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ON THE MOTION OF LIQUIDS ENCLOSED IN AEROSPACE VEHICLE TANKS UNDER WEAK GRAVITATIONAL FIELDS

Volume I
THEORETICAL CONSIDERATIONS

by

L. L. Fontenot
UNIDEV, INC.
Huntsville, Alabama

March 1970
Contract No. NAS8-21272
UNIDEV Report UR-00010

Submitted to

George C. Marshall Space Flight Center
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Page 8: Equation (1.8_1) should read:

$$"\bar{F}_{01} = -\int_\tau \left(\frac{d}{dt}\frac{\partial T_*}{\partial \bar{v}} + \bar{\omega}x\frac{\partial T_*}{\partial \bar{v}}\right)\rho d\tau + \int_\tau \bar{f}_1\, \rho d\tau - \sigma\int_S \kappa(\zeta)\,\bar{n}dS$$

$$= -\left(\frac{d}{dt}\frac{\partial T_1}{\partial \bar{u}} + \bar{\omega}x\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{F}_1 - \sigma\oint_\Gamma (\bar{t}x\bar{n})\, d\Gamma\ ,"$$

Page 9: Equation (1.8_2) should read:

$$"\bar{N}_{01} = -\int_\tau \left[(\bar{\ell}+\bar{r})\ x\ \left(\frac{d}{dt}\frac{\partial T_*}{\partial \bar{v}} + \bar{\omega}x\frac{\partial T_*}{\partial \bar{v}}\right)\right]\rho d\tau + \int_\tau [(\bar{\ell}+\bar{r})x\bar{f}_1]\ \rho d\tau$$

$$- \sigma\int_S \kappa(\zeta)[(\bar{\ell}+\bar{r})x\bar{n}]\ dS$$

$$= -\left(\frac{d}{dt}\frac{\partial T_1}{\partial \bar{\omega}} + \bar{\omega}x\frac{\partial T_1}{\partial \bar{\omega}} + \bar{u}x\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{N}_1 - \sigma\oint_\Gamma [(\bar{t}x\bar{n})x(\bar{\ell}+\bar{r})]\ d\Gamma\ ,"$$

Page 23: See 2.1. Paragraph following should read: "As a first step in the development of the small deviant equations of motion, the general equations of motion are specialized to the reference state of motion."

Page 33: The left-hand members of the expressions given by (2.34) should read "$\delta\overline{F}_{01}$ and $\delta\overline{N}_{01}$" respectively.

Page 42: Line 1 given in expression (2.53) should read:

"$F_z = M(a+g),\ F_y = 0,\ F_x = 0,\ N_x = 0,\ N_y = 0,\ N_z = 0$"

Page 47: Footnote 16 should read: "Weak Gravitational Fields."

Page 55: The sign preceding the right-hand member of expression (3.18) should read: "$\mp$"

Page 59: Equation (3.31_1) should read:

"$\omega = \omega_m(1+\varepsilon\lambda+\varepsilon^2\mu)$"

Page 62: Equation (3.40_2) should read:

$$"P_{2k} = -R^2\varepsilon^2 \iint_Q (\mathrm{grad}N_m \cdot \mathrm{grad}N_k)dxdy - \frac{\omega_m^2}{a+g} h_0\delta_{mk} + \frac{\omega_m^2}{a+g} R^2\varepsilon^2 \cos\gamma \oint_\Gamma N_m N_k\, dS,"$$

Page 63: The first expression given by (3.44) should read:

"$i\omega_m\varepsilon^2 b_k - i\varepsilon^2\omega_k B_k = P_{1k}$," Page 65: The last term in the right-hand member of (3.48) should read:

$$" \frac{(a+g)^2}{\omega_m^4} (Q/\oint_\Gamma dS) \oint_\Gamma (\frac{\partial N_m}{\partial s})^2 \, dS \Big] \Big\} ."$$

Page 65: The approximation given in (3.49) should be preceded by a minus - .

Page 65: Expression (3.50_1) for $i \neq j$ should read:

$$"B_{ij} = \frac{\omega_i \omega_j}{\omega_i^2 - \omega_j^2} \bar{\beta}_{ij} , \quad i \neq j"$$

Page 66: Expression (3.50_2) for $i \neq j$ should read:

$$"b_{ij} = \frac{\omega_i^2}{\omega_i^2 - \omega_j^2} \beta_{ij} , \quad i \neq j"$$

Page 73: The frequency correction factor following equation (3.71) should read:

$$"\varepsilon^2 \mu_i = - \frac{1}{2} \varepsilon^2 \beta_{ii} = \frac{\sigma}{2\rho(a+g)} k_i^2), "$$

Page 81 thru 82: The collection of coefficients a_i^*, b_i^*, α_i, A should be called "Expression (3.88)."

Page 82: Item (4) should read: "Compute the coefficients called for in (3.83) and (3.84), namely (3.57), (3.67), (3.75), (3.76), (3.77), (3.79) and/or (3.88);"

Page 87: The coefficient preceding $\ddot{\theta}$ in (3.90_4) should read: " - A "

Page 88: The last term in the right-hand member of coefficient b_i^* occurring in (3.98) should read:

$$" \frac{1}{\Omega\gamma_i} R\cos\gamma \left[R \oint_\Gamma y\, N_i\, dS - \Omega\gamma_i \left(\frac{R}{Q} \oint_\Gamma dS\right) b_i \right] , "$$

Page 88: Coefficient f_i^* occurring in (3.98) should read:

$$"f_i^* = \frac{R^2}{\Omega\gamma_i} \cos\gamma \oint_\Gamma \frac{\partial s}{\partial y} \frac{\partial N_i}{\partial s}\, dS,"$$

Page 93: Equation (4.6_4) should read:

$$"(\delta\overline{N}_{01})_x = T_1' + \frac{1}{B_0} T_x;"$$

Page 93: Equation (4.8_2) should read:

$$"F_y = - M \sum_{n=1}^{\infty} \gamma_n b_n \ddot{\lambda}_n - M \sum_{n=1}^{\infty} \gamma_n [e_n^* \ddot{\xi}_n + \alpha_3 f_n^* \xi_n],"$$

Page 97: Equation (4.14) should read:

$$"\varepsilon^2 \mu_n = - \frac{1}{2B_0} \beta_{nn}"$$

FOREWORD

This report was prepared by UNIDEV, INC., Huntsville, Alabama, under Contract NAS8-21272, for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under technical direction of the Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center, with Mr. Frank Bugg, acting as Project Manager. This document deals with theoretical aspects of the problem. The computational scheme is presented in Volume II.

Technical contributions and helpful suggestions were made by Dr. D. Lomen, University of Arizona.

This study was executed under the direction of Dr. L. L. Fontenot, Program Manager and Principal Investigator; Mr. T. S. Chandler developed the Digital Computer Program described in Volume II.

ABSTRACT

This document deals with the behavior of liquids enclosed in moving tanks under weak gravitational fields. The complete equations of motion of an aerospace vehicle with partially liquid-filled cavities are derived, and the contributions of the liquid motions to the overall system motion defined. The vehicle is slightly disturbed from an assumed reference state of motion, and the small perturbation equations of motion derived. The contributions of the small liquid motions to the overall perturbation vehicle motion are defined and isolated. Using boundary layer and perturbation theory, the behavior of a liquid enclosed in a tank undergoing planar motion is determined. The fundamental results obtained constitute a first order correction of the "high-g" theory ($B_0 = \infty$). The equations are specialized to rotationally symmetric vessels, wherein the axis of symmetry coincides with the vehicle acceleration vector, and then cast in a form suitable for digital implementation. The forces, moments and free boundary displacement are given in terms of non-dimensional coefficients, which are expressible by means of the geometry of the vessel, properties of the liquid, contact angle and parameters associated with the "high-g" eigenoscillation problem.

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INTRODUCTION

Problems concerning the behavior of liquids in moving tanks under conditions of weightlessness and weak gravitational fields have gained topical importance recently. Theoretical studies which have appeared in the open literature deal mostly with the eigenoscillations of liquids in stationary containers under somewhat restrictive assumptions.

The main objective of this study is to develop methods for determining the behavior of liquids enclosed in moving tanks, such as abound in aerospace vehicle application, under weak gravitational fields. A further objective is to develop a digital computer program to calculate the pertinent liquid parameters in a manner similar to / 1 /.

In section 1, certain fundamental results and equations from classical mechanics are used to derive the general equations of motion of an aerospace vehicle with a partially liquid-filled cavity under the influence of a general force field. The contributions of the liquid motion to the overall system motion is defined and isolated. Diverse forms of the equations of motion are presented. The fundamental equations are specialized to a classical central force field, and the motion of a vehicle in a circular orbit discussed.

In section 2, the general equations of motion are specialized to a reference state of motion in the form of constant linear accelerations and angular velocities. Then, the vehicle is slightly disturbed from the known reference state motion and the resulting perturbation equations of general vehicle motion obtained. The contributions of the small liquid motions to the overall perturbation vehicle motion are defined and isolated. Various simplified but significant flight conditions and configurations are presented.

In section 3, assuming that the surface tension is small in comparison with the effective mass forces, a method for determining the behavior of a liquid enclosed in a tank is developed for a vehicle undergoing planar motion. The approach is based on the fundamental work of / 2 / (which is presented in part herein), and constitutes a first order correction of the so-called "high-g" theory ($B_0 = \infty$). Then, the equations are specialized to rotationally symmetric tanks, wherein the axis of symmetry coincides with the vehicle acceleration vector, and cast in a form suitable for digital implementation.

In section 4, the equations are transformed to agree with / 4 /, which forms the basis of the NASA-MSFC computer program for the "high-g" problem. The forces, moments and free boundary displacement are given in terms of non-dimensional coefficients, which are expressible by means of the geometry of the vessel, properties of the liquid, contact angle and parameters associated with the "high-g" eigenoscillation problem. The parameters for a right circular cylinder are worked out in detail.

1. ON THE GENERAL EQUATIONS OF LIQUID-FILLED AEROSPACE VEHICLE MOTION

It is well known that the forces and moments resulting from the action of liquid motions on the walls of partially filled cavities play a major role in stability and control of launch vehicles. To predict and control the motions of a vehicle with cavities partially filled with liquid, it is therefore necessary to identify the contributions of the liquid motion to the overall system motion. Methods for determining liquid motions in axisymmetric tanks, connected mainly with the design of the Saturn vehicle, have been considerably developed /1,3 /. Yet, because of the diverse configurations and modes of operation of aerospace vehicles, the applicability of these methods is sometimes questionable. To put the problems and questions associated with these methods in their proper perspective, we must examine them in detail.

In most applied problems dealing with the stability and control of motions of a launch vehicle with liquid cavities, one is interested mainly in the motion of the vehicle; the motion of the liquids in the cavities is of interest only inasmuch as it affects that of the vehicle. To be sure, these two aspects of the single stability and control problem are interrelated: the motion of the body depends on that of the liquids and vice versa. It is, therefore, natural to begin our discussion with the general formulation of the equations governing the motion of an aerospace vehicle with cavities partially or completely filled with liquids.

1.1 General Equations Governing the Motion of Launch Vehicles with Cavities Partially or Completely Filled with Liquid Propellants

Accordingly, imagine a free launch vehicle with a simply-connected cavity partially or completely filled with an ideal, homogeneous and incompressible liquid (propellant). Regarding the vehicle and the liquid as a single mechanical system, it is easy to set up its equations of motion in some form. In general, three vector equations and one scalar equation define the motion of the system. For a single tank configuration, these are: a vector equation governing the translational motion of, say, the center of mass; a vector equation governing the rotational motion of the vehicle about the center of mass, or about the geocenter; a vector equation governing the liquid motion inside the tank; and a scalar equation describing the continuity of liquid motion[1]. Thus, for example, if the motion of the vehicle is specified by the translational velocity $\overline{u}$ and angular velocity $\overline{\omega}$, the equations of motion can be written[2]

$$\frac{d}{dt}\frac{\partial(T_0+T_1)}{\partial\overline{u}} + \overline{\omega} \times \frac{\partial(T_0+T_1)}{\partial\overline{u}} = \overline{F} + (\overline{F}_0+\overline{F}_1) \quad ,$$

---

1 In general, equations governing the control and guidance systems, engines, propulsion system, etc. also are required. The inclusion of these equations in our discussion are not necessary. The theory can be modified readily to include the effects of draining propellants / 7 /.

2 We assume that the motions of both the vehicle and the liquid are continuous, so that the coordinates of each particle of the liquid are continuous functions of their initial values and time.

$$\frac{d}{dt}\frac{\partial(T_0+T_1)}{\partial\overline{\omega}} + \overline{\omega} \times \frac{\partial(T_0+T_1)}{\partial\overline{\omega}} + \overline{u} \times \frac{\partial(T_0+T_1)}{\partial\overline{u}} = \overline{N} + (\overline{N}_0+\overline{N}_1) \quad ,$$

$$\frac{d}{dt}\frac{\partial T_*}{\partial\overline{v}} + \overline{\omega} \times \frac{\partial T_*}{\partial\overline{v}} = \overline{f}_1 - \frac{1}{\rho}\,\text{grad}\,(p-p_0) \quad , \tag{1.1}$$

$$\text{div}\ \overline{v} = 0 \quad ,$$

where T_i is the kinetic energy of the body ($i=0$) and of the liquid ($i=1$), T_* is the kinetic energy density of the liquid, $\overline{F}$ is the resultant vector of external forces applied to the system[3], ρ is the mass density of the liquid, $p(x,y,z,t)$ is the liquid pressure, p_0 is the constant gas pressure in the tank, $\overline{f}_i$ is the vector of body forces per unit mass acting on the body ($i=0$) and on the liquid ($i=1$), $\overline{F}_i$ is the resultant vector of mass forces, $\overline{N}_i$ is the moment of mass forces, $\overline{v}(x,y,z,t)$ is the velocity of the liquid relative to the tank. The velocity of the liquid is

$$\overline{V} = \overline{u} + \overline{\omega}\times(\overline{\ell}+\overline{r}) + \overline{v} \quad ,$$

$$\overline{v} = \frac{d\overline{r}}{dt}\,[4] \tag{1.2}$$

---

3 The externally applied forces may be functions of the state variables of the system or preassigned functions of time, oriented in any arbitrary fashion with respect to vehicle body axes.

4 Differentiation relative to the moving coordinates is indicated by $d(\ldots)/dt$.

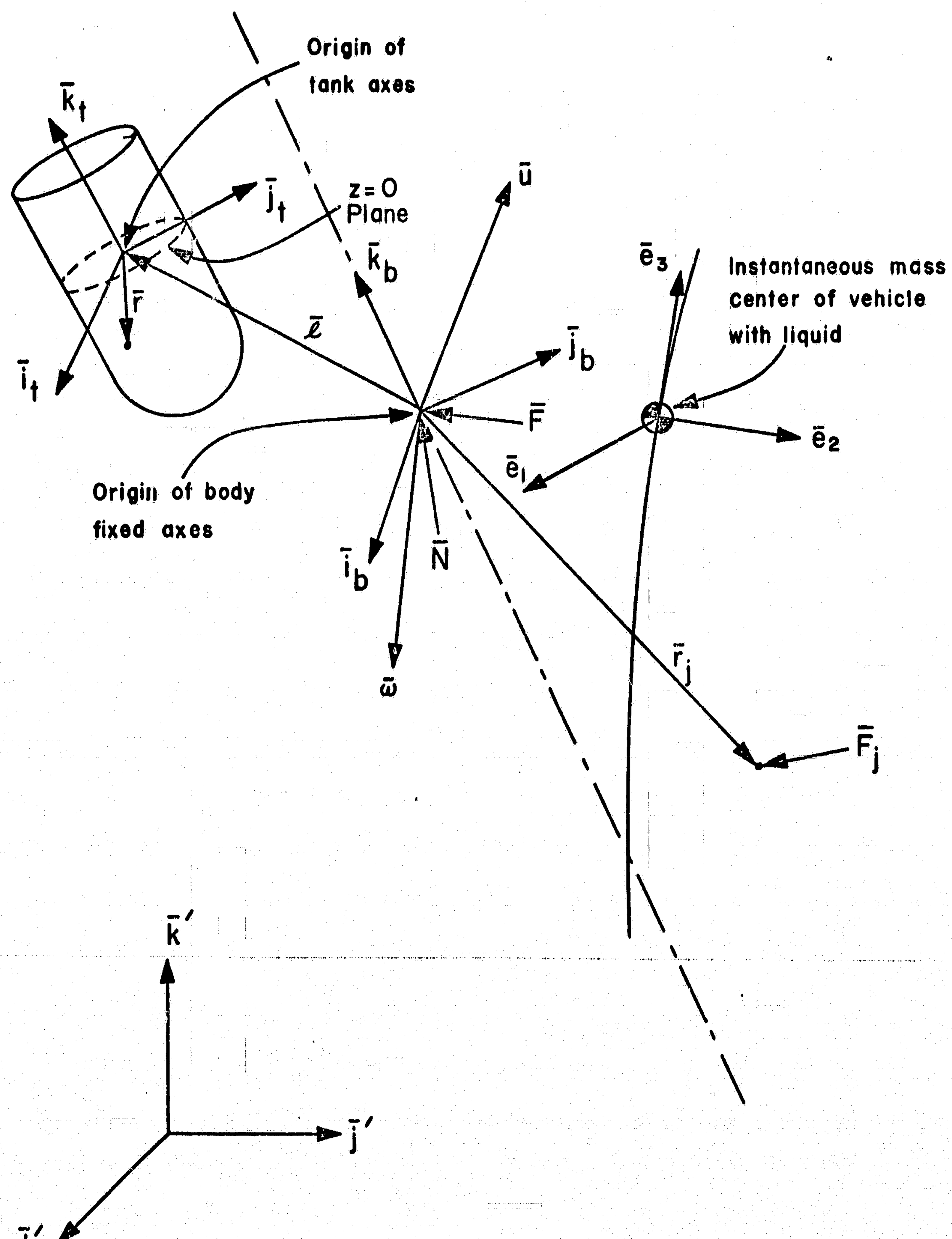

Figure 1. Geometry, Notation

where $\bar{r} = \bar{r}(x,y,z)$ is the position vector of the liquid relative to local tank axes, $\bar{\ell}$ is the position vector of the origin of liquid tank axes relative to vehicle body axes (Fig. 1). [For each additional tank, a separate vector and scalar equation is needed, i.e., (1.1_3) and (1.1_4).] Thus, for a single tank configuration, ten scalar equations determine the following ten unknowns: three components of the translational velocity $\bar{u}$, three components of the angular velocity $\bar{\omega}$ relative to inertial space, three components of the liquid velocity $\bar{v}$ relative to the vehicle tank, and, finally, the pressure inside the liquid enclosed by the tank.

1.2 Boundary Conditions

The liquid velocity relative to the cavity must satisfy certain kinematical conditions, namely, the component normal to the wetted surface(s) must vanish, and the component normal to the free boundary must equal the velocity of the surface normal to itself[5]. Thus, if we specify the equation of the free boundary in the form $z - \zeta(x,y,t) = 0$, we can write

$$v_n = \begin{cases} 0 & , \text{ at } \Sigma , \\ \zeta_t \cos(\bar{n},z) & , \text{ at } S^{6,7} , \end{cases} \tag{1.3}$$

---

[5] All surfaces are assumed to be piecewise smooth.

[6] $\zeta_t \equiv \partial\zeta/\partial t$.

[7] Equation (1.3), in conjunction with (1.1_4) and the divergence theorem yields the injunction $\int_S \zeta_t \cos(\bar{n},z)dS = \int_Q \zeta_t \, dxdy = 0$, where Q is the projection of S into the x,y-plane.

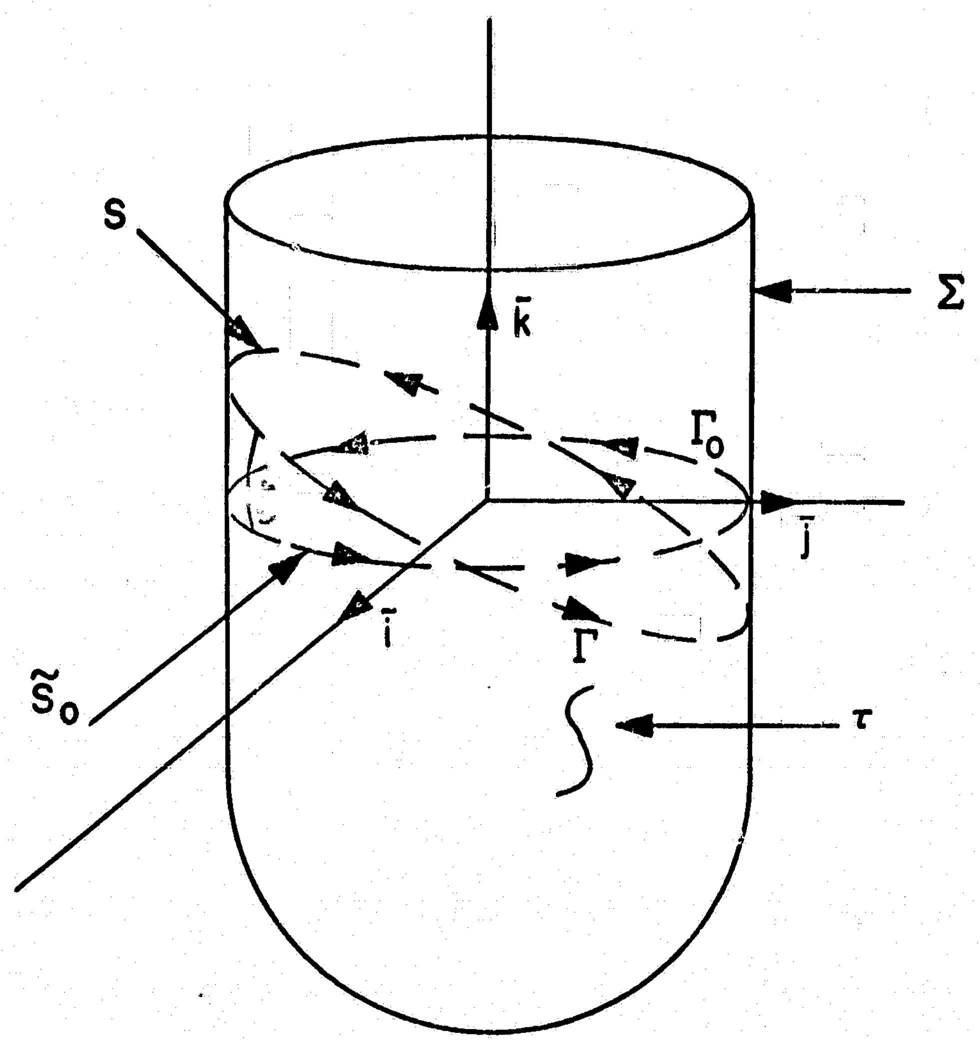

Figure 2. Cavity Geometry, Notation

where $\overline{n}$ is the unit vector along the outward drawn normal to the surface under consideration (see Fig. 2 for notations). Moreover, the pressure inside the cavity must satisfy a condition of thermodynamic equilibrium at the free boundary, which means that the discontinuity in pressure at the liquid-vapor interface is proportional to the product of the surface tension and twice the mean curvature of the surface. Now, the unit vector along the outward directed normal to S is

$$\overline{n} = \pm \frac{(-\zeta_x \overline{i}_t - \zeta_y \overline{j}_t + \overline{k}_t)}{(1+\zeta_x^2+\zeta_y^2)^{1/2}}, \tag{1.4}$$

where subscripts x,y here denote partial derivatives, so that this condition can be written

$$p - p_0 = \sigma\kappa(\zeta) = \pm\,\sigma \operatorname{div} \overline{n} = \pm\,\sigma \frac{\zeta_{xx}(1+\zeta_y^2) - 2\zeta_x\zeta_y\zeta_{xy} + \zeta_{yy}(1+\zeta_x^2)}{(1+\zeta_x^2+\zeta_y^2)^{3/2}}. \tag{1.5}$$

Here, the upper sign is taken if the liquid is below the surface $z=\zeta$, and the lower sign if it is above the surface $z=\zeta$. Finally, at points on the line of intersection of the free surface and the tank wall (the contour Γ), the normals to the two surfaces form a constant angle of contact which depends only on the material of the tank and properties of the liquid. If the equation of the wall is specified in the form $\Phi_\Sigma(x,y,z) = 0$, this condition can be written

$$\frac{\partial\Phi_\Sigma}{\partial z} - \frac{\partial\Phi_\Sigma}{\partial x}\zeta_x - \frac{\partial\Phi_\Sigma}{\partial y}\zeta_y = (1+\zeta_x^2+\zeta_y^2)^{1/2}|\operatorname{grad}\Phi_\Sigma|\cos\gamma\,, \tag{1.6}$$

or alternately,

$$\bar{n}_\Sigma \cdot \bar{n}_S = \cos\gamma \quad , \tag{1.7}$$

where $\bar{n}_\Sigma$ is the unit vector along the outward drawn normal to Σ and $\bar{n}_S$ the unit vector along the outward directed normal to S.

We note that condition (1.3_2) introduces yet another variable - the displacement of the free boundary ζ. But, because the physical condition at the liquid-vapor interface (1.5) also involves ζ, this variable can be eliminated in principle.

1.3 Forces and Torques Resulting From Action of Liquid

If the force resulting from the action of the liquid motion on the cavity surface Σ, and the moment about the origin of body axes of the forces exerted on the cavity surface Σ are required, we have

$$\bar{F}_{01} = -\int_\tau \left(\frac{d}{dt}\frac{\partial T_*}{\partial \bar{v}} + \bar{\omega}\times\frac{\partial T_*}{\partial \bar{v}}\right)\rho d\tau + \int_\tau \bar{f}_1 \, \rho d\tau - \sigma \int_S \kappa(\zeta)\,\bar{n} dS$$

$$= -\left(\frac{d}{dt}\frac{\partial T_1}{\partial \bar{u}} + \bar{\omega}\times\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{F}_1 - \sigma\int_\Gamma (\bar{t}\times\bar{n})\, d\Gamma \ ,$$

$$\bar{N}_{01} = -\int_\tau [(\bar{\ell}+\bar{r}) \times (\frac{d}{dt}\frac{\partial T_*}{\partial \bar{v}} + \bar{\omega}\times\frac{\partial T_*}{\partial \bar{v}})]\,\rho d\tau + \int_\tau [(\bar{\ell}+\bar{r})\times\bar{f}_1]\,\rho d\tau$$
$$- \sigma\int_S \kappa(\zeta)[(\bar{\ell}+\bar{r})\times\bar{n}]\, dS \tag{1.8}$$

$$= -\left(\frac{d}{dt}\frac{\partial T_1}{\partial \overline{\omega}} + \overline{\omega}x\frac{\partial T_1}{\partial \overline{\omega}} + \overline{u}x\frac{\partial T_1}{\partial \overline{u}}\right) + \overline{N}_1 - \sigma\int_\Gamma [(\overline{t}x\overline{n})x(\overline{\ell}+\overline{r})]\, d\Gamma ,$$

where $\overline{n}$ is the unit vector along the outward drawn normal to S, $\overline{t}$ is the tangent vector to contour Γ formed by the points of the line of intersection of the surfaces S and Σ, and τ is the volume of liquid occupying the tank.

1.4 Irrotational Liquid Motion

The problem can be simplified if the function u_1 representing body forces per unit mass acting on the liquid does not depend explicitly on time and if the motion of the liquid is irrotational. Thus, curl $\overline{V} = 0$, and the velocity relative to the tank can be written /3 /

$$\overline{v} = \mathrm{grad}\phi - \overline{\omega}x(\overline{\ell}+\overline{r}) , \tag{1.9}$$

and leads, as is easily seen, to the equations

$$\Delta\phi = 0, \quad \text{throughout } \tau , \tag{1.10}$$

$$\frac{\partial\phi}{\partial n} = \begin{cases} \overline{\omega}\cdot[(\overline{\ell}+\overline{r})x\overline{n}] & , \text{at } \Sigma , \\ \\ \overline{\omega}\cdot[(\overline{\ell}+\overline{r})x\overline{n}] + \zeta_t \cos(\overline{n},z) & , \text{at } S , \end{cases} \tag{1.11}$$

where Δ is the Laplace operator.

Introduction of the kinetic energy density

$$2T_* = \overline{u}\cdot\frac{\partial T_*}{\partial \overline{u}} + \overline{\omega}\cdot\frac{\partial T_*}{\partial \overline{\omega}} + \overline{v}\cdot\frac{\partial T_*}{\partial \overline{v}} = |\overline{u}+\overline{\omega}x(\overline{\ell}+\overline{r})+\overline{v}|^2 , \tag{1.12}$$

into (1.1_3) and subsequent application of the Cauchy-Lagrange integral, gives

$$\frac{p-p_0}{\rho} + \frac{\partial\phi}{\partial t} + u_1 + [\frac{d\bar{u}}{dt} + \bar{\omega}x\bar{u}]\cdot(\bar{\ell}+\bar{r}) + \frac{1}{2}v^2 - \frac{1}{2}|\bar{\omega}x(\bar{\ell}+\bar{r})|^2 = c(t) \quad , \tag{1.13}$$

where $\frac{d\bar{u}}{dt} + \bar{\omega}x\bar{u}$ is the acceleration of the origin of body axes. Thermodynamic equilibrium at the free boundary (1.5) can be written

$$\frac{\partial\phi}{\partial t} + u_1 + [\frac{d\bar{u}}{dt} + \bar{\omega}x\bar{u}]\cdot(\bar{\ell}+\bar{r}) + \frac{1}{2}v^2 - \frac{1}{2}|\bar{\omega}x(\bar{\ell}+\bar{r})|^2 + \frac{\sigma}{\rho}\kappa(\zeta) = c(t) \quad , \tag{1.14}$$

at $z=\zeta$.

Thus, for a single tank configuration, the number of variables are reduced to seven: three components of the translational velocity $\bar{u}$, three components of the angular velocity $\bar{\omega}$, and the scalar potential ϕ. The scalar potential is harmonic throughout the volume τ (1.10) and satisfies inhomogeneous Neumann conditions at the tank wall Σ (1.11_1) and a mixed condition at the liquid-vapor interface (1.11_2). If the cavity is completely filled, boundary conditions (1.11) with $\zeta\equiv 0$ may be satisfied by writing

$$\phi = \bar{\omega}\cdot\bar{\phi} \tag{1.15}$$

where $\bar{\phi}$ (Stokes' potential) is a vector whose components along the Cartesian axes are solutions of the Laplace equation (1.10) and satisfy boundary conditions,

$$\frac{\partial\bar{\phi}}{\partial n} = [(\bar{\ell}+\bar{r})\times\bar{n}], \quad \text{at } \Sigma+S \quad , \tag{1.16}$$

so that $\bar{\phi}$ depends solely on the shape of the cavity and not on its motion /3 /. The equation for the pressure in the liquid takes the form

$$\frac{p-p_0}{\rho} + \dot{\bar{\omega}}\cdot\bar{\phi} + u_1 + [\frac{d\bar{u}}{dt} + \bar{\omega}\times\bar{u}]\cdot(\bar{\ell}+\bar{r}) - \frac{1}{2}|\bar{\omega}\times(\bar{\ell}+\bar{r})|^2 = c(t). \tag{1.17}$$

1.5 Diverse Forms of the Equations of Motion

To facilitate further discussion, we examine more familiar forms of the general equations (1.1) et seq. Accordingly, the kinetic energy of the body and of the liquid can be written

$$\begin{aligned} 2T_0 &= \frac{\partial T_0}{\partial\bar{u}}\cdot\bar{u} + \frac{\partial T_0}{\partial\bar{\omega}}\cdot\bar{\omega} = \int |\bar{u}+\bar{\omega}\times\bar{r}'|^2 \, dm \quad , \\ 2T_1 &= 2\int_\tau T_* \, \rho d\tau \quad , \end{aligned} \tag{1.18}$$

where T_* is defined by (1.12), so that the equations of motion of the vehicle containing liquid (1.1_1) and (1.1_2) can be cast in the forms

$$\begin{aligned} M\ddot{\bar{R}}_* &= \bar{F} + (\bar{F}_0+\bar{F}_1) \quad , \\ I\cdot\dot{\bar{\omega}} + \bar{\omega}\times(I\cdot\bar{\omega}) + M(\bar{r}_*\times\ddot{\bar{R}}_*) &+ \int_\tau [(\bar{\ell}+\bar{r}-\bar{r}_*) \times (\frac{d\bar{v}}{dt} + 2\bar{\omega}\times\bar{v})] \, \rho d\tau = \\ &\bar{N} + (\bar{N}_0+\bar{N}_1) \quad , \end{aligned} \tag{1.19}$$

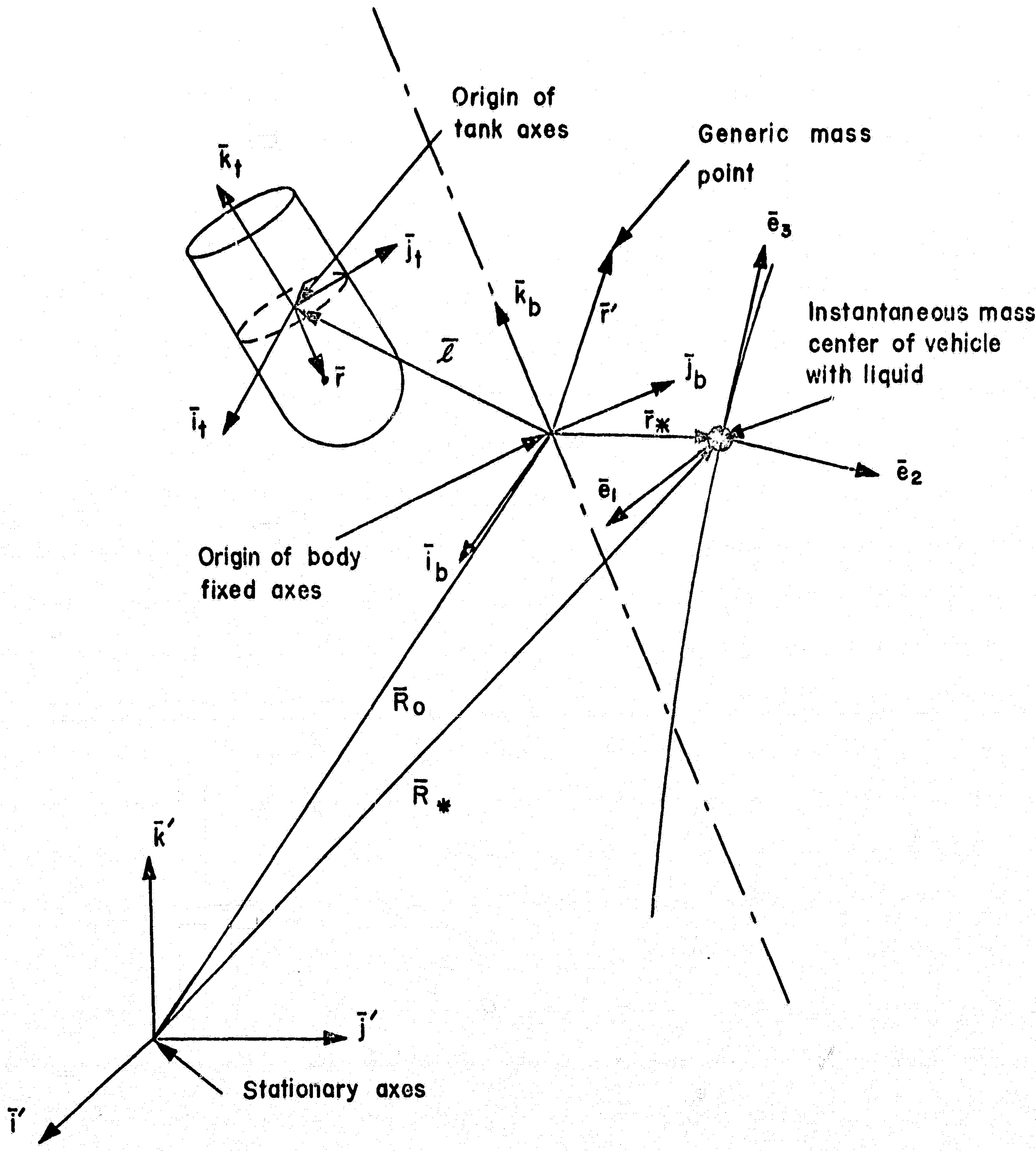

Figure 3. Position of mass center, coordinate axes

where $\bar{R}_*$ is the instantaneous position of the mass center of the vehicle relative to a system of stationary coordinate axes, $\bar{r}_*$ is the instantaneous position of the mass center relative to the system of axes rigidly attached to the vehicle (Fig. 3), M is the mass of the vehicle with liquid,

$$M = \int dm + \int_\tau \rho d\tau, \quad M\bar{r}_* = \int \bar{r}'\, dm + \int_\tau (\bar{\ell}+\bar{r})\, \rho d\tau, \quad \bar{F}_0 = \int \bar{f}_0\, dm,$$

$$\bar{F}_1 = \int_\tau \bar{f}_1\, \rho d\tau, \quad \bar{N}_0 = \int (\bar{r}' x \bar{f}_0)\, dm, \quad \bar{N}_1 = \int_\tau [(\bar{\ell}+\bar{r}) x \bar{f}_1]\, \rho d\tau, \tag{1.20}$$

$$\bar{F} = \sum_j \bar{F}_j, \quad \bar{N} = \sum_j (\bar{r}_j x \bar{F}_j),$$

$$\bar{u} = \dot{\bar{R}}_* - \dot{\bar{r}}_*, \quad \frac{d\bar{u}}{dt} + \bar{\omega} x \bar{u} = \ddot{\bar{R}}_* - \ddot{\bar{r}}_*$$

I is the inertia dyadic of the body with liquid and, therefore, is a point function of the mass elements and associated position vectors of the body and the liquid. I can be written in partitioned form

$$I = I_0 + I_1 - MI_* \quad , \tag{1.21}$$

where[8]

$$I_0 = \int [|\bar{r}'|^2 E - \bar{r}';\bar{r}']\, dm \quad ,$$

$$I_1 = \int_\tau [|\bar{\ell}+\bar{r}|^2 E - (\bar{\ell}+\bar{r});(\bar{\ell}+\bar{r})]\, \rho d\tau \quad , \tag{1.22}$$

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8 The indefinite or dyadic product is defined by $\bar{a};\bar{b}$.

$I_* = |\bar{r}_*|^2 E - \bar{r}_* ; \bar{r}_*$, (E is the idemfactor) .

I_0 is the momental dyadic of the solid whose components have constant values when referred to the system of moving coordinates, I_1 is the momental dyadic of the liquid whose components are functions of the form of the liquid inside the cavity, and I_* is the inertia tensor whose components are functions of the instantaneous position of the center of mass relative to the body axes.

Note that if $\bar{v} \equiv 0$, $\bar{r}_* \equiv 0$, expression (1.19_2) reduces to the classical Euler equation for general rigid body motion, as it should. Clearly, (1.19_2) represents the rotational equation of motion of the vehicle with liquid about its center of mass. To write the rotational equation of motion of the vehicle about the origin of the stationary system of coordinate axes, we note that the system kinetic energy $T = T_0 + T_1$ and angular momentum $\bar{D}$ about the origin of the stationary coordinate axes can be expressed as

$$T = T' + \bar{D}' \cdot \bar{\omega} + \frac{1}{2} \bar{\omega} \cdot I' \cdot \bar{\omega} ,$$

$$\bar{D} = \bar{D}' + I' \cdot \bar{\omega} \tag{1.23}$$

where

$$T' = \frac{1}{2} M \left[\left(\frac{dR_*}{dt}\right)^2 - \left(\frac{d\bar{r}_*}{dt}\right)^2 \right] + \frac{1}{2} \int_\tau v^2 \rho d\tau ,$$

$$\overline{D}' = M\,[\overline{R}_* \times \frac{dR_*}{dt}] + \int_\tau [(\overline{\ell}+\overline{r}-\overline{r}_*)\times\overline{v}]\,\rho d\tau, \tag{1.24}$$

$$I'\cdot\overline{\omega} = M\,[\overline{R}_*\times(\overline{\omega}\times\overline{R}_*)] + I\cdot\overline{\omega}\ ,$$

($\overline{D}'$ is the relative angular momentum about the origin of the stationary coordinate axes), so that from the fundamental equation of motion for the total angular momentum

$$I\cdot\dot{\overline{\omega}} + \overline{\omega}\times(I\cdot\overline{\omega}) + M(\overline{R}_*\times\ddot{\overline{R}}_*) + \int_\tau [(\overline{\ell}+\overline{r}-\overline{r}_*)\times(\frac{d\overline{v}}{dt} + 2\overline{\omega}\times\overline{v})]\,\rho d\tau =$$

$$\sum_j [(\overline{R}_*-\overline{r}_*+\overline{r}_j)\times\overline{F}_j] + \int [(\overline{R}_*-\overline{r}_*+\overline{r}')\times\overline{f}_0]\,dm + \int_\tau [(\overline{R}_*-\overline{r}_*+\overline{\ell}+\overline{r})\times\overline{f}_1]\,\rho d\tau, \tag{1.25}$$

which may be deduced equally well from (1.19) and (1.20). If the equation of rotational motion about the origin of the body axes is required, we have

$$(I_0+I_1)\cdot\dot{\overline{\omega}} + \overline{\omega}\times[(I_0+I_1)\cdot\overline{\omega}] + M(\overline{r}_*\times\ddot{\overline{R}}_0) + \int_\tau [(\overline{\ell}+\overline{r})\times(\frac{d\overline{v}}{dt} + 2\overline{\omega}\times\overline{v})]\,\rho d\tau =$$

$$\overline{N} + (\overline{N}_0+\overline{N}_1)\quad , \tag{1.26}$$

where

$$\ddot{\overline{R}}_0 = \ddot{\overline{R}}_* - \ddot{\overline{r}}_*\quad . \tag{1.27}$$

In particular, if the motion of the liquid is irrotational, (1.26) assumes the form

$$I_0\cdot\dot{\overline{\omega}} + \overline{\omega}x(I_0\cdot\overline{\omega}) + M(\overline{r}_*x\ddot{\overline{R}}_0) + \int_\tau [(\overline{\ell}+\overline{r}) \ x \ (\frac{d}{dt}\text{grad}\phi+\overline{\omega}x\text{grad}\phi)] \ \rho d\tau =$$

$$\overline{N} + (\overline{N}_0+\overline{N}_1) \qquad (1.28)$$

The forces and moments resulting from the action of the liquid motion on the cavity surface Σ (1.8) can be rewritten

$$\overline{F}_{01} = -\int_\tau \left\{\frac{d\overline{v}}{dt} + 2\overline{\omega}x\overline{v} + \dot{\overline{\omega}}x(\overline{\ell}+\overline{r}) + \overline{\omega}x[\overline{\omega}x(\overline{\ell}+\overline{r})]\right\} \rho d\tau - M_1\ddot{\overline{R}}_0$$

$$+ \int_\tau \overline{f}_1 \ \rho d\tau - \sigma\int_S \kappa\overline{n}dS \ , \qquad (1.29)$$

$$\overline{N}_{01} = -\left\{I_1\cdot\dot{\overline{\omega}} + \overline{\omega}x(I_1\cdot\overline{\omega}) - \ddot{\overline{R}}_0 \ x \int_\tau (\overline{\ell}+\overline{r}) \ \rho d\tau\right.$$

$$\left. + \int_\tau [(\overline{\ell}+\overline{r}) \ x \ (\frac{d\overline{v}}{dt} + 2\overline{\omega}x\overline{v})] \ \rho d\tau\right\} + \int_\tau [(\overline{\ell}+\overline{r})x\overline{f}_1] \ \rho d\tau$$

$$- \sigma\int_S \kappa \ [(\overline{\ell}+\overline{r})x\overline{n}] \ dS.$$

1.6 Vehicle Motion in an Inverse Square Law Field

The equations of the vehicle with liquid in an inverse square law field admit, under certain conditions, a first integral which will prove to be useful in control and stability investigations.

Now, in a classical inverse square law field, (1.19_1) and (1.25) take the forms

$$M\ddot{\overline{R}}_* = \overline{F} + (\overline{F}_0+\overline{F}_1) \quad , \qquad (1.30)$$

$$I\cdot\dot{\overline{\omega}} + \overline{\omega}x(I\cdot\overline{\omega}) + M(\overline{R}_*x\ddot{\overline{R}}_*) + \int_\tau [(\overline{\ell}+\overline{r}-\overline{r}_*) \times (\frac{d\overline{v}}{dt} + 2\overline{\omega}x\overline{v})]\ \rho d\tau =$$

$$\overline{N} + (R_*-\overline{r}_*)x\overline{F}$$

where

$$u_1 = -\frac{k}{\alpha_1},\quad \overline{f}_1 = -k\frac{\overline{\alpha}_1}{\alpha_1^3},\quad \overline{f}_0 = -k\frac{\overline{\alpha}_0}{\alpha_0^3}, \tag{1.31}$$

$$\overline{\alpha}_1 = \overline{R}_* - \overline{r}_* + \overline{\ell}+\overline{r},\quad \overline{\alpha}_0 = \overline{R}_* - \overline{r}_* + \overline{r}' ,$$

while the rotational equation about the center of mass becomes

$$I\cdot\dot{\overline{\omega}} + \overline{\omega}x(I\cdot\overline{\omega}) + \int_\tau [(\overline{\ell}+\overline{r}-\overline{r}_*) \times (\frac{d\overline{v}}{dt} + 2\overline{\omega}x\overline{v})]\ \rho d\tau = \overline{N} - \overline{r}_*x\overline{F}$$

$$- \overline{R}_* \times (\overline{F}_0+\overline{F}_1) . \tag{1.32}$$

If we specify that the body is isolated from applied forces, $\overline{F}=0$, or if the moment of applied forces about the origin of the system of stationary coordinate axes vanishes, $\overline{N}+(\overline{R}_*-\overline{r}_*)x\overline{F} = 0$, the angular momentum $\overline{D}$ about the origin of the stationary coordinate axes is conserved

$$\overline{D} = I\cdot\overline{\omega} + M(\overline{R}_*x\dot{\overline{R}}_*) + \int_\tau [(\overline{\ell}+\overline{r}-\overline{r}_*)x\overline{v}]\ \rho d\tau = \overline{c} , \tag{1.33}$$

and the kinetic energy takes the form

$$T = T' + \overline{\omega}\cdot(\overline{c} - \tfrac{1}{2}I'\cdot\overline{\omega}) , \tag{1.34}$$

which follows from (1.23) and (1.24). $\overline{c}$ is a constant vector fixed in space.

Now, the position of the vehicle and the liquid relative to the stationary system of coordinate axes $0'x'y'z'$ can be defined by the Lagrangian coordinates of the body q_j ($j = 1,2, \ldots . n \leq 6$) and relative x,y,z coordinates of the liquid, which are infinite in number[9]. Thus, because the mass forces occurring in (1.30_1) and (1.32) are functions of the Lagrangian coordinates of the vehicle and form of the liquid inside the cavity, the trajectory of the mass center (in the absence of applied forces) depends on the attitude and motion of the vehicle relative to the mass center.

1.7 Uniform Steady-State Rotation About Geocenter

If the applied forces and constraints are such that the vehicle with liquid performs a uniform steady-state rotation ω_0 as a single rigid body about a stationary axis $\bar{\nu}$ through the geocenter (circular orbit), the system is in equilibrium relative to the moving coordinate axes, and (1.1_1), (1.1_2) can be written

$$\bar{\omega} \times \frac{\partial(T_0+T_1)}{\partial \bar{u}} = \bar{F} + (\bar{F}_0+\bar{F}_1) \quad ,$$

$$\bar{\omega} \times \frac{\partial(T_0+T_1)}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial(T_0+T_1)}{\partial \bar{u}} = \bar{N} + (\bar{N}_0+\bar{N}_1) \quad , \tag{1.35}$$

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9 In general, when no assumptions are made on the motion of the liquid in the cavity other than the natural assumptions of continuity and compactness, the state of the system is described by an infinite number of variables.

where

$$\frac{d\bar{u}}{dt} = 0, \quad \frac{d\bar{\omega}}{dt} = 0, \quad \bar{v} \equiv 0, \quad \bar{u} = \bar{\omega}x(\bar{R}_{*}-\bar{r}_{*}) \ . \tag{1.36}$$

Combining (1.35_1) and (1.35_2) gives

$$\bar{\omega}x \frac{\partial(T_0+T_1)}{\partial\bar{\omega}} + \bar{u} \times \frac{\partial(T_0+T_1)}{\partial\bar{u}} + (\bar{R}_{*}-\bar{r}_{*}) \times \left[\bar{\omega} \times \frac{\partial(T_0+T_1)}{\partial\bar{u}}\right] = 0 \ , \tag{1.37}$$

for the rotational motion of the system. Now, (1.37) is equivalent to the vector equation

$$\bar{\omega} \times \left\{ I\cdot\bar{\omega} + M[\bar{R}_{*}x(\bar{\omega}x\bar{R}_{*})] \right\} = 0 \quad ,$$

that is,

$$I\cdot\bar{\omega} + M[\bar{R}_{*}x(\bar{\omega}x\bar{R}_{*})] = \bar{c} \tag{1.38}$$

where $\bar{c}$ is a constant vector fixed in space. Equation (1.38) can be deduced equally well from (1.33) with $\bar{v}\equiv 0$. Expression (1.35_1) is equivalent to the vector equation

$$M[\bar{\omega}x(\bar{\omega}x\bar{R}_{*})] = \bar{F} + (\bar{F}_0+\bar{F}_1) \tag{1.39}$$

which, when combined with (1.38) yields

$$M \frac{[\bar{\nu}x(\bar{\nu}x\bar{R}_{*})]}{I_{\nu\nu}^2} c_0^2 = \bar{F} + (\bar{F}_0+\bar{F}_1) \tag{1.40}$$

for the Lagrangian coordinates of the vehicle relative to the mass center in steady-state motion, where

$$\bar{c} = c_0\bar{\nu}, \qquad \bar{\omega} = \omega_0\bar{\nu}, \qquad \omega_0 = c_0/I_{\nu\nu},$$

$$I_{\nu\nu} = M\bar{\nu}\cdot[\bar{R}_* x(\bar{\nu} x \bar{R}_*)] + \bar{\nu}\cdot I\cdot\bar{\nu} \; . \tag{1.41}$$

$I_{\nu\nu}$ is the moment of inertia of the system about the axis $\bar{\nu}$.

Now $\bar{v}\equiv 0$ throughout the volume of the liquid τ_0 and from (1.3_2) $\zeta_t=0$, so that the free boundary displacement can be written

$$\zeta = - h_0 + h(x,y) \; , \tag{1.42}$$

where h_0 is a constant. Furthermore, from (1.9),

$$\text{grad}\phi = \bar{\omega}x(\bar{\ell}+\bar{r}) \; , \tag{1.43}$$

so that, analogous to (1.15) et seq., we can write

$$\phi = \bar{\omega}\cdot\bar{\phi} \tag{1.44}$$

with

$$\Delta\bar{\phi} = 0 \qquad , \quad \text{throughout } \tau_0 \; ,$$

$$\frac{\partial\bar{\phi}}{\partial n} = [(\bar{\ell}+\bar{r})x\bar{n}] \; , \quad \text{at } \Sigma+S_0 \; . \tag{1.45}$$

The equation for the pressure in the liquid is obtained from (1.13), noting (1.42), (1.44) and $\dot{\bar{\omega}}=0$,

$$\frac{p-p_0}{\rho} + u_1 + (\bar{\omega}x\bar{u})\cdot(\bar{\ell}+\bar{r}) - \frac{1}{2}|\bar{\omega}x(\bar{\ell}+\bar{r})|^2 = 0 \tag{1.46}$$

which gives

$$u_1 + (\bar{\omega}\times\bar{u})\cdot(\bar{\ell}+\bar{r}) - \frac{1}{2}|\bar{\omega}\times(\bar{\ell}+\bar{r})|^2 + \frac{\sigma}{\rho}\kappa(h) = 0 \quad , \tag{1.47}$$

at $z = -h_0 + h(x,y)$, for the determination of the liquid-vapor interface (1.42), if the liquid does not fill the cavity completely. With $\bar{\omega} = \omega_0\bar{\nu}$, (1.47) assumes the form

$$\omega_0^2[(\bar{\ell}+\bar{r})\cdot(\bar{\nu}\times[\bar{\nu}\times(\bar{R}_*-\bar{r}_*)]) - \frac{1}{2}|\bar{\nu}\times(\bar{\ell}+\bar{r})|^2 + u_1 + \frac{\sigma}{\rho}\kappa(h) = 0 \quad , \tag{1.48}$$

at $z = -h_0 + h(x,y)$. The boundary condition for (1.47) or (1.48) is provided by specifying the edge angle on contour Γ_0, condition (1.7), where the free surface touches the wall. Constant h_0 can be found by noting that the liquid volume is constant.

For a fixed value of the area parameter c_0 and equations of constraint, (1.40) and (1.48) determine the coordinates of the vehicle and the form of the free surface of the liquid in uniform steady-state rotation (circular orbit).

We now proceed to develop the small perturbation equations of motion of a space vehicle with liquid.

2. SMALL-PERTURBATION EQUATIONS OF LIQUID-FILLED AEROSPACE VEHICLE MOTION

The equations that govern the motion of the vehicle with liquid-filled tanks are non-linear, and do not admit general, exact solutions. Nevertheless, it is sometimes possible to obtain or approximate particular solutions, such as steady-state motion in a circular orbit derived previously. Then, if the system is assumed to be slightly disturbed from this known condition, the resulting small motion of deviation will be given by a set of linear differential equations. The usual method of construction of the equations of deviation is, briefly, as follows. Let the complete non-linear equations of motion of the vehicle with liquid be written for two states of motion: reference and disturbed. The reference or 'undisturbed' state describes the path of the center of mass, attitude of the vehicle, and the form of the shape of the free surface of the liquid[10]. The disturbed state describes the actual motion of the system, and consists of the reference motion plus small deviant motions therefrom. The difference between the equations of motion of the disturbed and reference states, upon neglecting all terms involving products of the deviant quantities or their derivatives, yields the equations of deviation.

The reference state of motion embodying the particular attitude that we wish the vehicle to have is, in general, one of non-uniformity. An approximation is made here, in the

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[10] Vehicle attitude control implies the existence at some time of reference axes embodying the particular attitude that we wish the vehicle to have. Clearly, if there were no such ideal attitude, we could allow the attitude to be that resulting from the action of the laws of motion, obviating the need for attitude control.

form of uniform linear and constant angular velocities[11]. With the foregoing approximations to the actual reference state of motion, and assuming further no velocity of liquid relative to the tank in the reference state of motion, we have

$$\ddot{\overline{R}}_* = \overline{a}, \quad \dot{\overline{\omega}} = 0, \quad \overline{v} = 0. \tag{2.1}$$

2.1 Reference State of Motion

As a first step in the development of the small deviant equations of deviant motion, the general equations of motion are specialized to the reference state of motion. Applying (2.1) to (1.19),

$$\begin{aligned} M\ddot{\overline{R}}_* &= \overline{F} + (\overline{F}_0+\overline{F}_1), \quad \ddot{\overline{R}}_* = \overline{a} \text{ (constant)}, \\ \overline{\omega}x(I\cdot\overline{\omega}) + M(\overline{r}_*x\ddot{\overline{R}}_*) &= \overline{N} + (\overline{N}_0+\overline{N}_1), \end{aligned} \tag{2.2}$$

which states that the system is in equilibrium relative to the moving coordinates attached to the body. Now, from (2.1) $\overline{v}\equiv 0$ throughout the volume of the liquid τ_0 and from (1.3_2) $\zeta_t=0$, so that the free boundary displacement can be written

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11 It is reasoned that for many practical problems involving alignment of reference axes with the local earth vertical, the line of sight to another vehicle or satellite, etc., reference axes linear accelerations and angular velocities will not vary appreciably during several cycles of short-period closed-loop attitude motion. Moreover, this approximation does not rule out rapid attitude changes due to commands.

$$\zeta = -h_0 + h(x,y) , \tag{2.3}$$

where h_0 is a constant. Furthermore, from (2.1) and (1.9),

$$\text{grad}\phi = \overline{\omega}\times(\overline{\ell}+\overline{r}) \tag{2.4}$$

so that analogous to (1.15) et seq., we can write

$$\phi = \overline{\omega}\cdot\overline{\phi}, \tag{2.5}$$

with

$$\begin{aligned} \Delta\overline{\phi} &= 0 \quad , \quad \text{throughout } \tau_0 \\ \frac{\partial\overline{\phi}}{\partial n} &= [(\overline{\ell}+\overline{r})\times\overline{n}] \quad , \quad \text{at } \Sigma+S_0 . \end{aligned} \tag{2.6}$$

The equation for the pressure in the liquid is obtained from (1.13), noting (2.1), (2.3) and (2.5),

$$\frac{p-p_0}{\rho} + u_1 + (\ddot{\overline{R}}_*-\ddot{\overline{r}}_*)\cdot(\overline{\ell}+\overline{r}) - \frac{1}{2}|\overline{\omega}\times(\overline{\ell}+\overline{r})|^2 = 0 , \tag{2.7}$$

which gives

$$u_1 + (\ddot{\overline{R}}_*-\ddot{\overline{r}}_*)\cdot(\overline{\ell}+\overline{r}) - \frac{1}{2}|\overline{\omega}\times(\overline{\ell}+\overline{r})|^2 + \frac{\sigma}{\rho}\kappa(h) = 0 , \tag{2.8}$$

at $z = -h_0+h$, for the determination of the liquid-vapor interface (2.3), if the liquid does not fill the cavity completely. The boundary condition for (2.8) is provided by specifying the edge angle on contour Γ_0, where the surface touches the wall,

$$\bar{n}_\Sigma \cdot \bar{n}_{S_0} = \cos\gamma \ . \tag{2.9}$$

The constant h_0 can be found by noting the volume of liquid is constant.

If the attitude of the vehicle is prescribed, then (2.2) and (2.8) define the force system and form of the liquid necessary to maintain that attitude. On the other hand, if the force system is prescribed, the same equations determine the resulting orientation of the vehicle and form of the liquid in the tank.

2.2 Perturbation Equations

In the reference state of motion defined by (2.1), the vehicle with a liquid-filled cavity can be associated with a certain solid vehicle, which we call transformed, consisting of the given vehicle and the frozen liquid with free surface (2.8). Let us find the small deviant motion of the transformed vehicle in passing from a position corresponding to steady state motion of the system to a neighboring perturbed position. We can conceive of this transition as proceeding in two stages: 1) displacement of the whole system, as a single rigid body, to a perturbed position; 2) deformation of the form $\zeta = -h_0+h$ of the liquid (by covering its free surface with a layer $\tau-\tau_0$ of zero volume) into a form $\zeta = -h_0+h+\eta$ with free surface (1.14).

The increment of any variable, say $\dot{\bar{r}}_*$, may then be represented in the form

$$\delta\dot{\bar{r}}_* = \delta_1\dot{\bar{r}}_* + \delta_2\dot{\bar{r}}_* , \tag{2.10}$$

where δ_1 denotes the increment when the whole system is displaced to the perturbed position as a single rigid body, and δ_2 is the increment due to subsequent deformation of the free surface of the liquid into the surface (1.14).

Let us denote by Q the region of the xy-plane bounded by the projection of a closed curve representing the locus of the intersection points of the surface (2.3) with the cavity walls Σ. Then, in the first approximation, we have

$$\begin{gathered}\int_{\tau-\tau_0} (\ldots)\,\rho d\tau \approx \iint_Q dxdy \int_{h-h_0}^{h-h_0+\eta} (\ldots)\,\rho dz,\\ \int_{\tau-\tau_0} \rho d\tau \approx \iint_Q \rho\,\eta\, dxdy ,\end{gathered} \tag{2.11}$$

the latter expression resulting from the incompressibility of the liquid, where τ_0 is the unperturbed liquid volume.

By definition

$$\bar{r}_* = \frac{1}{M} \left(\int \bar{r}'\, dm + \int_\tau (\bar{\ell}+\bar{r})\, \rho d\tau \right) , \tag{2.12}$$

so that, up to second order accuracy in η, we have

$$\begin{gathered}\delta\bar{r}_* = \delta_1\bar{r}_* + \delta_2\bar{r}_*\\ \delta_1\bar{r}_* = 0, \quad \delta_2\bar{r}_* = \frac{1}{M} \iint_Q \rho(\bar{\ell}+\bar{r})\,\eta\, dxdy ,\end{gathered} \tag{2.13}$$

where $\bar{\ell}+\bar{r}$ is taken at $z = h-h_0$. Similarly,

$$\dot{\bar{r}}_* = \frac{d\bar{r}_*}{dt} + \bar{\omega}\times\bar{r}_*, \qquad \frac{d\bar{r}_*}{dt} = \frac{1}{M}\int_\tau \bar{v}\,\rho d\tau \ , \tag{2.14}$$

with the result that,

$$\begin{aligned} \delta\dot{\bar{r}}_* &= \delta_1\dot{\bar{r}}_* + \delta_2\dot{\bar{r}}_* \ , \\ \delta_1\dot{\bar{r}}_* &= \delta\bar{\omega}\times\bar{r}_*, \qquad \delta_2\dot{\bar{r}}_* = \frac{1}{M}\int_{\tau_0} \delta\bar{v}\,\rho d\tau + \bar{\omega}\times\delta\bar{r}_* \ , \end{aligned} \tag{2.15}$$

where, to the same order of accuracy,

$$\begin{aligned} \delta\bar{v} &= \mathrm{grad}\,\delta\phi - \delta\bar{\omega}\times(\bar{\ell}+\bar{r}) \ , \\ \delta\phi &= \delta\bar{\omega}\cdot\bar{\phi} + \psi \ , \end{aligned} \tag{2.16}$$

for the perturbation velocity of the liquid relative to the cavity. Here $\bar{\phi}$ and ψ satisfy

$$\Delta\bar{\phi} = 0, \quad \text{throughout } \tau_0,$$

$$\frac{\partial\bar{\phi}}{\partial n} = \begin{cases} [(\bar{\ell}+\bar{r})\times\bar{n}], & \text{at } \Sigma, \\ [(\bar{\ell}+\bar{r})\times\bar{n}], & \text{at } S_0, \ (z = -h_0+h), \end{cases} \tag{2.17}$$

and

$$\Delta\psi = 0, \quad \text{throughout } \tau_0,$$

$$\frac{\partial\psi}{\partial n} = \begin{cases} 0 & , \text{at } \Sigma, \\ \eta_t \cos(\bar{n},z), & \text{at } S_0, \ (z = -h_0+h), \end{cases} \tag{2.18}$$

respectively. Note that terms involving products of the deviant quantities $\delta\overline{\omega}$, $\delta\overline{r}_{*}$, $\delta\overline{v}$, η, ψ or their derivatives were neglected in the derivation of (2.15,16,17,18).

On the free surface $z = -h_0+h+\eta$, we have for any function

$$q(x,y,z,t) = q(x,y,h-h_0,t) + \left.\frac{\partial q}{\partial z}\right|_{h-h_0} \eta + \ldots , \tag{2.19}$$

so that, in the first approximation,

$$\delta(p-p_0) = \delta_1(p-p_0) + \delta_2(p-p_0) , \tag{2.20}$$

for the increment of pressure, where

$$\begin{aligned} \delta_1(p-p_0) &= -\rho[\delta\dot{\overline{\omega}}\cdot\overline{\phi}+\delta\ddot{\overline{R}}_0\cdot(\overline{\ell}+\overline{r})-[\overline{\omega}\mathrm{x}(\overline{\ell}+\overline{r})]\cdot[\delta\overline{\omega}\mathrm{x}(\overline{\ell}+\overline{r})]+\delta_1 u_1] , \\ \delta_2(p-p_0) &= -\rho[\frac{\partial u_1}{\partial z} + \ddot{\overline{R}}_0\cdot\frac{\partial\overline{r}}{\partial z} - [\overline{\omega}\mathrm{x}(\overline{\ell}+\overline{r})]\cdot(\overline{\omega}\mathrm{x}\frac{\partial\overline{r}}{\partial z})]\eta-\rho\psi_t . \end{aligned} \tag{2.21}$$

But, from (1.5) $\delta(p-p_0) = \sigma\delta\kappa$, and it follows

$$\begin{aligned} &\psi_t + \delta\dot{\overline{\omega}}\cdot\overline{\phi} + \delta\ddot{\overline{R}}_0\cdot(\overline{\ell}+\overline{r}) - [\omega\mathrm{x}(\overline{\ell}+\overline{r})]\cdot[\delta\overline{\omega}\mathrm{x}(\overline{\ell}+\overline{r})] + \delta_1 u_1 + \\ &[\frac{\partial u_1}{\partial z} + \ddot{\overline{R}}_0\cdot\frac{\partial\overline{r}}{\partial z} - [\overline{\omega}\mathrm{x}(\overline{\ell}+\overline{r})]\cdot(\overline{\omega}\mathrm{x}\frac{\partial\overline{r}}{\partial z})]\eta + \frac{\sigma}{\rho}\,\delta\kappa = 0 , \end{aligned} \tag{2.22}$$

at $z = -h_0+h$, for thermodynamic equilibrium at the free surface of the liquid, where

$$\delta\kappa = \pm\,\mathrm{div}\left[\frac{\mathrm{grad}\eta}{[1+(\mathrm{grad}h)^2]^{1/2}} - \frac{(\mathrm{grad}\eta)\cdot(\mathrm{grad}h)}{[1+(\mathrm{grad}h)^2]^{3/2}}\,\mathrm{grad}h\right], \tag{2.23}$$

and the gradient is with respect to x,y. Recall that

$$\ddot{\bar{R}}_0 = \ddot{\bar{R}}_* - \ddot{\bar{r}}_* = \frac{d\bar{u}}{dt} + \bar{\omega}x\bar{u} , \tag{2.24}$$

whence

$$\delta\ddot{\bar{R}}_0 = \delta\ddot{\bar{R}}_* - \delta\ddot{\bar{r}}_* = \delta\frac{d\bar{u}}{dt} + \delta\bar{\omega}x\bar{u} + \bar{\omega}x\delta\bar{u} ,$$

$$\delta\ddot{\bar{r}}_* = \frac{1}{M}\int_{\tau_0}(\frac{\partial\,\delta\bar{v}}{\partial t} + 2\bar{\omega}x\delta\bar{v})\ \rho d\tau + \delta\dot{\bar{\omega}}x\bar{r}_* + \delta\bar{\omega}x(\bar{\omega}x\bar{r}_*) \tag{2.25}$$

$$+ \bar{\omega}x(\delta\bar{\omega}x\bar{r}_*) + \bar{\omega}x(\bar{\omega}x\delta\bar{r}_*) .$$

Expressions (2.24) and (2.25), in conjunction with the translational equations of the mass center, can be used to obtain other forms of (2.22). The increment $\delta_1 u_1$ occurring in (2.22) depends on the force field; for example, if the field is constant, $\delta_1 u_1 = 0$.

The equation of deviant motion of the center of mass of the system can be written

$$M\delta\ddot{\bar{R}}_* = \delta\bar{F} + \delta(\bar{F}_0+\bar{F}_1) \tag{2.26}$$

which follows from (1.19_1), (2.1) and (2.2), where

$$\delta\bar{F} = \sum_j \delta\bar{F}_j , \qquad \delta\bar{F}_0 = \int\delta_1\bar{f}_0\ dm,$$

$$\delta\bar{F}_1 = \int_{\tau_0}\delta_1\bar{f}_1\ \rho d\tau + \iint_Q dxdy\int_{h-h_0}^{h-h_0+\eta}\bar{f}_1\ \rho dz. \tag{2.27}$$

The increments of the applied forces $\delta\bar{F}_j$ may be functions of the deviant state of the vehicle or preassigned functions of time. The increments $\delta_1\bar{f}_0$ and $\delta_1\bar{f}_1$ depend on the force field; for example, in an inverse square law field defined in (1.31), we have

$$\delta_1\bar{f}_0 = -k\left[\frac{\delta\bar{\alpha}_0}{\alpha_0^3} - 3\,\frac{(\bar{\alpha}_0\cdot\delta\alpha_0)}{\alpha_0^5}\,\bar{\alpha}_0\right],$$

$$\delta_1\bar{f}_1 = -k\left[\frac{\delta\bar{\alpha}_1}{\alpha_1^3} - 3\,\frac{(\bar{\alpha}_1\cdot\delta\alpha_1)}{\alpha_1^5}\,\bar{\alpha}_1\right], \qquad (2.28)$$

$$\delta\bar{\alpha}_0 = \delta\bar{\alpha}_1 = \delta\bar{R}_0 = \delta\bar{R}_* - \delta\bar{r}_* \; .$$

Similarly, from (1.19_2) the equation of deviant rotational motion of the vehicle with liquid can be written

$$I\cdot\delta\dot{\bar{\omega}} + \bar{\omega}x(\delta I\cdot\bar{\omega})+\bar{\omega}x(I\cdot\delta\bar{\omega})+\delta\bar{\omega}x(I\cdot\bar{\omega})+M[(\delta\bar{r}_*x\ddot{\bar{R}}_*)+(\bar{r}_*x\delta\ddot{\bar{R}}_*)]$$

$$+\int_{\tau_0}[(\bar{\ell}+\bar{r}-\bar{r}_*)x(\frac{\partial\delta\bar{v}}{\partial t} + 2\bar{\omega}x\delta\bar{v})]\rho d\tau = \delta\bar{N} + \delta(\bar{N}_0+\bar{N}_1)\;, \qquad (2.29)$$

where

$$\delta\bar{N} = \sum_j \bar{r}_j x\delta\bar{F}_j\,, \qquad \delta\bar{N}_0 = \int \bar{r}'x\delta_1\bar{f}_0\, dm$$

$$\delta\bar{N}_1 = \int_{\tau_0}[(\bar{\ell}+\bar{r})x\delta_1\bar{f}_1]\rho d\tau + \iint_Q dxdy\int_{h-h_0}^{h-h_0+\eta}[(\bar{\ell}+\bar{r})x\bar{f}_1]\rho dz \qquad (2.30)$$

$$\delta I = \delta_2 I_1 - M\delta_2 I_*, \quad \delta_2 \tau_1 = \iint_Q [(\bar{\ell}+\bar{r})^2 E - (\bar{\ell}+\bar{r});(\bar{\ell}+\bar{r})]\eta\rho \; dxdy,$$

$$\delta_2 I_* = \frac{1}{M} \iint_Q [\bar{r}_* \cdot (\bar{\ell}+\bar{r})E - \bar{r}_*;(\bar{\ell}+\bar{r}) - (\bar{\ell}+\bar{r});\bar{r}_*]\eta\rho \; dxdy$$

In some investigations, it is convenient to write (2.27) and (2.30) in the form

$$M\delta\ddot{\bar{R}}_0 = -\int_{\tau_0} \left(\frac{\partial\delta\bar{v}}{\partial t} + 2\bar{\omega}x\delta\bar{v}\right)\rho d\tau - M(\delta\dot{\bar{\omega}}x\bar{r}_*) - M\delta\bar{\omega}x(\bar{\omega}x\bar{r}_*) - M\bar{\omega}x(\delta\bar{\omega}x\bar{r}_*)$$

$$- M\bar{\omega}x(\bar{\omega}x\delta\bar{r}_*) + \delta\bar{F} + \delta(\bar{F}_0+\bar{F}_1) \; ,$$

$$(2.31)$$

$$(I_0+I_1)\cdot\delta\dot{\bar{\omega}} + \delta\bar{\omega}x[(I_0+I_1)\cdot\bar{\omega}] + \bar{\omega}x[(I_0+I_1)\cdot\delta\bar{\omega}] + \bar{\omega}x(\delta I_1\cdot\bar{\omega}) +$$

$$M[(\bar{r}_*x\delta\ddot{\bar{R}}_*) + (\delta\bar{r}_*x\ddot{\bar{R}}_0)] + \int_{\tau_0} [(\bar{\ell}+\bar{r})x\left(\frac{\partial\delta\bar{v}}{\partial t} + 2\bar{\omega}x\delta\bar{v}\right)]\rho d\tau = \delta\bar{N} + \delta(\bar{N}_0+\bar{N}_1).$$

Thus, to determine the small motion of deviation of the vehicle, we have the following: a perturbation vector equation governing the translational motion of the center of mass (2.26); a perturbation vector equation governing the rotational motion of the vehicle (2.29); a scalar Laplace equation expressing continuity (2.18_1). The potential ψ associated with (2.18_1) satisfies vanishing Neumann conditions on the wetted walls of the vessel Σ (2.18_2) and an inhomogeneous Neumann condition involving the perturbation free boundary displacement η at the reference free surface (2.18_3). Thermodynamic equilibrium at the free boundary (2.22) furnishes the last link between the scalar potential ψ, free surface displacement η, and the remaining variables of the problem. Finally, with (1.7) the perturbation contact angle requirement can be written

$$\bar{n}_\Sigma \cdot \delta\bar{n}_{S_0} + \delta\bar{n}_\Sigma \cdot \bar{n}_{S_0} = 0, \quad \text{at } \Gamma_0, \tag{2.32}$$

which, in many significant applications, can be approximated by

$$\frac{\partial \eta}{\partial n} = 0, \quad \text{at } \Gamma_0. \tag{2.33}$$

Condition (2.33) is valid only if: (1) the vessel wall close to the free surface is vertical, i.e., the outward normal to the plane contour Γ in the xy-plane is equal to the outward normal to the wall; in this case h is a single-valued function of x,y and the liquid is everywhere below the free surface; (2) the angle of contact is constant, γ = const.; (3) $\partial h/\partial\Gamma = 0$, i.e., h is everywhere close to Γ; with this Γ_0 is a plane contour, and its projection on the xy-plane is the same as Γ. The last assumption holds if: (a) the wall close to Γ is a circular cylinder (in this case the vessel need not be rotationally symmetric); (b) the wall close to Γ is a plane; and (c) $\gamma = \pi/2$ whatever the vessel. To be consistent with these assumptions, we must take the upper sign in expression (2.24).

To continue, given initial conditions and externally applied forces, the Lagrangian coordinates of the vehicle, free surface displacement η, and potential ψ can all be determined in principle. Moreover, if the potential ψ and free surface displacement can be expanded in Fourier series - and this is possible in many significant applications - then we obtain 2n+6 second order ordinary differential equations for the determination of the Lagrangian coordinates of the vehicle, liquid velocity, and free surface displacement. With these known, the perturbation forces and moments resulting from the action of the liquid motion on the cavity can be computed from

$$\begin{aligned}
\overline{F}_{01} = &- \int_{\tau_0} \left\{ \frac{\partial \delta \overline{v}}{\partial t} + 2\overline{\omega} x \delta \overline{v} + \delta \dot{\overline{\omega}} x (\overline{\ell}+\overline{r}) + \overline{\omega} x [\delta \overline{\omega} x (\overline{\ell}+\overline{r})] + \delta \overline{\omega} x [\overline{\omega} x (\overline{\ell}+\overline{r})] \right\} \rho d\tau \\
&- \iint_Q \left\{ \overline{\omega} x [\overline{\omega} x (\overline{\ell}+\overline{r})] \right\} \eta \rho dxdy - M_1 \delta \ddot{\overline{R}}_0 + \int_{\tau_0} \delta \overline{f}_1 \rho d\tau \\
&+ \iint_Q dxdy \int_{h-h_0}^{h-h_0+\eta} \overline{f}_1 \, \rho dz \\
&- \sigma \iint_Q \delta\kappa \operatorname{grad}(z-h+h_0) dxdy + \sigma \iint_Q \kappa \operatorname{grad}\eta \, dxdy ,
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\overline{N}_{01} = &- \left\{ I_1 \cdot \delta \dot{\overline{\omega}} + \delta \overline{\omega} x (I_1 \cdot \overline{\omega}) + \overline{\omega} x (I_1 \cdot \delta \overline{\omega}) + \overline{\omega} x (\delta I_1 \cdot \overline{\omega}) - \int_{\tau_0} [\delta \ddot{\overline{R}}_0 x (\overline{\ell}+\overline{r})] \rho d\tau \right. \\
&\left. - \iint_Q [\ddot{\overline{R}}_0 x (\overline{\ell}+\overline{r})] \eta \rho dxdy + \int_{\tau_0} [(\overline{\ell}+\overline{r}) x (\frac{\partial \delta \overline{v}}{\partial t} + 2\overline{\omega} x \delta \overline{v})] \rho d\tau \right\} \\
&+ \int_{\tau_0} [(\overline{\ell}+\overline{r}) x \delta \overline{f}_1] \rho d\tau + \iint_Q dxdy \int_{h-h_0}^{h-h_0+\eta} [(\overline{\ell}+\overline{r}) x \overline{f}_1] \rho dz \\
&- \sigma \iint_Q \delta\kappa [(\overline{\ell}+\overline{r}) x \operatorname{grad}(z-h+h_0)] \, dxdy \quad .
\end{aligned}$$

$$+ \sigma \iint_Q \kappa[(\bar{\ell}+\bar{r})\times \mathrm{grad}\,\eta]\, dxdy.$$

Thus far, we have given a brief account of certain fundamental results from classical mechanics and hydrodynamics necessary to describe the motion of aerospace vehicles with partially liquid-filled tanks under rather general conditions. The basic equations derived were specialized to a reference state of motion in the form of constant linear and angular velocities. The vehicle then was slightly disturbed from the reference state of motion and the resulting small motion of deviation obtained. The perturbation motion is described by a set of linear integro-differential equations whose coefficients are functions of the reference orientation of the vehicle and form of the liquid free surface. These equations differ markedly from those used in previous aerospace vehicle dynamic and attitude control system studies. Many additional terms (of the same order of magnitude as these identified elsewhere and heretofore neglected) are retained in our equations. Moreover, the equations account for the complete motion of the vehicle with liquid, and therefore provide a sound theoretical basis for general investigations of stability and attitude control of liquid-filled aerospace vehicles and satellites. For certain flight situations, the perturbation equations can be simplified considerably.

2.3 Simplified Flight Conditions

Suppose that, in the reference state of motion, $\overline{\omega} \approx 0$; then, we have

$$M\ddot{\overline{R}}_{*} = \overline{F} + (\overline{F}_0+\overline{F}_1) \; ,$$

$$M(\overline{r}_{*}\times\ddot{\overline{R}}_{*}) = \overline{N} + (\overline{N}_0+\overline{N}_1) \; , \tag{2.35}$$

for the force and torque balance of the vehicle. The form of the shape of the free surface is determined from

$$u_1 + (\ddot{\overline{R}}_{*}-\ddot{\overline{r}}_{*})\cdot(\overline{\ell}+\overline{r}) + \frac{\sigma}{\rho}\,\kappa(h) = 0 \; , \tag{2.36}$$

at $z = -h_0+h$, subject to contact angle requirement

$$\overline{n}_{\Sigma}\cdot\overline{n}_{S_0} = \cos\gamma \; . \tag{2.37}$$

Considering a slight perturbation from the reference state of motion gives

$$M\delta\ddot{\overline{R}}_{*} = \delta\overline{F} + \delta(\overline{F}_0+\overline{F}_1) \; ,$$

$$I\cdot\delta\dot{\overline{\omega}} + M[(\delta\overline{r}_{*}\times\ddot{\overline{R}}_{*})+(\overline{r}_{*}\times\delta\ddot{\overline{R}}_{*})] + \int_{\tau_0} [(\overline{\ell}+\overline{r}-\overline{r}_{*})\times\frac{\partial\,\delta\overline{V}}{\partial t}]\,\rho d\tau \tag{2.38}$$

$$= \delta\overline{N} + \delta(\overline{N}_0+\overline{N}_1) \; ,$$

for the determination of the deviant translational and rotational motion of the system. On the other hand, we can express (2.38) as

$$M\delta\ddot{\bar{R}}_0 + M(\delta\dot{\bar{\omega}}x\bar{r}_*) + \int_{\tau_0} \frac{\partial\delta\bar{v}}{\partial t}\rho d\tau = \delta\bar{F} + \delta(\bar{F}_0+\bar{F}_1) ,$$

$$(I_0+I_1)\cdot\delta\dot{\bar{\omega}}+M[(\bar{r}_*x\delta\ddot{\bar{R}}_0)+(\delta\bar{r}_*x\ddot{\bar{R}}_0)] + \int_{\tau_0} [(\bar{\ell}+\bar{r})x\frac{\partial\delta\bar{v}}{\partial t}] = \qquad (2.39)$$

$$\delta\bar{N} + \delta(\bar{N}_0+\bar{N}_1) .$$

Thermodynamic equilibrium at the free boundary requires

$$\psi_t+\delta\dot{\bar{\omega}}\cdot\bar{\phi}+\delta\ddot{\bar{R}}_0\cdot(\bar{\ell}+\bar{r})+\delta_1 u_1+(\frac{\partial u_1}{\partial z} + \ddot{\bar{R}}_0\cdot\frac{\partial\bar{r}}{\partial z})\eta + \frac{\sigma}{\rho}\,\delta\kappa = 0, \qquad (2.40)$$

at $z = -h_0+h$, where

$$\delta\bar{v} = \text{grad}(\delta\bar{\omega}\cdot\phi+\psi) - \delta\bar{\omega}x(\bar{\ell}+\bar{r}) , \qquad (2.41)$$

and $\bar{\phi}$, ψ are determined from (2.17) and (2.18). The deviant forces and moments resulting from the action of the liquid motion on the cavity are given by

$$\delta\bar{F}_{01} = - \int_{\tau_0} [\frac{\partial\delta\bar{v}}{\partial t} + \delta\dot{\bar{\omega}}x(\bar{\ell}+\bar{r})]\rho d\tau - M_1\delta\ddot{\bar{R}}_0 + \int_{\tau_0} \delta\bar{f}_1\,\rho d\tau +$$
$$\iint_Q dxdy\int_{h-h_0}^{h-h_0+\eta} \bar{f}_1\,\rho dz - \sigma\iint_Q \delta\kappa\ \text{grad}(z-h+h_0)\ dxdy$$
$$+ \sigma\iint_Q \kappa\ \text{grad}\eta\ dxdy,$$

(2.42)

$$\delta\bar{N}_{01} = - \left\{ I_1\cdot\delta\dot{\bar{\omega}} - \int_{\tau_0} [\delta\ddot{\bar{R}}_0 x(\bar{\ell}+\bar{r})]\rho d\tau - \iint_Q [\ddot{\bar{R}}_0 x(\bar{\ell}+\bar{r})]\eta\ dxdy\right.$$
$$\left. + \int_{\tau_0} [(\bar{\ell}+\bar{r})x\frac{\partial\delta\bar{v}}{\partial t}]\rho d\tau\right\} + \int_{\tau_0} [(\bar{\ell}+\bar{r})x\delta\bar{f}_1]\rho d\tau +$$

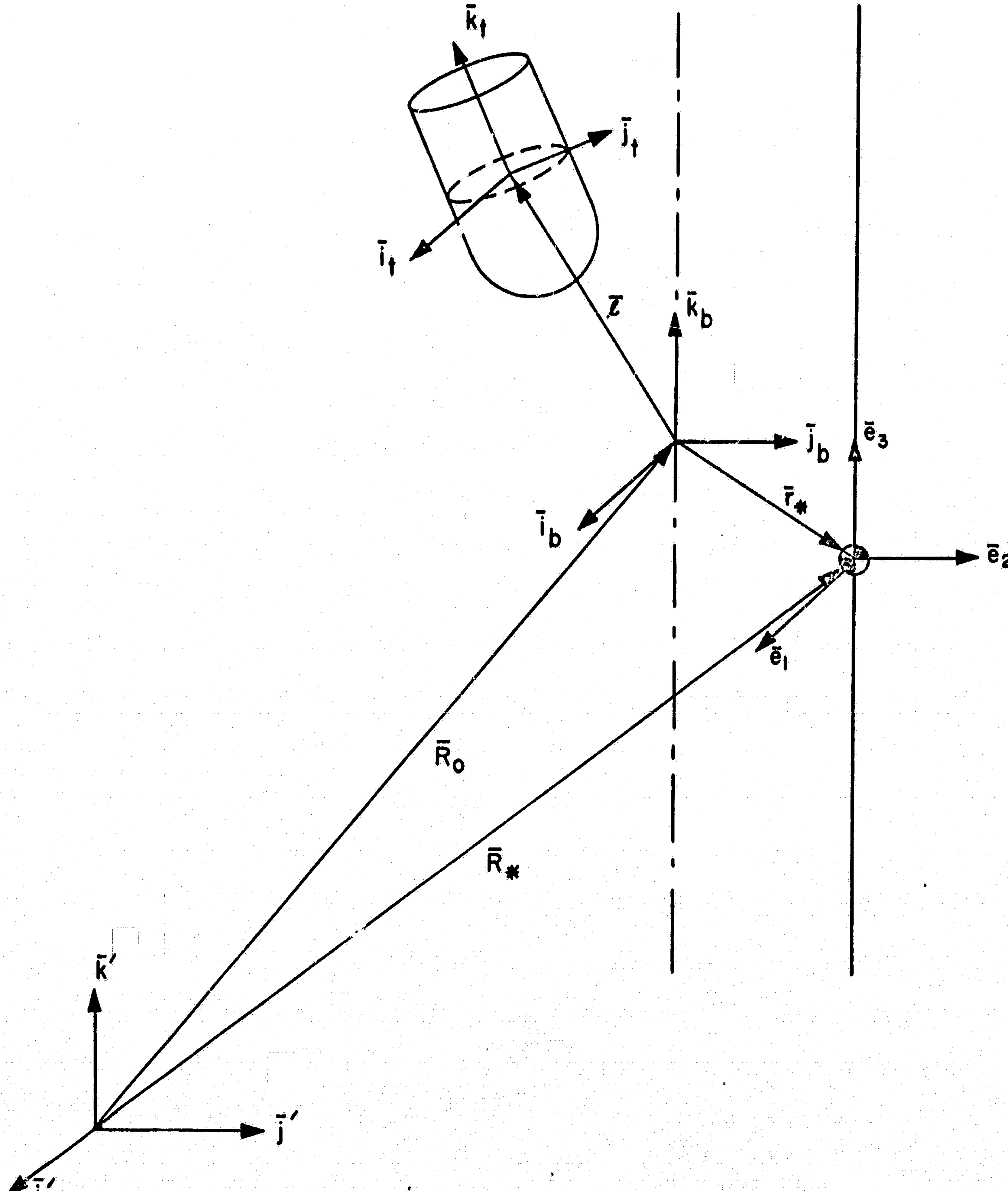

Figure 4. Simplified Flight Condition

$$\iint_Q dxdy \int_{h-h_0}^{h-h_0+\eta} [(\bar{\ell}+\bar{r})x\bar{f}_1]\ \rho dz$$

$$-\ \sigma \iint_Q \delta\kappa\ [(\bar{\ell}+\bar{r})x\mathrm{grad}(z-h+h_0)]\ dxdy$$

$$+\ \sigma \iint_Q \kappa\ [(\bar{\ell}+\bar{r})x\mathrm{grad}\eta]\ dxdy\ .$$

Clearly, (2.35-42) can be used in many flight situations wherein the angular rates are slowly varying functions of time. For example, suppose that, in the reference state of motion, the vehicle is constrained to move vertically upwards without rotation in a uniform gravitational field, Fig. 4. Then, we have

$$\bar{f}_1 = \bar{f}_0 = -\ \bar{g}, \quad u_1 = \bar{g}\cdot(\bar{\ell}+\bar{r}), \quad (\bar{F}_0+\bar{F}_1) = -\ M\bar{g},$$

$$(\bar{N}_0+\bar{N}_1) = -\ M(\bar{r}_* x\bar{g}), \quad \delta(\bar{F}_0+\bar{F}_1) = 0, \quad \delta(\bar{N}_0+\bar{N}_1) = -\ M(\delta\bar{r}_* x\bar{g})\ , \tag{2.43}$$

$$\delta\bar{r}_* = \frac{1}{M}\iint_Q (\bar{\ell}+\bar{r})\eta\ \rho\ dxdy\ ,$$

$$\ddot{\bar{R}}_* = a\bar{e}_3 = a\bar{k}_b, \quad \bar{g} = g\bar{e}_3 = g\bar{k}_b, \quad \bar{\omega} = 0, \quad \ddot{\bar{r}}_* = 0.$$

Using (2.35), we get

$$M(a+g)\bar{k}_b = \bar{F} = \sum_j \bar{F}_j\ ,$$

$$M(a+g)(\bar{r}_* x\bar{k}_b) = \bar{N} = \sum_j (\bar{r}_j x\bar{F}_j)\ , \tag{2.44}$$

for the force and torque balance of the vehicle. The form of the shape of the free surface is determined from

$$(a+g)\bar{k}_b\cdot(\bar{\ell}+\bar{r}) + \frac{\sigma}{\rho}\kappa(h) = 0 \;, \tag{2.45}$$

at $z = -h_0+h$, which follows from (2.36). The deviant translational and rotational equations of motion of the vehicle are

$$M\delta\ddot{\bar{R}}_* = \delta\bar{F} = \sum_j \delta\bar{F}_j \;,$$

$$I\cdot\delta\dot{\bar{\omega}} - (a+g)\iint_Q [\bar{k}_b x(\bar{\ell}+\bar{r})]\eta\,\rho\,dxdy + M(\bar{r}_* x\delta\ddot{\bar{R}}_*) + \tag{2.46}$$

$$\int_{\tau_0} [(\bar{\ell}+\bar{r}-\bar{r}_*)x\frac{\partial\delta\bar{v}}{\partial t}]\rho d\tau = \delta\bar{N} = \sum_j (\bar{r}_j x\delta\bar{F}_j) \;,$$

which follows from (2.38). Alternately, from (2.39) we can write

$$M\delta\ddot{\bar{R}}_0 + M(\delta\dot{\bar{\omega}}x\bar{r}_*) + \int_{\tau_0}\frac{\partial\delta\bar{v}}{\partial t}\rho d\tau = \delta\bar{F} = \sum_j \delta\bar{F}_j \;,$$

$$(I_0+I_1)\cdot\delta\dot{\bar{\omega}}-(a+g)\iint_Q [\bar{k}_b x(\bar{\ell}+\bar{r})]\eta\,\rho\,dxdy + M(\bar{r}_* x\delta\ddot{\bar{R}}_0) \tag{2.47}$$

$$+\int_{\tau_0} [(\bar{\ell}+\bar{r})x\frac{\partial\delta\bar{v}}{\partial t}]\,\rho d\tau = \delta\bar{N} = \sum_j (\bar{r}_j x\delta\bar{F}_j) \;.$$

Finally, from (2.40) we get

$$\psi_t+\delta\dot{\bar{\omega}}\cdot\bar{\phi}+\delta\ddot{\bar{R}}_0\cdot(\bar{\ell}+\bar{r})+(a+g)(\bar{k}_b\cdot\bar{k}_t)\eta + \frac{\sigma}{\rho}\delta\kappa = 0, \tag{2.48}$$

at $z = -h_0+h$, for thermodynamic equilibrium at the free boundary.

The equations defined in (2.47) can be written

$$M\delta\ddot{\overline{R}}_0 + \delta\dot{\overline{\omega}}\mathrm{x}\left(\int \overline{r}'dm + \int_{\tau_0} (\nabla \mathrm{x} \overline{\phi})\rho d\tau\right) + \int_{\tau_0} (\delta\dot{\overline{\omega}}\cdot\nabla)\overline{\phi}\rho d\tau +$$

$$\iint_Q \overline{r}\eta_{tt}\ \rho\ dxdy = \delta\overline{F}\ ,$$

$$(I_0+\widetilde{I})\cdot\delta\dot{\overline{\omega}} + M(\overline{r}_* \mathrm{x}\ \ddot{\overline{R}}_0) - (a+g) \iint_Q [\overline{k}_b \mathrm{x}(\overline{\ell}\mathrm{x}\overline{r})]\eta\ \rho\ dxdy + \tag{2.49}$$

$$\iint_Q \overline{\phi}\eta_{tt}\ \rho\ dxdy = \delta\overline{N}\ ,$$

which follows from (2.41), (2.17), (2.18) and the divergence theorem, where, if we specify $\overline{\phi} = (\phi_x,\phi_y,\phi_z) = \phi_i$,

$$\widetilde{I}_{ij} = \int_{\tau_0} \nabla\phi_i\cdot\nabla\phi_j\ \rho d\tau\ . \tag{2.50}$$

Similarly, the deviant forces and moments (2.42) can be written

$$\delta\overline{F}_{01} = -\ M_1\delta\ddot{\overline{R}}_0 - \delta\dot{\overline{\omega}}\mathrm{x}\left(\int_{\tau_0} (\nabla\mathrm{x}\overline{\phi})\rho d\tau\right) - \int_{\tau_0} (\delta\dot{\overline{\omega}}\cdot\nabla)\overline{\phi}\rho d\tau\ -$$

$$\iint_Q \overline{r}\eta_{tt}\ \rho\ dxdy + \sigma\left[\iint_Q \kappa\,\mathrm{grad}\eta\, dxdy\ -\right.$$

$$\left.\iint_Q \delta\kappa\ \mathrm{grad}(z-h+h_0)dxdy\right]\ , \tag{2.51}$$

$$\delta\overline{N}_{01} = -\ \widetilde{I}\cdot\delta\dot{\overline{\omega}} + \delta\ddot{\overline{R}}_0\mathrm{x}\left(\int_{\tau_0} (\overline{\ell}+\overline{r})\rho d\tau\right) + (a+g)\iint_Q [\overline{k}_b\mathrm{x}(\overline{\ell}+\overline{r})]\eta\rho\ dxdy$$

$$-\ \iint_Q \overline{\phi}\eta_{tt}\ \rho\ dxdy + \sigma\left[\iint_Q \kappa[(\overline{\ell}+\overline{r})\mathrm{x}\,\mathrm{grad}\eta]dxdy\ -\right.$$

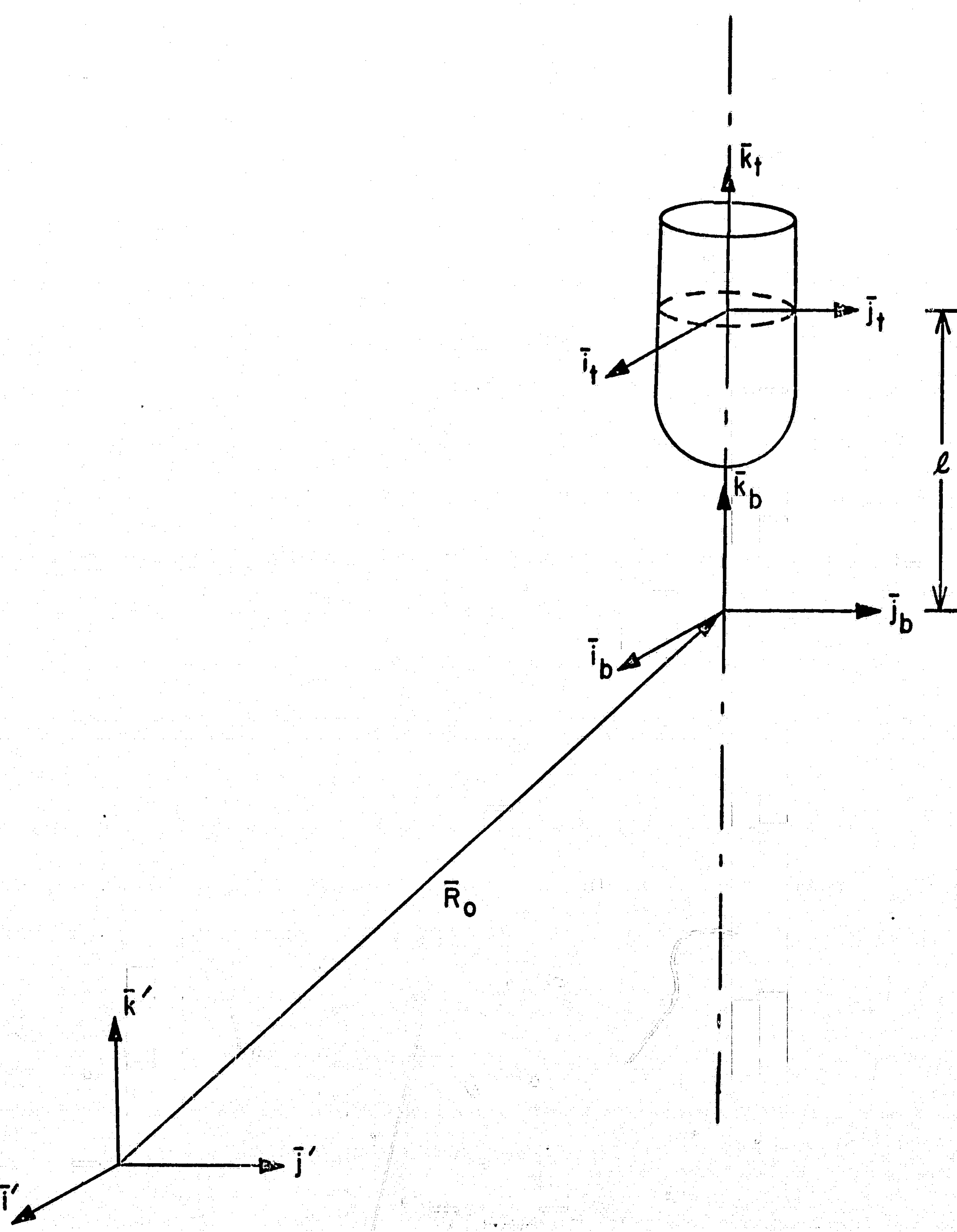

Figure 5. Planar Motion

$$\iint_Q \delta\kappa[(\bar{\ell}+\bar{r})\times\mathrm{grad}(z-h+h_0)]dxdy] \ .$$

Consider a vehicle which is constrained to move in the y,z plane, as indicated in Fig. 5. Moreover, assume that

$$(\bar{i}_b,\bar{j}_b,\bar{k}_b) = (\bar{i}_t,\bar{j}_t,\bar{k}_t), \quad \bar{\ell} = \ell\bar{k}_b, \tag{2.52}$$

with the result that, in the reference state,

$$F_z = M(a+g),\ F_y = 0,\ F_z = 0,\ N_x = 0,\ N_y = 0,\ N_z = 0$$

$$(a+g)(\ell-h_0+h) + \frac{\sigma}{\rho}\kappa(h) = 0, \quad z = -h_0 + h\ , \tag{2.53}$$

$$\bar{\phi} = \phi_*\bar{i}_b,\ \tilde{I} = \bar{i}_b\bar{i}_b\int_{\tau_0}(\nabla\phi_*)^2\,\rho d\tau\ ,\ = \bar{i}_b\bar{i}_b\,J$$

where ϕ_* is to be determined from

$$\Delta\phi_* = 0\ , \qquad \text{throughout } \tau_0\ ,$$

$$\frac{\partial\phi_*}{\partial n} = y\cos(\bar{n},z) - (\ell+z)\cos(\bar{n},y)\ , \quad \text{at } \Sigma\ , \tag{2.54}$$

$$\frac{\partial\phi_*}{\partial z} - \mathrm{grad}\phi_*\cdot\mathrm{grad}h - y - (\ell+h-h_0)\frac{\partial h}{\partial y} = 0, \quad \text{at } z = -h_0+h\ .$$

Hence, with $\delta\bar{\omega} = \dot{\theta}\bar{i}_b$, the small perturbation equations can be written

$$\delta F_x = 0,\quad \delta N_y = 0,\quad \delta N_z = 0,\quad (\delta\ddot{\bar{R}}_0)_x = 0$$

$$M(\delta\ddot{\bar{R}}_0)_y - \ddot{\theta}\left(\int z'dm + \int_{\tau_0}(\ell+z)\rho d\tau\right) + \iint_Q y\eta_{tt}\,\rho\,dxdy = \delta F_y$$

$$M(\delta\ddot{\vec{R}}_0)_z + \ddot{\theta}\left(\int y'dm + \int_{\tau_0} y\rho d\tau\right) + \iint_Q (h-h_0)\eta_{tt}\, \rho\, dxdy = \delta F_z \tag{2.55}$$

$$(I_0+J)\ddot{\theta} + (\delta\ddot{\vec{R}}_0)_z\left(\int y'dm + \int_{\tau_0} y\rho d\tau\right) - (\delta\ddot{\vec{R}}_0)_y\left(\int z'dm + \int_{\tau_0} (\ell+z)\rho d\tau\right) + (a+g)\iint_Q y\,\eta\,\rho\, dxdy + \iint_Q \phi_* \eta_{tt}\, \rho\, dxdy = \delta N_x$$

where ψ is to be determined from

$$\Delta\psi = 0\,, \qquad \text{throughout } \tau_0,$$

$$\frac{\partial\psi}{\partial n} = 0\,, \qquad \text{at } \Sigma, \tag{2.56}$$

$$\eta_t + \text{grad}h\cdot\text{grad}\psi - \frac{\partial\psi}{\partial z} = 0, \quad \text{at } z = -h_0 + h\,,$$

with the deviant thermodynamic equilibrium condition at the free boundary furnishing the last link between ψ, η, $\ddot{\theta}$, $(\delta\ddot{\vec{R}}_0)_z$ and $(\delta\ddot{\vec{R}}_0)_y$,

$$\psi_t + \phi_*\ddot{\theta} + (\delta\ddot{\vec{R}}_0)_y y + (\delta\ddot{\vec{R}}_0)_z(\ell+h-h_0) + (a+g)\eta + \frac{\sigma}{\rho}\,\delta\kappa = 0, \tag{2.57}$$

at $z = -h_0 + h$.

The forces and moments resulting from the deviant motion on the tank walls (2.51) become

$$(\delta\overline{F}_{01})_y = - M_1(\delta\ddot{\overline{R}}_0)_y + \ddot{\theta} \int_{\tau_0} (\ell+z)\rho d\tau - \iint_Q y\eta_{tt}\ \rho\ dxdy +$$

$$\sigma[\iint_Q \kappa\frac{\partial \eta}{\partial y}\ dxdy + \iint_Q \delta\kappa\ \frac{\partial(h-h_0)}{\partial y}\ dxdy] ,$$

$$(\delta\overline{F}_{01})_z = - M_1(\delta\ddot{\overline{R}}_0)_z - \ddot{\theta} \int_{\tau_0} y\ \rho d\tau - \iint_Q (h-h_0)\eta_{tt}\ \rho\ dxdy -$$

$$\sigma \iint_Q \delta\kappa\ dxdy , \tag{2.58}$$

$$(\delta\overline{N}_{01})_x = - J\ddot{\theta} + (\delta\ddot{\overline{R}}_0)_y \int_{\tau_0} (\ell+z)\rho d\tau - (\delta\ddot{\overline{R}}_0)_z \int_{\tau_0} y\ \rho d\tau -$$

$$(a+g) \iint_Q y\eta\rho\ dxdy - \iint_Q \phi_* \eta_{tt}\ \rho\ dxdy -$$

$$\sigma[\iint_Q \kappa(\ell+h-h_0)\frac{\partial \eta}{\partial y}\ dxdy +$$

$$\iint_Q \delta\kappa[y+(\ell+h-h_0)\frac{\partial(h-h_0)}{\partial y}]\ dxdy] .$$

In the absence of surface tension ($h = h_0 = \sigma = 0$), (2.56), (2.57) and (2.58) become

$$\Delta\psi = 0 \quad , \text{ throughout } \tau_0,$$

$$\frac{\partial\psi}{\partial n} = 0 \quad , \text{ at } \Sigma, \tag{2.59}$$

$$\eta_t - \frac{\partial\psi}{\partial z} = 0 \ , \text{ at } z = 0 ;$$

$$\psi_t + \phi_* \ddot{\theta} + (\delta\ddot{\bar{R}}_0)_y y + (a+g)\eta = 0, \quad \text{at } z = 0 \; ; \tag{2.60}$$

$$(\delta\bar{F}_{01})_y = - M(\delta\ddot{\bar{R}}_0)_y + \ddot{\theta} \int_{\tau_0} (\ell+z)\rho d\tau - \iint_Q y\eta_{tt} \, \rho \, dxdy \, ,$$

$$(\delta\bar{F}_{01})_z = - M_1(\delta\ddot{\bar{R}}_0)_z - \ddot{\theta} \int_{\tau_0} y \, \rho d\tau \, , \tag{2.61}$$

$$(\delta\bar{N}_{01})_x = - J\ddot{\theta} + (\delta\ddot{\bar{R}}_0)_y \int_{\tau_0} (\ell+z)\rho d\tau - (\delta\ddot{\bar{R}}_0)_z \int_{\tau_0} y \, \rho d\tau -$$

$$(a+g) \iint_Q y \, \eta\rho \, dxdy - \iint_Q \phi_* \eta_{tt} \, \rho \, dxdy \, ,$$

where ϕ_* is to be determined from

$$\Delta\phi_* = 0 \quad , \quad \text{throughout } \tau_0,$$

$$\frac{\partial \phi_*}{\partial n} = y\cos(\bar{n},z) - (\ell+z)\cos(\bar{n},y), \text{ at } \Sigma, \tag{2.62}$$

$$\frac{\partial \phi_*}{\partial z} - y = 0 \quad , \quad \text{at } z = 0 \, .$$

If $\int_{\tau_0} (\ell+z)\rho d\tau = \int_{\tau_0} y \, \rho d\tau = 0$, the system of equations (2.59-62) agree (except for notation) with / 4 /[12]. Subsequently, we shall reduce equations (2.53-58) to a form suitable for digital implementation similar to / 1 /[13].

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12 These equations form the basis for a so-called yaw plane model.

13 The equations in / 4 / form the basis for the current MSFC Slosh Program.

2.4 Role of Liquid Motion

While the perturbation equations of motion of a vehicle with liquid can be written in different forms, the contribution of the liquid to the overall system motion is generally governed by (2.8), (2.17), (2.18), (2.22) and (2.34). Indeed, the simplified flight conditions already given point this out very well.

In general, it is impossible to compute the coefficients required in (2.18), (2.22) and (2.34) without first determining the reference liquid-vapor interface $h(x,y)$, constant h_0 and the Stokes potential $\bar{\phi}$ [14]. In principle, $h(x,y)$ can be determined from the solution of (2.8) subject to the contact angle condition (2.9). Constant h_0 can be obtained by observing that the volume of liquid is constant. With h and h_0 known, $\bar{\phi}$ can be found from the differential system (2.17). Thus, (2.8) and (2.17) characterize the role of the liquid in the reference state of motion.

Clearly, the contribution of the liquid to the deviant motion of the vehicle is characterized by (2.18), (2.22) and (2.34). Now, (2.18) is an inhomogeneous Neumann problem, so that in principle, ψ can be expressed in terms of $\dot{\eta}_t$ via an integral operator. Subsequent substitution of this expression in (2.22) furnishes us with an integro-differential equation

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[14] Moreover, the inertial properties of the system depend on h, h_0, and $\bar{\phi}$.

for the determination of η in terms of $\delta\dot{\overline{\omega}}$ and $\delta\ddot{\overline{R}}_0$ [15]. With ψ and η known, the forces and moments resulting from the action of the deviant liquid motion on the cavity walls can be found from (2.34).

3. PLANAR MOTION OF LAUNCH VEHICLE (SMALL SURFACE TENSION)

In many significant applications, the surface tension is small compared with the effective mass forces[16]. Using this assumption, we now develop a method for determining the contributions of the liquid motion to the overall planar motion of a launch vehicle. We will cast the equations in a form suitable for digital implementation similar to / 1 /.

3.1 Basic Equations

Now, for planar motion of a launch vehicle, the role of the liquid is characterized by (2.53,54) and (2.56-58), which, without loss of generality, we rewrite

$$h + \frac{\sigma}{\rho(a+g)} \kappa(h) = 0, \quad \text{at } z = h - h_0 ; \tag{3.1}$$

$$\Delta\phi_* = 0, \quad \text{throughout } \tau_0,$$

$$\frac{\partial\phi_*}{\partial\nu} = y\cos(\overline{\nu},z) - (\ell+z)\cos(\overline{\nu},y), \quad \text{at } \Sigma, \tag{3.2}$$

$$\frac{\partial\phi_*}{\partial z} - \operatorname{grad}\phi_* \cdot \operatorname{grad}h = y + (\ell+h-h_0)\frac{\partial h}{\partial y}, \quad \text{at } z = h - h_0 ;$$

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15 $\delta\dot{\overline{\omega}}$ and $\delta\ddot{\overline{R}}_0$ may be expressed in terms of the Lagrangian coordinates of the body.

16 High Bond number conditions.

2

$$\Delta\psi = 0, \qquad \text{throughout } \tau_0 ,$$

$$\frac{\partial\psi}{\partial\nu} = 0, \qquad \text{at } \Sigma , \tag{3.3}$$

$$\eta_t + \text{grad}\psi\cdot\text{grad}h - \frac{\partial\psi}{\partial z} = 0, \quad \text{at } z = h - h_0 ;$$

$$\frac{1}{a+g}\psi_t + \eta + \frac{\sigma}{\rho(a+g)}\delta\kappa = -\frac{1}{a+g}[\phi_*\ddot{\theta}+(\delta\ddot{\bar{R}}_0)_y y+(\delta\ddot{\bar{R}}_0)_z h],$$

$$\text{at } z = h - h_0 ; \tag{3.4}$$

$$(\delta\bar{F}_{01})_y = -M_1(\delta\ddot{\bar{R}}_0)_y + \ddot{\theta}\int_{\tau_0}(\ell+z)\rho d\tau - \iint_Q y\eta_{tt}\,\rho\,dxdy +$$

$$\sigma[\iint_Q \kappa\frac{\partial\eta}{\partial y}\,dxdy + \iint_Q \delta\kappa\frac{\partial h}{\partial y}\,dxdy] , \tag{3.5}$$

$$(\delta\bar{F}_{01})_z = -M_1(\delta\ddot{\bar{R}}_0)_z - \ddot{\theta}\int_{\tau_0} y\rho d\tau - \iint_Q (h-h_0)\eta_{tt}\,\rho\,dxdy -$$

$$\sigma\iint_Q \delta\kappa\,dxdy ,$$

$$(\delta\bar{N}_{01})_x = -J\ddot{\theta} + (\delta\ddot{\bar{R}}_0)_y\int_{\tau_0}(\ell+z)\rho d\tau - (\delta\ddot{\bar{R}}_0)_z\int_{\tau_0} y\rho d\tau -$$

$$(a+g)\iint_Q y\eta\rho\,dxdy - \iint_Q \phi_x\eta_{tt}\,\rho\,dxdy -$$

$$\sigma[\iint_Q \kappa(\ell+h-h_0)\frac{\partial\eta}{\partial y}\,dxdy + \iint_Q \delta\kappa[y+(\ell+h-h_0)\frac{\partial h}{\partial y}]\,dxdy]$$

Here $\bar{\nu}$ is the inward normal to the wall.

To find solutions of (3.1-4), we employ boundary layer and perturbation methods similar to / 2 /, wherein the eigenoscillations of a fluid subject to small surface tension in a vessel were determined. Indeed, the results of / 2 / are fundamental, and we use them as a springboard for attacking our problem.

3.2 Reference Shape of Free Surface According to / 2 /

Following / 2 /, we introduce the dimensionless parameter

$$\varepsilon = \sqrt{\sigma} \,/\, (R \sqrt{\rho(a+g)}) , \tag{3.6}$$

where R is the characteristic linear dimension of the vessel; the parameter ε (= $1/B_0$, where B_0 is the Bond number) is assumed to be small. Equation (3.1) now becomes

$$h + R^2\varepsilon^2\kappa(h) = 0. \tag{3.7}$$

The boundary condition for this equation (the angle of contact between the wall and the free surface) is specified on contour Γ_0 which is not in general known in advance. With $\sigma=\varepsilon=0$, the solution has the form $h=0$, while Γ_0 transforms into the plane contour Γ along which the $z=0$ plane intersects the wall (Fig. 6).

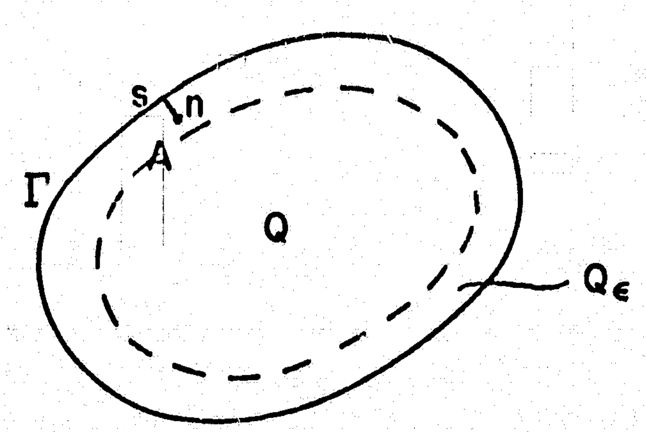

Figure 6. Contour Γ

We make the assumption that, for $0<\varepsilon\leq 1$, contour Γ_0 is close to Γ, while the solution of (3.1) differs markedly from zero in a narrow region close to Γ_0.

Let us introduce curvilinear orthogonal coordinates n,s in the xy plane where n is the distance along the normal from the given point A to contour Γ, and s is the arc length of Γ from the point taken as origin on the normal passing through A. Observe that coordinates n,s are not uniquely determined for points whose distance from Γ is of the order of its radius of curvature, and h(n,s) need not be single-valued. Nevertheless, we will employ coordinates n,s, since this simplifies the analysis, does not affect the final results and enables us to parallel / 2 /. We assume that contour Γ has no angular points and that its radius of curvature is everywhere much greater than ε. With these assumptions, the coordinates n,s are determined uniquely in the narrow region Q_t of the xy plane adjacent to Γ and of width $\sim\varepsilon$.

Thus, with the boundary layer method, we perform the extensions $h=\varepsilon H$, $n=\varepsilon u$, $s=s$. Because the edge angle is finite, we have on the boundary contour $h_n\approx H_u\sim 1$. Moreover, we have in Q_ε, $u\sim s\sim R$. We assume that H(u,s) and all its derivatives are finite (of order O(1)) in Q_ε. Hence, we have in Q_ε,

$$h\sim\varepsilon R,\quad h_n\sim 1,\quad h_s\sim\varepsilon,$$
$$h_{nn}\sim(R\varepsilon)^{-1},\quad h_{ns}\sim R^{-1},\quad h_{ss}\sim\varepsilon R^{-1}. \tag{3.8}$$

Passing to the variables n,s in (3.1) and observing (3.6), (3.8) and $(\partial n/\partial x)^2 + (\partial n/\partial y)^2 = 1$, we arrive at the equation

$$h = \pm \frac{R^2\varepsilon^2 h_{nn}}{(1+h_n^2)^{3/2}}, \tag{3.9}$$

which holds in Q_ε up to higher order terms. The upper sign is taken if the liquid is below the surface $z = h-h_0$, and the lower sign if it is above the surface $z = h-h_0$. No derivatives with respect to s occur in (3.9), so that we can write its first integral immediately,

$$\frac{h^2}{2} = \pm \frac{R^2\varepsilon^2}{(1+h_n^2)^{1/2}} + c_1 . \tag{3.10}$$

Remote from contour Γ, $h_n \to 0$, $h \ll \varepsilon$, and we have to take the upper sign in (3.10), because the liquid lies below the free surface. Thus, we have, to higher order terms, $c_1 = R^2\varepsilon^2$. At points where the free surface is vertical and the sign changes in (3.10), we have $h_n = \infty$, $|h| = R\varepsilon\sqrt{2}$.

With $c_1 = R^2\varepsilon^2$, we solve (3.10) for h_n,

$$h_n = \pm \frac{h\sqrt{4R^2\varepsilon^2-h^2}}{h^2-2R^2\varepsilon^2} .$$

For any possible combinations of angles of inclination of the wall and angles of contact, it can be shown that for $|h|<R\varepsilon\sqrt{2}$, h and h_n have opposite signs, while for $|h|>R\varepsilon\sqrt{2}$ their signs are the same. This determines the

the choice of the upper sign in the above formula for all possibilities,

$$h_n = \frac{h\sqrt{4R^2\varepsilon^2-h^2}}{h^2-2R^2\varepsilon^2} \, . \tag{3.11}$$

Observe that, by (3.11), $|h| \le 2R\varepsilon$. Integrating (3.11) we get

$$n = c - \sqrt{4R^2\varepsilon^2-h^2} - \frac{R\varepsilon}{2} \, \mathrm{LN}\left[\frac{1-\sqrt{1-(h/2R\varepsilon)^2}}{1+\sqrt{1-(h/2R\varepsilon)^2}}\right]. \tag{3.12}$$

The section of the free surface $z=\zeta=h(x,y)-h_0$ by the vertical z,n plane is shown in Fig. 7. Here, $\bar{n}$ is the inward normal to plane contour Γ in the $z=0$ plane, $\bar{\nu}$ is the inward normal to the wall, and the angle β between them is the angle of the slope of the wall to the vertical, while $\zeta\to -h_0$ as $n\to\infty$. At the intersection of the free boundary and the wall (point P), we have the contact angle condition

$$h_n = -\cot(\gamma+\beta) \, . \tag{3.13}$$

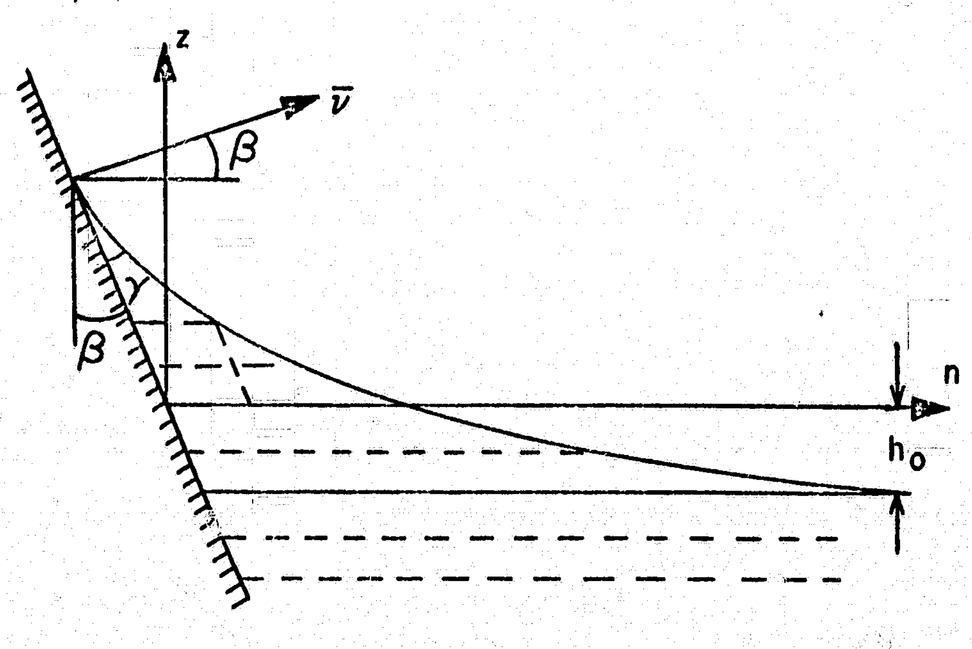

Figure 7. The zn Plane, Notations

From an inspection of Fig. 7 and substitution of (3.13) into (3.10), we find h on the wall (at point P)

$$h_1 = R\varepsilon \sqrt{2}\ \frac{\cos(\gamma+\beta)}{\sqrt{1+\sin(\gamma+\beta)}} , \tag{3.14}$$

where the sign of h is determined from the sign of $\cos(\gamma+\beta)$. Moreover, points of the wall satisfy $n = -z \tan\beta$ close to P, that is,

$$n = n_1 = (h_0-h_1)\tan\beta \ . \tag{3.15}$$

Substituting (3.14,15) in (3.12), we find the constant

$$c = \left[h_0 - \frac{R\varepsilon\sqrt{2}\ \cos(\gamma+\beta)}{\sqrt{1+\sin(\gamma+\beta)}}\right]\tan\beta + R\varepsilon \quad \sqrt{2[1+\sin(\gamma+\beta)]} +$$

$$\frac{R}{2}\ \mathrm{LN}\left[\frac{1-\sqrt{[1+\sin(\gamma+\beta)]/2}}{1+\sqrt{[1+\sin(\gamma+\beta)]/2}}\right].$$

Using (3.12) for $n\sim R$ and arguing as in / 2 /, it can be shown that function h and its derivatives at finite distances from the wall can be estimated as

$$h \sim R\varepsilon\ e^{-k/\varepsilon}, \quad k = O(1) > 0,$$

$$h_x \sim h_y \sim e^{-k/\varepsilon}, \quad h_{xx} \sim h_{yy} \sim h_{xy} \sim e^{-k/\varepsilon}(R\varepsilon)^{-1} ,$$

so that we can take $h\equiv 0$ outside Q_ε with an error less than any power of ε.

We find h_0 from the condition of incompressibility, namely,

$$\iint_Q (h-h_0)\, dxdy = c \qquad Qh_0 = \iint_Q h\, dxdy + c. \tag{3.16}$$

On the other hand, if we denote by S the area in the zn plane, bounded by the wall, the free surface and the straight line $z = -h_0$ (Fig. 7), we have

$$Qh_0 = \oint_\Gamma S(s)\, dS.$$

To find S, it is more convenient to integrate with respect to z, because h(n) need not be single-valued, while n(h) of (3.12) is single-valued,

$$S = \int_{-h_0}^{h_1-h_0} [n(h)-(-z\tan\beta)]dz = \int_0^{h_1} n(h)dh + \frac{1}{2}\tan(h_1^2-2h_1h_0)$$

$$= n_1h_1 - \int_0^{h_1} dh + \frac{1}{2}\tan(h_1^2-2h_1h_0) \; ,$$

upon integrating by parts. Applying (3.15), (3.11) and (3.14),

$$S = -\frac{1}{2}\tan h_1^2 - \int_0^{h_1} \frac{h^2-2R^2\varepsilon^2}{\sqrt{4R^2\varepsilon^2-h^2}}\, dh = -\frac{1}{2}\tan h_1^2 + \frac{1}{2} h_1 \sqrt{4R^2\varepsilon^2-h_1^2}$$

$$= R^2\varepsilon^2(\cos\gamma-\sin\beta)/\cos\beta \; ,$$

so that

$$Qh_0 = R^2\varepsilon^2 \oint_{\Gamma} \frac{\cos\gamma(s)-\sin\beta(s)}{\cos\beta(s)}\, ds \,, \tag{3.17}$$

which accounts for the dependence of the angle of contact and of the slope of the wall on the point of the wall.

Finally, we transform $\delta\kappa$ by passing to the variables n,s, using (3.1) and (3.8) and omitting small terms of higher order in ε,

$$\delta\kappa = \pm\,(1+h_n)^{-3/2}\,[\Delta\eta+h_n^2(\eta_{xx}n_y^2-2\eta_{xy}n_xn_y+\eta_{yy}n_x^2)] +$$

$$3(1+h_n^2)^{-1}\,h\,h_n\,(\eta_xn_x+\eta_yn_y)R^{-2}\varepsilon^{-2} \,, \tag{3.18}$$

where Δ is the Laplace operator in the xy plane, and the partial derivatives n_x, n_y are the direction cosines of the inward normal $\bar{n}$ to Γ, drawn from point $A(x,y)$, with respect to the coordinate axes. Outside Q_ε, allowing for choice of sign, (3.18) gives $\delta\kappa = -\Delta\eta$.

3.3 Eigenoscillations According to / 2 /

We now consider the eigenoscillations of a liquid in a vessel subject to the following additional assumptions: (1) the vessel wall close to the free surface is vertical, i.e., $\bar{\nu}=\bar{n}$, $\beta=0$ close to contour Γ; with this assumption, h is a single-valued function of x,y and the liquid is everywhere below the free surface; (2) the angle of contact is constant: γ = const.; (3) $\partial h/\partial s = 0$, i.e., $h = h(n)$ everywhere close to Γ; with this assumption, Γ_0 is a plane contour, and its projection on the xy plane is the same as Γ. The last assumption holds e.g. if: (a) the wall close to Γ is a circular cylinder (in this case the vessel does not even need to be rotationally symmetric), (b) the wall close to Γ is a plane, (c) $\gamma = \pi/2$ whatever the vessel. In most significant applications dealing with liquid motions in rotationally symmetric tanks of launch vehicles, the above assumptions hold.

With these assumptions, (2.32) on Γ_0 can be simplified greatly. To be sure, for points of the dynamic free surface

$$\frac{(\bar{k}_t - \mathrm{grad}h - \mathrm{grad}\eta)\cdot\bar{n}}{\sqrt{1+(\mathrm{grad}h+\mathrm{grad}\eta)^2}} = \cos\gamma \ , \qquad \text{on } \Gamma,$$

where the gradient is taken with respect to variables x,y. Recalling that $h = h(n)$, we can write

$$\frac{-h_n - \eta_n}{\sqrt{1+(h_n+\eta_n)^2+\eta_s^2}} = \cos\gamma \ , \qquad \text{on } \Gamma \ .$$

However, $-h_n(1+h_n^2)^{-1/2} = \cos\gamma$ by virtue of the conditions for the reference free surface. Thus, in the linear approximation with respect to η, we get $\partial\eta/\partial n = 0$ on Γ as indicated in (2.33).

The free vibration problem is governed by (3.3) and (3.4) with $\ddot{\theta} = (\delta\ddot{\bar{R}}_0)_y = (\delta\ddot{\bar{R}}_0)_z = 0$, that is,

$$\Delta\psi = 0 \quad , \qquad \text{throughout } \tau_0 \ ,$$

$$\frac{\partial\psi}{\partial\nu} = 0 \quad , \qquad \text{at } \Sigma \qquad\qquad (3.19)$$

$$\eta_t + \mathrm{grad}\psi \cdot \mathrm{grad}h - \frac{\partial \psi}{\partial z} = 0, \quad \text{at } z = h - h_0 ;$$

$$\frac{1}{a+g}\psi_t + \eta + R\varepsilon^2 \delta\kappa = 0, \quad \text{at } z = h - h_0 . \tag{3.20}$$

In addition, we have the condition

$$\frac{\partial \eta}{\partial n} = 0, \quad \text{on } \Gamma. \tag{3.21}$$

To solve the eigenoscillation problem, we put

$$\psi = e^{i\omega t}\,\Psi(x,y,z), \qquad \eta = e^{i\omega t}\,N(x,y) ,$$

where ω is the required eigenfrequency. The functions Ψ, N satisfy

$$\begin{aligned} \Delta\Psi &= 0 , \quad \text{throughout } \tau_0, \\ \frac{\partial \Psi}{\partial \nu} &= 0 , \quad \text{at } \Sigma; \end{aligned} \tag{3.22}$$

$$i\omega N + h_x\Psi_x + h_y\Psi_y - \Psi_z = 0, \quad \text{at } z = h - h_0 ; \tag{3.23}$$

$$i\omega(a+g)^{-1}\Psi + N + R^2\varepsilon^2\,\delta\kappa(N) = 0, \quad \text{at } z = h - h_0 ; \tag{3.24}$$

$$\frac{\partial N}{\partial n} = 0, \quad \text{on } \Gamma . \tag{3.25}$$

We wish to find the eigenfrequencies ω for which the linear homogeneous boundary value problem (3.22-25) for Ψ, N has a non-zero solution, and the characteristic functions Ψ, N themselves.

In the absence of surface tension ($\varepsilon=0$, $h=h_0=0$) the properties of the eigenoscillation problem are well known / 3 /. In this case there is a discrete spectrum of eigenfrequencies ω_m ($m=1,2,\ldots$) and corresponding eigenfunctions Ψ_m that form a complete and orthogonal system in the region bounded by the walls and the $z=0$ plane. We assume, in anticipation of the solution of (3.3-4), that there are no multiple frequencies. In the region Ω bounded by the walls and the $z=0$ plane, we have with $\sigma=0$, instead of (3.22-25),

$$\Delta\Psi_m = 0 \quad , \quad \text{throughout } \Omega,$$

$$\frac{\partial\Psi_m}{\partial\nu} = 0 \quad , \quad \text{at } \Sigma \; ; \tag{3.26}$$

$$i\omega_m N_m - \partial\Psi_m/\partial z = 0 \; , \quad \text{at } z = 0 \; ; \tag{3.27}$$

$$i\omega_m (a+g)^{-1} \, \Psi_m + N_m = 0 \quad , \quad \text{at } z = 0 \; ; \tag{3.28}$$

$$\partial N_m/\partial n = 0 \; , \quad \text{on } \Gamma \; . \tag{3.29}$$

Moreover, functions $N_m(x,y)$ form a complete and orthogonal system in Q. We subject them to the normalization condition

$$\iint_Q N_k N_\ell \, dxdy = \delta_{k\ell} \; . \tag{3.30}$$

The eigenfrequencies ω_m and functions Ψ_m, N_m of the undisturbed problem ($\sigma=\varepsilon=0$) will be assumed known. We seek the solution of the disturbed problem ($\varepsilon \neq 0$), close to the m-th eigenoscillation in the undisturbed case, by the method of perturbation theory / 2 /, confining ourselves to terms of the order of $O(\varepsilon^2)$

$$\omega = \omega_m(1+\varepsilon\lambda+\varepsilon\mu)$$

$$\Psi = \Psi_m + \sum_{k=1}^{\infty}(\varepsilon A_k+\varepsilon^2 B_k)\Psi_k, \quad N = N_m + \sum_{k=1}^{\infty}(\varepsilon a_k+\varepsilon^2 b_k)N_k. \tag{3.31}$$

The Laplace equation (3.22_1), the condition for Ψ on the walls (3.22_2) and the condition (2.25) on Γ are now satisfied. We must carry over conditions (3.23) and (3.24) into the region Q of the xy plane, substitute (3.31) in them and expand all the functions in series in the functions N_m of the orthonormalized system. Upon equating coefficients of like powers of ε, we obtain the required corrections to the frequency (λ,μ) and to the eigenfunctions (A_k, B_k, a_k, b_k).

Now, the Fourier coefficients of any function $g(x,y)$ defined in Q are given by

$$g_k = \iint_Q g\, N_k\, dxdy\ . \tag{3.32}$$

Thus, if $||g|| = O(\varepsilon^2)$, where the norm is taken in the sense of space L_2 (square summable), $|gh| = O(\varepsilon^2)$, and such function in conditions (3.23) and (3.24) can be omitted. Note that, because h and h_0 are zero outside Q_ε, while (3.8) hold outside Q_ε (being of order $O(\varepsilon)$), $||h|| \sim \varepsilon^2$, $||h_n|| \sim \varepsilon$.

Because the projection of the free surface on the z=0 plane is the same as Q, conditions (3.23) and (3.24) can be carried vertically into Q. Now, for any function f(x,y,z) on the free surface $z = h-h_0$, we have

$$f(x,y,z) = f(x,y,0) + f_z(x,y,0)\,(h-h_0) + \dots \quad , \tag{3.33}$$

where the norms of omitted terms are $< O(\varepsilon^2)$. Transforming Ψ, Ψ_x, Ψ_y, Ψ_z in accordance with (3.33), employing (3.18) and $\Psi_{zz} = -\Psi_{xx}-\Psi_{yy}$, condition (3.23) and (3.24) become

$$i\omega N-\Psi_z = h_0\Delta\Psi-h\Delta\Psi-\mathrm{grad}h\cdot\mathrm{grad}\Psi-h\,\mathrm{grad}h\cdot\mathrm{grad}\Psi_z \equiv P_1 \tag{3.34}$$

$$\frac{i\omega}{a+g}\Psi+N = \frac{i\omega}{a+g}h_0\Psi_z + R^2\varepsilon^2\Delta N - \frac{i\omega}{a+g}h\Psi_z - \frac{3hh_n}{1+h_n^2}\frac{\partial N}{\partial n} \equiv P_2 \; , \tag{3.35}$$

wherein terms with norm $O(\varepsilon^2)$ are discarded, and operators Δ, grad are taken with respect to x,y.

Since the walls are 'locally' vertical, we obtain from (3.27), (3.28), (3.29) and (3.31) for contour Γ

$$\frac{\partial\Psi_m}{\partial n} = \frac{\partial N_m}{\partial n} = \frac{\partial\Psi_{mz}}{\partial n} = \frac{\partial N}{\partial n} = \frac{\partial\Psi_z}{\partial n} = \frac{\partial\Psi}{\partial n} = 0 \; . \tag{3.36}$$

The derivatives of (3.36) are $O(\varepsilon)$ in Q_ε, and, therefore, the last term in P_2 of (3.35) is of order ε^2 in Q_ε and zero outside Q_ε, so that it can be neglected without loss of accuracy.

Likewise, since $h_s=0$, $h_n\sim 1$ in Q_ε in the present case, we conclude that $\text{grad}h\cdot\text{grad}\Psi_z\sim\varepsilon$ in Q_ε, with the result that the last term in P_1 of (3.34) can be omitted. The remaining terms in P_1 and P_2 have norms $O(\varepsilon^2)$; in particular $h_0\sim\varepsilon^2$. Hence, when substituting expansions (3.31) in P_1, P_2, we need only substitute the principal terms $N=N_m$, $\Psi=\Psi_m$, $\omega=\omega_m$. If we express Ψ_m and Ψ_{mz} in terms of N_m by means of (3.27) and (3.28), we get

$$
\begin{aligned}
P_1 &= \frac{ig}{\omega_m}\left[h_0\Delta N_m-\text{div}(h\,\text{grad}N_m)\right] ,\\
P_2 &= R^2\varepsilon^2\Delta N_m - \frac{\omega_m^2}{a+g}\,h_0N_m + \frac{\omega_m^2}{a+g}\,h\,N_m
\end{aligned}
\qquad (3.37)
$$

To compute the Fourier coefficients of P_1 and P_2 from (3.32), we note that, from (3.36),

$$
\begin{aligned}
\iint_Q N_k\Delta N_m\,dxdy &= -\iint_Q(\text{grad}N_m\cdot\text{grad}N_k)\,dxdy ,\\
\iint_Q N_k\text{div}(h\,\text{grad}N_m)\,dxdy &= -\iint_Q h(\text{grad}N_m\cdot\text{grad}N_k)\,dxdy .
\end{aligned}
\qquad (3.38)
$$

The integration over Q of functions which are non-zero only in Q_ε reduces to integration over n from 0 to ∞ (h decreases rapidly as $n\to\infty$) and to integration over the arc s of Γ. The functions N_m and grad N_m can be replaced here without loss of accuracy by their values on Γ. Note that, for $\beta=0$, we have from the equation preceding (3.17),

$$\int_0^\infty h\,dn = s = R^2\varepsilon^2\cos\gamma \qquad (3.39)$$

with (3.38), (3.39) and (3.30), we obtain the required Fourier coefficients of P_1, P_2 from (3.38)

$$P_{1k} = \frac{i(a+g)}{\omega_m}\left[-h_0\iint_Q(\text{grad}N_m\cdot\text{grad}N_k)dxdy + R^2\varepsilon^2\cos\gamma\oint_\Gamma\frac{\partial N_m}{\partial s}\frac{\partial N_k}{\partial s}\,dS\right], \tag{3.40}$$

$$P_{2k} = -R^2\varepsilon^2\iint_Q(\text{grad}N_m\cdot\text{grad}N_k)dxdy - \frac{\omega_m^2}{a+g}h_0\delta_{mk} + \frac{\omega_m^2}{a+g}R^2\varepsilon^2\cos\gamma\oint N_mN_k\,dS\ ,$$

where h_0 is given by (3.17) for $\beta=0$, $\gamma=$const.,

$$h_0 = \frac{R^2\varepsilon^2}{Q}\oint_\Gamma dS\cos\gamma\ . \tag{3.41}$$

Because the Fourier coefficients P_{1k} and P_{2k} of the right-hand sides of (3.34) and (3.35) are $O(\varepsilon^2)$, a homogeneous system of equations, which is satisfied by the zero solution, is obtained for the terms of order ε in expansions (3.31)[17]. Hence, $\lambda=A_k=a_k=0$ for all k in (3.31).

Using (3.27) and (3.28), we transform expansions (3.31) in the region Q,

$$\omega = \omega_m(1+\varepsilon^2\mu) \qquad \Psi = N_m+i(a+g)\varepsilon^2\sum_{k=1}^{\infty}\frac{B_k}{\omega_k}N_k\ ,$$
$$N = N_m+\varepsilon^2\sum_{k=1}^{\infty}b_kN_k \qquad \Psi_z = i\omega_mN_m+i\varepsilon^2\sum_{k=1}^{\infty}\omega_kB_kN_k\ . \tag{3.42}$$

17 In other words, because the disturbance is of the order of ε^2, the corrections to the eigenfrequencies and eigenfunctions are also of this order.

Substituting (3.42) into the left-hand sides of (3.34) and (3.35) and equating the Fourier coefficients of the left and right-hand sides of these equations, we get the algebraic expressions

$$i\omega_m\varepsilon^2 b_m + i\varepsilon^2\mu\omega_m - i\omega_m\varepsilon^2 B_m = P_{1m}$$

$$- \varepsilon^2\mu - \varepsilon^2 B_m + \varepsilon^2 b_m = P_{2m} \,, \tag{3.43}$$

for $k = m$, and

$$i\omega_m\varepsilon^2 b_k - i\varepsilon^2\omega_k b_k = P_{1k}, \qquad -\frac{\omega_m}{\omega_k}\varepsilon^2 B_k + \varepsilon^2 b_k = P_{2k} \,, \tag{3.44}$$

for $k \neq m$. System (3.43) and (3.44) yield

$$\varepsilon^2\mu = -\frac{iP_{1m} - \omega_m P_{2m}}{2\omega_m} \,, \qquad \varepsilon^2(B_m - b_m) = \frac{iP_{1m} - \omega_m P_{2m}}{2\omega_m}$$

$$\varepsilon^2 B_k = \frac{\omega_k(iP_{1k} + \omega_m P_{2k})}{\omega_k^2 - \omega_m^2} \,, \quad \varepsilon^2 b_k = \frac{i\omega_m P_{1k} + \omega_k^2 P_{2k}}{\omega_k^2 - \omega_m^2} \,. \tag{3.45}$$

The problem is now solved for the case of 'local' vertical walls. The non-zero coefficients of expansions (3.31) are expressible by means of (3.45), (3.40) and (3.41) in terms of ω_k, N_k, the angle of contact γ, the parameters of the tank and properties of the liquid. The coefficients B_k, b_k are given by (3.45) up to an arbitrary added term, which can be fixed by normalization of the eigenfunctions. Note that the corrections obtained are proportional to ε^2, i.e., to σ. Thus, we have

$$\varepsilon^2 B_k = \frac{\omega_k \omega_m}{\omega_k^2 - \omega_m^2} \left[\frac{(a+g)}{\omega_m^2} h_0 \iint_Q (\mathrm{grad} N_m \cdot \mathrm{grad} N_k)\, dxdy \right.$$

$$- \frac{(a+g)}{\omega_m^2} R^2 \varepsilon^2 \cos\gamma \oint_\Gamma \frac{\partial N_m}{\partial s} \frac{\partial N_k}{\partial s}\, dS - \frac{\omega_m^2}{a+g} h_0\, \delta_{mk}$$

$$- R^2 \varepsilon^2 \iint_Q (\mathrm{grad} N_m \cdot \mathrm{grad} N_k)\, dxdy$$

$$\left. + \frac{\omega_m^2}{a+g} R^2 \varepsilon^2 \cos\gamma \oint_\Gamma N_m N_k\, dS \right] \tag{3.46}$$

$$\varepsilon^2 b_k = \frac{\omega_k^2}{\omega_k^2 - \omega_m^2} \left[\frac{(a+g)}{\omega_k^2} h_0 \iint_Q (\mathrm{grad} N_m \cdot \mathrm{grad} N_k)\, dxdy \right.$$

$$- \frac{(a+g)}{\omega_k^2} R^2 \varepsilon^2 \cos\gamma \oint_\Gamma \frac{\partial N_m}{\partial s} \frac{\partial N_k}{\partial s}\, dS - \frac{\omega_m^2}{a+g} h_0\, \delta_{mk}$$

$$- R^2 \varepsilon^2 \iint_Q (\mathrm{grad} N_m \cdot \mathrm{grad} N_k)\, dxdy$$

$$\left. + \frac{\omega_m^2}{a+g} R^2 \varepsilon^2 \cos\gamma \oint_\Gamma N_m N_k\, dS \right], \tag{3.47}$$

$$\frac{\omega - \omega_m}{\omega_m} = \frac{\sigma}{2\rho(a+g)} \left\{ \iint_Q (\mathrm{grad} N_m)^2\, dxdy + \frac{\omega_m^2}{(a+g)Q} \left(\oint_\Gamma dS \right) \cos\gamma \left[1 - \right. \right.$$

$$\frac{(a+g)^2}{\omega_m^4} \iint_Q (\mathrm{grad} N_m)^2\, dxdy - \left(Q / \oint_\Gamma dS\right) \oint_\Gamma N_m^2\, dS +$$

$$\frac{(a+g)^2}{\omega_m^4} (Q/\int_\Gamma dS) \int_\Gamma (\frac{\partial N_m}{\partial s})^2 \, dS \Bigg] \Bigg\}. \tag{3.48}$$

Equation (3.48) is the correction to the m-th eigenfrequency.

We now have the mathematical and physical preliminaries necessary to solve (3.3-4).

3.4 Solution of the Non-Autonomous Equations

In light of the preceding results, we seek the solution of (3.3) and (3.4) close to the undisturbed case ($\sigma=\varepsilon=0$), by the method of perturbation theory, again confining ourselves to terms of the order of $O(\varepsilon^2)$,

$$\psi = - \sum_{i=1}^{\infty} \frac{\dot{q}(t)}{\omega_i(1+\varepsilon^2\mu_i)} \left[\Psi_i + \varepsilon^2 \sum_{j=1}^{\infty} B_{ji} \Psi_j\right]$$

$$\approx \sum_{i=1}^{\infty} \frac{\dot{q}(t)}{\omega_i} \left[(1-\varepsilon^2\mu_i)\Psi_i + \varepsilon^2 \sum_{j=1}^{\infty} B_{ji} \Psi_j\right], \tag{3.49}$$

$$\eta = \sum_{i=1}^{\infty} q_i(t) \left[N_i + \varepsilon^2 \sum_{j=1}^{\infty} b_{ji} N_j\right],$$

where

$$B_{ij} = \frac{\omega_i \omega_j}{\omega_i^2 - \omega_j^2} \overline{B}_{ij}, \quad i \neq j$$

$$= b_{ii} + \mu_i + \frac{a+g}{\omega_i^2} R^2 \cos\gamma \left[(\frac{1}{Q} \oint_\Gamma dS) \iint_Q (\mathrm{grad} N_i)^2 \, dxdy - \oint_\Gamma (\frac{\partial N_i}{\partial s})^2 \, dS \right], \quad i=j,$$

$$b_{ij} = \frac{\omega_i^2}{\omega_i^2-\omega_j^2}\,\overline{\beta}_{ij}\ , \quad i\neq j \tag{3.50}$$

$$= b_{ii}\ , \quad i=j\ ,$$

with

$$\beta_{ij} = -\,R^2 \iint\limits_Q (\mathrm{grad}N_i\cdot\mathrm{grad}N_j)\,dxdy - \frac{\omega_j^2}{a+g}\,R\cos\gamma\Bigg[\Big(\frac{R}{Q}\oint\limits_\Gamma dS\Big)\delta_{ij} -$$

$$\frac{(a+g)^2}{\omega_i^2\omega_j^2}\Big(\frac{R}{Q}\oint\limits_\Gamma dS\Big)\cdot$$

$$\iint\limits_Q (\mathrm{grad}N_i\cdot\mathrm{grad}N_j)\,dxdy + R\left(\frac{(a+g)^2}{\omega_i^2\omega_j^2}\oint\limits_\Gamma \frac{\partial N_i}{\partial s}\,\frac{\partial N_j}{\partial s}\,dS - \right.$$

$$\left.\oint\limits_\Gamma N_iN_j\ dS\right)\Bigg]\ , \tag{3.51}$$

$\overline{\beta}_{ij} = \beta_{ij}$ with ω_i replaced by ω_j,

$$\mu_i = -\,\frac{1}{2}\,\beta_{ii}\ ,$$

which follows from (3.46-48).

We subject the functions $N_i + \varepsilon^2 \sum\limits_{j=1}^{\infty} b_{ji}\, N_j$ to the normalization condition

$$\iint\limits_Q \Big[N_i+\varepsilon^2 \sum_{k=1}^{\infty} b_{ki}N_k\Big]\Big[N_j+\varepsilon^2 \sum_{\ell=1}^{\infty} b_{\ell j}N_\ell\Big]dxdy =$$

$$\delta_{ij}+\varepsilon^2R^2(\delta_{ij}-1)\iint\limits_Q (\mathrm{grad}N_i\cdot\mathrm{grad}N_j)\ dxdy\ ,$$

which, in light of (3.30), gives

$$b_{ii} = 0 \,,$$

$$b_{ij} + b_{ji} = - R^2 \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}N_j)\, dxdy \,, \tag{3.52}$$

to terms of the order of $O(\varepsilon^2)$. Note that (3.52_1) can be deduced equally well from (3.50_2) and (3.51). The eigen-functions ω_i and functions Ψ_i, N_i of the undisturbed problem ($\sigma=\varepsilon=0$) may be determined from (3.26-29), or equivalently from

$$\Delta\Psi_i = 0 \quad , \text{ throughout } \tau_0,$$

$$\frac{\partial\Psi_i}{\partial\nu} = 0 \quad , \text{ at } \Sigma,$$

$$\frac{\partial\Psi_i}{\partial z} + \omega_i N_i = 0 \;, \text{ at } \tilde{S}_0, \text{ i.e., at } z=0, \tag{3.53}$$

$$\omega_i (a+g)^{-1} \Psi_i + N_i = 0, \text{ at } \tilde{S}_0,$$

and will be assumed known.

With (3.49), the Laplace equation (3.31), the condition for ψ on the walls (3.3_2) and the condition $\partial\eta/\partial n = 0$ on Γ are satisfied. Continuing, we carry over conditions (3.3_3) and (3.4) into the region Q of the xy plane. Transforming ψ, ψ_x, ψ_y, ψ_z, ψ_t, ϕ_* in accordance with (3.33), using (3.18) and $\psi_{zz} = -\psi_{xx}-\psi_{yy}$, condition (3.3_3) and (3.4) become

$$\eta_t-\psi_z+\mathrm{grad}h\cdot\mathrm{grad}\psi+(h-h_0)\Delta\psi+h\,\mathrm{grad}h\cdot\mathrm{grad}\psi_z = 0,$$

$$\frac{1}{a+g}\,(\psi_t+(h-h_0)\psi_{zt})+\eta-R^2\varepsilon^2\Delta\eta+\frac{3hh_n}{1+h_n^2}\frac{\partial\eta}{\partial n} = -\frac{1}{a+g}\Big[(\phi_*+(h-h_0)\phi_{*z})\ddot{\theta} \tag{3.54}$$

$$+\; y(\delta\ddot{\bar{R}}_0)_y + h(\delta\ddot{\bar{R}}_0)_z\Big],$$

at z=0, wherein terms with norm $O(\varepsilon^2)$ are discarded, and operators Δ, grad are taken with respect to x,y. Using (3.53), we transform expansions (3.49) in the region Q

$$\psi = \sum_{i=1}^{\infty}\frac{\dot{q}_i(t)}{\omega_i(1+\varepsilon^2\mu_i)}\left[\frac{a+g}{\omega_i}N_i+(a+g)\varepsilon^2\sum_{j=1}^{\infty}\frac{B_{ji}}{\omega_j}N_j\right]$$

$$\approx \sum_{i=1}^{\infty}\frac{\dot{q}_i(t)}{\omega_i}\left[\frac{a+g}{\omega_i}(1-\varepsilon^2\mu_i)N_i+(a+g)\varepsilon^2\sum_{j=1}^{\infty}\frac{B_{ji}}{\omega_j}N_j\right],$$

$$\psi_z = \sum_{i=1}^{\infty}\frac{\dot{q}_i(t)}{\omega_i(1+\varepsilon^2\mu_i)}\left[\omega_iN_i+\varepsilon^2\sum_{j=1}^{\infty}\omega_jB_{ji}N_j\right] \tag{3.55}$$

$$\approx \sum_{i=1}^{\infty}\frac{\dot{q}_i(t)}{\omega_i}\left[\omega_i(1-\varepsilon^2\mu_i)N_i+\varepsilon^2\sum_{j=1}^{\infty}\omega_jB_{ji}N_j\right],$$

$$\eta = \sum_{i=1}^{\infty}q_i(t)\left[N_i+\varepsilon^2\sum_{j=1}^{\infty}b_{ji}N_j\right].$$

With (3.55), equation (3.54_1) is satisfied identically (to terms of the order of $O(\varepsilon^2)$). Substituting (3.55) into the left-hand side of (3.54_2), applying the method of Galerkin, i.e., taking the inner product of the left- and right-hand sides of the equation with respect to N_i, and neglecting terms $>O(\varepsilon^2)$, we arrive at the system of second order ordinary differential equations

$$(\ddot{q}_i+\omega_i^2 q_i)+\varepsilon^2 \sum_{j=1}^{\infty} (A_{ij}^* \ddot{q}_j+\omega_i^2 B_{ij}^* q_j) = -\frac{\omega_i^2}{a+g}\Big[\ddot{\theta}\iint_Q (\phi_*+(h-h_0)\phi_{*z})N_i \, dxdy$$

$$+ C_i^*(\delta\ddot{\bar{R}}_0)_y+\varepsilon^2 d_i^*(\delta\ddot{\bar{R}}_0)_z\Big] , \tag{3.56}$$

for the determination of the generalized coordinates q_i, where

$$A_{ij}^* = \frac{\omega_i^2}{\omega_i^2-\omega_j^2}\bar{\beta}_{ij} + \frac{\omega_i^2}{a+g} R^2\cos\gamma \oint_\Gamma N_i N_j \, dS , \qquad i\neq j ,$$

$$= \frac{a+g}{\omega_i^2} R^2\cos\gamma \left[\left(\frac{1}{Q}\oint_\Gamma dS\right) \iint_Q (\text{grad}N_i)^2 \, dxdy - \oint_\Gamma \left(\frac{\partial N_i}{\partial s}\right)^2 dS \right] +$$

$$\frac{\omega_i^2}{a+g} R^2\cos\gamma \left[\oint_\Gamma N_i^2 \, dS - \frac{1}{Q}\oint_\Gamma dS \right] , \qquad i=j , \tag{3.57}$$

$$B_{ij}^* = \frac{\omega_i^2}{\omega_i^2-\omega_j^2}\beta_{ij} + R^2 \iint_Q (\text{grad}N_i\cdot\text{grad}N_j) \, dxdy, \qquad i\neq j$$

$$= R^2 \iint_Q (\text{grad}N_i)^2 \, dxdy, \qquad i=j$$

$$C_i^* = \iint_Q y \, N_i \, dxdy,$$

$$d_i^* = R^2\cos\gamma \oint_\Gamma N_i \, dS ,$$

and

$$\phi_*(x,y,z) = \phi_*(x,y,0)+(h-h_0)\phi_{*z}(x,y,0) + \ldots\ldots\ldots, \quad \text{at } z=0.$$

To complete the coefficient of $\ddot{\theta}$ in the right-hand side of (3.56), we must find the solution of system (3.2).

To this end, we introduce the transformation

$$\phi_* = - y(\ell+z) + \psi_* \tag{3.58}$$

into (3.2), getting

$$\begin{aligned} &\Delta\psi_* = 0 \quad , \quad \text{throughout } \tau_0, \\ &\frac{\partial \psi_*}{\partial \nu} = 2y\cos(\bar{\nu},z), \quad \text{at } \Sigma, \\ &\psi_{xz} - \text{grad}h\cdot\text{grad}\psi_* = 2y, \quad \text{at } z=h-h_0. \end{aligned} \tag{3.59}$$

In the absence of surface tension ($\sigma=\varepsilon=0$), (3.59) reduces to the system

$$\begin{aligned} &\Delta\tilde{\psi}_* = 0 \quad , \quad \text{throughout } \Omega, \\ &\frac{\partial \tilde{\psi}_*}{\partial \nu} = \cos(\bar{\nu},z), \quad \text{at } \Sigma \\ &\tilde{\psi}_{*z} = 2y, \quad \text{at } \tilde{S}_0, \text{ i.e., at } z=0. \end{aligned} \tag{3.60}$$

Carrying over condition (3.59_3) into the region Q of the xy plane, we obtain

$$\psi_{*z}-2y = (h-h_0)\Delta\psi_*+(\text{grad}h\cdot\text{grad}\psi_*), \quad \text{at } z=0, \tag{3.61}$$

wherein terms with norm $O(\varepsilon^2)$ are neglected, and, as before, operators Δ, grad are taken with respect to x,y. We seek the solution of (3.59_1), (3.59_2) and (3.61) close to (3.60), restricting ourselves to terms of the order of $O(\varepsilon^2)$,

$$\psi_* = \widetilde{\psi}_* - \varepsilon^2 \sum_{i=1}^{\infty} \frac{\alpha_i}{\omega_i} \Psi_i \ , \tag{3.62}$$

where α_i is to be determined. With (3.62), the Laplace equation (3.59_1) and the condition for ψ_* on the walls (3.59_2) are satisfied. Using the properties of the eigenoscillation problem (3.53), we transform (3.62) in the region Q

$$\psi_* = \widetilde{\psi}_* + (a+g)\varepsilon^2 \sum_{i=1}^{\infty} \frac{\alpha_i}{\omega_i^2} N_i \ ,$$

$$\psi_{*z} = \widetilde{\psi}_{*z} + \varepsilon^2 \sum_{i=1}^{\infty} \alpha_i N_i \ , \quad (\widetilde{\psi}_{*z} = 2y). \tag{3.63}$$

Substituting (3.63) into (3.61), noting (3.60_3), we find

$$\varepsilon^2\alpha_i = R^2\varepsilon^2\cos\gamma \left[\left(\frac{1}{Q}\oint_\Gamma dS\right) \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}\widetilde{\psi}_*)\, dxdy - \oint_\Gamma \frac{\partial\widetilde{\psi}_*}{\partial s} \frac{\partial N_i}{\partial s}\, dS \right] . \tag{3.64}$$

Transforming (3.58) in accordance with (3.33), we get

$$\phi_*(x,y,z) = \phi_*(x,y,0)+(h-h_0)\phi_{*z}(x,y,0)+\ldots..$$

$$= -y(\ell+h-h_0)+\psi_*(x,y,0)+(h-h_0)\psi_{*z}(x,y,0)+\ldots. \ ,$$

so that, from (3.63),

$$\phi_*+(h-h_0)\phi_{*z} = y(h-h_0)+(\tilde{\psi}_*-\ell y)+\varepsilon^2 \sum_{i=1}^{\infty} \frac{a+g}{\omega_i^2} \alpha_i N_i \qquad (3.65)$$

to terms of the order of $O(\varepsilon^2)$. Substituting (3.65) into the right-hand side of (3.56), noting (3.64), discarding terms $>O(\varepsilon^2)$, we get

$$(\ddot{q}_i+\omega_i^2 q_i)+\varepsilon^2 \sum_{j=1}^{\infty} (A_{ij}^*\ddot{q}_j+\omega_i^2 B_{ij}^* q_j) = -\frac{\omega_i^2}{a+g}\left[(a^*+\varepsilon^2 b_i^*)\ddot{\theta}+c_i^*(\delta\ddot{\bar{R}}_0)_y+\varepsilon^2 d_i^*(\delta\ddot{\bar{R}}_0)_z\right], \qquad (3.66)$$

where

$$a_i^* = \iint_Q (\tilde{\psi}_*-\ell y)N_i \, dxdy, \qquad c_i^* = \iint_Q y \, N_i \, dxdy$$

$$b_i^* = \frac{a+g}{\omega_i^2} R^2\cos\gamma \left[\left(\frac{1}{Q}\oint_\Gamma dS\right) \iint_Q (\mathrm{grad}N_i\cdot\mathrm{grad}\tilde{\psi}_*)\, dxdy - \oint_\Gamma \frac{\partial\tilde{\psi}_*}{\partial s}\frac{\partial N_i}{\partial s}\, dS\right]$$

$$+ R^2\cos\gamma\left[\oint_\Gamma y \, N_i \, dS - \left(\frac{1}{Q}\oint_\Gamma dS\right)\iint_Q y\, N_i\, dxdy\right], \qquad (3.67)$$

The solution of the non-autonomous equations (3.3) and (3.4) can be completed by solving the system of differential equations (3.66). Again, observe that the corrections obtained here are proportional to ε^2, i.e., to σ. Clearly, we must solve for the q_j's to the same degree of accuracy. For example, consider the solution of the q_i's for a limiting condition of interest, namely that of a prismatic cylinder filled to a constant depth H with $\gamma=\pi/2$. Here, the eigenfunctions N_i satisfy the boundary value problem / 3 /,

$$\Delta N_i + k_i^2 N_i = 0, \quad \text{in } Q,$$

$$\frac{\partial N_i}{\partial n} = 0, \quad \text{on } \Gamma, \tag{3.68}$$

where k_i is connected with the frequency by

$$\omega_i^2 = k_i(a+g) \tanh k_i H. \tag{3.69}$$

The normalization condition (3.30) and (3.68_1) gives the additional result

$$\iint_Q (\text{grad}N_i \cdot \text{grad}N_j)\, dxdy = k_i^2\, \delta_{ij}. \tag{3.70}$$

Using (3.70), expression (3.66) with $\gamma=\pi/2$ assumes the form

$$\ddot{q}_i+\omega_i^2(1+\varepsilon^2 R^2 k_i^2)q_i = - \frac{\omega_i^2}{a+g}\left[a_i^*\ddot{\theta}+c_i^*(\delta\ddot{R}_0)_y\right] \tag{3.71}$$

Notice that the effective frequencies of this system are

$$\omega_i \sqrt{1+\varepsilon^2 R^2 k_i^2} = \omega_i(1 + \frac{1}{2}\, \varepsilon^2 R^2 k_i^2) + \text{term}>O(\varepsilon^2)$$

$$\approx \omega_i(1 + \frac{\sigma}{2\rho(a+g)}\, k_i^2) \quad ,$$

so that the frequency correction factor

$$\mu_i = - \frac{1}{2}\, \beta_{ii} = \frac{\sigma}{2\rho(a+g)}\, k_i^2 \ ,$$

which also may be deduced directly from (3.51). Note also that this is in agreement with Benjamin and Ursell / 6 / and / 3 /. To simplify the problem, consider the steady state solution of (3.71) with $\theta = \theta_0 e^{i\omega t}$, $(\delta\ddot{\bar{R}}_0)_y = 0$,

$$e^{-i\omega t} q_i = \frac{\theta_0 a_i^*}{a+g} \frac{\omega^2 \omega_i^2}{(\omega_i^2 - \omega^2) + \omega_i^2 R^2 k_i^2 \varepsilon^2} = \frac{\theta_0 a_i^*}{a+g} \frac{\omega^2 \omega_i^2}{\omega_i^2 - \omega^2} \left[1 - \varepsilon^2 \frac{\omega_i^2}{(\omega_i^2 - \omega^2)} R^2 k_i^2 \right]$$

$$+ \text{ terms } > O(\varepsilon^2). \tag{3.72}$$

On the other hand, using the method of perturbation theory to solve (3.71), close to the undisturbed solution (ε=0),

$$q_i = q_i' + \varepsilon^2 \xi_i \,, \tag{3.73}$$

we get

$$\ddot{q}_i' + \omega_i^2 q_i' = - \frac{\omega_i^2}{a+g} a_i^* \ddot{\theta} \,,$$

$$\ddot{\xi}_i + \omega_i^2 \xi_i = - R^2 k_i^2 q_i' \,,$$

which yield solutions

$$q_i' = \frac{\theta_0 a_i^*}{a+g} \frac{\omega^2 \omega_i^2}{\omega_i^2 - \omega^2} e^{i\omega t} \,, \qquad \xi_i = - \frac{\theta_0 a_i^*}{a+g} \frac{\omega_i^2 \omega^2}{(\omega_i^2 - \omega^2)^2} k_i^2 R^2 e^{i\omega t} \,. \tag{3.74}$$

Substituting (3.74) into (3.73), we get exactly the results of (3.72) (with terms $>O(\varepsilon^2)$ discarded) as we should. Subsequently, we will employ (3.73) to find solutions of (3.66).

We now express the forces and moment resulting from the action of the liquid motion on the cavity walls (3.5) in terms of ω_i, N_i, $\tilde{\psi}_*$, γ, q_i, $\ddot{q}_i$, parameters of the cavity, and properties of the liquid, confining ourselves, as before, to terms of the order of $O(\varepsilon^2)$. First, consider the perturbation force $(\delta\overline{F}_{01})_y$ in (3.5_1). The coefficient of $\ddot{\theta}$ in this expression can be determined in the following manner:

$$c = \rho\int_{\tau_0}(\ell+z)d\tau = \rho\int_{\tau_0}\frac{\partial}{\partial z}\frac{(\ell+z)^2}{2}\rho d\tau = -\rho\int_{\Sigma+S_0}\frac{(\ell+z)^2}{2}\cos(\overline{\nu},z)\,dS$$

$$= -\rho\int_{\Sigma}\frac{(\ell+z)^2}{2}\cos(\overline{\nu},z)dS + \rho\iint_{Q}\ell^2/2\;dxdy +$$

$$\rho\iint_{Q}[\ell(h-h_0)+\;h^2/2 - h\,h_0]\;dxdy$$

$$= -\rho\int_{\Sigma+\widetilde{S}_0}\frac{(\ell+z)^2}{2}\cos(\overline{\nu},z)dS + \ell\rho\iint_{Q}h\;dxdy - \rho h_0\ell\iint_{Q}dxdy +$$

$$\text{terms} > O(\varepsilon^2)$$

$$= \rho\int_{\Omega}(\ell+z)d\tau + \ell\rho\oint_{\Gamma}dS\int_0^\infty hdn - \ell h_0\rho Q + \text{terms} > O(\varepsilon^2) \qquad (3.75)$$

$$= \rho \int_{\Omega} (\ell+z)d\tau + \ell\rho\, [R^2\varepsilon^2\cos\gamma \oint_{\Gamma} dS - h_0 Q] + \text{terms} > O(\varepsilon^2)$$

$$\approx \rho \int_{\Omega} (\ell+z)d\tau = M_1\ell + \rho \int_{\Omega} z d\tau .$$

Clearly,

$$\iint_{Q} y\eta_{tt}\, dxdy = \rho \sum_{i=1}^{\infty} (c_i^* + \varepsilon^2 e_i^*)\ddot{q}_i \ ,$$

where

$$e_i^* = \sum_{j=1}^{\infty} b_{ji} c_j^* = - \sum_{\substack{j=1 \\ i\neq j}}^{\infty} \frac{\omega_i^2}{\omega_i^2 - \omega_j^2} \beta_{ji}\, c_j^*$$

$$= - \sum_{\substack{j=1 \\ i\neq j}}^{\infty} \left[\frac{\omega_i^2}{\omega_i^2-\omega_j^2} \beta_{ij} + R^2 \iint_{Q} (\text{grad}N_i \cdot \text{grad}N_j) dxdy \right] c_j^*$$

$$= = \sum_{\substack{j=1 \\ i\neq j}}^{\infty} B_{ij}^*\, c_j^* \ . \qquad (3.76)$$

Now,

$$\sigma \iint_{Q} \kappa \frac{\partial\eta}{\partial y}\, dxdy = - \frac{\sigma}{R^2\varepsilon^2} \iint_{Q} h \frac{\partial\eta}{\partial y}\, dxdy = - \frac{\sigma}{R^2\varepsilon^2} \oint_{\Gamma} \frac{\partial\eta}{\partial y}\, dS \int_{0}^{\infty} h\, dn$$

$$= - \sigma\cos\gamma \oint_{\Gamma} \frac{\partial\eta}{\partial y}\, dS = - \sigma\cos \oint_{\Gamma} \frac{\partial\eta}{\partial s} \frac{\partial s}{\partial y}\, dS \ .$$

$$= -\rho(a+g)\varepsilon^2\cos\gamma \sum_{i=1}^{\infty} f_i^* q_i(t) ,$$

where

$$f_i^* = R^2\cos\gamma \oint_\Gamma \frac{\partial s}{\partial y}\frac{\partial N_i}{\partial s}\, dS . \tag{3.77}$$

Finally,

$$\sigma \iint_Q \delta\kappa \frac{\partial h}{\partial y}\, dxdy = \sigma \iint_Q \left[-\Delta\eta + \frac{3}{R^2\varepsilon^2}\frac{h\, h_n}{1+h_n^2}\right] h_n \frac{\partial n}{\partial y}\, dxdy$$

$$= -\sigma \oint_\Gamma \Delta\eta \frac{\partial n}{\partial y}\, dS \int_0^\infty dh + \text{terms} > O(\varepsilon^2)$$

$$= -\sigma h_1 \oint_\Gamma \Delta\eta \frac{\partial n}{\partial y}\, dS > O(\varepsilon^2) \approx 0 .$$

Substituting the above formulae in (3.5_1), we get

$$(\delta\bar{F}_{01})_y = -M_1(\delta\ddot{\bar{R}}_0)_y + c\ddot{\theta} - \rho \sum_{i=1}^{\infty} [(c_i^* + \varepsilon^2 e_i^*)\ddot{q}_i + (a+g)\varepsilon^2 f_i^* q_i] . \tag{3.78}$$

Working to the same order of accuracy, we can show

$$\iint_Q (h-h_0)\eta_{tt}\, dxdy = \oint_\Gamma \eta_{tt}\, dS \int_0^\infty h\, dn \approx R^2\varepsilon^2\cos\gamma \oint_\Gamma \eta_{tt}\, dS$$

$$= \varepsilon^2 \sum_{i=1}^{\infty} d_i^* \ddot{q}_i ,$$

$$\sigma \iint_Q \delta\kappa \; dxdy > O(\varepsilon^2) \approx 0,$$

$$\rho \int_{\tau_0} y \; d\tau \approx D + \varepsilon^2 E,$$

$$J \approx A + \varepsilon^2 B,$$

$$\sigma \iint_Q \kappa(\ell+h-h_0) \frac{\partial\eta}{\partial y} \; dxdy \approx - \rho(a+g)\varepsilon^2 \sum_{j=1}^{\infty} \ell f_i^* \, q_i,$$

$$\sigma \iint_Q \delta\kappa \; [y+(\ell+h-h_0) \frac{\partial h}{\partial y}] \; dxdy \approx \rho(a+g)\varepsilon^2 \sum_{i=1}^{\infty} h_i^* \, q_i,$$

$$\iint_Q \phi_x(x,y,h-h_0)\eta_{tt} \; dxdy = \sum_{i=1}^{\infty} [a_i^*+\varepsilon^2(b_i^*+g_i^*)] \; \ddot{q}_i,$$

where

$$d_i^* = R^2 \cos\gamma \oint_\Gamma N_i \; dS,$$

$$g_i^* = - \sum_{\substack{j=1 \\ i\neq j}}^{\infty} B_{ij}^* \; a_j^*,$$

$$h_i^* = R^2 \iint_Q \frac{\partial N_i}{\partial y} \; dxdy, \qquad (3.79)$$

$$A = \rho \int_\Omega [y^2+(\ell+z)^2] d\tau - 4\rho \int_\Omega y^2 d\tau - 2\rho \int_{\Sigma+\tilde{S}_0} y\tilde{\psi}_* \cos(\bar{\nu},z) \; dS,$$

$$B = 2\rho \sum_{i=1}^{\infty} \alpha_1 \left[\frac{1}{\omega_i} \int_{\Sigma} y\Psi_i \cos(\bar{\nu},z)\, dS + \frac{a+g}{\omega_i^2} \iint_Q y\, N_i\, dxdy \right] +$$

$$\rho R^2 \cos\gamma \left[\oint_\Gamma y^2 dS - \left(\frac{1}{Q} \oint_\Gamma dS\right) \iint_Q y^2\, dxdy \right],$$

$$D = \rho \int_\Omega y\, d\tau,$$

$$E = \rho R^2 \cos\gamma \left[\oint_\Gamma y\, dS - \left(\frac{1}{Q} \oint_\Gamma dS\right) \iint_Q y\, dxdy \right],$$

so that equation (3.5_2) becomes

$$(\delta\bar{F}_{01})_z = - M_1(\delta\ddot{\bar{R}}_0)_z - (D+\varepsilon^2 E)\ddot{\theta} - \rho \sum_{i=1}^{\infty} \varepsilon^2 d_i^* \ddot{q}_i, \tag{3.80}$$

while the moment equation (3.5_3) assumes the form

$$(\delta\bar{N}_{01})_x = - (A+\varepsilon^2 B)\ddot{\theta} + c(\delta\ddot{\bar{R}}_0)_y - (D+\varepsilon^2 E)(\delta\ddot{\bar{R}}_0)_z$$
$$- \rho \sum_{i=1}^{\infty} \left\{ [a_i^* + \varepsilon^2(b_i^* + g_i^*)\ddot{q}_i + (a+g)[c_i^* + \varepsilon^2(h_i^* - \ell f_i^* + e_i^*)]q_i \right\}. \tag{3.81}$$

We apply the method of perturbation theory to (3.66), (3.77), (3.79) and (3.80) close to the undisturbed case, restricting ourselves to terms of the order of $O(\varepsilon^2)$,

$$q_i = q_i' + \varepsilon^2 \xi_i,$$
$$(\delta\bar{F}_{01})_y = (\delta\bar{F}_{01})_y' + \varepsilon^2 F_y, \tag{3.82}$$

$$(\delta\bar{F}_{01})_z = (\delta\bar{F}_{01})'_z + \varepsilon^2 F_z,$$

$$(\delta\bar{N}_{01})_x = (\delta\bar{N}_{01})'_x + \varepsilon^2 T_x,$$

with the result that

$$\ddot{q}'_i + \omega_i^2 q'_i = -\frac{\omega_i^2}{a+g}[a_i^*\ddot{\theta} + c_i^*(\delta\ddot{\bar{R}}_0)_y],$$

$$(\delta\bar{F}_{01})'_y = -M_1(\delta\ddot{\bar{R}}_0)_y + c\ddot{\theta} - \rho\sum_{i=1}^{\infty} c_i^*\ddot{q}'_i, \tag{3.83}$$

$$(\delta\bar{F}_{01})'_z = -M_1(\delta\ddot{\bar{R}}_0)_z - D\ddot{\theta},$$

$$(\delta\bar{N}_{01})'_x = -A\ddot{\theta} + c(\delta\ddot{\bar{R}}_0)_y - D(\delta\ddot{\bar{R}}_0)_z - \rho\sum_{i=1}^{\infty}[a_i^*\ddot{q}'_i + (a+g)c_i^*q'_j];$$

$$\ddot{\xi}_i + \omega_i^2\xi_i = -\sum_{j=1}^{\infty}(A_{ij}^*\ddot{q}'_j + \omega_i^2 B_{ij}^* q'_j) - \frac{\omega_i^2}{a+g}[b_i^*\ddot{\theta} + d_i^*(\delta\ddot{\bar{R}}_0)_z],$$

$$F_y = -\rho\sum_{i=1}^{\infty} c_i^*\ddot{\xi}_i - \rho\sum_{i=1}^{\infty}[e_i^*\ddot{q}'_i + (a+g)f_i^*q'_i],$$

$$F_z = -E\ddot{\theta} - \rho\sum_{i=1}^{\infty} d_i^*\ddot{q}'_i, \tag{3.84}$$

$$T_x = -B\ddot{\theta} - E(\delta\ddot{\bar{R}}_0)_z - \rho\sum_{i=1}^{\infty}[a_i^*\ddot{\xi}_i + (a+g)c_i^*\xi_i]$$

$$-\rho\sum_{i=1}^{\infty}[(g_i^* + b_i^*)\ddot{q}'_i + (a+g)(e_i^* + h_i^* - \ell f_i^*)g'_i].$$

Using (3.81), the free surface displacement (3.49_2) becomes

$$\eta = \sum_{i=1}^{\infty} N_i q_i'(t) + \varepsilon^2 \sum_{i\ 1}^{\infty} \left[N_i \xi_i(t) + \sum_{j=1}^{\infty} b_{ji} N_j q_i'(t) \right] \qquad (3.85)$$

The system of equations (3.83) and the first term in the right-hand side of (3.85) define the contribution of the liquid motion to the overall vehicle motion in the absence of surface tension ($\sigma=\varepsilon=0$). System (3.84) attenuated by ε^2 and the remaining terms in the right-hand side of (3.85) define the additional contribution of the liquid motion to the overall vehicle motion when surface tension and contact angle requirements are taken into account. Notice that the corrections obtained here are proportional to ε^2, i.e., to σ.

In some applications, it is convenient to transform the Stokes potential $\tilde{\psi}_*$,

$$\tilde{\psi}_* = 2y(\ell+z) - \tilde{\psi}_1, \qquad (3.86)$$

where $\tilde{\psi}_1$ satisfies

$$\Delta\tilde{\psi}_1, \quad \text{throughout } \Omega,$$

$$\frac{\partial\tilde{\psi}_1}{\partial\nu} = 2(\ell+z)\cos(\bar{\nu},y), \quad \text{at } \Sigma + \hat{S}_0(z=0). \qquad (3.87)$$

If we use $\tilde{\psi}_1$ instead of $\tilde{\psi}_*$ in the above expressions, the results remain essentially unchanged except for

$$a_i^* = \iint_Q (\ell y - \tilde{\psi}_1)\, dxdy$$

$$b_i^* = \frac{a+g}{\omega_i^2}\alpha_1 + R^2\cos\gamma\left[\oint_\Gamma y\, N_i\, dS - \left(\frac{1}{Q}\oint_\Gamma dS\right)\iint_Q y\, N_i\, dxdy\right]$$

$$\alpha_i = R^2\cos\gamma\left[\left(\frac{1}{Q}\oint_\Gamma dS\right)\left(2\ell\iint_Q \frac{\partial N_i}{\partial y}\, dxdy - \iint_Q (\mathrm{grad}N_i\cdot\mathrm{grad}\tilde{\psi}_1)\, dxdy\right)\right.$$

$$\left. - 2\ell\oint_\Gamma \frac{\partial y}{\partial s}\frac{\partial N_i}{\partial s}\, dS + \oint_\Gamma \frac{\partial\psi_1}{\partial s}\frac{\partial N_i}{\partial s}\, dS\right]$$

$$A = \rho\int_\Omega [y^2+(\ell+z)^2]d\tau - 4\rho\int_\Omega (\ell+z)^2 d\tau - 2\rho\int_{\Sigma+\hat{S}_0} (\ell+z)\tilde{\psi}_1\cos(\bar{\nu},y)\, dS.$$

To find the contribution of the liquid motion to the overall vehicle motion in an actual problem, we would proceed as follows:

(1) In the absence of surface tension ($\varepsilon=0$), solve the boundary value problem (3.53) subject to condition (3.30) for the eigenfrequencies ω_i and corresponding functions Ψ_i, N_i;

(2) In the absence of surface tension, solve the system (3.60) or (3.86) for the Stokes potential $\tilde{\psi}_*$ or $\tilde{\psi}_1$;

(3) Determine the coefficients given in (3.51) and (3.64) or (3.87_2);

(4) Compute the coefficients called for in (3.82) and (3.83), namely (3.57), (3.67), (3.75), (3.76), (3.77), (3.79) and/or (3.88);

(5) Solve the system of second order ordinary differential equations (3.83_1), thereby expressing the generalized coordinates q_i' in terms of the vehicle state;

(6) Solve the system of second order ordinary differential equations (3.84_1), thereby expressing the generalized coordinates ξ_i in terms of the vehicle state (assuming the q_i''s to be expressed in terms of the vehicle state). When attenuated by ε^2, the ξ_i's give the correction due to surface tension and contact angle effects;

(7) Express the forces and moment from (3.83_2), (3.83_3) and (3.83_4), in terms of the state of the vehicle via the q_i''s;

(8) Express the forces and moment from (3.84_2), (3.84_3) and (3.84_4), in terms of the state of the vehicle via the q_i''s and ξ_i's;

(9) Substitute the above forces and moment into (3.82) to obtain the total forces and moment;

(10) Substitute the total forces and moment thus obtained into the overall vehicle equations. On the other hand, we may substitute the quantities called for in (2.55) directly;

(11) The free boundary displacement η may be obtained from (3.85) if desired.

3.5 Rotationally Symmetric Cavities

Let us reduce our theory to the pragmatic case of rotationally symmetric tanks, such as abound in aerospace vehicle applications. Naturally, we assume the vessel to be symmetric about the axis of constant acceleration, the z axis. In what follows, it is convenient to let

$$z-L \xrightarrow{\text{Replace}} z\ ,$$

$$L+L_1 \longrightarrow \ell\ ,$$

$$2L,y+L^2\tilde{\psi}_1 \longrightarrow \psi_1, \qquad (3.89)$$

$$-\bar{\nu} \longrightarrow \bar{\nu}\ ,$$

where the new z is now reckoned from the undisturbed center of gravity of the liquid ($\varepsilon=0$), L is the distance between the undisturbed center of gravity and the free boundary, and L_1 is the distance between the center of gravity of the liquid ($\varepsilon=0$) and an arbitrary point along the vehicle axis (Fig. 8). The new Stokes potential satisfies

$$\Delta\tilde{\psi}_1 = 0 \quad , \quad \text{throughout } \Omega,$$

$$\frac{\partial\tilde{\psi}_1}{\partial\nu} = \frac{2z}{L^2}\cos(\bar{\nu},y) \quad , \quad \text{at } \Sigma + \tilde{S}_0(z=L). \qquad (3.90)$$

In addition, let

$$\sqrt{L/\Omega\gamma_i}\ N_i \xrightarrow{\text{Replace}} N_i\ ,$$

$$\frac{a+g}{\omega_i}\sqrt{L/\Omega\gamma_i}\ \Psi_i \longrightarrow \Psi_i\ ,$$

$$\sqrt{\Omega\gamma_i/L}\ q_i' \longrightarrow q_i'\ , \qquad (3.91)$$

$$\sqrt{\Omega\gamma_i/L}\ \xi_i \longrightarrow \xi_i\ ,$$

$$\frac{K_i}{L} \longrightarrow \frac{\omega_i^2}{a+g}\ ,$$

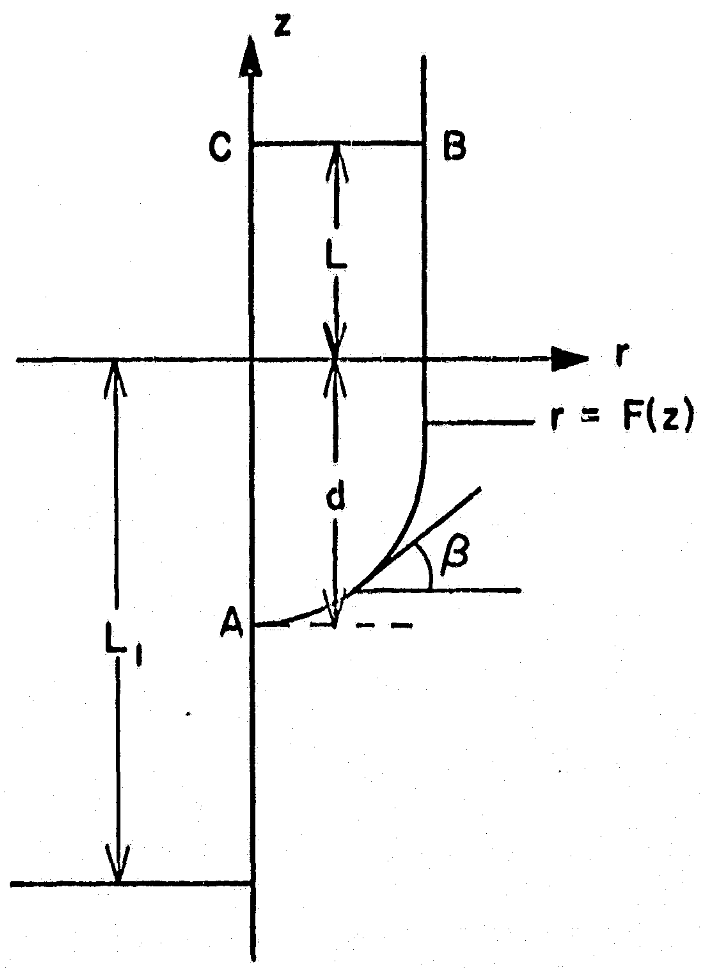

Figure 8. Rotationally Symmetric Vessel, Notation

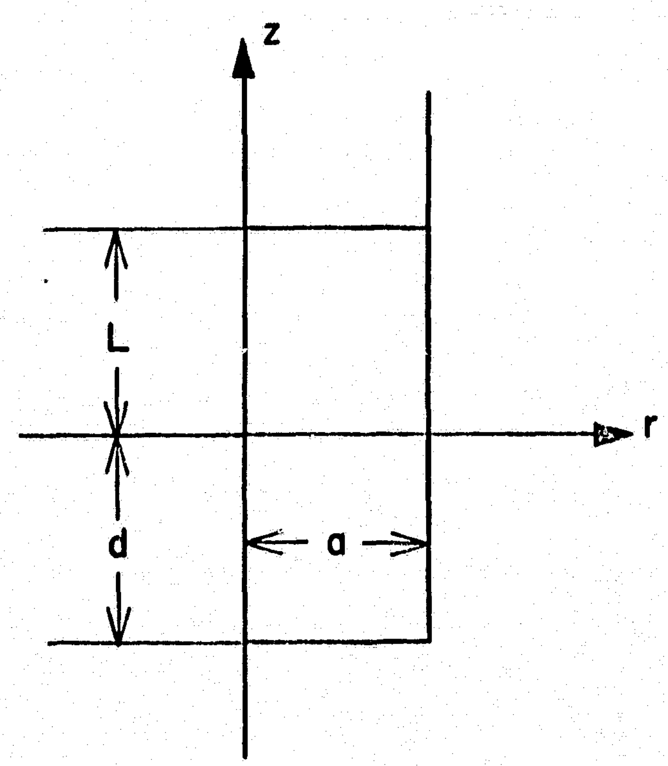

Figure 9. Circular Cylindrical Tank

where

$$\Omega = \int_\Omega d\tau \ , \tag{3.92}$$

and the new N_i is connected with γ_i by

$$\frac{\gamma_i \Omega}{L} = \iint_Q N_i^2 \, dxdy \ . \tag{3.93}$$

The new Ψ_i and N_i are determined from the boundary value problem

$$\begin{aligned}
\Delta\Psi_i &= 0 \quad , \quad \text{throughout } \Omega, \\
\frac{\partial\Psi_i}{\partial\nu} &= 0 \quad , \quad \text{at } \Sigma, \\
\frac{\partial\Psi_i}{\partial z} &= \frac{K_i}{L}\Psi_i \quad , \quad \text{at } \widetilde{S}_0 \ (z=L), \\
\Psi_i + N_i &= 0 \quad , \quad \text{at } z=L \ .
\end{aligned} \tag{3.94}$$

Using (3.89-94), our theory reduces to the following for rotationally symmetric tanks:

$$\begin{aligned}
q_i &= q_i' + \varepsilon^2 \xi_i \ , \\
(\delta\overline{F}_{01})_y &= (\delta\overline{F}_{01})_y' + \varepsilon^2 F_y \ , \\
(\delta\overline{F}_{01})_z &= (\delta\overline{F}_{01})_z' + \varepsilon^2 F_z \ , \\
(\delta\overline{N}_{01})_x &= (\delta\overline{N}_{01})_x' + \varepsilon^2 T_x \ ,
\end{aligned} \tag{3.95}$$

where

$$\ddot{q}_i' + \frac{a+g}{L} K_i q_i' = - K_i[a_i^*\ddot{\theta} + c_i^*(\delta\ddot{\bar{R}}_0)_y] ,$$

$$(\delta\bar{F}_{01})_y' = - M_1(\delta\ddot{\bar{R}}_0)_y + M_1 L_1\ddot{\theta} - M_1 \sum_{i=1}^{\infty} \gamma_i c_i^* \ddot{q}_i' ,$$

$$(\delta\bar{F}_{01})_z' = - M_1(\delta\ddot{\bar{R}}_0)_z , \tag{3.96}$$

$$(\delta\bar{N}_{01})_x' = - \Delta\ddot{\theta} + M_1 L_1(\delta\ddot{\bar{R}}_0)_y - M_1 \sum_{i=1}^{\infty} \gamma_i[a_i^*\ddot{q}_i' + (a+g)c_i^* q_i'] ;$$

$$\ddot{\xi}_i + \frac{a+g}{L} K_i \xi_i = - \sum_{j=1}^{\infty} (A_{ij}^*\ddot{q}_j' + \frac{a+g}{L} K_i B_{ij}^* q_j') - K_i b_i^*\ddot{\theta} ,$$

$$F_y = - M_1 \sum_{i=1}^{\infty} \gamma_i c_i^*\ddot{\xi}_i - M_1 \sum_{i=1}^{\infty} \gamma_i[e_i^*\ddot{q}_i' + (a+g) f_i^* q_i'] ,$$

$$F_z = 0 , \tag{3.97}$$

$$T_x = - B\ddot{\theta} - M_1 \sum_{i=1}^{\infty} \gamma_i[a_i^*\ddot{\xi}_i + (a+g)c_i^*\xi_i]$$

$$- M_1 \sum_{i=1}^{\infty} \gamma_i[(b_i^* + g_i^*)\ddot{q}_i' + (a+g)(e_i^* + h_i^* - (L+L_1)f_i^*)q_i'] ,$$

with

$$A_{ij}^* = \frac{K_i}{K_i - K_j}\bar{\beta}_{ij} + \frac{K_i}{\Omega\gamma_i} R^2\cos\gamma \oint_{\Gamma} N_i N_j \, dS , \quad i \neq j ,$$

$$= \frac{L^2 R}{K_i \gamma_i \Omega}\cos\gamma \left[\left(\frac{R}{Q}\oint_{\Gamma} dS\right) \iint_{Q} (\mathrm{grad} N_i)^2 \, dx dy - R \oint_{\Gamma} \left(\frac{\partial N_i}{\partial s}\right)^2 dS \right]$$

$$+ \frac{K_i}{\gamma_i \Omega} R\cos\gamma \left[R \oint_\Gamma N_i^2 \, dS - \frac{\gamma_i \Omega}{L} \left(\frac{R}{Q} \oint_\Gamma dS\right) \right] , \quad i=j ,$$

$$B_{ij}^* = \frac{K_i}{K_i - K_j} \beta_{ij} + \frac{R^2 L}{\Omega \gamma_i} \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}N_j) \, dxdy , \quad i \neq j$$

$$= \frac{R^2 L}{\Omega \gamma_i} \iint_Q (\mathrm{grad}N_i)^2 \, dxdy , \quad i=j ,$$

$$a_i^* = (L-L_1) b_i - L h_i ,$$

$$b_i^* = \frac{L}{\Omega \gamma_i K_i} R\cos\gamma \left[\left(\frac{R}{Q} \oint_\Gamma dS\right) \left(2L \iint_Q \frac{\partial N_i}{\partial y} \, dxdy - L^2 \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}\widetilde{\psi}_1) dxdy \right. \right.$$

$$\left. - 2LR \oint_\Gamma \frac{\partial y}{\partial s} \frac{\partial N_i}{\partial s} \, dS + L^2 R \oint_\Gamma \frac{\partial \widetilde{\psi}_1}{\partial s} \frac{\partial N_i}{\partial s} \, dS \right] + \frac{1}{\Omega \gamma_i} R\cos\gamma \left[R \int_\Gamma y \, N_i \, dS \right.$$

$$\left. - \Omega \gamma_i \left(\frac{R}{Q} \oint_\Gamma dS\right) b_i \right] , \tag{3.98}$$

$$c_i^* = b_i ,$$

$$e_i^* = - \sum_{\substack{j=1 \\ i \neq j}}^{\infty} B_{ij}^* \, c_j^* ,$$

$$f_i^* = \frac{R^2}{\Omega \gamma_i} \cos\gamma \int_\Gamma \frac{\partial s}{\partial y} \frac{\partial N_i}{\partial s} \, dS ,$$

$$g_i^* = - \sum_{\substack{j=1 \\ i \neq j}}^{\infty} B_{ij}^* \, a_j^* ,$$

$$h_i^* = \frac{R^2}{\Omega\gamma_i} \iint_Q \frac{\partial N_i}{\partial y}\, dxdy \;,$$

$$A = \rho \int_\Omega (y^2+z^2)d\tau - 4\rho \int_\Omega z^2 d\tau + 2\rho L^2 \int_{\Sigma+\widetilde{S}_0} z\widetilde{\psi}_1 \cos(\overline{\nu},y)\, dS + M_1 L_1^2 \;,$$

$$B = M_1 \left\{ \frac{R^2}{\Omega} \cos\gamma \left[\oint_\Gamma y^2 dS - \left(\frac{1}{Q} \oint_\Gamma dS\right) \iint_Q y^2 dxdy \right] + 2 \sum_{i=1}^{\infty} \gamma_i \ell_i^* \right\} \;,$$

$$\beta_{ij} = - \frac{R^2 L}{\Omega\gamma_i} \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}N_j) dxdy - \frac{K_j R}{L} \cos\gamma \left[\sqrt{\frac{\gamma_j}{\gamma_i}} \left(\frac{R}{Q} \oint_\Gamma dS\right)\delta_{ij} \right.$$

$$- \frac{L^3}{\Omega\gamma_i K_i K_j} \left(\frac{R}{Q} \oint_\Gamma dS\right) \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}N_j)\, dxdy \; +$$

$$\left. \frac{RL^3}{\Omega\gamma_i} \left(\frac{1}{K_i K_j} \oint_\Gamma \frac{\partial N_i}{\partial s} \frac{\partial N_j}{\partial s}\, dS - \frac{1}{L^2} \oint_\Gamma N_i N_j \, dS \right) \right] \;,$$

$$\ell_i^* = \frac{RL^2}{\nu\gamma_i} \cos\gamma \left[\left(\frac{R}{Q} \oint_\Gamma dS\right)\left(2L \iint_Q \frac{\partial N_i}{\partial y}\, dxdy - L^2 \iint_Q (\mathrm{grad}N_i \cdot \mathrm{grad}\widetilde{\psi}_1) dxdy\right) \right.$$

$$\left. - 2LR \oint_\Gamma \frac{\partial y}{\partial s} \frac{\partial N_i}{\partial s}\, dS + RL^2 \oint_\Gamma \frac{\partial \widetilde{\psi}_1}{\partial s} \frac{\partial N_i}{\partial s}\, dS \right] \left(\frac{1}{2}\, h_i + b_i\right)$$

$\overline{\beta}_{ij} = \beta_{ij}$ with K_i <u>replaced</u> by K_j ,

$$\Omega\gamma_i b_i = \iint_Q N_i \, y \, dxdy \;,$$

$$\frac{\Omega\gamma_i h_i}{L} = \iint_Q \widetilde{\psi}_1 \, N_i \, dxdy \;.$$

The free boundary displacement may be represented in the form

$$\eta = \eta' + \varepsilon^2 \zeta \ , \tag{3.99}$$

where

$$\eta' = \sum_{i=1}^{\infty} N_i q_i'(t) \ ,$$

$$\zeta = \sum_{i=1}^{\infty} \left[N_i \xi_i(t) + \sum_{\substack{j=1 \\ i \neq j}}^{\infty} B^{*}_{ij} N_j q_i'(t) \right] . \tag{3.100}$$

Finally, the frequency correction is given by the expression

$$\frac{K - K_i}{K_i} = -\frac{1}{2} \varepsilon^2 \beta_{ii} \ . \tag{3.101}$$

This concludes the degeneration of our theory to the case of rotationally symmetric cavities with nearly vertical walls in the vicinity of the free boundary. All the coefficients required in (3.95-3.97) and (3.99-3.101) are expressible by means of (3.98) in terms of K_i, N_i, Ψ_i, $\widetilde{\psi}_1$, angle of contact γ, parameters of the tank and properties of the fluid. The Stokes potential $\widetilde{\psi}_1$ is determined from the system (3.90), while K_i, N_i, Ψ_i are determined from the solution of boundary value problem (3.94). Again, observe that the corrections obtained here are proportional to ε^2, i.e., to $1/B_0$. Whenever $B_0 = \infty$, the above equations for rotationally symmetric vessels agree with / 4 / (except for trivial differences in notation). This is significant because / 4 / forms the theoretical basis for a digital computer routine developed for NASA-MSFC / 1 / (with $B_0 = \infty$), wherein N_i, Ψ_i, K_i and the coefficients called for in (3.96) may be computed for rotationally symmetric tanks of interest. This means that we can extend the digital routine to compute the coefficients required for high Bond number conditions ($B_0 \neq \infty$) with minimum additional effort.

4. TRANSFORMATION OF ROTATIONALLY SYMMETRIC EQUATIONS TO A FORM SUITABLE FOR DIGITAL IMPLEMENTATION ANALOGOUS TO / 4 /.

To extend the NASA-MSFC digital computer routine to include surface tension and contact angle effects, we must first cast the equations of rotationally symmetric cavities in the same setting and terminology of / 4 /.

4.1 Cylindrical Polar Coordinates; Changes in Notation

To this end, we refer the vessel to cylindrical coordinates defined by

$$
\begin{aligned}
x &= r\cos\theta, \\
y &= r\sin\theta, \\
z &= z
\end{aligned}
\tag{4.1}
$$

Because the tank is symmetric about the z axis, the undisturbed boundary surface of the liquid (with $\sigma=0$) enclosed by the cavity is a surface of revolution formed by revolving a curve, shown as ABC in Fig. 8, about the z axis. The origin 0 is located at the undisturbed center of gravity of the liquid.

The curve A-B is formed by the tank profile, and curve B-C is formed by the intersection of the quiescent free boundary (plane z=L) and a plane parallel to and including the z axis. The line AOC is given by the portion of the line r=0, which is interior to the liquid volume.

For cylindrical polar coordinates, the components of the outward directed normal are

$$\cos(\overline{\nu},x) = \sin\beta \cos\theta,$$
$$\cos(\overline{\nu},y) = \sin\beta \sin\theta, \tag{4.2}$$
$$\cos(\overline{\nu},z) = -\cos\beta,$$

where β is defined by

$$\tan\beta = \frac{dr}{dz} \tag{4.3}$$

on the curve ABC. The element of arc length is

$$dS^2 = dz^2 \csc^2\beta = dr^2 \sec^2\beta \tag{4.4}$$

To put our equations in the same setting and terminology of / 4 /, we let

$$\begin{array}{lll} \xi_i \xrightarrow{\text{Replace}} q_i' & \Phi_i(r,z)\sin\theta \rightarrow \Psi_i & \\ \lambda_i \longrightarrow \xi_i & \Phi_i(r,L)\sin\theta \rightarrow N_i & \\ T_1' \longrightarrow (\delta\ddot{\overline{N}}_{01})_x' & \Psi_1(r,z)\sin\theta \rightarrow \tilde{\Psi}_1 & \\ F_2' \longrightarrow (\delta\overline{F}_{01})_y' & F_3' \longrightarrow (\delta\overline{F}_{01})_z' + (\overline{F}_{01})_z & \\ \alpha_3 \longrightarrow (a+g) & -\ddot{D} \longrightarrow \ddot{\theta} & (4.5) \\ \alpha_2 \longrightarrow (\delta\ddot{\overline{R}}_0)_z & I_{11}' \longrightarrow A & \\ M \longrightarrow M_1 & a \longrightarrow R & \\ v \longrightarrow \Omega & 0 \longrightarrow (\delta\overline{F}_{01})_z' & \end{array}$$

4.2 Transformed Equations

With (4.1-5), the equations of rotationally symmetric cavities can be written

$$
\begin{aligned}
q_n &= \xi_n + \frac{1}{B_0}\lambda_n , \\
(\delta\bar{F}_{01})_y &= F_2' + \frac{1}{B_0}F_y , \\
(\delta\bar{F}_{01})_z &= F_3' + \frac{1}{B_0}F_z , \\
(\delta\bar{N}_{01})_x &= T_1' + \frac{1}{B_0}T_y ;
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\ddot{\xi}_n + \frac{\alpha_3}{L}K_n\xi_n &= -K_n\left\{[L(b_n-h_n)-L_1b_n](-\ddot{D})+\alpha_2 b_n\right\} , \\
F_3' &= -M\alpha_3 , \\
F_2' &= -M\alpha_2+ML_1(-\ddot{D})-M\sum_{n=1}^{\infty}\gamma_n b_n\ddot{\xi}_n , \\
T_1' &= ML_1\alpha_2-I_{11}'(-\ddot{D})-M\sum_{n=1}^{\infty}\gamma_n\left\{\alpha_3 b_n\xi_n+[L(b_n-h_n)-L_1b_n]\ddot{\xi}_n\right\} ;
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\ddot{\lambda}_n + \frac{\alpha_3}{L}K_n\lambda_n &= -\sum_{m=1}^{\infty}\left(A_{nm}^{*}\ddot{\xi}_m+\frac{\alpha_3}{L}K_nB_{nm}^{*}\xi_m\right) - K_nb_n^{*}(-\ddot{D}) , \\
F_y &= -M\sum_{n=1}^{\infty}\gamma_n b_n\ddot{\gamma}_n - M\sum_{n=1}^{\infty}\gamma_n[e_n^{*}\ddot{\xi}_n+\alpha_3 f_n^{*}\xi_n] , \\
T_x &= -B(-\ddot{D}) - M\sum_{n=1}^{\infty}\gamma_n\left\{\alpha_3 b_n\lambda_n+[L(b_n-h_n)-L_1b_n]\ddot{\lambda}_n\right\} \\
&\quad - M\sum_{n=1}^{\infty}\gamma_n\left[(b_n^{*}+g_n^{*})\ddot{\xi}_n+\alpha_3(e_n^{*}+h_n^{*}-(L+L_1)f_n^{*})\xi_n\right] , \\
F_z &= 0 ;
\end{aligned}
\tag{4.8}
$$

where the eigenvalues and functions satisfy the boundary value problem

$$\frac{\partial^2\Phi_n}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi_n}{\partial r} - \frac{1}{r^2}\Phi_n + \frac{\partial^2\Phi_n}{\partial z^2} = 0 \ , \qquad \text{(interior to ABCOA)} \ ,$$

$$\frac{\partial\Phi_n}{\partial z} = \frac{K_n}{L}\Phi_n \ , \qquad \text{(along BC)} \ , \qquad (4.9)$$

$$\sin\beta\,\frac{\partial\Phi_n}{\partial r} - \cos\beta\,\frac{\partial\Phi_n}{\partial z} = 0 \ , \qquad \text{(along AB)} \ ,$$

and the Stokes potential is determined from

$$\frac{\partial^2\Psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi_1}{\partial r} - \frac{1}{r^2}\Psi_1 + \frac{\partial^2\Psi_1}{\partial z^2} = 0, \qquad \text{(interior to ABCOA)}$$

$$(4.10)$$

$$\sin\beta\,\frac{\partial\Psi_1}{\partial r} - \cos\beta\,\frac{\partial\Psi_1}{\partial z} = \frac{2z}{L^2}\sin\beta, \qquad \text{(on ABC)} \ .$$

The coefficients called for in (4.8) are

$$A^*_{nm} = \frac{K_n}{K_n-K_m}\bar{\beta}_{nm} + \frac{\pi a^3 K_n}{v\gamma_n}\nu_{nm}\cos\gamma, \qquad n\neq m,$$

$$= \frac{\pi a L^2}{v\gamma_n K_n}(2\tau_{nn}-\nu_{nn})\cos\gamma + \frac{K_n a}{L}\left(\frac{\pi a^2 L}{v\gamma_n}\nu_{nn}-2\right)\cos\gamma, \qquad n=m,$$

$$(4.11)$$

$$B^*_{nm} = \frac{K_n}{K_n-K_m}\beta_{nm} + \frac{\pi a^2 L}{v\gamma_n}\tau_{nm}, \qquad n\neq m$$

$$= \frac{\pi a^2 L}{v\gamma_n}\tau_{nn}, \qquad n=m,$$

$$b_n^* = \frac{\pi a L^3}{v\gamma_n K_n}(\Phi_n(a,L)\Psi_1(a,L) - 2\rho_n)\cos\gamma + \frac{\pi a^4}{v\gamma_n}(\Phi_n(a,L) - \frac{2v\gamma_n}{\pi a^3} b_n)\cos\gamma$$

$$e_n^* = -\sum_{\substack{m=1\\ n\neq m}}^{\infty} B_{nm}^* b_m ,$$

$$f_n^* = \frac{\pi a^2}{v\gamma_n}\Phi_n(a,L)\cos\gamma,$$

$$g_n^* = -\sum_{\substack{m=1\\ n\neq m}}^{\infty} B_{nm}^* [L(b_m - h_m) - L_1 b_m] ,$$

$$h_n^* = \frac{\pi a^3}{v\gamma_n}\Phi_n(a,L) ,$$

$$\ell_n^* = \frac{\pi a L^4}{v\gamma_n}(\Phi_n(a,L)\Psi_1(a,L) - 2\rho_n)(b_n + \frac{1}{2}h_n)\cos\gamma$$

$$B = M\left[\frac{\pi a^5}{2v}\cos\gamma + 2\sum_{n=1}^{\infty}\gamma_n \ell_n^*\right] ,$$

where

$$\beta_{nm} = -\frac{\pi a^2 L}{v\gamma_n}\tau_{nm} - \frac{K_m a}{L}\left[2\sqrt{\frac{\gamma_m}{\gamma_n}}\,\delta_{nm} - \frac{2\pi L^3}{v\gamma_n K_m K_n}\tau_{nm}\right.$$
$$\left. + \frac{\pi L^3}{v\gamma_n}\left(\frac{1}{K_n K_m} - \frac{a^2}{L^2}\right)\nu_{nm}\right]\cos\gamma, \tag{4.12}$$

$\overline{\beta}_{nm} = \beta_{nm}$ with K_n <u>replaced</u> by K_m ,

$$\tau_{nm} = \int_C^B \left[\frac{\partial \Phi_n(r,L)}{\partial r} \frac{\partial \Phi_m(r,L)}{\partial r} + \frac{\Phi_n(r,L)\Phi_m(r,L)}{r^2} \right] r\, dr \ ,$$

$$\nu_{nm} = \Phi_n(a,L)\Phi_m(a,L) \ ,$$

$$\rho_n = \int_C^B \left[\frac{\partial \Phi_n(r,L)}{\partial r} \frac{\partial \Psi_1(r,L)}{\partial r} + \frac{\Phi_n(r,L)\Psi_1(r,L)}{r^2} \right] r\, dr.$$

The eigenfrequencies K_n, eigenfunctions Φ_n, Stokes potential Ψ_1, and the coefficients

$$b_n = \frac{\pi}{v\gamma_n} \int_C^B r^2 \ \Phi_n(r,L)\ dr \ ,$$

$$h_n = \frac{2\pi}{K_n v \gamma_n} \int_A^B zr \ \Phi_n(r,z)\ dz \ , \tag{4.13}$$

$$\gamma_n = \frac{\pi L}{v} \int_C^B r[\Phi_n(r,L)]^2\ dr \ ,$$

$$I_{11}' = \rho \int_{uv} (r^2 \sin^2\theta + z^2) dv - 4\rho \int_{uv} z^2 dv +$$

$$2\rho L^2 \pi \int_A^B zr\Psi_1(r,z) dz + ML_1^2$$

are parameters associated with the equations of motion in the absence of surface tension, and therefore can be found via the current NASA-MSFC digital computer program / 1 /.

To find the corrections due to surface tension and contact angle effects, we must compute the additional matrices and vectors in (4.11) which are expressible by means of (4.12) and (4.13). However, an examination of (4.11-13) and the digital computer program / 1 / shows that (4.11) can be formed readily if τ_{nm} and ρ_n are known. The computer program, therefore, was extended to compute these parameters and the resulting matrices and vectors in (4.11). In addition, provision for computing the frequency correction

$$\mu_i = - \frac{1}{2B_0} \beta_{ii} \tag{4.14}$$

was made. The procedure and program are described in / 8 /.

With the scalars, vectors and matrices in (4.11-13) known, the generalized coordinates ξ_n, λ_n can be expressed, in principle, in terms of the vehicle state α_2, $\ddot{D}$ via solutions of (4.7_1) and (4.8_1). The forces and moments, in turn, can be expressed completely in terms of the vehicle state via (4.7_2), (4.7_3), (4.8_2), (4.8_3) and finally (4.6_2-6_3). The resulting expressions then can be substituted into the overall equations of motion of the vehicle.

4.3 Parameters for a Cylindrical Tank

For a circular cylindrical tank with notation given in Fig. 9 , we find

$$\Phi_n(r,z) = \cosh \lambda_n(z+d)\, J_1(\lambda_n r)$$

$$\Psi_1(r,z) = \frac{1}{L^2}\left\{(L-d)r + \frac{4}{h}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]I_1(n\pi r/h)\cos(n\pi/h)(z+d)}{(n\pi/h)^3 I_1'(n\pi a/h)}\right\}$$

$$K_n = L\lambda_n \tanh \lambda_n h$$

$$\gamma_n = \frac{L}{2h}\left(1 - \frac{1}{\lambda_n^2 a^2}\right)[\cosh \lambda_n h\, J_1(\lambda_n a)]^2$$

$$\gamma_n b_n = \frac{\cosh \lambda_n h\, J_1(\lambda_n a)}{ah\lambda_n^2}$$

$$v\gamma_n K_n h_n = \frac{2\pi a}{\lambda_n} J_1(\lambda_n a)\left[L \sinh \lambda_n h - \frac{(\cosh\lambda_n h-1)}{\lambda_n}\right] \qquad (4.15)$$

$$\tau_{nm} = \frac{v\gamma_n}{L\pi}\lambda_n^2\,\delta_{mn} = (\lambda_n a)^2(\tfrac{h}{L})\gamma_n\,\delta_{mn}$$

$$\rho_n = \frac{v\gamma_n}{L\pi}\lambda_n^2\,h_n = (\lambda_n a)^2\,(\tfrac{h}{L})\gamma_n h_n$$

$$\frac{I_{11}'}{\rho} - vL_1^2 = \pi\left[\frac{a^4(L+d)}{4} - (L^3+d^3)a^2\right] + 2\pi a\left[\frac{a(L-d)(L^2-d^2)}{2} + \right.$$

$$\left.\frac{4}{h}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]^2 I_1(n\pi a/h)}{(n\pi/h)^5 I'(n\pi a/h)}\right],$$

where

$\lambda \sim$ roots of $J_1'(\lambda_n a) = 0$,

J_1, $I_1 \sim$ Bessel functions of the first and second kinds, of order one,

$h = L + d.$

In particular, if $\gamma = \pi/2$, we have

$$b_n^* = e_n^* = f_n^* = g_n^* = 0 \ ,$$

$$h_n^* = \frac{\pi a^3}{v\gamma_n} \cosh \lambda_n h \ J_1(\lambda_n a) \ ,$$

$$\beta_{nm} = \overline{\beta}_{nm} = -(\lambda_n a)^2 \delta_{mn} \ , \tag{4.16}$$

$$A_{nm}^* = 0 \ ,$$

$$B_{nm}^* = (\lambda_n a)^2 \delta_{mn} \ .$$

Equations (4.15) and (4.16) may be used to check out the digital computer program.

CONCLUSIONS

Using fundamentals from classical mechanics and hydrodynamics, the complete equations of motion of aerospace vehicles with partially liquid-filled cavities are derived herein. Also, the motions of the liquids in the tanks and their contributions to the overall vehicle motion are defined.

The fundamental equations are specialized to a reference state of motion in the form of constant linear accelerations and angular velocities. Then, the vehicle is slightly disturbed from the known reference state motion and the resulting small motion of deviation obtained. Moreover, the small motions of deviation of the liquids in the tanks and their contributions to the overall perturbation vehicle motion are defined. Several simplified but significant flight conditions are considered.

Assuming that the surface tension is small in comparison with the effective mass forces, a method for determining the behavior of liquids enclosed in tanks are developed for a launch vehicle undergoing planar motion. The approach is based on boundary layer and perturbation theory / 2 /, and constitutes a first order correction of the so-called "high-g' theory ($B_0 = \infty$). Then, the equations are specialized to rotationally symmetric tanks, wherein the axis of symmetry coincides with the vehicle acceleration vector. Finally, the equations describing the behavior of liquids in rotationally symmetric vessels (including surface tension and contact angle effects) are cast in a form suitable for digital implementation analogous to / 1 /.

This document is the theoretical basis of companion report UR-00010, Volume II, which describes the digital computer routine used to obtain the liquid parameters for sloshing in rotationally symmetric tanks, under moderate to high Bond number conditions ($B_0 = 10 - \infty$) at any given contact angle.

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