 - r_2\right)\left(1 - r_1\right)\right]\right\} = 1 , \tag{4-8}$$

$$4\rho^2\left\{\left(1 - r_1 - r_2\right)\left(1 - \omega_2^2\right)\left(r_1 - r_1 r_2\right) + \left(1 - r_2 - r_1\omega_2^2\right)\left[\omega_2^2 r_1 r_2 - \left(1 - r_2\right)\left(1 - r_1\right)\right]\right\} = 1 . \tag{4-9}$$

These expressions are rather unwieldy; however, they can be reduced with some algebraic sleight-of-hand. Write Equation 4-8 as

$$4\rho^2\Bigg(\left(1 - r_2 - r_1\omega_1^2\right) r_1 r_2\left(\omega_1^2 - \omega_2^2\right) + \Big\{\left(1 - r_2 - r_1\omega_1^2\right)\left[r_1 r_2\omega_2^2 - \left(1 - r_2\right)\left(1 - r_1\right)\right]$$

$$+ \left(1 - r_1 - r_2\right)\left(1 - \omega_1^2\right)\left(r_1 - r_1 r_2\right)\Big\}\Bigg) = 1 .$$

The term inside the brace can be written as

$$- r_1^2 r_2\omega_1^2\omega_2^2 + \left(1 - r_2\right) r_1 r_2\left(\omega_1^2 + \omega_2^2\right) + \left(1 - r_2\right)\left[- r_1^2 - 2r_1 r_2 + 2r_1 - 1 + r_2\right] , \tag{4-10}$$

and ω_1^2 and ω_2^2 are the roots of this equation, which can be written as

$$r_1 r_2\omega^4 + \left[3r_1^2 - 2r_1 r_2 - 2r_1 - 1 + r_2 - \left(\frac{9e^2}{2} + \frac{45e^4}{8}\right) r_1\left(1 - r_1\right)\right]\omega^2$$

$$+ 4\left(1 - r_1\right)\left(1 - r_2\right) + \left(\frac{9e^2}{2} + \frac{45e^4}{8}\right)\left(1 - r_1\right)\left(1 - r_2\right) = 0 .$$

Thus

$$\omega_1^2 + \omega_2^2 = \frac{-\left[3r_1^2 - 2r_1 r_2 - 2r_1 - 1 + r_2 - \left(\frac{9e^2}{2} + \frac{45e^4}{8}\right) r_1 (1 - r_1)\right]}{r_1 r_2},$$

and

$$\omega_1^2 \omega_2^2 = \frac{4(1 - r_1)(1 - r_2) + \left(\frac{9e^2}{2} + \frac{45e^4}{8}\right)(1 - r_1)(1 - r_2)}{r_1 r_2}.$$

When these expressions are substituted in expression 4-10, it becomes identically zero; thus Equation 4-8 can be written as

$$4\rho^2 \left(1 - r_2 - r_1 \omega_1^2\right) r_1 r_2 \left(\omega_1^2 - \omega_2^2\right) = 1,$$

and by a similar process Equation 4-9 becomes

$$4\sigma^2 \left(1 - r_2 - r_1 \omega_2^2\right) r_1 r_2 \left(\omega_2^2 - \omega_1^2\right) = 1.$$

These two equations can be solved to give

$$2\rho = \frac{i}{\left[\left(1 - r_2 - r_1 \omega_1^2\right) r_1 r_2 \left(\omega_2^2 - \omega_1^2\right)\right]^{1/2}}$$

and

$$2\sigma = \frac{i}{\left[\left(1 - r_2 - r_1 \omega_2^2\right) r_1 r_2 \left(\omega_1^2 - \omega_2^2\right)\right]^{1/2}}.$$

With these two substitutions, [D] becomes

$$
[D] = \begin{bmatrix}
\dfrac{\omega_1(r_1+r_2-1)}{\left[\left(1-r_2-r_1\omega_1^2\right) r_1 r_2\left(\omega_2^2-\omega_1^2\right)\right]^{1/2}} & \dfrac{\omega_2(r_1+r_2-1)}{\left[\left(1-r_2-r_1\omega_2^2\right) r_1 r_2\left(\omega_1^2-\omega_2^2\right)\right]^{1/2}} & 0 & 0 \\
0 & 0 & \dfrac{-\left(1-r_2-r_1\omega_1^2\right)^{1/2}}{\omega_1\left[r_1 r_2\left(\omega_2^2-\omega_1^2\right)\right]^{1/2}} & \dfrac{-\left(1-r_2-r_1\omega_2^2\right)^{1/2}}{\omega_2\left[r_1 r_2\left(\omega_2^2-\omega_1^2\right)\right]^{1/2}} \\
0 & 0 & \dfrac{\left(1-\omega_1^2\right) r_1\left(1-r_2\right)}{\omega_1\left[\left(1-r_2-r_1\omega_1^2\right) r_1 r_2\left(\omega_2^2-\omega_1^2\right)\right]^{1/2}} & \dfrac{\left(1-\omega_2^2\right) r_1\left(1-r_2\right)}{\omega_2\left[\left(1-r_2-r_1\omega_2^2\right) r_1 r_2\left(\omega_1^2-\omega_2^2\right)\right]^{1/2}} \\
\dfrac{\omega_1\left[\left(1-r_2\right)\left(1-r_1\right)-r_1 r_2\omega_1^2\right]}{\left[\left(1-r_2-r_1\omega_1^2\right) r_1 r_2\left(\omega_2^2-\omega_1^2\right)\right]^{1/2}} & \dfrac{\omega_2\left[\left(1-r_2\right)\left(1-r_1\right)-r_1 r_2\omega_2^2\right]}{\left[\left(1-r_2-r_1\omega_2^2\right) r_1 r_2\left(\omega_1^2-\omega_2^2\right)\right]^{1/2}} & 0 & 0
\end{bmatrix}
\tag{4-11}
$$

Thus the matrix [D] simplifies the roll-yaw portion of the Hamiltonian.

It is also possible to find new variables that replace q_2 and p_3, and eliminate the H_0 and H_1 terms. This time, consider the following portion of the original Hamiltonian:

$$H' = T_1(t) - 1 + p_3 T_1(t) + \frac{1}{2}\left(p_3^2 + \omega_3^2 q_3^2\right) .$$

Two successive canonical transformations, the first derived from a generating function:

$$F\left(q_3, p_3', t\right) = q_3 p_3' - b(t) p_3' ,$$

with

$$p_3 = \frac{\partial F}{\partial q} = p_3' ,$$

$$q_3' = \frac{\partial F}{\partial p_3'} = q - b(t) ,$$

and

$$H^* = H' - \dot{b} p' ,$$

and the second derived from a generating function

$$F^*\left(q_3'', p_3', t\right) = -p_3' q_3'' + c(t) q''$$

$$+ \int \left[b\dot{c} - \frac{1}{2}\left(c^2 + \omega^2 b^2\right) - T_1(t) + 1\right] dt ,$$

with

$$q_3' = -\frac{\partial F^*}{\partial p_3'} = q''$$

$$p_3'' = -\frac{\partial F}{\partial q_3''} = p_3' - c(t) ,$$

and

$$H^{**} = H^{*} + \dot{b}q'' + \left[b\dot{c} - \frac{1}{2}\left(c^{2} + \omega_{3}^{2} b^{2}\right) - T_{1}(t) + 1\right],$$

combine to give a transformation

$$q_3 = q_3'' + b(t),$$

$$p_3 = p_3'' + c(t),$$

and a new Hamiltonian

$$H^{**} = \frac{1}{2}\left(p_3''^{2} + \omega^{2} q_s''^{2}\right) + p_3''\left[c - \dot{b} + T_1(t)\right] + q_3''\left(\omega_3^{2} b + \dot{c}\right).$$

Now if $c(t)$ and $b(t)$ are chosen so that

$$c(t) - \dot{b}(t) + T_1(t) = 0,$$

and

$$\dot{c}(t) + \omega^{2} b(t) = 0,$$

i.e., if $c(t)$ and $b(t)$ are solutions to

$$\ddot{c}(t) + \omega^{2} c(t) = -\omega_3^{2} T_1(t),$$

and

$$\ddot{b}(t) + \omega^{2} b(t) = \dot{T}_1(t),$$

then the Hamiltonian H^{**} becomes

$$H^{**}\left(q_3'', p_3'', t\right) = \frac{1}{2}\left(p_3''^{2} + \omega_3^{2} q_3''^{2}\right).$$

with $T_1(t)$ defined by Equation 4-1, $b(t)$ becomes

$$b(t) = \sum_{i=1}^{4} b_i \sin(it), \tag{4-12}$$

where

$$b_1 = \frac{2e}{\omega^2 - 1} - \frac{e^3}{4\left(\omega^2 - 1\right)} ,$$

$$b_2 = \frac{5e^2}{\omega^2 - 4} - \frac{11e^4}{6\left(\omega^2 - 4\right)} ,$$

$$b_3 = \frac{39e^3}{4\left(\omega^2 - 9\right)} ,$$

$$b_4 = \frac{103e^4}{6\left(\omega^2 - 16\right)} .$$

Similarly

$$c(t) = \sum_{i=1}^{4} c_i \cos(it) , \tag{4-13}$$

where

$$c_1 = \frac{2e\,\omega_3^2}{\omega_3^2 - 1} - \frac{e^3\,\omega_3^2}{4\left(\omega_3^2 - 1\right)} ,$$

$$c_2 = \frac{5e^2\,\omega_3^2}{2\left(\omega_3^2 - 4\right)} - \frac{11e^4\,\omega_3^2}{12\left(\omega_3^2 - 4\right)} ,$$

$$c_3 = \frac{13e^3\,\omega_3^2}{4\left(\omega_3^2 - 9\right)} ,$$

$$c_4 = \frac{103e^4\,\omega_3^2}{24\left(\omega_3^2 - 16\right)} .$$

When the above transformations are all applied to the original Hamiltonian given by Equation 4-3, the new Hamiltonian has the form

$$H^{**}(q'', p'', t) = \sum_{i=0}^{3} H_i^{**}(q'', p'', t) ,$$

where

$$H_0^{**}(q'', p'', t) = 0 ,$$

$$H_1^{**}(q'', p'', t) = 0 ,$$

$$H_2^{**}(q'', p'', t) = \sum_{i=1}^{3} \frac{1}{2} \left(p_i''^2 + \omega_i^2 q_i''^2\right) ,$$

and H_3^{**} is obtained by making the following substitutions in $H_3(q, p, t)$:

$$q_1 = d_{11} q_1'' + d_{12} q_2'' ,$$

$$q_2 = d_{23} p_1'' + d_{24} p_2'' ,$$

$$p_1 = d_{33} p_1'' + d_{34} p_2'' ,$$

$$p_2 = d_{41} q_1'' + d_{42} q_2'' ,$$

$$q_3 = q_3'' + b(t) ,$$

$$p_3 = p_3'' + c(t) ,$$

where d_{ij} represents the element in the i^{th} row and the j^{th} column of the matrix $[D]$ defined by Equation 4-11, and $b(t)$, $c(t)$ are defined by Equations 4-12 and 4-13. Thus these new coordinates lead to a system which has no terms of order 0 or 1. The terms of order 2 will be removed, in the next section, by a final canonical transformation.

Solution of Second Order Equations

In the preceding section the original Hamiltonian was considerably simplified by a series of canonical transformations. The end result was a system in which the linearized equations represent three uncoupled harmonic oscillators. The solution to the linear equation is well known; however, when it is obtained by finding Hamilton's principal function S, then this function generates a contact transformation which will remove the second-order terms from the new Hamiltonian.

The process of finding the principal function for a Hamiltonian of the form

$$H^{**} = \sum_{i=1}^{3} \frac{1}{2}\left[(p_i'')^2 + \omega_i^2 (q_i'')^2\right]$$

follows directly from the solution for a single harmonic oscillator (Goldstein, Reference 32). The principal function $S(q'', \alpha, t)$ is

$$S = \sum_{i=1}^{3} \left\{ \int \left[2\alpha_i - \omega_i^2 q_i''^2 \right]^{1/2} dq_i - \alpha_i t \right\},$$

and the transformation defined by this function is obtained from

$$\beta_i = \frac{\partial S}{\partial \alpha_i} = \frac{1}{\omega_i} \arc\sin \frac{\omega_i q_i''}{\sqrt{2\alpha_i}} - t,$$

which is equivalent to

$$q_i'' = \frac{\sqrt{2\alpha_i}}{\omega_i} \sin \omega_i \left(t + \beta_i\right)$$

and

$$p_i'' = \frac{\partial S}{\partial q_i''} = \sqrt{2\alpha_i} \cos \omega_i \left(t + \beta_i\right).$$

The new Hamiltonian, H^{***}, is related to the old one by

$$H^{***} = H^{**} + \frac{\partial S}{\partial t} = H^{**} - \sum_{i=1}^{3} \alpha_i .$$

Thus the new Hamiltonian is

$$H^{***}(\alpha, \beta, t) = \sum_{i=0}^{3} H_i^{***}(\alpha, \beta, t), \tag{4-14}$$

where

$$H_0^{***}(\alpha, \beta, t) = 0,$$

$$H_1^{***}(\alpha, \beta, t) = 0,$$

$$H_2^{***}(\alpha, \beta, t) = 0,$$

and $H_3^{***}(\alpha, \beta, t)$ is obtained from Equation 4-3 by making the following substitutions in $H_3(q, p, t)$:

$$q_1 = \frac{\sqrt{2}\, d_{11}}{\omega_1} \sqrt{\alpha_1} \sin \omega_1 \left(t + \beta_1\right) + \frac{\sqrt{2}\, d_{12}}{\omega_2} \sqrt{\alpha_2} \sin \omega_2 \left(t + \beta_2\right) ,$$

$$q_2 = \sqrt{2}\, d_{23} \sqrt{\alpha_1} \cos \omega_1 \left(t + \beta_1\right) + \sqrt{2}\, d_{24} \sqrt{\alpha_2} \cos \omega_2 \left(t + \beta_2\right) ,$$

$$p_1 = \sqrt{2}\, d_{33} \sqrt{\alpha_1} \cos \omega_1 \left(t + \beta_1\right) + \sqrt{2}\, d_{34} \sqrt{\alpha_2} \cos \omega_2 \left(t + \beta_2\right) ,$$

$$p_2 = \frac{\sqrt{2}\, d_{41}}{\omega_1} \sqrt{\alpha_1} \sin \omega_1 \left(t + \beta_1\right) + \frac{\sqrt{2}\, d_{42}}{\omega_2} \sqrt{\alpha_2} \sin \omega_2 \left(t + \beta_2\right) ,$$

$$q_3 = \frac{\sqrt{2}}{\omega_3} \sqrt{\alpha_3} \sin \omega_3 \left(t + \beta_3\right) + c(t) ,$$

$$p_3 = \sqrt{2} \sqrt{\alpha_3} \cos \omega_3 \left(t + \beta_3\right) + b(t) . \tag{4-15}$$

Summary

In this chapter, the equations describing the attitude motion of a rigid gravity-gradient-stabilized spacecraft moving in a known eccentric orbit has been formulated from the Hamiltonian point of view. A series of canonical transformations, following Breakwell and Pringle (Reference 12) were developed; these considerably simplify the Hamiltonian. The formulation is extended to include terms to fourth-order in eccentricity.

The final form of the Hamiltonian given by Equation 4-14 can be easily interpreted. When only the linear terms are included (i.e., when H^{***} and all higher-order terms are deleted) the Hamiltonian equations

$$\dot{\beta}_i = \frac{\partial H^{***}(\alpha, \beta, t)}{\partial \alpha_i} = 0 \tag{4-16}$$

and

$$\dot{\alpha}_i = \frac{-\partial H^{***}(\alpha, \beta, t)}{\partial \beta_i} = 0$$

have the solutions

$$\beta_i(t) = \beta_i(0)$$

and

$$\alpha_i(t) = \alpha_i(0) .$$

The original variables of the problem (i.e., the q_i and p_i) are then given by Equations 4-15 as combinations of sinusoidal oscillations with constant amplitude and frequency. In fact, the amplitudes are determined by the initial conditions, and the frequencies are the eigen-frequencies or frequencies of the normal modes found from Equations 4-5 and 4-6. Unbounded motion only occurs when $\omega_3 = 1$, for in this case $b(t)$ and $c(t)$ are unbounded, as can be seen from Equations 4-12 and 4-13.

The situation changes when the nonlinear terms are included. The third-order part of the Hamiltonian, $H_3^{***}(\alpha, \beta, t)$ is not identically zero; therefore Equations 4-16 no longer have a constant value as a solution. The terms in these equations are small in magnitude, either because they are second-order in small displacements and momenta; because they involve a small parameter e; or for both reasons. Thus the motion may be regarded as consisting of sinusoidal oscillations with slowly varying amplitude and frequency. In the next chapter the method of averaging is applied to find regions where these small variations will have a significant effect.

CHAPTER 5

GENERAL PARAMETRIC AND NONLINEAR RESONANCES

Introduction

An approximate solution to the single-degree-of-freedom, linear, in-plane, pitch equation was developed in terms of asymptotic expansions in powers of a small parameter in Chapter 3. A second form of this general approach, called the method of averaging (or of Krylov-Bogoliubov) is applicable to multi-degree-of-freedom equations such as those of the previous chapter. The theory for both these methods of solution is given by Bogoliubov and Mitropolsky (Reference 27). A particular implementation of the method of averaging is used herein; it follows closely that of Breakwell and Pringle (Reference 12). The formulation has been extended to include all terms of second or third order in the generalized coordinates and through fourth order in eccentricity.

The approach outlined in Reference 27 is to consider the problem as defined by kinetic and potential energies (which are quadratic forms in the generalized coordinates) and to assign the nonlinearities to a perturbation term. This implies that both the nonlinear effects and the other components of the perturbing term are "small." The equations are transformed, first to normal coordinates and then again to reflect the harmonic solution of the linear portion of the problem. Once this is accomplished, the equations are in what Bogoliubov and Mitropolsky refer to as "the standard form," which is

$$\frac{dx_k}{dt} = \epsilon X_k\left(t, x_1, x_2, \cdots, x_n\right), \qquad k = 1, \cdots, n, \tag{5-1}$$

where ϵ is a small parameter and X_k may be represented by the sum

$$X_k\left(t, x_1, \cdots, x_n\right) = \sum_{v} \exp(ivt) X_{kv}\left(x_1, \cdots, x_n\right), \qquad k = 1, \cdots, n.$$

This is exactly the result of Chapter 4 in which the Hamiltonian is developed through third order in the generalized coordinates and momenta, and through fourth order in eccentricity. The transformations to normal coordinates q'', p'', and then to coordinates α, β (which reflect the simple harmonic motion associated with the unperturbed linear equations) are accomplished through the applications of canonical transformations.

In the specific case under discussion, Equation 5-1 is replaced by the six Equations 4-16 developed in the preceding chapter. The intent of this chapter is to find, by the method of averaging, the portions of the Lagrange region where the perturbing terms due to orbital eccentricity and nonlinear coupling have a significant effect on the motion.

The basic assumption of the principle of averaging is that the coordinates of the problem undergo a motion composed of a slowly varying term and small vibrational terms. That is, the time required for any of the variables to change appreciably is long compared with the oscillation period of the unperturbed coordinate. When this is the case, Equation 5-1 is replaced in the first approximation by an equation of the form.

$$\frac{d\xi_k}{dt} = \epsilon \underset{t}{M}\left[X_k\left(x_1, x_2, \cdots, x_n\right)\right], \qquad k = 1, \cdots, n,$$

where

$$x_k = \xi_k + \text{small vibrational terms},$$

$\underset{t}{M}$ is the operator of averaging with respect to time, and the ξ are held constant during the averaging. The theory can be continued to develop higher-order oscillations that reflect the effect of the small oscillations in x on the perturbing force and thus on the motion in general.

There are no constant terms in $H_3^{***}(\alpha, \beta, t)$; thus, when the Hamiltonian is averaged over time, its value is zero and the coordinates represent unperturbed motion except in a limited number of cases. The third-order Hamiltonian contains a number of terms that are themselves products of harmonic terms. For example, the term $1 - r_2/r_2 \; q_1 p_3 p_2$ that appears in $H_3(q, p, t)$ is transformed into

$$\frac{1-r_2}{r_2}\left[d_{11} d_{41} (q_1'')^2 + \left(d_{11} d_{42} + d_{12} d_{41}\right) q_1'' q_2'' + d_{12} d_{42} (q_2'')^2\right]\left[p_3'' + c(t)\right]$$

in $H_3^{**}(q'', p'', t)$ and then into

$$\frac{1-r_2}{r_2}\left[\frac{d_{11} d_{41}}{\omega_1^2} 2\alpha_1 \sin^2 \omega_1\left(t+\beta_1\right) + \left(d_{11} d_{42} + d_{12} d_{41}\right)\frac{2}{\omega_1 \omega_2}\sqrt{\alpha_1 \alpha_2}\, \sin\omega_1\left(t+\beta_1\right) \sin\omega_2\left(t+\beta_2\right)\right.$$

$$\left. + \frac{d_{12} d_{42}}{\omega_2^2} 2\alpha_2 \sin^2 \omega_2\left(t+\beta_2\right)\right]\left[\sqrt{2\alpha_3}\cos\omega_3\left(t+\beta_3\right) + c(t)\right]$$

in $H_3^{***}(\alpha, \beta, t)$. There are two types of terms in an expression such as that given above which may not integrate to zero. They become more apparent if it is noted that

$$\sin^2 \omega_1\left(t+\beta_1\right)\cos\omega_3\left(t+\beta_3\right) = \frac{1}{2}\cos\omega_3\left(t+\beta_3\right)$$

$$- \frac{1}{4}\left\{\cos\left[\left(2\omega_1 + \omega_3\right)t + 2\omega_1\beta_1 + \omega_3\beta_3\right] + \cos\left[\left(2\omega_1 - \omega_3\right)t + 2\omega_1\beta_1 - \omega_2\beta_3\right]\right\},$$

and that

$$\sin^2 \omega_1 \left(t + \beta_1\right) c(t) = \frac{1}{2} \sum_{i=1}^{4} c_i \cos(it)$$

$$- \frac{1}{4} \sum_{i=1}^{4} c_i \left\{ \cos\left[\left(2\omega_1 + i\right)t + 2\omega_1 \beta_1\right] + \cos\left[\left(2\omega_1 - i\right) + 2\omega_1 \beta_1\right] \right\} .$$

Thus the first term does not average to zero when $2\omega_1 = \omega_3$ and the second term does not average to zero when $2\omega_1 = 1, 2, 3,$ or 4. These are examples of internal and external resonances respectively. This nomenclature arises because the internal resonances are inherent in the system, even for a circular orbit. They originate in the expressions for the kinetic and potential energies, while the external resonances arise from commensurability of frequencies of the external forces and accelerations with the normal frequencies of the system. When the orbit is circular (i.e., when $e = 0$), all the coefficients a_i, b_i, and c_i are zero and there are no external resonances.

Breakwell and Pringle demonstrated the existence of two internal and four external resonances for spacecraft with inertia parameters in the Lagrange region. The internal resonances appeared for $\omega_3 \approx 2\omega_1$ and $\omega_3 \approx \omega_2 - \omega_1$. External resonances were found when ω_1 or $\omega_3 = 1/2$, when $\omega_3 = 1$, and when $\omega_2 - \omega_1 = 1/2$. A number of additional external resonances become apparent when terms through the fourth order in e are included in the expansions for $a(t)$, $b(t)$, and $c_2(t)$. Additional resonances, both external and internal, could also be obtained by inclusion of fourth-order terms in the Hamiltonian, or by consideration of higher-order approximations in the method of averaging. The algebra involved in either of the latter two steps would be almost prohibitive, and they are not included herein.

The method of averaging presents a set of first-order differential equations whose solutions define, within some degree of approximation, the time variation of the parameters $\left(\alpha_i, \beta_i\right)$ and thus the original variables $\left(q_i, p_i\right)$. However, it is not necessary to obtain the solution to these equations; also, the approximations in the analysis might cause misleading results. The analysis, instead, is intended to determine to what extent the variations are bounded and what conditions are required to excite significant changes in the unperturbed motion.

Resonant Frequencies

Any rigid gravity-gradient spacecraft with inertia parameters in the Lagrange region can be shown to be Liapunov-stable for a circular orbit, as shown by Pringle (Reference 16). However, there are inertia parameters for which the averaged third-order Hamiltonian is not zero. These statements do not conflict; for the Lagrange region, internal resonances have been shown (Reference 12) to produce bounded, but occasionally large, interchanges of energy between pitch, roll, and yaw motions. The situation changes markedly when the external resonances are considered for spacecraft in the Lagrange region. In these cases orbital eccentricity, however small, can excite unbounded oscillations in one or more of the spacecraft modes. This is just a three-dimensional generalization of the results obtained in Chapters 2 and 3 for the one-dimensional pitch-motion case.

In discussing the situations in which resonances occur, it is helpful to remember that the ω_i are the frequencies of the normal modes of the linearized equations of motion. They are obtained from Equations 4-5 and 4-6, which can be written with $\lambda = i\omega$ as

$$r_1 r_2 \omega_{1,2}^4 + \left[k r_1^2 - 2r_1 r_2 - (k-1) r_1 + r_2 - 1\right] \omega_{1,2}^2 + (k+1)\left(1 - r_1\right)\left(1 - r_2\right) = 0$$

and

$$\omega_3^2 - k\left(r_2 - r_1\right) = 0 .$$

Thus, since by definition $\omega_1 < \omega_2$, therefore

$$\omega_1 = \left[\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]^{1/2} , \tag{5-2}$$

$$\omega_2 = \left[\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]^{1/2} , \tag{5-3}$$

and

$$\omega_3 = \left[k\left(r_2 - r_1\right)\right]^{1/2} , \tag{5-4}$$

where

$$a = r_1 r_2 ,$$

$$b = k r_1^2 - 2r_1 r_2 - (k-1) r_1 + r_2 - 1 ,$$

$$c = (k+1)\left(1 - r_1\right)\left(1 - r_2\right) ,$$

and

$$k = 3\left(1 + \frac{3e^2}{2} + \frac{15e^4}{8}\right) .$$

When the Hamiltonian $H_3^{***}(\alpha, \beta, t)$ is expanded completely in the manner illustrated above, the combinations of frequencies shown in Table 5-1 are resonant.

Table 5-1

Resonant Frequency Combinations.

Resonance	Effect	Resonance	Effect
1. $\omega_3 - 1 = 0$	D	17. $\omega_3 + 2\omega_1 - 2 = 0$	D
2. $2\omega_1 - 1 = 0$	D	18. $\omega_3 - 2\omega_2 + 2 = 0$	I
3. $2\omega_3 - 1 = 0$	D	19. $\omega_3 + \omega_1 + \omega_2 - 2 = 0$	D
4. $\omega_2 - \omega_1 - 1 = 0$	N	20. $\omega_3 + \omega_2 - \omega_1 - 2 = 0$	N
5. $\omega_3 - 2\omega_1 = 0$	I	21. $\omega_3 - \omega_2 - \omega_1 + 2 = 0$	I
6. $\omega_3 + \omega_1 - \omega_2 = 0$	N	22. $2\omega_2 - 3 = 0$	D
7. $\omega_3 - 2\omega_1 + 1 = 0$	I	23. $2\omega_3 - 3 = 0$	D
8. $\omega_3 - 2\omega_1 - 1 = 0$	I	24. $\omega_1 + \omega_2 - 3 = 0$	D
9. $\omega_3 + 2\omega_1 - 1 = 0$	D	25. $\omega_3 + 2\omega_1 - 3 = 0$	D
10. $\omega_3 + \omega_1 - \omega_2 + 1 = 0$	N	26. $\omega_3 + 2\omega_2 - 3 = 0$	D
11. $\omega_3 + \omega_2 - \omega_1 - 1 = 0$	N	27. $\omega_3 - 2\omega_2 + 3 = 0$	I
12. $\omega_3 - \omega_2 - \omega_1 + 1 = 0$	I	28. $\omega_3 + \omega_1 + \omega_2 - 3 = 0$	D
13. $2\omega_1 - 2 = 0$	D	29. $\omega_3 + \omega_2 - \omega_1 - 3 = 0$	N
14. $2\omega_2 - 2 = 0$	D	30. $2\omega_2 - 4 = 0$	D
15. $2\omega_3 - 2 = 0$	D	31. $2\omega_2 + \omega_3 - 4 = 0$	D
16. $\omega_1 + \omega_2 - 2 = 0$	D	32. $\omega_1 + \omega_2 + \omega_3 - 4 = 0$	D

D indicates divergent oscillations result

I indicates an interchange of energy between the pitch and roll-yaw modes

N indicates no significant effect

The first six terms have been discussed in Reference 12 and with the exception of the linear pitch resonance case ($\omega_3 = 1$) are all either second-order in the coordinates and first-order in eccentricity or third-order in the coordinates. The next ten items are third-order in the coordinates and first-order in eccentricity or second-order in both coordinates and momenta. Items 17 through 24 are either third-order in eccentricity and second-order in the coordinates or the reverse. The resonances for items 25 through 30 are third-order in both coordinates and eccentricity or second-order in the coordinates and fourth-order in eccentricity. The resonances for items 31 and 32 are both third-order in the coordinates and fourth-order in eccentricity.

Each of these resonant-frequency combinations defines a line or point in the inertia space of the Lagrange region. These lines are plotted from numerical evaluation of Equations 5-2, 5-3, and 5-4 with $k = 3$ (i.e., for $e = 0$). The first six lines are shown in Figure 5-1; the next ten are shown in Figure 5-2, the following eight in Figure 5-3 and the last eight in Figure 5-4. The same combination of lines is shown in Figures 5-5 through 5-8 respectively where the equations are evaluated with $k = 3.189$ (i.e., for $e = 0.2$). The similarity of the figures demonstrates the relative insensitivity of the locations of most of the lines of resonance to variations in e.

Behavior Near Resonances

The Hamiltonian $H^{***}(\alpha, \beta, t)$ has terms with a steady non-zero component whenever any of the 32 resonant relationships of the preceding section are satisfied. Whenever this occurs, the averaged perturbation equations

$$\dot{\alpha}_i = -\frac{2}{2\beta_i} \underset{t}{M}\left[H^{***}(\alpha, \beta, t)\right]$$

and

$$\dot{\beta}_i = \frac{2}{2\alpha_i} \underset{t}{M}\left[H^{***}(\alpha, \beta, t)\right]$$

are not identically zero; thus, there are long-term changes in the values of the α_i and β_i. It is possible to determine the nature of these long-term variations without obtaining formal solutions for $\alpha_i(t)$ and $\beta_i(t)$.

The procedure is best explained with the aid of several examples. Consider the case where $2\omega_1 - 1 = \epsilon$; then the slowly varying terms in the Hamiltonian are

$$\underset{t}{M}\left[H^{***}(\alpha, \beta, t)\right] = \left\{\left[\frac{1-r_2}{r_2} \frac{d_{11}(d_{11}+d_{41})}{\omega_1^2} + d_{23}(d_{23}+d_{33})\right] c_1 + \frac{3(1-r_1)}{2} d_{23}^2 s_1 + \frac{3(1-r_2)}{2} \frac{d_{11} d_{23}}{\omega_1} \left[b_1 + \frac{1}{2}(s_1 b_2 - s_2 b_1)\right]\right\} \frac{\alpha_1}{2} \cos\left[(2\omega_1 - 1)t + 2\omega_1 \beta_1\right] \cdot \quad (5\text{-}7)$$

Each of the terms b_1, c_1, and s_1, are first-order or higher in e; thus Equation 5-7 can be written.

$$H^{***}(\alpha, \beta, t) = eK\alpha_1 \cos(\epsilon t + 2\omega_1 \beta_1) .$$

It is possible to obtain the same equations for α_1 and β_1 from a Hamiltonian that is independent of time and thus a constant of the motion, by considering new variables α_1^*, β_1^* such that

$$\alpha_1^*(t) = \alpha_1(t)$$

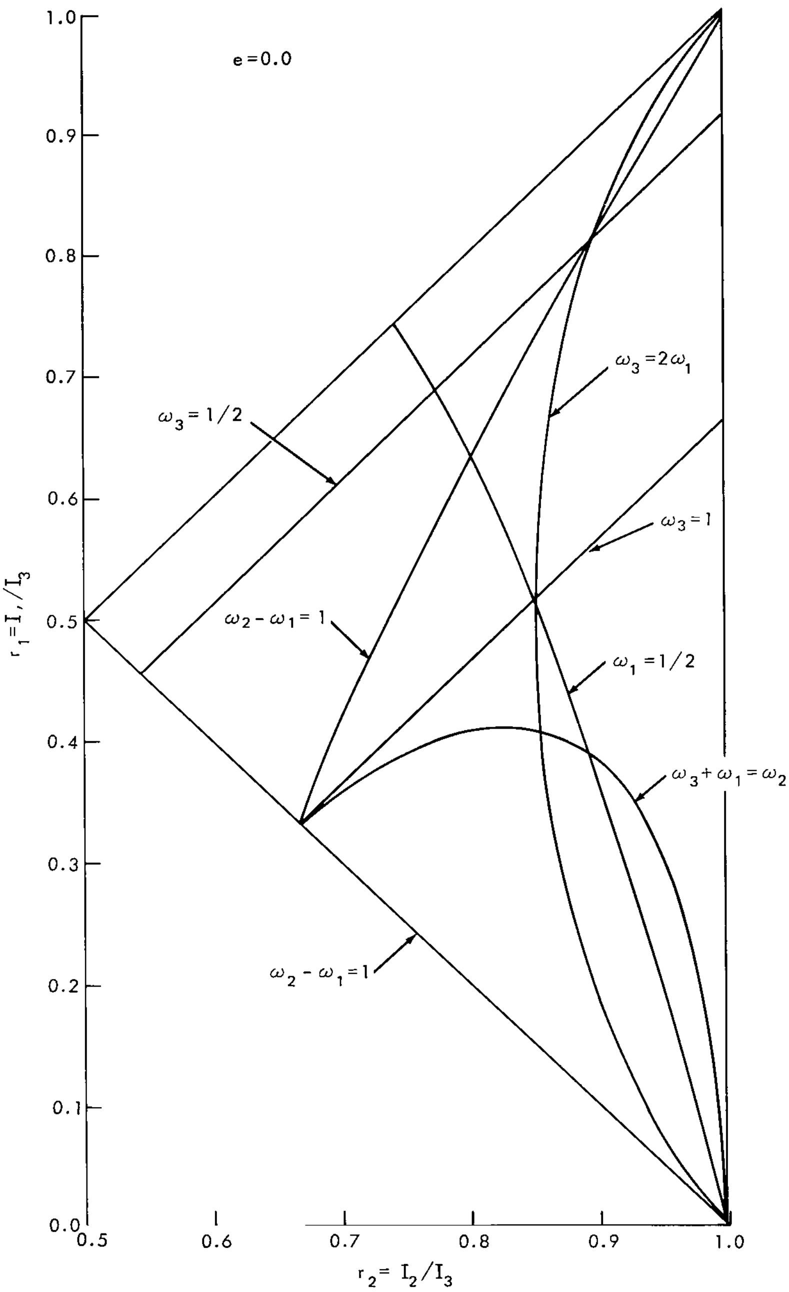

Figure 5-1—Resonant-frequency combinations 1-6 in Table 5-1 (e = 0).

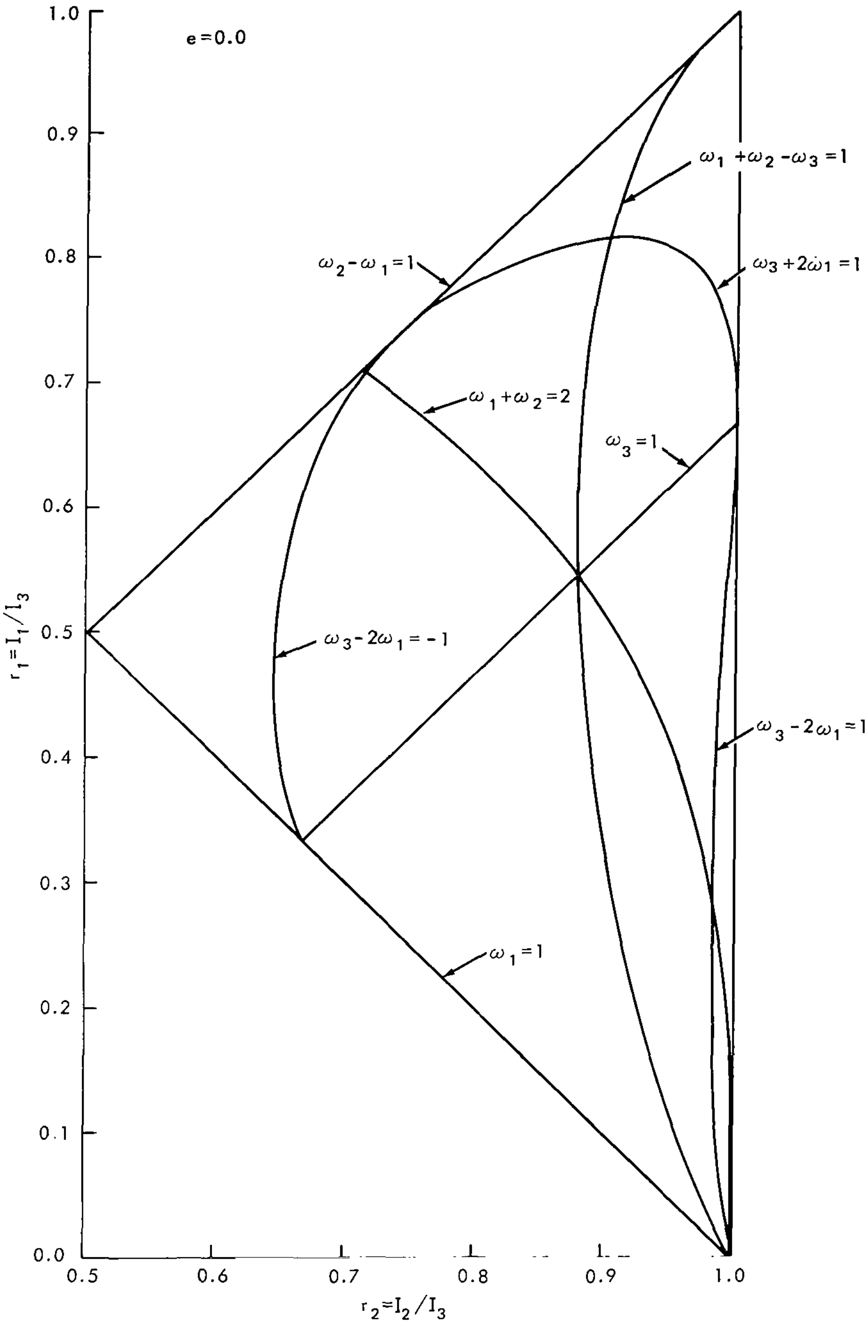

Figure 5-2—Resonant-frequency combinations 7-16 in Table 5-1 (e = 0).

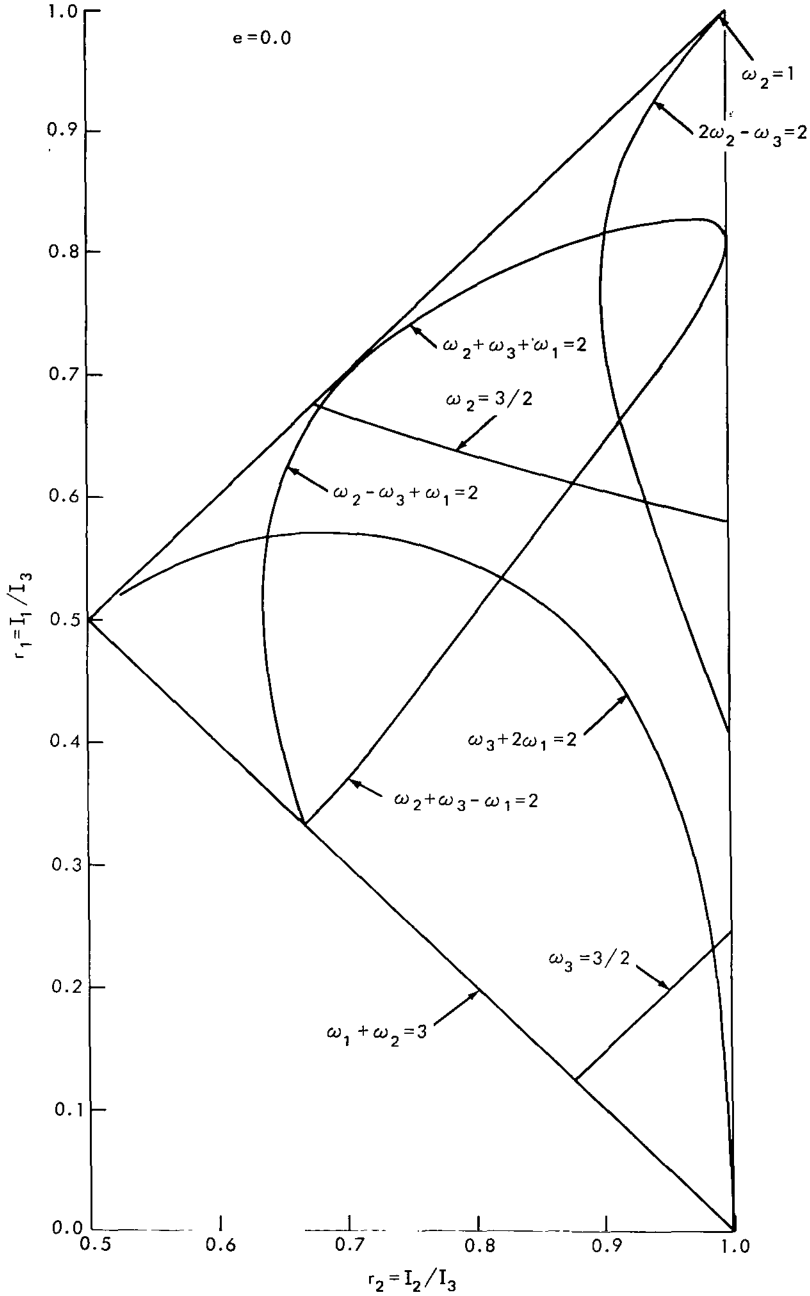

Figure 5-3—Resonant-frequency combinations 17–24 in Table 5-1 (e = 0).

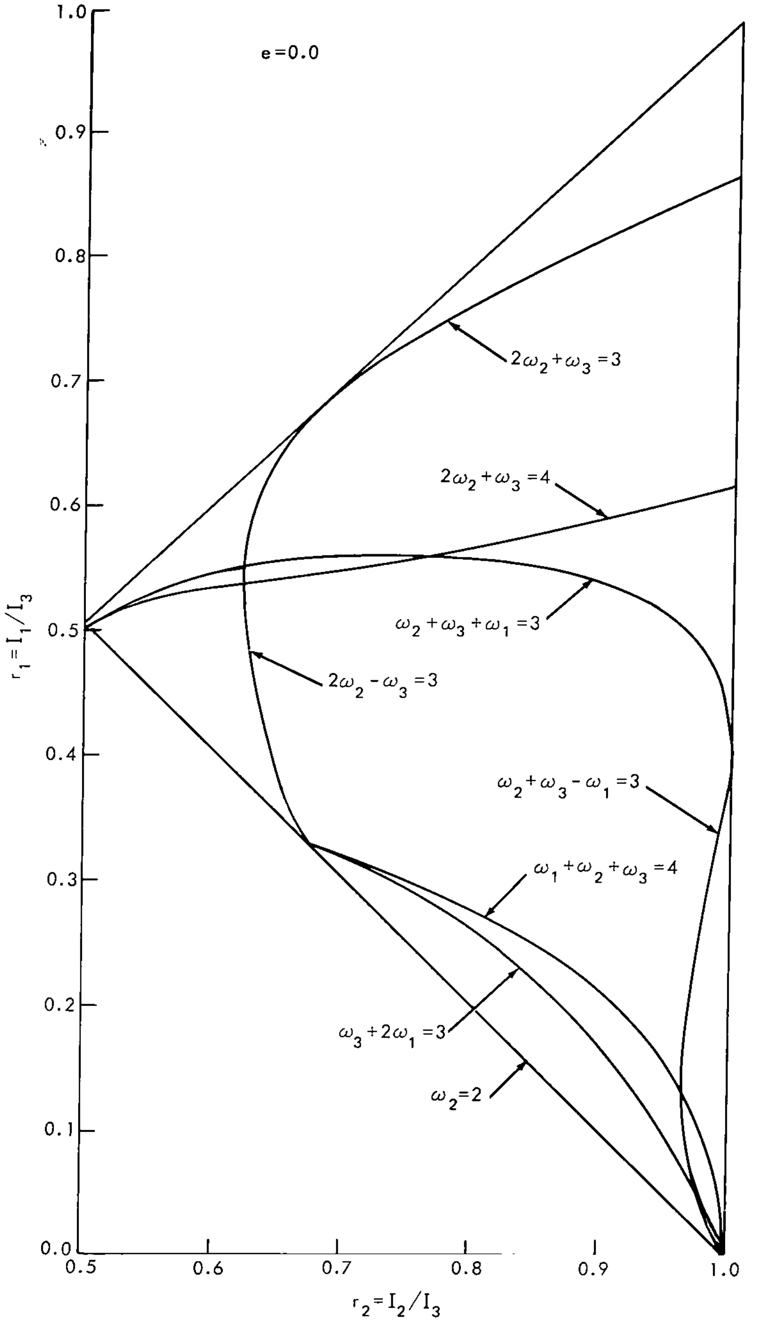

Figure 5-4—Resonant-frequency combinations 25-32 in Table 5-1 (e = 0).

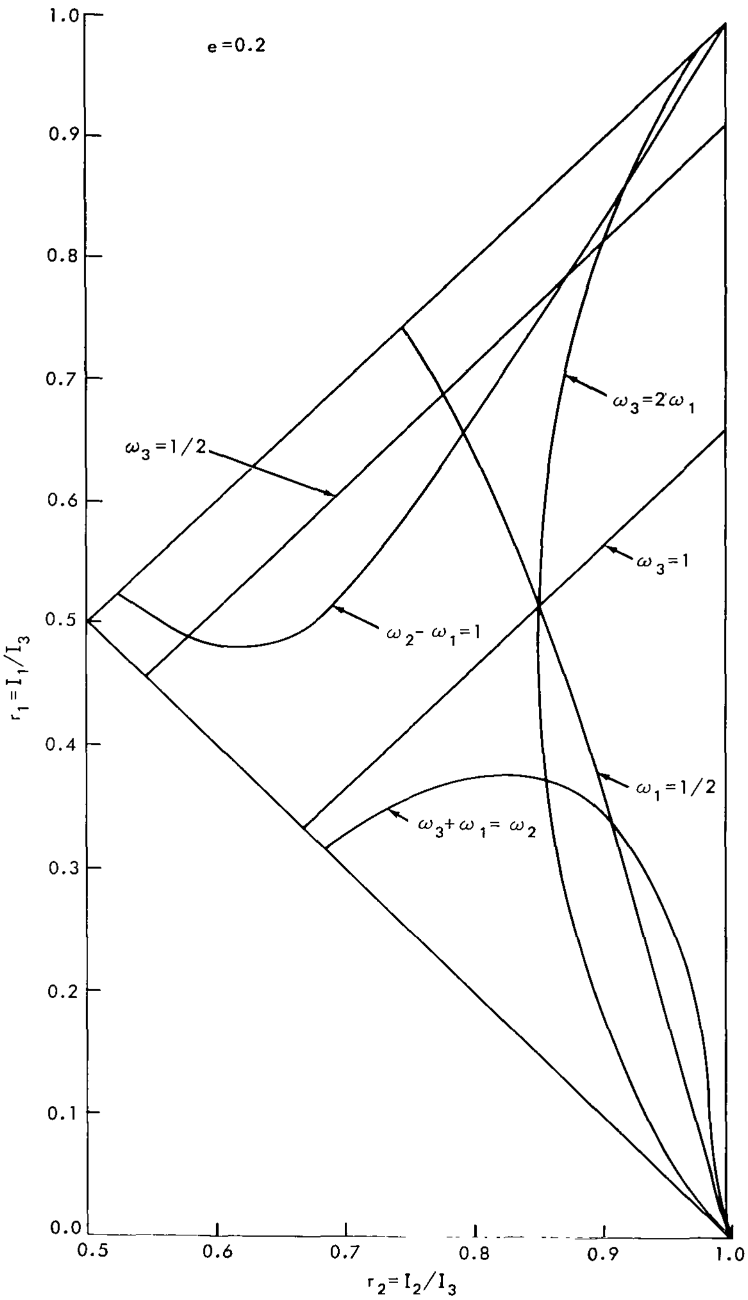

Figure 5-5—Resonant-frequency combinations 1-6 in Table 5-1 (e = 0.2).

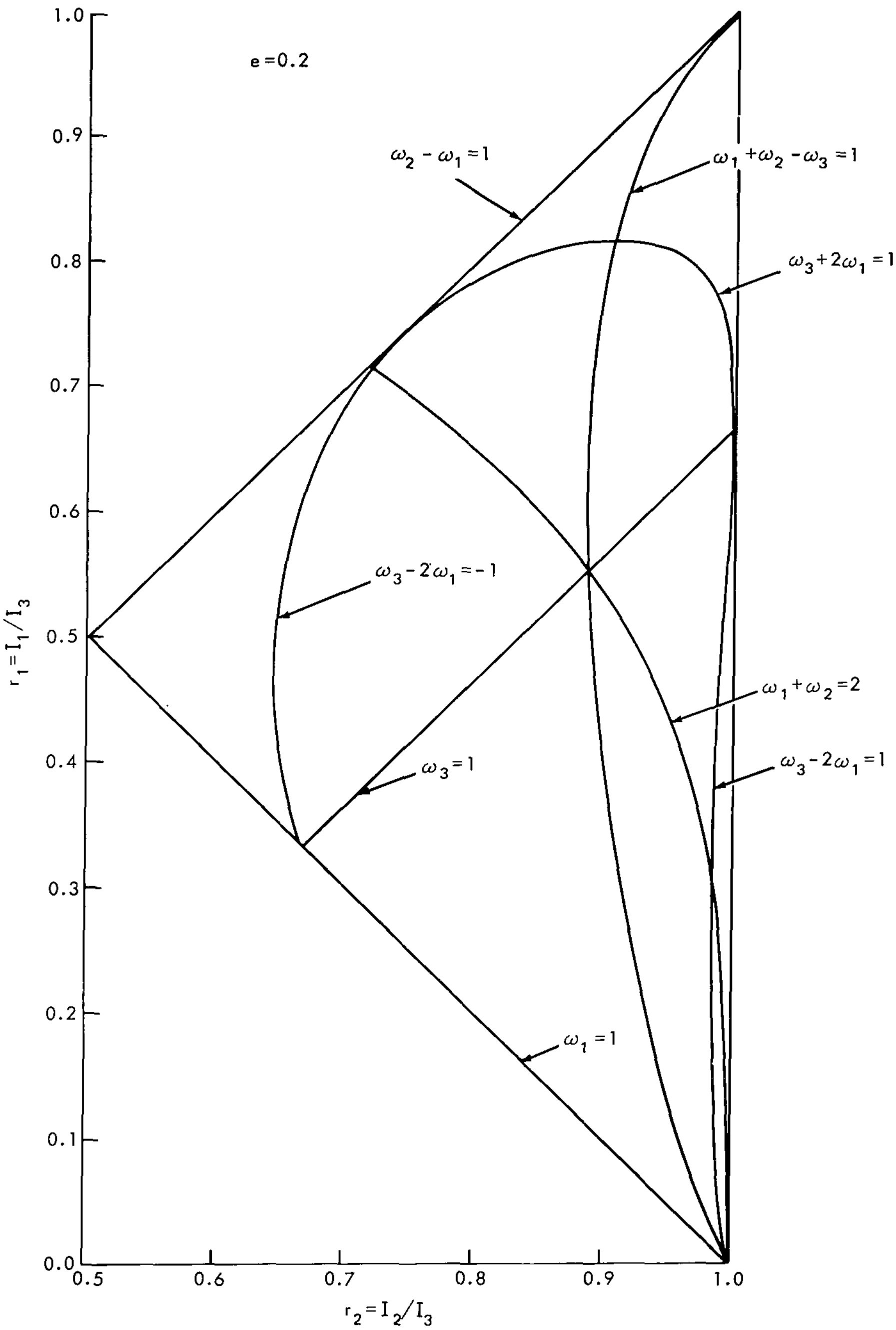

Figure 5-6—Resonant-frequency combinations 7-16 in Table 5-1 (e = 0.2).

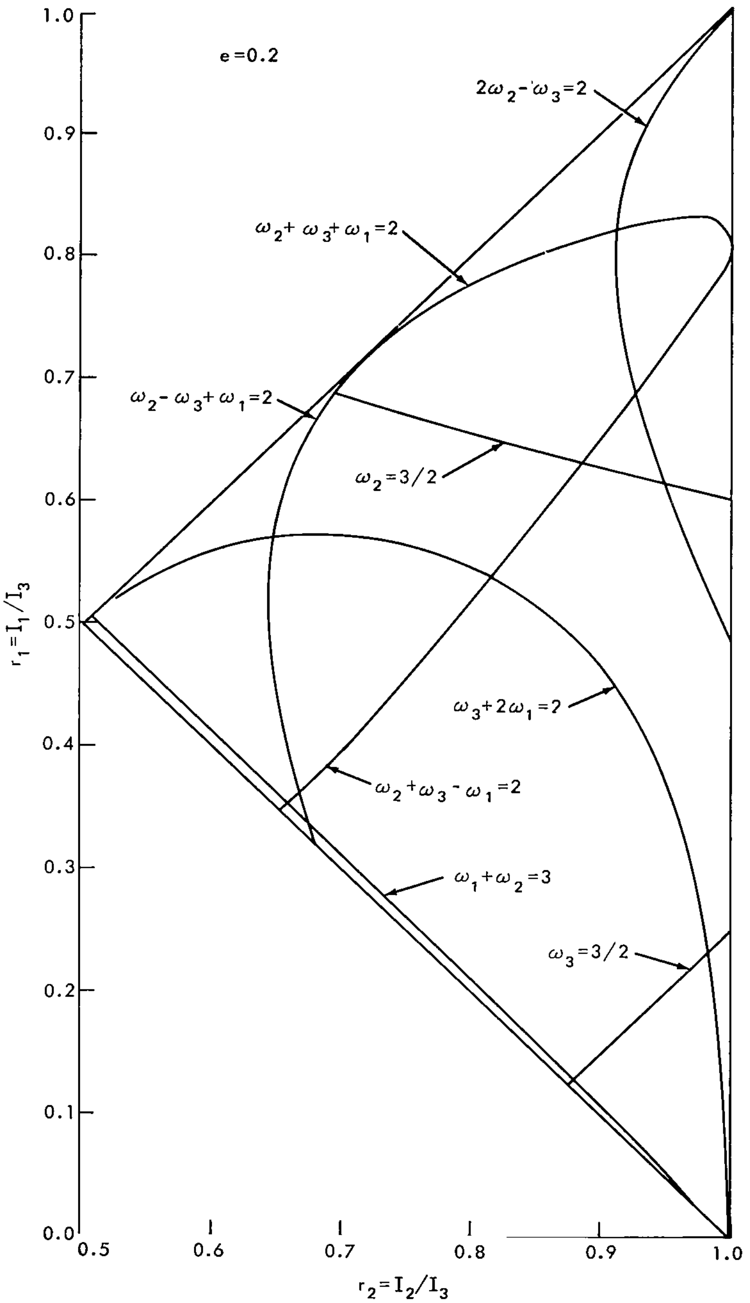

Figure 5-7—Resonant-frequency combinations 17-24 in Table 5-1 (e = 0.2).

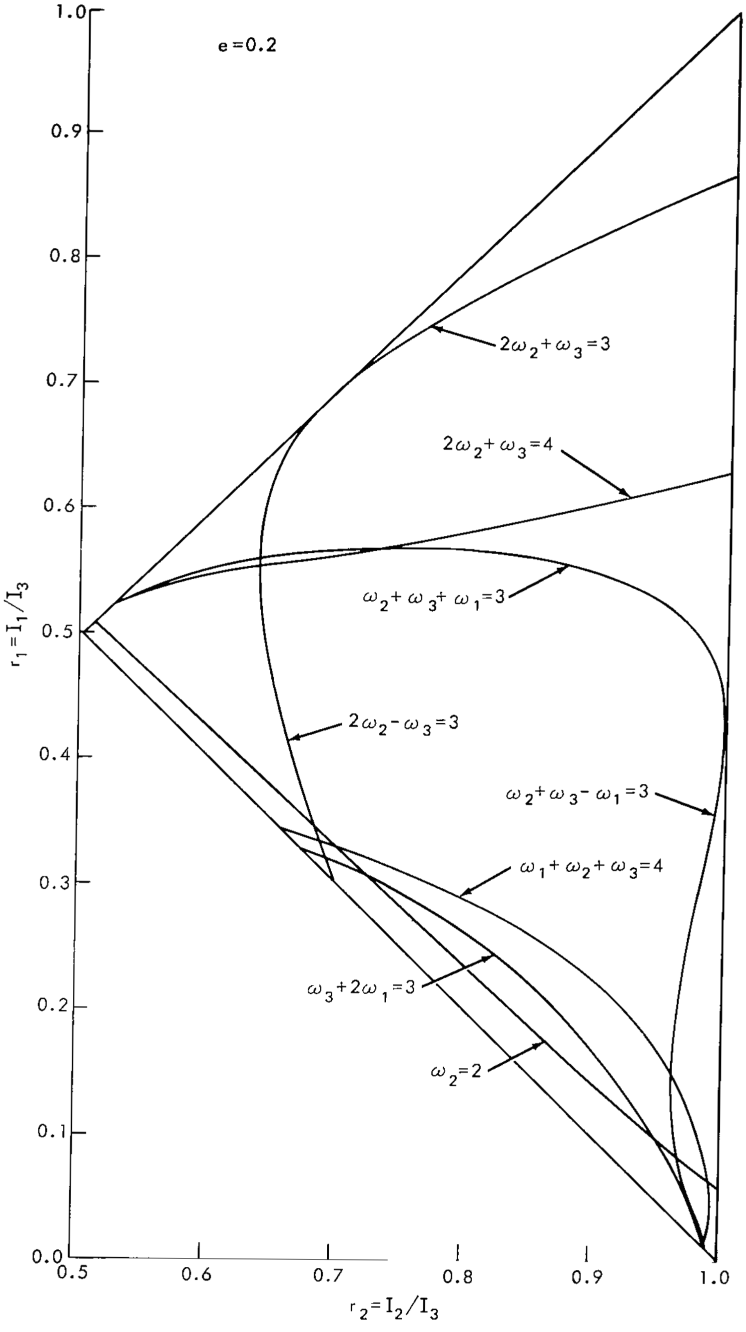

Figure 5-8—Resonant-frequency combinations 25-32 in Table 5-1 (e = 0.2).

and

$$\beta_1^*(t) = \beta_1(t) + \frac{\epsilon t}{2\omega_1}$$

together with a new Hamiltonian $H'(\alpha^*, \beta^*)$ defined as

$$H'(\alpha^*, \beta^*) = eK\,\alpha_1^* \cos\left(2\omega_1 \beta_1^*\right) + \frac{\epsilon \alpha_1^*}{2\omega_1} \cdot$$

The perturbation equations are

$$\dot{\alpha}_1^* = 2\omega_1\, eK\,\alpha_1^* \sin 2\omega_1 \beta_1$$

and

$$\dot{\beta}_1^* = eK \cos 2\omega_1 \beta_1 + \frac{\epsilon}{2\omega_1} \cdot$$

The nature of the solution to these equations depends on the magnitude of ϵ relative to e. When $|\epsilon/2\omega_1| > |eK|$, $\dot{\beta}_1^*$ is always positive for $\epsilon > 0$ and negative for $\epsilon < 0$; thus $\beta_1^*(t)$ increases monotonically for $\epsilon > 0$ and decreases monotonically for $\epsilon < 0$. In this case $\dot{\alpha}_1^*(t)$ alternates in sign and $\alpha_1^*(t)$ exhibits bounded oscillation.

$$\text{When } |eK| > \left|\frac{\epsilon}{2\alpha_1}\right|, \qquad \lim_{t\to\infty} \beta_1^*(t) = -\operatorname{arc\,cos} \frac{\epsilon}{2e\,\omega_1 K}$$

in such a manner that $\lim_{t\to\infty} \alpha_1^*(t) \to \infty$. This can be more easily seen from the phase plane considerations which will be discussed next.

The new Hamiltonian $H'(\alpha_1^*, \beta_1^*)$ is a constant of the motion, thus it is possible to draw lines of constant H in a phase space with coordinates α_1^* and $2\omega_1 \beta_1^*$. This is done in Figure 5-9 for the case $|eK| < |\epsilon/2\omega|$ and in Figure 5-10 for $|eK| > |\epsilon/2\omega_1|$. Figure 5-9 is drawn for $\epsilon > 0$ and Figure 5-10 for $K > 0$; it is only necessary to change the direction of arrowheads for the cases $\epsilon < 0$ and $K < 0$.

There is a third approach to the problem which also depends on the constancy of $H'(\alpha^*, \beta^*)$. Equating values of H at $t = 0$ and $t = t$ gives

$$\cos 2\omega_1 \beta_1^*(t) = \frac{\dfrac{H_0'}{eK} - \dfrac{\epsilon \alpha_1^*(t)}{2e\,\omega_1 K}}{\alpha_1^*(t)},$$

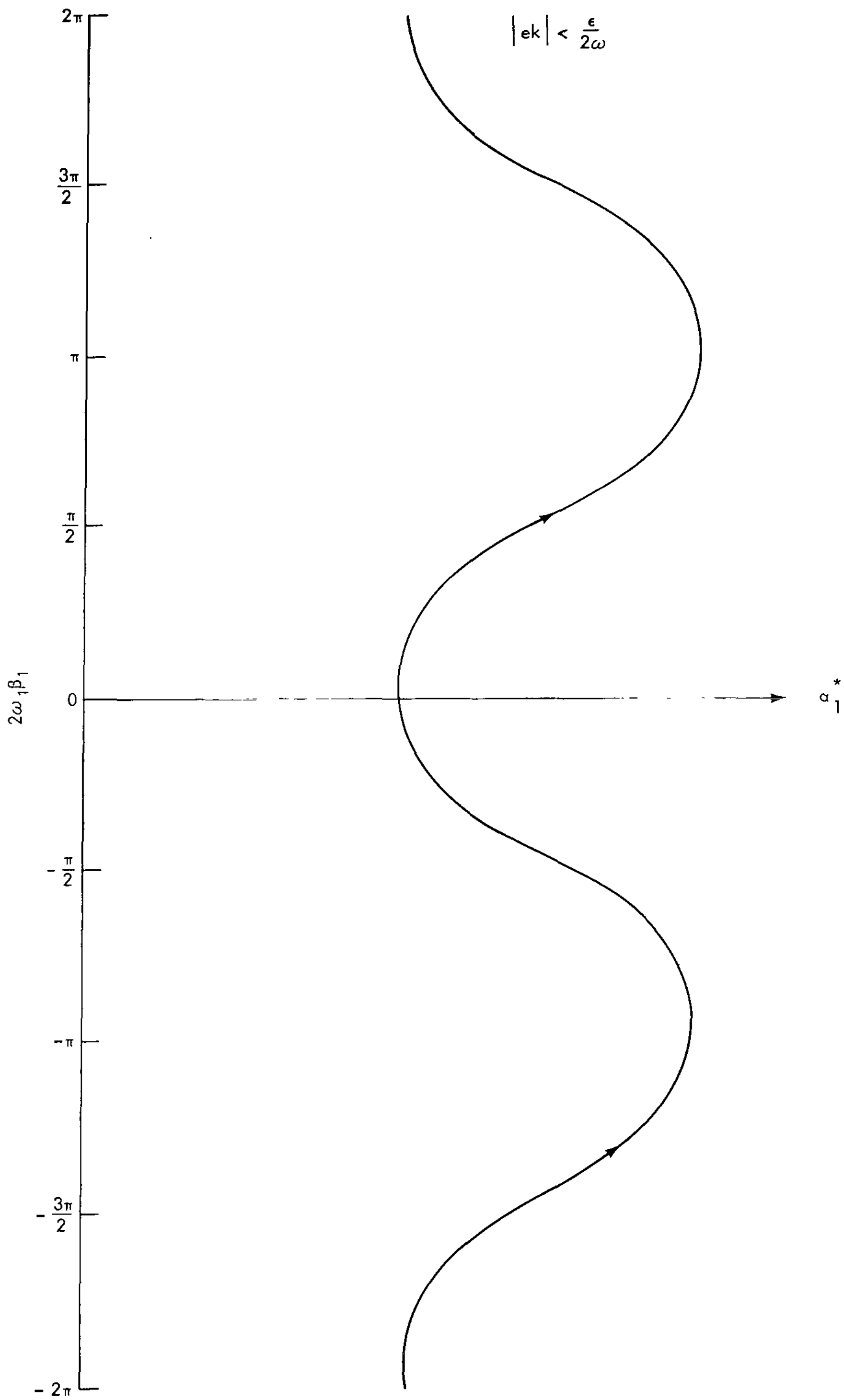

Figure 5-9—Phase plane illustrating bounded motion when $2\omega_1 - 1 = \epsilon$, $\epsilon > 2\omega_1$ eK.

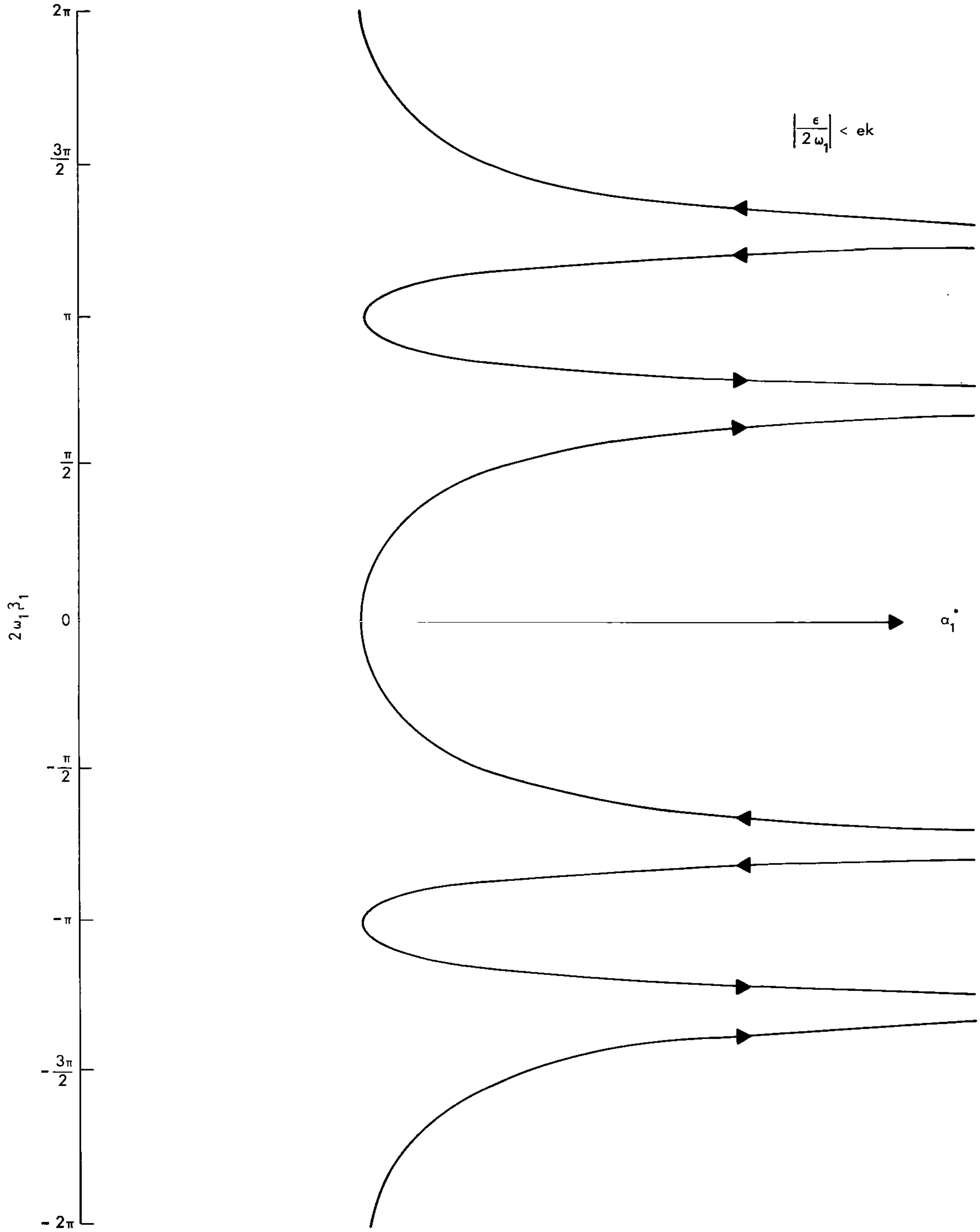

Figure 5-10—Phase plane illustrating unbounded motion when $2\omega_1 - 1 = \epsilon$, $\epsilon < 2\omega_1$ eK.

where $H_0' = H'(\alpha^*, \beta^*)$ evaluated at $t = 0$. However, $\dot{\alpha}_1(t)$ is zero only when $\cos 2\omega_1\beta_1 = \pm 1$ (i.e., when $\sin 2\omega_1\beta_1 = 0$). Thus $\dot{\alpha}_1(t) = 0$ only when

$$\pm\alpha_1^*(t) = \frac{H_0}{eK} - \frac{\epsilon\,\alpha_1^*(t)}{2e\,\omega_1 K} .$$

The two sides of this equation are plotted versus α_1^* (as firm and dotted lines, respectively) in Figure 5-11 for $\epsilon/2e\,\omega_1 K > 1$, which clearly reveals the extent to which $\alpha_1^*(t)$ is bounded. Figure 5-12 makes the same plots for $0 < \epsilon/2e\,\omega_1 K < 1$. In this case it is apparent that if $\dot{\alpha}_1^*(t)$ is ever positive it is always positive, and it can be seen from Equations 5-9 and 5-10 that $\dot{\alpha}_1^*(t)$ will become positive for any initial conditions.

Each of the three approaches leads to the same set of conclusions. The value of $\alpha_1^*(t)$ increases without bound for $2\omega_1$ sufficiently close to 1, and the minimum frequency mismatch (i.e. ϵ) required to guarantee bounded motion increases linearly with eccentricity. As has been noted before, the first conclusion means that the motion increases until higher-order terms become important.

Fortunately, this same analysis applies to a number of the other resonances shown in Table 5-1. In particular, when $2\omega_3 - 1 = \epsilon$, the slowly varying terms in the Hamiltonian have the form

$$\underset{t}{M}\left[H^{***}(\alpha, \beta, t)\right] = eK'\,\alpha_3 \cos\left[\left(2\omega_3 - 1\right)t + 2\omega_3\,\beta_3\right] ,$$

where K' can be obtained as a collection of terms in the same manner as was done above. Obviously the same conclusions apply to this case as did for the previous one. In fact, when any of the frequencies are such that $2\omega_i - j = \epsilon$, the Hamiltonian will have the form

$$\underset{t}{M}\left[H^{***}(\alpha, \beta, t)\right] = e^j\,K\alpha_i \cos\left[\left(2\omega_i - j\right)t + 2\omega_i\,\beta_i\right] ,$$

where K may again be obtained by collecting the appropriate terms in the expansion of $H^{***}(\alpha, \beta, t)$, and the width of the resonant band is proportional to e^j. Thus frequency combinations 13, 14, 15, 22, 23, and 30 all lead to unbounded motion if the appropriate value of ϵ is sufficiently small. This is the three-dimensional generalization of the parametric resonance phenomena studied in Chapters 2 and 3. These unbounded notions arise from the variation in the restoring torque with orbit position, rather than from the nonlinear terms.

Both the combinations $\omega_1 + \omega_2 = 2$, and $\omega_1 + \omega_2 = 3$, have long-term variations of the form

$$\underset{t}{M}\left[H(\alpha, \beta, t)\right] = Ke^j\sqrt{\alpha_1\,\alpha_2}\,\cos\left[\left(\omega_1 + \omega_2 - j\right)t + \omega_1\,\beta_1 + \omega_2\,\beta_2\right] .$$

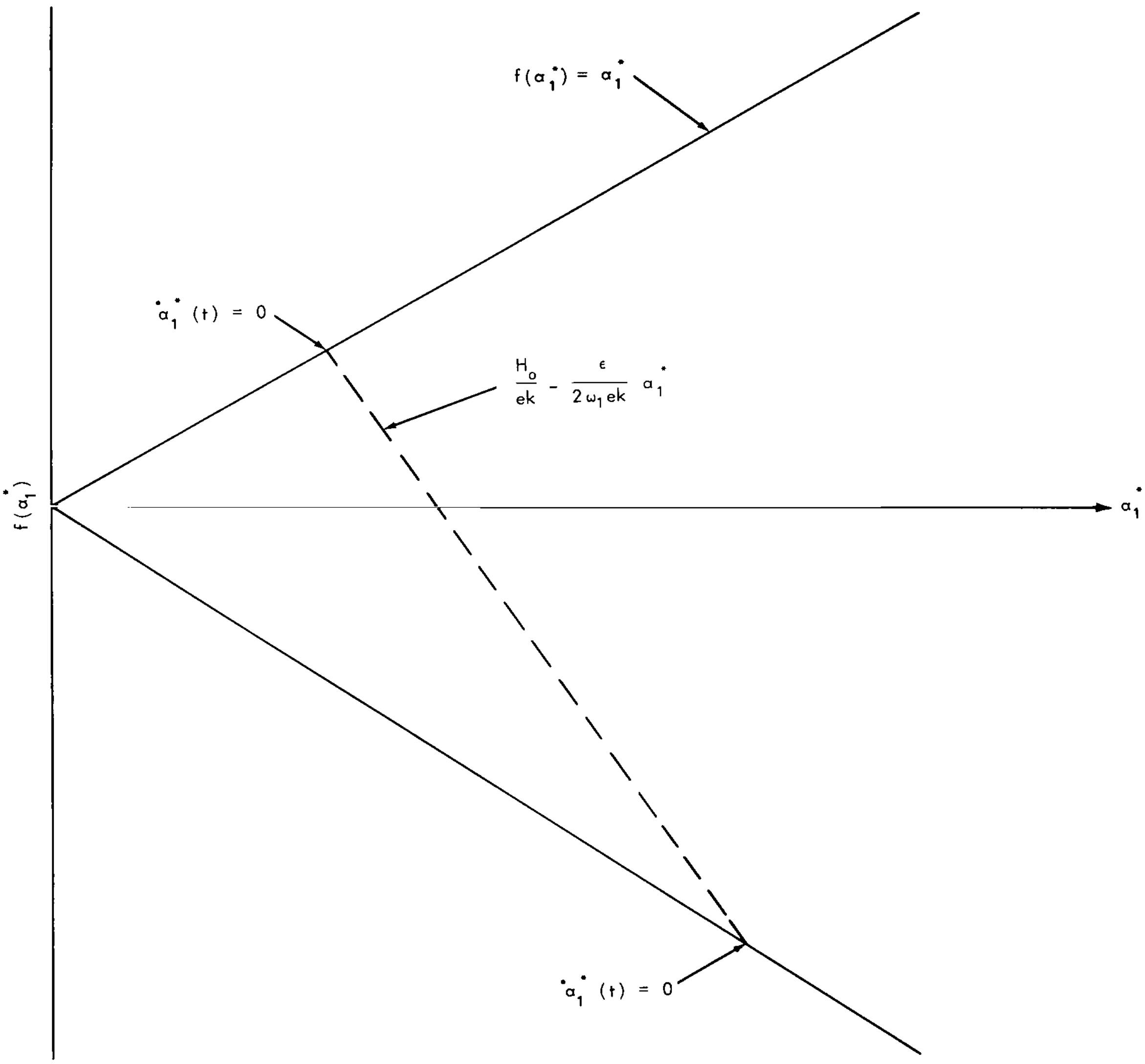

Figure 5-11—Inflection points for $\dot{\alpha}_1^*$, when $2\omega_1 - 1 = \epsilon$, $\epsilon < 2\omega_1 eK$.

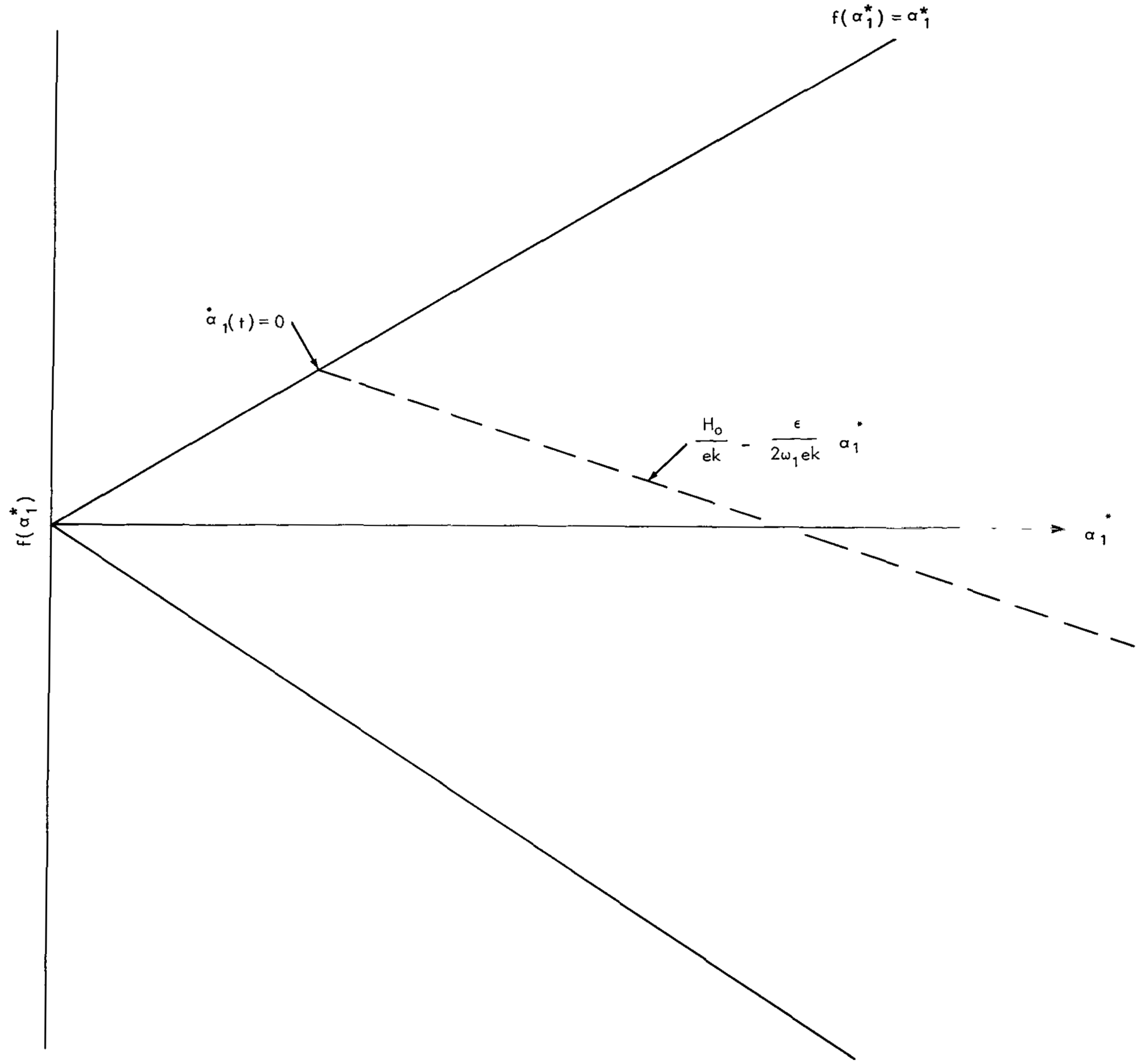

Figure 5-12—Inflection points for $\dot{\alpha}_1^*$, when $2\omega_1 - 1 = \epsilon$, $\epsilon < 2\omega_1$ eK.

Although this form differs from any that have been previously discussed, it is similarly applied. With $\omega_1 + \omega_2 - j = \epsilon$, the averaged Hamiltonian can be written as

$$H^{***}(\alpha, \beta, t) = Ke^j \sqrt{\alpha_1 \alpha_2} \cos\left[\omega_2 \beta_2 + \omega_1 \beta_1 + \epsilon t\right] .$$

The pitch variable α_3 is not included in the Hamiltonian, indicating that, except for small oscillations, it is constant. A new Hamiltonian which is also a constant of the motion can be developed by considering new variables

$$\alpha_1^* = \alpha_1 ,$$

and

$$\beta_1^* = \beta_1 + \frac{\epsilon t}{\omega_1} ,$$

and a new Hamiltonian,

$$H' = Ke^j \sqrt{\alpha_1^* \alpha_2} \cos\left[\omega_2 \beta_2 + \omega_1 \beta_1^*\right] + \frac{\epsilon \alpha_1^*}{\omega_1}$$

The canonical equations for $\dot{\alpha}_1^*(t)$ and $\dot{\alpha}_2(t)$ are

$$\dot{\alpha}_1^* = \omega_1 Ke^j \sqrt{\alpha_1^* \alpha_2} \sin\left[\omega_2 \beta_2 + \omega_1 \beta_1^*\right]$$

and

$$\dot{\alpha}_2 = \omega_2 Ke^j \sqrt{\alpha_1^* \alpha_2} \sin\left[\omega_2 \beta_2 + \omega_1 \beta_1^*\right] ;$$

thus,

$$\dot{\alpha}_2 - \frac{\omega_2}{\omega_1} \dot{\alpha}_1^* = 0$$

and

$$\alpha_2 = \frac{\omega_2}{\omega_1} \left(\alpha_1^* + C_1\right) .$$

The new Hamiltonian can be written with the above as

$$H' = Ke^{j}\sqrt{\frac{\omega_2}{\omega_1}}\sqrt{\alpha_1^*(\alpha_1^*+C)}\cos\left[\omega_2\beta_2+\omega_1\beta_1^*\right]+\frac{\epsilon\alpha_1}{\omega_1},$$

and $\dot{\alpha}_1^*(t)$ is only zero when

$$\pm\sqrt{\alpha_1^{*2}+\alpha_1^*C} = \frac{H_0-\dfrac{\epsilon\alpha_1}{\omega_1}}{Ke^{j}\sqrt{\dfrac{\omega_2}{\omega_1}}}.$$

A graphical construction of the two sides of this equation would be quite similar to Figures 5-11 and 5-12, except that $f(\alpha_1^*) = \sqrt{\alpha_1^{*2}+\alpha_1^*C_1}$ instead of $f(\alpha_1^*) = \alpha_1^*$. Thus, for any initial conditions such that $\alpha_1^*(0)$, $\alpha_2(0) \neq 0$, $\alpha_1^*(t)$ is unbounded if $|\epsilon/\sqrt{\omega_1\omega_2}\,Ke^{j}| < 1$. Since

$$\alpha_2(t) = \frac{\omega_2}{\omega_1}\alpha_1(t)+\frac{\omega_2 C_1}{\omega_1},$$

this means that both the roll-yaw modes are unbounded, to the order of approximation of this solution, when ϵ is sufficiently small.

The 19th, 28th, and 32nd combinations of frequencies in Table 5-1 are all of the form

$$\omega_1+\omega_2+\omega_3-j = 0.$$

The long-period Hamiltonian associated with each of these combinations of frequencies is of the form

$$H^{***}(\alpha,\beta,t) = Ke^{j}\sqrt{\alpha_1\alpha_2\alpha_3}\cos\left[\omega_1\beta_1+\omega_2\beta_2+\omega_3\beta_3+\epsilon t\right],$$

and the change of variables

$$\alpha_1^* = \alpha_1$$

$$\beta_1^* = \beta_1+\frac{\epsilon t}{\omega_1}$$

leads to a Hamiltonian which is a constant of the motion

$$H' = Ke^{j} \sqrt{\alpha_1^* \alpha_2 \alpha_3} \cos\left(\omega_1 \beta_1^* + \omega_2 \beta_2 + \omega_3 \beta_3\right) + \frac{\epsilon \alpha_1^*}{\omega_1} .$$

The derivatives of α_1^*, α_2, and α_3 are linearly related and

$$\dot{\alpha}_3 = \frac{\omega_3}{\omega_1} \dot{\alpha}_1^*$$

i.e.,

$$\alpha_3 = \frac{\omega_3}{\omega_1} \left(\alpha_1^* + C_1\right)$$

and

$$\dot{\alpha}_2 = \frac{\omega_2}{\omega_1} \dot{\alpha}_1^*$$

i.e.,

$$\alpha_2 = \frac{\omega_2}{\omega_1} \left(\alpha_1^* + C_2\right)$$

The Hamiltonian can now be written as

$$H' = Ke^{j} \sqrt{\frac{\omega_2 \omega_3}{\omega_1^2}} \sqrt{\alpha_1^* \left(\alpha_1^* + C_1\right)\left(\alpha_1^* + C_2\right)} \cos\left(\omega_1 \beta_1^* + \omega_2 \beta_2 + \omega_3 \beta_3\right) + \frac{\epsilon \alpha_1^*}{\omega_1}$$

and, similar to the previous case, $\dot{\alpha}_1^*(t)$ is zero only when

$$\pm\sqrt{\alpha_1^* \left(\alpha_1^* + C_1\right)\left(\alpha_1^* + C_2\right)} = \frac{H_0 + \dfrac{\epsilon \alpha_1^*}{\omega_1}}{Ke^{j} \dfrac{\sqrt{\omega_2 \omega_3}}{\omega_1}} .$$

In this case $\alpha_1^*(t)$ is unbounded for any set of initial conditions such that $\alpha_1^*(0)\, \alpha_2(0)\, \alpha_3(0) \neq 0$ if

$$\left|\frac{\epsilon}{Ke^{j} \sqrt{\omega_2 \omega_3}}\right| < \frac{d}{d\alpha_1^*} f\left(\alpha_1^*\right)\Big|_0$$

where

$$f(\alpha_1^*) = \sqrt{\alpha_1^*(\alpha_1^* + C_1)(\alpha_1^* + C_2)} .$$

Thus, when ϵ is sufficiently small, all three modes are excited and the pitch, roll, and yaw oscillations increase without bound.

In a similar manner the combinations of frequencies shown as items 5, 7, 8, 18, and 27 of Table 5-1 are all of the form

$$\omega_3 - 2\omega_i \pm j = 0 , \qquad i = 1, 2 \qquad j = 0, 1, 2, 3 ,$$

and the long-period part of the Hamiltonian has the form

$$H^{***}(\alpha, \beta, t) = Ke^j \alpha_1 \sqrt{\alpha_3} \cos(\omega_3 \beta_3 - 2\omega_i \beta_8 + \epsilon t) ,$$

and, for $j = 0$, this is the internal resonance case of Reference 12. The analysis proceeds with new variables,

$$\alpha_i^* = \alpha_i ,$$

$$\beta_i^* = \beta_i - \frac{\epsilon t}{\omega_i}$$

and a new Hamiltonian

$$H' = Ke^j \alpha_i^* \sqrt{\alpha_3} \cos(\omega_3 \beta_3 - 2\omega_i \beta_i^*) - \frac{\epsilon \alpha_i^*}{2\omega_i} .$$

Then

$$\dot{\alpha}_i^* = -2\omega_i Ke^j \alpha_i^* \sqrt{\alpha_3} \cos(\omega_3 \beta_3 - 2\omega_i \beta_i^*)$$

and

$$\dot{\alpha}_3 = \omega_3 Ke^j \alpha_i^* \sqrt{\alpha_3} \cos(\omega_3 \beta_3 - 2\omega_i \beta_i^*) .$$

Therefore

$$\dot{\alpha}_3 + \frac{\omega_3}{2\omega_i} \dot{\alpha}_i^* = 0$$

and

$$\alpha_3 + \frac{\omega_3}{2\omega_i} \alpha_i^* = C .$$

The new Hamiltonian can thus be written

$$H' = Ke^j \alpha_i^* \sqrt{C - \frac{\omega_3}{2\omega_i} \alpha_i^*} \cos\left(\omega_3 \beta_3 - 2\omega_i \beta_i^*\right) .$$

The process that was demonstrated above applies also to this Hamiltonian. Both the phase plane and the geometric construction for finding values of α_i^* for which $\dot{\alpha}_i^*(t) = 0$ are shown in Reference 12. In essence, the important features of this motion are that the sum of α_i^* and α_3 are constant as is α_j $(j \neq i)$ and,

$$\text{when } \left|\frac{\epsilon}{2\omega_i}\right| < \left|Ke^j \sqrt{C}\right| , \tag{5-5}$$

there is an appreciable periodic interchange of energy between the pitch, α_3, and roll-yaw α_i $(i = 1$ or $2)$ modes.

In fact, it is possible to obtain an analytic solution for $\alpha_i^*(t)$. The first step is to eliminate the β's from the equation for $\dot{\alpha}_i^*(t)$. The process is as follows:

$$\dot{\alpha}_i^*(t) = -2\omega_i Ke^j \alpha_i^* \sqrt{C - \frac{\omega_3}{2\omega_1} \alpha_i^*} \sin\left(\omega_3 \beta_3 - 2\omega_i \beta_i^*\right)$$

or

$$\left[\dot{\alpha}_i^*(t)\right]^2 = 4\omega_i^2 K^2 e^{2j} \alpha_i^{*2} \left(C - \frac{\omega_3}{2\omega_1} \alpha_i^*\right)\left[1 - \cos^2\left(\omega_3 \beta_3 - 2\omega_i \beta_i\right)\right] .$$

However,

$$Ke^j \alpha_i^* \sqrt{C - \frac{\omega_3}{2\omega_1} \alpha_i^*} \cos\left(\omega_3 \beta_3 - 2\omega_i \beta_i^*\right) = H_0' + \frac{\epsilon \alpha_i^*}{2\omega_i} ,$$

and therefore

$$\left[\dot{\alpha}_1^*(t)\right]^2 = 4\omega_i^2\left[K^2 e^{2j}\alpha_i^{*2}\left(C - \frac{\omega_3}{2\omega_i}\alpha_i^*\right) - \frac{\epsilon^2\alpha_i^{*2}}{4\omega_i^2} - \frac{\epsilon H_0}{\omega_i}\alpha_i^* - H_0^2\right]$$

or

$$t = \frac{1}{2\omega_i}\int_{\alpha_i^*(0)}^{\alpha_i^*(t)} \frac{d\alpha_i^*}{\sqrt{-\left[\frac{\omega_3 K^2 e^{2j}}{2\omega_i}\alpha_i^{*3} + \left(\frac{\epsilon^2}{4\omega_i^2} - CK^2 e^{2j}\right)\alpha_i^{*2} + \frac{\epsilon H_0}{\omega_i}\alpha_i^* + H_0^2\right]}}, \tag{5-6}$$

which is an elliptic integral of the first kind. This equation is applicable to any of the five cases listed above, whether they are internal or external resonances.

The internal resonance for $e = 0$ and $\omega_3 = 2\omega_1$ was discovered numerically by Kane (Reference 11). He considered a restricted form of three dimensional motion in which the roll and yaw angles were linearized and a circular orbit was assumed. Floquet theory was used to evaluate the stability of these equations at equal intervals over a parameter space (K_1, K_2) where $K_1 = -(1 - r_2)/r_1$ and $K_2 = (1 - r_1)/r_2$. The initial pitch angle was the third parameter of his solutions. The preceding analysis, when specialized to the case $i = 1$, $j = 0$, indicates that there will be a region about the line $\omega_3 = 2\omega_1$ whose width is proportional to the initial pitch amplitude and in which there will be a large interchange of energy between the pitch and roll-yaw modes of oscillation. The results of Kane's Floquet analysis for initial pitch angles of 5° and 10° are repeated here as Figures 5-13 and 5-14 where these are taken directly from his Figures 8 and 9 respectively. The results are replotted versus r_1 and r_2 in Figure 5-15 and as predicted in Reference 12 show very good agreement with the anticipated results. The scarcity of points at the two ends of the resonance line result from the non-uniformity of the test grid when it is transformed from (K_1, K_2) to (r_1, r_2), and could be removed by evaluating the Floquet analysis for a finer grid. There are also three points of unstable motion on Figures 5-14 and 5-15 which are not explained by the above. These points may be the result of higher order resonance terms not included in this analysis.

Kane also numerically integrated the exact equations of motion for several of his points of instability. His results show that these cases actually correspond to points at which the amplitude of the roll-yaw motion exhibits a slow but bounded oscillation as is apparent from the preceding analysis of the third order Hamiltonian. Breakwell and Pringle commented on this similarity and also demonstrated that the period of the variation in the amplitude of the roll-yaw oscillation obtained from Equation 5-6 was in good agreement with the result of Kane's numerical integration of the exact equations of motion.

The combinations of frequencies shown as items 9, 17, 25, 26, and 31 in Table 5-1 are all of the form

$$\omega_3 + 2\omega_i - j = 0\,, \qquad i = 1, 2 \qquad j = 1, \cdots 4\,;$$

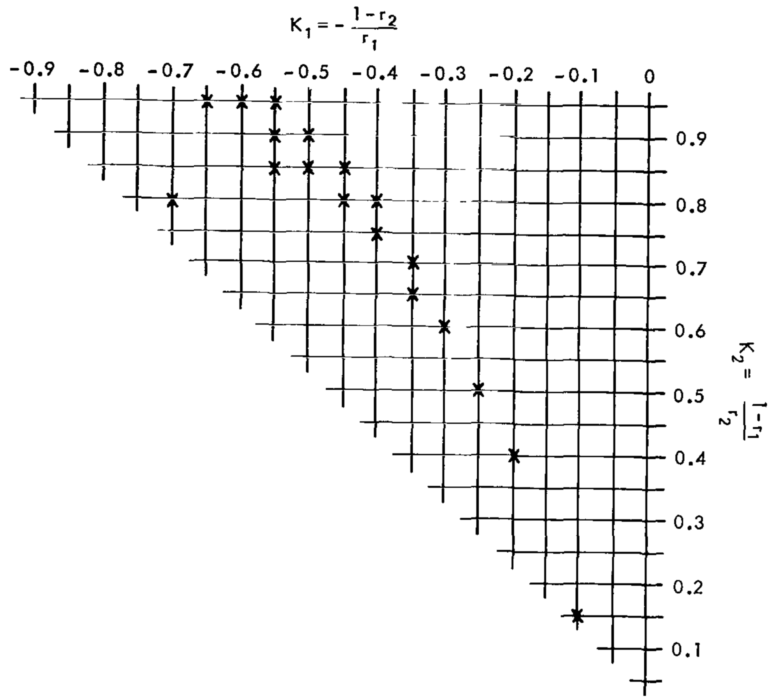

Figure 5-13—Instability chart for $\phi_0 = 5^\circ$ from Kane (Reference 11).

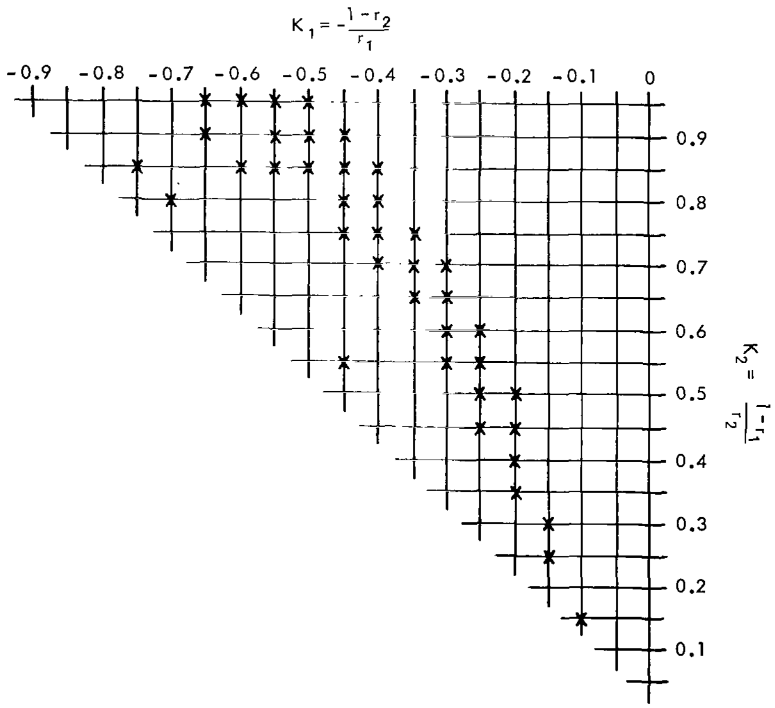

Figure 5-14—Instability chart for $\phi_0 = 10^\circ$ from Kane (Reference 11).

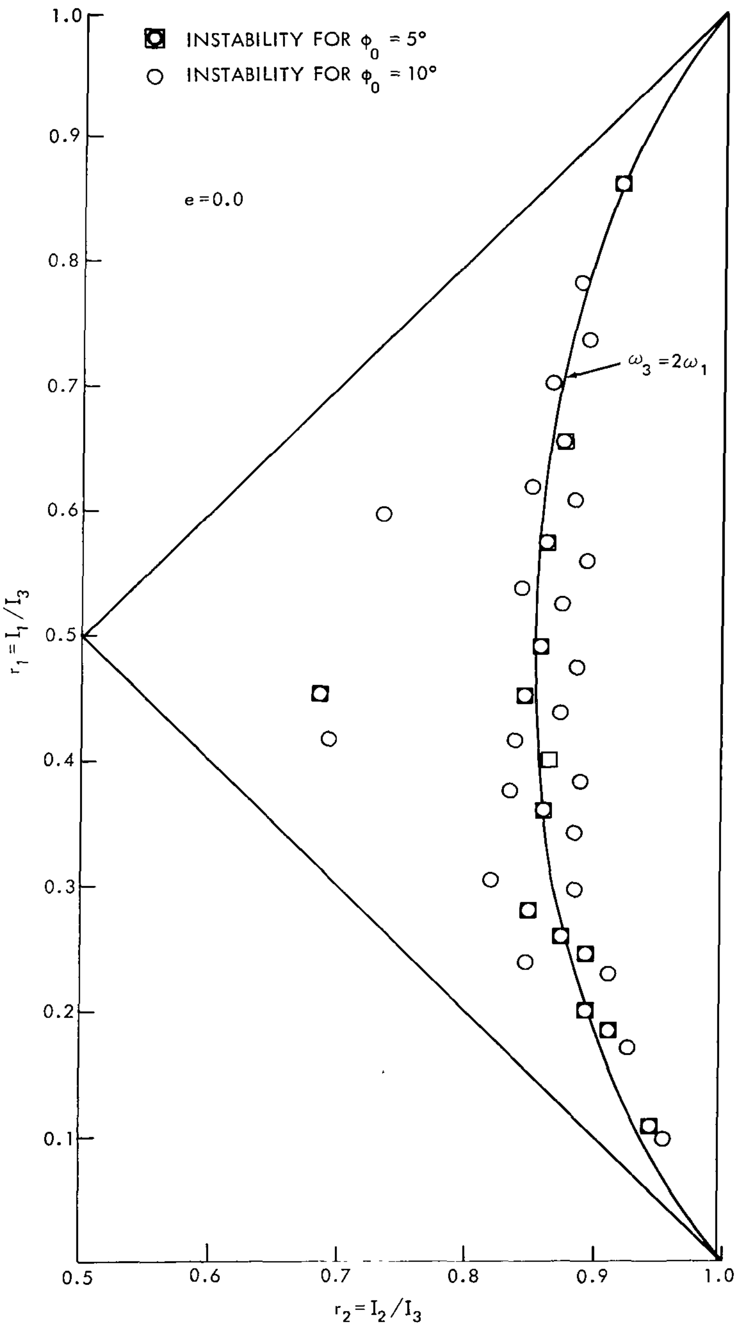

Figure 5-15—Comparison with Kane's numerical results (Reference 11).

and the form of the Hamiltonian associated with these terms is

$$H^{***}(\alpha, \beta, t) = Ke^{j} \alpha_i \sqrt{\alpha_3} \cos\left[\left(\omega_3 + 2\omega_i - j\right)t + \omega_3 \beta_3 + 2\omega_i \beta_i\right] .$$

This general form resembles what resulted when the resonance $\omega_3 \approx 2\omega_1$ was evaluated in the Delp region in Reference 12. When the approach is generalized, the following results are apparent. The difference between the pitch variable α_3 and the appropriate roll-yaw variable α_i is a constant, i.e.,

$$\alpha_3(t) - \frac{\omega_3}{2\omega_i} \alpha_i(t) = C .$$

When the frequency mismatch ϵ is large enough, bounded oscillations are possible for $\alpha_3(t)$ and $\alpha_i(t)$. However, if $|\epsilon/2\omega_1 Ke^{j}|$ is sufficiently small, the motion of both $\alpha_3(t)$ and $\alpha_i(t)$ is completely unbounded. Similarly the combinations of frequencies shown as items 6, 10, 11, 20, and 29 are all of the form

$$\omega_3 + \omega_i - \omega_j \pm k = 0 , \qquad i, j = 1 , \qquad i \neq j \qquad k = 1, 2, 3 ,$$

and they are all associated with terms in the Hamiltonian of form:

$$H^{***}(\alpha, \beta, t) = Ke^{k} \sqrt{\alpha_1 \alpha_2 \alpha_3} \cos\left[\left(\omega_3 + \omega_i - \omega_j \pm k\right)t + \omega_2 \beta_3 + \omega_i \beta_i - \omega_j \beta_j\right] .$$

In this case

$$\alpha_j + \frac{\omega_j}{\omega_i} \alpha_i = C_1 \quad \text{and} \quad \alpha_3 - \frac{\omega_3}{\omega_i} \alpha_i = C_2 .$$

The first of these two relations establishes that if the roll-yaw motion is initially small it will remain small; the second shows that the pitch motion will also remain small.

The frequency combinations shown as items 12 and 21 in Table 5-1 are both of the form

$$\omega_3 - \omega_1 - \omega_2 + j = 0 , \qquad j = 1, 2 ,$$

and the associated Hamiltonian has the form

$$H = Ke^{j} \sqrt{\alpha_1 \alpha_2 \alpha_3} \cos\left[\left(\omega_3 - \omega_1 - \omega_2 + j\right)t + \omega_3 \beta_3 - \omega_1 \beta_1 - \omega_2 \beta_2\right] .$$

The analysis for these two cases shows that

$$\alpha_2 - \frac{\omega_2}{\omega_1} \alpha_1 = c_1 ,$$

and

$$\alpha_3 + \frac{\omega_3}{\omega_1} \alpha_1 = c_2 .$$

Thus energy can be interchanged between the pitch α_3 and both of the roll yaw modes α_1 and α_2. All three modes are bounded and there will be an appreciable interchange only if $|\epsilon/\omega_1 \, Ke^j|$ is sufficiently small.

Application to DODGE Spacecraft

The results of the preceding analysis find an interesting application in the DODGE satellite (Department of Defense Gravity Experiment) of the Johns Hopkins University Applied Physics Laboratory. This spacecraft was designed to test several combinations of mechanical configurations and damping methods. The flight experience of DODGE has displayed a number of unexplained anomalies in the actual versus predicted attitude motion (Reference 33). This analysis does not purport to explain these anomalies; however it does illustrate some interesting features.

It is possible to vary the length of the inertia booms of the DODGE spacecraft and this feature has been exercised frequently during the first six months of the spacecraft's flight. The inertia parameters corresponding to two of the known configurations for DODGE are superimposed on Figures 5-16 and 5-17. At various times in the flight the configuration has been such as to map into other points in this same area. These two points straddle two of the lines of "unbounded" motion which were developed in the preceding section. It is possible that for some portions of the DODGE Mission the inertia parameters have been such as to plot exactly on one of these lines. Actually, no spacecraft with similar extensible booms is exactly rigid; we cannot exactly equate the normal frequencies ω_1, ω_2, and ω_3 with the telemetered boom lengths. Thus we cannot determine whether there have been times when the frequencies of the normal modes have exactly satisfied the criteria for resonance.

Even so, there are several points that should be considered. Although, as a result of the uncertainty mentioned above, we cannot state conclusively that the otherwise unexplained large amplitude motions of the DODGE spacecraft are at least partly due to the resonance phenomena described herein, it is possible. These resonances depend on frequencies of excitation associated with terms of the third and fourth order in e. However, as mentioned in the previous section, the magnitude of the possible disturbance does not depend on the value of e. A numerical simulation of the spacecraft equations of motion that includes terms of third or fourth order in e from the developments for the orbital parameters (as most do) cannot possibly model the effect of these resonances.

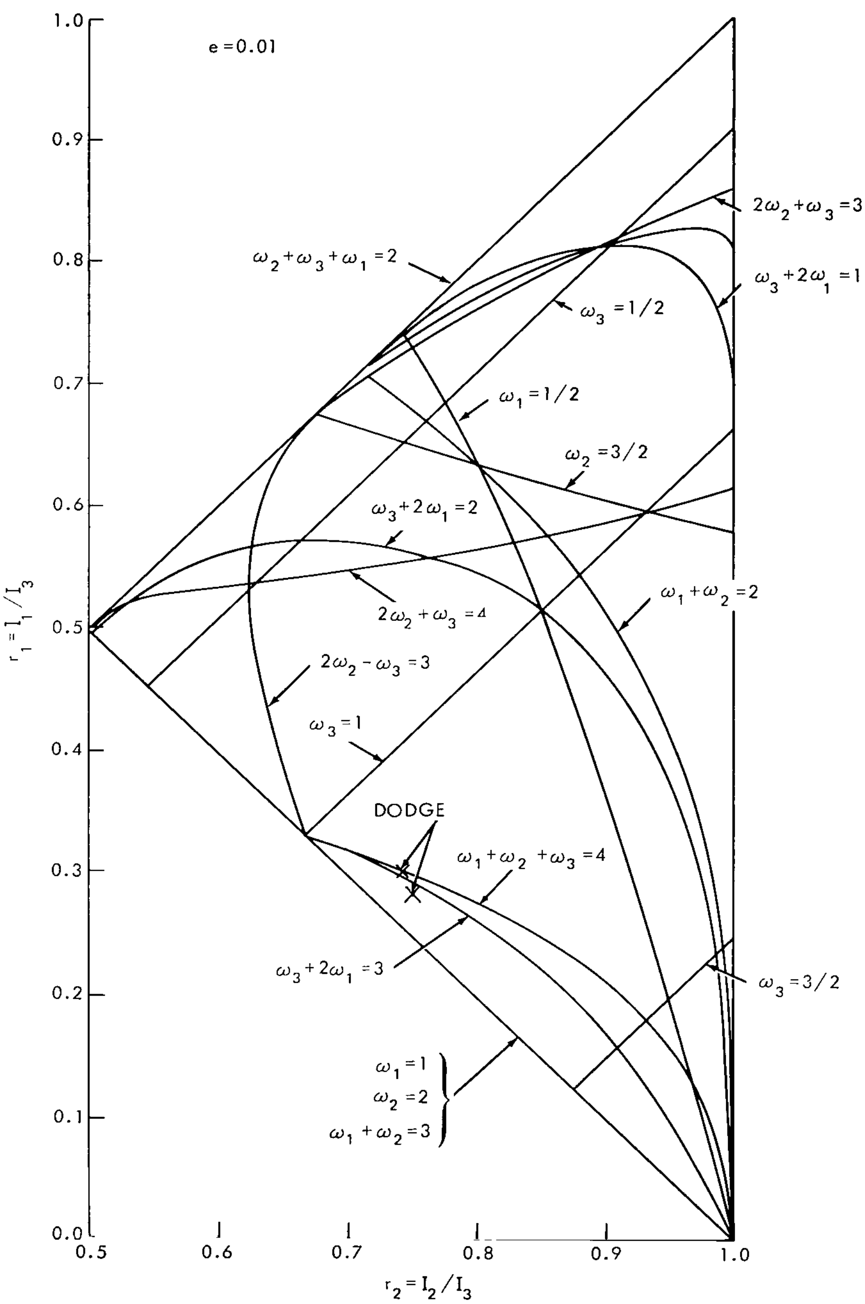

Figure 5-16—Resonance lines that yield unbounded motion.

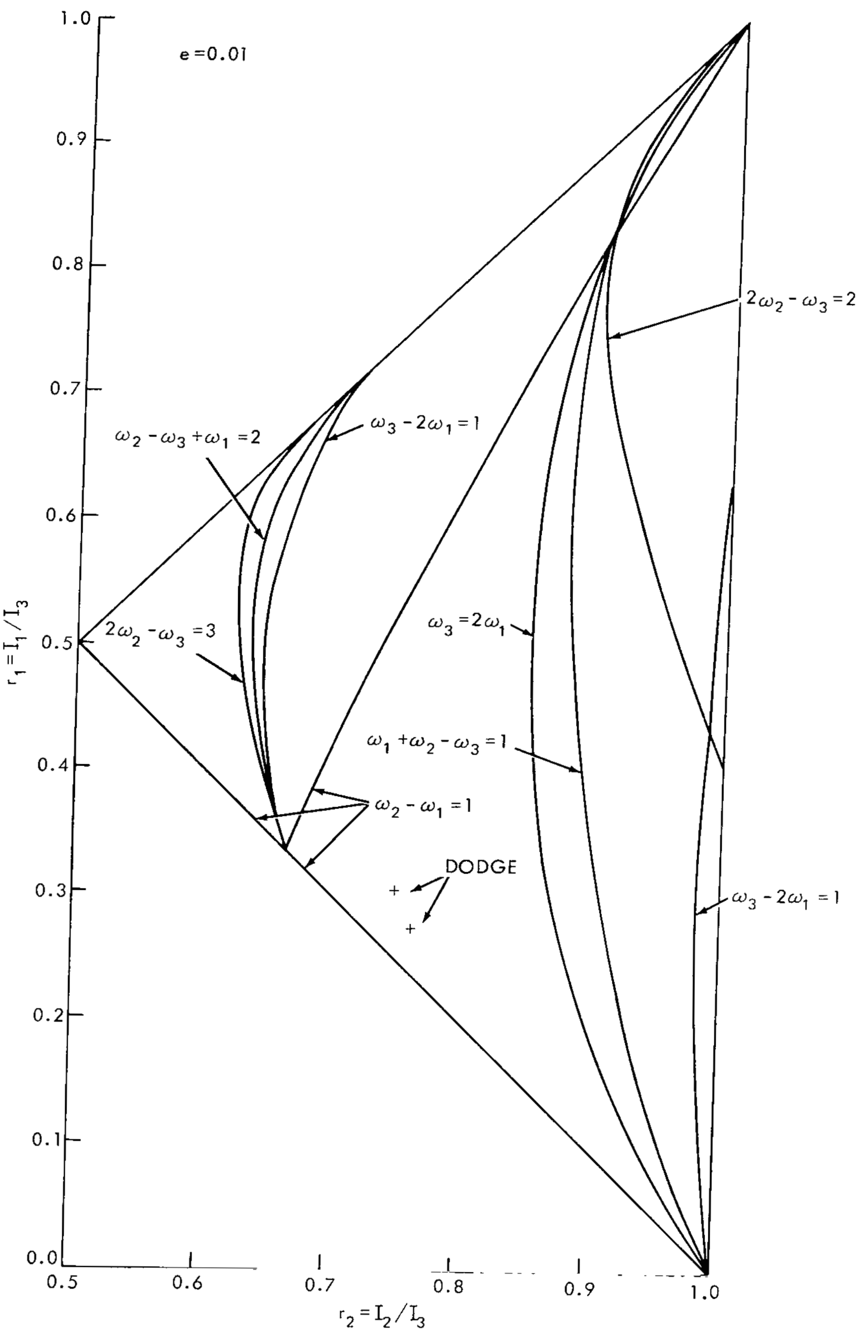

Figure 5-17—Resonance lines for large intermodal energy exchange.

The preferred method (this may sound like a paradox) of reducing any possible resonance effects for DODGE is to shorten the main booms. This unfortunately tends to decrease the pitch and roll restoring torques, but it moves the parameter point (r_1, r_2) toward a "safe" area of Figures 5-16 and 5-17 (see below). The other alternative—lengthening the booms—leads to a more planar configuration; more possible resonances exist for this configuration than for any other, and it should be avoided if possible.

Summary

The motion of a nominally stable gravity-gradient-stabilized spacecraft in a nearly circular orbit is often described as consisting of small oscillations of its normal modes plus small forced oscillations due to the orbital eccentricity. This description is known to be inadequate if the in-plane (pitch) period is the same as the orbital period ($\omega_3 = 1$); there is then a linear resonance between the in-plane mode and the first term in the expansion for the non-uniform rate of rotation of the local vertical. In other respects, too, the situation is much more complex. Resonance relationships exist whenever the normal frequencies are such that there are terms in the equations of motion for which

$$i\omega_1 \pm j\omega_2 \pm k\omega_3 = \pm m , \qquad i, j, k, m, = 0, 1, \cdots, n .$$

Thirty-two resonance combinations of frequencies for a single rigid body in a slightly eccentric orbit are shown in Table 5-1. These combinations were developed on the basis of a first-order solution to a set of equations expanded through the third order in the generalized coordinates and momenta and with terms through fourth order in e retained in the expansions for the orbital parameters. Additional terms would undoubtedly appear if a higher-order solution were developed, or if more terms were included in the equations of motion or of the orbital expansions.

Nineteen of the resonance combinations shown in Table 5-1 lead to unbounded increases in the amplitude of oscillation of one or more of the spacecraft's modes. Figure 5-16 plots the loci of inertias for which these resonances exist. Eight more of the combinations of frequencies lead to significant interchanges of energy between various modes of oscillation (e.g., a small pitch oscillation could change to a large roll-yaw oscillation and then back to a small pitch oscillation with a typical period for the interchange of twenty orbits). Figure 5-17 shows the loci of inertial combinations for which this occurs. The remaining five resonances have no significant effect on the motion.

Several comments on the analysis of the preceding section are in order. As has been noted previously, the phrase "unbounded motion" means that the motion increases to a point for which higher-order terms become significant. A measure of the closeness of the frequency resonance of the form $|\epsilon/Ke^j \omega|$ which is required to produce the various motions is developed for each situation. The term K is a constant which is obtained by collecting terms in the expansion of the Hamiltonian, as illustrated in the previous section for the case $2\omega_1 - 1 = 0$. As such the value of K is dependent on the order to which the solution is formulated. Thus, these conditions for bounded motion represent the relationship between ϵ and the order of ϵ for which the resonance becomes important, and not the exact value of ϵ, unless K includes higher-order terms.

When the inertia parameters of a spacecraft lie exactly on any of the lines of Figures 5-16 or 5-17, the motion will be resonant no matter how small e is. The rate of change of the appropriate mode will depend on e, but the final value will not. This was also demonstrated in Chapter 2, where each of the unstable regions exists for an arbitrarily small e. The dependency of the rate of divergence on e is indicated for the planar case by Figures 2-6 through 2-8.

CHAPTER 6

CONCLUSION

Summary

The analyses in the preceding chapters have demonstrated a number of differences between the motion of a rigid gravity-gradient-stabilized spacecraft as determined by a linearized analysis for a circular orbit and as determined by a more complete analysis for the same spacecraft in a slightly eccentric orbit. Prior investigation, both analytical and numerical, had previously indicated the importance of considering the various effects studied herein. The major contribution of this dissertation is the specific demonstration—in a manner suitable for use in spacecraft design—of the effect and location of both parametric and nonlinear resonances.

There are a large number of resonances that can affect the motion of a gravity-gradient-stabilized spacecraft. Several possible resonances occur for inertia parameters in the portion of parameter space at present considered most desirable for spacecraft design. Many of the resonances only occur for eccentric orbits, and several are the result of terms which are third- or fourth-order in eccentricity. The rate at which a particular resonance affects spacecraft motion is a function of the magnitude of the eccentricity as is the width of the band in parameter space for which the resonance occurs. However, in any given resonant situation the peak magnitude of the oscillation does not depend on the value of the eccentricity, although for the energy-interchange types of resonance it does depend on the initial conditions.

Recommendations

The existence of resonances, such as those studied in the preceding chapters, has consequences that should be considered in the design of any gravity-gradient-stabilized spacecraft. In the conceptual design stage, configurations which deliberately utilize an interchange type of resonance to transfer energy from one mode to another to enhance damping (e.g. Pringle (Reference 34)) should be considered. Once any configuration has been selected, the parameter selection process should specifically include consideration of all the types of resonance.

In general, much of the analysis of a given spacecraft is based on the results of a digital simulation of the equations of motion. These simulations frequently retain the nonlinear character of the equations of motion as is desirable. However, if the expansions for the radius vector and true anomaly are limited to first-order terms in eccentricity, the numerical simulations cannot demonstrate the effect of many of the pertinent resonances. Terms through the fourth order in eccentricity should therefore be included in the simulations even though in nonresonant situations they have little effect.

Extensions

The analysis of this dissertation could be extended in several directions: it could include higher-order terms in the Hamiltonian, or continue the method of averaging to second order. However, the return from either of these steps is not likely to justify the effort.

The analysis could also be extended to cover several types of spacecraft presently being considered. The equations could be quickly modified to include the effect of a constant-speed inertia wheel as has been suggested for improved yaw response in a number of spacecraft. Or the effects of damping on the motion of an essentially rigid spacecraft (achieved through use of hysteresis rods or a greaseball type of damper) could be considered. The order of the system could be extended to include two- or three-body spacecraft, thus including configurations such as the Vertistat.

The analysis of the fourth and fifth chapters follows a formal procedure which is applicable to a large class of mechanical systems whose motion consists of "small oscillations." The Hamiltonian for any such system can be written as the sum of terms of increasing order in the coordinates and momenta (i.e. $H = H_0 + H_1 + \cdots + H_n$). It may be possible to determine generalized criteria for parametric and nonlinear resonances from the form of the third and higher order terms in the Hamiltonian without requiring a complete analysis such as the preceding. For example if in the second order Hamiltonian there exists a coordinate pair q_i, p_i such that there are no terms in $q_i\, q_j$, $q_i\, p_j$ or $p_i\, p_j$ for $i \neq j$, then the linear equations for the $\dot{q}_i$ and $\dot{p}_i$ are independent of the remaining variables. There may be similar criteria for the third order Hamiltonian which determine the existence or absence of the various resonances.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, November 5, 1969
039-01-01-01-51

Appendix A

Basic Equations of Motion

The motion of a rigid spacecraft in an inverse square force field can be completely specified by six coordinates. The position of the center of mass can be specified by three coordinates r, ρ, v and the attitude of the body about its center of mass by the Euler angles ϕ, θ, and ψ. The equations of motion for this six-degree-of-freedom problem can be obtained directly from the potential and kinetic energies via the Lagrangian formalism.

The kinetic energy of the spacecraft is

$$T = \frac{1}{2} m_s \left[\dot{r}^2 + (r\dot{\rho})^2 + (r \cos \rho \dot{v})^2\right] + \frac{1}{2} \left[I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right]$$

where

$$\Omega_1 = \dot{\psi} - (\dot{\phi} + \dot{v}) \sin (\theta + \rho) ,$$

$$\Omega_2 = (\dot{\theta} + \dot{\rho}) \cos \psi + (\dot{\phi} + \dot{v}) \cos (\theta + \rho) \sin \psi ,$$

and

$$\Omega_3 = - (\dot{\theta} + \dot{\rho}) \sin \psi + (\dot{\phi} + \dot{v}) \cos (\theta + \rho) \cos \psi .$$

The potential energy of the spacecraft is

$$V = -\frac{\mu_e}{r} \left\{ m + \frac{I_1 + I_3 - I_2}{2r^2} + \frac{3}{2r^2} \left[\left(I_2 - I_1\right) (\cos \phi \cos \theta)^2 + \left(I_2 - I_3\right) (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)^2 \right] \right\} .$$

The three Lagrangian equations for the coordinates describing the motion of the center-of-mass of the spacecraft are

$$m_s \ddot{r} - m_s r \dot{\rho}^2 - m_s r \cos^2\rho \, \dot{v}^2 - \frac{\mu}{r^2}\left\{ m_s + 3\,\frac{I_1 + I_3 - I_2}{2r^2} \right.$$

$$\left. + \frac{9}{2r^2}\left[(I_2 - I_1)(\cos\phi\cos\theta)^2 + (I_2 - I_3)(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)^2 \right] \right\} = 0 \, ,$$

$$\frac{d}{dt}\left\{ m_s r^2 \cos^2\rho\, \dot{v} - I_1 \left[\dot{\psi} - (\dot{\phi} + \dot{v}) \sin(\theta + \rho) \right] \sin(\theta + \rho) \right.$$

$$+ I_2 \left[(\dot{\theta} + \dot{\rho}) \cos\psi + (\dot{\phi} + \dot{v}) \cos(\theta + \rho) \sin\psi \right] \cos(\theta + \rho) \sin\psi$$

$$\left. + I_3 \left[- (\dot{\theta} + \dot{\rho}) \sin\psi + (\dot{\phi} + \dot{v}) \cos(\theta + \rho) \cos\psi \right] \cos(\theta + \rho) \cos\psi \right\} = 0$$

and

$$\frac{d}{dt}\left(m_s r^2 \dot{\rho} + I_2 \omega_2 \cos\psi - I_3 \omega_3 \sin\psi \right) + m_s r^2 \dot{v}^2 \cos\rho \sin\rho$$

$$+ I_1 \omega_1 (\dot{\phi} + \dot{v}) \cos(\theta + \rho) + I_2 \omega_2 (\dot{\phi} + \dot{v}) \sin\psi \sin(\theta + \rho)$$

$$+ I_3 \omega_3 (\dot{\phi} + \dot{v}) \cos\psi \cos(\theta + \rho) = 0 \, .$$

The three preceding equations all contain terms of sharply contrasting magnitudes since the inertias I_1, I_2, and I_3 are all of the order $m_s \ell^2$ where ℓ is the linear dimension of the spacecraft. The ratio $(\ell / r)^2$ is on the order of 10^{-12} for a gravity gradient spacecraft, thus the equations for the orbital coordinates can be simplified to the form

$$\ddot{r} - r \dot{\rho}^2 - r \left[(\cos\rho\, \dot{v})^2 + \dot{\rho}^2 \right] + \frac{\mu_e}{r^2} = 0$$

$$\frac{d}{dt}\left[r^2 \cos^2\rho\, \dot{v} \right] = 0$$

$$\frac{d}{dt}\left[r^2 \dot{\rho} \right] = 0$$

and these are the well known equations for the motion of a point mass subject to an inverse square gravitational force. Actually Bėletskii demonstrated in Reference 9 that the relativistic correction due to the velocity of a typical gravity stabilized spacecraft is several orders larger then the terms deleted above, although it too can be safely ignored. The coordinate system can be chosen such that $\rho = \dot{\rho} = 0$ in the above equation leaving the two familiar equations

$$\ddot{r} - r\,\dot{v}^2 + \frac{\mu_e}{r^2} = 0,$$

and

$$\frac{d}{dt}\left[r^2\,\dot{v}\right] = 0.$$

The equations for the remaining coordinates are

$$I_1\left[\ddot{\psi} - (\ddot{\phi} + \ddot{v})\sin\theta - (\dot{\phi} + \dot{v})\,\dot{\theta}\cos\theta\right] + \left(I_3 - I_2\right)\left[(\dot{\phi} + \dot{v})\cos\theta\cos 2\psi\right.$$

$$\left. + \frac{1}{2}(\dot{\phi} + \dot{v})^2\cos^2\theta + \sin^2 2\psi + \frac{1}{2}\,\dot{\theta}^2\sin 2\psi\right]$$

$$+ \frac{3\mu_e}{2r^3}\left(I_3 - I_2\right)\left(\sin^2\phi\sin 2\psi + \cos^2\phi\sin^2\theta\sin 2\psi\right.$$

$$\left. - \sin 2\phi\sin\theta\cos 2\psi\right) = 0,$$

$$\left(I_2\cos^2\psi + I_3\sin^2\psi\right)\ddot{\theta} + \left(I_3 - I_2\right)\left[\dot{\theta}\dot{\psi}\sin 2\psi - \frac{1}{2}(\ddot{\phi} + \dot{v})\cos\theta\sin 2\psi\right.$$

$$\left. + \frac{1}{2}(\dot{\phi} + \dot{v})\,\dot{\theta}\sin\theta\sin 2\psi - (\dot{\phi} + \dot{v})\,\dot{\psi}\cos\theta\cos 2\psi\right] - I_1\left[\frac{1}{2}(\dot{\phi} + \dot{v})^2\sin 2\theta\right.$$

$$\left. - (\dot{\phi} + \dot{v})\,\dot{\psi}\cos\theta\right] + \frac{1}{2}\left(I_2 - I_3\right)(\dot{\phi} + \dot{v})\,\dot{\theta}\sin 2\theta\sin 2\psi$$

$$+ \frac{1}{2}\left(I_2\sin^2\psi + I_3\cos^2\psi\right)(\dot{\phi} + \dot{v})^2\sin 2\theta$$

$$+ \frac{3\mu_e}{2r^3}\left[\left(I_2 - I_1\right)\cos^2\phi\sin 2\theta - \left(I_2 - I_3\right)\left(\cos^2\phi\sin 2\theta\cos^2\psi\right.\right.$$

$$\left.\left. + \frac{1}{2}\sin 2\phi\cos\theta\sin 2\psi\right)\right] = 0,$$

and

$$\left(I_3 \cos^2\psi + I_2 \sin^2\psi\right) \left[(\ddot{\phi} + \ddot{v}) \cos^2\theta - (\dot{\phi} + \dot{v})\,\dot{\theta} \sin 2\theta\right]$$

$$+ I_1 \left[(\ddot{\phi} + \ddot{v}) \sin^2\theta + (\dot{\phi} + \dot{v})\,\dot{\theta} \sin 2\theta - \ddot{\psi} \sin\theta - \dot{\psi}\,\dot{\theta} \cos\theta\right]$$

$$+ \frac{1}{2}\left(I_2 - I_3\right) \left[\ddot{\theta} \cos\theta \sin 2\psi - \dot{\theta}^2 \sin\theta \sin 2\psi\right.$$

$$\left. + 2\dot{\theta}\dot{\psi} \cos\theta \cos 2\psi + (\dot{\phi} + \dot{v})\,\dot{\psi} \cos^2\theta \sin 2\psi\right]$$

$$+ \frac{3\mu}{2r^2} \left[\left(I_2 - I_1\right) \sin 2\phi \cos^2\theta - \left(I_2 - I_3\right) \left(\cos 2\phi \sin\theta \sin 2\psi\right.\right.$$

$$\left.\left. + \sin 2\phi \sin^2\psi - \sin 2\phi \sin^2\theta \cos^2\psi\right)\right] = 0 .$$

Appendix B

Auxiliary Formulas Relating to Orbital Mechanics

A collection of formulas which are applicable to the motion of a point mass in an inverse square gravitational field are given below.

Force of Gravity

$$-F = \frac{\gamma Mm}{r^2} = \frac{K^2 m}{pr^2} = \frac{\mu m}{r^2}$$

Motion in an Elliptical Orbit

$$r = \frac{p}{1 + e \cos v}$$

$$p = b\left[1 - e^2\right]^{1/2} = \frac{b^2}{a}$$

$$r^2 \frac{dv}{dt} = K$$

$$KT = \int_0^T r^2 \frac{dv}{dt} = 2\pi ab$$

$$\frac{K}{ab} = \frac{2\pi}{T} = \omega_0$$

$$\omega_0^2 = \frac{K^2 a}{a^3 b^2} = \frac{\mu}{a^3} = n^2$$

Useful Derivatives

$$\frac{dv}{dt} = \frac{K}{r^2} = \frac{K(1 + e \cos v)^2}{p^2}$$

$$\frac{d}{dv}\left[\frac{dv}{dt}\right] = \frac{d\dot{v}}{dv} = \frac{-2eK \sin v}{rp}$$

Change of Independent Variable

$$\frac{d[\ ]}{dt} = \frac{d[\ ]}{dv}\frac{dv}{dt} = \frac{K}{r^2}\frac{d[\ ]}{dv} \,,$$

$$\frac{d^2[\ ]}{dt^2} = \frac{d^2[\ ]}{dv^2}\left(\frac{dv}{dt}\right)^2 + \frac{d[\ ]}{dv}\frac{d\dot{v}}{dv}\frac{dv}{dt}$$

$$= \frac{K^2}{r^4}\frac{d^2[\ ]}{dv^2} - \frac{2eK^2 \sin v}{pr^3}\frac{d[\ ]}{dt}$$

$$= \frac{\mu(1 + e\cos v)}{r^3}\frac{d^2[\ ]}{dv^2} \quad \frac{\mu 2e \sin v}{r^3}\frac{d[\ ]}{dv}$$

Appendix C

Development of Asymptotic Expansion Formulas to the Third Order

The method of application of asymptotic expansions to problems of nonlinear oscillations which is used herein was developed by the Russian authors N. M. Krylov and N. M. Bogoliubov in Reference 26. A more thorough explanation of the method is given by N. M. Bogoliubov and Y. A. Mitropolsky in Reference 27, and readers interested in the theory underlying the development of the formulas which follow should consult the latter reference. A considerable simplification in the development results from the restricted form of the disturbing function for this problem.

The intent of this approach is to find a solution to the differential equation

$$\frac{d^2 x}{dt^2} + \omega^2 x = \epsilon f(t, x) \tag{C-1}$$

in the form

$$x = a \cos \psi + \sum_{n=1}^{m-1} \epsilon^n u_n (a, \psi, t) \tag{C-2}$$

where a and ψ are variables obtained from the equations

$$\frac{da}{dt} = \sum_{n=1}^{m} \epsilon^n A_n (a) \tag{C-3}$$

and

$$\frac{d\psi}{dt} = \omega + \sum_{n=1}^{m} \epsilon^n B_n (a) . \tag{C-4}$$

The perturbing function which is needed for this application is of the form

$$\epsilon f(t, x) = \sum_{n=1}^{4} \epsilon^n \left[f_{n1} (t) + f_{n2} (t) x \right] . \tag{C-5}$$

Formally the problem becomes one of finding functions $A_n(a)$, $B_n(a)$, $u_n(0, \psi, t)$ $n = 1, \cdots, m$ such that Equation C-2, with a and ψ defined by Equations C-3 and C-4, satisfies Equation C-1 to an order ϵ^{m+1}. This is accomplished by differentiating Equation C-2 twice with the aid of Equation C-3 and C-4, and substituting the result into the left hand side of Equation C-1. Then the right hand side of Equation C-1 is also expanded by substituting Equation C-2 for x in $f(t, x)$. Terms of equal magnitude in ϵ are then equated to give a series of differential equations from which the desired functions are obtained. The resulting equations are all of the form

$$\omega^2 \frac{\partial^2 u_m}{\partial \psi^2} + 2\omega \frac{\partial^2 u_m}{\partial \psi \partial t} + \frac{\partial^2 u_m}{\partial t^2} - \omega^2 u_m$$

$$= F_m + 2\omega A_m \sin\psi + 2a\omega B_m \cos\psi \qquad \text{(C-6)}$$

where

$$F_1 = f_{11}(t) + f_{12}(t)\, a\cos\psi ,$$

$$\begin{aligned} F_2 = {} & f_{21}(t) + f_{22}(t)\, a\cos\psi + f_{12}(t)\, u_1 \\ & + \left(aB_1^2 - A_1 \frac{dA_1}{da}\right)\cos\psi + aA_1\left(\frac{dB_1}{da} + 2A_1 B_1\right)\sin\psi \\ & - 2\omega B_1 \frac{\partial^2 u_1}{\partial\psi^2} - 2A_1 \frac{\partial^2 u_1}{\partial a\,\partial t} - 2B_1 \frac{\partial^2 u_1}{\partial\psi\,\partial t} - 2\omega A_1 \frac{\partial^2}{\partial a} , \end{aligned}$$

$$\begin{aligned} F_3 = {} & f_{31}(t) + f_{32}(t)\, a\cos\psi + f_{22}(t)\, u_1 + f_{12}(t)\, u_2 \\ & + \left(2a B_1 B_2 - A_1 \frac{dA_2}{da} - A_2 \frac{dA_1}{da}\right)\cos\psi + \left(aA_1 \frac{dB_2}{da} + aA_2 \frac{dB_1}{da}\right. \\ & \left. + 2A_1 B_2 + 2A_2 B_1\right)\sin\psi - A_1\left(\frac{dA_1}{da}\frac{\partial u_1}{\partial a} + \frac{dB_1}{da}\frac{\partial u_1}{\partial\psi}\right) \\ & - A_1^2 \frac{\partial^2 u_1}{\partial a^2} - \left(B_1^2 + 2\omega B_2\right)\frac{\partial^2 u_1}{\partial\psi^2} - \left(2A_1 B_1 + 2\omega A_2\right)\frac{\partial^2 u_1}{\partial a\,\partial\psi} - 2A_2 \frac{\partial^2 u_1}{\partial a\,\partial t} \\ & - 2B_2 \frac{\partial^2 u_1}{\partial\psi\,\partial t} - 2A_1 \frac{\partial^2 u_2}{\partial a\,\partial t} - 2\omega A_1 \frac{\partial^2 u_2}{\partial a\,\partial\psi} - 2\omega B_1 \frac{\partial^2 u_2}{\partial\psi^2} - 2B_1 \frac{\partial^2 u_2}{\partial\psi\,\partial t} \end{aligned}$$

and

$$F_4 = f_{41}(t) + f_{42}(t)\, a\cos\psi + f_{32}(t)\, u_1 + f_{22}(t)\, u_2 + f_{12}(t)\, u_1$$

$$+ \left(2aB_1B_3 + aB_2^2 - A_1 \frac{dA_3}{da} - A_2 \frac{dA_2}{da} - A_3 \frac{dA_1}{da}\right)\cos\psi$$

$$+ \left(aA_1 \frac{dB_3}{da} + aA_2 \frac{dB_2}{da} + aA_3 \frac{dB_1}{da} + 2A_1B_3 + 2A_2B_2 + A_3B_1\right)\sin\psi$$

$$- \left(A_1 \frac{dA_2}{da} + A_2 \frac{dA_1}{da}\right)\frac{\partial u_1}{\partial a} - \left(A_1 \frac{dB_2}{da} + A_2 \frac{dB_1}{da}\right)\frac{\partial u_1}{\partial \psi}$$

$$- 2A_1A_2 \frac{\partial^2 u_1}{\partial a^2} - \left(2B_1B_2 + 2\omega B_3\right)\frac{\partial^2 u_1}{\partial \psi^2} - 2\left(A_1B_2 + A_2B_1 + \omega A_3\right)\frac{\partial^2 u_1}{\partial a\,\partial\psi}$$

$$- 2B_3 \frac{\partial^2 u_1}{\partial t\,\partial\psi} - 2A_3 \frac{\partial^2 u_1}{\partial t^2} - A_1 \frac{dA_1}{da}\frac{\partial u_2}{\partial a} - A_1 \frac{dB_1}{da}\frac{\partial u_2}{\partial \psi}$$

$$- A_1^2 \frac{\partial^2 u_2}{\partial a^2} - \left(B_1^2 + 2\omega B_2\right)\frac{\partial^2 u_2}{\partial \psi^2} - \left(2A_1B_1 + 2\omega A_2\right)\frac{\partial^2 u_2}{\partial a\,\partial\psi}$$

$$- 2A_2 \frac{\partial^2 u_2}{\partial a\,\partial t} - 2B_2 \frac{\partial^2 u_2}{\partial\psi\,\partial t} - 2\omega B_1 \frac{\partial^2 u_3}{\partial \psi^2} - 2\omega_1 A_1 \frac{\partial^2 u_3}{\partial a\,\partial\psi}$$

$$- 2A_1 \frac{\partial^2 u_3}{\partial a\,\partial t} - 2B_1 \frac{\partial^2 u_3}{\partial\psi\,\partial t} \; .$$

The first equation is solved for u_1, A_1, and B_1 as follows. Assume that u_1 and F_1 can be written as double Fourier series,

$$u_1(a, \psi, t) = \sum_n \sum_m u^1_{nm}(a) \exp\left[i(nt + m\psi)\right] ,$$

and

$$F_1(a, \psi, t) = \sum_n \sum_m F^1_{nm}(a) \exp\left[i(nt + m\psi)\right] ,$$

equate the coefficients of equal harmonics on both sides of the first of Equations C-6 to obtain

$$u^1_{nm} = \frac{F^1_{nm}}{\omega^2 - (n + m\omega)^2} \qquad \omega^2 - (n + m\omega)^2 \neq 0$$

and require that

$$2\omega A_1 \sin\psi + 2a\omega B_1 \cos\psi + \sum_n \sum_m F_{nm} \exp\left[i(nt + m\psi)\right] = 0$$

for all n, m such that $\omega^2 - (n + m\omega)^2 = 0$. This development is only valid for the "nonresonance case" which is characterized by a requirement that

$$\omega \neq \frac{p}{q}$$

for integer values of p and q (other than p = q = 1) such that

$$\frac{p}{q} = \frac{n}{1 - m}$$

for values of n and m that occur in the expansion of F_1 as a Fourier series. (A similar set of formulas, which are valid for the "resonance case," are developed later in this appendix.) The only terms in F^1_{nm} for which $\omega^2 - (n + m\omega)^2 = 0$, subject to the above restriction, occur for $n = 0$, $m = \pm 1$. This part of F_1 can be written as

$$F_1(a, \psi, t) = h_1(a) \sin\psi + h_2(a) \cos\psi$$

and $A_1(a)$ and $B_1(a)$ are

$$A_1(a) = \frac{h_1(a)}{2\omega}$$

and

$$B_1(a) = \frac{h_2(a)}{2a\omega} \cdot$$

The values of A_1, B_1, and u_1 obtained from the above process can be used to expand $F_2(a, \psi, t)$ as a Fourier series from which A_2, B_2, and u_2 can be found and this process can be continued to any desired degree of approximation.

The first order solution to Equation C-1 is expressed as

$$x = a \cos \psi$$

where

$$\frac{da}{dt} = \epsilon A_1 (a)$$

and

$$\frac{d\psi}{dt} = \omega + \epsilon B_1 (a) ,$$

the m^{th} order solution is

$$x = a \cos \psi + \sum_{n=1}^{m-1} \epsilon^n u_n (a, \psi, t)$$

where

$$\frac{da}{dt} = \sum_{n=1}^{m} \epsilon^n A_n (a)$$

and

$$\frac{d\psi}{dt} = \omega + \sum_{n=1}^{m} \epsilon^n B_n (a) .$$

Several changes are made in the above process for the "resonance case" which occur when $\omega \approx p/q$. In this case Δ is defined by

$$\epsilon\Delta \equiv \omega^2 - \left(\frac{p}{q}\right)^2 ,$$

Equation C-1 is rewritten in the form

$$\frac{d^2 x}{dt^2} + \left(\frac{p}{q}\right)^2 x = \epsilon \left[f(t, x) - \Delta x\right] ,$$

and there is a change of variables from ψ to θ where

$$\psi \equiv \frac{p}{q} t + \theta .$$

The form of the solution becomes

$$x = a \cos\left(\frac{p}{q} t + \theta\right) + \sum_{n=1}^{m-1} \epsilon^n u_n (a, \theta, t)$$

where a and θ are obtained from

$$\frac{da}{dt} = \sum_{n=1}^{m} \epsilon^n A_n (a, \theta)$$

and

$$\frac{d\theta}{dt} = \sum_{n=1}^{m} \epsilon^n B_n (a, \theta) .$$

The disturbing function is still in the form of Equation C-5 and the set of Equations C-6 is replaced by

$$\frac{\partial^2 u_1}{\partial t^2} + \left(\frac{p}{q}\right)^2 u_1 = F_1 (a, \psi, t) + 2\frac{p}{q} A_1 \sin\psi + \left(2\frac{p}{q} B_1 - \Delta\right) a \cos\psi ,$$

$$\frac{\partial^2 u_2}{\partial t^2} + \left(\frac{p}{q}\right)^2 u_2 = F_2 (a, \psi, t) + \left[2A_1 B_1 + a\frac{\partial B_1}{\partial a} A_1 + a\frac{\partial B_1}{\partial \theta} B_1 + 2\frac{p}{q} A_2\right] \sin\psi + \left[-\frac{\partial A_1}{\partial a} A_1 - \frac{\partial A_1}{\partial \theta} B_1 + a B_1^2 + 2a\frac{p}{q} B_2\right] \cos\psi$$

and

$$\frac{\partial^2 u_3}{\partial t^2} + \left(\frac{p}{q}\right)^2 u_3 = F_3(a, \psi, t) + \left[2A_1 B_2 + 2A_2 B_1 + a \frac{\partial B_1}{\partial a} A_2 + a \frac{\partial B_2}{\partial a} A_1 + a \frac{\partial B_1}{\partial \theta} B_2 + a \frac{\partial B_2}{\partial \theta} B_1 + 2 \frac{p}{q} A_3\right] \sin\psi$$

$$+ \left[2aB_1 B_2 - \frac{\partial A_1}{\partial a} A_2 - \frac{\partial A_2}{\partial a} A_1 - \frac{\partial A_1}{\partial \theta} B_2 - \frac{\partial A_2}{\partial \theta} B_1 + 2 \frac{p}{q} a B_3\right] \cos\psi$$

where

$$F_1(a, \psi, t) = f_{11}(t) + f_{12}(t) a \cos\psi ,$$

$$F_2(a, \psi, t) = f_{21}(t) + f_{22}(t) a \cos\psi + \left[f_{12}(t) - \Delta\right] u_1(a, \psi, t) - 2A_1 \frac{\partial^2 u_1}{\partial a \partial t} - 2B_1 \frac{\partial^2 u_1}{\partial \theta \partial t} ,$$

and

$$F_3(a, \psi, t) = f_{31}(t) + f_{32}(t) a \cos\psi + f_{22}(t) u_1(a, \psi, t) + \left[f_{12}(t) - \Delta\right] u_2(a, \psi, t) - \left(\frac{\partial A_1}{\partial a} A_1 + \frac{\partial A_1}{\partial \theta} B_1\right) \frac{\partial u_1}{\partial a}$$

$$- \left(\frac{\partial B_1}{\partial a} A_1 + \frac{\partial B_1}{\partial \theta} B_1\right) \frac{\partial u_1}{\partial \theta} - A_1^2 \frac{\partial^2 u_1}{\partial a^2}$$

$$- 2A_1 B_1 \frac{\partial^2 u_1}{\partial a \partial \theta} - 2A_2 \frac{\partial^2 u_1}{\partial a \partial t} - 2A_1 \frac{\partial^2 u_2}{\partial a \partial t} - B_1^2 \frac{\partial^2 u_1}{\partial \theta^2} - 2B_2 \frac{\partial^2 u_1}{\partial \theta \partial t} - 2B_1 \frac{\partial^2 u_2}{\partial \theta \partial t} .$$

The rest of the procedure is quite similar to that for the nonresonant case. However this time u^1_{nm} is

$$u^1_{nm} = \frac{F^1_{nm}}{\left(\frac{p}{q}\right)^2 - \left(n + m\frac{p}{q}\right)^2}$$

whenever

$$\left(\frac{p}{q}\right)^2 - \left(n + m\frac{p}{q}\right)^2 \neq 0$$

and $u^1_{nm} = 0$ otherwise. The previous inequality could also be written as

$$nq + (m \pm 1)\, p \neq 0$$

and there are a number of values of n, m for which it is not satisfied (e.g., when $p = 1$, $q = 2$ it fails for $n = 1$, $m = -1$). The terms in $F_1\,(a, \psi, t)$ for which the inequality fails can be written in the form

$$h_1\,(a, \psi, t) \sin\psi + h_2\,(a, \psi, t) \cos\psi$$

and by equating the harmonics in $\sin\psi$ and $\cos\psi$,

$$A_1\,(a, \psi, t) = \frac{q}{2p} h_1\,(a, \psi, t)$$

and

$$B_1\,(a, \psi, t) = \frac{q}{2p}\left[\Delta - \frac{h_2\,(a, \psi, t)}{a}\right] .$$

As in the nonresonant case this process can be continued to obtain A_n, B_n, and u_n to any desired order.

REFERENCES

1. Cajori, F., "Newton's Principia, Motte's Translation Revised," Book III Proposition XXXVIII, Berkeley, California: University of California Press, 1934.

2. Routh, E. J., "Dynamics of a System of Rigid Bodies," Advanced Part, New York: Dover Publications, Inc., 1955, p. 376.

3. Plummer, H. C., "An Introductory Treatise on Dynamical Astronomy," Chapter XXIII, "Libration of the Moon," (first published in 1918), New York: Dover Publ., 1960.

4. Baker, R. M. L., "Librations on a Slightly Eccentric Orbit," *ARS J.*, 30(1):124-126, Jan. 1960.

5. Schrello, D. M., "The Effect of Aerodynamic Torques on the Angular Motion of an Artificial Satellite," Ph. D. Dissertation, Ohio State University, 1960.

6. DeBra, D. B., "The Large Attitude Motions and Stability, Due to Gravity, of a Satellite with Passive Damping in an Orbit of Arbitrary Eccentricity About an Oblate Body," Ph. D. Dissertation, Stanford University, SUDAER No. 126, 1962.

7. Beletskii, V. V., "The Translation-Rotational Motion of a Rigid Body in a Newtonian Field of Force, Artificial Earth Satellites, No. 3," Acad. Sci. USSR Press, 1959, translated by Consultants Bureau, New York.

8. Beletskii, V. V., "Librations on an Eccentric Orbit," NASA Technical Translation F-8504, Washington, D. C., 1963.

9. Beletskii, V. V., "Motion of an Artificial Satellite About its Center of Mass," NASA Technical Translation F-429, U.S. Dept. Commerce, Springfield, Va., 1966.

10. Chernous'ko, F. L., "On the Motion of a Satellite About its Center of Mass Under the Action of Gravitational Moments," translated from *PMM* 27(3):474-483, 1963.

11. Kane, T. R., "Attitude Stability of Earth-Pointing Satellites," *AIAA J.*, 3(4):726-731, April 1965.

12. Breakwell, J. V., and Pringle, R., Jr., "Nonlinear Resonance Affecting Gravity-Gradient Stability," in "Astrodynamics" (ed. Michal Lunc), Proc. XVIth International Astronautical Congress, Athens, 1965. Paris: Gauthier-Villars, 1966, pp. 305-325.

13. Struble, R. A., "Nonlinear Differential Equations," New York: McGraw-Hill, 1962.

14. Cesari, L., "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," par. 6 and 8, Berlin: Springer-Verlag, 1959.

15. Stoker, J. J., "Stability of Continuous Systems," in "Dynamic Stability of Structures," New York: Pergamon Press, 1967, pp. 45-52.

16. Pringle, Ralph, Jr., "On the Capture, Stability, and Passive Damping of Artificial Satellites," NASA Contractor Report 139, 1964. (Also Ph. D. Dissertation, Stanford University, 1964.)

17. Moran, J. P., "Effects of Plane Librations on the Orbital Motion of a Dumbbell Satellite," *Ars J.*, 31(8):1089-1096, Aug. 1961.

18. Shechter, H. B., "Satellite Librations on an Elliptic Orbit," Rand Memorandum RM-3632-PR, May 1963.

19. Moulton, F. R., "An Introduction to Celestial Mechanics," Second Revised Edition, New York: Macmillan Co., 1960, p. 171.

20. Mathieu, E., "Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique," J. de Math. Pures et Appliquées (J. de Liouville), 13:137, 1868.

21. McLachlan, N. W., "Theory and Application of Mathieu Functions," New York: Dover Publ., 1964.

22. Stoker, J. J., "Nonlinear Vibrations in Mechanical and Electrical Systems," New York: Interscience Publ., 1950, p. 115.

23. National Bureau of Standards, "Tables Relating to Mathieu Functions," New York: Columbia University Press, 1961.

24. Abramowitz, M., and Stegun, I., Editors, "Handbook of Mathematical Functions," National Bureau of Standards Applied Mathematics Series 55, Third printing, 1965. (Figures 20.8, 20.9 and 20.10.)

25. Cayley, A., "Tables of the Development of Functions in the Theory of Elliptic Motion," *Memoirs of the Royal Astron. Soc.*, 29:191-306, 1861.

26. Krylov, N. M., and Bogoliubov, N. M., "Introduction to Nonlinear Mechanics," Princeton: Princeton University Press, 1947.

27. Bogoliubov, N. M., and Mitropolsky, Y. A., "Asymptotic Methods in the Theory of Non-Linear Oscillations," Delhi-6, India: Hindustan Publishing Corporation, 1961. (Available through Gordon and Breach Science Publishers, New York.)

28. Liu, Han-Shou, "Satellite Libration on an Elliptic Orbit," NASA Goddard Space Flight Center Technical Memorandum X-56173, 1965.

29. Pars, L. A., "A Treatise on Analytical Dynamics," New York: John Wiley and Sons, 1965, pp. 523-526.

30. Wylie, C. R., Jr., "Advanced Engineering Mathematics," Theorem 7, Sec. 1.3, New York: McGraw-Hill, 2nd Edition, 1960.

31. Whittaker, E. T., "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," Sec. 192, London: Cambridge University Press, 4th Edition, 1960.

32. Goldstein, H., "Classical Mechanics," Sec. 9-2, Cambridge, Mass.: Addison-Wesley, 1950.

33. Mobley, F., "Gravity Gradient Stabilization Results from the Dodge Satellite," paper presented at the 2nd AIAA Communications Satellite Systems Conference, San Francisco, Calif., April 8-10, 1968.

34. Pringle, R., Jr., "On the Exploitation of Nonlinear Resonance in Damping an Elastic Dumbbell Satellite," paper presented at the AIAA Guidance, Control and Flight Dynamics Conference, Huntsville, Ala., Aug. 14-16, 1967.

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