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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
INSTRUMENTATION LABORATORY
CAMBRIDGE, MASS. 02139

C. S. DRAPER
DIRECTOR

69-341L-1
25 April 1969

TO: Mr. Thomas B. Murtagh/FM8
NASA Manned Spacecraft Center
Advanced Mission Design Branch
Houston, Texas 77058

FROM: Donald C. Fraser
Mail Station 60
MIT Instrumentation Laboratory
75 Cambridge Parkway
Cambridge, Massachusetts 02142

SUBJECT: First Quarterly Progress Report Under Contract NAS-9-6024

- 1) Pursuant to the Technical Report Provisions and the schedule in Paragraph 6.1 of the referenced contract the First Quarterly Progress Report is hereby submitted. This covers the period of work from 1 February 1969 to 1 May 1969.
- 2) Nineteen copies of this report are enclosed. In addition copies have been distributed as indicated below.

Very truly yours,

Donald C. Fraser

Dr. Donald C. Fraser
Program Manager

Enclosures: (19)

cc: NASA Manned Spacecraft Center:

J. D. Wilcox/JC-33 (2)
J. Funk/FM-8 (2)
J. T. Wheeler/BM-7 (2)

MIT Instrumentation Laboratory:

R. H. Battin	S. R. Croopnick
J. C. Deckert	D. G. Hoag
J. H. Lanning	L. E. Larson
J. E. Potter	P. C. Pollock
W. S. Widnall	Central Files (2)

Letter of Transmittal Only:

D. Dolan
E. Howard

FACILITY FORM 602

N71-14050

(ACCESSION NUMBER)

44

(PAGES)

CR-108708

(NASA CR OR TMX OR AD NUMBER)

(THRU)

63

(CODE)

21

(CATEGORY)

CL-102702

First Quarterly Progress Report, Contract NAS -9-9024

1 February 1969 - 1 May 1969

INTRODUCTION

This quarterly progress report to the technical monitor of the referenced contract is organized into three main sections. Section I presents a scheme for assuring that the covariance matrix of state estimation errors used in the navigational state estimation problem remains greater than or equal to the true covariance matrix. A formal proof of the important properties of this scheme is included. Section II gives some representative results from a simulation of the use of this scheme on an Earth orbital mission. Section III, which is the doctoral thesis proposal of Mr. Steven Croopnick, describes an approach to the problem of improving the estimate of the state of a vehicle which is on an orbital mission about a parent body whose gravitational anomalies are initially unknown.

It is expected that during the next reporting period work will continue or commence along several paths:

- 1) Application of optimal control theory to choose the free parameter of the filter technique presented in this report
- 2) Incorporation of measurements into the filter scheme presented herein with the objective of reducing errors in the face of gravity anomalies
- 3) Identification of state variables in terms of mission objectives
- 4) Attempt to develop a method for using correlation between consecutive orbits to aid in the prediction problem

The following people have contributed to the effort summarized
in this First Quarterly Progress Report.

R. H. Battin
S. R. Croopnick
J. C. Deckert
D. C. Fraser
J. H. Lanning
J. E. Potter

I. A Method for Assuring that the Predicted State Errors are Conservative.

The straight forward application of modern optimal linear filter theory to practical navigation problems such as those encountered on the Apollo lunar mission can have serious problems in that the gains may tend to zero with the result that new information is essentially discarded. This can lead to the development of a large discrepancy between the actual state estimation errors and those which the filter predicts. It is the purpose of this section to present a method for assuring that the predicted error between actual and estimated states remains greater than or equal to the actual error when the cause of the error is a random driving force. This driving force for the navigation problems under study could be the result of such phenomena as gravity anomalies, atmospheric drag, solar wind, etc. The advantage of keeping the predicted error at least equal to the actual error is that navigation measurements are not improperly disregarded because the gains do not tend toward zero.

Another property of the method to be presented here is that it is not only conservative in that the predicted error is always at least as large as the actual error; but if the associated free parameter is chosen properly it is the least conservative of all conservative forms. Thus the difference between the predicted error and actual error is the smallest of that which results from any conservative scheme. The advantage gained through this property is that if it is at all possible to keep the gains up without making them unnecessarily large, this method will do it at least as well as any other.

Let us begin by hypothesizing a state equation:

$$\dot{\underline{x}} = F \underline{x} + \underline{u}$$

where $\underline{u}$ is an unknown driving force. If the estimate is extrapolated in the usual way:

$$\dot{\hat{\underline{x}}} = F \hat{\underline{x}}$$

the error in the estimate propagates by:

$$\dot{\underline{e}} = \underline{F}\underline{e} + \underline{u} \quad (1)$$

where

$$\underline{e} = \underline{x} - \hat{\underline{x}}$$

Forming the covariance matrix of state estimation errors we have:

$$\begin{aligned} \underline{P} &= \overline{\underline{e}\underline{e}^T} \\ \dot{\underline{P}} &= \overline{\underline{e}\dot{\underline{e}}^T} + \overline{\dot{\underline{e}}\underline{e}^T} \end{aligned} \quad (2)$$

where the overbar denotes ensemble average.

Substitution of Eq. (1) into Eq. (2) yields:

$$\dot{\underline{P}} = \underline{F}\underline{P} + \underline{P}\underline{F}^T + \overline{\underline{e}\underline{u}^T} + \overline{\underline{u}\underline{e}^T} \quad (3)$$

Since the cross correlations of $\underline{u}$ and $\underline{e}$ are in practice difficult to determine whereas the covariances $\overline{\underline{u}\underline{u}^T}$ and $\overline{\underline{e}\underline{e}^T}$ are usually better known, the problem is posed:

Given the covariances $\overline{\underline{u}\underline{u}^T}$ and $\overline{\underline{e}\underline{e}^T}$, how do we choose the matrix $\underline{M}$ such that

$$\underline{M} \geq \overline{\underline{u}\underline{e}^T} + \overline{\underline{e}\underline{u}^T} \quad (4)$$

and $\underline{M}$ is minimal in that there is no other matrix $\underline{M}'$ which satisfies inequality (4) and is smaller than the chosen $\underline{M}$?

Unfortunately this problem does not have a unique solution. It will be shown below that all matrices, $\underline{M}$, which are both conservative and

minimal have the form:

$$M = \frac{1}{\lambda} Q + \lambda P \quad ; \quad Q = \overline{u u^T}$$

where the positive scalar, λ , lies in a range to be specified below.

The conservative property is demonstrated as follows:

$$(\sqrt{\lambda} \underline{e} - \frac{1}{\sqrt{\lambda}} \underline{u}) (\sqrt{\lambda} \underline{e} - \frac{1}{\sqrt{\lambda}} \underline{u})^T \geq 0$$

$$\lambda \overline{e e^T} - \overline{u e^T} - \overline{e u^T} + \frac{1}{\lambda} \overline{u u^T} \geq 0$$

thus

$$\lambda \overline{e e^T} + \frac{1}{\lambda} \overline{u u^T} \geq \overline{u e^T} + \overline{e u^T}$$

or

$$\lambda P + \frac{1}{\lambda} Q \geq \overline{u e^T} + \overline{e u^T}$$

The following theorem provides an algebraic definition of a conservative matrix.

Theorem I

$$C \geq \overline{e f^T} + \overline{f e^T} \quad (5)$$

for all random vectors $\underline{e}$ and $\underline{f}$ such that

$$\overline{e e^T} = A \quad ; \quad \overline{f f^T} = B \quad (6)$$

if and only if

$$\underline{x}^T C \underline{x} \geq 2 \sqrt{(\underline{x}^T A \underline{x})(\underline{x}^T B \underline{x})} \quad \text{for all } \underline{x} \quad (7)$$

Proof:

If (assume Eq. (7) and prove Eq. (5))

Let $\xi = \underline{x}^T \underline{e}$, $\eta = \underline{x}^T \underline{f}$ (scalars)

then Eq. (7) implies $\underline{x}^T \underline{C} \underline{x} \geq 2 \sqrt{\xi^2 \eta^2}$ (8)

By the Schwartz inequality

$$\xi \eta \leq \sqrt{\xi^2 \eta^2} \quad (9)$$

Substituting Eq. (9) into Eq. (8) yields

$$\begin{aligned} \underline{x}^T \underline{C} \underline{x} &\geq 2 \overline{\xi \eta} = 2 \overline{\underline{x}^T \underline{e} \underline{f}^T \underline{x}} = \overline{\underline{x}^T \underline{e} \underline{f}^T \underline{x}} + \overline{\underline{x}^T \underline{f} \underline{e}^T \underline{x}} \\ &= \underline{x}^T (\underline{e} \underline{f}^T + \underline{f} \underline{e}^T) \underline{x} \end{aligned}$$

Only if (Prove that Eqs. (5) and (6) imply (7))

Construct $\underline{e}$ and $\underline{f}$ as follows.

Let $\underline{u}$ be a random variable with covariance $\overline{\underline{u} \underline{u}^T} = \underline{I}$

and let $\underline{e} = \sqrt{\underline{A}} \underline{u}$, $\underline{f} = \sqrt{\underline{B}} \underline{T} \underline{u}$

where $\sqrt{\underline{A}}$ is a matrix with the property that $\sqrt{\underline{A}} (\sqrt{\underline{A}})^T = \underline{A}$ and $\underline{T}$ is an orthogonal matrix (i.e. $\underline{T} \underline{T}^T = \underline{I}$) which will be chosen below. Then

$$\underline{x}^T (\underline{e} \underline{f}^T + \underline{f} \underline{e}^T) \underline{x} = 2 \underline{x}^T \sqrt{\underline{A}} \underline{T}^T (\sqrt{\underline{B}})^T \underline{x} \quad (10)$$

The proof will be completed by choosing $\underline{T}$ such that

$$\underline{x}^T \sqrt{\underline{A}} \underline{T}^T (\sqrt{\underline{B}})^T \underline{x} = \sqrt{(\underline{x}^T \underline{A} \underline{x})(\underline{x}^T \underline{B} \underline{x})} \quad (11)$$

for by Eq. (5)

$$\underline{x}^T C \underline{x} \geq \underline{x}^T (\underline{e} \underline{f}^T + \underline{f} \underline{e}^T) \underline{x} \quad (12)$$

and Eq. (7) follows from Eqs. (10), (11), and (12).

Construction of T

In terms of the vectors

$$\underline{y} = (\sqrt{A})^T \underline{x}, \quad \underline{z} = (\sqrt{B})^T \underline{x}$$

Eq. (11) requires that

$$\underline{z}^T T \underline{y} = |\underline{z}| |\underline{y}|$$

which will in turn be satisfied if

$$T \frac{\underline{y}}{|\underline{y}|} = \frac{\underline{z}}{|\underline{z}|} \quad (13)$$

Define $\underline{u}_1 = \frac{\underline{y}}{|\underline{y}|}$ and $\underline{u}_2 = \frac{\underline{z}}{|\underline{z}|}$

Assume that $\underline{u}_2 \neq \pm \underline{u}_1$ (If equal, use $T = \pm 1$)

and define $\underline{u}_3 = \frac{1}{\beta} [\underline{u}_2 - (\underline{u}_2 \cdot \underline{u}_1) \underline{u}_1]$

with $\beta = |\underline{u}_2 - (\underline{u}_2 \cdot \underline{u}_1) \underline{u}_1|$

Now $\beta^2 = 1 - (\underline{u}_2 \cdot \underline{u}_1)^2$

and $\beta \neq 0$ since $\underline{u}_2 \cdot \underline{u}_1 \neq \pm 1$

Let

$$\alpha = \underline{u}_2 \cdot \underline{u}_1$$

then

$$\alpha^2 + \beta^2 = 1$$

and

$$\underline{u}_2 = \alpha \underline{u}_1 + \beta \underline{u}_3$$

(13a)

Finally

$$\underline{u}_1 \cdot \underline{u}_3 = 0, \quad |\underline{u}_1| = |\underline{u}_3| = 1$$

Define

$$P = I - \underline{u}_1 \underline{u}_1^T - \underline{u}_3 \underline{u}_3^T$$

Observe

$$P = P^T, \quad P^2 = P, \quad P \underline{u}_1 = P \underline{u}_3 = 0$$

By Eq. (13a)

$$P \underline{u}_2 = 0$$

Define

$$R = \alpha (\underline{u}_1 \underline{u}_1^T + \underline{u}_3 \underline{u}_3^T) + \beta (\underline{u}_3 \underline{u}_1^T - \underline{u}_1 \underline{u}_3^T)$$

Then

$$R \underline{u}_1 = \alpha \underline{u}_1 + \beta \underline{u}_3 = \underline{u}_2$$

$$RR^T = \alpha^2 (\underline{u}_1 \underline{u}_1^T + \underline{u}_3 \underline{u}_3^T) + \beta^2 (\underline{u}_1 \underline{u}_1^T + \underline{u}_3 \underline{u}_3^T) \\ = I - P$$

$$PR^T = RP = 0$$

Let

$$T = P + R$$

Then

$$T \underline{u}_1 = P \underline{u}_1 + R \underline{u}_1 = R \underline{u}_1 = \underline{u}_2$$

and

$$TT^T = P^2 + RR^T = I$$

Thus we see that Eq. (13) holds for the T constructed above.

Q.E.D.

The following theorem shows that minimal conservative approximations to the cross correlation $\overline{\underline{c} \underline{f}^T} + \overline{\underline{f} \underline{c}^T}$ must have the form

$$M = \lambda P + \frac{1}{\lambda} Q$$

Theorem II

Given non zero symmetric matrices $A, B \geq 0$

and a symmetric matrix C such that

$$\underline{x}^T C \underline{x} \geq 2 \sqrt{(\underline{x}^T A \underline{x})(\underline{x}^T B \underline{x})} \quad \text{for all } \underline{x} \quad (14)$$

Then there exists a scalar $\lambda > 0$ such that

$$C \geq \lambda A + \frac{1}{\lambda} B \quad (15)$$

Proof:

$$\text{Let } T = \{ \underline{x} \mid \underline{x}^T A \underline{x} > 0, \underline{x}^T B \underline{x} > 0 \}$$

and for all $\underline{x} \in T$

$$L(\underline{x}) = \frac{\frac{1}{2} \underline{x}^T C \underline{x}}{\sqrt{(\underline{x}^T A \underline{x})(\underline{x}^T B \underline{x})}} \quad (16)$$

Then by Eq. (14)

$$L(\underline{x}) \geq 1 \quad (17)$$

Also define

$$L(C) = \inf_{\underline{x} \in T} L(\underline{x}) \quad (18)$$

(note that T is non-empty from the theorem hypothesis)

Lemma A: There exists a vector $\underline{x}_0 \in T$ such that

$$L(\underline{x}_0) = L(c)$$

Proof of Lemma A:

Let $\{\underline{x}_n\}$ be a sequence of vectors in T such that

$$L(\underline{x}_n) \rightarrow L(c) \quad \text{as } n \rightarrow \infty \quad (19)$$

(we know this sequence exists by the definition of infimum)

In order to avoid the indeterminate situation $L(\underline{x}) = c/0$ let V_0 be the orthogonal complement of the intersection of the null spaces of A , B and C . V_0 is thus a linear subspace of a Euclidean space and is therefore a closed set. Let $\underline{y}_n$ be the orthogonal projection of $\underline{x}_n$ onto V_0 .

By the definition of $\{\underline{y}_n\}$, we have

$$\underline{x}_n = \gamma_n \frac{\underline{y}_n}{|\underline{y}_n|} + \underline{j}$$

where

$$A\underline{j} = B\underline{j} = C\underline{j} = \underline{0}$$

thus

$$L(\underline{x}_n) = \frac{1}{2} \frac{\gamma_n^2}{|\underline{y}_n|^2} \underline{y}_n^T C \underline{y}_n = L\left(\frac{\underline{y}_n}{|\underline{y}_n|}\right)$$

$$\sqrt{\frac{\gamma_n^4}{|\underline{y}_n|^4} (\underline{y}_n^T A \underline{y}_n)(\underline{y}_n^T B \underline{y}_n)}$$

and by Eq. (19)

$$L\left(\frac{\underline{y}_n}{|\underline{y}_n|}\right) \rightarrow L(c) \quad (20)$$

Let S be the intersection of the unit sphere in the Euclidean space with V_0 . S is closed since the intersection of two closed sets is closed, and is bounded because it is a subset of the unit sphere. Therefore S is a compact set.

Since the infinite sequence $\left\{ \frac{y_n}{\|y_n\|} \right\}$ lies in the compact set S , the sequence must have a convergent subsequence by the Bolzano-Weierstrass theorem.

Let $\underline{x}_0$ be the limit of the convergent subsequence. Then $\underline{x}_0 \in V_0$. Since $L(\underline{x})$ is continuous on T the lemma will follow by Eq. (20) if it can be shown that $\underline{x}_0 \in T$.

Let

$$\underline{x}_0^T A \underline{x}_0 = a, \quad \underline{x}_0^T B \underline{x}_0 = b, \quad \underline{x}_0^T C \underline{x}_0 = c$$

If

$$a > 0, \quad b > 0$$

(21)

then

$$\underline{x}_0 \in T$$

We will prove Eq. (21) by contradiction

Assume $a = 0$. Since $A \geq 0$ this implies

$$A \underline{x}_0 = A^T \underline{x}_0 = \underline{0}$$

From Eq. (16) we have

$$c = 2 L(c) \sqrt{ab}$$

Since $L(c)$ is bounded it follows that $c = 0$

Since $C \geq 0$ this implies

$$C \underline{x}_0 = C^T \underline{x}_0 = \underline{0}$$

Since $\underline{x}_0 \in V_0$ and $\underline{x}_0 \neq 0$

$$b > 0$$

Since $A \neq 0$ we may choose a $\underline{v}$ such that

$$\underline{v}^T A \underline{v} > 0$$

Then

$$L(\underline{x}_0 + \epsilon \underline{N}) = \frac{1}{2} \frac{\epsilon \underline{N}^T C \underline{N}}{\sqrt{(\underline{N}^T A \underline{N}) [\underline{x}_0 + \epsilon \underline{N}]^T B [\underline{x}_0 + \epsilon \underline{N}]}}$$

$$\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

This contradicts Eq. (17) and the same result ensues if b is assumed to be zero at the outset.

$$\therefore a > 0, \quad b > 0 \quad \text{Q.E.D.}$$

Lemma B:

$$\lambda A + \frac{1}{\lambda} B \leq \frac{1}{L(c)} C$$

where

$$\lambda = \sqrt{\frac{\underline{x}_0^T B \underline{x}_0}{\underline{x}_0^T A \underline{x}_0}}$$

(Lemma B implies Theorem II since by Eq. (17)

$$L(c) \geq 1 \quad \text{and thus} \quad C \geq \frac{C}{L(c)} \quad)$$

The strategy for the proof of Lemma B will be the following:

Choose any vector $\underline{x}$.

Let $\underline{y}$ be a particular vector in the space spanned by $\underline{x}$ and $\underline{x}_0$.

Lemma B will then be proved for all $\underline{z}$ of the form

$$\underline{z} = \gamma \underline{x}_0 + \mu \underline{y}$$

Since the choice of $\underline{x}$ is completely arbitrary, this results in general proof of Lemma B.

Proof of Lemma B

Let $D = \frac{C}{L(c)}$ and define

$$a = \underline{x}_0^T A \underline{x}_0, \quad b = \underline{x}_0^T B \underline{x}_0, \quad g = \underline{x}_0^T D \underline{x}_0$$

Then by Lemma A

$$g = 2\sqrt{ab} \quad (22)$$

and $a, b, g > 0$

Let $\underline{x}$ be any vector and define

$$\underline{y} = \underline{x} - r \underline{x}_0 \quad (23)$$

where

$$r = \frac{\underline{x}_0^T D \underline{x}}{g}$$

Then

$$\underline{x}_0^T D \underline{y} = 0$$

Define

$$\tilde{D} = \begin{bmatrix} g & 0 \\ 0 & g\delta \end{bmatrix} = [\underline{x}_0, \underline{y}]^T D [\underline{x}_0, \underline{y}]$$

$$\tilde{A} = a \begin{bmatrix} 1 & e \\ e & \alpha \end{bmatrix} = [\underline{x}_0, \underline{y}]^T A [\underline{x}_0, \underline{y}]$$

$$\tilde{B} = b \begin{bmatrix} 1 & f \\ f & \beta \end{bmatrix} = [\underline{x}_0, \underline{y}]^T B [\underline{x}_0, \underline{y}]$$

Let $\underline{z}$ be defined by

$$\underline{z} = \gamma \underline{x}_0 + \mu \underline{y}$$

From Eq. (16) and Eq. (17) we have

$$\underline{z}^T C \underline{z} \geq 2L(c) \sqrt{(\underline{z}^T A \underline{z})(\underline{z}^T B \underline{z})}$$

Thus by the definition of D we have

$$\underline{z}^T D \underline{z} \geq 2 \sqrt{(\underline{z}^T A \underline{z})(\underline{z}^T B \underline{z})} \quad (24)$$

Lemma C

For Eq. (24) to hold for all γ and μ the following must be true.

$$f = -e \quad ; \quad 2\delta \geq \alpha + \beta$$

Proof of Lemma C

Substituting the definition of $\tilde{A}$, $\tilde{B}$, $\tilde{D}$, $\underline{y}$ and $\underline{z}$ into Eq. (24) yields

$$\begin{aligned} & \gamma^2 \underline{x}_0^T D \underline{x}_0 + \gamma\mu (\underline{x}_0^T D \underline{y} + \underline{y}^T D \underline{x}_0) + \mu^2 \underline{y}^T D \underline{y} \geq \\ & 2 \sqrt{(\gamma^2 \underline{x}_0^T A \underline{x}_0 + \gamma\mu (\underline{x}_0^T A \underline{y} + \underline{y}^T A \underline{x}_0) + \mu^2 \underline{y}^T A \underline{y})(\gamma^2 \underline{x}_0^T B \underline{x}_0 + \gamma\mu (\underline{x}_0^T B \underline{y} + \underline{y}^T B \underline{x}_0) + \mu^2 \underline{y}^T B \underline{y})} \\ & \gamma(\gamma^2 + \mu^2 \delta) \geq 2\sqrt{ab} \sqrt{(\gamma^2 + 2\gamma\mu e + \mu^2 \alpha)(\gamma^2 + 2\gamma\mu f + \mu^2 \beta)} \end{aligned}$$

Substituting Eq. (22) yields

$$\gamma^2 + \mu^2 \delta \geq \sqrt{(\gamma^2 + 2\gamma\mu e + \mu^2 \alpha)(\gamma^2 + 2\gamma\mu f + \mu^2 \beta)}$$

Squaring both sides and collecting terms yields

$$\mu^4 (\delta^2 - \alpha\beta) - 2\gamma\mu^3 (\alpha f + e\beta) + \gamma^2 \mu^2 (2\delta - 4ef - \alpha - \beta) - 2\gamma^3 \mu (e + f) \geq 0 \quad (25)$$

We know that the above function has a local minimum at the point $\mu = 0$. Setting the first derivative to zero at $\mu = 0$ yields

$$e = -f \quad (26)$$

Substituting Eq. (26) into Eq. (25) and dividing by μ^2 yields

$$\mu^2 (\delta^2 - \alpha\beta) + 2\mu\gamma e(\alpha - \beta) + \gamma^2 (2\delta - 4e^2 - \alpha - \beta) \geq 0 \quad (27)$$

Define

$$\Delta = 2\delta - \alpha - \beta \quad (28)$$

Since

$$\delta = \frac{\underline{y}^T D \underline{y}}{\underline{x}_0^T D \underline{x}_0} \text{ implies } \delta \geq 0$$

we have

$$\Delta \geq -(\alpha + \beta) \quad (29)$$

Also Eq. (27) may be written

$$m_1(\Delta) + m_2(\Delta)\gamma\mu + m_3(\Delta)\gamma^2\mu^2 \geq 0 \quad (30)$$

with

$$m_1(\Delta) = \Delta + 4e^2 \quad (31)$$

$$m_2(\Delta) = 2e(\alpha + \beta) \quad (32)$$

$$m_3(\Delta) = \frac{1}{4} [(\alpha - \beta)^2 + 2(\alpha + \beta)\Delta + \Delta^2] \quad (33)$$

In order for Eq. (30) to hold for all μ we must have

$$m_1(\Delta) \geq 0 \quad (34)$$

$$\gamma^2 m_3(\Delta) \geq 0 \text{ for all } \gamma \text{ implies } m_3(\Delta) \geq 0 \quad (35)$$

$$\gamma^2 (4m_1(\Delta)m_3(\Delta) - m_2^2(\Delta)) \geq 0 \text{ for all } \gamma$$

$$\text{implies } 4m_1(\Delta)m_3(\Delta) - m_2^2(\Delta) \geq 0 \quad (36)$$

Substituting Eq. (31), Eq. (32) and Eq. (33) into Eq. (36) yields

$$4s m_4(s) \geq 0 \quad (37)$$

with

$$m_4(s) = m_3(s) + e^2 [2(\alpha + \beta) + s] \quad (38)$$

By Eqs. (29) and (35)

$$m_4(s) \geq e^2 (\alpha + \beta) \quad (39)$$

Since $\alpha, \beta \geq 0$, $m_4(s) \geq 0$ and by Eq. (37) either

$$s \geq 0 \quad (40)$$

or else

$$m_4(s) = 0$$

which implies that

$$e^2 (\alpha + \beta) = 0 \quad (41)$$

If Eq. (40) holds, then by Eq. (28)

$$2s \geq \alpha + \beta \quad (42)$$

If $e = 0$ by Eq. (31) $m_1(s) = s$ and by Eq. (34) $s \geq 0$ and Eq. (42) holds by Eq. (28)

If $\alpha + \beta = 0$ then Eq. (42) holds since $s \geq 0$. Thus Eq. (42) always holds. (End of proof of Lemma C)

Now by Eq. (23) we have

$$\underline{x} = \underline{y} + \Gamma \underline{x}_0$$

and thus

$$\underline{x}^T D \underline{x} = g(r^2 + \delta) \quad (43)$$

Also by definition

$$\lambda = \sqrt{\frac{b}{a}}$$

and

$$\underline{x}^T (\lambda A + \frac{1}{\lambda} B) \underline{x} = \sqrt{\frac{b}{a}} (r^2 + 2er + \alpha)a + \sqrt{\frac{a}{b}} (r^2 + 2rf + \beta)b$$

By Eq. (22) and Eq. (26)

$$\underline{x}^T (\lambda A + \frac{1}{\lambda} B) \underline{x} = g(r^2 + \frac{\alpha + \beta}{2})$$

and by Eq. (42) and Eq. (43)

$$\underline{x}^T D \underline{x} \geq \underline{x}^T (\lambda A + \frac{1}{\lambda} B) \underline{x} \quad Q.E.D.$$

Range of Values of λ

We want to extremize the quantity

$$\lambda = \sqrt{\frac{\underline{x}^T B \underline{x}}{\underline{x}^T A \underline{x}}} \quad (44)$$

Since λ is non negative, λ^2 will be extremized if λ is extremized. Thus we want to extremize

$$\lambda^2 = \frac{\underline{x}^T B \underline{x}}{\underline{x}^T A \underline{x}} \quad (45)$$

Setting the differential with respect to $\underline{x}$ of Eq. (45) to zero yields

$$\frac{2 \underline{x}^T A \underline{x} \underline{x}^T B - 2 \underline{x}^T B \underline{x} \underline{x}^T A}{(\underline{x}^T A \underline{x})^2} = 0$$

Assuming $A > 0$ and dividing yields

$$\underline{x}^T B - \frac{\underline{x}^T B \underline{x}}{\underline{x}^T A \underline{x}} \underline{x}^T A = \underline{0}^T$$

Transposing yields

$$B \underline{x} = \frac{\underline{x}^T B \underline{x}}{\underline{x}^T A \underline{x}} A \underline{x}$$

Premultiplying by A^{-1} yields

$$A^{-1} B \underline{x} = \frac{\underline{x}^T B \underline{x}}{\underline{x}^T A \underline{x}} \underline{x} = \lambda^2 \underline{x}$$

Thus the condition on λ^2 for it to be extremized is that λ^2 be equal to an eigenvalue of $A^{-1}B$. Therefore the extremes of λ^2 are given by the maximum and minimum eigenvalues of $A^{-1}B$.

In the next report it is planned to provide a proof that M chosen by

$$M = \lambda P + \frac{1}{\lambda} Q$$

is minimal if and only if λ lies within the interval bounded by the minimum and maximum eigenvalues of $P^{-1}Q$.

Section I - Summary

1) Theorem I has demonstrated that a necessary and sufficient algebraic condition on a symmetric matrix C for the matrix inequality

$$C \geq \overline{e e^T} + \overline{u u^T}$$

to be valid for all $\underline{x}$ is given by

$$\underline{x}^T C \underline{x} \geq 2 \sqrt{(\underline{x}^T P \underline{x})(\underline{x}^T Q \underline{x})} \quad \text{for all } \underline{x} \quad (46)$$

where

$$\overline{e e^T} = P \quad \overline{u u^T} = Q$$

Eq. (46) provides an algebraic criterion for determining if an approximation to the cross correlations of $\underline{u}$ and $\underline{e}$ is conservative.

2) Theorem II shows that a minimal approximation must have the form

$$M = \lambda P + \frac{1}{\lambda} Q$$

i.e. There is no other matrix C which satisfies Eq. (46) such that

$$C < M$$

II Simulation Results

The purpose of this section is to describe some typical simulation results wherein the errors predicted by the method of Section I are compared to "known errors" for an Earth orbital mission. The standard chosen here is a model of the geopotential of the Earth* which includes trigonometric terms in longitude up to order 10 and Legendre functions in latitude up to order 10. A circular polar orbit of 500,000 feet altitude is used for the presentation of the results herein. All simulations start from zero initial conditions; the covariance matrix Q is approximated by using the geopotential model of the Earth to compute the mean squared value of the disturbing acceleration over three periods. All terms in the model except the J_2 term were considered to be perturbation sources.

Use of the technique of Section I requires a choice of the free parameter, λ . Four candidate algorithms for the choice of λ are presented here, together with simulation results. It is expected that a fifth method, which is to treat λ as the control variable in an optimal control problem, will be presented in the next progress report. The cost function (to be minimized) for this control problem will be some linear combination of the state errors at some selected future time.

Method 1

Method 1 is motivated by one of the results of Section I. Specifically, it was shown in Section I that the choice of λ^2 can be constrained to lie between the minimum and maximum eigenvalues of $P^{-1}Q$. This method of choosing λ is:

$$\lambda = \sqrt{\frac{\text{tr.}(Q_{22})}{\text{tr.}(P_{22})}}$$

*The model and computer program used is due to Dr. J. H. Lanning of the MIT Instrumentation Laboratory.

where a state defined as

$$\underline{X} = \begin{bmatrix} \underline{r} \\ \underline{v} \end{bmatrix}$$

is assumed and P and Q are partitioned as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix}$$

By computing λ in this way an eigenvalue problem is avoided and yet λ^2 in most cases will be of the order of the principal non zero eigenvalues of $P^{-1}Q$.

Figure 1 illustrates a comparison of the RMS position error predicted by this method with the actual rms position deviation from the nominal circular orbit. Figure 2 gives the same for rms velocity errors and Figure 3 gives the time history of $1/\lambda$ for this method. Figures 4 and 5 are enlarged versions of Figures 1 and 2 for the first half orbit period.

Two comments are appropriate at this point. First, this method predicts the errors to within a factor of two for about the first half period. Second, as can be seen in the enlarged view, the predicted error is briefly less than the actual error--a result which is theoretically impossible as shown in Section I. There are two reasons for this. First, the covariance matrix Q is only an approximation to the actual covariance matrix of acceleration perturbations. Second, in order to begin the integration process with an infinite value of λ as this method calls for, a starting artifice was used.

The need for some artificial way of starting the solution is caused by the fact that λ is infinite initially. The approach used to generate these results is to approximate the trajectory by rectilinear motion until a reasonable sample period for numerically integrating the differential equations is possible. In this case the rectilinear approximation is used for the first 200 seconds, after which numerical integration along the circular reference trajectory begins. By assuming

rectilinear motion the following analytic solution for P is obtainable:

$$P = \begin{bmatrix} \frac{8t^4}{9} I & \frac{8t^3}{6} I \\ \frac{8t^3}{6} I & \frac{8t^2}{3} I \end{bmatrix}$$

where I is the 3 x 3 unit matrix and

$$Q_{22} = \frac{8}{3} I$$

Method 2

Method 2 is motivated by the desire to control period changes since knowledge of period is crucial for prediction in the future. Period is a function of semimajor axis which is directly related to energy. By controlling errors in the direction of the velocity vector we then have an approximate method for controlling period changes. Thus choose λ by:

$$\lambda = \sqrt{\frac{\underline{v}^T Q_{22} \underline{v}}{\underline{v}^T E_{22} \underline{v}}}$$

Figures 6 through 10 correspond directly with Figures 1 through 5 except λ is chosen by Method 2. Note they exhibit essentially the same behavior. The same comments are applicable to this case as are to Method 1.

Method 3

Method 3 is again designed to minimize the change in energy for the same reasons as given above for Method 2. The difference is that λ is chosen to minimize the rate of change of mean squared energy uncertainty.

The result is that

$$\frac{1}{\lambda} = \sqrt{\frac{\underline{\xi}^T P \underline{\xi}}{\underline{\xi}^T Q \underline{\xi}}}$$

where

$$\underline{\dot{z}} = \begin{bmatrix} \frac{\mu \underline{r}}{r^3} \\ \underline{v} \end{bmatrix}$$

Note that this time the contribution of changes in position to changes in energy are accounted for. The vector $\underline{\dot{z}}$ is determined by

$$\underline{\dot{z}} = \frac{\partial h}{\partial \underline{x}}^T$$

where

$$h = \frac{v^2}{2} - \frac{\mu}{r}$$

and

$$\underline{x} = \begin{bmatrix} \underline{r} \\ \underline{v} \end{bmatrix}$$

Figures 11 through 15 correspond directly to Figures 1 through 5 except that λ is chosen by Method 3. Unfortunately there is little improvement using this method. The reason is identified below in Method 4 and considerable improvement is obtained.

Method 4

All three methods presented above give predicted errors which are far too large after several periods. An explanation for this lies in the fact that the total energy has not been constrained. The motivation of this method is to constrain the energy. This is done by replacing the along path component of velocity by a new state variable which is the total energy.

(1) Energy for a Rotating Planet

Let $\underline{r}$ be the position of the vehicle relative to the center of the planet and let $\dot{\underline{r}}$ be the rate of change of the position of the vehicle as calculated in a coordinate system rotating with the planet. Assuming that the planet center is not accelerating, the inertial velocity of the vehicle, $\underline{v}$, is

$$\underline{v} = \dot{\underline{r}} + \underline{\omega} \times \underline{r}$$

where $\underline{\omega}$ is the angular velocity of the planet about its axis. The massless kinetic energy of the vehicle is

$$T = \frac{1}{2} |\underline{v}|^2$$

or

$$T = \frac{1}{2} |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2$$

Let V be the massless gravitational potential of the planet. If $\underline{r}$ is expressed in a coordinate system which rotates with the planet, then

$$V = V(\underline{r})$$

that is, the potential depends only on the position of the vehicle and not on time.

Based on these assumptions, the Lagrangian function for the vehicle is

$$L = T - V = \frac{1}{2} |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2 - V(\underline{r})$$

and L does not depend explicitly on time. Because L is time invariant, the Hamiltonian function

$$h = \frac{\partial L}{\partial \dot{\underline{r}}} \cdot \dot{\underline{r}} - L$$

is a constant.¹⁸

In our case,

$$\frac{\partial L}{\partial \dot{\underline{r}}} = (\dot{\underline{r}} + \underline{\omega} \times \underline{r})^T = \underline{v}^T$$

and

$$h = \frac{1}{2} |\underline{v}|^2 - \underline{\omega} \cdot (\underline{r} \times \underline{v}) + V \quad (1-1)$$

Note that $\underline{r} \times \underline{v}$ is the massless angular momentum of the vehicle. If the gravitational potential of the planet is not symmetric about its axis, the angular momentum of the vehicle is not constant and its variation makes up for the change in the sum of vehicle kinetic and potential energy due to planet rotation.

If a perturbing force $\underline{f}$ not included in the potential V is added, the equation of motion of the vehicle is

$$\frac{d\underline{v}}{dt} = - \left(\frac{\partial V}{\partial \underline{r}} \right)^T + \underline{f}$$

Using the relationship that

$$\frac{d\underline{v}}{dt} = \frac{\partial V}{\partial \underline{r}} (\underline{v} - \underline{\omega} \times \underline{r})$$

yields

$$\frac{dh}{dt} = (\underline{v} - \underline{\omega} \times \underline{r}) \cdot \underline{f} \quad (1-2)$$

(2) Use of Energy as a State Variable

For the present example of a satellite orbiting the earth with the J_2 gravitational potential terms set to zero, the component of the potential which is assumed to be known is μ/r . This known potential is symmetric about the earth's spin axis so that the rotation of the earth can be ignored and we can set $\underline{\omega} = 0$. (We will not always want to ignore $\underline{\omega}$, since for the moon the triaxial ellipsoid potential would be taken as the known component of the gravitational field.)

Thus

$$h = \frac{1}{2} |\underline{v}|^2 - \frac{\mu}{r}$$

and

$$\frac{dh}{dt} = \underline{v} \cdot \underline{f}$$

Let us now replace the vehicle velocity magnitude in the state vector with h . In the final equations of motion this replacement has the effect of eliminating the component of perturbing force parallel to the vehicle velocity vector.

Let $\underline{u}_1$, $\underline{u}_2$ and $\underline{u}_3$ be orthogonal unit vectors with $\underline{u}_1$ parallel to the vehicle velocity vector on the reference trajectory and let $\underline{\Omega}$ be the angular velocity of the $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ coordinate frame. Let the position and velocity perturbations be

$$\delta \underline{r} = \delta r_1 \underline{u}_1 + \delta r_2 \underline{u}_2 + \delta r_3 \underline{u}_3$$

$$\delta \underline{v} = \delta v_1 \underline{u}_1 + \delta v_2 \underline{u}_2 + \delta v_3 \underline{u}_3$$

Then δh has the form

$$\delta h = v \delta v_1 + \frac{\mu}{r^2} \delta r$$

where v is the vehicle velocity on its reference trajectory.

Solving for δv_1 yields

$$\delta v_1 = \frac{1}{v} \left[\delta h - \frac{\mu}{r^2} \delta r \right]$$

The variational equations for the perturbations in the rotating $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ coordinate system are

$$\frac{d \delta \underline{v}}{dt} = -\underline{\Omega} \times \underline{v} + G \delta \underline{r} + \underline{f}$$

$$\frac{d \underline{r}}{dt} = \underline{v} - \underline{\Omega} \times \underline{r}$$

where G is the gravity gradient matrix

$$G = -\frac{\mu}{r^3} [\underline{I} - 3 \underline{u}_r \underline{u}_r^T]$$

with

$$\underline{u}_r = \frac{1}{|\underline{r}|} \underline{r}$$

Using Eq. (1) to eliminate δv_1 from the variational equations yields

$$\delta \dot{r}_1 = -\frac{\mu}{v r^2} \delta r + \Omega_3 \delta r_2 - \Omega_2 \delta r_3 + \frac{1}{v} \delta h$$

$$\delta \dot{r}_2 = -\Omega_3 \delta r_1 + \Omega_1 \delta r_3 + \delta v_2$$

$$\delta \dot{r}_3 = \Omega_2 \delta r_1 - \Omega_1 \delta r_2 + \delta v_3$$

$$\begin{aligned}\delta \dot{N}_2 = & G_{21} \delta r_1 + G_{22} \delta r_2 + G_{23} \delta r_3 + \frac{\Omega_3 \mu}{N r^2} \delta r \\ & + \Omega_1 \delta N_3 - \frac{\Omega_3}{N} \delta h + f_2\end{aligned}$$

$$\begin{aligned}\delta \dot{N}_3 = & G_{31} \delta r_1 + G_{32} \delta r_2 + G_{33} \delta r_3 - \frac{\Omega_2 \mu}{N r^2} \delta r \\ & - \Omega_1 \delta N_2 + \frac{\Omega_2}{N} \delta h + f_3\end{aligned}$$

$$\delta \dot{h} = N f_1$$

where

$$\delta r = \frac{1}{|r|} r \cdot \delta r$$

Rather than employing the above equation for $\delta \dot{h}$, δh will be treated as a random forcing function. Since the perturbing gravitational force is conservative it comes from a potential V_p , that is

$$\underline{f} = \left(\frac{\partial V_p}{\partial \underline{r}} \right)^T$$

Letting $\tilde{h}$ denote the complete energy of the vehicle including the perturbing potential V_p the last variational equation is replaced by

$$\delta \dot{\tilde{h}} = 0$$

since $\tilde{h}$ is truly a constant and δh is replaced in the other equations by

$$\delta h = \delta \tilde{h} - V_p$$

where V_p is now thought of as a forcing function.

For the case of a circular orbit, let $\underline{u}_1$ be the radial, $\underline{u}_2$ the tangential and $\underline{u}_3$ the out of plane unit vectors. Then

$$\underline{\Omega} = -\Omega \underline{u}_3$$

where Ω is the orbital angular velocity

$$\Omega = \sqrt{\frac{\mu}{r^3}}$$

Also, in this case

$$G_{21} = 0, \quad G_{22} = \frac{2\mu}{r^3}, \quad G_{23} = 0$$

$$G_{31} = G_{32} = 0, \quad G_{33} = -\frac{\mu}{r^3}$$

and

$$\delta r = \delta r_2$$

and the variational equations become

$$\delta \dot{r}_1 = -2\Omega \delta r_2 + \frac{1}{r} \delta \tilde{h} = \frac{1}{r} V_p$$

$$\delta \dot{r}_2 = \Omega \delta r_1 + \delta \dot{r}_2$$

$$\delta \dot{r}_3 = \delta \dot{r}_3$$

$$\delta \dot{r}_2 = \Omega^2 \delta r_2 + \frac{1}{r} \delta \tilde{h} - \frac{1}{r} V_p + f_2$$

$$\delta \dot{r}_3 = -\Omega^2 \delta r_3 + f_3$$

$$\delta \dot{\tilde{h}} = 0$$

This system of differential equations is linear with constant coefficients. Ignoring the $\tilde{h}$ equation, which is not driven by a random force, the characteristic roots of the equations are 0 and $\pm i$. For the first computer runs it was decided to adjust the parameter λ to minimize the effect of the random forces on the mode of the equations corresponding to the characteristic root zero. This requires that λ be chosen to minimize the quantity

$$\underline{U}^T \left(\lambda P + \frac{1}{\lambda} Q \right) \underline{U}$$

where

$$\underline{U} = \begin{bmatrix} \omega \\ 0 \\ 0 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$$

With this choice of λ an analytic solution for the rms position error was obtained. The result is:

$$\text{RMS position error} = \sqrt{t \left(K_c \overline{\delta v^2} + \frac{1}{3} K_f \overline{\delta f^2} \right)}$$

where

$$K_c = [19\theta + 3\sin 2\theta - 24\sin \theta] / \Omega v^2$$

$$K_f = [14\theta + \sin 2\theta - 16\sin \theta] / 2\Omega^3$$

$$\theta = \Omega t$$

$$\begin{aligned} \overline{\delta v^2} &= \text{mean squared potential deviation} \\ \overline{\delta f^2} &= \text{mean squared perturbing force.} \end{aligned}$$

Figure 16 depicts a comparison of the predicted RMS position error using this method with the actual position deviations. Note that now after three full orbits the actual and predicted errors lie within a factor of two--a substantial gain over the first three methods. The fact that the predicted error is less than the actual error is once again due to the fact that the mean squared values involved were constant and set at the value obtained by averaging the actual deviations over three orbits.

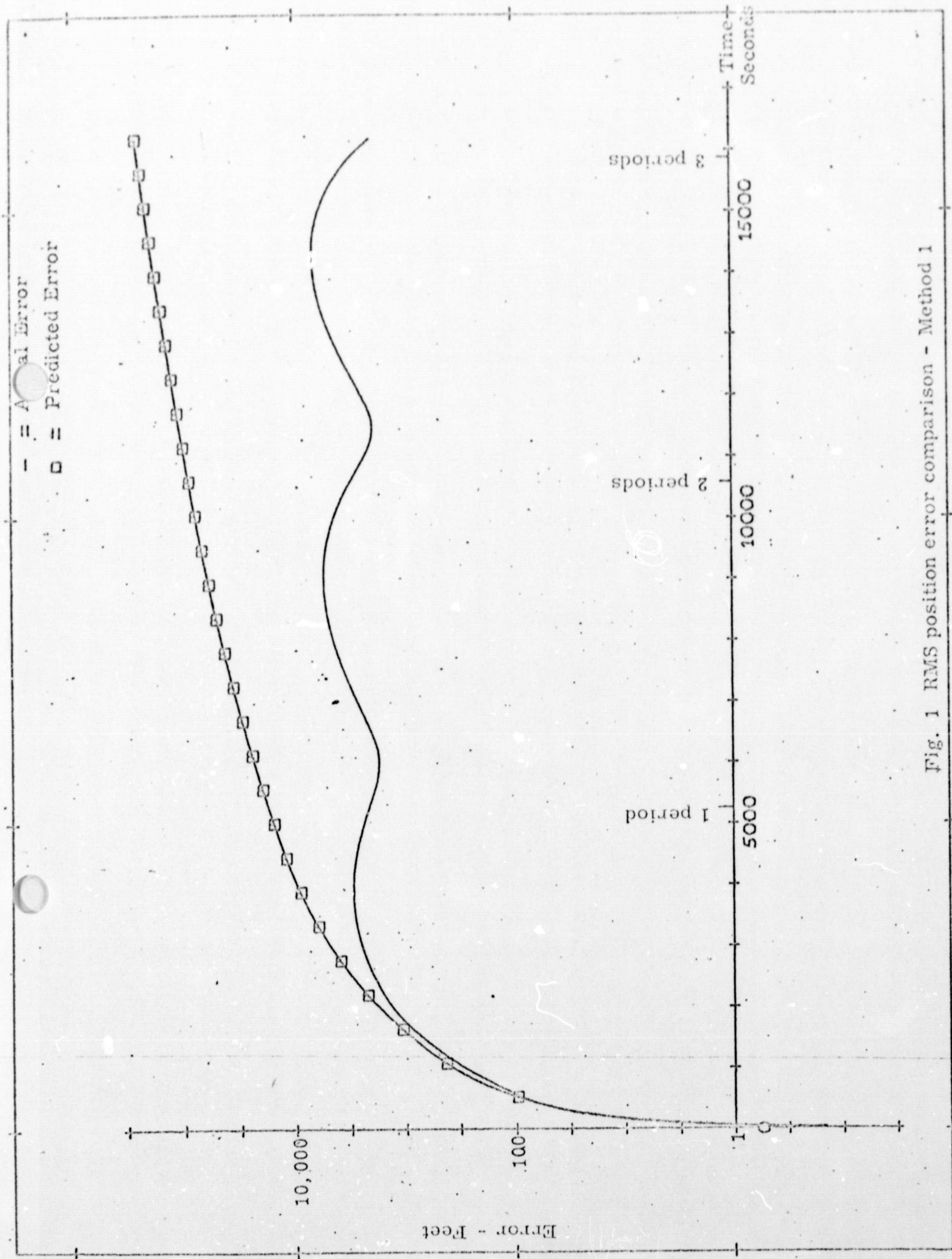


Fig. 1 RMS position error comparison - Method 1

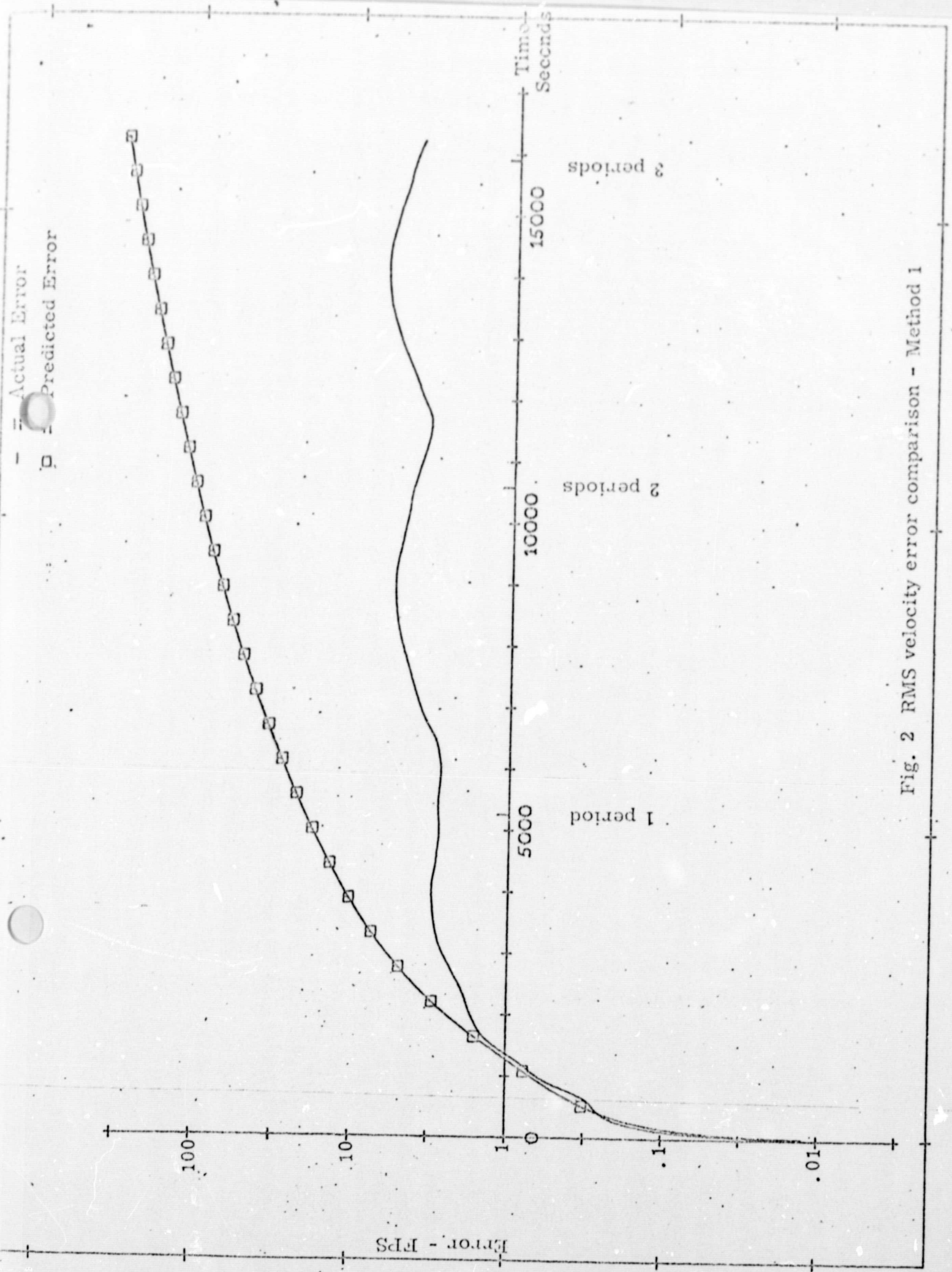


Fig. 2 RMS velocity error comparison - Method 1

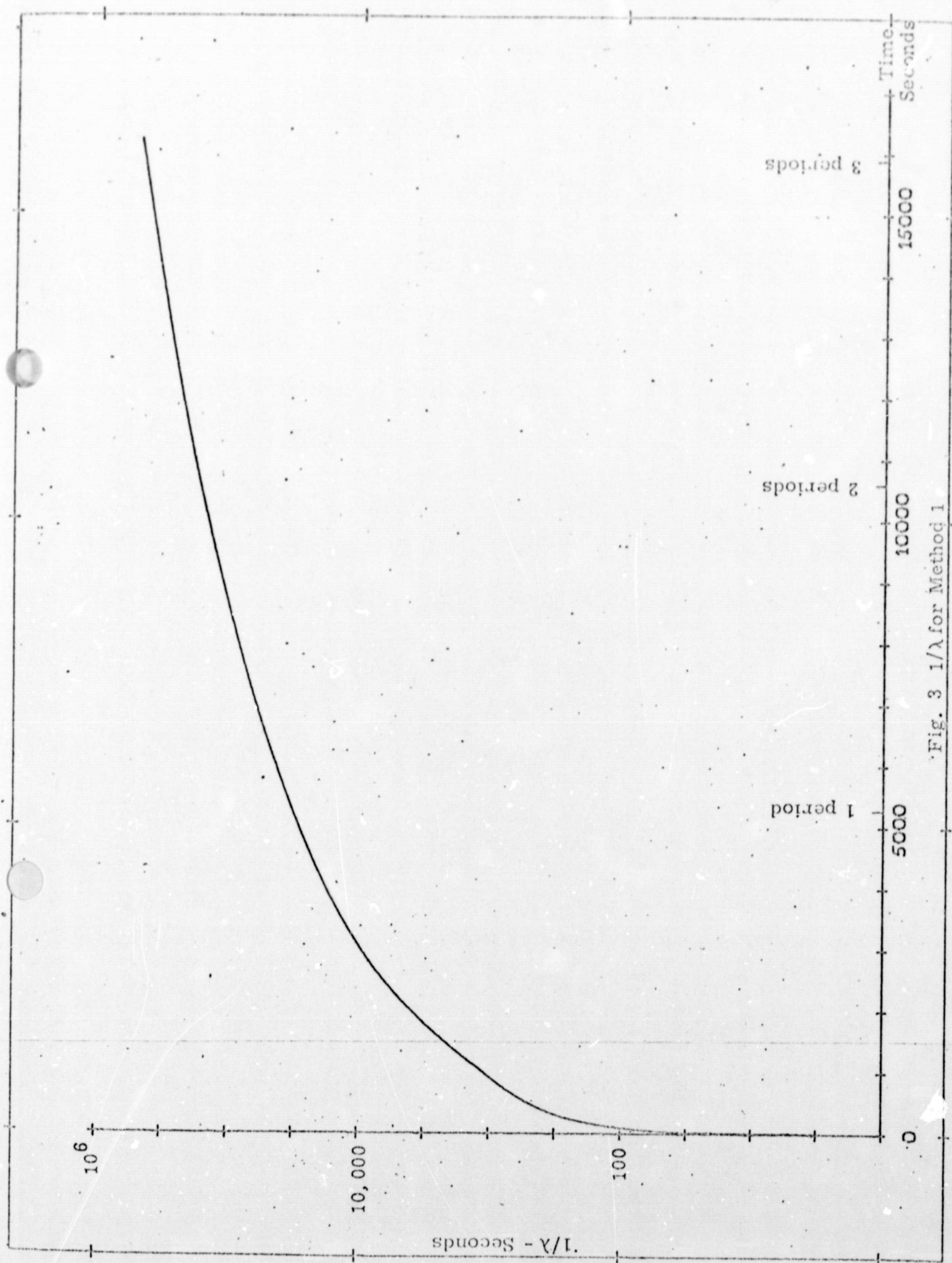


Fig. 3 $1/\lambda$ for Method 1

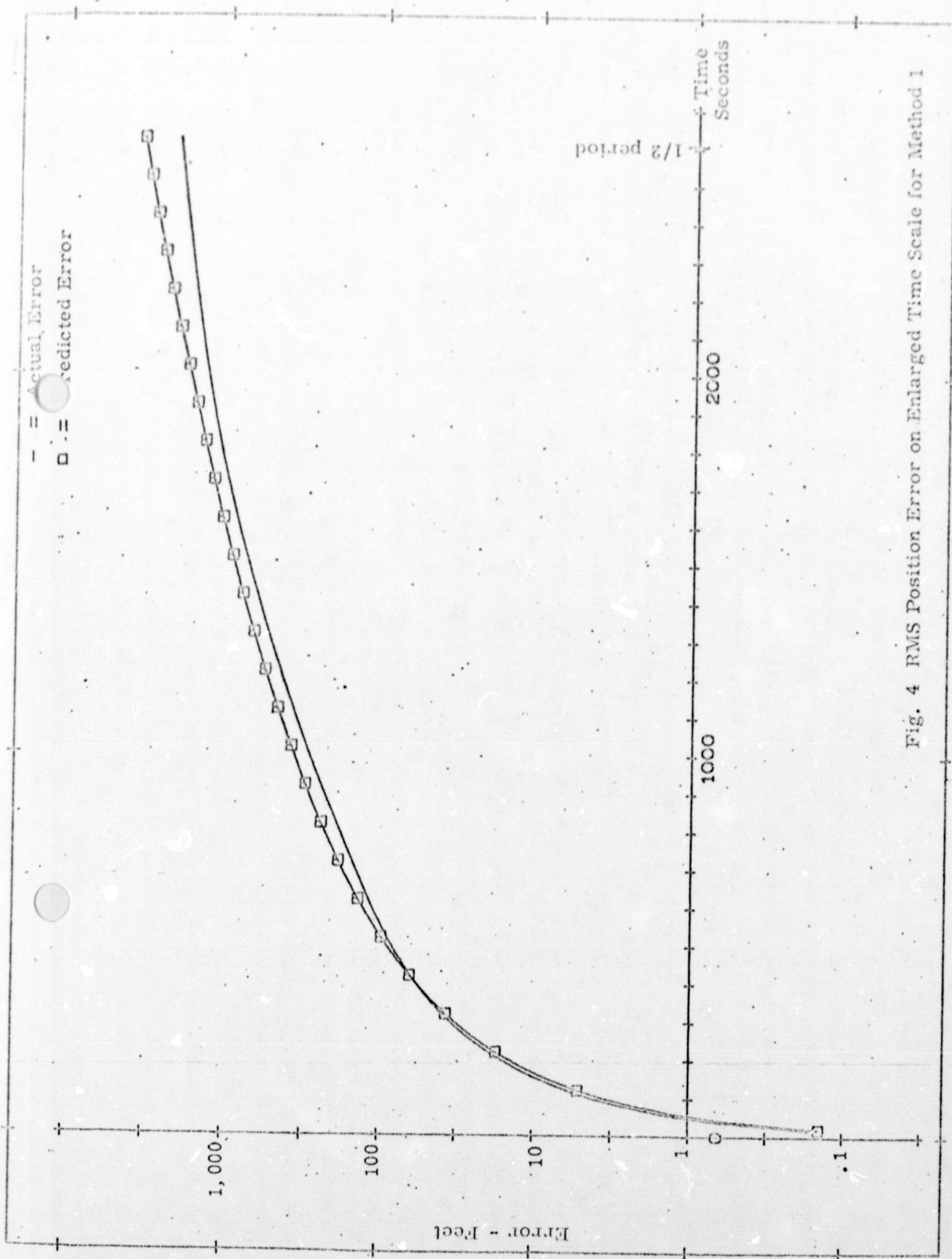


Fig. 4 RMS Position Error on Enlarged Time Scale for Method 1

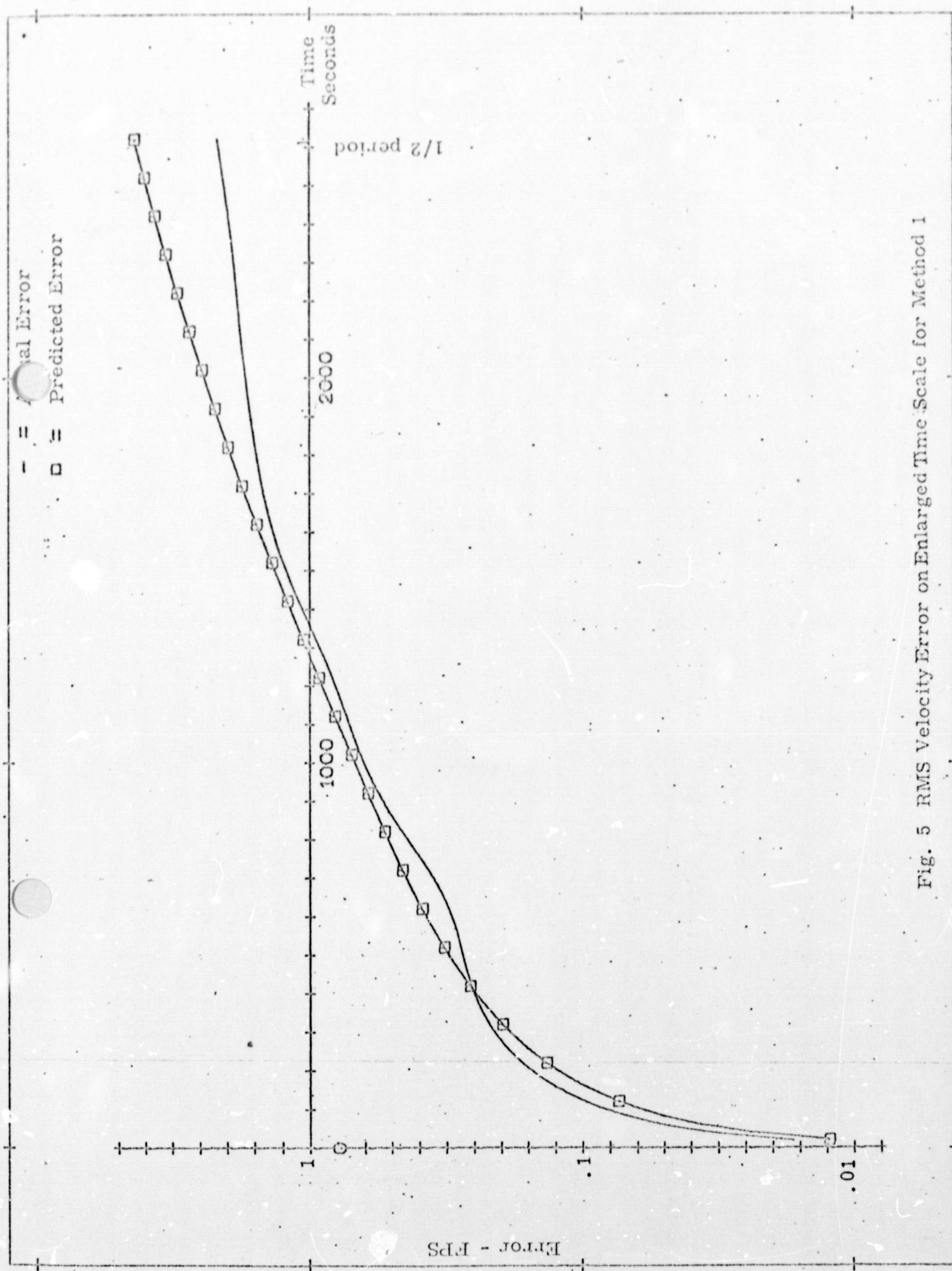


Fig. 5 RMS Velocity Error on Enlarged Time Scale for Method 1

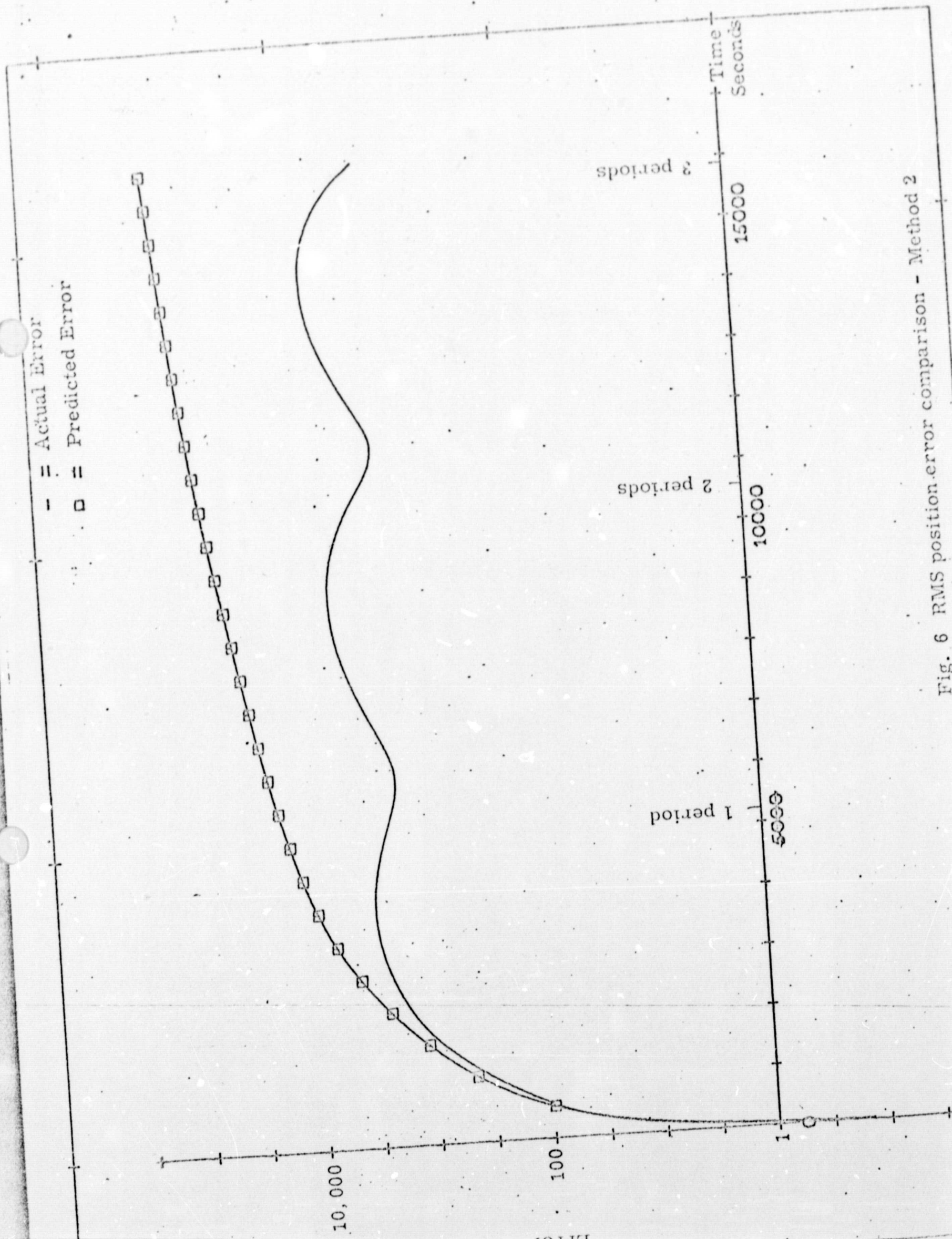


Fig. 6 RMS position error comparison - Method 2

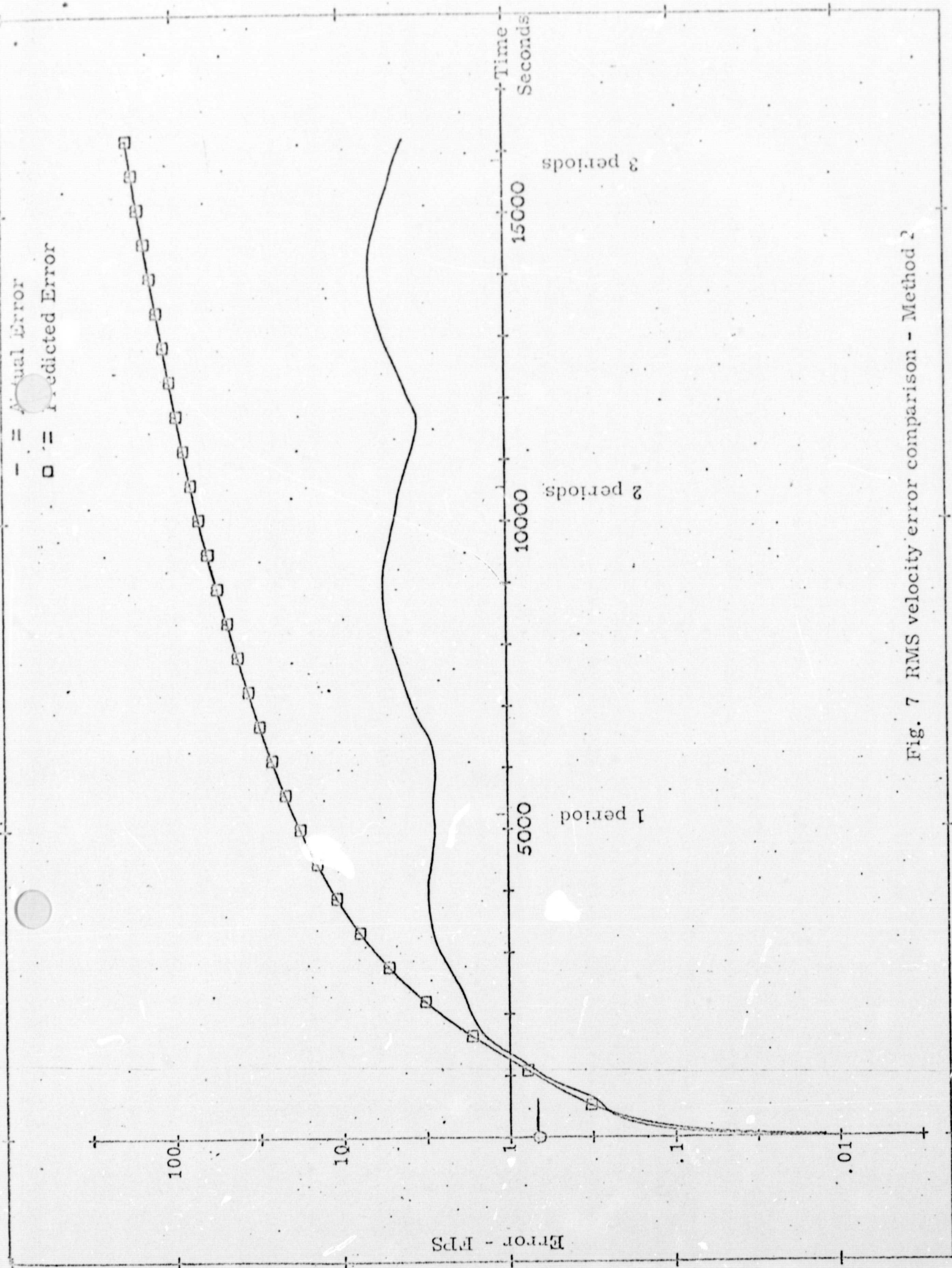


Fig. 7 RMS velocity error comparison - Method 2

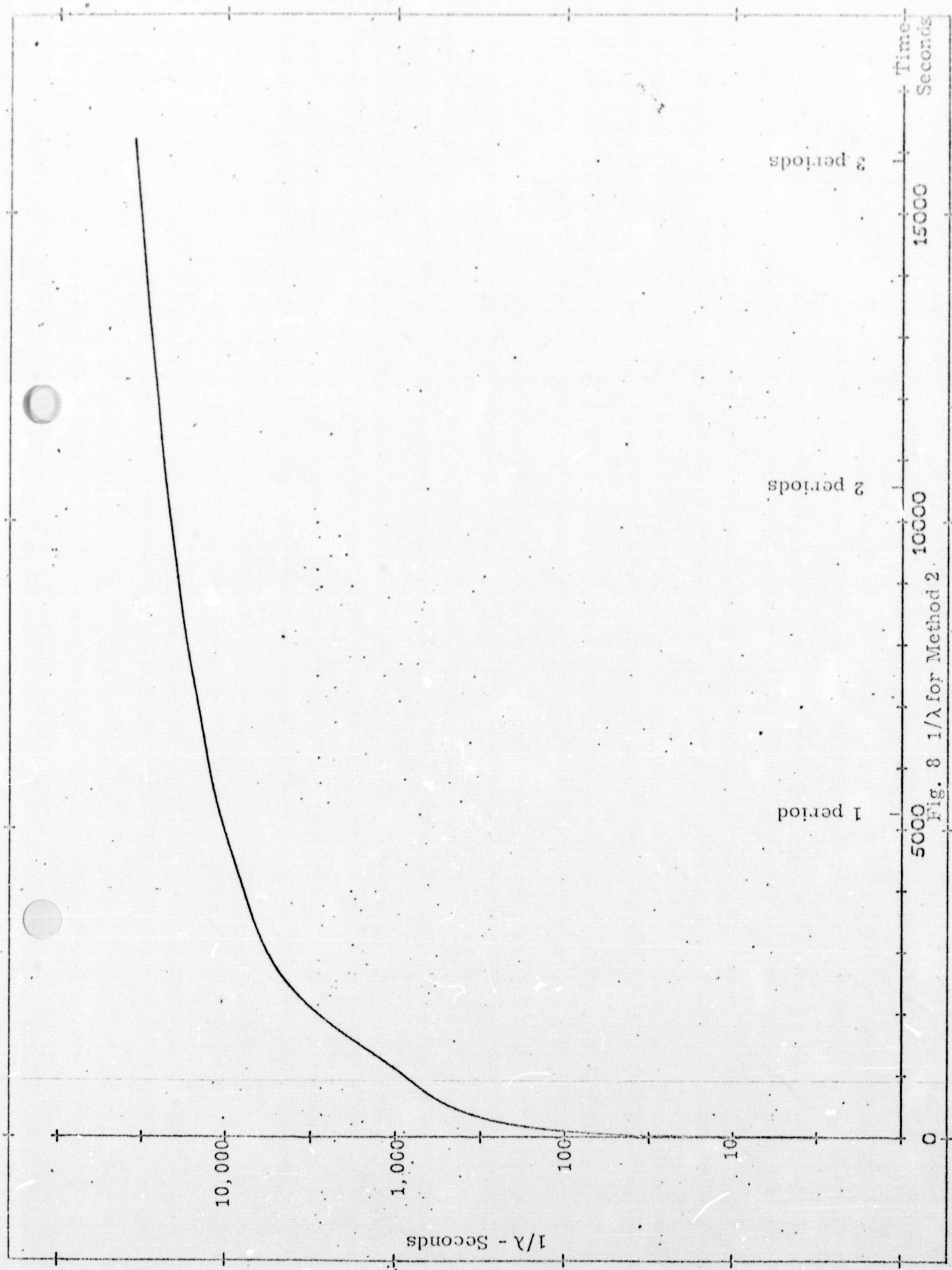


Fig. 8 $1/\lambda$ for Method 2.

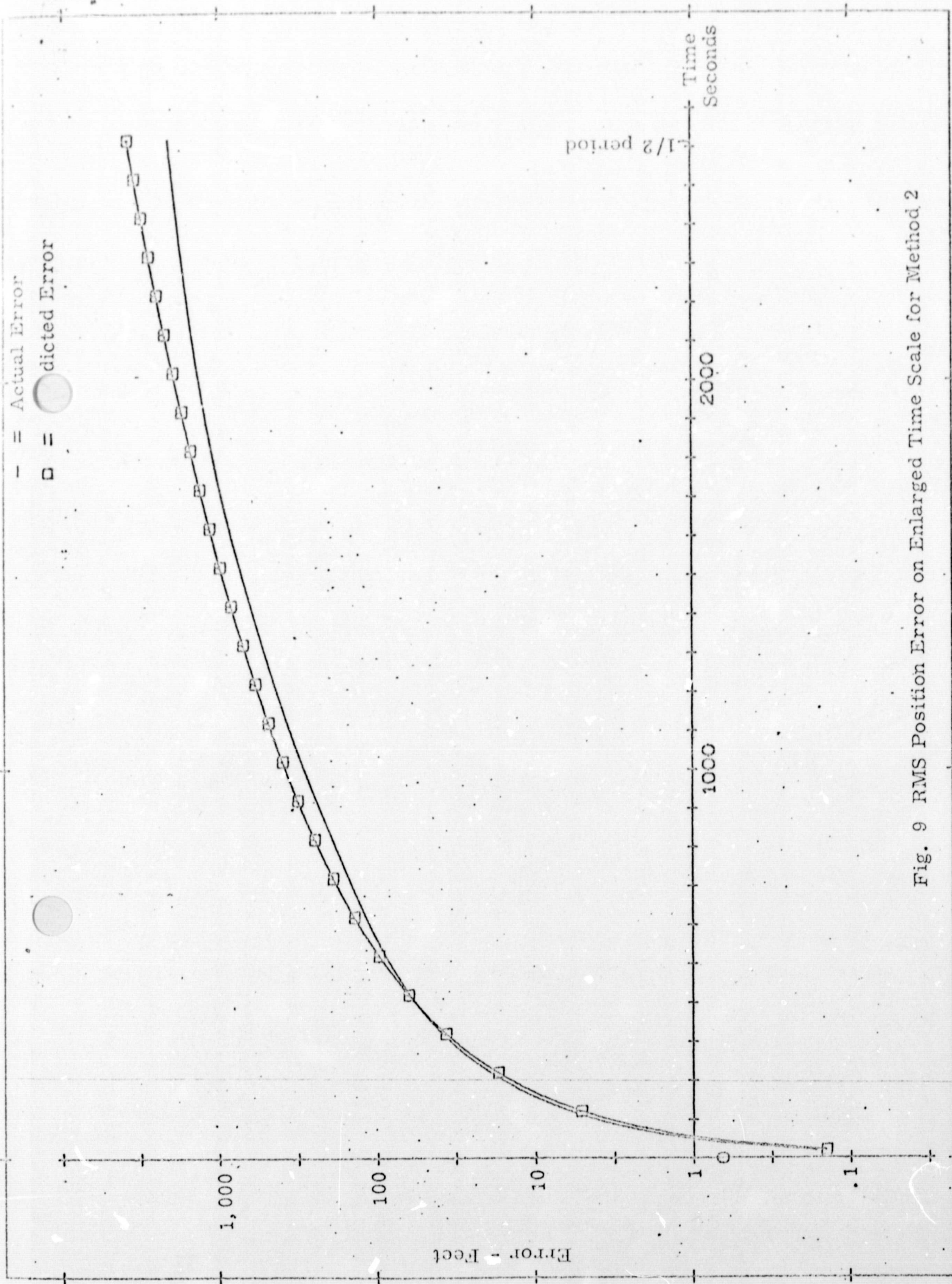


Fig. 9 RMS Position Error on Enlarged Time Scale for Method 2

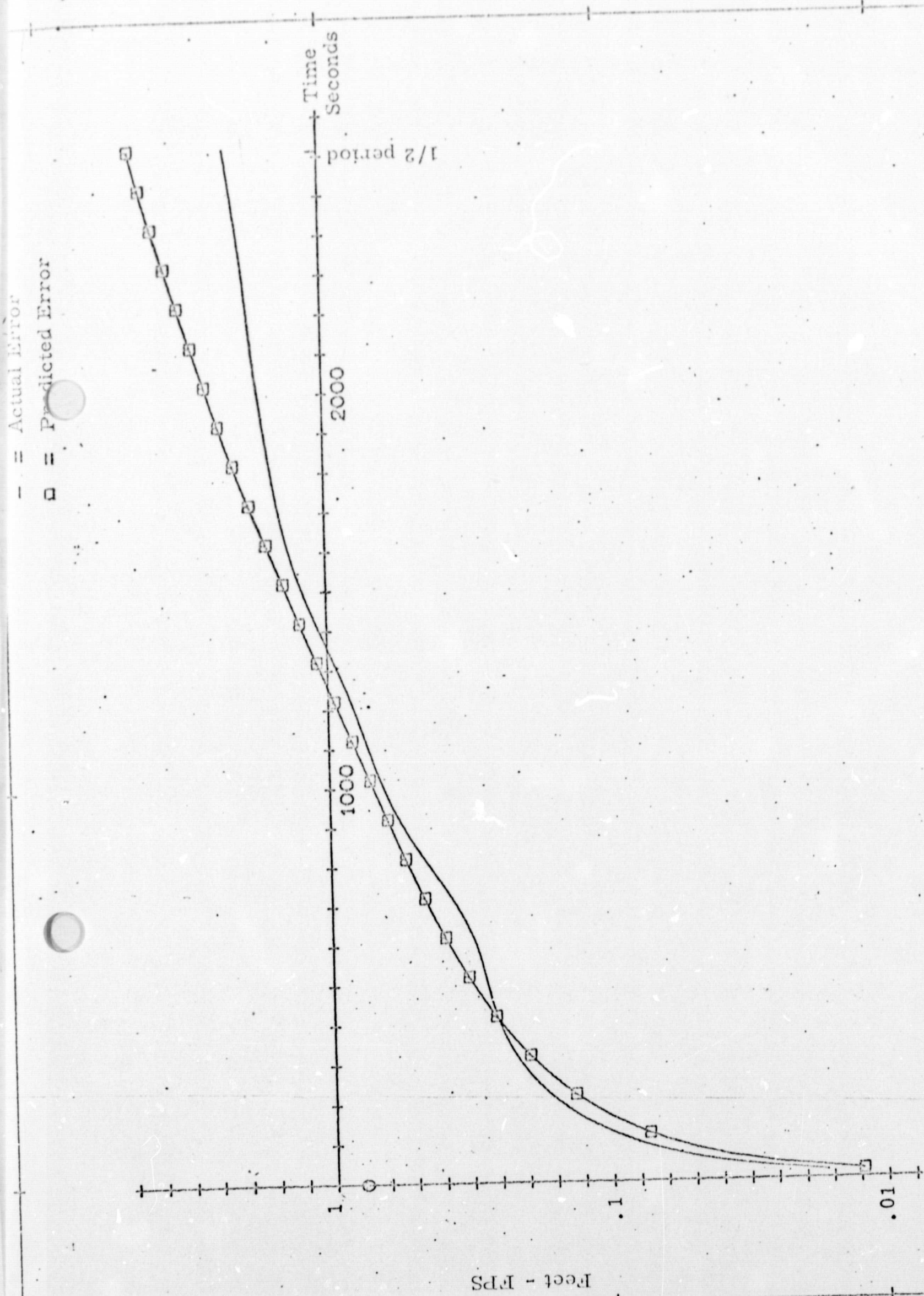


Fig. 10 RMS Velocity Error on Enlarged Time Scale for Method 2

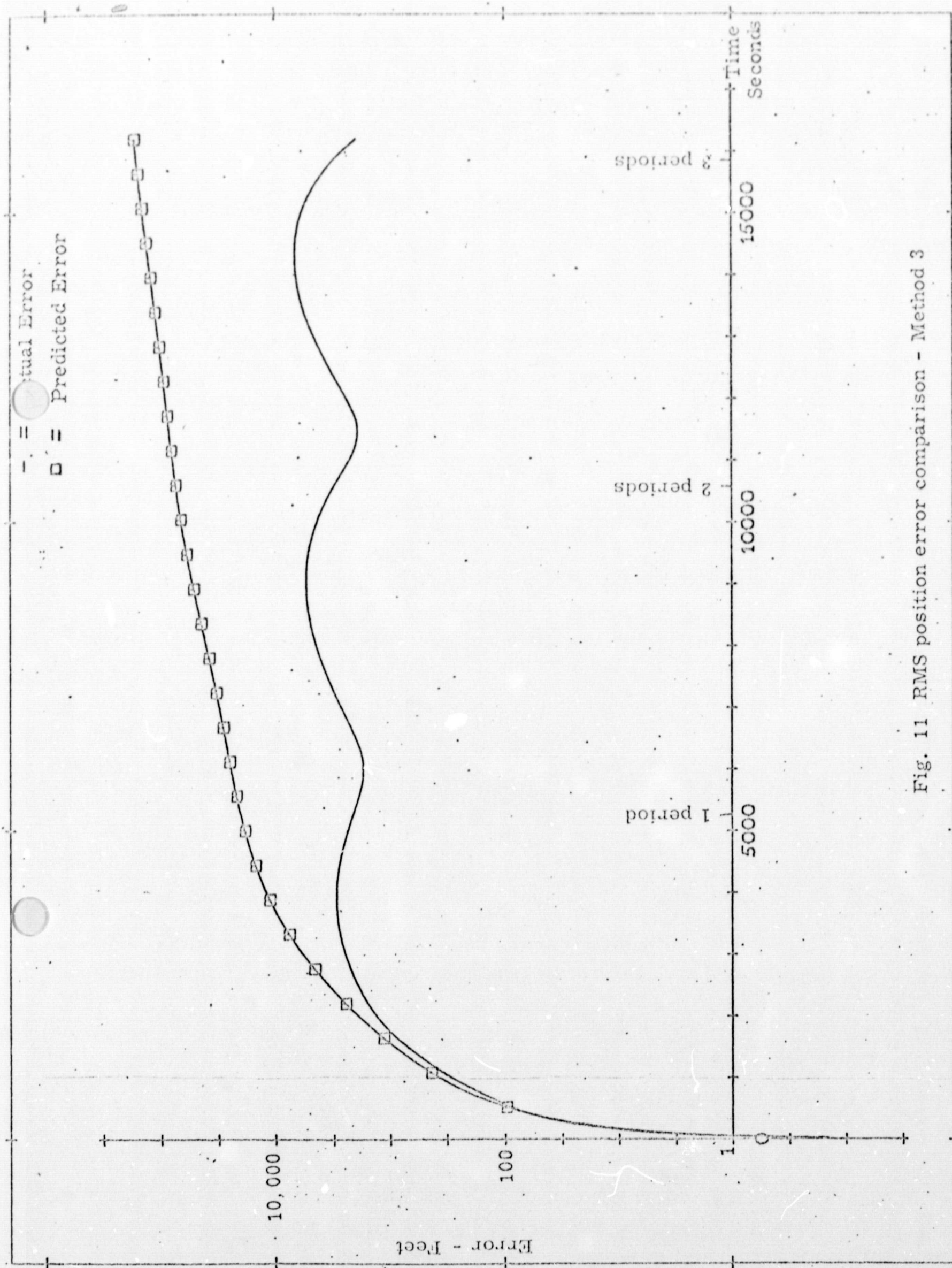


Fig. 11 RMS position error comparison - Method 3

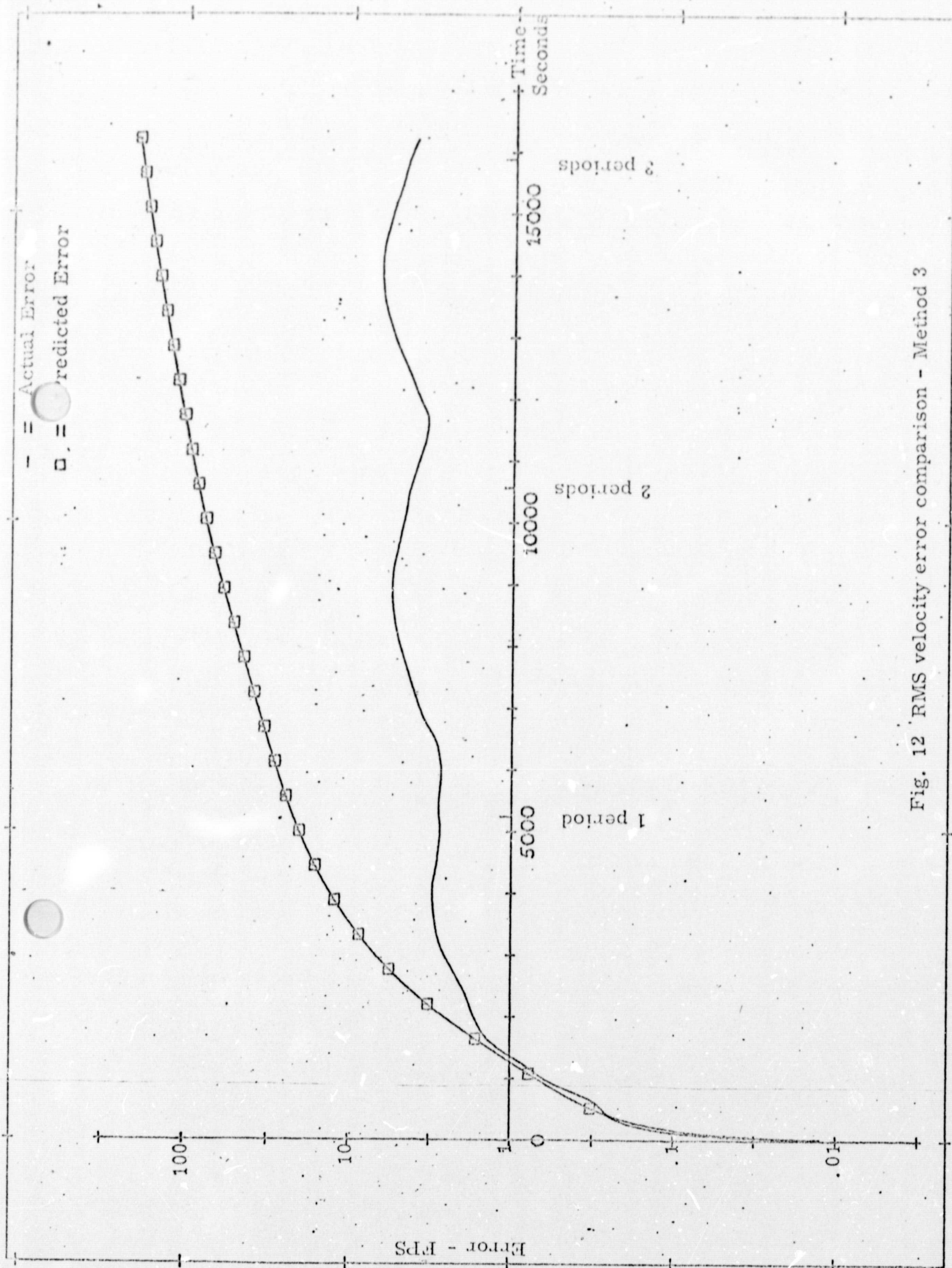


Fig. 12 RMS velocity error comparison - Method 3

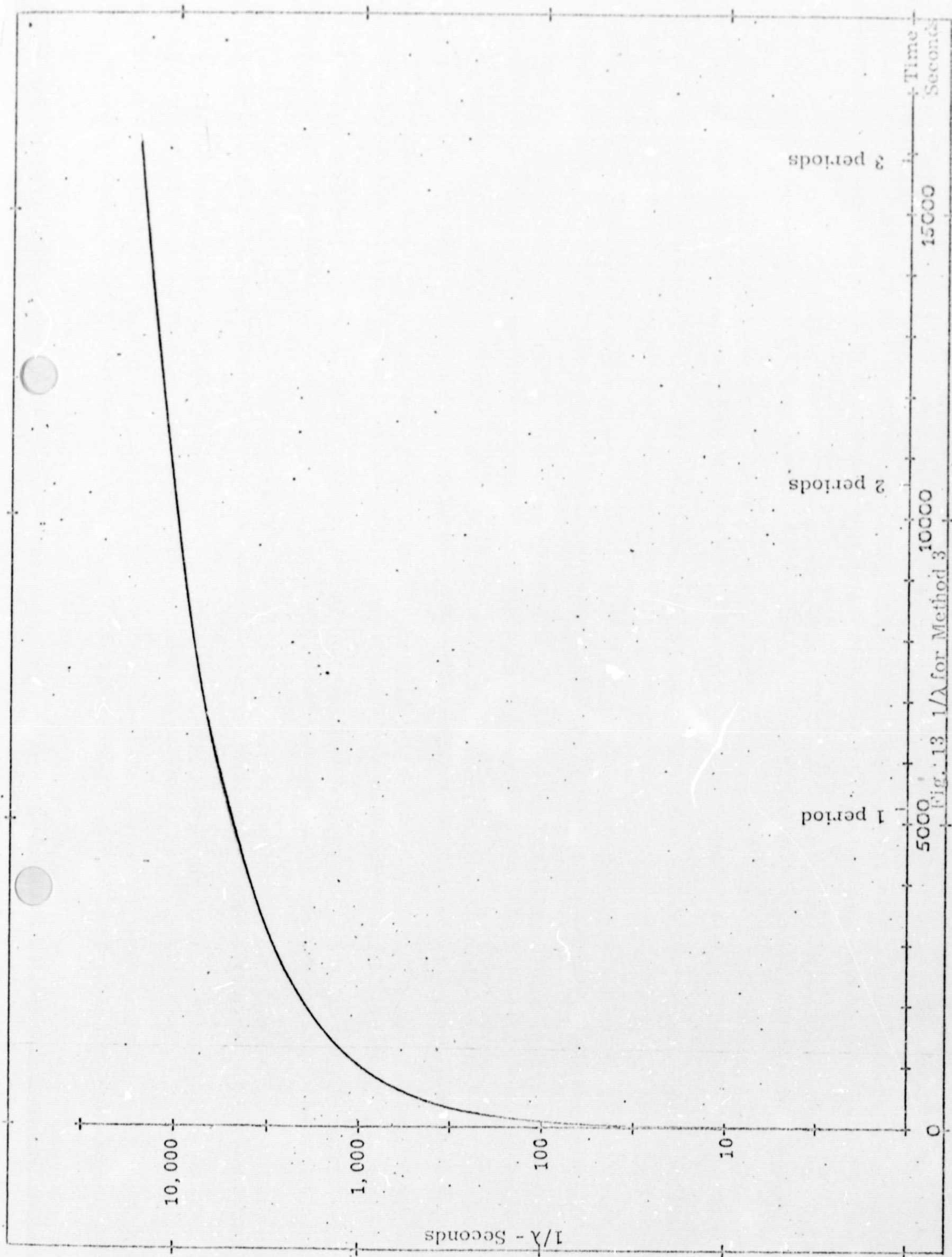


Fig. 13 $1/\lambda$ for Method 3

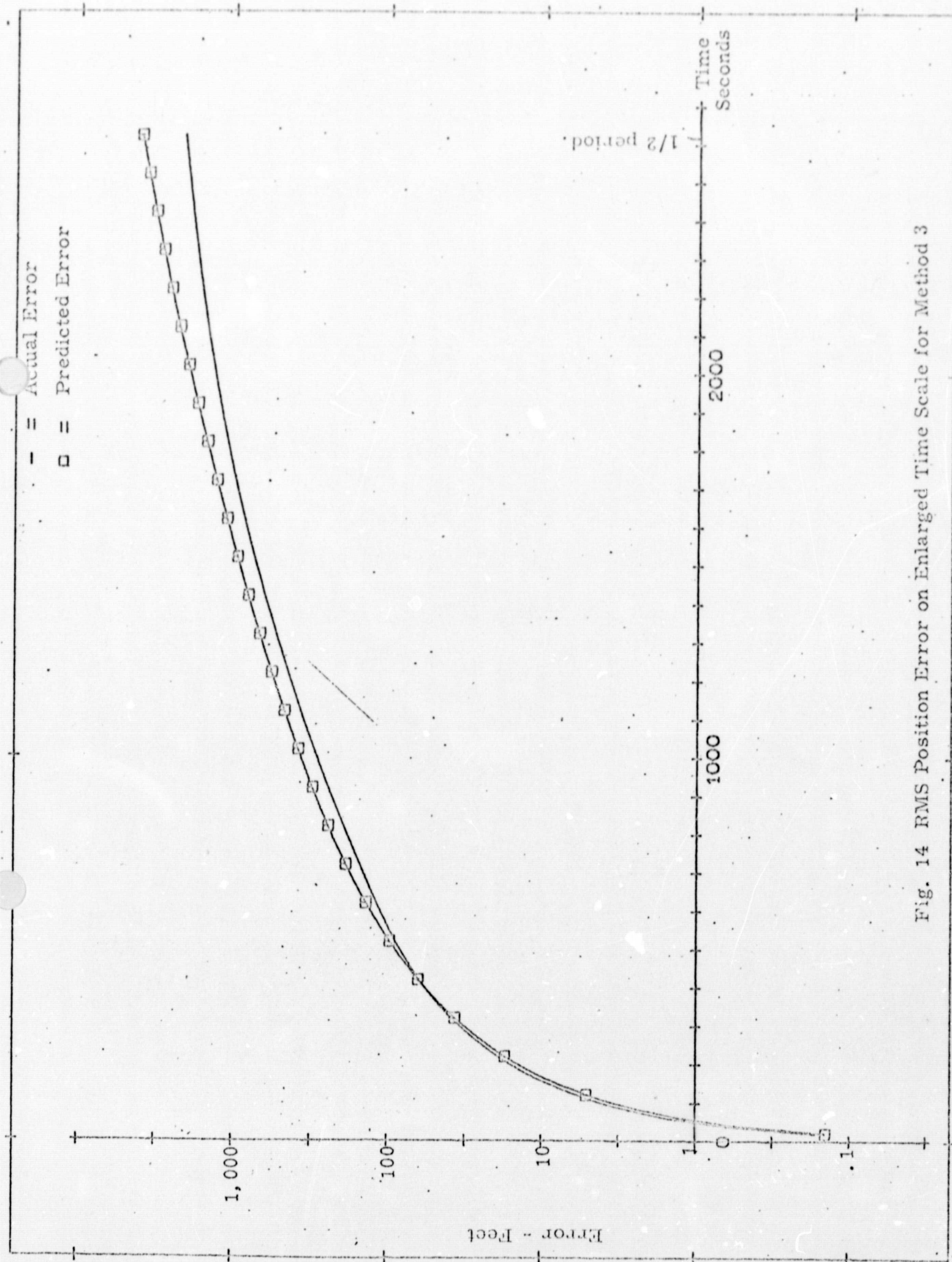


Fig. 14 RMS Position Error on Enlarged Time Scale for Method 3

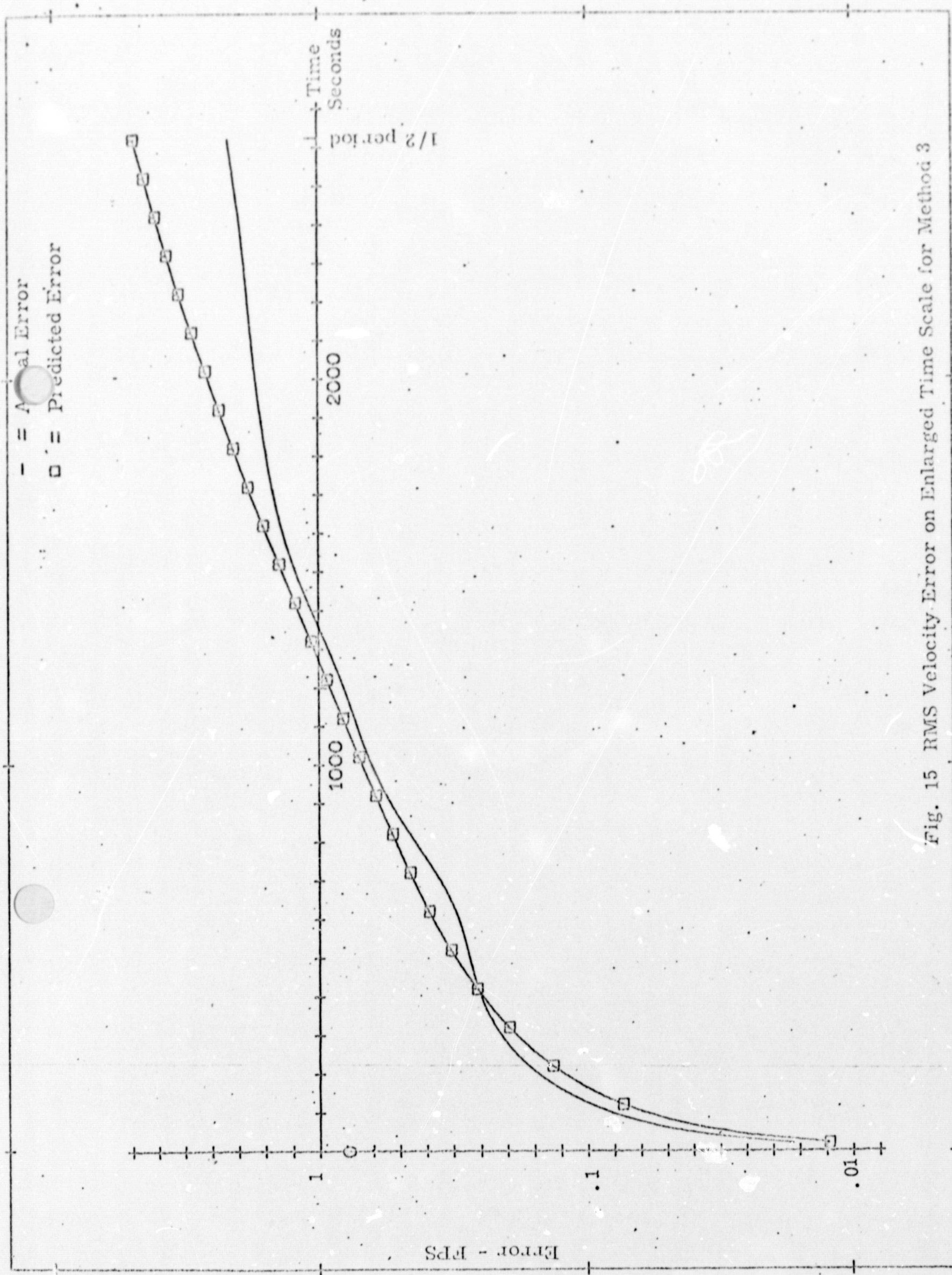


Fig. 15 RMS Velocity Error on Enlarged Time Scale for Method 3

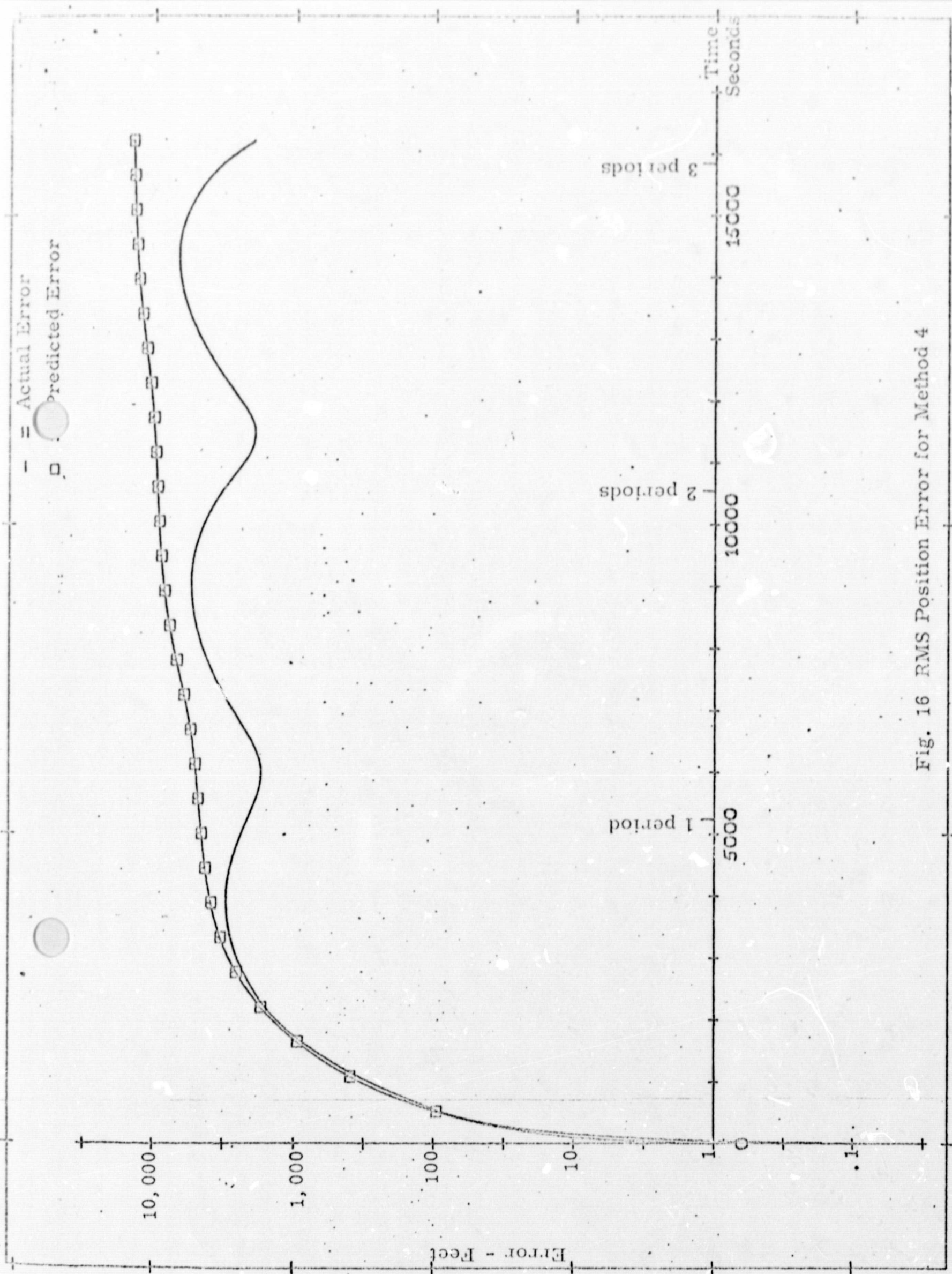


Fig. 16 RMS Position Error for Method 4

III. Orbit Prediction in the Presence of Gravitational Anomalies

This section consists of a description of a parallel effort which represents the doctoral thesis work of Mr. S. R. Croopnick. This section consists entirely of his thesis proposal.

Thesis ProposalORBIT PREDICTION IN THE PRESENCE
OF GRAVITATIONAL ANOMALIES

by

Steven Roy Croopnick

April 22, 1969

ABSTRACT

The classical means of describing a body's gravitational field is with an infinite series expansion in spherical harmonics, which lends itself well to most orbit determination and prediction problems. The difficulty with this description, however, is that it isn't efficient in modeling local gravitational anomalies which have an increasing effect on the path of a spacecraft as the distance from the spacecraft to the body decreases. This thesis will investigate the fundamental modeling problems which give rise to these errors and consider new gravitational models to reduce these uncertainties. Promising cases will be simulated on a computer and compared to the present methods of orbital prediction.

Doctoral Thesis Committee

Dr. Richard H. Battin

Prof. Walter McKay

Prof. Wallace E. VanderVelde(Chairman)

PRIOR WORK - The basic equations of motion of a spacecraft in orbit about a celestial body are (considering only gravitational accelerations):

$$\text{and, } \begin{aligned} \dot{\underline{r}} &= \underline{v} \\ \dot{\underline{v}} &= \underline{g} \end{aligned}$$

where $\underline{r}$ = position of spacecraft with respect to body

$\underline{v}$ = velocity of spacecraft with respect to body

and $\underline{g}$ = total gravitational acceleration acting on body

The gravitation acceleration $\underline{g}$ may be expressed as a sum of terms involving $\underline{g}_0$, G , $\underline{a}_{do}$ and G_a as shown in Fig. 1, where $\underline{g}_0$ is the gravitation acceleration along a nominal path as calculated from the best available model, G is its associated gravity gradient matrix, $\underline{a}_{do}$ is the best estimate of the disturbing acceleration as obtained from measurement residuals and G_a is the anomalous gravity gradient. Classically, the method of estimating $\underline{g}$ consists of collecting observational data on the orbital motion of a spacecraft about a body, and then filtering the data to obtain the coefficients of an infinite series. This infinite series is an expansion of the gravitational potential of the body in spherical harmonics. The gradient of this potential, when evaluated along a nominal path will yield $\underline{g}_0$ and its associated gravity gradient matrix G_0 . When the actual measurements are compared to the expected measurements, their difference, the measurement residual, may then be obtained. This residual is due to the difference between the true state of the spacecraft and the predicted state, which in turn, is due to the disturbing acceleration terms in the equations of motion.

The disturbing acceleration is due to the difference between the estimated mass distribution and the actual mass distribution of the body. Let $\delta\rho(r', \phi', \lambda')$ represent this difference where (r', ϕ', λ') are the spherical coordinates of a point inside the body. There exist four possible ways of assigning $\delta\rho(r', \phi', \lambda')$.

1. Distributed mass (3-D)
2. Surface(or subsurface) mass layer (2-D)
3. Line mass distribution (1-D)
4. Point masses

Historically, $\delta p(r', \phi', \lambda')$ has been lumped into the general expression for the gravitational potential by computing more terms in the series expansion. The residuals aren't significantly reduced by the introduction of higher order harmonics as depicted in Fig. 2. The difficulty is that the higher frequency harmonics needed to account for the residuals take large numbers of coefficients to produce. The C_{44} term, for example, may be considered as due to positive and negative surface mass concentrations as depicted in Fig. 3. It is apparent that this representation cannot produce the higher frequency terms necessary to model the disturbing accelerations although it is excellent for large r due to the $(\frac{1}{r})^m$ dependence. Further, it stands to reason that an appropriate assignment of the C 's and S 's to surface mass layers yields no new information if the measurement residuals aren't used.

An attempt at reconciling the disturbing accelerations with a grid of mass points on the surface of the moon was the subject of reference 2. In this study, the disturbing accelerations (resulting in measurement residuals) were attributed to mass points on the surface of the moon directly below the spacecraft's track (see Fig. 4). The results produced were interesting and indicated large mass concentrations (mascons) below the ringed maria, but are of limited usefulness for orbit prediction because:

1. data arcs are treated independently
2. the statistics weren't considered
3. the measurement geometry wasn't included
4. the masses along the track were the only ones considered to disturb the motion
5. correlation between masses on neighboring tracks wasn't allowed
6. data can't be inverted

MOTIVATION - The need to use a more sophisticated model for the gravitational potential became evident with the start of the recent lunar orbiter flights. An examination of the measurement residuals¹ shows that there typically are large disturbing accelerations acting on the lunar orbiter near pericenter. These measurement residuals result

from a difference in what the measurement was expected or predicted to be, based on the current model of the moon's gravitational field, and the actual measurement. It is reasonable to assume, that since a spacecraft makes its closest approach to a planet at pericenter, and the gravitational force obeys an inverse square law, that the disturbing accelerations would be greatest when the spacecraft was closest to the planet, especially, if they were caused by local mass anomalies in the terrain.

Accounting for these disturbing accelerations, and including them in the model would allow one to do a better job of navigation in many circumstances. For example, if a mission were planned about the equator of a body, and test flights were flown in the neighborhood of the expected mission path, it may be possible to collect data in this region, thereby improving the knowledge of the local gravitational field. Statistical arguments may then be employed to further assess the confidence in one's estimate of the gravitational anomalies obtained through modeling.

RESEARCH OBJECTIVE - The major objective of this thesis is to improve orbit determination and prediction by utilizing the information contained in measurement residuals. This, hopefully, will come about from the following two fold effort.

- I. Investigate the origin of measurement residuals
- II. Adjust the model of a gravitation potential to account for the residuals

INITIAL ATTACK - The first problem will be approached by examining the errors committed in determining the gravitational potential of a body whose potential is unknown by solving for the coefficients in a series expansion in spherical harmonics. By assuming a potential for the body, it will be possible to generate the "true" trajectory and compare it to the predicted trajectory based on the estimated gravitational potential. It will be of special interest to examine the source of errors incurred during low altitude regions and associate them with the higher harmonics. At the same time, it would be useful to study the statistics of the estimated

coefficients and how these coefficients are affected by local gravitational anomalies.

The second problem will involve adjusted models of the body's analytical mass distribution. The intent is to infer certain components of the disturbing acceleration from the measurement residuals, and then attribute them to errors in the current estimate of the mass distribution of the body. A promising model is depicted in Fig. 5. The error in the mass distribution is attributed to the suborbital track through the use of a line density of mass $\delta\rho(\lambda)$ and a line density of "gravitational dipoles" $\underline{P}(\lambda)$. The line density of mass will yield the in plane disturbing accelerations while the gravitational dipoles will yield the out of plane components. Implicit in this investigation are the questions of uniqueness and of sufficiency in assigning these line densities. The following concepts pertinent to this model will be studied:

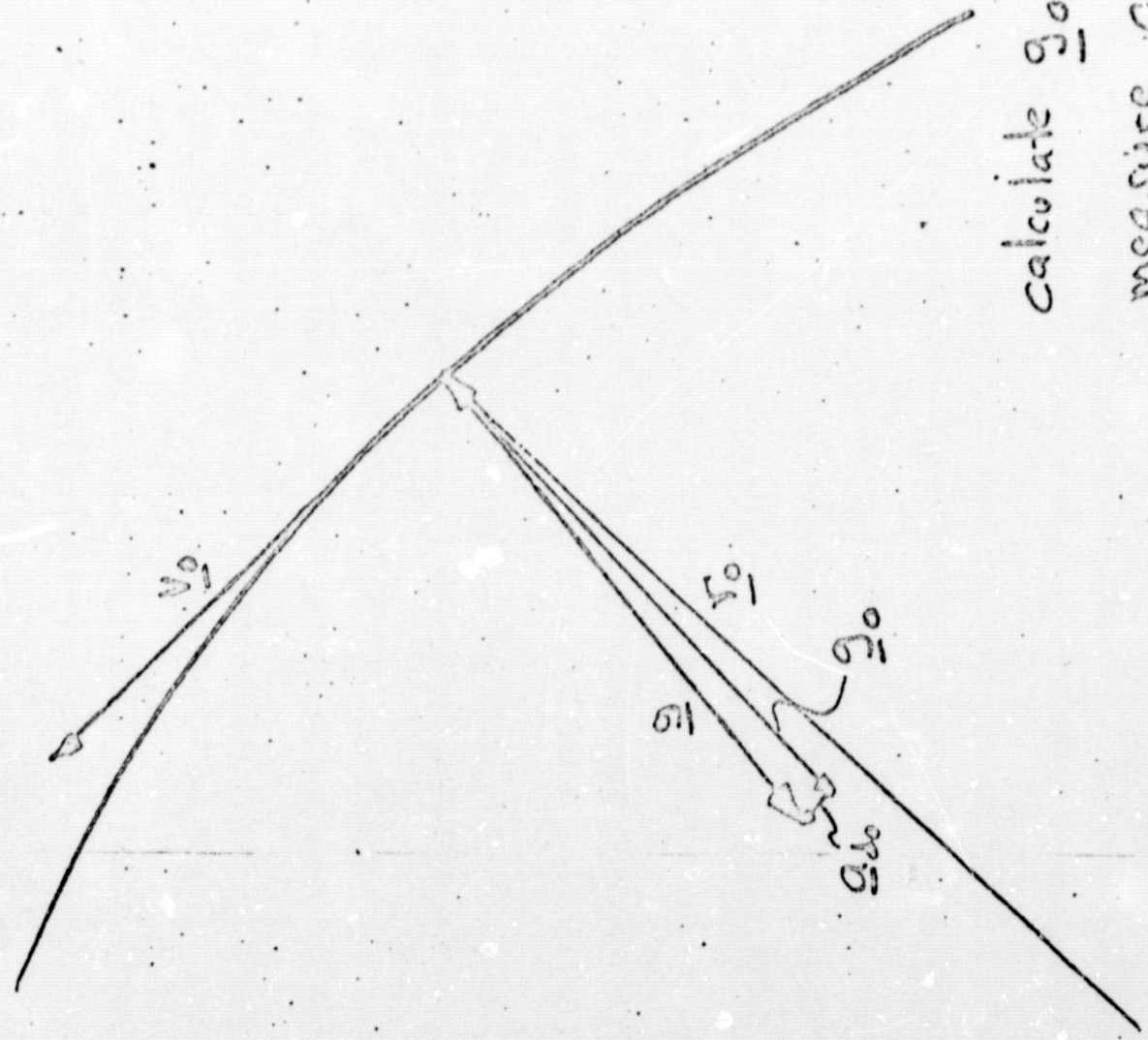
- 1) uniqueness and sufficiency
- 2) correlations between mass distributions along various orbit tracks
- 3) the gravitational error covariance matrix
- 4) long term orbit prediction
- 5) linearization techniques for paths flown close to measurement points including the region of application

Finally, depending on the outcome of the initial investigations on this or on other models, test cases will be simulated on the computer for a relatively well known gravitational field. In the case of the earth, a conceivable test would be to improve on an early estimate of the gravitational potential through appropriate modeling and then comparing these results to the actual path based on the current potential model for various orbital geometries.

Egn. of Motion

$$\dot{\mathbf{r}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = \mathbf{g} = \mathbf{g}_0 + \mathbf{g}_{d0}$$



calculate $\mathbf{g}_0, \mathbf{g}$

measure $\mathbf{g}_{d0}$

model $\mathbf{G}_a$

Fig. 1

$$\dot{\mathbf{v}} = \mathbf{g} = \mathbf{g}_0 + \mathbf{g}_{d0}$$

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \mathbf{g}_0 + \mathbf{g}_{d0}$$

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \mathbf{g}_0 + \mathbf{g}_{d0}$$

Distributed Mass

$$\phi = \frac{1}{r} \left[1 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{r}{R} \right)^m P_n^m(\sin \phi) (C_{nm} \cos n\lambda + S_{nm} \sin n\lambda) \right]$$

4,4 gravity field



Doppler Residual, cps

7,7



11,11

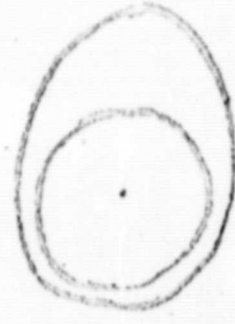


minutes

LO-III

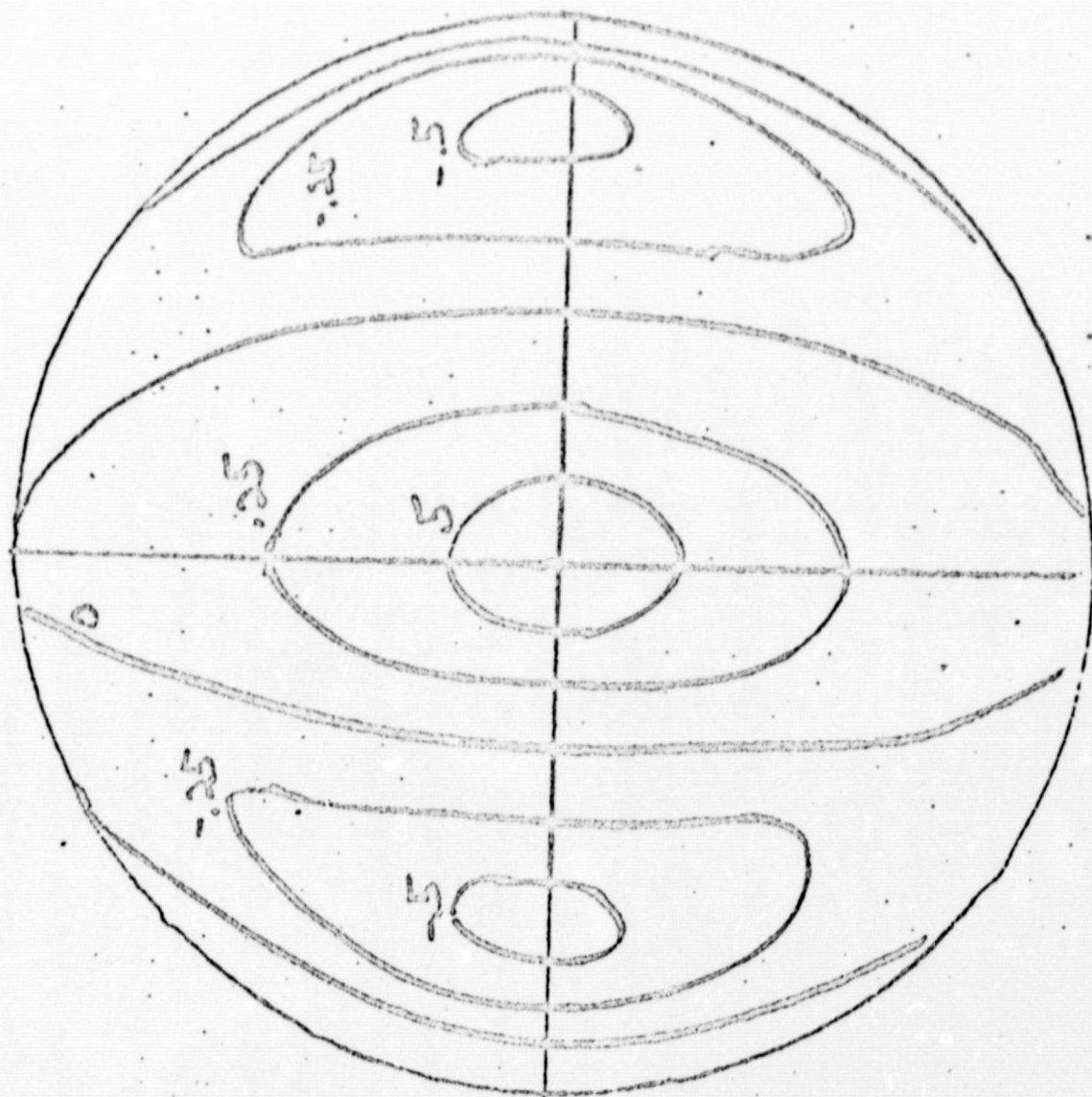
Compton, NASA, Langley

Fig. 2



Surface Layer

$$C_{44} = 10^{-6}$$



elevation in km

Muller, Siogren

JPL

Fig. 3

Point Masses (Mascons):

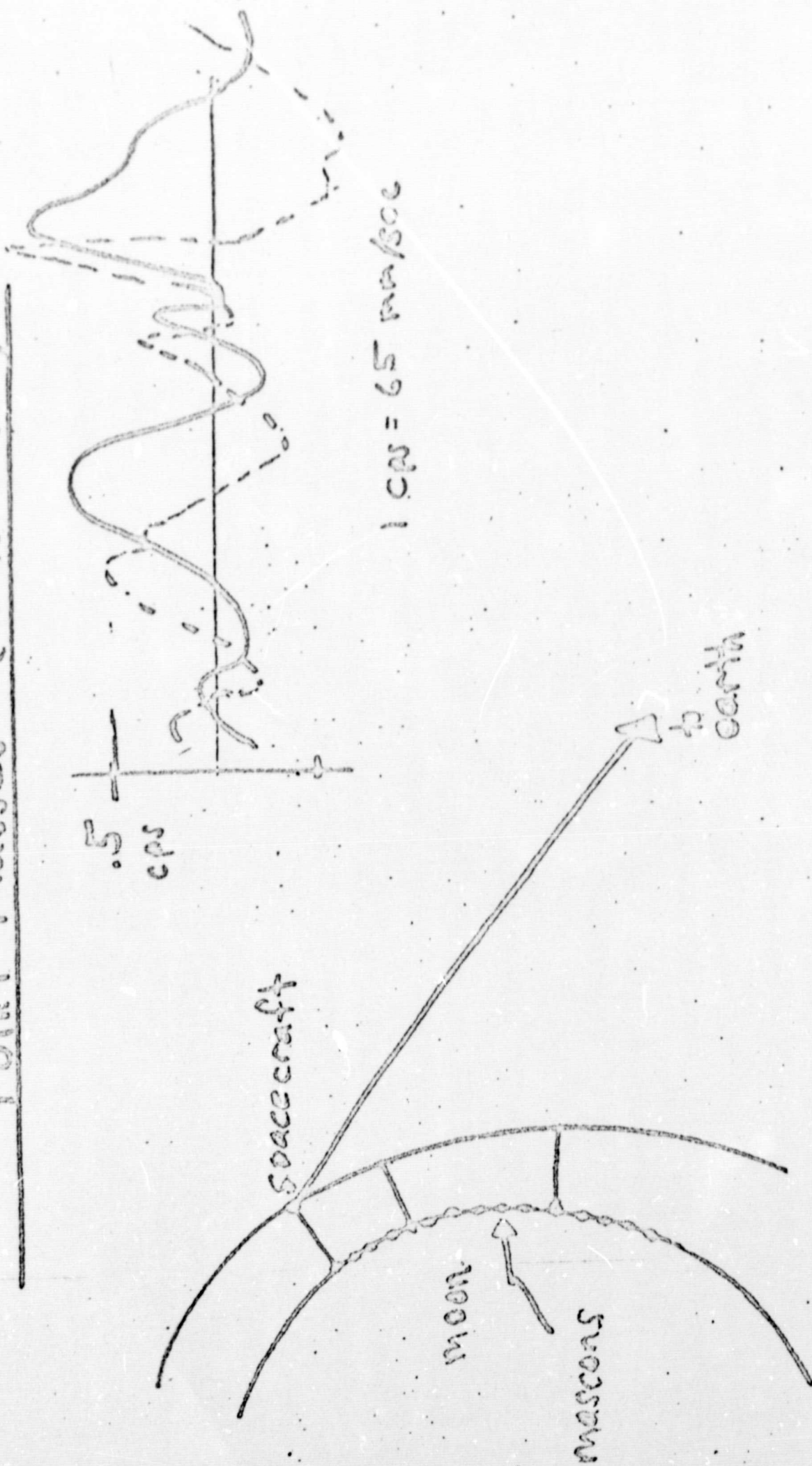


Fig. 4

P.M. Muller JPL
W.L. Sjogren

Line Mass Density (Gravitational Dipole)

$$\rho(\lambda) \left[\frac{\text{mass}}{\text{length}} \right] \sim \frac{1}{\lambda^2}$$

$$\text{gravitational dipole density} \Rightarrow \bar{\rho}(\lambda) \left[\text{mass} \right] \sim \frac{1}{\lambda^3}$$

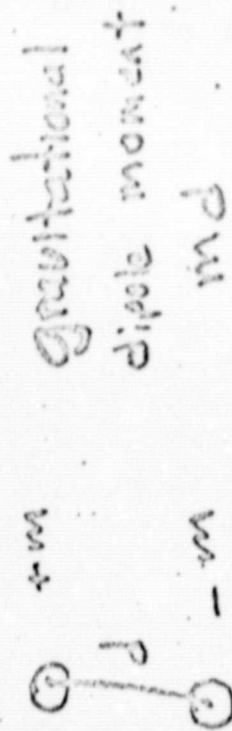
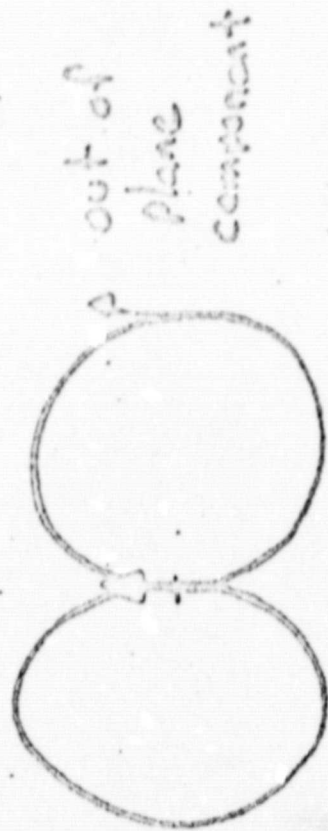
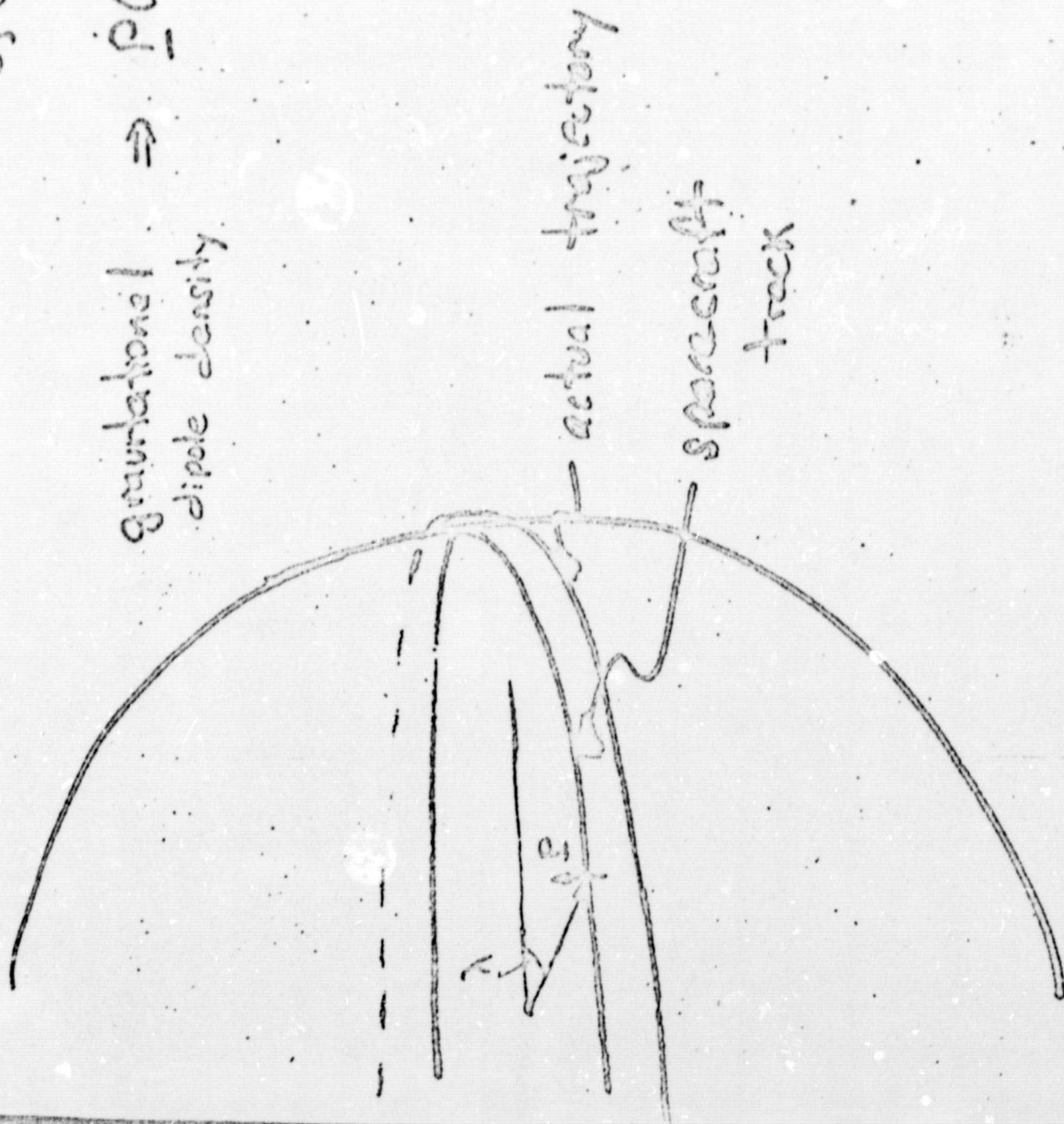


Fig. 5

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