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PM-104

AN EXAMINATION OF SIMPLE COORDINATE TRANSFORMATIONS IN THE Z-PLANE AND W-PLANE FOR ROOT LOCUS ANALYSIS OF SAMPLED-DATA SYSTEMS

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April 1970

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SUMMARY

Simple transformations of the Z-plane and W-plane coordinates to S-plane coordinates are examined in an attempt to facilitate root locus analysis of sampled-data systems. First, constant damping ratio, constant damping factor, and constant frequency loci are mapped onto the Z-plane and W-plane. From these, sampling frequency boundaries are specified wherein interpretation in the Z-plane or W-plane becomes identical to that in the S-plane. The implication is a savings in the analysis time if the system frequencies of interest are within the specified boundaries. Although the sampling frequency limit in the Z-plane is too large for practical considerations, that in the W-plane appears low enough to warrant its usefulness.

INTRODUCTION

The root locus technique has long been useful in analyzing the behavior of continuous systems. Discrete systems, however, generally introduce transcendental functions in S thus precluding a simple root locus analysis in the S-plane.

The transformation given by

$$Z = e^{ST} \tag{1}$$

where Z is a complex variable and T is the sampling period, permits the root locus technique to be extended readily to sampled-data systems. This results from the fact that the characteristic equation of the discrete system becomes algebraic in Z and only a finite number of poles and zeros enter the problem. Consequently, a magnitude and angle criterion can be established and a locus generated in the Z-plane. The procedure is similar to that in the S-plane for continuous systems. The point of departure occurs in the stability boundary and the interpretation of the locus in the Z-plane. The stability boundary for discrete systems is the unit circle in the Z-plane instead of the imaginary axis. A sampled-data system is stable if all the roots of the system characteristic equation lie inside the unit circle in the Z-plane.

The interpretation of the locus in the Z-plane can be facilitated with the aid of constant damping ratio, constant frequency and constant damping factor loci.

The locations of the characteristic equation roots in Z with respect to the unit circle can be determined by use of the Schur-Cohn criterion or Jury's stability test. Unlike the Routh-Hurwitz criterion for continuous systems, these tests do not indicate how many roots are outside the unit circle or if any of the roots are located on the unit circle, and stability cannot be determined by inspection of the characteristic equation. Further, the Schur-Cohn test relies on the evaluation of higher order determinants.

The Routh-Hurwitz criterion can be extended to sampled-data systems by means of the bilinear transformation given by

$$Z = \frac{1 + W}{1 - W} \tag{2}$$

where W is a complex variable. This transformation maps the interior of the unit circle in the Z-plane onto the left-half of the W-plane and the exterior of the unit circle onto the right-half of the W-plane. As a result of the transformation, the stability boundary for discrete systems is transferred back to the imaginary axis and the Routh-Hurwitz criterion can now be applied to sampled-data systems.

The W transformation is used primarily in frequency response analysis and as the medium for the application of the Routh-Hurwitz criterion to sampled-data systems. Root trends in the W-plane can also be examined with the aid of constant damping ratio and constant frequency loci.

SYMBOLS

c(t)	output signal of sampled-data system
e(t)	error signal of sampled-data system
G(S)	Laplace transform of forward loop transfer function of sampled-data system
GH(Z)	Z-transform of open loop transfer function of sampled-data system
GH(W)	W-transform of open loop transfer function of sampled-data system
H(S)	Laplace transform of feedback loop transfer function of sampled-data system

Im( )	imaginary part of ( )
j	$\sqrt{-1}$
K	open loop gain of sampled-data system
Re( )	real part of ( )
r(t)	input signal of sampled-data system
S	Laplace transform variable
T	sampling period of sampled-data system
W	bilinear transformation variable
Z	Z-transform variable
σ	real part of S
μ	real part of W
γ	imaginary part of W
ω	imaginary part of S
ω_s	sampling frequency

S, Z, AND W-PLANE CORRESPONDENCE

This section describes the mapping of the S-plane onto the Z-plane and the Z-plane onto the W-plane.

Equation (1) maps the left half of the primary strip in the S-plane into the unit circle of the Z-plane. This mapping is shown in Figure 1. The correspondence is shown by the shaded areas. If the remaining complementary periodic strips in the S-plane are considered they can be mapped individually into the interior of the unit circle by the Z transformation until the entire left half of the S-plane is mapped into the interior of the unit circle in the Z-plane.

The W transformation given by Eq. (2) maps the inside of the Z-plane unit circle onto the entire left half of the W-plane for each periodic strip in the left half of the S-plane. This is also shown in Figure 1. Algebraically, the transformation converts the original characteristic equation in Z into a ratio of two polynomials in W of the same or lesser order depending on the existence of roots at the origin of the Z-plane. The condition for stability becomes the absence of any roots of the W function in the right half of the W-plane.

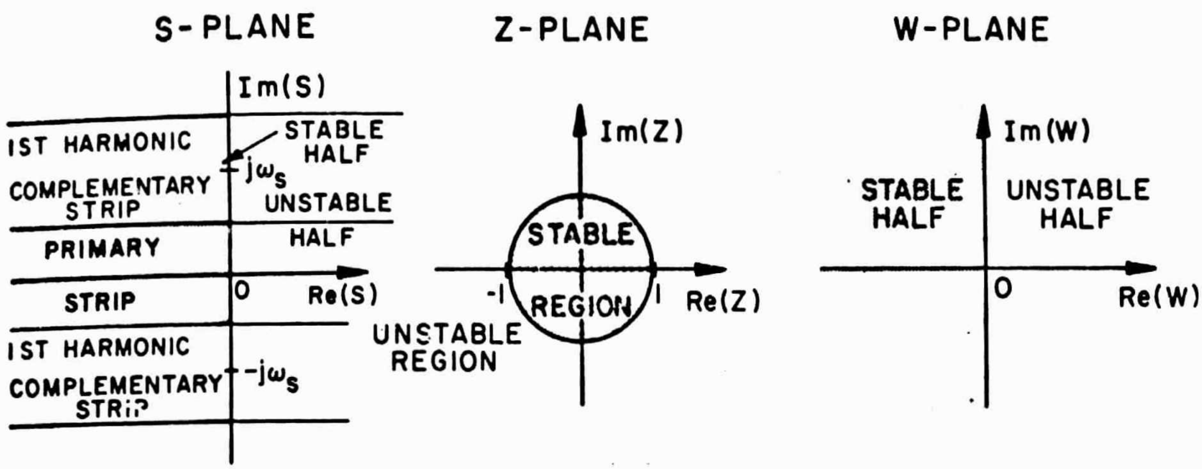

Figure 1.- The mapping of the primary strip in the left half of the S-plane into the interior of the unit circle of the Z-plane and onto the left half of the W-plane

Z-Plane

A Simple Coordinate Transformation in the Z-Plane.- Relative stability of sampled-data systems can be observed in the Z-plane by studying the locations of the roots of the characteristic equation with respect to constant damping factor, constant frequency, and constant damping ratio loci. Some typical constant damping factor and constant frequency loci are shown in Figure 2. The modified S-plane coordinates, namely σT and ωT, are chosen since they apply to sampled-data systems with any sampling period.

Figure 2 reveals that, near the +1 point in the Z-plane, the modified S-plane coordinates are almost linearly mapped on the Z-plane. Thus a transformation from the Z-plane to the S-plane can be accomplished by a simple coordinate change in the Z-plane. Roots near the +1 point can now be easily interpreted with respect to the simplified S-plane coordinates.

The error between the actual S-plane coordinates and the simplified S-plane coordinates depends on the sampling frequency. In order to keep the error less than 3 percent, a sampling frequency of at least 120 times the highest system frequency of interest is required. This is depicted by the boundary given by $\sigma T = 0.05$ and $\omega T = 0.05$. However, the sampling frequency requirement is too restrictive for practical considerations. Therefore, any simplifications in the Z-plane do not appear to be warranted.

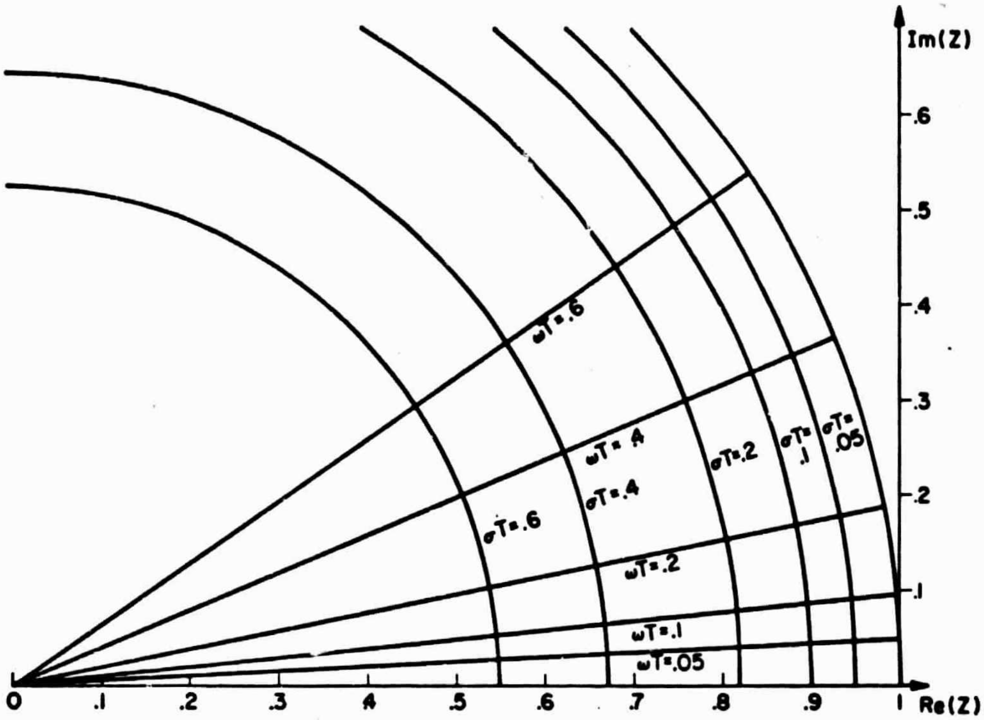

Figure 2.- Mapping of the S-plane coordinates onto the Z-plane

W-Plane

W-Plane Root Locus.- A root locus can be generated in the W-plane similar to that in the S-plane or Z-plane. To illustrate the procedure for arriving at a root locus in the W-plane, consider the discrete feedback system depicted in Figure 3. The characteristic equation for this system is of the form

$$1 + GH(Z) = 0 \tag{3}$$

With the substitution of Eq. (2) the characteristic equation becomes

$$1 + GH(W) = 0 \tag{4}$$

where GH(W) means that $\frac{1 + W}{1 - W}$ has been substituted for Z in the function GH(Z). The construction of the W-plane root locus now follows from conventional rules.

Absolute stability of the sampled-data system simply requires that all roots of the characteristic equation in W lie in the

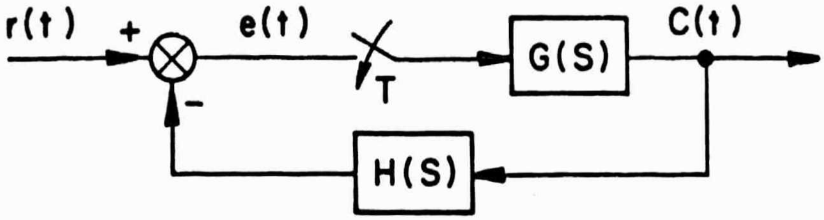

Figure 3.- Discrete feedback system

left-half of the W-plane and can be observed from the root locus plots. Relative stability can also be observed from the root loci by studying the locations of the roots of the characteristic equation with respect to constant damping factor, constant frequency and constant damping ratio loci.

Mapping of Constant Damping Factor, Constant Frequency, and Constant Damping Ratio Loci onto the W-Plane.- The relationship between the two complex variables, S and W, is derived in reference (1) and is given by

$$W = \tanh \frac{ST}{2} \tag{5}$$

where T is the sampling period.

Letting $S = \sigma + j\omega$ and $W = \mu + j\gamma$ and substituting into equation 5 gives

$$\mu + j\gamma = \tanh \left[\frac{(\sigma + j\omega)T}{2}\right] . \tag{6}$$

Using fundamental trigonometric and hyperbolic identities, the relationships between the S-plane and W-plane coordinates thus become

$$\mu = \frac{\sinh\ (\sigma T)}{\cosh(\sigma T) + \cos(\omega T)} \tag{7}$$

$$\gamma = \frac{\sin\ (\omega T)}{\cosh(\sigma T) + \cos(\omega T)} \tag{8}$$

The mapping of the modified S-plane coordinates, σT and ωT, is shown in Figure 4 in the second quadrant of the W-plane. The remaining quadrants are mirror images of each other. The 0.7 constant damping ratio locus is also shown in Figure 4.

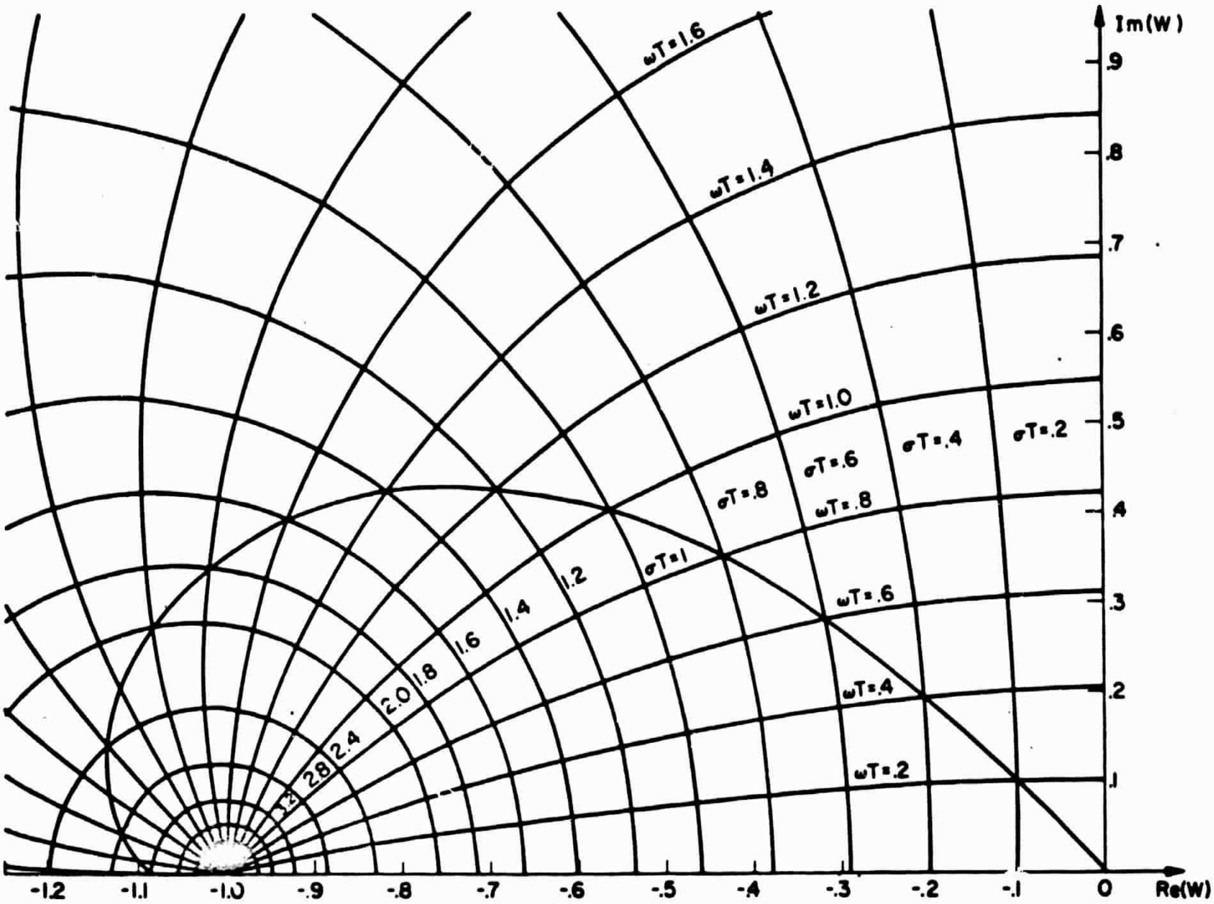

Figure 4.- Mapping of the S-plane coordinates and the .7 damping ratio locus onto the W-plane

<u>Transient Response Indication from W-Plane Roots</u>.- If the design of a particular sampled-data system is specified in terms of a certain minimum settling time, then all the characteristic equation roots of the system must lie to the left of the respective modified constant damping locus (i.e., σT) in the W-plane. This is equivalent to requiring all the roots in the S-plane to lie left of the constant damping locus given by $S = \sigma$.

If a certain damping ratio is specified, all the characteristic equation roots must lie within the decaying spiral of the given damping ratio in the W-plane. The constant damping ratio loci of Figure 4 apply only to the primary strip in the S-plane. The spiral actually continues to decay to -1 in the W-plane as the constant damping ratio loci transverse the higher complementary strips in the S-plane. A typical constant damping ratio locus that transverses the primary strip and first complementary strip is shown in Figure 5 in the S-plane and W-plane. The second and third quadrants are shown for clarity. Care must therefore be exercised (as in the Z-plane) when specifying the damping ratio requirements in the W-plane for roots that lie in the complementary strips.

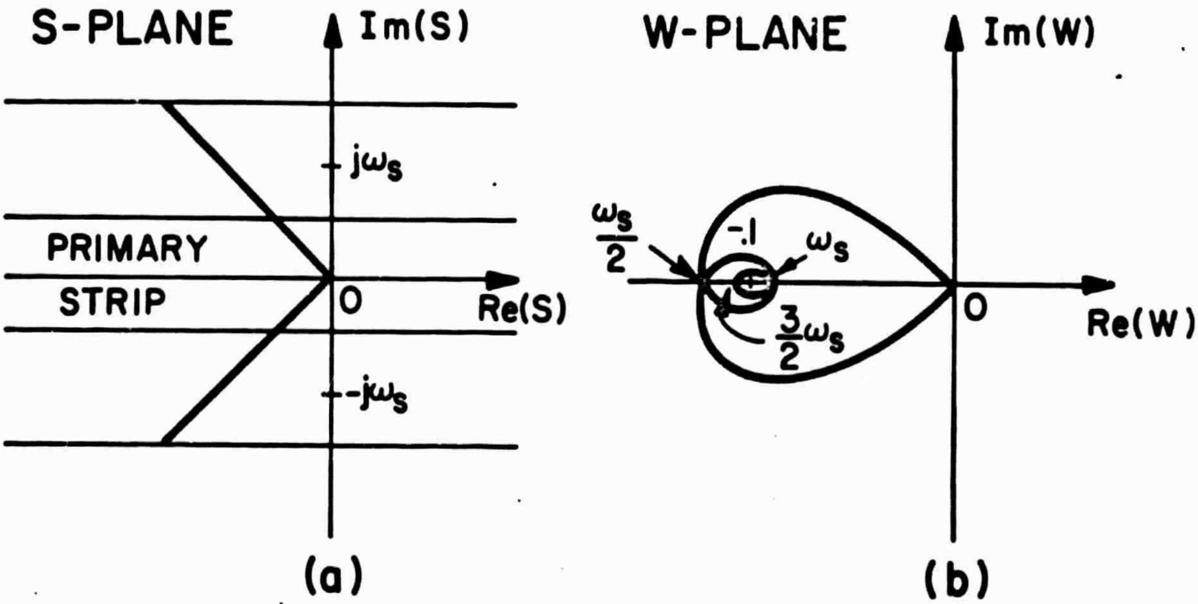

Figure 5.- (a) A constant damping ratio locus transversing the primary and 1st complementary strips in the S-plane (b) The corresponding locus in the W-plane (this locus has been expanded for clarity)

The relative stability of a sampled-data system can now be investigated by superimposing the loci of figure 4 on the W-plane system root locus plots.

A Simple Coordinate Transformation in the W-Plane.- Figure 4 reveals that, near the origin of the W-plane, the modified S-plane coordinates are almost linearly mapped on the W-plane. This follows directly from Eqs. (7) and (8). When $\sigma T \ll 1$ and $\omega T \ll 1$, Eqs. (7) and (8) reduce to Eqs. (9) and (10) respectively.

$$\mu \cong \frac{\sigma T}{2} \tag{9}$$

$$\gamma \cong \frac{\omega T}{2} \tag{10}$$

In effect, transformation from the W-plane to the S-plane can be accomplished by a simple coordinate change in the W-plane. Thus roots near the origin can now be easily interpreted with respect to the simplified S-plane coordinates.

The 3 percent error boundary is now depicted by the constant loci given by $\sigma T = 0.4$ and $\omega T = 0.4$. This corresponds to a sampling frequency of only 15 times the highest system frequency of interest. When sampling at 30 times the highest system frequency of interest, the error is almost undetectable. The simplification, of course, still preserves information on stability for all frequencies.

In summary, root trends of discrete control systems can be analyzed adequately in the W-plane when the sampling frequency is greater than 15 times the highest frequency of interest. Once the open loop transfer function is represented in Z, the bilinear transformation is used to change the variable to W. The root locus is then performed in the W-plane and interpreted with the simplified S-plane coordinates.

The alternative to this method is to perform the root locus in the Z-plane and either invert the results to the S-plane by means of the partial fraction expansion method or interpret the results directly in the Z-plane with constant frequency and constant damping ratio loci superimposed. Inversion involves a step which is more complicated and would require more computer storage and time if a digital computer is used in the design process. Direct interpretation requires a careful consideration of the constant frequency, constant damping factor, and constant damping ratio loci for proper interpretation (i.e., damping, gain sensitivity) and can be especially trying for one familiar with conventional root locus analysis but with little experience in the Z-plane.

CONCLUSION

Simple transformations of the Z-plane and W-plane coordinates to S-plane coordinates have been examined to facilitate root locus analysis of sampled-data control systems.

Previously the bilinear transformation that relates the W-plane to the Z-plane was used primarily in frequency response analyses and as the medium for applying the Routh-Hurwitz criterion to sampled-data systems. With the mapping of constant frequency constant damping factor, and constant damping ratio loci onto the W-plane the system root locus can now be interpreted directly in the W-plane.

Sampling frequency boundaries were determined wherein interpretation in the Z-plane or W-plane becomes identical to that in the S-plane. The implication is a savings in work and time in arriving at physically meaningful results if the system frequencies of interest are within the specified boundaries. Although the sampling frequency limit in the Z-plane is too large for practical considerations, that in the W-plane appears low enough to warrant its usefulness.

EXAMPLE

Consider the closed loop system of Figure 6. It represents a second order continuous plant with a simple lag compensation. The error signal is sampled and reconstructed with a zero-order hold.

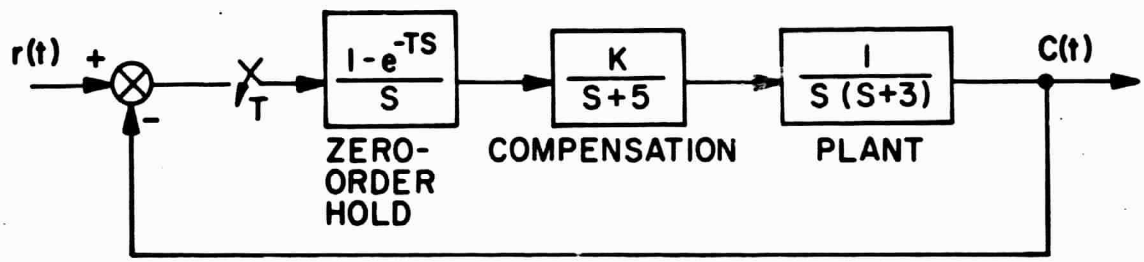

Figure 6.- Sampled-data system with continuous plant and compensation, and zero-order hold.

The root locus of this system, with K as the gain parameter, was generated in the Z-plane, W-plane, and S-plane with the aid of a general-purpose digital computer program. Sampling periods of 0.01, 0.1 and 1.0 second were examined. The Z-plane representation is shown in Figure 7; the W-plane representations in Figures 8, 9, and 10; and the S-plane representation in Figure 11. The 3 percent error boundaries are indicated on the Z-plane and W-planes. For ease of comparison the simplified S coordinates in each W-plane were scaled to conform with those in the S-plane.

In the Z-plane, only the locus with sampling period of 0.01 second lies within the 3 percent error boundary. The simple coordinate change here thus transforms the Z-plane into the S-plane. However, the sampling frequency is extremely high compared to the highest system frequency to warrant the simplification.

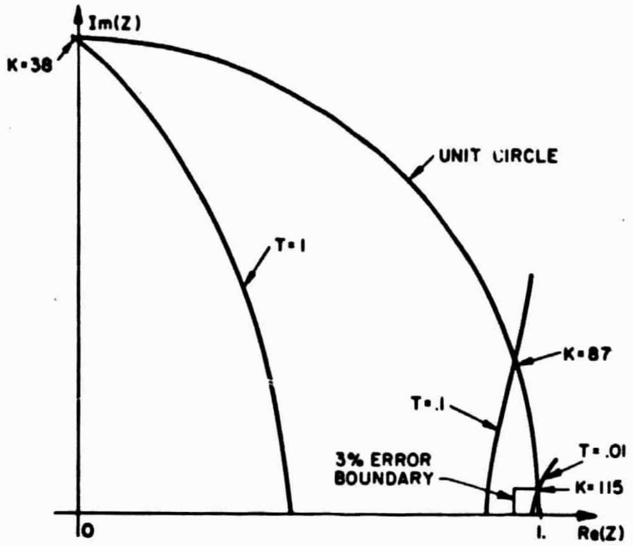

Figure 7.- Z-plane root locus branches of sampled-data system for different sampling periods

On the other hand, in the W-plane the system locus lies within the 3 percent error boundary when the sampling period is 0.1 second. (The sampling frequency is now about 20 times the highest system frequency.) For these cases, the loci in the W-plane are identical in interpretation to their respective loci in the S-plane and the simple coordinate change thus transforms the W-plane into the S-plane.

When the sampling period is 1.0 second the locus transcends the 3 percent error boundary. The locus outside this boundary is no longer subject to simple interpretation except for stability. This is substantiated by the distortion of the locus in the Z-plane and W-plane outside the 3 percent error boundary from the corresponding locus in the S-plane.

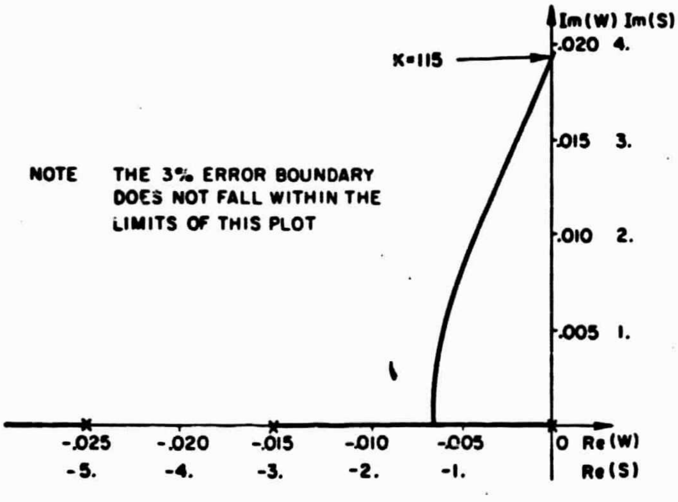

Figure 8.- W-plane root locus of sampled-data system, with simplified s coordinates (T = 0.1 second)

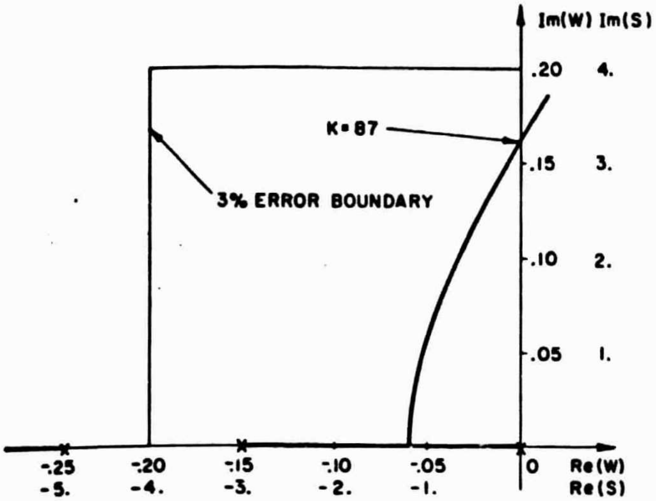

Figure 9.- W-plane root locus of sampled-data system, with simplified s coordinates (T = .1 second)

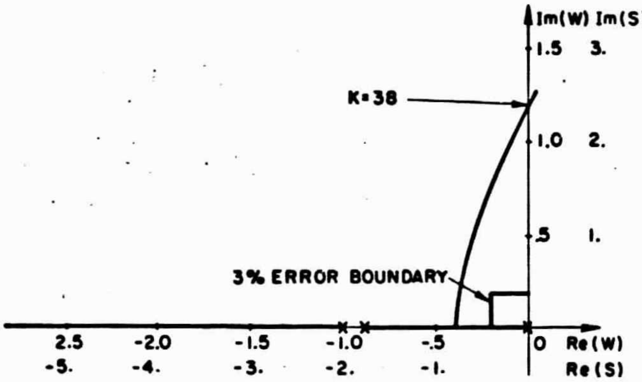

Figure 10.- W-plane root locus of sampled-data system, with simplified s coordinates (T = 1. second)

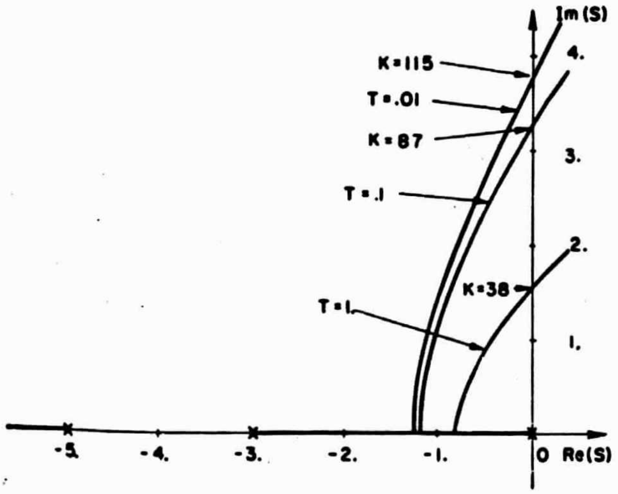

Figure 11.- S-plane root locus of sampled-data system for different sampling periods (Primary Strip)

GENERAL REFERENCES

1. Kuo, B. C.: Analysis and Synthesis of Sampled Data Control Systems. Prentice-Hall, Inc., 1963.

2. Truxal, J. G.: Automatic Feedback Control System Synthesis. McGraw-Hill, Inc., 1955.

3. Lindorff, D. P.: Theory of Sampled Data Control Systems. John Wiley & Sons, Inc., 1965.

4. Saucedo, R.; and Schiring, E. E.: Introduction to Continuous and Digital Control Systems. MacMillan Company, 1968.

5. Tou, J. T.: Digital and Sampled Data Control Systems. McGraw-Hill, Inc., 1959.

6. Savant, C. J., Jr.: Control System Design. McGraw-Hill, Inc., 1964.

7. Raven, F. H.: Automatic Control Engineering. McGraw-Hill, Inc., 1961.

8. Ragazzini, J. R.; and Franklin, G. F.: Sampled Data Control Systems. McGraw-Hill, Inc., 1958.

9. Lee, J. F. L.: A Digital Adaptive Flight Control System for Flexible Missiles. Presented at the 7th Symposium on Ballistic Missile and Space Technology (Colorado Springs, Colorado), 1962.