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ON INDUCED BIREFRINGENCE IN VISCOELASTIC MATERIALS

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16. Abstract

This report provides a theoretical framework for the description of induced birefringence in materials with long-range memory. The basic constitutive assumptions are that the stress and dielectric properties are given by functions of the history of the strain; no assumptions of linearity or smoothness are made for these constitutive functionals. The consequences of material symmetry and frame-indifference are examined in detail, and reduced forms of the constitutive equations are given for several classes of motions of isotropic materials. Among the motions studied are those locally equivalent to extensions, shears, and sheared extensions. Several optical experiments are suggested.

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PREFACE

The goal of the research reported here has been to develop a phenomenological theory of induced birefringence which is general enough to account for observed memory effects and non-linearities in the dependence of dielectric properties on the history of deformation. The theory we give has a mathematical structure closely related to that of Noll's dynamical theory of "simple materials". In that theory, a material is characterized by a functional $\mathfrak{S}$ which gives the stress when the history of the strain is specified. Here, in addition to $\mathfrak{S}$, we have, for each material, a functional $\mathfrak{N}$ which relates the dielectric properties of the material to the history of the strain. For certain broad classes of motions, the requirements of material symmetry and frame-indifference greatly simplify the forms of $\mathfrak{S}$ and $\mathfrak{N}$. Much of this report is concerned with optical applications of such "reduction theorems", all of which are obtained without invoking integral expansions or other special hypotheses of smoothness for material response.

In conversations held in 1960 and 1961, B. D. Coleman and R. A. Toupin [1962, 2] discussed the possibility of developing a theory of induced birefringence which could employ the mathematical machinery then being developed in non-linear continuum mechanics. Sections 3, 4, 6, 10, and 21c of this report draw heavily on unpublished notes which Coleman wrote in the Fall of 1961 as a summary of his work with Toupin.

The classes of motions we discuss in Chapter IV, particularly Sections 14 and 15, are chosen as much for their practical nature as for their mathematical simplicity. Such motions were used in the experimental program carried out in the Department of Aeronautics and Astronautics of the University of Washington, under Grant No. NsG-401 from the National Aeronautics and Space Administration.[#] E. H. Dill's work was supported by that grant, while B. D. Coleman's research was partly supported by the Air Force Office of Scientific Research through Contract AF 49(638)541 and Grant AF 68-1538 to the Mellon Institute of Carnegie-Mellon University.

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[#] See Dill & Fowlkes [1966, 2] and Fowlkes [1969, 1].

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I. INTRODUCTION

1. Fresnel's Theory of Double Refraction

Rays of light generally change direction on passing from one transparent substance into another. Such deflection of light at an interface is called refraction. For some materials, under certain circumstances, such as water at rest, or glass in an undistorted state, a single incident ray gives rise to only one ray upon refraction. A ray of light incident upon a crystal, however, is generally split into two rays upon refraction. The two refracted rays are polarized at right angles to each other and have different speeds of propagation. Substances in which such splitting occurs are said to be doubly refracting or birefringent. Many materials which are not birefringent in undistorted states become birefringent upon deformation. In this essay we study the theory of such induced birefringence.

The propagation of light in a birefringent body can be described by the theory of Fresnel [1827, 1], which we here attempt to outline.[#] In the theory it is assumed that light propagates in transverse harmonic waves. One associates with each point $\underset{\sim}{x}$ of the material a symmetric, positive definite, tensor $\underset{\sim}{L}$, which we may call the Fresnel tensor. If we choose a Cartesian coordinate system (ξ, η, ζ), with origin at $\underset{\sim}{x}$ and axes along the proper vectors $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$ of $\underset{\sim}{L}$, then the equation,

$$v_1^2\xi^2 + v_2^2\eta^2 + v_3^2\zeta^2 = 1, \tag{1.1}$$

[#] A history of the subject is given by Whittaker [1951, 1].

with v_i^2 the proper number of $\underset{\sim}{L}$ corresponding to $\underset{\sim}{k}_i$, describes a figure called the <u>Fresnel ellipsoid</u>. The positive numbers v_1, v_2, v_3 are called the <u>principal wave speeds</u>, and the vectors $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$ are called the <u>principal axes of refraction</u>. The tensor $\underset{\sim}{L}$, and hence the quadric (1.1) depend, in general, on the material and its deformation history. Consider now a plane, harmonic, light wave at $\underset{\sim}{x}$. The plane normal to the direction of propagation of this wave (i.e. the "wave front") intersects the Fresnel ellipsoid in an ellipse $\underline{\underline{E}}$. Fresnel's theory requires that the amplitude vector of the wave be directed along one of the axes of $\underline{\underline{E}}$. There are thus, in general, two harmonic waves possible for a given direction of propagation: each has its amplitude vector directed along one axis of $\underline{\underline{E}}$. The lengths of the semi-axes of $\underline{\underline{E}}$ are the reciprocals of the speeds of these two harmonic waves.

Consider now a single, fixed, Cartesian coordinate system (x,y,z), and suppose that the body $\mathcal{B}$ under consideration has the form of a strip bounded by the parallel planes $z = +\ell/2$, $z = -\ell/2$. We permit the Fresnel tensor $\underset{\sim}{L}$ to vary with x and y in $\mathcal{B}$ but assume that $\underset{\sim}{L}$ is independent of z and is everywhere such that it has a proper vector $\underset{\sim}{k}_3$ parallel to the z-axis. When a plane harmonic wave propagating along the z-axis enters $\mathcal{B}$, it is, according to Fresnel's theory, split into two harmonic waves with mutually perpendicular amplitude vectors. These two waves continue to propagate in the z-direction, but with different speeds v_1, v_2. Each wave has its amplitude parallel to a proper vector of $\underset{\sim}{L}$, and the speed of the wave equals the square root of the corresponding proper number of $\underset{\sim}{L}$. Upon leaving $\mathcal{B}$ the two waves have a relative retardation r given by

$$r = \ell(n_1 - n_2), \tag{1.2}$$

where the numbers

$$n_i = \frac{c}{v_i}, \qquad i = 1,2, \tag{1.3}$$

are called principal indices of refraction. Here c is the velocity of light *in vacuo*. During its traverse of $\mathcal{B}$, each wave experiences an absolute retardation r_i given by

$$r_i = \ell(n_i - 1), \qquad i = 1,2. \tag{1.4}$$

Of course, the relative retardation r is just the difference of the absolute retardations:

$$r = r_1 - r_2. \tag{1.5}$$

We have here assumed that $\mathcal{B}$ is such that one principal axis of refraction, $\underset{\sim}{k}_3$, is everywhere parallel to the z-axis of the system (x,y,z). One can determine r and the direction of the two remaining principal axes, $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, by using a plane polariscope, i.e. by placing $\mathcal{B}$ between two crossed polarizing devices which can be rotated in planes parallel to the (x,y)-plane in such a way that the polarizing axes of the devices remain at right angles. Whenever the polarizer, i.e. the polarizing device between $\mathcal{B}$ and the light source, has its polarizing axis parallel to $\underset{\sim}{k}_1$ or $\underset{\sim}{k}_2$, each light wave transmitted by the polarizer passes through $\mathcal{B}$ with no change in the direction of its amplitude vector. Such a light wave is blocked by the second polarizing device, called the analyzer. If $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ vary from point to point in the (x,y)-plane, then the locus of the points in this plane for which either $\underset{\sim}{k}_1$ or $\underset{\sim}{k}_2$ is parallel to an axis of the polarizer appears as a dark figure when viewed through the analyzer. This dark figure, which depends on the orientation of the polarizer, usually takes the form of a pair of curved lines, called _isoclinic lines_ or _isoclines_. By rotating the polarizer and analyzer in unison, a family of such lines, called the _isoclinic fringe pattern_, is obtained. Observation of this pattern enables one to determine the axes of refraction for $\mathcal{B}$ as functions of x and y. Whenever the axis of the polarizer is not parallel to a principal axis of refraction, each of the two harmonic waves transmitted by $\mathcal{B}$ is resolved by the analyzer into two waves: a component parallel and a component perpendicular to the polarizing axis. Of course, only components along the axis of the analyzer pass through it. Thus, the analyzer transmits two waves with colinear, but oppositely directed amplitude vectors of equal magnitude; these two transmitted waves will interfere (i.e. cancel) whenever the relative retardation r obeys the formula

$$r = N\lambda, \tag{1.6}$$

with N an integer and λ the wavelength of the light _in vacuo_. For each integer N, the locus of (x,y)-values for which $r = N\lambda$ forms a figure which appears dark when viewed through the analyzer; this dark figure, which often is a curved line or set of curved lines, is referred to as the N'th _isochromatic line_ or _isochromatic fringe_. The family of all isochromatic fringes (i.e. $N = 0, \pm 1, \ldots$) is called the _isochromatic fringe pattern_. Through observations of the isochromatic fringe pattern, the relative retardation r can be determined at several values of x and y. It is clear from equation (1.2) that if the thickness ℓ of $\mathcal{B}$ is known, measurement of r yields the difference Δ between the indices of refraction n_1, n_2:

$$\Delta \stackrel{\text{def}}{=\!=} n_1 - n_2 = \frac{r}{\ell} . \tag{1.7}$$

The number Δ is called the <u>birefringence</u> of $\mathcal{B}$ (at x and y) for propagation in the direction $\underset{\sim}{k}_3$.

The above description of the properties of isoclinic and isochromatic lines is summarized and extended in a single formula which, for given values of λ and ℓ, describes the variation of the intensity $\underline{I}$ of the light transmitted by the analyzer as a function of the birefringence Δ and the angle ϕ between $\underset{\sim}{k}_1$ and the axis of the polarizer:#

$$\underline{I} = A \sin^2(2\phi)\sin^2\left(\frac{\pi\ell}{\lambda}\Delta\right) . \tag{1.8}$$

Here A is a constant which depends upon the loss of light through reflection, absorption, and scattering. It is clear from (1.8) that $\underline{I}$ vanishes wherever ϕ equals 0° or 90° and wherever $\ell\Delta/\lambda$ is an integer; i.e. on the isoclinic and the isochromatic lines. Under appropriate circumstances one can use (1.8) to determine the birefringence from intensity measurements at values of x and y which are not on an isochromatic fringe. An often simpler method of obtaining Δ away from the isochromatic fringe pattern involves the insertion of a compensator between $\mathcal{B}$ and the analyzer. The compensator, when properly aligned with respect to $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$, gives an additional and controllable relative retardation R to the light reaching the analyzer; one adjusts the compensator so that the total relative retardation, r + R, equals an integral multiple of λ, and then calculates Δ, using (1.7).

#Cf. e.g. Born & Wolfe [1959, 1] §14.4.3.

The indices of refraction n_1 and n_2, at given values of x and y, can be measured with an interferometer. As usually employed, such a device splits the incident wave into two waves; one is polarized along a principal axis of refraction for $\mathcal{B}$ and is passed through $\mathcal{B}$ to incur a retardation given by equation (1.4). The second wave is directed through a medium which induces a controllable retardation. The two waves are then recombined and are found to interfere whenever the difference in their retardations is an integral multiple of λ. Thus, if λ is known, r_1 and r_2 can be determined. Once r_1, r_2, and the thickness ℓ are measured, the indices of refraction n_1 and n_2 may be calculated from (1.4).

2. Classical Theories of Induced Birefringence

a. Photoelasticity

Most transparent solid materials which are not birefringent in stress-free states become birefringent when deformed. When the Fresnel tensor is determined by the present configuration, and is independent of the past history of deformation, this phenomenon is called the photoelastic effect.# For isotropic materials the photoelastic effect can be described by assuming that each principal direction of stretch is a principal axis of refraction and that the principal wave speeds v_i (and therefore the principal indices of refraction $n_i \overset{\text{def}}{=} c/v_i$) depend upon only the principal extensions ϵ_i. Employing the assumed isotropy of the material, it can be shown that, in the limit of infinitesimal deformations, this

#The phenomenon was apparently first observed by Brewster [1816, 1].

dependence must have the form:

$$\left.\begin{aligned} n_1 &= n^o + C_1\epsilon_1 + C_2(\epsilon_1+\epsilon_2+\epsilon_3), \\ n_2 &= n^o + C_1\epsilon_2 + C_2(\epsilon_1+\epsilon_2+\epsilon_3), \\ n_3 &= n^o + C_1\epsilon_3 + C_2(\epsilon_1+\epsilon_2+\epsilon_3). \end{aligned}\right\} \qquad (2.1)$$

These relations, which may be called the <u>constitutive equations of linear photoelasticity</u> agree, for small deformations, with those proposed by Neumann [1841, 1]. The numbers n^o, C_1, and C_2 are material constants; of course, n^o is the refractive index of the unstrained material. It follows from (2.1) that the birefringence for wave propagation along $\underset{\sim}{k}_3$ is given by the equation

$$\Delta \stackrel{\text{def}}{=\!=} n_1 - n_2 = C_1(\epsilon_1 - \epsilon_2), \qquad (2.2)$$

which is sometimes called the "strain-optic law".

For an isotropic elastic material subject to an infinitesimal deformation, the principal axes of stress lie along the principal directions of stretch, and the principal stresses σ_i are given by

$$\left.\begin{aligned} \sigma_1 &= 2\mu\epsilon_1 + \lambda(\epsilon_1+\epsilon_2+\epsilon_3), \\ \sigma_2 &= 2\mu\epsilon_2 + \lambda(\epsilon_1+\epsilon_2+\epsilon_3), \\ \sigma_3 &= 2\mu\epsilon_3 + \lambda(\epsilon_1+\epsilon_2+\epsilon_3), \end{aligned}\right\} \qquad (2.3)$$

where λ and μ are called the <u>Lamé constants</u>. For solids these material constants are expected to obey the inequalities

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu > 0. \qquad (2.4)$$

When such is the case, each principal axis of stress is a principal axis of refraction and the principal indices of refraction can be regarded as functions of the principal stresses. Indeed, (2.1), (2.3), and (2.4) yield

$$\left.\begin{aligned} n_1 &= n^o + c_1\sigma_1 + c_2(\sigma_1+\sigma_2+\sigma_3), \\ n_2 &= n^o + c_1\sigma_2 + c_2(\sigma_1+\sigma_2+\sigma_3), \\ n_3 &= n^o + c_1\sigma_3 + c_2(\sigma_1+\sigma_2+\sigma_3), \end{aligned}\right\} \tag{2.5}$$

where c_1 and c_2 are material constants related to C_1, C_2, λ, and μ. The equations (2.5) are, in essence, the photoelastic relations proposed by Maxwell.# It follows from (2.5) that the birefringence for propagation along $\underset{\sim}{k}_3$ obeys the equation

$$\Delta = c_1(\sigma_1 - \sigma_2), \tag{2.6}$$

which is often called the "stress-optic law".

In the most common way of testing the photoelastic relations, one subjects a layer of material to plane stress.## It is usually found that the material parameters occurring here, such as n^o and c_1, depend on the frequency of the light, an effect called dispersion. We do not discuss dispersion in this essay, but instead focus our attention on

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#Maxwell [1853, 1], unfamiliar with the earlier work of Neumann [1841, 1], who had taken strain as the independent variable, formulated a theory of photoelasticity in terms of stress. For small deformations of elastic materials, the theories of Neumann and Maxwell become identical. Nevertheless, in proposing that it is the stress that determines birefringence, Maxwell innocently unleashed a controversy which occupies much of the subsequent literature on the subject.

##For an historical survey see Coker and Filon [1931, 1].

phenomena which can be studied using light of a single frequency. First, it is known that the equations (2.1) and (2.5) can hold only for small strains. Second, even in the limit of infinitesimal strains, it is often observed that the birefringence varies in time during intervals of constant stress[#] or strain; such <u>optical creep</u> or <u>optical relaxation</u> is indicative of a dependence of birefringence on the <u>past history</u> of deformation.

b. Streaming Birefringence

It has long been known that viscous fluids can exhibit birefringence when flowing.[##] As we shall show in a forthcoming paper,[###] for an incompressible simple fluid with fading memory,[####] in the limit of slow motion, each proper vector of the rate of deformation tensor $\mathbf{D}$,[####] is also a principal axis of refraction, and the principal indices of

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[#] Optical creep was observed in gelatine by von Bjerkén [1891, 1]; Rossi [1910, 1] made an extensive study of the phenomenon in celluloid; cf. Coker & Filon [1931, 1] §3.34.

[##] Maxwell [1873, 1] observed the effect in Canada balsam in 1866, and Mach [1872, 1] independently observed it in glass.

[###] The matter was treated also in the work Coleman & Toupin referred to earlier [1962, 2].

[####] The concepts of "fading memory" and "slow motion" employed here are those used by Coleman & Noll [1960, 1] in proving the retardation theorem.

[#####] $\mathbf{D}$, also called the <u>stretching tensor</u>, is equal to the symmetric part of the velocity gradient.

refraction are determined by the proper numbers of $\underset{\sim}{D}$ through linear equations of the form

$$n_i = n^\circ + 2\bar{\eta} d_i, \qquad i = 1,2,3, \tag{2.7}$$

with n° and $\bar{\eta}$ material constants. This yields the relation

$$\Delta = 2\bar{\eta}(d_1 - d_2) \tag{2.8}$$

for the birefringence $\Delta = n_1 - n_2$.

For quantitative experimental studies of streaming birefringence, one usually employs Couette flow. In this circular flow of a fluid confined between coaxial cylinders, at each point $\underset{\sim}{x}$, one proper vector of $\underset{\sim}{D}$, say $\underset{\sim}{d}_3$, is parallel to the common axis of the cylinders, and the remaining two form angles of 45° with the radial vector to $\underset{\sim}{x}$. One studies this flow with a polariscope arranged so that the light is propagating in the direction $\underset{\sim}{d}_3$; the smallest of the angles formed by the isoclinic lines# with the axis of the polarizer is usually denoted by χ and called the <u>extinction angle</u>. Now, if each proper vector of $\underset{\sim}{D}$ is also an axis of refraction, as is the case in the "slow flow" approximation behind equation (2.7), then the extinction angle should be 45°. Steady flow experiments on many "non-Newtonian fluids" show that χ is

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#In Section 21 we remark that if the gap between the inner and outer cylinders is small, then each isocline is a straight line lying along a radius vector.

not, in general, equal to 45°, but approaches this value as the rate of shear tends to zero.#

In the same limit of slow motions for which (2.7) holds, the stress in a general incompressible simple fluid with fading memory obeys the constitutive relations of a Navier-Stokes fluid;## i.e. the principal axes of stress equal the proper vectors of $\underset{\sim}{D}$, and the principal stresses obey the equations

$$\sigma_i \quad = \quad -p + 2\eta d_i, \qquad i = 1,2,3, \tag{2.9}$$

where p is a hydrostatic pressure and η is a positive material constant called the <u>viscosity</u>. It follows that, in the limit of slow motions, each principal axis of stress is also a principal axis of refraction, and the birefringence Δ is given by a relation of the form

$$\Delta \quad = \quad b(\sigma_1 - \sigma_2) \tag{2.10}$$

with $b = \bar{\eta}/\eta$.

Although in the limiting cases of small deformations of solids and slow flow of fluids the principal axes of stress are expected to lie along principal axes of refraction, there is no theoretical principle or

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#Departures of χ from 45° were apparently first observed and studied by Kundt [1881, 1]; solutions of collodion in alcohol and ether were among the substances for which large departures were observed. It appears that De Metz [1888, 1] [1906, 1] made the first quantitative measurements of the birefringence $n_1 - n_2$ in Couette flow.

##Cf. [1960, 1].

reliable experimental evidence indicating that such an elementary rule holds in general.#

We here seek a simple theory of birefringence in viscoelastic materials consistent with the general principles of mechanics and electromagnetism and with the observed phenomenon of optical creep. Furthermore, we want the theory to be sufficiently general to enable discussion not only of photoelasticity for solids but also of streaming birefringence for non-Newtonian fluids (at arbitrary rates of shear). Hence, we must drop the linear approximations which are made in classical theories of mechanics and must also allow the past kinematical history to influence present optical and mechanical behavior.

Any theory which accounts for past deformations of the medium is beset with difficulties not encountered in the classical theory of photoelasticity; these difficulties arise from the fact that motion must be discussed; i.e. the medium cannot be assumed to be always stationary as is done in photoelasticity. Of course, we shall here neglect relativistic effects of order u^2/c^2, where u is the speed of the material points and c is the speed of light *in vacuo*. Even in this approximation, the Maxwell-Lorentz equations for the electromagnetic field in moving media are considerably more complicated than for the case of stationary media. A simple theory of birefringence emerges only if one can safely neglect the dragging of light by the moving dielectric medium. Fortunately, for the fluid speeds ordinarily encountered in viscoelastic flows, such dragging is truly negligible.

#Coleman & Toupin [1962, 1]; see Sections 15 and 21 of this report.

II. BASIC ASSUMPTIONS

3. Concepts from Electromagnetic Theory

Fresnel's theory of double refraction in transparent media can be shown to rest upon the foundation of Maxwell's electromagnetic theory of light.[#] In the terminology of electromagnetic theory, induced birefringence in an isotropic material is a consequence of the dependence of dielectric properties upon the history of deformation.

As we are here concerned with non-magnetic, non-conducting, electrically polarizable media, we may write the basic field equations of electromagnetic theory in the form

$$\left.\begin{aligned} \frac{\partial \mathbf{d}}{\partial t} &= \operatorname{curl} \mathbf{h}, \qquad & \operatorname{div} \mathbf{d} &= 0, \\ \frac{\partial \mathbf{b}}{\partial t} &= -\operatorname{curl} \mathbf{e}, \qquad & \operatorname{div} \mathbf{b} &= 0, \end{aligned}\right\} \tag{3.1}$$

with

$$\mathbf{d} = \epsilon_o \mathbf{e} + \mathbf{p}, \qquad \mathbf{h} = \mu_o^{-1} \mathbf{b} + \mathbf{u} \times \mathbf{p}. \tag{3.2}$$

Here $\mathbf{e}$ is the <u>electric field</u>, $\mathbf{b}$ is the <u>magnetic flux density</u>, $\mathbf{p}$ is the <u>polarization density</u>, and $\mathbf{u}$ is the velocity of the medium. The two vectors $\mathbf{d}$ and $\mathbf{h}$, given by (3.2), are called, respectively, the <u>electric displacement</u> and the <u>magnetic field</u>. The positive numbers ϵ_o and μ_o are fundamental constants which depend upon only the choice of units and obey

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[#]Maxwell [1865, 1] §§102-105, [1873, 2]; see also Born & Wolf [1959, 1] and Truesdell & Toupin [1960, 2].

the relation $\mu_o \epsilon_o = c^{-2}$, with c the speed of light _in vacuo_. A saltus $[\underset{\sim}{h}]$, $[\underset{\sim}{d}]$, etc. experienced by $\underset{\sim}{h}$, $\underset{\sim}{d}$, $\underset{\sim}{e}$, or $\underset{\sim}{b}$ across a surface with unit normal $\underset{\sim}{\nu}$, must be compatible with the following _jump conditions_:

$$\left.\begin{aligned} \underset{\sim}{\nu} \times [\underset{\sim}{h}] &= \underset{\sim}{0}, & \underset{\sim}{\nu}\cdot[\underset{\sim}{d}] &= 0, \\ \underset{\sim}{\nu} \times [\underset{\sim}{e}] &= \underset{\sim}{0}, & \underset{\sim}{\nu}\cdot[\underset{\sim}{b}] &= 0. \end{aligned}\right\} \tag{3.3}$$

Now, the simplest electromagnetic theory of light in moving media is based upon the constitutive equation#

$$\underset{\sim}{p} = \epsilon_o \underset{\sim}{Z}(\underset{\sim}{e} + \underset{\sim}{u} \times \underset{\sim}{b}), \qquad \text{with} \quad \underset{\sim}{Z} = \underset{\sim}{K} - \underset{\sim}{1}; \tag{3.4}$$

here $\underset{\sim}{1}$ is the unit tensor and $\underset{\sim}{K}$ is a symmetric, positive definite, linear transformation, called the _dielectric tensor_ and assumed to be independent of $\underset{\sim}{e}$, $\underset{\sim}{b}$, and $\underset{\sim}{u}$. (In subsequent chapters we shall discuss the way $\underset{\sim}{K}$ is related to the history of the deformation.) On substituting (3.4) into (3.2) we find that

$$\left.\begin{aligned} \mu_o \underset{\sim}{h} &= \underset{\sim}{b} + \underline{\underline{c^{-2}\underset{\sim}{u} \times \underset{\sim}{Z}(\underset{\sim}{u}\times\underset{\sim}{b})}} + \underline{c^{-2}\underset{\sim}{u} \times \underset{\sim}{Z}\underset{\sim}{e}}\,, \\ \underset{\sim}{d} &= \epsilon_o \underset{\sim}{K}\underset{\sim}{e} + \underline{\epsilon_o \underset{\sim}{Z}(\underset{\sim}{u}\times\underset{\sim}{b})}. \end{aligned}\right\} \tag{3.5}$$

The doubly underlined term here is $O(u^2/c^2)$, and may be safely neglected when the speed of the medium is small compared with the velocity of light. The singly underlined terms give rise to the dragging of light by the medium.## It is hoped that for the analysis of light waves in slowly

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#Cf. Truesdell & Toupin [1960, 2] p. 737.

##Cf. ibid. p. 740.

moving media, not only the doubly, but also the singly underlined terms can be neglected. In the present essay we make such an approximation and take

$$\left.\begin{aligned} \underset{\sim}{h} &= \mu_o^{-1}\underset{\sim}{b}, \\ \underset{\sim}{d} &= \epsilon_o \underset{\sim}{K}\underset{\sim}{e}, \end{aligned}\right\} \tag{3.6}$$

as our starting <u>electromagnetic</u> <u>constitutive</u> <u>equations</u>.

From a more general point of view, our starting assumptions (3.6) involve approximations beyond the neglect of relativistic effects and the dragging of light. Since we have assumed $\underset{\sim}{K}$ to be independent of $\underset{\sim}{e}$ and $\underset{\sim}{b}$ (and their past histories) and have implicitly set the electric current and the magnetization equal to zero, application of our theory should be restricted to weak fields (i.e. to light waves of "normal" intensity) and to media which neither absorb strongly nor rotate light. Materials obeying constitutive relations of the form (3.6) are called <u>perfect</u> <u>dielectrics</u>.

If the dielectric tensor $\underset{\sim}{K}$ is constant in space and time, then Fresnel's construction, discussed in Section 1, gives a correct description of a single plane harmonic electromagnetic wave governed by the equations (3.1) and (3.6), provided one identifies the Fresnel tensor $\underset{\sim}{L}$ with $c^2\underset{\sim}{K}^{-1}$:

$$\underset{\sim}{L} = c^2\underset{\sim}{K}^{-1}, \qquad \underset{\sim}{K} = c^2\underset{\sim}{L}^{-1}. \tag{3.7}$$

That is, the principal axes of refraction $\underset{\sim}{k}_i$ are the proper vectors of $\underset{\sim}{K}$, and the principal wave speeds v_i are related as follows to the proper numbers κ_i of $\underset{\sim}{K}$:

$$v_i^2 = c^2 \kappa_i^{-1}, \qquad i = 1,2,3. \tag{3.8}$$

By (3.8) and (1.3), each principal index of refraction n_i obeys the simple formula

$$n_i^2 = \kappa_i, \qquad i = 1,2,3. \tag{3.9}$$

The problem of describing a plane electromagnetic wave propagating in the direction $\underset{\sim}{k}_3$ through the strip $\mathcal{B}$ of Section 1 may be solved by applying the jump conditions (3.2) to the incident, reflected, and transmitted waves at each surface of $\mathcal{B}$. The exact solution is complicated;[#] but if the material is such that there is but a small loss of light by reflection at the entering and exiting surfaces, then the conclusions of Section 1 give a very good approximation to the exact solution. Thus, under the conditions expected in applications, a plane polarized wave incident normal to $\mathcal{B}$ <u>is</u> resolved into two waves which have mutually perpendicular amplitude vectors along $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ and which, during their traverse of $\mathcal{B}$, experience a relative retardation r given by (1.2).

The optical properties of a perfect dielectric are completely determined when its dielectric tensor $\underset{\sim}{K}$ is specified. However, any invertible function of $\underset{\sim}{K}$, such as the Fresnel tensor $\underset{\sim}{L}$, will do equally well. It appears to us that the theory of induced birefringence takes its simplest form if one works with the <u>refraction tensor</u> $\underset{\sim}{N}$, defined as

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[#]See Dill & Fowlkes [1966, 2] and Born & Wolf [1959, 1] §7.6.1.

the positive definite square root of $\underset{\sim}{K}$:

$$\underset{\sim}{N} \overset{\text{def}}{=\!=} \underset{\sim}{K}^{1/2}. \tag{3.10}$$

By (3.7) and (3.9), the proper vectors of $\underset{\sim}{N}$ are the principal axes of refraction $\underset{\sim}{k}_i$, while the proper numbers of $\underset{\sim}{N}$ are the principal indices of refraction n_i. Of course, $\underset{\sim}{N}$ determines $\underset{\sim}{L}$ and $\underset{\sim}{K}$ through the relations

$$\underset{\sim}{L} = c^2\underset{\sim}{N}^{-2}, \qquad \underset{\sim}{K} = \underset{\sim}{N}^2. \tag{3.11}$$

4. Kinematics and the Refraction Tensor

In non-relativistic field theories, a <u>body</u> $\mathscr{B}$ is a smooth, three-dimensional manifold whose elements X are called <u>material points</u> or <u>particles</u>. A <u>configuration</u> of $\mathscr{B}$ is a smooth homeomorphism of $\mathscr{B}$ onto a region in (three-dimensional) Euclidean space $\mathscr{E}$. A <u>motion</u> of $\mathscr{B}$ is a one-parameter family of configurations, the parameter being, of course, the time. A motion is described by expressing the position $\underset{\sim}{x}$ at time t of a particle X as a function $\underset{\sim}{\chi}$ of t and the position $\underset{\sim}{X}$ occupied by X in some reference configuration $\mathscr{K}$ of $\mathscr{B}$:

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X},t). \tag{4.1}$$

The gradient of the function $\underset{\sim}{\chi}$ with respect to $\underset{\sim}{X}$ is tensor $\underset{\sim}{F}$ called the <u>deformation gradient</u>:

$$\underset{\sim}{F} = \nabla_{\underset{\sim}{X}}\underset{\sim}{\chi}(\underset{\sim}{X},t). \tag{4.2}$$

For a given motion, $\underset{\sim}{F}$ depends not only on X and t but also on the choice of the reference configuration $\mathcal{R}$. For the same motion, particle, and time, the deformation gradient $\underset{\sim}{F}'$ relative to some other reference configuration $\mathcal{R}'$ is

$$\underset{\sim}{F}' = \underset{\sim}{F}\underset{\sim}{G}^{-1}, \qquad \text{i.e.} \quad \underset{\sim}{F} = \underset{\sim}{F}'\underset{\sim}{G}. \tag{4.3}$$

Here $\underset{\sim}{G}$ is the deformation gradient of $\mathcal{R}'$ relative to $\mathcal{R}$; i.e. if we write $\underset{\sim}{X}' = \underset{\sim}{f}(\underset{\sim}{X})$, where $\underset{\sim}{X}'$ and $\underset{\sim}{X}$ are the positions occupied by X in the configurations $\mathcal{R}'$ and $\mathcal{R}$, respectively, then

$$\underset{\sim}{G} = \nabla_{\underset{\sim}{X}}\underset{\sim}{f}(\underset{\sim}{X}); \tag{4.4}$$

to indicate this briefly one may use the notation,

$$\mathcal{R}' = \underset{\sim}{G}\mathcal{R}. \tag{4.5}$$

The important relation (4.3), which follows directly from the chain rule for differentiation of composite functions, may be stated in the following easily remembered form: If C_i, i = 1,2,3, are any three configurations of $\mathcal{B}$, and if $\underset{\sim}{F}_{(i,j)}$ is the deformation gradient for the configuration C_i taking C_j as the reference configuration, then

$$\underset{\sim}{F}_{(3,1)} = \underset{\sim}{F}_{(3,2)}\underset{\sim}{F}_{(2,1)}. \tag{4.6}$$

Since configurations are smooth homeomorphisms, deformation gradients are invertible tensors; hence, in (4.2) and (4.4) we have $\det \underset{\sim}{F} \neq 0$ and $\det \underset{\sim}{G} \neq 0$.

In a motion, the <u>history of the deformation gradient</u> at X up to time t is a tensor-valued function on $[0,\infty)$ defined by

$$\underset{\sim}{F}^t(s) \quad = \quad \underset{\sim}{F}(X,t-s), \qquad 0 \leq s < \infty. \tag{4.7}$$

We now lay down a constitutive assumption which broadly generalizes the starting hypotheses of the classical theories of induced birefringence discussed in Section 2. We assume that the dielectric tensor $\underset{\sim}{K}$ at time t is determined by a function $\mathfrak{K}$ of the history $\underset{\sim}{F}^t$:

$$\underset{\sim}{K}(t) \quad = \quad \mathfrak{K}(\underset{\sim}{F}^t). \tag{4.8}$$

In view of (3.10), this is equivalent to assuming that the refraction tensor $\underset{\sim}{N}$ is determined by $\underset{\sim}{F}^t$:

$$\underset{\sim}{N}(t) \quad = \quad \mathfrak{N}(\underset{\sim}{F}^t). \tag{4.9}$$

The functions $\mathfrak{K}$ and $\mathfrak{N}$, which are related by the equation

$$\mathfrak{N}(\underset{\sim}{F}^t) \quad = \quad \mathfrak{K}(\underset{\sim}{F}^t)^{1/2}, \tag{4.10}$$

are called <u>constitutive functionals</u>. Since the function $\underset{\sim}{F}^t$ depends upon the choice of reference configuration, so also do the functionals $\mathfrak{K}$ and $\mathfrak{N}$.

The values of $\mathfrak{K}$ are assumed to be symmetric positive-definite tensors, and the square root shown in (4.10) is to be interpreted as the positive definite square root. Hence specification of $\mathfrak{K}$ is equivalent to specification of $\mathfrak{N}$, and the values of $\mathfrak{N}$ are symmetric, positive definite tensors.

Because we assume the equations (3.6), with $\underset{\sim}{K}$ given by (4.8), our theory leads to familiar linear field equations for the electromagnetic variables $\underset{\sim}{b}$ and $\underset{\sim}{e}$. Since, however, we do not assume that either $\mathfrak{K}$ or $\mathfrak{N}$ is a linear functional, the theory is, in general, non-linear when one considers the influence of present and past deformations on optical behavior.

5. The Stress Tensor

Consistent with our linear treatment of the electromagnetic field, we suppose that the field is so weak that it has no effect on the motion of the material. We assume that Cauchy's stress principle is valid and that the equations of motion have the classical form

$$\operatorname{div} \underset{\sim}{S} + \rho \underset{\sim}{f} = \rho \ddot{\underset{\sim}{x}}, \tag{5.1}$$

with ρ the density of mass, $\underset{\sim}{f}$ the body force per unit mass, and $\underset{\sim}{S}$ the stress tensor. When it is assumed that the electromagnetic field does not influence the motion, the principle of angular momentum, together with (5.1) and the usual assumptions regarding the absence of distributed moments and couples, implies that Cauchy's stress tensor is symmetric:

$$\underset{\sim}{S} = \underset{\sim}{S}^T. \tag{5.2}$$

We here take (5.2) as a postulate. Furthermore, as is usual in modern continuum mechanics, we assume that $\underset{\sim}{S}$ is determined by a function $\mathfrak{S}$ of the history of the deformation gradient:

$$\underset{\sim}{S}(t) = \mathfrak{S}(\underset{\sim}{F}^t). \tag{5.3}$$

The constitutive functional $\mathfrak{S}$ occurring here, like $\mathfrak{N}$ and $\mathfrak{K}$, depends upon the choice of the reference configuration.

6. Changes of Frame

In classical mechanics a change of frame is a one-parameter family of transformations $\underset{\sim}{x} \rightarrow \underset{\sim}{x}_*$ of Euclidean space $\mathcal{E}$ onto itself, with time the parameter, such that at each instant the mapping $\underset{\sim}{x} \rightarrow \underset{\sim}{x}_*$ preserves distances.# It can be proved that every change of frame must be of the form

$$\underset{\sim}{x}_* = \underset{\sim}{c}(t) + \underset{\sim}{Q}(t)[\underset{\sim}{x}-\underset{\sim}{q}] \tag{6.1}$$

where, for each time t, $\underset{\sim}{c}(t)$ is a point in $\mathcal{E}$ and $\underset{\sim}{Q}(t)$ is an orthogonal tensor; $\underset{\sim}{q}$ is a point in $\mathcal{E}$ which can be taken independent of t. It follows that under a change of frame the vectorial difference $\underset{\sim}{u}$ of two points is transformed into a vector $\underset{\sim}{u}_*$ equal to $\underset{\sim}{Q}(t)\underset{\sim}{u}$; i.e., at each instant t

$$\underset{\sim}{u} \overset{\text{def}}{=\!=} \underset{\sim}{x} - \underset{\sim}{y} \quad \longrightarrow \quad \underset{\sim}{u}_* \overset{\text{def}}{=\!=} \underset{\sim}{x}_* - \underset{\sim}{y}_* , \tag{6.2}$$

where

$$\underset{\sim}{x}_* - \underset{\sim}{y}_* = \underset{\sim}{Q}(t)(\underset{\sim}{x}-\underset{\sim}{y}). \tag{6.3}$$

Thus, since $\underset{\sim}{Q}(t)$ is orthogonal, a change of frame preserves not only distances, but also inner products and hence angles, and, in particular,

#Cf. Noll [1958, 1].

the unit normal $\underset{\sim}{n}$ to a surface is transformed into

$$\underset{\sim}{n}_* = \underset{\sim}{Q}(t)\underset{\sim}{n}. \tag{6.4}$$

A change of frame transforms a motion $\underset{\sim}{\chi}$ into a new motion $\underset{\sim}{\chi}_*$, given by

$$\underset{\sim}{\chi}_*(\underset{\sim}{X},t) = \underset{\sim}{Q}(t)[\underset{\sim}{\chi}(\underset{\sim}{X},t) - \underset{\sim}{q}] + \underset{\sim}{c}(t). \tag{6.5}$$

Therefore, by (4.2), at each particle X and time t, the deformation gradient $\underset{\sim}{F}$ is transformed into

$$\underset{\sim}{F}_*(X,t) = \underset{\sim}{Q}(t)\underset{\sim}{F}(X,t), \tag{6.6}$$

and the history $\underset{\sim}{F}^t$ of $\underset{\sim}{F}$ is carried into the function $\underset{\sim}{F}_*^t$ given by

$$\underset{\sim}{F}_*^t(s) = \underset{\sim}{Q}(t-s)\underset{\sim}{F}^t(s) = \underset{\sim}{Q}^t(s)\underset{\sim}{F}^t(s). \tag{6.7}$$

It is assumed, in classical mechanics, that contact forces transform as point differences# under changes of frame; i.e. if $\underset{\sim}{s}$ is the stress vector, then $\underset{\sim}{s} \to \underset{\sim}{s}_*$, where

$$\underset{\sim}{s}_* = \underset{\sim}{Q}(t)\underset{\sim}{s}. \tag{6.8}$$

Since $\underset{\sim}{s} = \underset{\sim}{S}\underset{\sim}{n}$, it follows from (6.4) and (6.5) that

$$\underset{\sim}{s}_* = \underset{\sim}{S}_*\underset{\sim}{n}_* \quad \text{with} \quad \underset{\sim}{S}_* = \underset{\sim}{Q}(t)\underset{\sim}{S}\underset{\sim}{Q}(t)^{-1}. \tag{6.9}$$

This last formula gives the transformation rule for the stress tensor; i.e. $\underset{\sim}{S}(X,t) \to \underset{\sim}{S}_*(X,t)$, where

$$\underset{\sim}{S}_*(X,t) = \underset{\sim}{Q}(t)\underset{\sim}{S}(X,t)\underset{\sim}{Q}(t)^{-1}. \tag{6.10}$$

We here assume, further, that, under a change of frame, $\underset{\sim}{e}$ and $\underset{\sim}{d}$ transform

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#By "point differences" we mean "elements of the translation space of $\mathcal{E}$".

as point differences, while $\underset{\sim}{b}$ and $\underset{\sim}{h}$ transform as "axial vectors" (or the polar vectors of skew tensors); i.e. $\underset{\sim}{e} \to \underset{\sim}{e}_*$, $\underset{\sim}{d} \to \underset{\sim}{d}_*$, $\underset{\sim}{b} \to \underset{\sim}{b}_*$, and $\underset{\sim}{h} \to \underset{\sim}{h}_*$ with

$$\left.\begin{aligned}
\underset{\sim}{e}_*(X,t) &= \underset{\sim}{Q}(t)\underset{\sim}{e}(X,t),\\
\underset{\sim}{d}_*(X,t) &= \underset{\sim}{Q}(t)\underset{\sim}{d}(X,t),\\
\underset{\sim}{b}_*(X,t) &= [\det \underset{\sim}{Q}(t)]\underset{\sim}{Q}(t)\underset{\sim}{b}(X,t),\\
\underset{\sim}{h}_*(X,t) &= [\det \underset{\sim}{Q}(t)]\underset{\sim}{Q}(t)\underset{\sim}{h}(X,t).
\end{aligned}\right\} \tag{6.11}$$

In classical continuum physics, the idea that material properties should be independent of the observer, or frame of reference, is rendered mathematical by assuming the following postulate.

Principle of Material Frame-Indifference:# If a given process is compatible with a constitutive assumption, then all processes obtained from that process by changes of frame must be compatible with the same constitutive assumption.

We here employ this principle. In so doing, we impose on our theory the space-time structure of classical mechanics and, further, make assumptions which imply that the simultaneous spinning of a dielectric and its electromagnetic field has no effect upon the polarization of the dielectric. The use of classical space-time seems appropriate to a theory which is to be applied only in situations involving "small speeds".##

#Also called the "Principle of Objectivity"; see Noll [1958, 1].

##See the discussion after (3.5).

Let us consider first the electromagnetic constitutive equations (3.6). If $\dot{\underset{\sim}{h}}$, $\underset{\sim}{b}$, $\underset{\sim}{d}$, and $\underset{\sim}{e}$ obey these equations, and if $\underset{\sim}{e}_*$, $\underset{\sim}{d}_*$, $\underset{\sim}{b}_*$, and $\underset{\sim}{h}_*$ are given by (6.11), then $\underset{\sim}{b}_*$ and $\underset{\sim}{h}_*$ obey $(3.6)_1$, i.e.

$$\underset{\sim}{h}_* = \mu_0^{-1}\underset{\sim}{b}_*, \tag{6.12}$$

while $\underset{\sim}{e}_*$ and $\underset{\sim}{d}_*$ obey the equation

$$\underset{\sim}{d}_* = \epsilon_o \underset{\sim}{K}_* \underset{\sim}{e}_* \quad \text{with} \quad \underset{\sim}{K}_* = \underset{\sim}{Q}(t)\underset{\sim}{K}\underset{\sim}{Q}(t)^{-1}. \tag{6.13}$$

Thus, the defining equations for a perfect dielectric (3.6) are compatible with the principle of material frame-indifference if and only if we assume that for changes of frame the dielectric tensor obeys the same transformation rule as the stress tensor; i.e. $\underset{\sim}{K} \to \underset{\sim}{K}_*$, where

$$\underset{\sim}{K}_*(X,t) = \underset{\sim}{Q}(t)\underset{\sim}{K}(X,t)\underset{\sim}{Q}(t)^{-1}. \tag{6.14}$$

It follows that the refraction tensor $\underset{\sim}{N}$, defined in (3.10), also obeys this rule; i.e. $\underset{\sim}{N} \to \underset{\sim}{N}_*$, where

$$\underset{\sim}{N}_*(X,t) = \underset{\sim}{Q}(t)\underset{\sim}{N}(X,t)\underset{\sim}{Q}(t)^{-1}. \tag{6.15}$$

Now, frame-indifference requires that the constitutive functionals $\mathfrak{N}$ and $\mathfrak{S}$ in (4.9) and (5.3) be such that these equations remain valid whenever $\underset{\sim}{N}(t)$, $\underset{\sim}{S}(t)$, and $\underset{\sim}{F}^t$ are replaced by their transforms $\underset{\sim}{N}_*(t)$, $\underset{\sim}{S}_*(t)$, and $\underset{\sim}{F}_*^t$ under a change of frame:

$$\underset{\sim}{N}_*(t) = \mathfrak{N}(\underset{\sim}{F}_*^t), \qquad \underset{\sim}{S}_*(t) = \mathfrak{S}(\underset{\sim}{F}_*^t). \tag{6.16}$$

Substitution of (6.7), (6.15), and (6.10) into (6.16) yields

$$\underset{\sim}{Q}(t)\underset{\sim}{N}(t)\underset{\sim}{Q}(t)^{-1} = \mathfrak{N}(\underset{\sim}{Q}^t\underset{\sim}{F}^t), \qquad \underset{\sim}{Q}(t)\underset{\sim}{S}(t)\underset{\sim}{Q}(t)^{-1} = \mathfrak{S}(\underset{\sim}{Q}^t\underset{\sim}{F}^t), \tag{6.17}$$

and, in view of (4.9) and (5.3), we have

$$\left.\begin{aligned} \underset{\sim}{Q}^t(0)\mathfrak{N}(\underset{\sim}{F}^t)\underset{\sim}{Q}^t(0)^{-1} &= \mathfrak{N}(\underset{\sim}{Q}^t\underset{\sim}{F}^t), \\ \underset{\sim}{Q}^t(0)\mathfrak{S}(\underset{\sim}{F}^t)\underset{\sim}{Q}^t(0)^{-1} &= \mathfrak{S}(\underset{\sim}{Q}^t\underset{\sim}{F}^t). \end{aligned}\right\} \tag{6.18}$$

The principle of material frame-indifference implies that these relations hold for every function $\underset{\sim}{Q}^t$ whose values are orthogonal tensors and for every history $\underset{\sim}{F}^t$.[#] Conversely, if the equations (6.18) hold as identities, the constitutive relations (4.9) and (5.3) are preserved under changes of frame. It is obvious that a similar remark holds for the functional $\mathfrak{K}$ and the constitutive equation (4.8).

Functionals which obey the identity (6.18) for each orthogonal-tensor-valued function $\underset{\sim}{Q}^t$ are called <u>objective functionals</u>. Thus, the principle of material frame-indifference is here <u>equivalent</u> to the assertion that the constitutive equations (3.6), (4.8), and (5.3) are such that $\mathfrak{K}$ in (4.8) and $\mathfrak{S}$ in (5.3) are objective functionals. Of course, $\mathfrak{K}$ is objective if and only if $\mathfrak{N}$ is.

[#] More precisely, (6.18) holds for every history $\underset{\sim}{F}^t$ in the domain $\mathcal{D}$ of $\mathfrak{N}$ and $\mathfrak{S}$ and every orthogonal-tensor-valued function $\underset{\sim}{Q}^t$ such that $\underset{\sim}{Q}^t\underset{\sim}{F}^t$ is in $\mathcal{D}$. In Coleman & Noll's theory of fading memory [1960, 1] [1961, 2], if $\underset{\sim}{F}^t$ is a history in $\mathcal{D}$ and if $\underset{\sim}{Q}^t(s)$ is orthogonal for each s, then, for $\underset{\sim}{Q}^t\underset{\sim}{F}^t$ to be in $\mathcal{D}$, it suffices that the function $\underset{\sim}{Q}^t$ be measurable.

III. GENERAL PROPERTIES OF CONSTITUTIVE FUNCTIONALS

7. Consequences of Material Frame-Indifference

The basic constitutive equations of our theory may be summarized as follows:

$$\left.\begin{aligned} \underset{\sim}{N}(t) &= \mathfrak{N}(\underset{\sim}{F}^t), \qquad \underset{\sim}{S}(t) = \mathfrak{S}(\underset{\sim}{F}^t), \\ \underset{\sim}{h}(t) &= \mu_0^{-1}\underset{\sim}{b}(t), \\ \underset{\sim}{d} &= \epsilon_o \underset{\sim}{K}(t)\underset{\sim}{e}(t), \quad \text{with} \quad \underset{\sim}{K}(t) = \underset{\sim}{N}(t)^2, \end{aligned}\right\} \tag{7.1}$$

where $\underset{\sim}{N}(t)$, called the <u>refraction tensor</u>, is positive definite and symmetric, and $\underset{\sim}{S}(t)$, the <u>stress tensor</u>, is symmetric. The constitutive functionals $\mathfrak{N}$ and $\mathfrak{S}$ are objective; i.e. obey the relations

$$\mathfrak{N}(\underset{\sim}{Q}^*\underset{\sim}{F}^*) = \underset{\sim}{Q}^*(0)\mathfrak{N}(\underset{\sim}{F}^*)\underset{\sim}{Q}^*(0)^{-1}, \qquad \mathfrak{S}(\underset{\sim}{Q}^*\underset{\sim}{F}^*) = \underset{\sim}{Q}^*(0)\mathfrak{S}(\underset{\sim}{F}^*)\underset{\sim}{Q}^*(0)^{-1}, \tag{7.2}$$

for every history $\underset{\sim}{F}^*$ and every function $\underset{\sim}{Q}^*$ whose values are orthogonal tensors.

The general solution of identities of the type (7.2) is known# and has found applications in continuum mechanics. To present the solution, let us note that, by the polar decomposition theorem, the (non-singular) deformation gradient tensor $\underset{\sim}{F}$ can be written in two ways as the product of a symmetric, positive-definite tensor and an orthogonal

#Cf. Noll [1958, 1].

tensor:

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R}. \tag{7.3}$$

The orthogonal tensor $\underset{\sim}{R}$ and the symmetric, positive-definite tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ in these decompositions are uniquely determined by $\underset{\sim}{F}$ and obey the relations

$$\left.\begin{aligned} \underset{\sim}{U}^2 &= \underset{\sim}{F}^T\underset{\sim}{F} \overset{\text{def}}{=\!=} \underset{\sim}{C}, \\ \underset{\sim}{V}^2 &= \underset{\sim}{F}\underset{\sim}{F}^T \overset{\text{def}}{=\!=} \underset{\sim}{B}, \\ \underset{\sim}{V} &= \underset{\sim}{R}\underset{\sim}{U}\underset{\sim}{R}^{-1}; \end{aligned}\right\} \tag{7.4}$$

$\underset{\sim}{R}$ is called the <u>rotation tensor</u>, while $\underset{\sim}{U}$ and $\underset{\sim}{V}$, respectively, are called the <u>right</u> and <u>left stretch tensors</u>; $\underset{\sim}{C}$ and $\underset{\sim}{B}$ are called the <u>right</u> and <u>left Cauchy-Green tensors</u>. The functions $\underset{\sim}{R}^t$, $\underset{\sim}{U}^t$, and $\underset{\sim}{C}^t$, defined by

$$\underset{\sim}{R}^t(s) = \underset{\sim}{R}(X,t-s), \quad \underset{\sim}{U}^t(s) = \underset{\sim}{U}^t(X,t-s), \quad \underset{\sim}{C}^t(s) = \underset{\sim}{C}(X,t-s), \quad 0 \le s < \infty, \tag{7.5}$$

are the <u>histories of the rotation tensor, the right stretch tensor</u>, and <u>the right Cauchy-Green tensor</u> at X up to time t.

Clearly, $\underset{\sim}{R}^t(s)^{-1}$ is, for each s, an orthogonal tensor, and if, in (7.2), we put $\underset{\sim}{F}^*(s) = \underset{\sim}{F}^t(s)$ and $\underset{\sim}{Q}^*(s) = \underset{\sim}{R}^t(s)^{-1}$, we obtain

$$\mathfrak{N}(\underset{\sim}{U}^t) = \underset{\sim}{R}(t)^{-1}\mathfrak{N}(\underset{\sim}{F}^t)\underset{\sim}{R}(t), \qquad \mathfrak{S}(\underset{\sim}{U}^t) = \underset{\sim}{R}(t)^{-1}\mathfrak{S}(\underset{\sim}{F}^t)\underset{\sim}{R}(t). \tag{7.6}$$

Hence, $(7.1)_{1\&2}$ can be written#

$$\underset{\sim}{N}(t) = \underset{\sim}{R}(t)\mathfrak{N}(\underset{\sim}{U}^t)\underset{\sim}{R}(t)^{-1}, \qquad \underset{\sim}{S}(t) = \underset{\sim}{R}(t)\mathfrak{S}(\underset{\sim}{U}^t)\underset{\sim}{R}(t)^{-1}; \tag{7.7}$$

#The equation $(7.7)_2$ was first obtained by Noll [1958, 1].

i.e. a knowledge of the history of the right stretch tensor and the present value of the rotation tensor suffices to determine the present values of the refraction and stress tensor.

It is often convenient to employ the functionals $\tilde{\mathfrak{N}}$, $\tilde{\mathfrak{S}}$, $\tilde{\tilde{\mathfrak{N}}}$, $\tilde{\tilde{\mathfrak{S}}}$ defined by the equations

$$\left.\begin{aligned} \tilde{\mathfrak{N}}(\underset{\sim}{C}^*) &= \mathfrak{N}(\underset{\sim}{C}^{*\frac{1}{2}}), & \tilde{\mathfrak{S}}(\underset{\sim}{C}^*) &= \mathfrak{S}(\underset{\sim}{C}^{*\frac{1}{2}}), \\ \tilde{\tilde{\mathfrak{N}}}(\underset{\sim}{C}^*) &= \underset{\sim}{C}^*(0)^{-\frac{1}{2}}\,\tilde{\mathfrak{N}}(\underset{\sim}{C}^*)\underset{\sim}{C}^*(0)^{-\frac{1}{2}}, & \tilde{\tilde{\mathfrak{S}}}(\underset{\sim}{C}^*) &= \underset{\sim}{C}^*(0)^{-\frac{1}{2}}\,\tilde{\mathfrak{S}}(\underset{\sim}{C}^*)\underset{\sim}{C}^*(0)^{-\frac{1}{2}}, \end{aligned}\right\} \quad (7.8)$$

which hold for each function $\underset{\sim}{C}^*$ on $[0,\infty)$ whose values $\underset{\sim}{C}^*(s)$ are symmetric, positive-definite tensors. Here $\underset{\sim}{C}^{*\frac{1}{2}}$ denotes the function whose value for each s, $\underset{\sim}{C}^*(s)^{\frac{1}{2}}$, is the positive-definite square root of $\underset{\sim}{C}^*(s)$, and $\underset{\sim}{C}^*(0)^{-\frac{1}{2}}$ is the inverse of $\underset{\sim}{C}^*(0)^{\frac{1}{2}}$. In terms of these functionals, (7.7) can be written

$$\underset{\sim}{N}(t) = \underset{\sim}{R}(t)\tilde{\mathfrak{N}}(\underset{\sim}{C}^t)\underset{\sim}{R}(t)^{-1}, \qquad \underset{\sim}{S}(t) = \underset{\sim}{R}(t)\tilde{\mathfrak{S}}(\underset{\sim}{C}^t)\underset{\sim}{R}(t)^{-1} \qquad (7.9)$$

or, equivalently,#

$$\underset{\sim}{N}(t) = \underset{\sim}{F}(t)\tilde{\tilde{\mathfrak{N}}}(\underset{\sim}{C}^t)\underset{\sim}{F}(t)^T, \qquad \underset{\sim}{S}(t) = \underset{\sim}{F}(t)\tilde{\tilde{\mathfrak{S}}}(\underset{\sim}{C}^t)\underset{\sim}{F}(t)^T. \qquad (7.10)$$

One may write (7.9) in the form

$$\tilde{\underset{\sim}{N}}(t) = \tilde{\mathfrak{N}}(\underset{\sim}{C}^t), \qquad \tilde{\underset{\sim}{S}}(t) = \tilde{\mathfrak{S}}(\underset{\sim}{C}^t), \qquad (7.11)$$

where

$$\tilde{\underset{\sim}{N}}(t) \overset{\text{def}}{=} \underset{\sim}{R}(t)^{-1}\underset{\sim}{N}(t)\underset{\sim}{R}(t) \quad \text{and} \quad \tilde{\underset{\sim}{S}}(t) \overset{\text{def}}{=} \underset{\sim}{R}(t)^{-1}\underset{\sim}{S}(t)\underset{\sim}{R}(t) \qquad (7.12)$$

#The equation $(7.10)_2$ was obtained by Green & Rivlin [1957, 1].

are called, respectively, the rotated refraction tensor and the rotated stress tensor.

8. Material Symmetry

As we have already mentioned, the constitutive functionals $\mathfrak{N}$ and $\mathfrak{S}$ are affected by the choice of reference configuration. Since the motion determines $\underset{\sim}{N}$ and $\underset{\sim}{S}$ independently of the reference configuration, it follows from (4.3) that if $\mathcal{R}' = \underset{\sim}{G}\mathcal{R}$, then for each motion, at each time t,

$$\mathfrak{N}_{\mathcal{R}'}(\underset{\sim}{F}^{t'}) = \mathfrak{N}_{\mathcal{R}}(\underset{\sim}{F}^{t}), \qquad \mathfrak{S}_{\mathcal{R}'}(\underset{\sim}{F}^{t'}) = \mathfrak{S}_{\mathcal{R}}(\underset{\sim}{F}^{t}) \tag{8.1}$$

where

$$F^{t'}(s)\underset{\sim}{G} = \underset{\sim}{F}^{t}(s), \qquad 0 \leq s < \infty, \tag{8.2}$$

and the subscripts on $\mathfrak{N}$ and $\mathfrak{S}$ show the choice of reference. Thus, if we put $\underset{\sim}{F}^{*} = \underset{\sim}{F}^{t'}$, we obtain the following general formulae:

$$\mathfrak{N}_{\mathcal{R}'}(\underset{\sim}{F}^{*}) = \mathfrak{N}_{\mathcal{R}}(\underset{\sim}{F}^{*}\underset{\sim}{G}), \qquad \mathfrak{S}_{\mathcal{R}'}(\underset{\sim}{F}^{*}) = \mathfrak{S}_{\mathcal{R}}(\underset{\sim}{F}^{*}\underset{\sim}{G}) \quad \text{for } \mathcal{R}' = \underset{\sim}{G}\mathcal{R}. \tag{8.3}$$

The relations (8.3) hold for every history $\underset{\sim}{F}^{*}$ and hence determine $\mathfrak{N}_{\mathcal{R}'}$ and $\mathfrak{S}_{\mathcal{R}'}$ when $\mathfrak{N}_{\mathcal{R}}$ and $\mathfrak{S}_{\mathcal{R}}$ are known.

If $\rho_{\mathcal{R}}$ and $\rho_{\mathcal{R}'}$ are the mass densities in two reference configurations $\mathcal{R}$ and $\mathcal{R}'$ with $\mathcal{R}' = \underset{\sim}{G}\mathcal{R}$, then

$$\frac{1}{\rho_{\mathcal{R}'}} = |\det \underset{\sim}{G}| \frac{1}{\rho_{\mathcal{R}}} . \tag{8.4}$$

We now take over and extend some definitions employed by Noll [1958, 1]. The group $\mathcal{G}_{\mathcal{R}}$ of unimodular tensors $\underset{\sim}{H}$ for which

$$\mathfrak{M}_{\underset{\sim}{H}\mathcal{R}} = \mathfrak{M}_{\mathcal{R}}, \qquad \mathfrak{S}_{\underset{\sim}{H}\mathcal{R}} = \mathfrak{S}_{\mathcal{R}} \tag{8.5}$$

is called the <u>symmetry group</u> (of the material at X) <u>relative to the reference configuration</u> $\mathcal{R}$.# The elements of $\mathcal{G}_{\mathcal{R}}$ may be interpreted as the deformation gradients of those changes of reference configuration, starting from $\mathcal{R}$, which are "undetectable" in the sense that they preserve both the mass density and the constitutive functionals $\mathfrak{M}$ and $\mathfrak{S}$. By (8.3), a unimodular tensor $\underset{\sim}{H}$ is in $\mathcal{G}_{\mathcal{R}}$ if and only if

$$\mathfrak{M}_{\mathcal{R}}(\underset{\sim}{F}^{*}\underset{\sim}{H}) = \mathfrak{M}_{\mathcal{R}}(\underset{\sim}{F}^{*}) \quad \text{and} \quad \mathfrak{S}_{\mathcal{R}}(\underset{\sim}{F}^{*}\underset{\sim}{H}) = \mathfrak{S}_{\mathcal{R}}(\underset{\sim}{F}^{*}) \tag{8.6}$$

for all histories $\underset{\sim}{F}^{*}$.

The symmetry group $\mathcal{G}_{\mathcal{R}}$ for a given material depends upon the reference $\mathcal{R}$. In fact, it is easily shown that if $\mathcal{R}' = \underset{\sim}{G}\mathcal{R}$, then $\mathcal{G}_{\mathcal{R}'}$ equals the conjugate, $\underset{\sim}{G}\mathcal{G}_{\mathcal{R}}\underset{\sim}{G}^{-1}$, of $\mathcal{G}_{\mathcal{R}}$ under $\underset{\sim}{G}$; i.e. $\underset{\sim}{H}'$ is in $\mathcal{G}_{\mathcal{R}'}$ if and only if $\underset{\sim}{H}' = \underset{\sim}{G}\underset{\sim}{H}\underset{\sim}{G}^{-1}$ for some $\underset{\sim}{H}$ in $\mathcal{G}_{\mathcal{R}}$.

If there exist reference configurations $\mathcal{R}$ for which the corresponding symmetry groups $\mathcal{G}_{\mathcal{R}}$ contain the full orthogonal group $\mathcal{O}$ ## as a subgroup, then the material is called <u>isotropic</u> and these reference configurations $\mathcal{R}$ with $\mathcal{G}_{\mathcal{R}} \supset \mathcal{O}$ are said to be <u>undistorted</u>. If, for some $\mathcal{R}$,

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$\underset{\sim}{H}$ is <u>unimodular</u> if and only if $|\det \underset{\sim}{H}| = 1$. Noll [1958, 1] called $\mathcal{G}_{\mathcal{R}}$ <u>the isotropy group relative to</u> $\mathcal{R}$.

i.e. the group of all orthogonal tensors.

$\mathcal{O}$ contains $\mathcal{I}_{\mathcal{R}}$ as a subgroup, then the material is said to be a solid. Hence, each isotropic solid has configurations $\mathcal{R}$ for which $\mathcal{I}_{\mathcal{R}} = \mathcal{O}$. If $\mathcal{I}_{\mathcal{R}}$ is the unimodular group $\mathcal{U}$,[#] then the material is a fluid; it is easily shown that if $\mathcal{I}_{\mathcal{R}} = \mathcal{U}$ for one reference configuration $\mathcal{R}$, then $\mathcal{I}_{\mathcal{R}'} = \mathcal{U}$ for every other reference configuration $\mathcal{R}'$, and thus every reference configuration of a fluid is undistorted.

There is a theorem of group theory which states that, if a group $\mathcal{G}$ is a subgroup of the unimodular group $\mathcal{U}$ and contains the orthogonal group $\mathcal{O}$ as a subgroup, then either $\mathcal{G} = \mathcal{U}$ or $\mathcal{G} = \mathcal{O}$.[##] Hence, each isotropic material is either a solid or a fluid.

9. Consequences of Material Isotropy

It can be shown[###] that for each isotropic material, whether solid or fluid, there exist two functionals, $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{S}}$, such that if $\underset{\sim}{R}(t)$ is the rotation tensor and $\underset{\sim}{U}^t$ the history of the right stretch tensor, both taken relative to an undistorted reference configuration $\mathcal{R}$, then

$$\underset{\sim}{N}(t) = \underset{\sim}{R}(t)\bar{\mathfrak{N}}(\nu \underset{\sim}{U}^t)\underset{\sim}{R}(t)^{-1}, \qquad \underset{\sim}{S}(t) = \underset{\sim}{R}(t)\bar{\mathfrak{S}}(\nu \underset{\sim}{U}^t)\underset{\sim}{R}(t)^{-1}, \tag{9.1}$$

where

$$\nu = \rho_{\mathcal{R}}^{-1/3}; \tag{9.2}$$

[#] i.e. the group of all unimodular tensors.

[##] Brauer [1965, 1], Noll [1965, 3].

[###] Coleman [1968, 1].

and $\rho_{\mathcal{R}}$ is the mass density in the configuration $\mathcal{R}$; the functionals $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{S}}$ are independent of the choice of $\mathcal{R}$, provided $\mathcal{R}$ is undistorted; furthermore, $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{S}}$ obey the relations

$$\bar{\mathfrak{N}}(\underset{\sim}{Q}\underset{\sim}{U}^{*}\underset{\sim}{Q}^{-1}) = \underset{\sim}{Q}\bar{\mathfrak{N}}(\underset{\sim}{U}^{*})\underset{\sim}{Q}^{-1}, \qquad \bar{\mathfrak{S}}(\underset{\sim}{Q}\underset{\sim}{U}^{*}\underset{\sim}{Q}^{-1}) = \underset{\sim}{Q}\bar{\mathfrak{S}}(\underset{\sim}{U}^{*})\underset{\sim}{Q}^{-1} \tag{9.3}$$

for each orthogonal tensor $\underset{\sim}{Q}$ and each history $\underset{\sim}{U}^{*}$ in the domain of $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{S}}$.

It is a consequence of (9.1) that for an isotropic material the constitutive equations $(7.1)_{1\&2}$ may be written

$$\tilde{\underset{\sim}{N}}(t) = \hat{\mathfrak{N}}(\underset{\sim}{U}^{t};\rho_{\mathcal{R}}), \qquad \tilde{\underset{\sim}{S}}(t) = \hat{\mathfrak{S}}(\underset{\sim}{U}^{t};\rho_{\mathcal{R}}), \tag{9.4}$$

where $\underset{\sim}{R}(t)$ and $\underset{\sim}{U}^{t}$ are taken relative to an arbitrary undistorted reference configuration $\mathcal{R}$ and

$$\tilde{\underset{\sim}{N}}(t) \stackrel{\text{def}}{=} \underset{\sim}{R}(t)^{-1}\underset{\sim}{N}(t)\underset{\sim}{R}(t), \qquad \tilde{\underset{\sim}{S}}(t) \stackrel{\text{def}}{=} \underset{\sim}{R}(t)^{-1}\underset{\sim}{S}(t)\underset{\sim}{R}(t) \tag{9.5}$$

are the <u>rotated refraction tensor</u> and <u>rotated stress tensor</u> relative to $\mathcal{R}$. It follows from (9.3) that, for each orthogonal tensor $\underset{\sim}{Q}$, the constitutive functionals $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{S}}$ obey the relations

$$\hat{\mathfrak{N}}(\underset{\sim}{Q}\underset{\sim}{U}^{*}\underset{\sim}{Q}^{-1};\rho_{\mathcal{R}}) = \underset{\sim}{Q}\hat{\mathfrak{N}}(\underset{\sim}{U}^{*};\rho_{\mathcal{R}})\underset{\sim}{Q}^{-1}, \qquad \hat{\mathfrak{S}}(\underset{\sim}{Q}\underset{\sim}{U}^{*}\underset{\sim}{Q}^{-1};\rho_{\mathcal{R}}) = \underset{\sim}{Q}\hat{\mathfrak{S}}(\underset{\sim}{U}^{*};\rho_{\mathcal{R}})\underset{\sim}{Q}^{-1} \tag{9.6}$$

as identities in $\rho_{\mathcal{R}}$ and $\underset{\sim}{U}^{*}$.

Functionals which obey identities of the type (9.3) and (9.6), for each orthogonal tensor $\underset{\sim}{Q}$, are called <u>isotropic functionals</u>. The isotropy of the functionals $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{S}}$ may be expressed as follows: For each constant orthogonal tensor $\underset{\sim}{Q}$,

$$\underset{\sim}{U}^{t} \to \underset{\sim}{Q}\underset{\sim}{U}^{t}\underset{\sim}{Q}^{-1} \implies \tilde{\underset{\sim}{N}}(t) \to \underset{\sim}{Q}\tilde{\underset{\sim}{N}}(t)\underset{\sim}{Q}^{-1} \quad \text{and} \quad \tilde{\underset{\sim}{S}}(t) \to \underset{\sim}{Q}\tilde{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1}. \tag{9.7}$$

Here $A \to B$ means "A replaced by B", and $\Longrightarrow$, as usual, denotes implication.

An alternative form of the equations (9.4) is

$$\tilde{\mathbf{N}}(t) = \tilde{\mathfrak{N}}(\mathbf{C}^t; \rho_{\mathcal{R}}), \qquad \tilde{\mathbf{S}}(t) = \tilde{\mathfrak{S}}(\mathbf{C}^t; \rho_{\mathcal{R}}), \tag{9.8}$$

where $\mathbf{C}^t$ is the history of the right Cauchy-Green tensor relative to an arbitrary undistorted reference configuration $\mathcal{R}$, and $\tilde{\mathbf{N}}(t)$ and $\tilde{\mathbf{S}}(t)$ are again given by (9.5). The functionals $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{S}}$, like $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{S}}$, are isotropic; i.e. for each orthogonal $\mathbf{Q}$,

$$\tilde{\mathfrak{N}}(\mathbf{Q}\mathbf{C}^*\mathbf{Q}^{-1}; \rho_{\mathcal{R}}) = \mathbf{Q}\tilde{\mathfrak{N}}(\mathbf{C}^*; \rho_{\mathcal{R}})\mathbf{Q}^{-1}, \qquad \tilde{\mathfrak{S}}(\mathbf{Q}\mathbf{C}^*\mathbf{Q}^{-1}; \rho_{\mathcal{R}}) = \mathbf{Q}\tilde{\mathfrak{S}}(\mathbf{C}^*; \rho_{\mathcal{R}})\mathbf{Q}^{-1} \tag{9.9}$$

identically in $\rho_{\mathcal{R}}$ and $\mathbf{C}^*$. Thus, when $\mathbf{Q}$ is orthogonal,

$$\mathbf{C}^t \to \mathbf{Q}\mathbf{C}^t\mathbf{Q}^{-1} \Longrightarrow \tilde{\mathbf{N}}(t) \to \mathbf{Q}\tilde{\mathbf{N}}(t)\mathbf{Q}^{-1} \quad \text{and} \quad \tilde{\mathbf{S}}(t) \to \mathbf{Q}\tilde{\mathbf{S}}(t)\mathbf{Q}^{-1}. \tag{9.10}$$

There is a striking formal similarity between the equations (9.8) and (7.11). It will be recalled, however, that (7.11) holds for all substances covered by the basic assumptions laid down in Chapter II, while (9.8) is appropriate to isotropic materials; the constitutive functionals in (7.11) are not, in general, isotropic, but the functionals in (9.8) are. Furthermore, in (7.11) no simple rule is implied for the dependence of $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{S}}$ on the choice of the reference configuration $\mathcal{R}$, while here we see that, for isotropic materials, if $\mathcal{R}$ is chosen to be undistorted, the corresponding functionals can depend on $\mathcal{R}$ only through the density $\rho_{\mathcal{R}}$. (In fact, the dependence on $\rho_{\mathcal{R}}$ must be compatible with the formulae (9.1) and (9.2).)

Let $\underset{\sim}{1}^{\dagger}$ be the (constant) function on $[0,\infty)$ with value $\underset{\sim}{1}$; i.e.

$$\underset{\sim}{1}^{\dagger}(s) = \underset{\sim}{1}, \qquad 0 \leq s < \infty. \tag{9.11}$$

Of course, for a body which has never left its reference configuration $\mathcal{R}$, or has been subjected to only rigid rotations from $\mathcal{R}$, $\underset{\sim}{C}^{t} = \underset{\sim}{1}^{\dagger}$. Employing (9.10), it is easy to show that, if $\mathcal{R}$ is undistorted, then

$$\underset{\sim}{C}^{t} = \underset{\sim}{1}^{\dagger} \implies \underset{\sim}{N}(t) = n^{\circ}(\rho_{\mathcal{R}})\underset{\sim}{1} \quad \text{and} \quad \underset{\sim}{S}(t) = -p(\rho_{\mathcal{R}})\underset{\sim}{1}. \tag{9.12}$$

Thus, for an isotropic material which has always been in a single undistorted configuration $\mathcal{R}$ or has experienced only rotations from $\mathcal{R}$, the velocity of light is the same in all directions, while the stress is a hydrostatic pressure; this velocity and pressure are determined by $\rho_{\mathcal{R}}$.

10. General Properties of Fluids

For a fluid, every configuration is undistorted, and hence, in each of the formulae of the previous section, one can let $\mathcal{R}$ be the configuration at time t. When such a choice is made, we have $\underset{\sim}{F}(t) = \underset{\sim}{1}$, $\underset{\sim}{R}(t) = \underset{\sim}{1}$, $\tilde{\underset{\sim}{N}}(t) = \underset{\sim}{N}(t)$, $\tilde{\underset{\sim}{S}}(t) = \underset{\sim}{S}(t)$, and (9.1) reduces to

$$\underset{\sim}{N}(t) = \bar{\mathfrak{N}}(\nu(t)\underset{\sim}{U}_{t}^{t}), \qquad \underset{\sim}{S}(t) = \bar{\mathfrak{S}}(\nu(t)\underset{\sim}{U}_{t}^{t}), \qquad \text{with} \qquad \nu(t) = \rho(t)^{-1/3}. \tag{10.1}$$

Here $\underset{\sim}{U}_{t}^{t}$ is a function on $[0,\infty)$ such that $\underset{\sim}{U}_{t}^{t}(s)$ is the right stretch tensor at time t-s relative to the configuration at time t. The functionals $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{S}}$ depend on only the fluid under consideration, and, of course, obey the identities (9.3) for each orthogonal tensor $\underset{\sim}{Q}$. The same choice of reference causes (9.8) to reduce to

$$\underset{\sim}{N}(t) = \tilde{\mathfrak{N}}(\underset{\sim}{C}_t^t; \rho(t)), \qquad \underset{\sim}{S}(t) = \tilde{\mathfrak{S}}(\underset{\sim}{C}_t^t; \rho(t)), \tag{10.2}$$

where $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{S}}$ obey the identities (9.9) for each orthogonal $\underset{\sim}{Q}$, and $\underset{\sim}{C}_t^t$ is such that

$$\underset{\sim}{C}_t^t(s) = \underset{\sim}{U}_t^t(s)^2 = \underset{\sim}{F}_t^t(s)^T \underset{\sim}{F}_t^t(s), \qquad 0 \leq s < \infty. \tag{10.3}$$

The function $\underset{\sim}{F}_t^t$, called the <u>history of the relative deformation gradient</u>, is computed as follows. Let $\underset{\sim}{\chi}_t(\underset{\sim}{x}, \tau)$ be the position in space at time τ of the particle which has the position $\underset{\sim}{x}$ at time t. The <u>relative deformation gradient</u> at time τ (i.e. the deformation gradient at time τ relative to the configuration at time t) is the tensor

$$\underset{\sim}{F}_t(\tau) = \nabla_{\underset{\sim}{x}} \underset{\sim}{\chi}_t(\underset{\sim}{x}, \tau), \tag{10.4}$$

and we have

$$\underset{\sim}{F}_t^t(s) = \underset{\sim}{F}_t(t-s), \qquad 0 \leq s < \infty. \tag{10.5}$$

The function $\underset{\sim}{C}_t^t$ is called the <u>history of the right relative Cauchy-Green tensor</u>. Clearly

$$\underset{\sim}{\chi}_t(\underset{\sim}{x}, t) = \underset{\sim}{x},$$

and hence

$$\underset{\sim}{F}_t(t) = \underset{\sim}{1} \quad \text{and} \quad \underset{\sim}{C}_t^t(0) = \underset{\sim}{F}_t^t(0) = \underset{\sim}{1}. \tag{10.6}$$

The isotropy of the functionals in (10.1) and (10.2) is expressed in the following assertion which is true only for fluids: For each orthogonal tensor $\underset{\sim}{Q}$,

$$\underset{\sim}{U}_t^t \rightarrow \underset{\sim}{Q}\underset{\sim}{U}_t^t\underset{\sim}{Q}^{-1} \implies \underset{\sim}{N}(t) \rightarrow \underset{\sim}{Q}\underset{\sim}{N}(t)\underset{\sim}{Q}^{-1} \quad \text{and} \quad \underset{\sim}{S}(t) \rightarrow \underset{\sim}{Q}\underset{\sim}{S}(t)\underset{\sim}{Q}^{-1} \tag{10.7}$$

and, similarly,

$$\underset{\sim}{C}_t^t \rightarrow \underset{\sim}{Q}\underset{\sim}{C}_t^t\underset{\sim}{Q}^{-1} \implies \underset{\sim}{N}(t) \rightarrow \underset{\sim}{Q}\underset{\sim}{N}(t)\underset{\sim}{Q}^{-1} \quad \text{and} \quad \underset{\sim}{S}(t) \rightarrow \underset{\sim}{Q}\underset{\sim}{S}(t)\underset{\sim}{Q}^{-1}. \tag{10.8}$$

It follows from (10.2) and (10.3) that, as expected,

$$\underset{\sim}{C}_t^t = \underset{\sim}{1}^{\dagger} \implies \underset{\sim}{N}(t) = n^\circ(\rho(t))\underset{\sim}{1} \quad \text{and} \quad \underset{\sim}{S}(t) = -p(\rho(t))\underset{\sim}{1}. \tag{10.9}$$

Thus, for a fluid which has been subjected to only rigid rotations, regardless of what the present configuration is, the index of refraction, n°, is independent of direction and the stress is a hydrostatic pressure, $-p\underset{\sim}{1}$; both n° and p are determined once the density is specified.#

#For an incompressible fluid, however, p is arbitrary, in the sense that it is not determined by the local motion alone. See Section 17.

IV. PARTICULAR MOTIONS OF ISOTROPIC MATERIALS

11. Sheared Extensions

The constitutive equations (9.8) for an isotropic material often can be simplified if something is known in advance about the motion the body is undergoing. In this chapter we describe the reduced forms taken by the functionals $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{S}}$ in a broad class of motions called "sheared extensions". Although the discussions of this section and Sections 12 and 13 are valid for general isotropic materials, whether solid or fluid, we expect that the results we give here will find their main application in the study of solids. The reductions we describe for $\tilde{\mathfrak{S}}$ were derived by Coleman [1968, 1], and we summarize his results without repeating the proofs. Since $\tilde{\mathfrak{N}}$ obeys the same identities as $\tilde{\mathfrak{S}}$, the theorems of Coleman can be applied also to $\tilde{\mathfrak{N}}$.

Let $\underset{\sim}{F}(t-s)$ be the deformation gradient at a particle X at time t-s relative to an undistorted reference configuration $\mathcal{R}$. One says that the history of X up to time t is a <u>sheared extension</u> if, for all $s \geq 0$,

$$\underset{\sim}{F}(t-s) = \underset{\sim}{P}(t-s)\underset{\sim}{M}(t-s) \tag{11.1}$$

where $\underset{\sim}{P}(t-s)$ is an orthogonal tensor for each s, and $\underset{\sim}{M}$ is such that there exists an orthonormal basis $\underset{\sim}{h}_i$, independent of s, relative to which the components of $\underset{\sim}{M}(t-s)$ have the form

$$[\underset{\sim}{M}(t-s)] = \begin{bmatrix} \beta_1(t-s) & 0 & 0 \\ \zeta(t-s) & \beta_2(t-s) & 0 \\ 0 & 0 & \beta_3(t-s) \end{bmatrix}, \qquad \beta_i(t-s) > 0; \qquad (11.2)$$

$\underset{\sim}{h}_i$ is called the canonical basis of the sheared extension. In the special case in which $\zeta(t-s) \equiv 0$, the motion is called an extension; if $\beta_1(t-s) \equiv \beta_2(t-s) \equiv \beta_3(t-s) \equiv 1$, the motion is called a shear; we discuss these two important special cases in Sections 12 and 13 below. First, however, we give some results valid for general sheared extensions.

In a given sheared extension, the histories of the rotation tensor $\underset{\sim}{R}$ and the right Cauchy-Green tensor $\underset{\sim}{C}$ can be calculated from $\underset{\sim}{P}$ and $\underset{\sim}{M}$. Using the relation

$$\underset{\sim}{C}(t-s) = \underset{\sim}{F}(t-s)^T \underset{\sim}{F}(t-s) \qquad (11.3)$$

one may easily show that the history of a particle up to time t is a sheared extension if and only if there exists an orthonormal basis $\underset{\sim}{h}_i$, independent of s, relative to which

$$[\underset{\sim}{C}(t-s)] = \begin{bmatrix} \gamma_1^2(t-s) & \xi(t-s) & 0 \\ \xi(t-s) & \gamma_2^2(t-s) & 0 \\ 0 & 0 & \gamma_3^2(t-s) \end{bmatrix}, \qquad (11.4)$$

$$\gamma_1(t-s)^2 > \frac{\xi(t-s)^2}{\gamma_2(t-s)^2}, \qquad \gamma_i(t-s) > 0$$

for all $s \geq 0$; this basis $\underset{\sim}{h}_i$ is, of course, the canonical basis of the sheared extension; furthermore,

$$\xi = \zeta\beta_2, \qquad \gamma_1 = \sqrt{\beta_1^2 + \zeta^2}, \qquad \gamma_2 = \beta_2, \qquad \gamma_3 = \beta_3, \tag{11.5}$$

or, equivalently,

$$\zeta = \xi/\gamma_2, \qquad \beta_1 = \sqrt{\gamma_1^2 - \xi^2/\gamma_2^2}, \qquad \beta_2 = \gamma_2, \qquad \beta_3 = \gamma_3. \tag{11.6}$$

The following theorem gives the reduced forms taken by the equations (9.8) when the local motion is a sheared extension:# If, relative to an undistorted reference configuration $\mathcal{R}$, the history up to time t of an isotropic material is a sheared extension with canonical basis $\underset{\sim}{h}_i$, then the matrices of the components, relative to $\underset{\sim}{h}_i$, of the rotated refraction tensor $(9.5)_1$ and the rotated stress tensor $(9.5)_2$, have the forms

$$[\tilde{\underset{\sim}{N}}(t)] = \begin{bmatrix} \tilde{N}_{11}(t) & \tilde{N}_{12}(t) & 0 \\ \tilde{N}_{12}(t) & \tilde{N}_{22}(t) & 0 \\ 0 & 0 & \tilde{N}_{33}(t) \end{bmatrix}, \tag{11.7}$$

$$[\tilde{\underset{\sim}{S}}(t)] = \begin{bmatrix} \tilde{S}_{11}(t) & \tilde{S}_{12}(t) & 0 \\ \tilde{S}_{12}(t) & \tilde{S}_{22}(t) & 0 \\ 0 & 0 & \tilde{S}_{33}(t) \end{bmatrix}; \tag{11.8}$$

moreover, six scalar-valued material functionals, $\mathcal{J}_N$, $\mathcal{A}_N$, $\mathcal{t}_N$, and $\mathcal{J}_S$, $\mathcal{A}_S$, $\mathcal{t}_S$, determine the non-zero components in (11.7) and (11.8)

#Cf. [1968, 1] Thm. 7.

as follows:

$$\left.\begin{aligned}\tilde{N}_{12}(t) &= \mathscr{Z}_N(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{N}_{11}(t) &= \mathscr{S}_N(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{N}_{22}(t) &= \mathscr{S}_N(\xi^t,\gamma_2^t,\gamma_1^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{N}_{33}(t) &= \mathscr{T}_N(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathscr{R}}),\end{aligned}\right\} \tag{11.9}$$

$$\left.\begin{aligned}\tilde{S}_{12}(t) &= \mathscr{Z}_S(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{S}_{11}(t) &= \mathscr{S}_S(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{S}_{22}(t) &= \mathscr{S}_S(\xi^t,\gamma_2^t,\gamma_1^t,\gamma_3^t;\rho_{\mathscr{R}}),\\ \tilde{S}_{33}(t) &= \mathscr{T}_S(\xi^t,\gamma_1^t\ \gamma_2^t\ \gamma_3^t;\rho_{\mathscr{R}}),\end{aligned}\right\} \tag{11.10}$$

with

$$\xi^t(s) = \xi(t-s), \quad \gamma_i^t(s) = \gamma_i(t-s), \quad i = 1,2,3, \quad 0 \le s < \infty. \tag{11.11}$$

These functionals $\mathscr{Z}_I$, $\mathscr{S}_I$, $\mathscr{T}_I$ $(I = N,S)$ are independent of the directions $\underset{\sim}{h}_i$, depend on $\mathscr{R}$ only through the density $\rho_{\mathscr{R}}$, and obey the identities

$$\left.\begin{aligned}\mathscr{Z}_I(\xi^*,\gamma_2^*,\gamma_1^*,\gamma_3^*;\rho_{\mathscr{R}}) &= \mathscr{Z}_I(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}),\\ \mathscr{Z}_I(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}) &= -\mathscr{Z}_I(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}),\\ \mathscr{S}_I(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}) &= \mathscr{S}_I(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}),\\ \mathscr{T}_I(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}) &= \mathscr{T}_I(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}),\\ \mathscr{T}_I(\xi^*,\gamma_2^*,\gamma_1^*,\gamma_3^*;\rho_{\mathscr{R}}) &= \mathscr{T}_I(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathscr{R}}),\end{aligned}\right\} \tag{11.12}$$

for all functions ξ^*, γ_1^*, γ_2^*, γ_3^* in their domain of definition. Furthermore,

$$\mathscr{t}_I(0^\dagger,\gamma_3^*,\gamma_2^*,\gamma_1^*;\rho_{\mathcal{R}}) = \mathscr{s}_I(0^\dagger,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}) \tag{11.13}$$

where $0^\dagger$ is the function on $[0,\infty)$ with constant value 0:

$$0^\dagger(s) = 0, \qquad 0 \leq s < \infty. \tag{11.14}$$

12. Motions of Extension

Since the tensor $\underset{\sim}{R}$ in (7.3) is an orthogonal tensor, the right and left stretch tensors, $\underset{\sim}{U}$ and $\underset{\sim}{V}$, have the same proper numbers λ_i; these numbers are called <u>principal stretch ratios</u>. The proper vectors of $\underset{\sim}{U}$ and $\underset{\sim}{V}$, $\underset{\sim}{u}_i$ and $\tilde{\underset{\sim}{u}}_i$, are called, respectively, the <u>right</u> and <u>left principal directions of stretch</u> and obey relations of the form

$$\tilde{\underset{\sim}{u}}_i = \underset{\sim}{R}\underset{\sim}{u}_i. \tag{12.1}$$

The proper numbers σ_i and proper vectors $\underset{\sim}{s}_i$ of the stress tensor $\underset{\sim}{S}$ are called <u>principal stresses</u> and <u>principal axes of stress</u>. The principal stresses are also the proper numbers of the rotated stress $\tilde{\underset{\sim}{S}} = \underset{\sim}{R}^{-1}\underset{\sim}{S}\underset{\sim}{R}$, while the principal axes of stress are related as follows to the proper vectors $\tilde{\underset{\sim}{s}}_i$ of $\tilde{\underset{\sim}{S}}$:

$$\underset{\sim}{s}_i = \underset{\sim}{R}\tilde{\underset{\sim}{s}}_i(t). \tag{12.2}$$

Similarly, the proper numbers n_i of $\underset{\sim}{N}$, i.e. the <u>principal indices of</u>

refraction, equal the proper numbers of $\tilde{\mathbf{N}} = \mathbf{R}^{-1}\mathbf{N}\mathbf{R}$, and the principal axes of refraction $\mathbf{k}_i$, i.e. the proper vectors of $\mathbf{N}$, are determined by $\mathbf{R}$ and the proper vectors $\tilde{\mathbf{n}}_i$ of $\tilde{\mathbf{N}}$ through the formula

$$\mathbf{k}_i = \mathbf{R}\tilde{\mathbf{n}}_i. \tag{12.3}$$

If the right principal directions of stretch are constant in time at a particle X, then the motion of X is an extension. More precisely, one says that the history of X up to t is an extension if there exists an orthonormal basis $\mathbf{u}_i$ (independent of s) relative to which the matrix of the components of $\mathbf{U}(t-s)$ has the form

$$[\mathbf{U}(t-s)] = \begin{bmatrix} \lambda_1(t-s) & 0 & 0 \\ 0 & \lambda_2(t-s) & 0 \\ 0 & 0 & \lambda_3(t-s) \end{bmatrix}, \tag{12.4}$$

for all $s \geq 0$. The vectors $\mathbf{u}_i$ are then obviously right principal directions of stretch and the numbers $\lambda_i(t-s)$ are principal stretch ratios. Furthermore, it is obvious that, in the terminology of Section 11, an extension can be defined to be a sheared extension with $\zeta(t-s) = 0$ for all $s \geq 0$. For an extension, $\mathbf{P}(t-s)$ in (11.1) can be set equal to $\mathbf{R}(t-s)$; $\mathbf{M}(t-s)$ then equals $\mathbf{U}(t-s)$, the basis $\mathbf{h}_i$ equals the basis $\mathbf{u}_i$, while, in (11.4), $\xi(t-s) = 0$ and $\gamma_i(t-s) = \lambda_i(t-s)$ for all $s \geq 0$.

The main reduction theorem for motions of extension states the following.# *If, relative to an undistorted configuration $\mathscr{R}$, the history up to t of an isotropic material is an extension, then at time t each left principal direction of stretch $\tilde{\mathbf{u}}_i(t)$ is also a principal axis of*

#See Coleman [1968, 1] Thm. 5.

refraction $\underset{\sim}{k}_i(t)$ and a principal axis of stress $\underset{\sim}{s}(t)$; moreover, there are two scalar-valued material functionals, $\mathscr{f}_N$ and $\mathscr{f}_S$, which determine the principal indices of refraction $n_i(t)$ and the principal stresses $\sigma_i(t)$ as follows:

$$\left.\begin{aligned} n_1(t) &= \mathscr{f}_N(\lambda_1^t,\lambda_2^t,\lambda_3^t;\rho_{\mathscr{R}}), \\ n_2(t) &= \mathscr{f}_N(\lambda_2^t,\lambda_1^t,\lambda_3^t;\rho_{\mathscr{R}}), \\ n_3(t) &= \mathscr{f}_N(\lambda_3^t,\lambda_1^t,\lambda_2^t;\rho_{\mathscr{R}}), \end{aligned}\right\} \tag{12.5}$$

$$\left.\begin{aligned} \sigma_1(t) &= \mathscr{f}_S(\lambda_1^t,\lambda_2^t,\lambda_3^t;\rho_{\mathscr{R}}), \\ \sigma_2(t) &= \mathscr{f}_S(\lambda_2^t,\lambda_1^t,\lambda_3^t;\rho_{\mathscr{R}}), \\ \sigma_3(t) &= \mathscr{f}_S(\lambda_3^t,\lambda_1^t,\lambda_2^t;\rho_{\mathscr{R}}), \end{aligned}\right\} \tag{12.6}$$

where λ_i^t, given by

$$\lambda_i^t(s) = \lambda_i(t-s), \qquad i = 1,2,3, \qquad 0 \leq s < \infty. \tag{12.7}$$

is the history up to t of the principal stretch ratio associated with the right principal direction of stretch $\underset{\sim}{u}_i$. The functionals $\mathscr{f}_N$ and $\mathscr{f}_S$ are independent of the directions of stretch, depend on $\mathscr{R}$ only through $\rho_{\mathscr{R}}$, and obey the identity

$$\mathscr{f}_I(\alpha^*,\beta^*,\gamma^*;\rho_{\mathscr{R}}) = \mathscr{f}_I(\alpha^*,\gamma^*,\beta^*;\rho_{\mathscr{R}}), \qquad I = N,S, \tag{12.8}$$

for all triplets of positive functions $\alpha^*,\beta^*,\gamma^*$ on $[0,\infty)$.

It follows from (7.4) that the proper vectors of $\underset{\sim}{U}$ coincide with those of $\underset{\sim}{C}$ while the proper vectors of $\underset{\sim}{V}$ coincide with those of $\underset{\sim}{B}$. Thus, to see whether the history $\underset{\sim}{F}^t$ at a particle X has been an extension one need merely observe whether the proper vectors of $\underset{\sim}{C}(t-s) = \underset{\sim}{F}(t-s)^T\underset{\sim}{F}(t-s)$ are independent of s at X; if they are, if the material at X is isotropic, and if the reference configuration is undistorted, then each proper vector of $\underset{\sim}{B}(t) = \underset{\sim}{F}(t)\underset{\sim}{F}(t)^T$ will be both a principal axis of refraction and a principal axis of stress at X.

The functionals $\mathscr{f}_I$ in (12.5) and (12.6) are determined by the functionals $\mathscr{s}_I$ and $\mathscr{t}_I$ in (11.9) and (11.10) through the relations,

$$\mathscr{f}_I(\alpha^*,\beta^*,\gamma^*;\rho_{\mathscr{R}}) = \mathscr{s}_I(0^\dagger,\alpha^*,\beta^*,\gamma^*;\rho_{\mathscr{R}}) = \mathscr{t}_I(0^\dagger,\gamma^*,\beta^*,\alpha^*;\rho_{\mathscr{R}}), \quad I = N,S, \tag{12.9}$$

which hold for all positive functions α^*, β^*, γ^* on $[0,\infty)$.

13. Shearing Motions

The history of a particle X up to time t is a <u>shear</u> if, for all $s \geq 0$,

$$\underset{\sim}{F}(t-s) = \underset{\sim}{P}(t-s)\underset{\sim}{M}(t-s), \tag{13.1}$$

where $\underset{\sim}{P}(t-s)$ is orthogonal and there exists an orthonormal basis $\underset{\sim}{h}_i$, independent of s, relative to which the components of $\underset{\sim}{M}$ have the form

$$[\underset{\sim}{M}(t-s)] = \begin{bmatrix} 1 & 0 & 0 \\ \zeta(t-s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{13.2}$$

Thus a shear is a sheared extension with

$$\beta_1(t-s) \equiv \beta_2(t-s) \equiv \beta_3(t-s) \equiv 1. \tag{13.3}$$

Clearly, for a shear, the equations (11.5) reduce to

$$\xi = \zeta, \qquad \gamma_1^2 = 1 + \zeta^2, \qquad \gamma_2 = \gamma_3 = 1, \tag{13.4}$$

and, by (11.4), the history of X up to t is a shear if and only if, for some orthonormal basis $\underset{\sim}{h}_i$, independent of s,

$$[\underset{\sim}{C}(t-s)] = \begin{bmatrix} 1+\zeta(t-s)^2 & \zeta(t-s) & 0 \\ \zeta(t-s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{13.5}$$

$\underset{\sim}{h}_i$ is, of course, the basis used in (13.2), and is called the <u>canonical basis</u> for the shear. The parameter $\zeta(t-s)$ is called the <u>amount of shear</u> at time t-s. The equation (13.4), (or (13.3)) implies that $\rho(t-s) \equiv \rho_{\mathcal{R}}$. When the history is known to be a shear, specification of the canonical basis $\underset{\sim}{h}_i$, density ρ, and the real-valued function ζ^t with

$$\zeta^t(s) = \zeta(t-s), \qquad 0 \leq s < \infty, \tag{13.6}$$

determines, by (13.5) and (9.8), both the rotated refraction tensor $\tilde{\underset{\sim}{N}}(t)$ and the rotated stress tensor $\tilde{\underset{\sim}{S}}(t)$. Indeed, the following proposition is a direct consequence of Coleman's theorem on sheared extensions stated in the last paragraph of Section 11.# If, relative to an undistorted

---

#Cf. [1968, 1] Thm. 8. Following that article, we here consider shearing motions relative to an arbitrary undistorted reference configuration, and the proposition we state is valid for both compressible solids and compressible fluids. For incompressible fluids, or whenever the present configuration is undistorted and the density is not a parameter, the proposition yields as corollaries Coleman and Noll's reduction theorems for steady viscometric flows [1959, 2,3] and unsteady lineal shearing flows [1961, 1] §5, pp. 694-699.

reference configuration with density ρ, the motion of an isotropic material is a shear with canonical basis $\underset{\sim}{h}_i$ and amount of shear $\zeta(t-s)$, for $s \geq 0$, then the matrices of the components, with respect to $\underset{\sim}{h}_i$, of $\underset{\sim}{\tilde{N}}(t)$ and $\underset{\sim}{\tilde{S}}(t)$ have the forms (11.7) and (11.8) with

$$\left.\begin{aligned} \tilde{N}_{12}(t) &= \mathscr{Y}_N(\zeta^t;\rho), \\ \tilde{N}_{ii}(t) &= \mathscr{W}_N^{(i)}(\zeta^t;\rho), \qquad i = 1,2,3, \end{aligned}\right\} \tag{13.7}$$

$$\left.\begin{aligned} \tilde{S}_{12}(t) &= \mathscr{Y}_S(\zeta^t;\rho), \\ \tilde{S}_{ii}(t) &= \mathscr{W}_S^{(i)}(\zeta^t;\rho), \qquad i = 1,2,3, \end{aligned}\right\} \tag{13.8}$$

here $\mathscr{Y}_I$ and $\mathscr{W}_I^{(i)}$, $I = N,S$, are material functionals obeying the identities

$$\left.\begin{aligned} \mathscr{Y}_I(-\zeta^*;\rho) &= -\mathscr{Y}_I(\zeta^*;\rho), \\ \mathscr{W}_I^{(i)}(-\zeta^*;\rho) &= \mathscr{W}_I^{(i)}(\zeta^*;\rho), \qquad i = 1,2,3. \end{aligned}\right\} \tag{13.9}$$

The material functionals $\mathscr{Y}_I$, $\mathscr{W}_I^{(i)}$, occurring here are determined by the functionals $\mathscr{Z}_I$, $\mathscr{S}_I$, $\mathscr{T}_I$ in (11.9) and (11.10) through the relations

$$\left.\begin{aligned} \mathscr{Y}_I(\zeta^*;\rho) &= \mathscr{Z}_I\left(\zeta^*,\sqrt{1^\dagger+\zeta^{*2}},1^\dagger,1^\dagger;\rho\right), \\ \mathscr{W}_I^{(1)}(\zeta^*;\rho) &= \mathscr{S}_I\left(\zeta^*,\sqrt{1^\dagger+\zeta^{*2}},1^\dagger,1^\dagger;\rho\right), \\ \mathscr{W}_I^{(2)}(\zeta^*;\rho) &= \mathscr{S}_I\left(\zeta^*,1^\dagger,\sqrt{1^\dagger+\zeta^{*2}},1^\dagger;\rho\right), \\ \mathscr{W}_I^{(3)}(\zeta^*;\rho) &= \mathscr{T}_I\left(\zeta^*,\sqrt{1^\dagger+\zeta^{*2}},1^\dagger,1^\dagger;\rho\right), \end{aligned}\right\} \tag{13.10}$$

where I = N,S and

$$1^{\dagger}(s) \equiv 1, \quad \text{i.e.} \quad \sqrt{1^{\dagger} + \zeta^{*2}(s)} = \sqrt{1 + \zeta^{*}(s)^2} \quad \text{for} \quad 0 \leq s < \infty.$$

The non-singular tensor $\underset{\sim}{M}$ of (13.1) and (13.2) has the polar decomposition

$$\underset{\sim}{M} = \underset{\sim}{Q}\underset{\sim}{U} \tag{13.11}$$

with $\underset{\sim}{Q}$ orthogonal and $\underset{\sim}{U}$ positive definite and symmetric. An elementary calculation shows that, relative to the canonical basis $\underset{\sim}{h}_i$,

$$[\underset{\sim}{Q}] = \frac{1}{\sqrt{1+\zeta^2/4}} \begin{bmatrix} 1 & -\zeta/2 & 0 \\ \zeta/2 & 1 & 0 \\ 0 & 0 & \sqrt{1+\zeta^2/4} \end{bmatrix} \tag{13.12}$$

and

$$[\underset{\sim}{U}] = \frac{1}{\sqrt{1+\zeta^2/4}} \begin{bmatrix} 1+\zeta^2/2 & \zeta/2 & 0 \\ \zeta/2 & 1 & 0 \\ 0 & 0 & \sqrt{1+\zeta^2/4} \end{bmatrix}. \tag{13.13}$$

It follows from (13.1) and (13.10) that in the polar decomposition of $\underset{\sim}{F}$ into the rotation tensor $\underset{\sim}{R}$ and the right stretch tensor $\underset{\sim}{U}$, i.e. $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U}$, we have $\underset{\sim}{R} = \underset{\sim}{P}\underset{\sim}{Q}$, and $\underset{\sim}{U}$ is the tensor in (13.11) and (13.13). In particular, at time t,

$$\underset{\sim}{R}(t) = \underset{\sim}{P}(t)\underset{\sim}{Q}(t). \tag{13.14}$$

Since $\underset{\sim}{Q}$ is shown explicitly in (13.12), when $\underset{\sim}{P}$ in (13.1) is specified $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ can be calculated from $\underset{\sim}{\tilde{N}}(t)$ and $\underset{\sim}{\tilde{S}}(t)$, i.e. from (13.7) and

(13.8); for, by (9.5),

$$\underset{\sim}{N}(t) = \underset{\sim}{R}(t)\underset{\sim}{\tilde{N}}(t)\underset{\sim}{R}(t)^{-1}, \qquad \underset{\sim}{S}(t) = \underset{\sim}{R}(t)\underset{\sim}{\tilde{S}}(t)\underset{\sim}{R}(t)^{-1}. \tag{13.15}$$

For example, when $\underset{\sim}{P} = \underset{\sim}{1}$, the components of $\underset{\sim}{N}(t)$ with respect to $\underset{\sim}{h}_i$ are

$$\left.\begin{aligned}
N_{12} = N_{21} &= \frac{\tilde{N}_{12}(1-\zeta^2/4) + (\tilde{N}_{11}-\tilde{N}_{22})\zeta/2}{1+\zeta^2/4}, \\
N_{11} &= \frac{\tilde{N}_{11} - \tilde{N}_{12}\zeta + \tilde{N}_{22}\zeta^2/4}{1+\zeta^2/4}, \\
N_{22} &= \frac{\tilde{N}_{22} + \tilde{N}_{12}\zeta + \tilde{N}_{11}\zeta^2/4}{1+\zeta^2/4}, \\
N_{33} &= \tilde{N}_{33}, \\
N_{23} = N_{32} = N_{31} &= N_{13} = 0,
\end{aligned}\right\} \tag{13.16}$$

where $\zeta = \zeta(t)$, and the numbers $\tilde{N}_{ij} = \tilde{N}_{ij}(t)$ are given in (13.7). Of course, similar expressions hold for the components of $\underset{\sim}{S}(t)$.

14. Homogeneous Extension

a. General Description

In this and the following two sections we discuss some global motions in which optical measurements appear feasible for compressible isotropic solids# and for which the history of each particle is an extension or a sheared extension. In each case we assume that the chosen reference configuration is undistorted.

A homogeneous extension is a motion for which there exists a fixed Cartesian coordinate system such that, for all t,

$$x_1 = \lambda_1(t)X_1, \quad x_2 = \lambda_2(t)X_2, \quad x_3 = \lambda_3(t)X_3, \quad \lambda_i(t) > 0, \tag{14.1}$$

where x_1, x_2, x_3 are the coordinates at time t of the particle which has the coordinates X_1, X_2, X_3 in the reference configuration. It is clear that, in such a motion, $\underset{\sim}{U} = \underset{\sim}{V} = \underset{\sim}{F}$, $\underset{\sim}{R} = \underset{\sim}{1}$, and each particle experiences the same history; moreover, this history is, for each t, an extension, in the sense of Section 12, with the right and left principal directions of stretch parallel to the coordinate directions. Hence, by the reduction theorem stated in Section 12, each coordinate direction is also a principal axis of stress, $\underset{\sim}{s}_i$, and a principal axis of polarization, $\underset{\sim}{k}_i$. The principal indices of refraction and principal stresses are, at each instant, the same at each particle and are given by (12.5) and (12.6). The principal stresses σ_1, σ_2, σ_3 and the principal stretch ratios λ_1, λ_2, λ_3 are accessible to measurement by mechanical means, and each principal index

---

#Some motions more appropriate to incompressible fluids are discussed in Chapter V.

of refraction n_i, $i = 1,2,3$, may be determined if one can measure the absolute retardation of polarized light traveling in a coordinate direction x_j $(j \neq i)$ with the axis of polarization pointing in the direction x_i. However, apparatus to simultaneously determine all of the quantities σ_1, σ_2, σ_3, n_1, n_2, n_3 for arbitrary histories λ_1^t, λ_2^t, λ_3^t has not been devised. Even if one exploits the symmetry conditions (12.8), it does not at present appear feasible to completely determine the functional $\not{f}_N$. Before discussing below the limited class of tests actually attempted for solid bodies, we mention here a difficulty which occurs in all attempts to study homogeneous extensions: General motions of the form (14.1) cannot be achieved by boundary loads alone.# Of course, every static state of homogeneous strain is a possible equilibrium configuration, and it would be possible to achieve arbitrary histories λ_1^t, λ_2^t, λ_3^t in homogeneous extension if inertia could be neglected. Experimentors believe that inertia can be neglected for a large class of histories called "quasistatic histories". Whether or not a history is "quasistatic" depends on not only the degree of precision sought but also the size of the specimen being studied.

#See Truesdell & Noll [1965, 4] pp. 61-63. It follows, however, from a result of Coleman & Truesdell [1965, 2], that all <u>isochoric</u> motions of the form (14.1) can be obtained in <u>incompressible</u> materials through application of appropriate tractions at the bounding surfaces.

b. Biaxial Stress

A motion of extension with one of the principal stresses zero, say

$$\sigma_3(\tau) = 0 \tag{14.2}$$

for all $\tau \leq t$, obeys, by (12.6), the equation

$$\mathscr{f}_S(\lambda_3^\tau, \lambda_1^\tau, \lambda_2^\tau; \rho_{\mathscr{R}}) = 0 \tag{14.3}$$

for $\tau \leq t$. Suppose now that this equation can be solved for λ_3 as a function of time, when λ_1^t and λ_2^t are given, i.e. that there exists a functional $\mathscr{a}$ such that (14.3) implies

$$\lambda_3(\tau) = \mathscr{a}(\lambda_1^\tau, \lambda_2^\tau; \rho_{\mathscr{R}}), \tag{14.4}$$

for $\tau \leq t$. When (14.4) holds, (12.5) and (12.6) yield n_1, n_2, σ_1, σ_2 as functions of the histories λ_1^t and λ_2^t:

$$\left.\begin{aligned} n_1(t) &= \mathscr{b}_N(\lambda_1^t, \lambda_2^t; \rho_{\mathscr{R}}), \\ n_2(t) &= \mathscr{b}_N(\lambda_2^t, \lambda_1^t; \rho_{\mathscr{R}}), \\ \sigma_1(t) &= \mathscr{b}_S(\lambda_1^t, \lambda_2^t; \rho_{\mathscr{R}}), \\ \sigma_2(t) &= \mathscr{b}_S(\lambda_2^t, \lambda_1^t; \rho_{\mathscr{R}}), \end{aligned}\right\} \tag{14.5}$$

where the material functionals $\mathscr{b}_N$ and $\mathscr{b}_S$ are derived from $\mathscr{f}_N$ and $\mathscr{f}_S$ through elimination of λ_3^t via (14.4).

In the laboratory one attempts to achieve the condition (14.2) by stretching, in the directions x_1, x_2, a thin sheet of material held so that the x_3-axis is normal to the sheet, and σ_3 is negligible. The stretches λ_1, λ_2, λ_3 and the stresses σ_1, σ_2 can be measured directly. Employing a plane polariscope mounted with its optic axis perpendicular to the sheet, one may determine the relative retardation for polarized light traveling in the direction x_3 and hence obtain the birefringence $n_1 - n_2$. Measurement (with an interferometer) of the absolute retardation of such light waves yields the indices of refraction n_1 and n_2. Thus, the functionals $\mathcal{a}$, $\mathcal{b}_N$, and $\mathcal{b}_S$ can be determined for those quasistatic histories λ_1^t, λ_2^t which the apparatus can impose on the specimen.

c. Tensile Tests

A tensile test is a homogeneous extension with

$$\sigma_2(\tau) = \sigma_3(\tau) = 0 \tag{14.6}$$

for all τ. By (12.6), the stretches then obey the relations

$$\left.\begin{aligned} \mathcal{f}_S(\lambda_2^\tau, \lambda_1^\tau, \lambda_3^\tau; \rho_{\mathcal{R}}) &= 0, \\ \mathcal{f}_S(\lambda_3^\tau, \lambda_1^\tau, \lambda_2^\tau; \rho_{\mathcal{R}}) &= 0, \end{aligned}\right\} \tag{14.7}$$

for all τ. If we assume that, for each history λ_1^τ, (14.7) has a solution for $\lambda_2(\tau)$ and $\lambda_3(\tau)$, then this solution must be such that $\lambda_2(\tau) = \lambda_3(\tau)$ and, therefore, can be written in the form

$$\lambda_2(\tau) = \lambda_3(\tau) = \mathcal{d}(\lambda_1^\tau; \rho_{\mathcal{R}}). \tag{14.8}$$

Furthermore, (12.5) and (12.6) then yield

$$\left.\begin{aligned} n_1(t) &= \mathcal{E}_N(\lambda_1^t;\rho_{\mathcal{R}}), \\ n_3(t) = n_2(t) &= \mathcal{G}_N(\lambda_1^t;\rho_{\mathcal{R}}), \\ \sigma_1(t) &= \mathcal{E}_S(\lambda_1^t;\rho_{\mathcal{R}}), \end{aligned}\right\} \tag{14.9}$$

where the functionals $\mathcal{E}_N$, $\mathcal{G}_N$, and $\mathcal{E}_S$ are obtained from $\mathcal{F}_N$ and $\mathcal{F}_S$ through elimination of λ_2^t and λ_3^t.

Experimentors seek to obtain the condition (14.6) by applying an axial load to the ends of a rectangular rod held so that the x_3-axis is normal to one free surface and the x_2-axis to another. The axis of the rod then lies parallel to the x_1-direction and the length of the rod is proportional to λ_1. The stretches λ_1, λ_2, λ_3 and the stress σ_1 are easily measured. The indices of refraction n_1 and n_2 can be obtained by measuring the absolute retardation of polarized light propagating in the x_3-direction (with the amplitude vector pointing in x_1- and x_2-directions). Thus, the functionals $\mathcal{E}_N$, $\mathcal{G}_N$, and $\mathcal{E}_S$ can be determined for quasistatic histories λ^t.

15. Rectilinear Sheared Extensions

The discussion of the previous section can be generalized without great difficulty. A motion for which there exists a fixed Cartesian coordinate system in which, at all times t,

$$x_1 = x_1(X_1,t), \quad x_2 = x_2(X_1,X_2,t), \quad x_3 = x_3(X_3,t), \quad \frac{\partial x_i}{\partial X_i} > 0, \tag{15.1}$$

can be called a rectilinear sheared extension.[#] The Cartesian components of $\underset{\sim}{F}$ here have the matrix

$$[\underset{\sim}{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & 0 & 0 \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial x_3}{\partial X_3} \end{bmatrix}, \qquad (15.2)$$

and the history of each particle is, therefore, a sheared extension with the canonical basis $\underset{\sim}{h}_i$ equal to the natural basis of the Cartesian system, with $\underset{\sim}{P}$ in (11.1) equal to $\underset{\sim}{1}$, and with

$$\zeta = \frac{\partial x_2}{\partial X_1}, \qquad \beta_1 = \frac{\partial x_1}{\partial X_1}, \qquad \beta_2 = \frac{\partial x_2}{\partial X_2}, \qquad \beta_3 = \frac{\partial x_3}{\partial X_3}, \qquad (15.3)$$

or, by (11.5),

$$\xi = \left(\frac{\partial x_2}{\partial X_1}\right)\left(\frac{\partial x_2}{\partial X_2}\right), \qquad \gamma_1 = \sqrt{\left(\frac{\partial x_1}{\partial X_1}\right)^2 + \left(\frac{\partial x_2}{\partial X_1}\right)^2}, \qquad \gamma_2 = \frac{\partial x_2}{\partial X_2}, \qquad \gamma_3 = \frac{\partial x_3}{\partial X_3}. \qquad (15.4)$$

Hence, by the reduction theorem stated in Section 11, at each particle the components of $\tilde{\underset{\sim}{N}}(t)$ and $\tilde{\underset{\sim}{S}}(t)$ relative to $\underset{\sim}{h}_i$ are given by the equations (11.7)-(11.11). It is not difficult to see that, with respect to $\underset{\sim}{h}_i$, $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ here have matrices similar to those given for $\tilde{\underset{\sim}{N}}(t)$ and $\tilde{\underset{\sim}{S}}(t)$ in (11.7) and (11.8):

$$[\underset{\sim}{N}(t)] = \begin{bmatrix} N_{11}(t) & N_{12}(t) & 0 \\ N_{12}(t) & N_{22}(t) & 0 \\ 0 & 0 & N_{33}(t) \end{bmatrix}, \qquad (15.5)$$

[#]Coleman [1968, 1], pp. 471-472, has derived, for such motions, reduced forms of the equations of motion.

$$
[\underset{\sim}{S}(t)] = \begin{bmatrix} S_{11}(t) & S_{12}(t) & 0 \\ S_{12}(t) & S_{22}(t) & 0 \\ 0 & 0 & S_{33}(t) \end{bmatrix}, \tag{15.6}
$$

and, furthermore,

$$
n_3(t) = N_{33}(t) = \tilde{N}_{33}(t), \qquad \sigma_3(t) = S_{33}(t) = \tilde{S}_{33}(t). \tag{15.7}
$$

Since the proper numbers n_i of $\underset{\sim}{N}$ equal the proper numbers of $\underset{\sim}{\tilde{N}}$, and the proper numbers σ_i of $\underset{\sim}{S}$ equal the proper numbers of $\underset{\sim}{\tilde{S}}$, it follows from (11.7) that

$$
\left.\begin{aligned} 2n_1 &= \tilde{N}_{11} + \tilde{N}_{22} + \sqrt{(\tilde{N}_{11} - \tilde{N}_{22})^2 + 4\tilde{N}_{12}^2}\ , \\ 2n_2 &= \tilde{N}_{11} + \tilde{N}_{22} - \sqrt{(\tilde{N}_{11} - \tilde{N}_{22})^2 + 4\tilde{N}_{12}^2}\ , \end{aligned}\right\} \tag{15.8}
$$

if the proper vectors of $\underset{\sim}{N}$ are labeled so that# $n_1 \geq n_2$. The birefringence for light propagating in the direction x_3 is

$$
n_1 - n_2 = \sqrt{(\tilde{N}_{11} - \tilde{N}_{22})^2 + 4\tilde{N}_{12}}\ , \tag{15.9}
$$

with $\tilde{N}_{11}$, $\tilde{N}_{22}$, and $\tilde{N}_{12}$ given by (11.9). Of course, analogous expressions hold for the principal stresses σ_1 and σ_2.

Continuing to denote the principal axes of stress and refraction by $\underset{\sim}{k}_i$ and $\underset{\sim}{s}_i$, respectively, we observe that $\underset{\sim}{k}_3$ and $\underset{\sim}{s}_3$ both point in the direction of the coordinate axis x_3, while $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ all lie in a plane parallel to the (x_1,x_2)-plane. When the local history does not

#One does not always use this labeling.

reduce to pure extension, there is no reason to suppose that $\underset{\sim}{k}_i = \underset{\sim}{s}_i$ for $i = 1,2$.# Let χ be the angle $\underset{\sim}{k}_1$ makes with the x_1-axis. Then

$$\tan 2\chi(t) = \frac{2N_{12}(t)}{N_{11}(t) - N_{22}(t)}, \tag{15.10}$$

From (7.12),

$$\underset{\sim}{N}(t) = \underset{\sim}{R}(t)\underset{\sim}{\tilde{N}}(t)\underset{\sim}{R}(t)^{-1}, \tag{15.11}$$

and $\underset{\sim}{R}$ is determined by (15.2) through the general equations (7.3), (7.4). Hence the proper vectors $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$ are determined by the motion and the material functionals in (11.9), but the resulting expressions are too cumbersome to give here.

A sheared extension with

$$S_{33}(\tau) = 0 \tag{15.12}$$

for all $\tau \leq t$, obeys, by (15.7) and (11.10), the relation

$$\mathcal{t}_S(\xi^\tau, \gamma_1^\tau, \gamma_2^\tau, \gamma_3^\tau; \rho_{\mathcal{R}}) = 0 \tag{15.13}$$

for $\tau \leq t$. Let us assume that (15.13) has a solution in the sense that

$$\gamma_3(\tau) = \mathcal{c}(\xi^\tau, \gamma_1^\tau, \gamma_2^\tau; \rho_{\mathcal{R}}), \qquad \tau \leq t. \tag{15.14}$$

#Indeed, it can be easily shown that in the theory of linear viscoelastic materials the principal axes of stress and refraction need not coincide in sheared extensions. The tests of Fowlkes, reported by Dill and Fowlkes [1966, 1] and Fowlkes [1969, 1], demonstrated a case for which the principal axes of stress, principal directions of stretch, and the principal axes of refraction are all distinct.

When such is the case, (11.9) and (11.10) yield

$$\left.\begin{aligned} \tilde{N}_{12}(t) &= \dot{\mathscr{f}}_N(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \\ \tilde{N}_{11}(t) &= \mathscr{k}_N(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \\ \tilde{N}_{22}(t) &= \mathscr{k}_N(\xi^t, \gamma_2^t, \gamma_1^t; \rho_{\mathscr{R}}), \\ \tilde{N}_{33}(t) &= \mathscr{l}_N(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \end{aligned}\right\} \tag{15.15}$$

and

$$\left.\begin{aligned} \tilde{S}_{12}(t) &= \dot{\mathscr{f}}_S(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \\ \tilde{S}_{11}(t) &= \mathscr{k}_S(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \\ \tilde{S}_{22}(t) &= \mathscr{k}_S(\xi^t, \gamma_1^t, \gamma_2^t; \rho_{\mathscr{R}}), \end{aligned}\right\} \tag{15.16}$$

where the material functionals $\dot{\mathscr{f}}_N$, $\mathscr{k}_N$, $\mathscr{l}_N$, $\dot{\mathscr{f}}_S$, $\mathscr{k}_S$ are derived from $\mathscr{z}_N$, $\mathscr{s}_N$, $\mathscr{t}_N$, $\mathscr{z}_S$, $\mathscr{s}_S$ through elimination of γ_3^t.

One may attempt to generate rectilinear sheared extensions obeying (15.12) by deforming a thin sheet held so that the sheet remains in a plane normal to the x_3-axis and no external tractions are exerted on the sheet except at the edges where it is gripped. If the sheet is thin enough that $\underset{\sim}{S}$ is independent of x_3, then the condition about external tractions yields

$$S_{13} = S_{23} = S_{33} = 0,$$

at all values of x_1 and x_2 away from the edges, in accord with (15.6) and (15.12). Of course it may not be easy to insure that the motion corresponds to a non-trivial shear superposed on a simple extension, i.e. that the equations (15.1) hold with $\partial x_2/\partial X_1 \neq 0$. Through use of appropriate grips and sufficiently slow loadings, one may be able to achieve, at least near the center of the sheet, a homogeneous, rectilinear sheared extension, i.e. a motion of the form

$$x_1 = \beta_1(t)X_1, \qquad x_2 = \beta_2(t)X_2 + \zeta(t)X_1, \qquad x_3 \doteq \beta_3(t)X_3. \qquad (15.17)$$

If one rules a rectangular grid on the surface of the sheet before deformation, study of photographs of the grid taken during a motion will indicate whether the motion is actually of the form (15.17) in some region and, if so, will yield the values of ζ_1, β_1, β_2, and β_3. If the sheet is viewed through a plane polariscope with its optic axis in the x_3-direction, then observation of the isoclinic lines will yield the angle χ given by (15.10) and (15.15). Indeed, when the polarizing axes of the polariscope are so oriented that a given point is on an isoclinic line, then χ at that point equals the smallest of the angles which the axis of the analyzer makes with the coordinate axes, x_1, x_2. Study of the isochromatic fringe pattern yields the birefringence given by (15.9) and (15.15). The individual indices of refraction in (15.8) can be obtained with an interferometer.

16. Extension, Torsion, and Inflation of a Tube

Let us now suppose that in its undistorted reference configuration the body under consideration has the form of a hollow circular tube. We employ a fixed cylindrical coordinate system with the z-axis along the common axis of the cylinders bounding the tube. The coordinates of a particle in the reference configuration are written Z, Θ, R, and if, for all t, the present coordinates z, θ, r are given by an expression of the form

$$z = z(Z,t), \qquad \theta = \Theta + \phi(Z,t), \qquad r = r(R,t), \tag{16.1}$$

then we say that the motion of the tube is a <u>simultaneous extension, torsion, and inflation</u>. In such a motion the body remains a circular tube at all times. With each particle X we can associate the orthonormal basis of unit vectors $\underset{\sim}{e}_Z$, $\underset{\sim}{e}_\Theta$, $\underset{\sim}{e}_R$, pointing along the coordinate axes at the place Z, Θ, R occupied by X in the reference configuration. Clearly, the basis $\underset{\sim}{e}_Z$, $\underset{\sim}{e}_\Theta$, $\underset{\sim}{e}_R$, albeit it varies from particle to particle, is constant in time in each motion. An elementary calculation shows that if the motion obeys (16.1) for all t, then at each instant the matrix of the components of $\underset{\sim}{C}$ relative to $\underset{\sim}{e}_Z$, $\underset{\sim}{e}_\Theta$, $\underset{\sim}{e}_R$ is

$$[\underset{\sim}{C}] = \begin{bmatrix} \left(\frac{\partial z}{\partial Z}\right)^2 + \left(r\,\frac{\partial \phi}{\partial Z}\right)^2 & \frac{r^2}{R}\,\frac{\partial \phi}{\partial Z} & 0 \\ \frac{r^2}{R}\,\frac{\partial \phi}{\partial Z} & \left(\frac{r}{R}\right)^2 & 0 \\ 0 & 0 & \left(\frac{\partial r}{\partial R}\right)^2 \end{bmatrix}. \tag{16.2}$$

Comparing (16.2) with (11.4), we see that, in a simultaneous extension, torsion, and inflation of a tube, the history of each particle is a sheared extension with

$$\xi = \frac{r^2}{R}\,\frac{\partial \phi}{\partial Z}, \qquad \gamma_1 = \sqrt{\left(\frac{\partial z}{\partial Z}\right)^2 + \left(r\,\frac{\partial \phi}{\partial Z}\right)^2}, \qquad \gamma_2 = \frac{r}{R}, \qquad \gamma_3 = \frac{\partial r}{\partial R},$$

i.e. with

$$\zeta = r\frac{\partial\phi}{\partial Z}, \qquad \beta_1 = \frac{\partial z}{\partial Z}, \qquad \beta_2 = \frac{r}{R}, \qquad \beta_3 = \frac{\partial r}{\partial R},$$

and with $\underset{\sim}{e}_Z$, $\underset{\sim}{e}_\Theta$, $\underset{\sim}{e}_R$ a canonical basis. Thus, in such a motion, at each time t, the components of $\underset{\sim}{\tilde{N}}(t)$ and $\underset{\sim}{\tilde{S}}(t)$, with respect to $\underset{\sim}{e}_Z$, $\underset{\sim}{e}_\Theta$, $\underset{\sim}{e}_R$, are given by (11.7)-(11.10) for each particle.

Optical experiments can be done with light directed radially through the wall of the tube. Since, however, the functions ξ^t and γ_i^t here depend on r, the birefringence for light traveling in the radial direction,

$$\Delta(t) = n_1(t) - n_2(t),$$

also depends on r, and there is no simple relation between $\Delta(t)$ and the observed relative retardation. Nevertheless, experimenters assume, with some justification, that if the tube is sufficiently thin, the change in the deformation gradient with radius can be neglected, and the relation (1.2) can be employed. When such is the case, the test of a thin walled tube in a motion of the form (16.1) provides a means of studying sheared extension and, of course, in the absence of torsion, an alternative to the stretch of a thin sheet, described in Section 14b.

V. PARTICULAR MOTIONS OF INCOMPRESSIBLE FLUIDS

17. Properties of Incompressible Materials

a. General Relations

The motions of incompressible materials are subject to the kinematical constraint ρ = const., which, when the reference configuration is sensibly chosen, reduces to

$$|\det \underset{\sim}{F}| \equiv |\det \underset{\sim}{C}| \equiv 1. \tag{17.1}$$

Furthermore, the stress at a point in an incompressible material is determined by the history $\underset{\sim}{F}^t$ only to within an arbitrary hydrostatic pressure; that is, for such materials, the deviator $_D\underset{\sim}{S}(t)$, defined by

$$_D\underset{\sim}{S} \stackrel{\text{def}}{=\!=} \underset{\sim}{S} - \frac{1}{3}(\operatorname{tr} \underset{\sim}{S})\underset{\sim}{1}, \tag{17.2}$$

is a function of $\underset{\sim}{F}^t$, but $\underset{\sim}{S}(t)$ is not. The local history $\underset{\sim}{F}^t$ does determine the refraction tensor $\underset{\sim}{N}(t)$ completely, but in this section we are more interested in the deviator $_D\underset{\sim}{N}$ of $\underset{\sim}{N}$:

$$_D\underset{\sim}{N} \stackrel{\text{def}}{=\!=} \underset{\sim}{N} - \frac{1}{3}(\operatorname{tr} \underset{\sim}{N})\underset{\sim}{1}. \tag{17.3}$$

Of course, if $\underset{\sim}{N}(t)$ is a function of $\underset{\sim}{F}^t$, then so also is $_D\underset{\sim}{N}(t)$. Thus, for incompressible materials we have

$$_D\underset{\sim}{N}(t) = {}_D\mathfrak{N}(\underset{\sim}{F}^t), \qquad _D\underset{\sim}{S}(t) = {}_D\mathfrak{S}(\underset{\sim}{F}^t). \tag{17.4}$$

b. Incompressible Fluids

The argument which gave us (10.2) tells us that for an incompressible fluid the relations (17.4) can be written

$$_D\underset{\sim}{N}(t) = {}_D\tilde{\mathfrak{N}}(\underset{\sim}{C}_t^t), \qquad _D\underset{\sim}{S}(t) = {}_D\tilde{\mathfrak{S}}(\underset{\sim}{C}_t^t); \tag{17.5}$$

$\underset{\sim}{C}_t^t$ is the history of the right Cauchy-Green tensor relative to the present configuration and may be calculated as shown in (10.3)-(10.5). The condition that the motion be isochoric implies that

$$|\det \underset{\sim}{C}_t^t(s)| \equiv 1 \quad \text{for} \quad 0 \leq s < \infty. \tag{17.6}$$

Since the functionals $_D\tilde{\mathfrak{N}}$ and $_D\tilde{\mathfrak{S}}$ are isotropic, incompressible fluids obey the following analogue of (10.8):

$$\underset{\sim}{C}_t^t \to \underset{\sim}{Q}\underset{\sim}{C}_t^t\underset{\sim}{Q}^{-1} \implies {}_D\underset{\sim}{N}(t) \to \underset{\sim}{Q}\,{}_D\underset{\sim}{N}(t)\underset{\sim}{Q}^{-1} \quad \text{and} \quad {}_D\underset{\sim}{S}(t) \to \underset{\sim}{Q}\,{}_D\underset{\sim}{S}(t)\underset{\sim}{Q}^{-1}. \tag{17.7}$$

Indeed, a fluid is an isotropic material with the property that every configuration of it is undistorted. The entire theory of Chapter IV applies to fluids, and, moreover, for a fluid one may choose the undistorted reference configuration $\mathcal{R}$ to be the present configuration, i.e. the configuration at time t, as we have done in (17.5). This choice of reference configuration yields $\tilde{\underset{\sim}{N}}(t) = \underset{\sim}{N}(t)$ and $\tilde{\underset{\sim}{S}}(t) = \underset{\sim}{S}(t)$ and results in a simplification of the theory of Chapter IV.

We note that, for an incompressible fluid, (10.9) becomes

$$\underset{\sim}{C}_t^t = \underset{\sim}{1}^{\dagger} \implies \underset{\sim}{N}(t) = n^\circ\underset{\sim}{1} \quad \text{and} \quad \underset{\sim}{S}(t) = -p\underset{\sim}{1} \tag{17.8}$$

with n° a positive material constant and p a number to be determined by the dynamical equations and boundary data. As a trivial consequence of (17.8) we have

$$\underset{\sim}{C}_t^t = \underset{\sim}{1}^\dagger \implies {}_D\underset{\sim}{N}(t) = {}_D\underset{\sim}{S}(t) = \underset{\sim}{0}. \tag{17.9}$$

The reduced constitutive equations which describe the behavior of incompressible fluids in shears, extensions, and sheared extensions, are given below.

18. Isochoric Sheared Extension

Suppose that, at a particular time t at a particle X of a fluid, there exists an orthonormal basis $\underset{\sim}{h}_i$ relative to which the components of the right relative Cauchy-Green tensor $\underset{\sim}{C}_t(t-s)$ have the form

$$\left.\begin{aligned}
[\underset{\sim}{C}_t(t-s)] &= \begin{bmatrix} \gamma_1^2(t-s) & \xi(t-s) & 0 \\ \xi(t-s) & \gamma_2^2(t-s) & 0 \\ 0 & 0 & \gamma_3^2(t-s) \end{bmatrix} \\
\gamma_3^2(t-s)\left[\gamma_2^2(t-s)\gamma_1^2(t-s) - \xi^2(t-s)\right] &\equiv 1, \qquad \gamma_i(t-s) > 0
\end{aligned}\right\} \tag{18.1}$$

for all $s \geq 0$. When such is the case we say that the history of X up to time t is an <u>isochoric sheared extension with canonical basis</u> $\underset{\sim}{h}_i$.

The arguments which yield the theorem stated in Section 11 yield also the following remark: If the history up to time t of an incompressible fluid is an isochoric sheared extension with canonical basis $\underset{\sim}{h}_i$, then the matrices of the components, relative to $\underset{\sim}{h}_i$, of $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ have the forms

$$[\underset{\sim}{N}(t)] = \begin{bmatrix} N_{11}(t) & N_{21}(t) & 0 \\ N_{12}(t) & N_{22}(t) & 0 \\ 0 & 0 & N_{33}(t) \end{bmatrix}, \tag{18.2}$$

$$[\underset{\sim}{S}(t)] = \begin{bmatrix} S_{11}(t) & S_{21}(t) & 0 \\ S_{12}(t) & S_{22}(t) & 0 \\ 0 & 0 & S_{33}(t) \end{bmatrix}, \tag{18.3}$$

and there are six scalar-valued material functionals $\underline{\underline{\mathfrak{z}}}_N$, $\underline{\underline{\mathfrak{s}}}_N$, $\underline{\underline{\mathfrak{t}}}_N$ and $\underline{\underline{\mathfrak{z}}}_S$, $\underline{\underline{\mathfrak{s}}}_S$, $\underline{\underline{\mathfrak{t}}}_S$, such that

$$\left.\begin{aligned} N_{12}(t) &= \underline{\underline{\mathfrak{z}}}_N(\xi^t,\gamma_1^t,\gamma_2^t), \\ N_{11}(t) - N_{22}(t) &= \underline{\underline{\mathfrak{s}}}_N(\xi^t,\gamma_1^t,\gamma_2^t), \\ N_{22}(t) - N_{33}(t) &= \underline{\underline{\mathfrak{t}}}_N(\xi^t,\gamma_1^t,\gamma_2^t), \\ \\ S_{12}(t) &= \underline{\underline{\mathfrak{z}}}_S(\xi^t,\gamma_1^t,\gamma_2^t), \\ S_{11}(t) - S_{22}(t) &= \underline{\underline{\mathfrak{s}}}_S(\xi^t,\gamma_1^t,\gamma_2^t), \\ S_{22}(t) - S_{33}(t) &= \underline{\underline{\mathfrak{t}}}_S(\xi^t,\gamma_1^t,\gamma_2^t), \end{aligned}\right\} \tag{18.4}$$

with

$$\xi^t(s) = \xi(t-s), \qquad \gamma_i^t(s) = \gamma_i(t-s), \qquad i = 1,2. \tag{18.5}$$

The constitutive functionals $\underline{\underline{\mathfrak{z}}}_I$, $\underline{\underline{\mathfrak{s}}}_I$, $\underline{\underline{\mathfrak{t}}}_I$ $(I = N,S)$ are independent of the basis $\underset{\sim}{h}_i$, are determined by the functionals ${}_D\tilde{\mathfrak{N}}$ and ${}_D\tilde{\mathfrak{S}}$ in (17.5), and obey the relations,

$$\left.\begin{aligned}
\underline{\underline{\mathfrak{z}}}_I(\xi^*,\gamma_2^*,\gamma_1^*) &= \underline{\underline{\mathfrak{z}}}_I(\xi^*,\gamma_1^*,\gamma_2^*),\\
\underline{\underline{\mathfrak{z}}}_I(-\xi^*,\gamma_1^*,\gamma_2^*) &= -\underline{\underline{\mathfrak{z}}}_I(\xi^*,\gamma_1^*,\gamma_2^*),\\
\underline{\underline{\mathfrak{s}}}_I(-\xi^*,\gamma_1^*,\gamma_2^*) &= \underline{\underline{\mathfrak{s}}}_I(\xi^*,\gamma_1^*,\gamma_2^*),\\
\underline{\underline{\mathfrak{t}}}_I(-\xi^*,\gamma_1^*,\gamma_2^*) &= \underline{\underline{\mathfrak{t}}}_I(\xi^*,\gamma_1^*,\gamma_2^*),\\
\underline{\underline{\mathfrak{t}}}_I(0^{\dagger},\gamma_1^*,\gamma_2^*) &= \underline{\underline{\mathfrak{s}}}_I(0^{\dagger},\gamma_1^*,\gamma_2^*),
\end{aligned}\right\}\tag{18.6}$$

for all functions ξ^*, γ_1^*, γ_2^* in their domain.

Employing (18.2) to calculate the proper numbers n_i of $\underset{\sim}{N}(t)$, we find that the birefringence Δ for light traveling in the direction $\underset{\sim}{h}_3$ is

$$\Delta \overset{\text{def}}{=\!=} n_1 - n_2 = \sqrt{(N_{11}-N_{22})^2 + 4N_{12}^2} = \sqrt{\underline{\underline{\mathfrak{s}}}_N(\xi^t,\gamma_1^t,\gamma_2^t)^2 + 4\,\underline{\underline{\mathfrak{z}}}_N(\xi^t,\gamma_1^t,\gamma_2^t)^2}\tag{18.7}$$

if the proper vectors of $\underset{\sim}{N}$ are labeled such that $n_1 \geq n_2$.# It follows from (18.2) that the principal axes of refraction associated with n_1 and n_2, i.e. $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$, lie in the same plane as the vectors $\underset{\sim}{h}_1$ and $\underset{\sim}{h}_2$. Furthermore, if we let χ be the angle from $\underset{\sim}{h}_1$ to $\underset{\sim}{k}_1$, then

$$\cot 2\chi = \frac{N_{11} - N_{22}}{2N_{12}} = \frac{\underline{\underline{\mathfrak{s}}}_N(\xi^t,\gamma_1^t,\gamma_2^t)}{2\,\underline{\underline{\mathfrak{z}}}_N(\xi^t,\gamma_1^t,\gamma_2^t)}\,.\tag{18.8}$$

#In Sections 20 and 21 we employ a different, and somewhat more elaborate, convention for distinguishing $\underset{\sim}{k}_1$ from $\underset{\sim}{k}_2$. (See the paragraph containing eqs. (20.6)-(20.8).) In the system of labeling of those sections, it is possible to have n_1 less than n_2.

19. Isochoric Extension

Suppose the motion of an incompressible fluid is such that at X at time t there is an orthonormal basis $\underset{\sim}{u}_i$ relative to which

$$\left.\begin{aligned} [\underset{\sim}{C}_t(t-s)] &= \begin{bmatrix} \lambda_1^2(t-s) & 0 & 0 \\ 0 & \lambda_2^2(t-s) & 0 \\ 0 & 0 & \lambda_3^2(t-s) \end{bmatrix}, \\ \lambda_1(t-s)\lambda_2(t-s)\lambda_3(t-s) &= 1, \qquad \lambda_i(t-s) > 0, \end{aligned}\right\} \qquad (19.1)$$

for all $s \geq 0$. In this case we say that the history of X up to t is an <u>isochoric extension</u>, and we note that, for each $s \geq 0$, each vector $\underset{\sim}{u}_i$ $(i = 1,2,3)$ is a right principal direction of stretch for the configuration at time t-s with the configuration at time t taken as reference. Furthermore, the arguments which yield the theorem stated in Section 12 here tell us that, at time t, each of the basis vectors $\underset{\sim}{u}_i$ is also a principal axis of refraction $\underset{\sim}{k}_i$ and a principal axis of stress $\underset{\sim}{s}_i$; for each incompressible fluid there are two scalar-valued functionals $\underset{=}{\mathscr{f}}_N$ and $\underset{=}{\mathscr{f}}_S$ such that

$$\left.\begin{aligned} n_i(t) - n_j(t) &= \underset{=}{\mathscr{f}}_N(\lambda_i^t, \lambda_j^t) \\ \sigma_i(t) - \sigma_j(t) &= \underset{=}{\mathscr{f}}_S(\lambda_i^t, \lambda_j^t), \qquad i,j = 1,2,3, \qquad i \neq j \end{aligned}\right\} \qquad (19.2)$$

where $n_i(t)$ and $\sigma_i(t)$ are, respectively, the principal index of refraction and the principal stress, at time t, associated with the axis $\underset{\sim}{u}_i = \underset{\sim}{k}_i = \underset{\sim}{s}_i$.

The following identities hold for all positive functions α^*, β^* on $[0,\infty)$:

$$\underset{=}{\mathcal{f}}_I(\alpha^*,\beta^*) = -\underset{=}{\mathcal{f}}_I(\beta^*,\alpha^*), \qquad I = N,S. \tag{19.3}$$

Hence

$$\underset{=}{\mathcal{f}}_I(\alpha^*,\alpha^*) = 0; \tag{19.4}$$

that is,

$$\lambda_1(t-s) \equiv \lambda_2(t-s) \implies n_1(t) = n_2(t) \text{ and } \sigma_1(t) = \sigma_2(t). \tag{19.5}$$

The functionals $\underset{=}{\mathcal{f}}_I$ are determined by the functionals $\underset{=}{\mathcal{s}}_I$ and $\underset{=}{\mathcal{t}}_I$ in (18.4) through the relations

$$\underset{=}{\mathcal{f}}_I(\alpha^*,\beta^*) = \underset{=}{\mathcal{s}}_I(0^\dagger,\alpha^*,\beta^*) = \underset{=}{\mathcal{t}}_I(0^\dagger,\alpha^*,\beta^*), \qquad I = N,S. \tag{19.6}$$

20. Shearing Flows

A shearing flow or "generalized viscometric flow" is one for which there exists, at each particle X and time t, an orthonormal basis $\underset{\sim}{h}_i$, relative to which

$$[\underset{\sim}{C}_t(t-s)] = \begin{bmatrix} 1+\zeta(t-s)^2 & \zeta(t-s) & 0 \\ \zeta(t-s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20.1}$$

for all $s \geq 0$. It follows from the results given in Section 18, that,#

#See also Coleman & Noll [1961, 1] §5; Truesdell & Noll [1965, 4] §106.

in a shearing flow of an incompressible fluid, the components, relative to $\underset{\sim}{h}_i$, of $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ obey the equations,

$$\left.\begin{aligned}
N_{12}(t) &= \underset{=}{\mathscr{Y}}_N(\zeta^t), \\
N_{11}(t) - N_{22}(t) &= \underset{=}{\mathscr{W}}_N^{(1)}(\zeta^t), \\
N_{22}(t) - N_{33}(t) &= \underset{=}{\mathscr{W}}_N^{(2)}(\zeta^t), \\
N_{13}(t) = N_{23}(t) &= 0, \\
S_{12}(t) &= \underset{=}{\mathscr{Y}}_S(\zeta^t), \\
S_{11}(t) - S_{22}(t) &= \underset{=}{\mathscr{W}}_S^{(1)}(\zeta^t), \\
S_{22}(t) - S_{33}(t) &= \underset{=}{\mathscr{W}}_S^{(2)}(\zeta^t), \\
S_{13}(t) = S_{23}(t) &= 0,
\end{aligned}\right\} \tag{20.2}$$

where $\underset{=}{\mathscr{Y}}_N$ and $\underset{=}{\mathscr{Y}}_S$ are odd functions, while $\underset{=}{\mathscr{W}}_N^{(1)}$, $\underset{=}{\mathscr{W}}_N^{(2)}$, $\underset{=}{\mathscr{W}}_S^{(1)}$, and $\underset{=}{\mathscr{W}}_S^{(2)}$ are even functions; that is, for each real-valued function ζ^* on $[0,\infty)$,

$$\left.\begin{aligned}
\underset{=}{\mathscr{Y}}_I(-\zeta^*) &= -\underset{=}{\mathscr{Y}}_I(\zeta^*), \\
\underset{=}{\mathscr{W}}_I^{(i)}(-\zeta^*) &= \underset{=}{\mathscr{W}}_I^{(i)}(\zeta^*), \qquad I = N,S, \quad i = 1,2.
\end{aligned}\right\} \tag{20.3}$$

Furthermore, (17.9) yields

$$\underset{=}{\mathscr{Y}}_I(0^\dagger) = \underset{=}{\mathscr{W}}_I^{(1)}(0^\dagger) = \underset{=}{\mathscr{W}}_I^{(2)}(0^\dagger) = 0, \qquad I = N,S. \tag{20.4}$$

It is obvious that the material functionals in Section 18 determine $\mathcal{Y}_I$ and $\mathcal{W}_I^{(i)}$ through the relations

$$
\left.\begin{aligned}
\mathcal{Y}_I(\zeta^*) &= \mathcal{J}_I\left(\zeta^*, \sqrt{1^\dagger + \zeta^{*2}}, 1^\dagger\right), \\
\mathcal{W}_I^{(1)}(\zeta^*) &= \mathcal{S}_I\left(\zeta^*, \sqrt{1^\dagger + \zeta^{*2}}, 1^\dagger\right), \\
\mathcal{W}_I^{(2)}(\zeta^*) &= \mathcal{E}_I\left(\zeta^*, \sqrt{1^\dagger + \zeta^{*2}}, 1^\dagger\right),
\end{aligned}\right\} \tag{20.5}
$$

Let us index the proper vectors of $\underset{\sim}{N}$ so that the counterclockwise[#] angle χ from $\underset{\sim}{h}_1$ to $\underset{\sim}{k}_1$ satisfies the relation $0 \leq \chi < \pi/2$. Then χ is determined by the relation

$$
\cot 2\chi = \frac{N_{11} - N_{22}}{2N_{12}}, \tag{20.6}
$$

and the birefringence is given by

$$
\Delta = n_1 - n_2 = \pm\sqrt{(N_{11} - N_{22})^2 + 4N_{12}^2} = (N_{11} - N_{22})\cos 2\chi + 2N_{12}\sin 2\chi. \tag{20.7}
$$

Employing (20.2), we find that

$$
\left.\begin{aligned}
\cot 2\chi &= \frac{\mathcal{W}_N^{(1)}(\zeta^t)}{2\mathcal{Y}_N(\zeta^t)}, \\
\Delta = \pm\sqrt{\mathcal{W}_N^{(1)}(\zeta^t)^2 + 4\mathcal{Y}_N(\zeta^t)^2} &= \mathcal{W}_N^{(1)}(\zeta^t)\cos 2\chi + 2\mathcal{Y}_N(\zeta^t)\sin 2\chi.
\end{aligned}\right\} \tag{20.8}
$$

It follows from (20.3) and (20.8) that Δ and $\chi - \frac{\pi}{4}$ are odd functions of ζ^t; i.e. the transformation $\zeta^t \to -\zeta^t$ induces the transformations $\Delta(t) \to -\Delta(t)$ and $\chi(t) \to \frac{\pi}{2} - \chi(t)$.

[#] Throughout this report, when we say that an angle in the $(\underset{\sim}{k}_1, \underset{\sim}{k}_2)$-plane is measured "counterclockwise" or "clockwise", it is to be taken for granted that the plane is viewed from the side toward which $\underset{\sim}{k}_3$ points.

We see no reason to believe that the ratio $\underline{\underline{\mathscr{w}}}_N^{(1)}(\zeta^t)/\underline{\underline{\mathscr{y}}}_N(\zeta^t)$ equals the ratio $\underline{\underline{\mathscr{w}}}_S^{(1)}(\zeta^t)/\underline{\underline{\mathscr{y}}}_S(\zeta^t)$ for each history ζ^t of an arbitrary incompressible fluid. That is, in spite of occasional claims to the contrary, there does not appear to be a general, sound, symmetry argument indicating that the principal axes of stress $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ are parallel to the principal axes of refraction $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$ in all shearing flows of all incompressible fluids.

A <u>viscometric flow</u># is a shearing flow for which the function ζ in (20.1) has the simple form

$$\zeta(t-s) = -\kappa s, \qquad 0 \leq s < \infty, \tag{20.9}$$

with κ a number called the <u>rate of shearing</u>. Let us define six functions, of the rate of shearing, τ_N, $\sigma_N^{(1)}$, $\sigma_N^{(2)}$ and τ_S, $\sigma_S^{(1)}$, $\sigma_S^{(2)}$, by the relations

$$\tau_I(\kappa) = \underline{\underline{\mathscr{y}}}_I\overset{\infty}{\underset{s=0}{(-\kappa s)}}, \qquad \sigma_I^{(i)}(\kappa) = \underline{\underline{\mathscr{w}}}_I^{(i)}\overset{\infty}{\underset{s=0}{(-\kappa s)}}, \qquad I = N,S, \quad i = 1,2. \tag{20.10}$$

Here, by $\underline{\underline{\mathscr{y}}}_I\overset{\infty}{\underset{s=0}{(-\kappa s)}}$ and $\underline{\underline{\mathscr{w}}}_I^{(i)}\overset{\infty}{\underset{s=0}{(-\kappa s)}}$ we mean the values of the functionals $\underline{\underline{\mathscr{y}}}_I$ and $\underline{\underline{\mathscr{w}}}_1^{(i)}$ at the <u>function</u> ζ^t defined by $\zeta^t(s) \equiv -\kappa s$. The functions τ_I and $\sigma_I^{(i)}$, mapping the real numbers into the real numbers, are material

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#The definition of a "viscometric flow" employed here is that of Coleman [1962, 1]; Truesdell & Noll [1965, 4] call such motions "steady viscometric flows". Coleman, Markovitz, & Noll [1966, 1] give a detailed exposition of the theory of these flows and of modern experiments in viscometry.

functions, determined by the fluid under consideration.[#] It is clear from (20.8) that in a viscometric flow

$$\left.\begin{aligned} \Delta &= \Delta(\kappa) = \pm\sqrt{\sigma_N^{(1)}(\kappa)^2 + 4\tau_N(\kappa)^2}\,, \\ \chi &= \chi(\kappa) = \frac{1}{2}\cot^{-1}\frac{\sigma_N^{(1)}(\kappa)}{2\tau_N(\kappa)}\,. \end{aligned}\right\} \tag{20.11}$$

As in the case of general shearing flows, there is no argument of symmetry implying coincidence of $\underset{\sim}{s}_i$ and $\underset{\sim}{k}_i$, for i = 1 and 2, in viscometric flows of arbitrary incompressible fluids.

The relations (20.3) and (20.4) yield[##]

$$\tau_I(-\kappa) = -\tau_I(\kappa), \qquad \sigma_I^{(i)}(-\kappa) = \sigma_I^{(i)}(\kappa), \tag{20.12}$$

$$\tau_I(0) = 0, \qquad \sigma_I^{(i)}(0) = 0, \tag{20.13}$$

for I = N,S and i = 1,2. The remark made after (20.8) here implies that Δ and $\chi - \frac{\pi}{4}$ are odd functions of κ.

[#]The function $\sigma_S^{(1)}$ does <u>not</u> equal the function σ_1 introduced by Coleman & Noll [1959, 2&3] and subsequently employed in several works, e.g. [1961, 1] [1962, 2] [1965, 4] [1966, 1]. Our present τ_S, $\sigma_S^{(1)}$, and $\sigma_S^{(2)}$ are related as follows to the viscometric functions τ, σ_1, and σ_2 discussed in the cited references:

$$\tau_S(\kappa) = \tau(\kappa), \qquad \sigma_S^{(1)}(\kappa) = \sigma_2(\kappa) - \sigma_1(\kappa), \qquad \sigma_S^{(2)}(\kappa) = \sigma_2(\kappa).$$

[##]Cf. Coleman & Noll [1959, 2].

21. Couette Flow

Of the various shearing flows that can be obtained in the laboratory, Couette flow is the one most widely used for studying induced birefringence. This flow lends itself readily to the determination of the material functionals $\underset{\approx}{\mathcal{Y}}_N$ and $\underset{\approx}{\mathcal{W}}_N$ in (20.2).

a. General Theory

In a Couette flow the velocity field has the contravariant components,

$$\dot{r} = 0, \qquad \dot{\theta} = \omega(r,t), \qquad \dot{z} = 0, \tag{21.1}$$

in a cylindrical coordinate system r, θ, z, and the fluid is contained between two coaxial cylinders, located at $r = R_1$ and $r = R_2$, which rotate about their common axis $(r = 0)$ with angular velocities $\Omega_1(t)$ and $\Omega_2(t)$.

It is easy to show that any motion obeying (21.1) is a shearing flow with

$$\zeta(t-s) = r\int_t^{t-s} \frac{\partial}{\partial r}\,\omega(r,\tau)d\tau = \zeta^t(s), \qquad 0 \le s < \infty. \tag{21.2}$$

Moreover, in such a flow, for each particle X and time t,

$$\underset{\sim}{h}_1 = \underset{\sim}{e}_r, \qquad \underset{\sim}{h}_2 = \underset{\sim}{e}_\theta, \qquad \underset{\sim}{h}_3 = \underset{\sim}{e}_z, \tag{21.3}$$

where $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_z$ is the orthonormal basis of unit vectors pointing along the coordinate directions r, θ, z, at the point occupied by X at time t.

Thus, (20.2) here becomes the assertion that the physical components of $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ in the coordinate system r, θ, z (i.e. the components of $\underset{\sim}{N}(t)$ and $\underset{\sim}{S}(t)$ with respect to $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_z$) obey the relations

$$\left.\begin{aligned} N_{r\theta}(t) &= \underset{\equiv}{\mathscr{Y}}_N(\zeta^t), \\ N_{rr}(t) - N_{\theta\theta}(t) &= \underset{\equiv}{\mathscr{W}}_N^{(1)}(\zeta^t), \\ N_{\theta\theta}(t) - N_{zz}(t) &= \underset{\equiv}{\mathscr{W}}_N^{(2)}(\zeta^t), \\ N_{rz}(t) = N_{\theta z}(t) &= 0, \end{aligned}\right\} \tag{21.4}$$

$$\left.\begin{aligned} S_{r\theta}(t) &= \underset{\equiv}{\mathscr{Y}}_S(\zeta^t), \\ S_{rr}(t) - S_{\theta\theta}(t) &= \underset{\equiv}{\mathscr{W}}_S^{(1)}(\zeta^t), \\ S_{\theta\theta}(t) - S_{zz}(t) &= \underset{\equiv}{\mathscr{W}}_S^{(2)}(\zeta^t), \\ S_{rz}(t) = S_{\theta z}(t) &= 0. \end{aligned}\right\} \tag{21.5}$$

It follows from (21.5) that in a Couette flow of an incompressible fluid the dynamical equations reduce to the scalar functional-differential equation

$$\rho r^3 \frac{\partial}{\partial t}\,\omega(r,t) = \frac{\partial}{\partial r}\,[r^2 \underset{\equiv}{\mathscr{Y}}_S(\zeta^t)] \tag{21.6}$$

with ζ^t given by (21.2). One may presume that solution of (21.6) with appropriate initial data and the adherence conditions,

$$\omega(R_1,t) = \Omega_1(t), \qquad \omega(R_1,t) = \Omega_2(t), \tag{21.7}$$

uniquely determines the angular velocity ω as a function of r and t, but a general theory to this effect, valid for non-linear functionals $\underset{\approx}{\mathcal{Y}}_S$ and non-steady motions, is lacking.

If the relative gap $(R_2 - R_1)/R_1$ is very small and $\Omega_1(t)$, $\Omega_2(t)$, $\dot{\Omega}_1(t)$, and $\dot{\Omega}_2(t)$ are small for all t, one expects that ω should be such that the rate of shear, $r\frac{\partial\omega}{\partial r}$, is independent of r and given, to a high degree of accuracy, by the first term on the right in the formula

$$r\frac{\partial}{\partial r}\omega(r,t) = \frac{R_1(\Omega_2(t) - \Omega_1(t))}{R_2 - R_1} + O\left(\frac{R_2 - R_1}{R_1}\right), \qquad (21.8)$$

which yields,

$$\zeta^t(s) = \frac{R_1}{R_2 - R_1}\int_t^{t-s}\Omega(\tau)d\tau + O\left(\frac{R_2 - R_1}{R_1}\right), \qquad (21.9)$$

with

$$\Omega = \Omega_2 - \Omega_1. \qquad (21.10)$$

Thus, when the term $O\left((R_2 - R_1)/R_1\right)$ in (21.8) can be neglected, i.e. when the "small gap approximation" is valid, the history ζ^t is independent of r at each time t, and so also are all the terms in (21.4) and (21.5), as well as the principal axes of refraction $\underset{\sim}{k}_i$ and the principal axes of stress $\underset{\sim}{s}_i$.

Optical measurements on Couette flow are usually performed with a plane polariscope mounted with its axis parallel to the axis of the bounding cylinders. Thus the light travels perpendicular to the planes z = const.; these planes contain the circular streamlines, two of the axes of stress $(\underset{\sim}{s}_1, \underset{\sim}{s}_2)$, and two of the axes of refraction $(\underset{\sim}{k}_1, \underset{\sim}{k}_2)$. The observed isoclinic lines are the locus of points at which $\underset{\sim}{k}_1$ or $\underset{\sim}{k}_2$

is parallel to a polarizing axis of the analyzer. When (21.8) holds, $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ are independent of r, and as one sees in Figure 21.1, each isocline is a straight radial line. Indeed, in this case, at each instant t the isoclines form a cross (called the "cross of isocline") whose two branches obey equations of the form

$$\theta = -\chi(t), \quad \theta = \frac{\pi}{2}-\chi(t), \quad \theta = \pi-\chi(t), \quad \theta = \frac{3\pi}{2}-\chi(t), \quad 0 \leq \chi(t) < \frac{\pi}{2}, \tag{21.11}$$

with θ measured counterclockwise from the direction of the polarizing axis of the polarizer. χ is called the <u>extinction angle</u> and may be defined to be the clockwise angle from the axis of the polarizer to that arm of the cross which is nearest to it. In view of Figure 21.1, however, χ is also the counterclockwise angle from $\underset{\sim}{e}_r$ to $\underset{\sim}{k}_1$ and, by (21.3), χ here has the same meaning as in the previous section. Thus, the equations (20.6)-(20.8) yield

$$\chi(t) = \frac{1}{2}\cot^{-1}\frac{N_{rr}(t) - N_{\theta\theta}(t)}{2N_{r\theta}(t)} = \frac{1}{2}\cot^{-1}\frac{\underset{\approx}{w}_N^{(1)}(\zeta^t)}{2\underset{\approx}{y}_N(\zeta^t)}. \tag{21.12}$$

If the polariscope is employed to measure the relative retardation of polarized light passing through the fluid in the z-direction, then the birefringence obtained is given by the formula

$$\Delta = n_1 - n_2 = \pm\sqrt{(N_{rr} - N_{\theta\theta})^2 + 4N_{r\theta}^2} \tag{21.13}$$

and, according to (20.6)-(20.8), or (21.4),

$$\Delta(t) = \pm\sqrt{\underset{\approx}{w}_N^{(1)}(\zeta^t)^2 + 4\underset{\approx}{y}_N(\zeta^t)^2} = \underset{\approx}{w}_N^{(1)}(\zeta^t)\cos 2\chi + 2\underset{\approx}{y}_N(\zeta^t)\sin 2\chi. \tag{21.14}$$

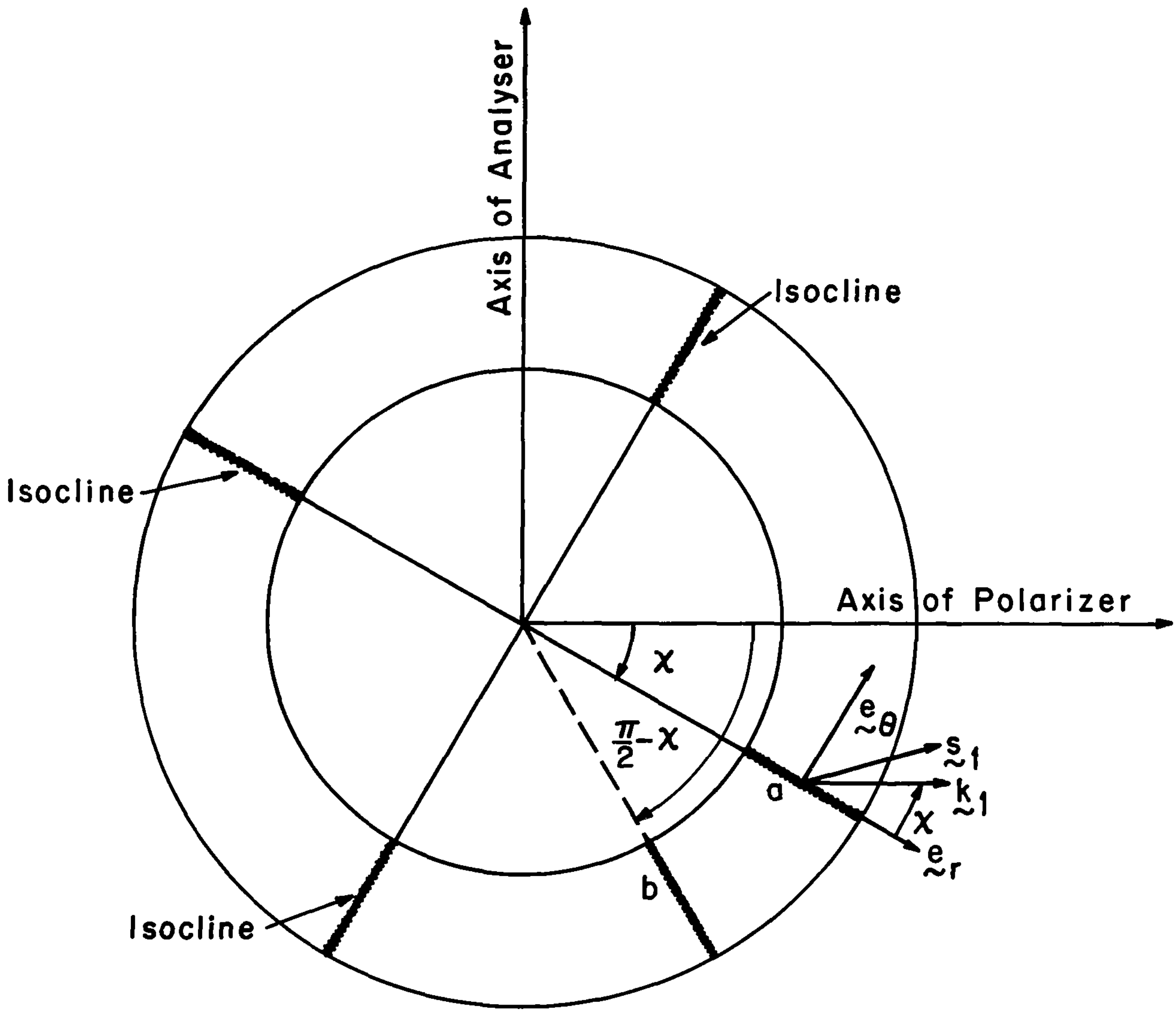

Fig. 21.1 Geometry of the Cross of Isocline

In practice the Couette apparatus must be of finite length, and the flow (21.1) cannot be maintained near the ends. Thus there are difficulties in determining the length ℓ to be used in (1.7) for calculating Δ, but since experimenters have found ways of overcoming analogous problems in viscometry,[#] these difficulties do not appear insuperable.

Since

$$2N_{r\theta} = \Delta \sin 2\chi, \qquad N_{rr} - N_{\theta\theta} = 2N_{r\theta} \cot 2\chi,$$

it is evident from (21.12) and (21.14) that when the dimensions of the apparatus and the speeds of rotation of the bounding cylinders are such that the "small gap approximation" of (21.8) and (21.9) is valid, measurement of both the extinction angle $\chi(t)$ and the birefringence[##] $\Delta(t)$ for a history Ω^t of Ω, i.e.

$$\Omega^t(s) = \Omega(t-s), \qquad 0 \leq s < \infty, \tag{21.15}$$

permits calculation of $\underset{=}{\mathscr{Y}}_N(\zeta^t)$ and $\underset{=}{\mathscr{W}}_N(\zeta^t)$ from the relations

$$\left.\begin{aligned} \underset{=}{\mathscr{Y}}_N(\zeta^t) &= \frac{1}{2}\Delta(t)\sin 2\chi(t) \\ \underset{=}{\mathscr{W}}_N^{(1)}(\zeta^t) &= 2\underset{=}{\mathscr{Y}}_N(\zeta^t)\cot 2\chi(t); \end{aligned}\right\} \tag{21.16}$$

ζ^t is here given by (21.9). Thus, one can, in principle, determine the functionals $\underset{=}{\mathscr{Y}}_N$ and $\underset{=}{\mathscr{W}}_N^{(1)}$ experimentally.

[#]See, for example, the survey [1966, 1].

[##]Of course, intensity measurements and eq. (1.8) yield only $|\Delta|$.

b. Oscillatory Couette Flow

If ω in (21.1) is such that there exists a number $\tau > 0$ for which

$$\omega(r,t+\tau) = \omega(r,t), \qquad \omega\left(r,t+\frac{\tau}{2}\right) = -\omega(r,t), \tag{21.17}$$

for all r in $[R_1,R_2]$ and all $t > -\infty$, and if τ is the smallest positive number for which this is the case, then we say that the Couette flow <u>oscillates symmetrically with period</u> τ.

It is clear from (21.2) that (21.17) yields

$$\zeta^t = \zeta^{t+\tau}, \qquad \zeta^{t+\tau/2} = -\zeta^t. \tag{21.18}$$

But, in view of (21.4), (21.5), and the identities (20.3), the equations (21.18) yield

$$\left.\begin{aligned} N_{r\theta}(t+\tau) &= N_{r\theta}(t), \qquad N_{r\theta}(t+\tau/2) = -N_{r\theta}(t), \\ N_{rr}(t+\tau/2) - N_{\theta\theta}(t+\tau/2) &= N_{\theta\theta}(t) - N_{rr}(t), \\ N_{\theta\theta}(t+\tau/2) - N_{zz}(t+\tau/2) &= N_{\theta\theta}(t) - N_{zz}(t), \end{aligned}\right\} \tag{21.19}$$

and similar relations for the components of $\underset{\sim}{S}$.[#] It follows from the remark made after (20.8), that the equations (21.18) imply

$$\left.\begin{aligned} \chi(t+\tau/2) &= \frac{\pi}{2} - \chi(t), \qquad \Delta(t+\tau/2) = -\Delta(t), \\ \chi(t+\tau) &= \chi(t), \qquad \Delta(t+\tau) = \Delta(t). \end{aligned}\right\} \tag{21.20}$$

[#] Vid. Coleman & Noll [1961, 1] pp. 698, 699.

Thus, Δ and χ are here periodic functions of time with period τ; furthermore, in Figure 21.1, an isocline which lies on the line a at time t will be on the line b at time $t+\tau/2$.

c. Steady Couette Flow

If (21.1) takes the form

$$\dot{r} = 0, \qquad \dot{\theta} = \omega(r), \qquad \dot{z} = 0 \tag{21.21}$$

with ω independent of t for all t, then the motion is called a <u>steady Couette flow</u>. Of course, in such a flow, Ω_1, Ω_2, and hence $\Omega = \Omega_1 - \Omega_2$, must be held constant for all t.

Here the equation (21.2) reduces to

$$\zeta^t(s) = -\kappa s, \qquad 0 \leq s < \infty, \tag{21.22}$$

where

$$\kappa = \kappa(r) = r\,\frac{d\omega(r)}{dr}\,. \tag{21.23}$$

Since the equation (21.22) is the same as the equation (20.9), each steady Couette flow is a viscometric flow with a rate of shear κ that is a function of r. It follows from (21.8) that

$$\kappa = \frac{R_1}{R_2 - R_1}\,\Omega + O\left(\frac{R_2 - R_1}{R_1}\right). \tag{21.24}$$

Thus, in the "small gap approximation", κ is independent of r.

The equations (21.4) and (21.5) here become

$$\left.\begin{aligned} N_{r\theta} &= \tau_N(\kappa), \\ N_{rr} - N_{\theta\theta} &= \sigma_N^{(1)}(\kappa), \\ N_{\theta\theta} - N_{zz} &= \sigma_N^{(2)}(\kappa), \\ N_{rr} = N_{\theta z} &= 0, \end{aligned}\right\} \tag{21.25}$$

$$\left.\begin{aligned} S_{r\theta} &= \tau_S(\kappa), \\ S_{rr} - S_{\theta\theta} &= \sigma_S^{(1)}(\kappa), \\ S_{\theta\theta} - S_{zz} &= \sigma_S^{(2)}(\kappa), \\ S_{r\theta} = S_{\theta z} &= 0; \end{aligned}\right\} \tag{21.26}$$ #

with all terms independent of t. The equations (21.12) and (21.14) take the forms shown in (20.11), while (21.16) reduces to

$$\left.\begin{aligned} \tau_N(\kappa) &= \frac{1}{2}\,\Delta(\kappa)\sin 2\chi(\kappa), \\ \sigma_N^{(1)}(\kappa) &= 2\tau_N(\kappa)\cot 2\chi(\kappa). \end{aligned}\right\} \tag{21.27}$$

Hence, if one measures both Δ and χ as functions of κ, one can calculate the functions τ_N and $\sigma_N^{(1)}$. We see no reason to expect $\sigma_N^{(1)}$ to be proportional to $\sigma_S^{(1)}$.##

#The equations (21.26) were derived, at this level of generality, by Coleman & Noll [1959, 2].

##$\sigma_S^{(1)}$ can be measured directly with apparatus of the type described by Padden & DeWitt [1954, 1]. Coleman, Markovitz, & Noll [1966, 1] describe various methods experimenters have employed to obtain τ_S, $\sigma_S^{(1)}$, and $\sigma_S^{(2)}$.

It follows from the last sentence of Section 20 that

$$\Delta(-\kappa) = -\Delta(\kappa), \qquad \chi(-\kappa) = \frac{\pi}{2} - \chi(\kappa). \tag{21.28}$$

Thus, the transformation $\kappa \rightarrow -\kappa$ changes the sign of the birefringence and results in a change in the position of an isoclinic line from a to b in Figure 21.1. When $\chi(\kappa) = \pi/4$ (i.e. if $\sigma_N^{(1)}(\kappa)/\tau_N(\kappa) = 0)^{\#}$, the cross of isocline remains invariant under such a transformation.

$^{\#}\chi(\kappa)$ does equal $\pi/4$ in the "limit of slow flow" mentioned in Section 2b.

VI. REFERENCES

1816 [1] Brewster, D.: Phil. Trans. Roy. Soc. (London) 106, 156-178.

1827 [1] Fresnel, A.: Mém. de l'Acad. Sci. (Paris) 7, 45-162; see also Ouvres, Vol. 2, 479-596.

1841 [1] Neumann, F. E.: Abh. Akad. Wissenschaften Berlin, Part II, 1-254.

1853 [1] Maxwell, J. C.: Trans. Roy. Soc. Edinburgh 20, 87-120 (1853/1854) = Sci. Papers, Vol. 1, 30-79.

1865 [1] Maxwell, J. C.: Phil. Trans. Roy. Soc. (London) 155, 459-512 = Sci. Papers, Vol. 1, 526-597.

1872 [1] Mach, E.: Ann. D. Physik u. Chemie, Ser. II, 146, 313-316.

1873 [1] Maxwell, J. C.: Proc. Roy. Soc. (London) 22, 46-47 = Sci. Papers, Vol. 2, 379-380.

[2] Maxwell, J. C.: A Treatise on Electricity and Magnetism. 2 vols., Oxford: University Press.

1881 [1] Kund, A.: Ann. d. Physik u. Chemie, Ser. III, 13, 110-133.

1888 [1] Metz, G. de: Ann. d. Physik u. Chemie, Ser. III, 35, 497-507.

1891 [1] Bjerkén, P. von: Ann. d. Physik u. Chemie, Ser. III, 43, 808-816.

1906 [1] Metz, G. de: La Double Refraction Accidentelle dans les Liquides, Paris: Gauthier-Villars.

1910 [1] Rossi, P.: Rend. Accad. delle Scienze Fisiche e Matematiche (Soc. Reale di Napoli), Ser. III, 16, 206-215.

1931 [1] Coker, E. G., & L. N. G. Filon: A Treatise on Photo-Elasticity. (Revised by H. T. Jessop, 1957) Cambridge: University Press.

1951 [1] Whittaker, E.: A History of the Theories of Aether and Electricity, Vol. 1, London: T. Nelson & Sons.

1954 [1] Padden, F. J., & T. W. DeWitt: J. Appl. Phys. 25, 1086-1091.

1957 [1] Green, A. E., & R. S. Rivlin: Arch. Rational Mech. Anal. 1, (1957/8), 1-21.

1958 [1] Noll, W.: Arch. Rational Mech. Anal. 2 (1958/9), 197-226.

1959 [1] Born, M., & E. Wolfe: Principles of Optics (Second Edition, 1964) New York: MacMillan.

[2] Coleman, B. D., & W. Noll: Arch. Rational Mech. Anal. 3, 289-303.

[3] Coleman, B. D., & W. Noll: J. Appl. Phys. 30, 1508-1512.

1960 [1] Coleman, B. D., & W. Noll: Arch. Rational Mech. Anal. 6, 355-370.

[2] Truesdell, C., & R. A. Toupin: The Classical Field Theories, in the Encyclopedia of Physics, Vol. III/1, ed. by S. Flügge, Berlin, Göttingen, Heidelberg: Springer.

1961 [1] Coleman, B. D., & W. Noll: Ann. N. Y. Acad. Sci. 89, 672-714.

[2] Coleman, B. D., & W. Noll: Rev. Mod. Phys. 33, 239-249; errata, ibid. 36, 1103.

1962 [1] Coleman, B. D.: Arch. Rational Mech. Anal. 9, 273-300.

[2] Coleman, B. D., & R. A. Toupin: Trans. Soc. Rheology 6, 313 (abstract only).

1965 [1] Brauer, R.: Arch. Rational Mech. Anal. 18, 97-99.

[2] Coleman, B. D., & C. Truesdell: Z. angew. Math. Mech. 45, 547-551.

1965 [3] Noll, W.: Arch. Rational Mech. Anal. 18, 100-102.

[4] Truesdell, C., & W. Noll: The Non-Linear Field Theories of Mechanics, in the Encyclopedia of Physics, Vol. III/3, ed. by S. Flügge, Berlin, Heidelberg, New York: Springer.

1966 [1] Coleman, B. D., H. Markovitz, & W. Noll: Viscometric Flows of Non-Newtonian Fluids, Berlin, Heidelberg, New York: Springer.

[2] Dill, E. H., & C. W. Fowlkes: Photoviscoelasticity, NASA Contractor Report, CR-444.

1968 [1] Coleman, B. D.: Proc. Roy. Soc. (London) A 306, 449-476.

1969 [1] Fowlkes, C. W.: Photoviscoelastic Model Testing, NASA Contractor Report, CR-1289.

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