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ON THE STRUCTURE OF MULTIVARIABLE SYSTEMS

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1. Introduction

The primary purpose of this paper is to state and prove a structure theorem for time invariant multivariable linear systems. The theorem can be used for controller design and synthesis and is applied here to the problems of realization ([1]) and decoupling ([2], [3]).

We consider systems of the form

$$(1) \qquad \dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underline{u} \quad , \quad \underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x}$$

where $\underset{\sim}{x}$ is an n-vector, called the state, $\underset{\sim}{u}$ is an m-vector, called the input, $\underset{\sim}{y}$ is a p-vector, called the output, and $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are constant matrices of the appropriate dimension. We assume that the matrices $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are of full rank. Now, it is well-known ([4], [5]) that if the pair $\{\underset{\sim}{A},\underset{\sim}{B}\}$ is controllable, then there is a Lyapunov transformation $\underset{\sim}{Q}$ such that the system

$$(2) \qquad \dot{\underset{\sim}{z}} = \underset{\sim}{Q}\,\underset{\sim}{A}\,\underset{\sim}{Q}^{-1}\underset{\sim}{z} + \underset{\sim}{Q}\,\underset{\sim}{B}\,\underset{\sim}{u} \quad , \quad \underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{Q}^{-1}\underset{\sim}{z}$$

is in "companion" form. The systems (1) and (2) are equivalent and have the same transfer matrix $\underset{\sim}{T}(s)$. In section 2, we shall show that if state variable feedback of the form $\underset{\sim}{u} = \underset{\sim}{F}\,\underset{\sim}{x} + \underset{\sim}{w}$ (or $\underset{\sim}{u} = \underset{\sim}{F}\,\underset{\sim}{Q}^{-1}\underset{\sim}{z}+\underset{\sim}{w}$) is applied to (1) (or (2)), then the resulting transfer matrix $\underset{\sim}{T}_{\underset{\sim}{F}}(s)$ is of the form $\hat{\underset{\sim}{C}}\,\underset{\sim}{S}(s)\underset{\sim}{\delta}_{\underset{\sim}{F}}^{-1}(s)\hat{\underset{\sim}{B}}_m$ where $\hat{\underset{\sim}{C}},\hat{\underset{\sim}{B}}_m$ are constant matrices, $\underset{\sim}{S}(s)$ is a matrix of single term monic polynomials in s, and $\underset{\sim}{\delta}_{\underset{\sim}{F}}(s)$ is a matrix of polynomials in s whose coefficients depend on $\underset{\sim}{A}+\underset{\sim}{B}\,\underset{\sim}{F}$.

This result is generalized to systems which are not completely controllable in section 3 and applied to the problems of realization (section 4) and decoupling (section 5).

2. A Structure Theorem for Controllable Systems

Suppose that the system (1) is completely controllable. Let $\underset{\sim}{K} = [\underset{\sim}{B}, \underset{\sim}{A}\,\underset{\sim}{B}, \ldots, \underset{\sim}{A}^{n-1}\underset{\sim}{B}]$. Then the $n \times nm$ matrix $\underset{\sim}{K}$ has rank n and it is possible to define a "lexicographic" basis for R_n consisting of the first n linearly independent columns of $\underset{\sim}{K}$ possibly reordered (cf. [5]). We let $\underset{\sim}{L}$ be the matrix whose columns are the elements of the "lexicographic" basis so that

$$(3) \qquad \underset{\sim}{L} = [\underset{\sim}{b}_1, \underset{\sim}{A}\,\underset{\sim}{b}_1, \ldots, \underset{\sim}{A}^{\sigma_1-1}\underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{A}^{\sigma_2-1}\underset{\sim}{b}_2, \ldots, \underset{\sim}{A}^{\sigma_m-1}\underset{\sim}{b}_m]$$

where $\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_m$ are the columns of $\underset{\sim}{B}$. Setting

$$(4) \qquad d_0 = 0, \quad d_k = \sum_{i=1}^{k} \sigma_i \qquad k = 1,2,\ldots,m$$

and letting $\underset{\sim}{\ell}'_k$ be the d_k-th row of $\underset{\sim}{L}^{-1}$, we can see that the matrix $\underset{\sim}{Q}$ given by

$$(5) \qquad \underset{\sim}{Q} = \begin{bmatrix} \underset{\sim}{\ell}'_1 \\ \underset{\sim}{\ell}'_1\underset{\sim}{A} \\ \vdots \\ \underset{\sim}{\ell}'_1\underset{\sim}{A}^{\sigma_1-1} \\ \vdots \\ \underset{\sim}{\ell}'_m\underset{\sim}{A}^{\sigma_m-1} \end{bmatrix}$$

generates a Lyapunov transformation for which (2) is in "companion" form ([4], [5]). More precisely, if we let $\hat{\underset{\sim}{A}} = \underset{\sim}{Q}\,\underset{\sim}{A}\,\underset{\sim}{Q}^{-1}$, $\hat{\underset{\sim}{B}} = \underset{\sim}{Q}\,\underset{\sim}{B}$, and $\hat{\underset{\sim}{C}} = \underset{\sim}{C}\,\underset{\sim}{Q}^{-1}$, then (2) becomes

$$\dot{\underset{\sim}{z}} = \hat{\underset{\sim}{A}}\,\underset{\sim}{z} + \hat{\underset{\sim}{B}}\,\underset{\sim}{u}\,, \quad \underset{\sim}{y} = \hat{\underset{\sim}{C}}\,\underset{\sim}{z} \tag{6}$$

where $\hat{\underset{\sim}{A}} = (\hat{a}_{ij})$ is a block-matrix of the form

$$\hat{\underset{\sim}{A}} = \begin{bmatrix} \hat{\underset{\sim}{A}}_{11} & \cdots & \hat{\underset{\sim}{A}}_{1m} \\ \hat{\underset{\sim}{A}}_{21} & \cdots & \hat{\underset{\sim}{A}}_{2m} \\ \vdots & & \\ \hat{\underset{\sim}{A}}_{m1} & \cdots & \hat{\underset{\sim}{A}}_{mm} \end{bmatrix} \tag{7}$$

with $\hat{\underset{\sim}{A}}_{ii}$ a $\sigma_i \times \sigma_i$ companion matrix given by

$$\hat{\underset{\sim}{A}}_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ \hat{a}_{d_i,d_{i-1}+1} & \hat{a}_{d_i,d_{i-1}+2} & & \hat{a}_{d_i,d_i-1} & \hat{a}_{d_i,d_i} \end{bmatrix} \tag{8}$$

and $\hat{\underset{\sim}{A}}_{ij}$ a $\sigma_i \times \sigma_j$ matrix given by

$$\hat{\underset{\sim}{A}}_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ \hat{a}_{d_i,d_{j-1}+1} & & \hat{a}_{d_i,d_j} \end{bmatrix} \tag{9}$$

for $i \neq j$ and with $\hat{\mathbf{B}} = (\hat{b}_{ij})$ an $n \times m$ matrix given by

$$(10) \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdot & \vdots \\ 1 & \hat{b}_{d_1,2} & \hat{b}_{d_1,3} & & \hat{b}_{d_1,m} \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \hat{b}_{d_2,3} & & \hat{b}_{d_2,m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

We now have

PROPOSITION 2.1 *Let* $\mathbf{u} = \mathbf{F}\,\mathbf{x} + \mathbf{w} = \hat{\mathbf{F}}\,\mathbf{z} + \mathbf{w}$ *where* $\hat{\mathbf{F}} = \mathbf{F}\,\mathbf{Q}^{-1}$. *Then the transfer matrices of the systems* $\dot{\mathbf{x}} = (\mathbf{A}+\mathbf{B}\,\mathbf{F})\mathbf{x}+\mathbf{B}\,\mathbf{w}$, $\mathbf{y} = \mathbf{C}\,\mathbf{x}$ *and* $\dot{\mathbf{z}} = (\hat{\mathbf{A}}+\hat{\mathbf{B}}\,\hat{\mathbf{F}})\,\mathbf{z}+\hat{\mathbf{B}}\,\mathbf{w}$, $\mathbf{y} = \hat{\mathbf{C}}\,\mathbf{z}$ *are the same.*

Proof: Simply note that $\mathbf{C}(s\mathbf{I}-\mathbf{A}-\mathbf{B}\,\mathbf{F})^{-1}\mathbf{B} = \mathbf{C}\,\mathbf{Q}^{-1}\mathbf{Q}(s\mathbf{I}-\mathbf{A}-\mathbf{B}\,\mathbf{F})^{-1}\mathbf{Q}^{-1}\mathbf{Q}\,\mathbf{B} = \mathbf{C}\,\mathbf{Q}^{-1}[(s\mathbf{I}-\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^{-1}-\mathbf{Q}\,\mathbf{B}\,\mathbf{F}\,\mathbf{Q}^{-1})]^{-1}\mathbf{Q}\,\mathbf{B} = \hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}\hat{\mathbf{B}}$.

Since $\hat{\mathbf{B}}$ as given by (10) has zero rows except for the d_1-th, d_2-th,...,d_m-th rows, we need only calculate the corresponding columns of $(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}$ in order to obtain the transfer matrix $\mathbf{T}_{\mathbf{F}}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A}-\mathbf{B}\,\mathbf{F})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}\hat{\mathbf{B}}$. Moreover, $\hat{\mathbf{B}}\,\hat{\mathbf{F}}$ has zero rows except for the d_1-th, d_2-th,...,d_m-th rows and so $\hat{\mathbf{A}}+\hat{\mathbf{B}}\,\hat{\mathbf{F}}$ is again a block matrix of exactly the same form as $\hat{\mathbf{A}}$. In other words, $\hat{\mathbf{A}}+\hat{\mathbf{B}}\,\hat{\mathbf{F}} = (\Phi_{ij})$ is a block matrix of the form

$$(11)\qquad \hat{A}+\hat{B}\hat{F} = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1m} \\ \Phi_{21} & \cdots & \Phi_{2m} \\ \vdots & & \\ \Phi_{m1} & \cdots & \Phi_{mm} \end{bmatrix}$$

where Φ_{ii} is a $\sigma_i \times \sigma_i$ companion matrix given by

$$(12)\qquad \Phi_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \phi_{d_i,d_{i-1}+1} & \phi_{d_i,d_{i-1}+2} & \cdots & \phi_{d_i,d_i-1} & \phi_{d_i,d_i} \end{bmatrix}$$

and Φ_{ij} is a $\sigma_i \times \sigma_j$ matrix given by

$$(13)\qquad \Phi_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ \phi_{d_i,d_{j-1}+1} & \phi_{d_i,d_{j-1}+2} & \cdots & \phi_{d_i,d_j} \end{bmatrix}$$

for $i \neq j$. These two simple observations are basic to the structure theorem 2.2.

THEOREM 2.2 Suppose that the pair (A,B) is controllable and let $T_F(s) = C(sI-A-BF)^{-1}B$ be the transfer matrix of the system $\dot{x} = (A+BF)x+Bw$, $y = Cx$. Then

$$T_F(s) = \hat{C}\, S(s)\,\delta_F^{-1}(s)\,\hat{B}_m \tag{14}$$

where $\hat{C} = C\,Q^{-1}$, $S(s)$ *is the* $n \times m$ *matrix given by*

$$S(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ s^{\sigma_1-1} & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & s^{\sigma_2-1} & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & s^{\sigma_m-1} \end{bmatrix} \tag{15}$$

$\delta_F(s)$ *is the* $m \times m$ *matrix* $(\delta_{F,ij}(s))$ *with entries given by* $\delta_{F,ii}(s) = \det(sI_{\sigma_i} - \Phi_{ii})$ *and* $\delta_{F,ij}(s) = -\phi_{d_i,d_{j-1}+1} - s\phi_{d_i,d_{j-1}+2} - \cdots - s^{\sigma_i-1}\phi_{d_i,d_j}$ *for* $i \neq j$, *and* $\hat{B}_m$ *is the* $m \times m$ *matrix given by*

$$\hat{B}_m = \begin{bmatrix} 1 & \hat{b}_{d_1 2} & \cdots & \hat{b}_{d_1 m} \\ 0 & 1 & \cdots & \hat{b}_{d_2 m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \tag{16}$$

where $\hat{B} = Q\,B = (\hat{b}_{ij})$.

Proof: In view of proposition 2.1, we need only show that $\hat{C}(sI - \hat{A} - \hat{B}\,\hat{F})^{-1}\hat{B} = \hat{C}\,S(s)\,\delta_F^{-1}(s)\,\hat{B}_m$. To do this, it will be sufficient to show that

$$(sI-\hat{A}-\hat{B}\hat{F})^{-1}\hat{B} = S(s)\delta_F^{-1}(s)\hat{B}_m \tag{17}$$

or, equivalently, that

$$(sI-\hat{A}-\hat{B}\hat{F})S(s) = \hat{B}\hat{B}_m^{-1}\delta_F(s) \tag{18}$$

But (18) is an immediate consequence of the definitions of $S(s)$ and $\delta_F(s)$. Thus the theorem is established.

This seemingly innocuous and easily proved theorem has, as we shall see, a number of significant consequences. For a beginning, we have

COROLLARY 2.3 *The matrices* $\hat{C}$, $S(s)$ *and* $\hat{B}_m$ *are invariant under state variable feedback (i.e. do not depend on* F*). Moreover, only the* d_k*-th,* $k = 1,\ldots,m$, *rows of* $\hat{A} = Q A Q^{-1}$ *can be altered by state variable feedback.*

Proof: An immediate consequence of (15), (16) and the definition of $\delta_F(s)$.

COROLLARY 2.4 *Let* $p = m$ *and* $C^*(s) = \hat{C}S(s)$. *Then the inverse system ([6]) to (1) exists if and only if* $C^*(s)$ *is nonsingular.*

Proof: The inverse system exists if and only if the transfer matrix $T(s)$ is nonsingular and so the corollary follows from the theorem as $\hat{B}_m$ and $\delta_Q(s)$ are nonsingular.

COROLLARY 2.5 *Let* $\Delta_F(s) = \det(s\underset{\sim}{I}-\underset{\sim}{A}-\underset{\sim}{B}\,\underset{\sim}{F})$. *Then* $\Delta_F(s) = \det(\underset{\sim}{\delta}_F(s))$ *and if* $p = m$

$$\det \underset{\sim}{T}_F(s) = (\det \underset{\sim}{C}^*(s))/\Delta_F(s) \tag{19}$$

where $\underset{\sim}{T}_F(s) = \underset{\sim}{N}_F(s)/\Delta_F(s)$ (*i.e.* $\underset{\sim}{N}_F(s)$ *is the numerator of the transfer matrix*).

Proof: By the definition of $\underset{\sim}{T}_F(s)$, we have $\underset{\sim}{T}_F(s) = \underset{\sim}{N}_F(s)/\Delta_F(s)$. It follows from the theorem that

$$\frac{\underset{\sim}{N}_F(s)}{\Delta_F(s)} = \frac{\underset{\sim}{C}^*(s)\underset{\sim}{D}_F(s)\hat{\underset{\sim}{B}}_m}{\det(\underset{\sim}{\delta}_F(s))} \tag{20}$$

where $\underset{\sim}{\delta}_F^{-1}(s) = \underset{\sim}{D}_F(s)/\det(\underset{\sim}{\delta}_F(s))$. However, $\Delta_F(s)$ and $\det(\underset{\sim}{\delta}_F(s))$ are both monic polynomials of degree n and the entries in $\underset{\sim}{N}_F(s)$ are polynomials of at most degree $n-1$. It follows that $\Delta_F(s) = \det(\underset{\sim}{\delta}_F(s))$ and hence, that (19) holds (since $\det(\underset{\sim}{\delta}_F^{-1}(s)) = 1/\det(\underset{\sim}{\delta}_F(s))$ and $\det \hat{\underset{\sim}{B}}_m = 1$).

COROLLARY 2.6 $\underset{\sim}{\delta}_F(s) = \underset{\sim}{\delta}_O(s)-\hat{\underset{\sim}{B}}_m\hat{\underset{\sim}{F}}\,\underset{\sim}{S}(s)$.

Proof: From (18), it follows that $\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}}_m^{-1}\underset{\sim}{\delta}_O(s) - \hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{F}}\,\underset{\sim}{S}(s) = \hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}}_m^{-1}\,\underset{\sim}{\delta}_F(s)$. Equating the nonzero rows in this equality gives us the corollary.

We observe that entirely analogous results can be obtained for observable systems by a consideration of the dual system ([1], [7])

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}'\underset{\sim}{x} + \underset{\sim}{C}'\underset{\sim}{y}\,, \quad \underset{\sim}{y} = \underset{\sim}{B}'\underset{\sim}{x} \tag{21}$$

which is controllable if and only if (1) is observable. While we shall not derive the results for observable systems here, we shall use them without further ado in the sequel.

3. A General Structure Theorem

Consider the system (1) and again let $K = [B, A B, \ldots, A^{n-1} B]$. However, we no longer assume that (1) is controllable and so, the $n \times nm$ matrix K has rank r with $r \leqq n$. To obtain a structure theorem in this general context, we shall consider a controllable extension of (1) and apply theorem 2.2. With this in mind, we let $q = n-r$ and W be the r-dimensional subspace of R_n spanned by the columns of K. Denoting the orthogonal complement of W by $W^{\perp}$ so that $R_n = W \oplus W^{\perp}$ and letting $\beta_1, \ldots, \beta_q$ be a basis of $W^{\perp}$, we consider the system

$$(22) \qquad \dot{x} = A x + B_e v, \quad y = C x$$

where B_e is the $n \times (m+q)$ matrix given by $B_e = [B\, \beta_1 \cdots \beta_q]$. The system (22) is controllable and there is a Lyapunov transformation Q_e which carries (22) into block companion form. We note that Q_e is a nonsingular $n \times n$ matrix. It follows that the system

$$(23) \qquad \dot{z} = \hat{A} z + \hat{B} u, \quad y = \hat{C} z$$

where $\hat{A} = Q_e^{-1} A Q_e$, $\hat{B} = Q_e B$, and $\hat{C} = C Q_e^{-1}$ is equivalent to (1). Moreover, the matrix $\hat{A}$ is in block companion form, the last $n-r$ rows of $\hat{B}$ are 0, and the lower left-hand $n-r \times r$ block of $\hat{A}$ is 0. Thus, the last $n-r$ rows of $\hat{A}$ cannot be altered by state variable feedback of the form $u = \hat{F} z + w$. We now have:

THEOREM 3.1 *Let* $\mathbf{T}_F(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A}-\mathbf{B}\,\mathbf{F})^{-1}\mathbf{B}$ *be the transfer matrix of the system* $\dot{\mathbf{x}} = (\mathbf{A}+\mathbf{B}\,\mathbf{F})\mathbf{x}+\mathbf{B}\,\mathbf{w}$, $\mathbf{y} = \mathbf{C}\,\mathbf{x}$. *Then*

$$\mathbf{T}_F(s) = \frac{\hat{\mathbf{C}}\,\mathbf{S}(s)\Delta_{F,u}(s)\boldsymbol{\delta}^{-1}_{F,c}(s)\hat{\mathbf{B}}_m}{\Delta_{F,u}(s)} \tag{24}$$

where $\hat{\mathbf{C}} = \mathbf{C}\,\mathbf{Q}_e^{-1}$, $\mathbf{S}(s)$ *is the* $n \times m$ *matrix given by*

$$\mathbf{S}(s) = \left[\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
s & 0 & & 0 \\
\vdots & \vdots & & \vdots \\
s^{\sigma_1-1} & 0 & & 0 \\
0 & 1 & & 0 \\
\vdots & \vdots & & \vdots \\
0 & s^{\sigma_2-1} & & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & & s^{\sigma_m-1} \\
\hdashline
\vdots & \vdots & & \vdots \\
0 & 0 & & 0
\end{array}\right] \tag{25}$$

(*with* $\mathbf{b}_1, \mathbf{A}\,\mathbf{b}_1, \ldots, \mathbf{A}^{\sigma_1-1}\mathbf{b}_1, \ldots, \mathbf{A}^{\sigma_m-1}\mathbf{b}_m$ *a "lexicographic" basis of the range of* $\mathbf{K}$ *so that* $\sum_{i=1}^{m}\sigma_i = r$), $\Delta_{F,u}(s) = \det \boldsymbol{\delta}_{F,u}(s)$, $\boldsymbol{\delta}_F(s)$ *is the* $(m+q) \times (m+q)$ *matrix* $(\delta_{F,ij}(s))$ *with entries given by* $\delta_{F,ii}(s) = \det(s\mathbf{I}-\boldsymbol{\Phi}_{ii})$ *and* $\delta_{F,ij}(s) = -\phi_{d_i,d_{j-1}+1} - \cdots - s^{\sigma_1-1}\phi_{d_i,d_j}$ *for* $i \neq j$

where $d_k = \sum_{i=1}^{k} \sigma_i$, $\sigma_i = 1$ *for* $i = m+1,\ldots,m+q$, *and* $\hat{\mathbf{A}}+\hat{\mathbf{B}}\,\hat{\mathbf{F}} = (\phi_{ij}) = [\boldsymbol{\Phi}_{ij}]$ *so that*

$$(26)\qquad \boldsymbol{\delta}_{\mathbf{F}}(s) = \left[\begin{array}{ccc|ccc} \delta_{F,11}(s) & \cdots & \delta_{F,1m}(s) & \delta_{F,1m+1}(s) & \cdots & \delta_{F,1m+q}(s) \\ \vdots & & & \vdots & & \vdots \\ \delta_{F,m1}(s) & \cdots & \delta_{F,mm}(s) & & & \\ \hline & & & \delta_{F,m+1m+1}(s) & \cdots & \delta_{F,m+1m+q}(s) \\ & \mathbf{0} & & \vdots & & \\ & & & \delta_{F,m+qm+1}(s) & \cdots & \delta_{F,m+qm+q}(s) \end{array}\right]$$

$$= \left[\begin{array}{c|c} \boldsymbol{\delta}_{\mathbf{F},c}(s) & \boldsymbol{\delta}_{\mathbf{F},cu}(s) \\ \hline \mathbf{0} & \boldsymbol{\delta}_{\mathbf{F},u}(s) \end{array}\right]^{+}$$

and where $\hat{\mathbf{B}}_m$ *is the* $m \times m$ *matrix consisting of the nonzero rows of* $\hat{\mathbf{B}}$.

Proof: Clearly we need only show that $\hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}\hat{\mathbf{B}} = \dfrac{\hat{\mathbf{C}}\,\mathbf{S}(s)\Delta_{\mathbf{F},u}(s)\boldsymbol{\delta}^{-1}_{\mathbf{F},c}(s)\hat{\mathbf{B}}_m}{\Delta_{\mathbf{F},u}(s)}$ where $\hat{\mathbf{F}} = \mathbf{F}\,\mathbf{Q}_e^{-1}$. We shall do this by considering the completely controllable system

$$(27)\qquad \dot{\mathbf{z}} = \hat{\mathbf{A}}\,\mathbf{z} + \hat{\mathbf{B}}_e\mathbf{v}, \quad \mathbf{y} = \hat{\mathbf{C}}\,\mathbf{z}$$

with $\hat{\mathbf{B}}_e = \mathbf{Q}_e\mathbf{B}_e$ and applying theorem 2.2.

[+] $\boldsymbol{\delta}_{\mathbf{F},cu}(s)$ involves only constant terms and the off-diagonal terms in $\boldsymbol{\delta}_{\mathbf{F},u}(s)$ are constant.

Let $\hat{\mathbf{F}}_e = \mathbf{F}_e\mathbf{Q}_e^{-1}$ where $\mathbf{F}_e = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}$ so that $\hat{\mathbf{F}}_e = \begin{bmatrix} \hat{\mathbf{F}} \\ \mathbf{0} \end{bmatrix}$. Since Since $\mathbf{B}_e = [\mathbf{B}\ \boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_q]$, we have, by the definition of $\mathbf{Q}_e$,

$$\hat{\mathbf{B}}_c = \left[\begin{array}{c|c} \hat{\mathbf{B}} & \mathbf{0} \\ \cline{2-2} & \mathbf{I}_q \end{array}\right]$$

and $\hat{\mathbf{B}}_e\hat{\mathbf{F}}_e = \hat{\mathbf{B}}\,\hat{\mathbf{F}}$. It follows that $(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\hat{\mathbf{F}}) = (s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}_e\hat{\mathbf{F}}_e)$ and hence, that the transfer matrix of (27) under the feedback $\mathbf{v} = \mathbf{F}_e\mathbf{x}+\mathbf{w}$ is given by $\hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}\hat{\mathbf{B}}_e$. However, (27) is controllable and thus, by theorem 2.2,

$$\hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}}-\hat{\mathbf{B}}\,\hat{\mathbf{F}})^{-1}\hat{\mathbf{B}}_e = \hat{\mathbf{C}}\,\mathbf{S}_e(s)\boldsymbol{\delta}_{\mathbf{F}}^{-1}(s)\hat{\mathbf{B}}_{e,m+q} \tag{28}$$

where $\mathbf{S}_e(s)$ is given by

$$\mathbf{S}_e(s) = \left[\begin{array}{c|c} \mathbf{S}(s) & \mathbf{0} \\ \cline{2-2} & \mathbf{I}_q \end{array}\right]$$

and $\mathbf{B}_{e,m+q}$ is the $m+q \times m+q$ matrix given by

$$\hat{\mathbf{B}}_{e,m+q} = \left[\begin{array}{c|c} \hat{\mathbf{B}}_m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_q \end{array}\right]$$

By equating the appropriate blocks in (28) and noting that

$$(29)\qquad \underset{\sim}{\delta}_{\underset{\sim}{F}}^{-1}(s)=\frac{\left[\begin{array}{c|c}(\det \underset{\sim}{\delta}_{\underset{\sim}{F},u}(s))\operatorname{adj} \underset{\sim}{\delta}_{\underset{\sim}{F},c}(s) & -(\operatorname{adj} \underset{\sim}{\delta}_{\underset{\sim}{F},c}(s))\underset{\sim}{\delta}_{\underset{\sim}{F},cu}(s)(\operatorname{adj} \underset{\sim}{\delta}_{\underset{\sim}{F},u} \\ \hline \underset{\sim}{0} & (\det \underset{\sim}{\delta}_{\underset{\sim}{F},c}(s))\operatorname{adj} \underset{\sim}{\delta}_{\underset{\sim}{F},u}(s)\end{array}\right.}{\det \underset{\sim}{\delta}_{\underset{\sim}{F},u}(s)\ \det \underset{\sim}{\delta}_{\underset{\sim}{F},c}(s)}$$

where adj( ) denotes the adjoint of a matrix, we deduce (24). Thus, the theorem is established.

COROLLARY 3.2 $\Delta_{\underset{\sim}{F},u}(s)$ *is independent of* $\underset{\sim}{F}$ *and the uncontrollable poles of the system* $\dot{\underset{\sim}{x}} = (\underset{\sim}{A}+\underset{\sim}{B}\ \underset{\sim}{F})\underset{\sim}{x}+\underset{\sim}{B}\ \underset{\sim}{w}$, $\underset{\sim}{y} = \underset{\sim}{C}\ \underset{\sim}{x}$ *are the zeros of* $\Delta_{\underset{\sim}{F},u}(s)[= \Delta_{\underset{\sim}{0},u}(s)]$.

Corollary 3.2 is simply a statement of the fact that the uncontrollable poles cannot be altered by state variable feedback. We also note that the factorization (24) involves the well-known pole-zero cancellation of the uncontrollable portion of the system ([8]).

COROLLARY 3.3 *The matrices* $\hat{\underset{\sim}{C}}$, $\underset{\sim}{S}(s)$ *and* $\hat{\underset{\sim}{B}}_m$ *are invariant under state variable feedback.*

COROLLARY 3.4 *Let* $p = m$ *and* $\underset{\sim}{C}^*(s) = \hat{\underset{\sim}{C}}\ \underset{\sim}{S}(s)$. *Then the inverse system to* (1) *exists if and only if* $\underset{\sim}{C}^*(s)$ *is nonsingular.*

COROLLARY 3.5 *Let* $p = m$ *and let* $\Delta_{\underset{\sim}{F}}(s) = \det \underset{\sim}{\delta}_{\underset{\sim}{F}}(s)$. *Then* $\det(\underset{\sim}{T}_{\underset{\sim}{F}}(s)) = (\det \underset{\sim}{C}^*(s))(\Delta_{\underset{\sim}{F},u}(s))/\Delta_{\underset{\sim}{F}}(s)$ where $\Delta_{\underset{\sim}{F}}(s) = \Delta_{\underset{\sim}{F},u}(s)\Delta_{\underset{\sim}{F},c}(s)$.

We again observe that entirely analogous results can be obtained for systems which are not observable by a consideration of the dual system (21). We use these results without further ado in the sequel.

4. The Problem of Realization

We now apply the structure theorem to obtain an algorithm for solving the problem of realization ([1], [9]). More precisely, we consider the following

REALIZATION PROBLEM: *Let* $\underset{\sim}{T}(s)$ *be a* $p \times m$ *matrix whose entries* $T_{ij}(s)$ *are rational functions of* s. *Suppose that* $T_{ij}(s) = n_{ij}(s)/d_{ij}(s)$ *where* $n_{ij}(s)$ *and* $d_{ij}(s)$ *are relatively prime and degree* $n_{ij}(s) <$ degree $d_{ij}(s)$. *Then, determine a triple* $\{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C}\}$ *of matrices such that*

$$(30) \qquad \underset{\sim}{T}(s) = \underset{\sim}{C}(s\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}$$

and $\{\underset{\sim}{A},\underset{\sim}{B}\}$ *is controllable and* $\{\underset{\sim}{A},\underset{\sim}{C}\}$ *is observable. Such a triple is called a minimal realization of* $\underset{\sim}{T}(s)$ ([1],[9]).

Kalman and Ho ([9]) proved that the realization problem has a solution and provided a constructive procedure for determining a minimal realization. Here, we present an alternate constructive algorithm for determining minimal realizations. A computer program has been developed for applying the algorithm.

The basic steps in the algorithm are

STEP 1 Calculation of the least common multiple of the denominator polynomials $\{d_{1j}(s),\ldots,d_{pj}(s)\}$ in each column of $\underset{\sim}{T}(s)$.

STEP 2 Construction of a standard controllable realization $\{\underset{\sim}{A}_c,\underset{\sim}{B}_c,\underset{\sim}{C}_c\}$ (not necessarily minimal).

STEP 3 Construction of a minimal realization by applying a suitable transformation to $\{\underset{\sim}{A}_c', \underset{\sim}{C}_c', \underset{\sim}{B}_c'\}$.

We shall examine each of these steps in detail paying particular attention to step 2.

Now let $g_j(s)$ be the least common multiple of the denominator polynomials $\{d_{1j}(s), \ldots, d_{pj}(s)\}$ (which are assumed, for convenience, to be monic). Let h_j denote the degree of $g_j(s)$ and let $\underset{\sim}{T}^*(s)$ be the $p \times m$ matrix given by

$$(31) \qquad \underset{\sim}{T}^*(s) = \begin{bmatrix} n_{11}^*(s)/g_1(s) & \cdots & n_{1m}^*(s)/g_m(s) \\ \vdots & & \\ n_{p1}^*(s)/g_1(s) & \cdots & n_{pm}^*(s)/g_m(s) \end{bmatrix}$$

where $n_{ij}^*(s) = n_{ij}(s) g_j(s)/d_{ij}(s)$. In other words, $\underset{\sim}{T}^*(s)$ is obtained from $\underset{\sim}{T}(s)$ by multiplying each numerator $n_{ij}(s)$ by $g_j(s)/d_{ij}(s)$ and replacing each denominator $d_{ij}(s)$ by $g_j(s)$. The construction of $\underset{\sim}{T}^*(s)$ completes step 1.

Let $n_1 = \sum_{j=1}^{m} h_j$ and $p_k = \sum_{1}^{k} h_j$. Since $g_j(s)$ is the least common multiple of $\{d_{1j}(s), \ldots, d_{pj}(s)\}$ and degree $n_{ij}(s) <$ degree $d_{ij}(s)$ and the $d_{ij}(s)$ are assumed monic, we have

$$(32) \qquad g_j(s) = s^{h_j} + \gamma_{j1} s^{h_j - 1} + \cdots + \gamma_{jh_j}$$

$$(33) \qquad n_{ij}^*(s) = \nu_{ij1} s^{h_j - 1} + \nu_{ij2} s^{h_j - 2} + \cdots + \nu_{ijh_j}$$

for all i,j and suitable constants γ_{jk}, ν_{ijk} Let $\underset{\sim}{A}_{c,j}$ be a companion matrix corresponding to $g_j(s)$ so that

$$(34)\qquad \utilde{A}_{c,j} = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ -\gamma_{jh_j} & -\gamma_{jh_j-1} & \cdots & -\gamma_{j1} \end{bmatrix}$$

and let $\utilde{A}_c$ be the $n_1 \times n_1$ block diagonal matrix given by

$$(35)\qquad \utilde{A}_c = \begin{bmatrix} \utilde{A}_{c,1} & & \utilde{0} \\ & \utilde{A}_{c,2} & \\ & & \ddots & \\ \utilde{0} & & & \utilde{A}_{c,m} \end{bmatrix}$$

If $\utilde{B}_c$ is the $n_1 \times m$ matrix with zero entries in all but the p_k-th rows, each of which is zero except for a 1 in the k-th column, then the pair $\{\utilde{A}_c, \utilde{B}_c\}$ is controllable. We now have

PROPOSITION 4.1 <u>Let</u> $\utilde{C}_c$ <u>be the</u> $m \times n_1$ <u>matrix given by</u>

$$(36)\qquad \utilde{C}_c = \left[\begin{array}{cccc|ccc|cc} \nu_{11h_1} & \nu_{11h_1-1} & \cdots & \nu_{111} & \nu_{12h_2} & \cdots & \nu_{121} & \cdots & \nu_{1m1} \\ \nu_{21h_1} & \nu_{21h_1-1} & \cdots & \nu_{211} & \nu_{22h_2} & \cdots & \nu_{221} & & \nu_{2m1} \\ \vdots & \vdots & & \vdots & \vdots & & & & \vdots \\ \nu_{p1h_1} & \nu_{p1h_1-1} & & \nu_{p11} & \nu_{p2h_2} & \cdots & \nu_{p21} & & \nu_{pm1} \end{array}\right]$$

<u>Then</u> $\{\utilde{A}_c, \utilde{B}_c, \utilde{C}_c\}$ <u>is a controllable realization of</u> $\utilde{T}(s)$.

Proof: Since $\{\mathbf{A}_c,\mathbf{B}_c\}$ is controllable, it follows from the structure theorem 2.2 and the definitions of $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{C}_c$, that

$$(37)\qquad \mathbf{C}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{B}_c = \mathbf{C}^*_c(s)\boldsymbol{\delta}_c^{-1}(s)\hat{\mathbf{B}}_{c,m}$$

where $\hat{\mathbf{B}}_{c,m} = \mathbf{I}_m$, $\boldsymbol{\delta}_c^{-1}(s) = \mathrm{diag}[1/g_1(s),\ldots,1/g_m(s)]$, and $\mathbf{C}^*_c(s) = (n^*_{ij}(s))$. Since $n^*_{ij}(s)/g_j(s) = n_{ij}(s)/d_{ij}(s)$, we deduce that $\mathbf{C}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{B}_c = (n_{ij}(s)/d_{ij}(s)) = \mathbf{T}(s)$. Thus, the proposition is established. This proposition completes the description of step 2.

As regards step 3, we consider the triple $\{\mathbf{A}'_c,\mathbf{C}'_c,\mathbf{B}'_c\}$ and apply a Lyapunov transformation $\mathbf{Q}_e$ of the type used in section 3 to it. Letting n be the rank of $[\mathbf{C}'_c\ \mathbf{A}'_c\mathbf{C}'_c\cdots\mathbf{A}_c^{n_1-1}\ \mathbf{C}'_c]$ and setting $\hat{\mathbf{A}}'_c = \mathbf{Q}_e\mathbf{A}'_c\mathbf{Q}_e^{-1}$, $\hat{\mathbf{C}}'_c = \mathbf{Q}_e\mathbf{C}'_c$, $\hat{\mathbf{B}}'_c = \mathbf{B}'_c\mathbf{Q}_e^{-1}$, we have

$$(38)\qquad \hat{\mathbf{C}}'_c = \begin{bmatrix} \mathbf{C}' \\ \mathbf{0}_{n_1-n,p} \end{bmatrix}, \qquad \mathbf{A}'_c = \left[\begin{array}{c|c} \mathbf{A}' & * \\ \mathbf{0}_{n_1-n,n} & * \end{array}\right]$$

and $\hat{\mathbf{B}}'_c = [\mathbf{B}'\ \ *_{mn_1-n}]$ where $\mathbf{C}'$ is $n \times p$, $\mathbf{A}'$ is $n \times n$ and $\mathbf{B}'$ is $m \times n$. Since $\mathbf{T}(s) = \mathbf{C}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{B}_c$; it follows that $\mathbf{T}'(s) = \hat{\mathbf{B}}'_c(s\mathbf{I}-\hat{\mathbf{A}}'_c)^{-1}\hat{\mathbf{C}}'_c = \mathbf{B}'(s\mathbf{I}-\mathbf{A}')^{-1}\mathbf{C}'$ or, equivalently, that $\mathbf{T}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$. Thus, $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ is a realization of $\mathbf{T}(s)$. But $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ is both controllable and observable and hence, is a minimal realization ([9]). The triple $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ is in "observable canonical form". The actual available program also produces a minimal realization in "controllable canonical form" as well as all the relevant Lyapunov transformations. A sample of the computer program printout for an example of Kalman's ([1] p. 182) is given in the appendix. A detailed write up and listing of the program can be obtained from the authors.

5. The Problem of Decoupling

We now apply the structure theorem to obtain some results related to the problem of decoupling. This problem has been examined previously by a number of authors (e.g. [2], [3]) and a number of relevant questions have been resolved. Here, our main emphasis will be on the question of pole assignability. More precisely, consider the following

DECOUPLING PROBLEM *Let* $\underset{\sim}{\dot{x}} = \underset{\sim}{A}\,\underset{\sim}{x}+\underset{\sim}{B}\,\underset{\sim}{u}$, $\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x}$ *be an m-input, m-output system. Does there exist a pair of matrices* $(\underset{\sim}{F},\underset{\sim}{G})$ *such that the transfer matrix*

$$(39) \qquad \underset{\sim}{C}(s\underset{\sim}{I}-\underset{\sim}{A}-\underset{\sim}{B}\,\underset{\sim}{F})^{-1}\underset{\sim}{B}\,\underset{\sim}{G} = \underset{\sim}{T}_{F,G}(s)$$

is diagonal and nonsingular? (*i.e. does the state variable feedback* $\underset{\sim}{u} = \underset{\sim}{F}\,\underset{\sim}{x}+\underset{\sim}{G}\,\underset{\sim}{w}$ *"decouple" the system?*).

A necessary and sufficient condition for the existence of a decoupling pair was first given in [2]. In particular, it has been shown that the system

$$(40) \qquad \underset{\sim}{\dot{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{u}, \quad \underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x}.$$

can be decoupled if and only if $\underset{\sim}{B}^*$ is nonsingular where $\underset{\sim}{B}^*$ is the $m \times m$ matrix given by

$$(41) \qquad \underset{\sim}{B}^* = \begin{bmatrix} \underset{\sim}{c}_1 \underset{\sim}{A}^{f_1} \underset{\sim}{B} \\ \vdots \\ \underset{\sim}{c}_m \underset{\sim}{A}^{f_m} \underset{\sim}{B} \end{bmatrix}$$

with $\underset{\sim}{c}_i$, the i-th row of $\underset{\sim}{C}$, and $f_i = \min[\{j\colon \underset{\sim}{c}_i \underset{\sim}{A}^j \underset{\sim}{B} \neq 0\}, n-1]$. $\underset{\sim}{B}^*$ and the f_i can also be characterized in the following way (cf. [3]): let $\underset{\sim}{T}_{F,G,i}(s)$ be the i-th row of the transfer matrix $\underset{\sim}{T}_{F,G}(s)$; then $f_i = \min[\{j\colon \lim_{s \to \infty} s^{j+1} \underset{\sim}{T}_{F,G,i}(s) \neq \underset{\sim}{0}\}, n-1]$ and $\underset{\sim}{B}^* \underset{\sim}{G} = \lim_{s \to \infty} \underset{\sim}{\Delta}(s) \underset{\sim}{T}_{F,G}(s)$ where $\underset{\sim}{\Delta}(s)$ is a diagonal matrix with entries s^{f_i+1}. It can be shown ([2], [3]) that $\underset{\sim}{B}^*$ and the f_i are invariant under state variable feedback.

Here, we shall use the structure theorem to answer the following questions:

QUESTION 1 Assuming that (40) can be decoupled, what is the maximum number of closed loop poles which can be arbitrarily specified while simultaneously decoupling the system?

QUESTION 2 Assuming that (40) can be decoupled, which closed loop poles are invariant under decoupling state variable feedback?

QUESTION 3 How can a decoupling pair which specifies the maximum number of closed loop poles be implemented?

While these questions are to some degree resolved in [2] and

[3], we provide a complete and elementary answer to them here.

Let $\underset{\sim}{T}(s)$ be the transfer matrix of (40). Then $\underset{\sim}{T}(s) = \underset{\sim}{C}^*(s) \frac{\Delta_u(s)}{\Delta_u(s)} \underset{\sim}{\delta}^{-1}_{\underset{\sim}{Q},c}(s)\hat{\underset{\sim}{B}}_m$ where $\underset{\sim}{C}^*(s) = \hat{\underset{\sim}{C}}\,\underset{\sim}{S}(s)$ by the structure theorem 3.1. We recall that $\underset{\sim}{C}^*(s)$ and $\Delta_u(s)$ are invariant under state variable feedback. Now we let $p_i(s)$ be the greatest common divisor of the polynomials which are the entries in the i-th row $\underset{\sim}{C}^*_i(s)$ of $\underset{\sim}{C}^*(s)$. We note that $p_i(s)$ is invariant under state variable feedback. We let r_i be the degree of $p_i(s)$ and we use the notation ∂_p to denote the degree of a polynomial (thus, $r_i = \partial_{p_i}$). We now have

THEOREM 5.1 *Suppose that the system (40) can be decoupled. Then* (i) *the maximum number* ν *of closed loop poles which can be arbitrarily specified while decoupling is given by*

$$\nu = \sum_{i=1}^{m} (r_i+f_i+1) \tag{42}$$

and (ii) *the invariant poles under decoupling feedback are the zeros of* $\Delta_u(s)$ *and* $\{\det \underset{\sim}{C}^*(s)\}/\prod_{i=1}^{m} p_i(s)$.

Proof: Let $\{\underset{\sim}{F},\underset{\sim}{G}\}$ be any decoupling pair. Then $\underset{\sim}{T}_{F,G}(s) = \underset{\sim}{C}(s\underset{\sim}{I}-\underset{\sim}{A}-\underset{\sim}{B}\,\underset{\sim}{F})^{-1}\underset{\sim}{B}\,\underset{\sim}{G}$ is a diagonal matrix with entries $n_{ii}(s)/d_{ii}(s)$ where $n_{ii}(s)$ and $d_{ii}(s)$ are relatively prime. We note that, since $f_i = \min\{j: \lim_{s\to\infty} s^{j+1}\underset{\sim}{T}_{F,G,i}(s) \neq \underset{\sim}{0}\}$,[+] $\partial_{n_{ii}} = \partial_{d_{ii}}-f_i-1$. It follows from corollary 3.5 and the definition of the $p_i(s)$ that

---

[+] Note that $\underset{\sim}{B}^*$ is nonsingular .

$$\prod_{i=1}^{m} \frac{n_{ii}(s)}{d_{ii}(s)} = \prod_{i=1}^{m} p_i(s) \det \underset{\sim}{C}^*_{II}(s) \frac{\Delta_u(s)}{\Delta_{\underset{\sim}{F}}(s)} \det \underset{\sim}{G} \tag{43}$$

where $\underset{\sim}{C}^*(s)$ is the matrix with rows $\underset{\sim}{C}^*_{II,i}(s) = \frac{1}{p_i(s)} \underset{\sim}{C}^*_i(s)$. Since $\Delta_{\underset{\sim}{F}}(s) = \Delta_u(s)\Delta_{\underset{\sim}{F},c}(s)$, we have

$$\partial_{F,c} = \sum_{i=1}^{m} (r_i+f_i+1)+\partial^*_{II} \tag{44}$$

where ∂^*_{II} is the degree of $\det \underset{\sim}{C}^*_{II}(s)$ and $\partial_{F,c}$ is the degree of $\Delta_{\underset{\sim}{F},c}(s)$. Now, it is clear from theorem 3.1 that

$$\underset{\sim}{T}_{\underset{\sim}{F},\underset{\sim}{G},i}(s)\underset{\sim}{G}^{-1}\hat{\underset{\sim}{B}}{}_m^{-1}\delta_{\underset{\sim}{F},c}(s) = \underset{\sim}{C}^*_i(s) \tag{45}$$

and hence, that $n_{ii}(s)$ is a common divisor of the entries in $\underset{\sim}{C}^*_i(s)$ (since $n_{ii}(s)$ and $d_{ii}(s)$ are relatively prime). In other words, $n_{ii}(s)$ <u>must divide</u> $p_i(s)$ and so, $\partial_{n_{ii}} \leq r_i$. Since no more than $\sum_{i=1}^{m} \partial_{d_{ii}}$ poles are assignable through $\{\underset{\sim}{F},\underset{\sim}{G}\}$ and $\sum_{i=1}^{m} \partial_{d_{ii}} = \sum_{i=1}^{m} (\partial_{n_{ii}}+f_i+1)$, we deduce that at most $\nu = \sum_{i=1}^{m} (r_i+f_i+1)$ poles are assignable while decoupling.

Writing $\underset{\sim}{T}_{\underset{\sim}{F},\underset{\sim}{G}}(s)$ as a diagonal matrix with entries $q_{ii}(s)/\Delta_{\underset{\sim}{F}}(s) = n_{ii}(s)/d_{ii}(s)$, we deduce that $q_{ii}(s)$ must divide $p_i(s)\Delta_{\underset{\sim}{F}}(s)$ or, equivalently, that

$$\frac{q_{ii}(s)}{\Delta_{\underset{\sim}{F}}(s)} = \frac{p_i(s)}{q_i(s)} \tag{46}$$

for $i = 1,\ldots,m$ and polynomials $q_i(s)$ with $\partial_{q_i} = r_i+f_i+1$. It follows that $\det \mathbf{T}_{\mathbf{F},\mathbf{G}}(s) = \prod_{i=1}^{m} (p_i(s))/\prod_{i=1}^{m} (q_i(s))$ and hence, from (43) that

$$\Delta_{\mathbf{F}}(s) = \det \mathbf{C}^*_{II}(s)\ \Delta_u(s)\det \mathbf{G} \prod_{i=1}^{m} q_i(s) \tag{47}$$

$$= \frac{\det \mathbf{C}^*(s)}{\prod_{i=1}^{m} p_i(s)}\, \Delta_u(s)\det \mathbf{G} \prod_{i=1}^{m} q_i(s)$$

Since $\mathbf{C}^*_{II}(s)$ is invariant under decoupling feedback, it follows that the zeros of $\Delta_u(s)$ and $\det \mathbf{C}^*_{II}(s)$ are invariant poles under decoupling feedback.

Thus, to complete the proof we need only construct a decoupling pair $\{\mathbf{F},\mathbf{G}\}$ such that the resulting polynomials $q_i(s)$ are arbitrary polynomials of degree r_i+f_i+1. To begin with, we note that the transfer matrix $\mathbf{T}(s) = \mathbf{C}^*(s)\frac{\Delta_u(s)}{\Delta_u(s)}\,\boldsymbol{\delta}^{-1}_{\mathbf{Q},c}(s)\hat{\mathbf{B}}_m = \mathbf{P}(s)\mathbf{C}^*_{II}(s)\frac{\Delta_u(s)}{\Delta_u(s)}\,\boldsymbol{\delta}^{-1}_{\mathbf{Q},c}(s)\hat{\mathbf{B}}_m$ where $\mathbf{P}(s)$ is a diagonal matrix with entries $p_i(s)$. Setting

$$\mathbf{T}_{II}(s) = \mathbf{C}^*_{II}(s)\frac{\Delta_u(s)}{\Delta_u(s)}\,\boldsymbol{\delta}^{-1}_{\mathbf{Q},c}(s)\hat{\mathbf{B}}_m \tag{48}$$

we can easily see that $r_i+f_i = \min\{j: \lim_{s \to \infty} s^{j+1} T_{II,i}(s) \neq 0\}$ and that $B^*_{II} = \lim_{s \to \infty} \Delta_{II}(s) T_{II}(s) = B^*$ where $\Delta_{II}(s)$ is a diagonal matrix with entries $s^{r_i+f_i+1}$ (Note that the $p_i(s)$ are monic). Moreover, as $C^*(s)$ is given by $\hat{C}\,S(s)$ and $p_i(s)$ is the greatest common divisor of the entries in $C^*_i(s)$, we can write $C^*_{II}(s)$ in the form $\hat{C}_{II}S(s)$ for some constant matrix $\hat{C}_{II}$ (where $S(s)$ is given by (25)). In other words, $T_{II}(s)$ is the transfer matrix of the system $\dot{x} = A\,x+B\,u$, $y_{II} = C_{II}x$ where $C_{II} = \hat{C}_{II} = \hat{C}_{II}Q$ (and Q is the Lyapunov transformation corresponding to (40)). Since $P(s)$ is diagonal, it will be sufficient to construct a decoupling pair $\{F,G\}$ for the system

$$\dot{x} = A\,x+B\,u\,,\quad y_{II} = C_{II}x \tag{49}$$

such that the closed loop poles are arbitrarily placed. However, letting $d_i = r_i+f_i$ and applying the synthesis procedure of [2] p. 655, we find that (49) can be decoupled and all its closed loop poles assigned. To be more explicit, if $q_i(s) = s^{d_i+1} - \sum_{j=0}^{d_i} m^i_j s^j$,$^+$ then the decoupling pair is given by

$$F = B^{*-1}\Big[\sum_{0}^{d} M_k C_{II} A^k - A^*\Big]\,,\quad G = B^{*-1} \tag{50}$$

where $d = \max d_i$, the M_k are diagonal matrices with entries m^i_k (i.e. $M_k = \mathrm{diag}[m^1_k,\ldots,m^m_k]$), and $A^* = (C_{II,i}A^{d_i+1})$ (i.e. the i-th row of A^* is given by $C_{II,i}A^{d_i+1}$). This completes the proof.

$^+$ Clearly, it is enough to consider the case of a monic $q_i(s)$.

References

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Appendix

A sample of the computer print-out for an example of Kalman's ([1]) is given here. The transfer matrix is given by

$$\mathfrak{T}(s) = \begin{bmatrix} \frac{3(s+3)(s+5)}{(s+1)(s+2)(s+4)} & \frac{6(s+1)}{(s+2)(s+4)} & \frac{2s+7}{(s+3)(s+4)} & \frac{2s+5}{(s+2)(s+3)} \\ \frac{2}{(s+3)(s+5)} & \frac{1}{s+3} & \frac{2(s+5)}{(s+1)(s+2)(s+3)} & \frac{8(s+2)}{(s+1)(s+3)(s+5)} \\ \frac{2(s^2+7s+18)}{(s+1)(s+3)(s+5)} & \frac{-2s}{(s+1)(s+3)} & \frac{1}{s+3} & \frac{2(5s^2+27s+34)}{(s+1)(s+3)(s+5)} \end{bmatrix}$$

([1] p. 182).

```
THE NUMERATOR OF THE MATRIX TRANSFER FUNCTION

S** 2   [illegible]   [illegible]   0.000000   0.000000
S** 1   20.[illegible]   0.[illegible]   2.330000   2.000000
S** 0   [illegible]   [illegible]   3.063000   0.000000

S** 1   [illegible]   [illegible]   2.000000   0.030000
S** 0   [illegible]   [illegible]   10.030003  10.000000

S** 2   [illegible]   [illegible]   0.000000   10.030000
S** 1   [illegible]   [illegible]   [illegible]   54.000000
S** 0   36.[illegible]   0.[illegible]   1.[illegible]   60.750000


THE DENOMINATOR OF THE MATRIX TRANSFER FUNCTION

S** 3   [illegible]   [illegible]   0.000000   0.000000
S** 2   7.[illegible]   1.[illegible]   1.000000   1.370000
S** 1   [illegible]   [illegible]   7.360000   5.000000
S** 0   0.[illegible]   [illegible]   12.300000   6.000000

S** 3   [illegible]   [illegible]   1.300000   1.330000
S** 2   1.[illegible]   [illegible]   6.060000   9.330000
S** 1   [illegible]   [illegible]   11.300000   23.300000
S** 0   13.[illegible]   3.[illegible]   6.000000   15.900000

S** 3   [illegible]   [illegible]   6.[illegible]   1.300000
S** 2   9.[illegible]   1.0[illegible]   [illegible]   9.056000
S** 1   23.[illegible]   [illegible]   1.000000   23.036000
S** 0   15.[illegible]   3.3[illegible]   3.000000   15.360000

THE STANDARD CONTROLLABLE REPRESENTATION OF THE SYSTEM

THE A MATRIX

[illegible]
```

```
THE B MATRIX

[illegible]

THE C MATRIX

[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  0.100000 02  0.420000 02  0.300000 02
[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  0.250000 02  0.400000 02  0.170000 02
[illegible]
[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  [illegible]  0.140000 02  0.700000 01
[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  [illegible]  0.320000 02  0.800000 01
[illegible]
[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  [illegible] -0.160000 02 -0.120000 02
[illegible]  [illegible]  [illegible]  [illegible]  [illegible]  0.130000 03  0.170000 03  0.740000 02
[illegible]

THE LYAPUNOV TRANSFORMATION 0

[illegible]

DETERMINANT =    [illegible]
```

```
A MINIMAL REALIZATION OF THE SYSTEM IN OBSERVABLE CANONICAL FORM

THE A MATRIX

 0.562070-15  -0.639652-13  -0.835730 01  -0.427440-14  -0.817530 00  [illegible]   [illegible]
 0.100000 01  -0.566360-15  -0.160740 02  -0.155430-14   0.470430 00  -0.860970-14   0.900210-01
 0.555360-12   0.100000 01  -0.701852 01  -0.777160-15  -0.552090-13  -0.110570-14  -0.932890-14
 0.222040-15   0.133230-14   0.146390 01   0.366170-14  -0.336630 01  -0.133230-14   0.881040 00
 0.399790-13  -0.721090-14   0.117310 01   0.100000 01  -0.159770 01  -0.133230-14   0.101360 01
 0.444200-15  -0.115460-13  -0.436780 01  -0.110400-13  -0.127320 00   0.888100-15  -0.538410 01
 0.222040-15   0.111020-14  -0.280450 01  -0.255350-14  -0.490140 00   0.100000 01  -0.630900 01

THE B MATRIX

 0.442750 02   0.395270 01   0.339040 01   0.460700 01
 0.240550 02   0.121110 02   0.703690 01   0.903790 01
 0.300000 01   0.600000 01   0.200000 01   0.200000 01
 0.683550 01  -0.665680 01  -0.260360 00  -0.944470 01
 0.240420 01   0.545840 01   0.107412 01   0.165410 01
-0.117540 02   0.332870 01   0.182520 01   0.232700 02
-0.203790 01  -0.160790 02  -0.169130 01   0.730170 01

THE C MATRIX

-0.247940-15  -0.693090-17   0.100000 01   0.262410-15  -0.333670-15  [illegible]   [illegible]
-0.173470-17   0.143110-16  -0.820070 00   0.142750-15   0.100000 01   0.110410-15  -0.279290-15
-0.170770-15   0.277590-15   0.134570 01   0.944150-16  -0.944090-15   0.444900-15   0.189800 01


THE LIAPUNOV TRANSFORMATION Q

-0.311180-01   0.303070-01   0.135040 00  -0.180130 00   0.279180-01  -0.181950 00   0.197470 00
 0.303070-01   0.130400 00  -0.114400 01   0.775190-01   0.345950 00   0.157670 00  -0.384400 00
-0.867000-02   0.612920-01  -0.132030 00  -0.776380-01  -0.932700-01   0.113790-01  -0.693460-01
 0.612920-01  -0.132030 00   0.344130 00  -0.632240-01   0.203400 00  -0.253760-01   0.179400 00
 0.113370-01   0.165030-01  -0.682100 00   0.193150 00   0.471400 00   0.133490 00  -0.171490 00
 0.165030-01  -0.682100 00   0.500910 01   0.621040 00  -0.180970 01  -0.171090 00   0.121400 01
 0.902070-02  -0.782090-01   0.985610 00  -0.773080-01   0.190720 00  -0.101310 00   0.287100 00

DETERMINANT =   -0.675770-06

THE INVERSE TRANSFORMATION Q**-1

 0.155610 02   0.642750 00  -0.132790 03   0.345470 01  -0.683610 02   0.538600 01  -0.347400 02
 0.932720 02   0.249350 02  -0.340990 01   0.121110 02  -0.165000 01   0.793600 01  -0.367900 00
 0.116510 02   0.300000 01   0.107930 00   0.600000 01   0.342100 01   0.700040 01  -0.187400 00
 0.918890 00   0.803650 01  -0.641700 02  -0.663680 01  -0.147390 02  -0.260350 00  -0.950190 01
 0.115420 02   0.248420 01   0.613300 01   0.494040 01   0.495730 01   0.165610 01  -0.969410 01
-0.367390 01  -0.117590 02   0.111170 02   0.332070 01   0.899510 02   0.108590 01   0.227470 00
-0.164900 02  -0.203700 01   0.970090 00  -0.160790 02  -0.134360 01  -0.169140 01   0.810490 00

A MINIMAL REALIZATION OF THE SYSTEM IN CONTROLLABLE CANONICAL FORM

THE A MATRIX

 0.222040-15   0.100000 01   0.666820-15  -0.222140-15  -0.499800-15   0.166530-15  -0.888180-15
 0.650123 01  -0.165423 00   0.244235 01   0.357160 01   0.128100 01   0.131610 01  -0.130050 01
 0.177640-14   0.277100-15   0.665330-16   0.164400 01   0.832670-15   0.300020-15  -0.199800-14
 0.419670 01   0.105630 00   0.491780 01  -0.222190 01   0.237610 01   0.358360 00   0.297870 00
-0.177600-14   0.888100-15  -0.147930-13  -0.133260-14  -0.117200-14   0.100000 01   0.355370-14
-0.145690 02  -0.366100 01  -0.417260 02  -0.190400 02  -0.150100 02  -0.830970 01   0.563650-01
 0.229560 01  -0.821570-14   0.214990 01  -0.888100-15   0.119000 01   0.888133-15  -0.377190 01

THE B MATRIX

 0.166530-15  -0.111320-15  -0.555110-16   0.000000-38
 0.100000 01   0.133230-14   0.666130-15   0.630720 00
 0.971450-16  -0.555110-16  -0.555110-16  -0.111020-15
 0.888160-15   0.100000 01   0.222040-15   0.830400 00
 0.177600-14   0.222000-15  -0.277500-16  -0.166330-15
 0.177640-14  -0.000000-38   0.100000 01   0.289710 01
-0.111020-14  -0.666130-15  -0.166530-15   0.100000 01

THE C MATRIX

 0.110510 02   0.300000 01   0.167530 02   0.600000 01   0.352100 01   0.200000 01  -0.187090 00
 0.169440 01  -0.799090-14  -0.770000 00   0.100000 01   0.164160 01  -0.200210-14  -0.834580 00
 0.110000 01   0.200000 01   0.154390 02  -0.200000 01   0.339420 01   0.100000 01   0.750110 01
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