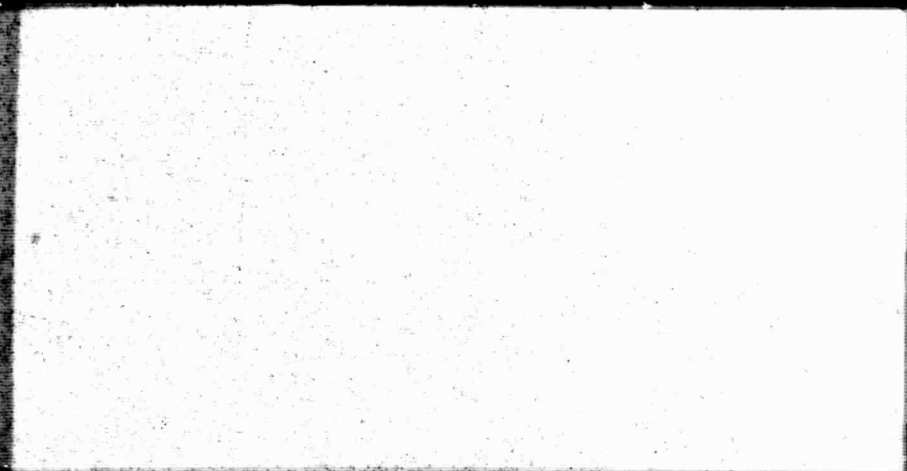


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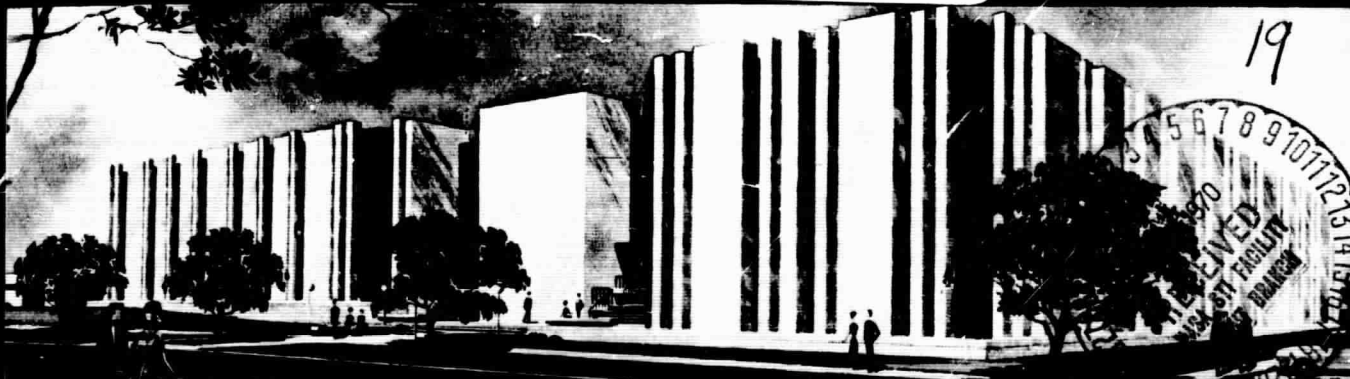
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STATISTICAL SIMULATION PROCEDURES *DRF*

by

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1. INTRODUCTION AND REVIEW OF THE LITERATURE

Often it is not practical to use a theoretical approach to solve statistical distribution problems. As an alternative, Monte Carlo sampling is one technique that is very useful in estimating distribution functions and their parameters.

With the rising importance of statistics, and the advent of the computer, Monte Carlo methods are just now being investigated. Although definitions of Monte Carlo sampling may differ, it is usually associated with random sampling used to solve deterministic or probabilistic problems. In this research, Monte Carlo methods are used to approximate the cumulative distribution function

$$F(h) = P[H(X_1, X_2, \dots, X_m) \leq h] , \quad (1.1)$$

where $H(X_1, X_2, \dots, X_m)$ is a function of m random variables with density function $f_1(x)$ and cumulative distribution function

$$F_1(x) = \int_{-\infty}^x f_1(t) dt . \quad (1.2)$$

If one observation is drawn from each variate in a

given sample, then (1.1) may be approximated by generating a number of random samples of $H(x_1, x_2, \dots, x_m)$, determining their empirical frequency distribution, and then using this distribution to determine $\hat{F}(h)$.

While there are many methods available to reduce the variance of the above estimation procedure, stratified sampling is considered by Cochran [2] and various other authors as one of the best methods. This technique is used in this research to arrive at a more precise Monte Carlo estimate of the desired parameter; that is, more precise than an estimate found by simple Monte Carlo simulation. Section 2 describes the properties of stratified sampling.

The discussion above has introduced the basis for this research effort. Stratified Monte Carlo sampling is used to estimate given functions of a multivariate statistic; Monte Carlo sampling because a theoretical solution is not practical, and stratified sampling because it is an effective variance-reduction technique compared to estimation by simple random sampling.

A stratified Monte Carlo procedure for estimation of distribution functions of univariate statistics is proposed by Ringer and Suharto [11]. The principle is to partition the range of each marginal distribution

into a desired number of subranges, and then sample randomly from the possible combinations of these subranges. With these sample values, the joint distribution function of the variates is estimated. Likewise, various functions of this joint distribution may be estimated. Also presented are various properties of the estimation procedure, and its relative efficiency compared to simple random sampling. This procedure is extended to the multivariate case in Section 2, and becomes the technique used in estimation of the two problems of Section 4 and Section 5.

Section 3 contains a second estimation procedure, by Somerville [12], extended in this research to the multivariate case. Somerville approaches the problem of distribution estimation in a similar but unique view to that of Ringer and Suharto. He proposes a numerical procedure for the specific problem of estimating cumulative distribution functions, and goes to some length to show his procedure "superior in general for small calculations" to Monte Carlo techniques. Various efficiency equations are presented. However, the actual application of Somerville's procedures shows them to be similar in many respects to those of Ringer and Suharto [11]. The parallel methods of both can therefore be weighed

and compared. It should be noted that there are, of course, different aspects germane to each paper. A thorough discussion is included in Section 3.

Hammersley and Hanscomb [6] cover a wide range of Monte Carlo principles and procedures, presenting efficiencies of the procedures compared to simple random sampling. Of particular interest is the section on stratified sampling as a variance-reducing technique, the conclusion being that stratified sampling is about ten times as efficient as "crude Monte Carlo" sampling (simple random sampling). Parameter estimation is thoroughly discussed, but estimation of distribution functions is overlooked.

Widely referred to is the book edited by Meyer [9], which includes a paper on variance-reduction techniques by Kahn [7]. Kahn proposes six techniques, including stratified sampling. He states that while stratified sampling has a rather limited scope of applications, it is still one of the most important techniques. Unfortunately, Kahn's discussion of stratified Monte Carlo sampling is of a general and limited nature.

Other works that add insight into multivariate distribution estimation include those by Haber [3,4] on Monte Carlo quadrature (discussed in Section 3), and by

Hammersley [5] on methods for solving multivariate problems. Spanier and Gelbard [13] present an encompassing view of Monte Carlo methods, and also approach the question of optimum combinations of variables.

Two illustrative problems are investigated using the extension of the Ringer and Suharto method as the estimation procedure. The problem approached in Section 4 involves estimation of parameters in a television amplifier circuit, defined in a paper by Bosinoff and Jacobs [1]. Simulating values of the input variables, estimates of these parameters are computed in transfer equations and presented in Section 4. Comparison of the precision of stratified to simple random sampling is also made.

The problem of Section 5 involves estimation of the distribution function of

$$\int_0^1 |Y(t)| dt, \quad (1.3)$$

where the stochastic variate $Y(t)$ is distributed $N(0, t-t^2)$. Speed and Smith [14] derive the limiting statistic defined by (1.3) from a given Riedwyl statistic.

2. THE STRATIFIED MONTE CARLO PROCEDURE

The methodology of Ringer and Suharto [11] is extended to the multivariate case in this section. The procedures formulated are then used in approximating the cumulative distribution functions associated with two problems; one practical, the other theoretical in origin. This extension of stratified Monte Carlo allows the variate ranges to be partitioned into any desired number of subranges, and samples are drawn from all possible combinations of these subranges. The resulting pseudo-random observations can then be arranged in a frequency array that forms an estimated cumulative distribution function.

2.1 Estimation Procedure

Consider a multivariate problem in which the distribution of Y must be determined, where Y , a function of the random variables $X_1, X_2, \dots, X_m$, is defined by

$$Y = H(X_1, X_2, \dots, X_m) . \quad (2.1)$$

Much theory in mathematical statistics is intended to do this, however, in practice the function of H is often so complicated that theoretical procedures are impractical. Thus, with the advent of the electronic

computer, numerical or simulation procedures for approximating this distribution function have been developed.

For the extension of the Stratified Monte Carlo (S.M.C.) procedure, assume that the joint distribution, as well as the marginal distributions of the i th variables $X_1, X_2, \dots, X_m$, are known, and

$$G_Y(h) = P[Y = H(X_1, X_2, \dots, X_m) \leq h] \quad (2.2)$$

must be approximated.

Now, partition the X_i -variate range into k_i subranges. The stratified sampling procedure is one in which a subrange of each variate X_i is chosen and a pseudo-random number is selected with proper distribution from this subrange. The resulting vectors will represent pseudo-random samples from the joint distribution of $X_1, X_2, \dots, X_m$.

Denote the probability of sampling from subrange j of the X_i variate by P_{j_i} . Then the probability of a given stratum is given as

$$P(j_1, j_2, \dots, j_m) = \prod_{i=1}^m P_{j_i} \quad (2.3)$$

For convenience, this probability will be denoted $p(j_i)$. The definition of "stratum" is now given as a particular combination of subranges of elements of the sampled ob-

ervation vector.

The following example will help to clarify the above discussion. Given that there are two variates, with $k_1 = 3$, $k_2 = 2$, $p_{1_1} = \frac{1}{6}$, $p_{2_1} = \frac{1}{2}$, $p_{3_1} = \frac{1}{3}$, $p_{1_2} = \frac{3}{4}$ and $p_{2_2} = \frac{1}{4}$, the possible subrange combinations (i.e., strata) and associated probabilities are:

Table 2.1
Stratification of Two Variates

	Variate		
	1	2	
Stratum	Subrange		$p(j_1)$
1	I	I	$\frac{3}{24}$
2	II	I	$\frac{9}{24}$
3	III	I	$\frac{6}{24}$
4	I	II	$\frac{1}{24}$
5	II	II	$\frac{3}{24}$
6	III	II	$\frac{2}{24}$

As the example points out, the total number of strata is simply the product of the number of subranges in each variate, and is given by

$$M = \prod_{i=1}^m k_i . \quad (2.4)$$

M can be defined as the reciprocal of $p(j_i)$.

The total number of Monte Carlo samples in the entire experiment with a given allocation s , denoted N_s , is equal to

$$N_s = p(j_i)N, \quad (2.5)$$

where N is the total number of Monte Carlo samples.

Clearly,

$$\sum_{s=1}^M N_s = N \sum_{s=1}^M p(j_i) = N \quad (2.6)$$

Define

$$F_s(h) = P[H(X_1, X_2, \dots, X_m) \leq h | X_1, X_2, \dots, X_m \in s] \quad (2.7)$$

as the conditional distribution of $H(X_1, X_2, \dots, X_m)$ for a given stratum s . If we denote the observations chosen for a given stratum as $(x_{j_1}, x_{j_2}, \dots, x_{j_m})$, then $\hat{F}_s(h)$ is found by using the proportion of the N_s Monte Carlo samples such that $H(x_{j_1}, x_{j_2}, \dots, x_{j_m}) \leq h$. For simulation purposes, the formula for $\hat{F}_s(h)$ may simply be stated as

$$\hat{F}_s(h) = \frac{\text{Number of samples where } H \leq h}{N_s}$$

for a given stratum s .

For $\hat{F}_S(h)$ to be considered a true proportion, it must satisfy the assumptions of the binomial distribution function. These assumptions are: There are two possible outcomes in each of a fixed number of independent trials, with the probability of these outcomes remaining constant from trial to trial.

It can be seen that the above assumptions are met. There can be only two possible outcomes, either $H(x_{j_1}, x_{j_2}, \dots, x_{j_m})$ is less than or equal to h , or it is greater than h . The probability that a sample is less than or equal to h does remain constant, as the only change in a given trial is the random observations chosen. The number of N_S trials is clearly constant, and the trials are independent due to the random selection process. Thus, $\hat{F}_S(h)$ can be considered as a binomial type estimator of the parameter $F_S(h)$. It follows that the properties of the binomial estimator p also apply to $\hat{F}_S(h)$, and

$$E[\hat{F}_S(h)] = F_S(h) \quad . \quad (2.9)$$

$\hat{F}_S(h)$ is then an unbiased estimator.

Let the cumulative distribution of $Y = H(X_1, X_2, \dots, X_m)$ be defined as

$$F(h) = P[Y \leq h] \quad . \quad (2.10)$$

Using conditional probability laws,

$$\begin{aligned}
 F(h) &= \sum_{s=1}^M F_s(h) \cdot \text{Prob}[s] \\
 &= \sum_{s=1}^M F_s(h) \cdot p(j_i) \quad . \quad (2.11)
 \end{aligned}$$

The simplest and one of the best forms of this procedure is to partition the variates into equiprobable subranges under proportional allocation. With p_{j_i} now a constant for each variate, $p(j_i)$ also becomes a constant, call it p . The formulas being derived here are therefore simplified. Specifically, (2.11) becomes

$$F(h) = p \sum_{s=1}^M F_s(h) \quad . \quad (2.12)$$

It has been pointed out that $\hat{F}_s(h)$ is an unbiased estimator of $F_s(h)$, and is actually used for all calculations. Likewise, $\hat{F}(h)$ is an unbiased estimator of $F(h)$. This can be shown as follows:

$$\begin{aligned}
 E[\hat{F}(h)] &= E\left[\sum_{s=1}^M \hat{F}_s(h) \cdot p(j_i)\right] \\
 &= \sum_{s=1}^M p(j_i) \cdot E[\hat{F}_s(h)] \\
 &= \sum_{s=1}^M p(j_i) F_s(h) \\
 &= F(h) \quad .
 \end{aligned}$$

When simulating with equiprobable subranges, $F(h)$ can easily be estimated by the following procedure: Sum the proportion $\hat{F}_s(h)$ for all strata, and divide by the number of strata. This will give an estimate of the total number of observations less than the given point h in the experiment.

The variance of $\hat{F}(h)$ can be found in the general case as follows:

$$\begin{aligned}
 v[\hat{F}(h)] &= v\left[\sum_{s=1}^M \hat{F}_s(h) \cdot p(j_i)\right] \\
 &= \sum_{s=1}^M p^2(j_i) \cdot v[\hat{F}_s(h)] \\
 &= \sum_{s=1}^M p^2(j_i) \cdot \frac{A_s}{N_s} \\
 &= \sum_{s=1}^M p^2(j_i) \cdot \frac{A_s}{p(j_i) \cdot N} \\
 &= \frac{1}{N} \sum_{s=1}^M p(j_i) F_s(h) [1 - F_s(h)]. \quad (2.13)
 \end{aligned}$$

where $A_s = F_s(h)[1 - F_s(h)]$.

The unbiased estimate of the variance follows from (2.13), and is given by

$$v[\hat{F}(h)] = \sum_{s=1}^M p(j_i) \frac{\hat{F}_s(h)[1 - \hat{F}_s(h)]}{N_s - 1} \cdot (2.14)$$

We wish to compare the variance of $\hat{F}(h)$ with the variance of an estimate found by simple random sampling. Under random sampling, denote the sample size by N , and the proportion of samples for which $H(x_1, x_2, \dots, x_n) \leq h$ by $\tilde{F}(h)$. Then, due to the fact that $\tilde{F}(h)$ has the properties of a sample proportion,

$$V_r[\tilde{F}(h)] = \frac{F(h)[1-F(h)]}{N} . \quad (2.15)$$

Thus, the reduction in variance due to stratified sampling is given as follows:

$$\begin{aligned} V_r - V &= \frac{1}{N}F(h) - \frac{1}{N}F^2(h) - \left[\frac{1}{N} \sum_{s=1}^M p(j_i) F_s(h)(1-F_s(h)) \right] \\ &= \frac{1}{N} [F(h) - F^2(h) - F(h) + \sum_{s=1}^M p(j_i) \cdot F_s^2(h)] \\ &= \frac{1}{N} \left[\sum_{s=1}^M p(j_i) F_s^2(h) - F^2(h) \right] \\ &= \frac{1}{N} \left[\sum_{s=1}^M p(j_i) F_s(h) - F(h) \right]^2 \\ &= \frac{1}{N} \sum_{s=1}^M p(j_i) [F_s(h) - F(h)]^2 . \end{aligned} \quad (2.16)$$

It is clear that (2.16) can never be negative. Therefore, stratified sampling will always give a more precise estimate than random sampling.

Note that (2.13) can be simplified for equiprobable subranges, and is given by

$$V_p[\hat{F}(h)] = \frac{P}{N} \sum_{s=1}^M F_s(h)[1-F_s(h)] . \quad (2.17)$$

2.2 Optimum Allocation

Optimum allocation takes into account the cost factor in sampling. Using this allocation, we can either

- a) Fix a cost for the experiment and minimize the variance for this cost

or

- b) Fix the desired variance and minimize the cost for this variance.

The procedure here will be to minimize the variance of the estimator, denoted $V[\hat{F}(h)]$ for a given cost C .

The total cost (Cochran [2]) is defined as

$$C = C_0 + cn = C_0 + C_1 , \quad (2.18)$$

where C_0 is an overhead charge for the entire experiment, c is the cost of drawing one random observation, and n is the total number of observations drawn. Note that c remains constant over all strata and all variates.

To derive the optimum sample number for each strata,

Neyman allocation is used (Cochran [2]). The number of samples with the allocation s is given by

$$N_{s,opt} = \frac{p(j_i) \cdot \sqrt{A_s}}{\sum_{s=1}^M p(j_i) \sqrt{A_s}} N, \quad (2.19)$$

and for equiprobable subranges becomes

$$N_{s,opt} = \frac{\sqrt{A_s}}{\sum_{s=1}^M \sqrt{A_s}}, \quad (2.20)$$

where $A_s = F_s(h)[1-F_s(h)]$

Now, with the optimum sample size per strata determined, the variance of $\hat{F}(h)$ can be derived from the more general optimum variance formula (Cochran [2, p. 108]).

Thus,

$$V_o[\hat{F}(h)] = \left(\sum_{s=1}^M p(j_i) \sqrt{A_s} \right)^2 / N, \quad (2.21)$$

and for equiprobable subranges is

$$V_o[\hat{F}(h)] = p^2 \left(\sum_{s=1}^M \sqrt{A_s} \right)^2 / N. \quad (2.22)$$

It is of interest to compare the variance given by (2.13) to the variance of optimum allocation, given by (2.21). The relative precision of (2.13) to (2.21) is

$$\frac{v_o}{V} = \frac{\left(\sum_{s=1}^M p(j_i) \sqrt{A_s} \right)^2}{\sum_{s=1}^M p(j_i) A_s} \quad (2.23)$$

With equiprobable subranges, this becomes

$$\frac{v_o}{V} = \frac{p \left(\sum_{s=1}^M \sqrt{A_s} \right)^2}{\sum_{s=1}^M A_s} \quad (2.24)$$

It should be pointed out that in order for optimum allocation to be effective, $\hat{F}(h)$ must be close to zero or one. Therefore, in most cases, the simplicity and convenience of the procedures derived in 2.1 make that method the preferred one.

3. COMPARISONS WITH OTHER QUADRATURES

A second estimation procedure is extended to the multivariate case in this section. Somerville [12] has proposed a numerical procedure for deriving the distribution of a statistic, a procedure that is similar in many respects to the Ringer-Suharto method. The variate ranges are again partitioned into subranges, a value for each subrange is derived, all possible combinations of these values are found, and an estimated cumulative distribution function is formed. While the Ringer-Suharto method is the one actually used to solve the problems of Sections 4 and 5, the Somerville method is of great value as an alternate tool with which to estimate distribution functions. These two methods are compared in the latter part of this section. Also, Somerville's method is compared to a method proposed by Haber [3].

3.1 Somerville's Estimation Procedure

Assume that the marginal distributions of the X_i input variates are known. Let there be n variates, each with a cumulative distribution function $F_i(x)$ defined on the interval $x_i = a_i$ to $x_i = b_i$. Let $x_1, x_2, \dots, x_n$ be a random sample drawn from the variates. Of course, some

of these variates may have the same distribution. Thus, while there are n variates, the number of variates with different distributions is usually less than n . Note that the subscript on the observation defines the variate. For example: Given that there are two different distributions and a total sample size of six, then, if it is desired to sample equally from the two distributions, x_1, x_2 and x_3 are from the first distribution, and x_4, x_5 and x_6 are from the second distribution. Therefore, while there are six variates, there are actually only two unique variates.

Let $g(x_1, x_2, \dots, x_n)$ be a given function of the sample with cumulative distribution function $H(g)$. Then

$$P[g \leq h] = \int_{R: g \leq h} \dots \int dF_1(x_1) \cdot dF_2(x_2) \dots dF_n(x_n) . \quad (3.1)$$

Partition each of the variate ranges into r_i subranges such that $a_i = a_{0_i} < a_{1_i} < a_{2_i} < \dots < a_{r_{1_i}} = b_i$. Then compute the "weight", or probability, to be assigned to each of the subranges of a given variate. The formula to be used is

$$w_{j_i} = \int_{a_{j_i-1}}^{a_{j_i}} dF_i(x) . \quad (3.2)$$

For clarity, the subscripts will be defined. The sub-

script i denotes the variate and the subscript j denotes the subrange in a given variate.

Choose r_i values for each X_i , denoted by y_{j_i} , such that $a_{j_i-1} \leq y_{j_i} \leq a_{j_i}$. Define

$$B_r = \sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} C_r(y_{j_1}, y_{j_2}, \dots, y_{j_n}) w_{j_1} w_{j_2} \cdots w_{j_n}, \quad (3.3)$$

where

$$\begin{aligned} C_r(y_{j_1}, y_{j_2}, \dots, y_{j_n}) &= 1 \text{ if } g(y_{j_1}, y_{j_2}, \dots, y_{j_n}) \leq h \\ &= 0 \text{ otherwise.} \end{aligned}$$

Define $\bar{w} = \text{Max}(w_{j_i})$ and $\underline{w} = \text{Min}(w_{j_i})$, and $f(x)$ as the frequency function of x . Then, the following theorem, proven by Somerville, may be stated:

Theorem.

If $f(x)$ is a continuous function or discontinuous and non-zero for a countably infinite number of values of x , then

$$P[g \leq h] = \lim_{r_i \rightarrow \infty} B_r. \quad (3.4)$$

$$\begin{aligned} r_i &\rightarrow \infty \\ \bar{w} &= 0 \end{aligned}$$

If $f(x)$ is discrete and non-zero for a finite number N of values x , then

$$P[g \leq h] = B_n \text{ if } w > 0, \quad (3.5)$$

and may be approximated for large values of $r < N$ by B_r .

3.2 Application

Following Somerville's suggested application of the theorem, the procedure for continuous frequency functions is given below:

- i) Select r_i , the number of subranges for $f_i(x)$.
- ii) Choose $w_{j_i} = \frac{1}{r_i}$ and find $a_{0_i} < a_{1_i} < a_{2_i} < \dots < a_{r_{i_i}}$.
- iii) Compute

$$y_{j_i} = r_i \int_{a_{j_i-1}}^{a_{j_i}} x F_i(x) dx. \quad (3.6)$$

- iv) Compute all possible values of $g(x_1, x_2, \dots, x_n)$, where each x_i takes on in turn each of the r_i values of y_{j_i} .
- v) The $P[g \leq h]$ can be estimated by the proportion of the computed g 's that are less than or equal to h .
- vi) The estimation of $H(g)$ may be found by the following: Rank the computed values of g and the number of times it was computed in

step iv). This gives the approximate cumulative frequency function of g .

The suggested procedure is essentially the same for discrete functions. The a_{j_i} 's are chosen as possible values of x , and

$$w_{j_i} = \sum_{x_i=a_{j_i-1}}^{a_{j_i}} f_i(x) , \quad i = 1, 2, \dots, r_i , \quad (3.7)$$

where a_{j_i} is the smallest value of x greater than a_{j_i} for which $f(x) \neq 0$. Also, the a_{j_i} 's should be chosen such that the values of w_{j_i} are approximately equal to $\frac{1}{r_i}$.

3.3 Comparison to Stratified Monte Carlo

Both of the methods presented in this research provide an alternative approach to theoretic procedures in solving statistical distribution problems. The Ringer-Suharto method relies on repeated sampling from the stratified variates to form a function of their joint distribution. The Somerville method is a numerical procedure which can be characterized as combining both estimation and theoretical techniques. This basic difference leads to the main dissimilarity in the two methods. Somerville's method determines what could be

called the midpoint of each of the subranges by (3.6). If these subranges are chosen to be equiprobable, then the y_{ij} values are in fact the expected value of X_i for the given subrange. The function $g(x_1, x_2, \dots, x_m)$ is then evaluated at these midpoint values. On the other hand, the Ringer-Suharto method simulates random values of the X_i in each of the chosen subranges. The random observations are then used to calculate $g(x_1, x_2, \dots, x_m)$. Apart from the obvious difference in the two approaches, also notice that the Somerville method gives a much smaller number of unique values of g . Thus, Somerville's is a numerical quadrature technique, whereas the extension of the Ringer-Suharto technique is actually an attempt to evaluate the cumulative distribution function of a random variable.

The Somerville method does not require the subranges to be equiprobable. However, in actual application of his technique, Somerville suggests that the subranges be of equal probability. In general, subranges of equal probability is considered advantageous in both methods.

Further analysis of these methods can be achieved by introducing a third method referred to by Somerville. The procedure, by Haber, will now be presented.

3.4 Haber's Procedure

Haber [3] presents a modified Monte Carlo quadrature method for evaluating integrals of the type

$$I = \int_A f \, dv, \quad (3.8)$$

where A is a region in k -dimensional Euclidian space. The method involves a form of stratified sampling, with the region A split into subranges in the following manner: A is the cartesian product of k one-dimensional intervals (a^i, b^i) , $i = 1, 2, \dots, k$. Each interval is partitioned into n_i subintervals such that $a^i = a_{i,0} < a_{i,1} < a_{i,2} < \dots < a_{i,n_i} = b^i$. These points are chosen so as to make all the subintervals multiples of the smallest one. Thus, A is divided into $n = n_1 \cdot n_2 \cdot \dots \cdot n_k$. Denoting these subintervals $A_1, A_2, \dots, A_n$ (in some systematic order), the number of the n points $x_1, x_2, \dots, x_n$ in each A_i is proportional to that subinterval's size. It can be seen that the simplest, and in many cases the best, division of the region A is to partition each interval (a^i, b^i) into equal subintervals.

The "pseudo-random" values $x_1, x_2, \dots, x_n$ are calculated from some numerical formula chosen by the researcher. These values can be likened to the midpoint

values employed in Somerville's method. In Haber's example [3, p. 366], for instance, the points used were

$$x_n = ([\frac{n(n+1)}{2}\sqrt{2}], [\frac{n(n+1)}{2}\sqrt{3}], [\frac{n(n+1)}{2}\sqrt{5}], [\frac{n(n+1)}{2}\sqrt{7}]) . \quad (3.9)$$

The integral (3.8) is then estimated by

$$J = \frac{|A|}{N} \sum_{i=1}^N f(x_i) , \quad (3.10)$$

where $|A|$ denotes the volume of A .

Clearly, Haber's procedure is a combination of stratified Monte Carlo and numerical techniques. It does produce an estimate of I which is slightly lower in variance than that of simple Monte Carlo. Haber can be considered one of the pioneers in this area, and his procedures are indeed some of the first introduced. This should be kept in mind when comparisons to other methods are made.

Somerville presents a detailed discussion of the accuracy of his technique compared to both the simple Monte Carlo technique and the stratified Monte Carlo technique proposed by Haber. While the conclusions below are of a somewhat general nature, note that there are various formulas upon which the comparisons are

made.

First, a comparison with the simple Monte Carlo technique will be made. Concerning the amount of work needed for equivalent accuracies, the particular statistic and its distribution will determine the ratio of work for the two techniques. Other conclusions are:

- 1) For small sample size, Somerville's method is superior. Accuracy of the Monte Carlo technique increases with sample size.
- 2) For very small sample size and number of subranges, the work of Somerville's method is less than the Monte Carlo method. Somerville states, however, that "for a number of statistics and distributions $F(x)$, even relatively small values of r will result in near prohibitive amounts of work".

Thus, the above suggests that for small problems and calculations Somerville's method is superior. Otherwise, the Monte Carlo method should be used.

Now the comparison is made to Haber's stratified Monte Carlo method. Indeed, due to the fact that Haber's method is a combination of the two methods discussed

above, there is very little difference from those conclusions stated above. Somerville shows the relative efficiency of his method to Haber's for various values of r and n . In the general case, the size of the problem and calculations will determine the procedure to be used, with Haber's procedure preferred for larger problems.

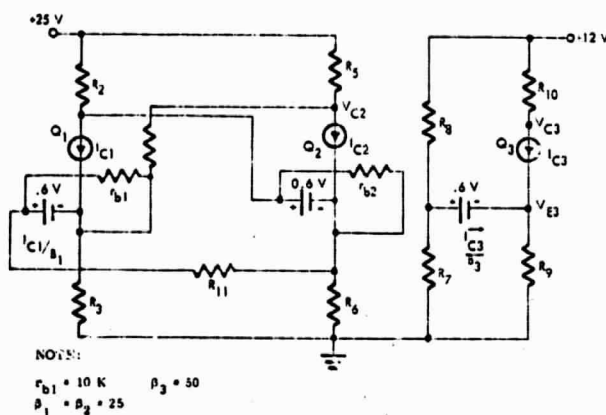
4. ELECTRONICS EXAMPLE

4.1 Technical Background

The problem to be considered in this section involves estimation of the joint cumulative distribution of output parameters of a television amplifier circuit, given input variables which obey specified probability distributions. This problem has been approached by Bosinoff and Jacobs [1], who used the Simple Monte Carlo technique to simulate the estimates. This research uses the Stratified Monte Carlo technique in an effort to improve the precision of the estimates.

Sixteen input variables are used to determine values of nine specific voltages and currents in the circuit. Thirteen of the input variables are actually components in the circuit, and the other three variables are factors affecting certain components. All sixteen variables have known distributions. From these distributions Monte Carlo samples are drawn and used in transfer functions to arrive at the estimated values. The use of stratified Monte Carlo procedures will insure more precise estimates than those of simple Monte Carlo. Figure 4.1 shows the circuit to be simulated.

Figure 4.1 Television Amplifier Circuit



The transfer functions to be used are given in matrix form below. While the equations are arranged in one nine by nine matrix in [1, p. 466], they have been partitioned into two smaller matrices here. The need for this partitioning will become evident later. These matrices, denoted A and B, are given by

$$A = \left[\begin{array}{ccc|c} A_1 & & & A_2 \\ \hline \frac{1}{R_{10}} & 0 & 1 & \frac{12}{R_{10}} \\ 0 & \frac{-1}{R_9} & (1 + \frac{1}{\beta_3}) & 0 \\ 0 & (\frac{1}{R_8} + \frac{1}{R_7}) & \frac{1}{\beta_3} & (\frac{11.4}{R_8} - \frac{0.6}{R_7}) \end{array} \right] \quad (4.1)$$

and

$$B = \begin{bmatrix} \frac{1}{R_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_4} & (\frac{1}{R_3} + \frac{1}{R_4}) & 0 & (-1 - \frac{1}{\beta_1}) & 0 & \frac{1}{\beta_2} & (\frac{25}{R_2} - \frac{0.6}{r_{b2}}) \\ 0 & (\frac{1}{R_4} + \frac{1}{R_5}) & -\frac{1}{R_4} & 0 & 0 & 1 & 0 & \frac{0.6}{r_{b1}} \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{25}{R_5} \\ 0 & 0 & -\frac{1}{R_{11}} & (\frac{1}{R_6} + \frac{1}{R_{11}}) & \frac{1}{\beta_1} & 0 & 0 & (-\frac{0.6}{r_{b1}} - \frac{0.6}{R_{11}}) \\ 0 & 0 & -\frac{1}{R_{11}} & (\frac{1}{R_6} + \frac{1}{R_7}) & 0 & (-1 - \frac{1}{\beta_2}) & 0 & (\frac{0.6}{R_{11}} + \frac{0.6}{r_{b2}}) \end{bmatrix} \quad (4.2)$$

To solve the transfer equations, the following must be done:

$$[A_1] \cdot \begin{bmatrix} V_{C3} \\ V_{E3} \\ I_{C3} \end{bmatrix} = [A_2] \quad , \quad (4.3)$$

$$[B_1] \cdot \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{E1} \\ V_{E2} \\ I_{C1} \\ I_{C2} \end{bmatrix} = [B_2] \quad (4.4)$$

It is desired to estimate the joint cumulative distribution function of the nine voltages and currents. However, of greater importance are the marginal cumulative distribution functions. With these functions, many characteristics of the circuit may be analyzed. Such information as the following can be derived:

a) The Monte Carlo samples provide the probability with which a specific output variate remains

in an acceptable control range. These acceptable boundary conditions must be met for proper operation of the circuit.

(b) The reliability of the entire circuit can be found as a result of the probabilities calculated above.

(c) The expected output, and variation, of any output variate can be estimated. Thus, the sensitivity and potential output of the circuit is also established.

(d) Possible changes in the circuit may be examined. This can be done by changing one or more of the input variables (e.g., a resistor's value). The ease of changing these variables when simulating makes this a very advantageous aspect of the Monte Carlo technique.

The nine output parameters will then be estimated by the Stratified Monte Carlo technique. Properties of the input variables are given below. The Monte Carlo observations are drawn from each of these variables, and values of the output parameter calculated using the observations in the transfer equations.

Table 4.1

Properties of the Input Variables

Variate	μ	σ	Lower Limit	Upper Limit
R_1	1,000	29	950	1,050
R_2	10,000	290	9,500	10,500
R_3	220	6	209	231
R_4	5,600	162	5,320	5,880
R_5	1,500	43	1,425	1,575
R_6	1,500	43	1,425	1,575
R_7	2,200	64	2,090	2,310
R_8	22,000	640	20,900	23,100
R_9	470	14	446	493
R_{10}	330	10	313	346
R_{11}	100,000	2887	95,000	105,000
β_1	20	4	10	38
β_2	20	4	10	38
r_{b1}	10,000	2887	5,000	15,000
r_{b2}	10,000	2887	5,000	15,000
β_3	75	26	30	120

Note: All distributions uniform except β_1 and β_2 which are truncated normal distributions.

4.2 Simulation Results

Two separate cases were investigated, the only difference being the number of subranges in each variate. The first simulation is described as follows: Input variables $R_1, R_3, r_{b1}, r_{b2}, \beta_1$, and β_2 were not partitioned. All of the other variables were partitioned into two subranges each with equal probability. Therefore, there were $2^{10} = 1024$ possible strata. Each possible stratum was sampled once. Simple random sampling took place in the six variates not partitioned, and in each subrange of the given stratum. Each time the sixteen observations were drawn from the variates, an estimate of the nine parameters was calculated using the transfer functions. These estimates were arranged in frequency arrays to form estimated cumulative distribution functions. To summarize, $k_i = 2$ for $i = 2, 4, \dots, 11, 16$, $N_s = 1$, and $N = M = 1024$. Five values of $F(h)$ were selected, $F(h) = .01, .10, .50, .90, .99$. $\hat{h}$ was calculated for these values of $F(h)$.

Now, the entire experiment above was repeated ten times. A measure of the precision of $\hat{F}(h)$ is given by the variation among the ten estimates. For the output variates selected, Tables 4.2 thru 4.6 present these

Table 4.2 Electronics Problem Simulation (V_{C3})

Case 1

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.116107 x 10 ²	.222756 x 10 ⁻⁴	.116090 x 10 ²	.778711 x 10 ⁻⁵	2.86
.10	.116405	.019138	.116401	.266622	0.72
.50	.116822	.010840	.116825	.071600	1.51
.90	.117205	.036557	.117208	.156889	2.33
.99	.117461	.033432	.117445	.285122	1.17

Table 4.3 Electronics Problem Simulation (V_{E3})

Case 1

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.381742	.517067 $\times 10^{-5}$	.381389	.398820 $\times 10^{-5}$	1.30
.10	.411175	.459186	.410665	.289403	1.59
.50	.462427	.135016	.462366	.137394	0.98
.90	.516090	.855973	.516282	.331403	2.58
.99	.549877	.637205	.550947	.369709	1.00

Table 4.4 Electronics Problem Simulation (I_{C3})

Case 1

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var	
.01	.786625 x 10^{-3}	.282933 x 10^{-10}	.786782 x 10^{-3}	.115702 x 10^{-10}	2.45
.10	.856450	.256597	.857189	.153871	1.67
.50	.969355	.051767	.969939	.041193	1.26
.90	1.093970	.197664	1.092440	.173798	1.14
.99	1.175990	.893328	1.179175	.325035	2.75

Table 4.5 Electronics Problem Simulation (V_{C2})

Case 1

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var	
.01	.856440 x 10 ¹	.259819 x 10 ⁻¹	.852126 x 10 ¹	.116855 x 10 ⁻¹	2.22
.10	.979099	.049765	.980027	.062278	0.80
.50	1.168578	.047606	1.169089	.033915	1.40
.90	1.297381	.013208	1.298896	.010302	1.28
.99	1.374930	.081555	1.375080	.025597	3.19

Table 4.6 Electronics Problem Simulation (I_{C2})

Case 1

Prob Level h	Std. M.C.		Strat. M.C		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.536282 x 10 ⁻²	.261677 x 10 ⁻⁸	.535357 x 10 ⁻²	.368939 x 10 ⁻⁸	0.71
.10	.593239	.085200	.593956	.089589	0.95
.50	.696824	.153290	.595981	.143572	1.07
.90	.851900	.345381	.851611	.324374	1.06
.99	.943925	.810209	.946719	.532489	1.52

variances for each of the five values of h . Also presented are the means of the ten repetitions for the five values of $\hat{F}(h)$. In order to compare the precision of these estimates to those of simple Monte Carlo simulation, the relative efficiency of the two is also given in the tables. The variances of the estimates under simple Monte Carlo are divided by the variances of the estimates under stratified Monte Carlo. That is,

$$\text{Rel. Eff.} = \frac{V_{\text{simple M.C.}}}{V_{\text{strat. M.C.}}} \quad (4.5)$$

Of course, the relative efficiency must be greater than one for the estimates of the Stratified Monte Carlo technique to be considered more precise than those of simple Monte Carlo.

Table 4.7

Mean Values of the Electronics Problem Output
Variates, Case 1, Stratified M.C.

V_{C3}	V_{E3}	I_{C3}	V_{C1}
$.116792 \times 10^2$	.461915	$.970100 \times 10^{-3}$	$.116924 \times 10^2$

V_{C2}	V_{E1}	V_{E2}	I_{C1}
$.114914 \times 10^2$	.648594	$.110924 \times 10^2$	$.901027 \times 10^{-3}$

I_{C2}
$.707002 \times 10^{-2}$

It was noticed that the coefficient matrix in [1, p. 466] is block diagonal, and can be partitioned into (4.1) and (4.2). This allows the number of subranges of the variables estimating V_{C3} , V_{E3} , and I_{C3} to be chosen independent of the number of subranges of the other variables. As the number of subranges increases, the estimates become more precise. It was therefore decided to partition the variables of matrix A (i.e., R_7, R_8, R_9, R_{10} , and β_3) into four equiprobable subranges, and to use simple random sampling for all other input variables. It can be seen that while the precision of V_{C3} , V_{E3} , and I_{C3} will increase, the precision of the other six variate's estimates will not increase. The total number of strata is $4^5 = 1024$. Each stratum is sampled once. The values of $F(h)$ remain the same, as does the number of repetitions of the experiment (ten). Therefore, the only change is the partitioning of the ranges of the sixteen input variables. Estimates similar to these for Case 1 are presented in Tables 4.8, 4.9, and 4.10. Estimates of the other six output variates are not given, as they are in fact found by simple random sampling.

Table 4.8 Electronics Problem Simulation (V_{C3})

Case 2

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.116107 x 10 ²	.222756 x 10 ⁻⁴	.116094 x 10 ²	.461511 x 10 ⁻⁵	4.83
.10	.116405	.019138	.116404	.104278	1.84
.50	.116822	.010840	.116824	.036989	2.93
.90	.117205	.036567	.117210	.117778	3.10
.99	.117461	.033432	.117448	.354322	0.94

Table 4.9 Electronics Problem Simulation (V_{E3})

Case 2

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.381742	.517067 x 10 ⁻⁵	.382661	.753775 x 10 ⁻⁵	0.69
.10	.411175	.459186	.410682	.171080	2.68
.50	.462427	.135016	.462381	.096589	1.40
.90	.516090	.855973	.516092	.104406	8.20
.99	.549877	.637205	.549905	.221565	2.88

Table 4.10 Electronics Problem Simulation (I_{C3})

Case 2

Prob Level h	Std. M.C.		Strat. M.C.		Rel. Eff.
	Mean	Est. Var.	Mean	Est. Var.	
.01	.786625 x 10 ⁻³	.282933 x 10 ⁻¹⁰	.790302 x 10 ⁻³	.129681 x 10 ⁻¹⁰	2.18
.10	.856450	.256597	.855414	.095030	2.70
.50	.969355	.051767	.971124	.033381	1.55
.90	1.093967	.197664	1.090862	.091537	2.16
.99	1.175988	.893328	1.175332	.268644	3.32

4.3 Discussion

The relative efficiencies in the tables above indicate the degree to which stratification has improved the estimate's variance. The parameters were estimated ten times. In the limiting case, the relative efficiency of stratified Monte Carlo to simple Monte Carlo can never be less than one (refer to (2.16)). However, with only ten repetitions, random variation causes the relative efficiency to be less than one in a few cases. In most cases, the stratified Monte Carlo technique produced a significant reduction in variance. Also, Case 2 produced estimates that were again more precise than Case 1.

Of course, the number of subranges and variates partitioned was somewhat arbitrary. Other possible combinations, and a larger number of subranges, are expected to produce even better results than these. It has been shown in this research that stratified Monte Carlo is an effective variance-reduction technique.

5. STOCHASTIC PROCESS EXAMPLE

5.1 The Problem

Riedwyl's goodness of fit statistics with unknown parameters are investigated in a paper by Speed and Smith [14]. One Riedwyl statistic proposed is the following: Let $X_1, \dots, X_n$ be a random sample from an absolutely continuous distribution function $F(x)$, and define $F_n(x)$ to be the proportion of X_i less than or equal to x . If $d_i = F(Z_i) - F_n(Z_i), i=1, \dots, n$, where $Z_i = F^{-1}(i/n)$, then the statistic is defined as

$$S_n = \sum_{j=1}^n |d_j| \quad (5.1)$$

S_n is asymptotically ($n \rightarrow \infty$) a normal stochastic process, the limiting statistic given by

$$S_n / \sqrt{n} \rightarrow \int_0^1 |Y(t)| dt = X \quad (5.2)$$

as $n \rightarrow \infty$, where $Y(t)$ is distributed $N(0, t-t^2)$. Given that $E[|Y(t)|] = [2(t-t^2)/\pi]^{1/2}$, it follows that the mean value of X is

$$\int_0^1 [2(t-t^2)/\pi]^{1/2} dt = (\sqrt{\pi/32}) \quad (5.3)$$

5.2 Simulation Results

It is desired to estimate the distribution function of X and its mean value. The first approach taken made use of the definitive form of an integral. That is, X may be approximated by

$$X \doteq \sum_{i=1}^n \frac{i}{n} |Y(t_i)|, \quad (5.4)$$

where $t_i = \frac{i}{n}$. Clearly, larger values of n (i.e., more subranges of smaller region) will give an increasingly better approximation of X .

This method was used in simulation of the desired estimates. However, for reasonable values of n , poor results were obtained. The estimate of the mean value was converging very slowly to $(\sqrt{\pi/32})$ as n was increased. Therefore, this method was considered inadequate for estimation in this case.

After investigation of other possible techniques, Gauss's formula for numerical integration (defined in [10]) was chosen as the technique with which to estimate X . The principle of Gauss's formula is to obtain the optimum subdivision of the range of integration, the values of the function at these subdivision points, and the coefficients to multiply the

functional values to yield the value of the integral. The procedure is fully described in [10].

X was simulated for $n = 2, \dots, 10, 14$, and 15 .

As n increased, the estimates of the mean value converged quickly to $(\sqrt{\pi/32})$. For each of the n values, the distribution of X was also estimated. The value of n also signifies the number of subintervals into which the integral range was partitioned.

Stratified values of $|Y(t)|$ were drawn in each subinterval, $Y(t)$ being partitioned into two equiprobable subranges. Therefore, there were 2^n possible strata. The sample size was $2^{10} = 1024$ for values of n up to and including ten. N_s then becomes 2^{10-n} . The sample size was $2^{14} = 16,384$ for $n = 14$, and $2^{15} = 32,768$ for $n = 15$. N_s was one in these cases.

Table 5.1 presents $\hat{h}$ for $F(h) = .01, .05, .10, .25, .50, .75, .90, .95$, and $.99$, at each of the values of n . Also presented for each n is the expected mean value for the Gaussian approximation (μ_n) and the estimated mean value ($\bar{x}$). It can be seen that $\bar{x}$ converges quickly to $(\sqrt{\pi/32}) = 0.313328$.

5.3 Discussion

This example accomplished three goals. First, the

Table 5.1 Values of $\hat{F}(h)$ and $\bar{x}$ for a Riedwyl Limiting Statistic

		n											
		2	3	4	5	6	7	8	9	10	14	15	
$\hat{F}(h)$	h												
	.01	.0118	.0624	.0410	.0855	.0614	.0956	.0767	.1018	.0840	.1085	.1436	
	.05	.0352	.1242	.0817	.1408	.0952	.1542	.1161	.1609	.1347	.1533	.1886	
	.10	.0587	.1705	.1020	.1740	.1205	.1836	.1477	.2003	.1685	.1891	.2156	
	.25	.1408	.2786	.1732	.2735	.1965	.2618	.2107	.2692	.2192	.2428	.2785	
	.50	.2817	.4639	.2850	.4172	.2979	.3693	.2973	.3677	.3038	.3056	.3505	
	.75	.4694	.6800	.4375	.5609	.4076	.4964	.3919	.4761	.3968	.3862	.4315	
	.90	.6455	.8807	.5799	.7267	.5427	.6138	.4943	.5845	.4898	.4578	.5034	
	.95	.7980	1.0814	.7019	.8152	.6018	.7018	.5652	.6436	.5405	.5116	.5574	
	.99	1.0445	1.3130	.8645	1.0142	.7623	.8680	.7080	.7716	.6504	.6101	.6563	
Mean	$\bar{x}$	.3250	.4937	.3149	.4271	.3136	.3891	.3074	.3765	.3111	.3126	.3529	
	μ_n	.3257	.4948	.3152	.4278	.3139	.3969	.3136	.3794	.3135	.3133	.3538	

distribution of X was estimated. Representative values of h give estimates of the cumulative distribution function at those points. Second, Gauss's formula for numerical integration was highlighted in this simulation. The technique is relatively simple to employ, and yields good estimates for small values of n . It is an effective quadrature method in combination with simulation methods of the Monte Carlo type. Third, the Stratified Monte Carlo method was used to reduce the variance of $\hat{F}(h)$ and $\bar{x}$ estimates. Precise estimates are an asset of any simulation technique. Thus, the Stratified Monte Carlo method is a desired part of simulations of this type.

6. CONCLUSIONS

The Ringer-Suharto method has been extended to the multivariate case and applied to two problems. The method involves repeated Monte Carlo sampling from stratified variates to form an estimated joint cumulative distribution function. It has been shown in Section 2 by (2.16) that the Stratified Monte Carlo technique gives more precise estimates than those of simple Monte Carlo. This fact was verified by the results of the television amplifier problem, where stratification of ten of the sixteen input variates into two subranges each (Case 1) produced significant reductions in variance. In Case 2, stratification of five specific input variates into four subranges each showed even larger reductions in variance. These estimates are also unbiased, an important property for an estimator to have. It is apparent that stratified Monte Carlo is a practical and effective variance-reduction technique.

Somerville [12] presents a similar technique, also extended to the multivariate case in this research. The given variates are again stratified, midpoint values of each subrange are derived, and all possible combinations of these values are calculated according to the desired function of the variates. A joint cumulative

distribution function is formed by these points arranged in order. The procedure does not rely on random sampling from the subranges, but rather on only one derived value from each subrange. Thus, it cannot be considered a true computer simulation technique, but rather a numerical quadrature. The value of this procedure lies in estimating cumulative distribution functions from a small number of variates partitioned into only a few subranges. For larger problems requiring the speed and efficiency of the electronic computer, the Stratified Monte Carlo technique is desired.

Haber [3] presents a numerical integration technique that incorporates concepts of both the procedures discussed above. Each dimension of the k -dimensional region of integration is partitioned into subranges. Sample values found by a given formula are derived for each subrange, the number derived being proportional to the subrange's size. A point estimate of the joint cumulative distribution function is found by drawing a sample from these subranges and computing the value with these observations. Like the Ringer-Suharto method, this method is most efficient for larger computer problems.

The television amplifier circuit parameters were

estimated using the Stratified Monte Carlo procedure. Significant reductions in variance of the estimates were observed. As stated above, the number of subranges in each input variate directly affected the reduction in variance. The author is unaware of the formula for the exact improvement in precision. While simple Monte Carlo sampling is known to have accuracy of the order $(1/\sqrt{n})$, it is conjectured that stratified Monte Carlo sampling has accuracy at least of the order $(1/n)$ when every variate range is partitioned into two or more subranges. This is one concept that should be investigated in a later paper.

The distribution and mean of the stochastic process problem were also estimated by use of the Stratified Monte Carlo procedure. Gauss's formula for numerical integration was utilized as the function from which the tabulations were made. Satisfactory results were obtained with only modest partitioning of the integral range ($n = 2$ up to $n = 15$), with $Y(t)$ being divided into only two subranges. For more partitioning of the integral range, say $n = 25$, and a larger number of subranges of $Y(t)$, the estimates will asymptotically approach the true parameters. Other numerical quadrature techniques may also be used with stratified Monte Carlo

to produce satisfactory results. The problem to be solved, and the asymptotic efficiency desired, are among the factors that will determine the quadrature technique used. Note that whichever technique is chosen, stratified Monte Carlo procedures will increase the precision of the estimates. Thus, while the speed of convergence must be considered, precision of these estimates is also of prime importance.

Other fields of application of the Stratified Monte Carlo technique include business and engineering. Not unlike the television circuit problem, many engineering problems involve estimation of joint cumulative distribution functions of many input variates with given distributions. With the rising use of computer techniques, the scientific application is a very useful one. The engineering field should be considered as one that could most effectively use this technique.

The business field also makes great use of estimating functions of many input variates. For instance, economic theory relies heavily on such measures as production, employment, and capacity. These and many other economic measures are estimated from combinations of a tremendous number of factors. With such a large number of known input parameters, the Stratified Monte Carlo

technique is of great value as a practical, fast, and efficient method.

As with any technique, there are disadvantages inherent with the procedure. Time and dimension limitations are incurred with very large problems. Storage size and speed of the computer used will automatically provide the researcher with limits on the problem approached. Also affecting these limits is the actual program techniques utilized. Care must be taken to make the simulation program as efficient as possible.

Affecting the scope of application is the fact that both the joint distribution function and the marginals of the input variables must be known. If independence of the variables can be assumed, then only the marginal distributions need be known. This disadvantage is minimal, however, as the distributions are usually known, or can be easily obtained. This is especially evident in applications where the variates have been used repeatedly, with only their parameters changing in any given point in time.

The relative contribution of the input variates should be taken into account when using the Stratified Monte Carlo procedure. It has been shown that more subranges gives more precise estimates, with considera-

tion of the fact that certain variates may contribute more to an estimate's variance than others. This was evident in the television circuit problem.

Various methods may be used to determine an input variate's relative contribution. Inspection of the function to be used provides a basic measure. By "statistical differentials", a variate's contribution to the overall variation may be approximated by computing

$$V[Y] = \sum_{i=1}^n \frac{\partial H}{\partial X_i} | V[X_i] , \quad (6.1)$$

where $Y = H(X_1, X_2, \dots, X_n)$, and X_i , $i=1, 2, \dots, n$, are the input variates. This formula provides a useful technique for determining which variates contribute most to the variation of Y ; thus, which ranges should be more highly subdivided.

In conclusion, stratified Monte Carlo is an effective technique when theoretical derivations become impractical. The amount of stratification determines the gain in accuracy over simple Monte Carlo estimates. The technique may be used for a wide range of applications, and should be considered one of the better simulation techniques available.

BIBLIOGRAPHY

- [1] Bosinoff, I. and Jacobs, R. (1963). Using transfer function in reliability productions, Ninth National Symposium on Reliability and Quality Control. Institute of Radio Engineers, Inc., New York, New York, 455-469.
- [2] Cochran, W. (1963). Sampling Techniques. John Wiley and Sons, Inc., New York, New York, 87-153.
- [3] Haber, S. (1966). A modified Monte Carlo quadrature, Mathematics of Computation, 20, 361-368.
- [4] Haber, S. (1968). A combination of Monte Carlo and classical methods for evaluating multiple integrals, Bulletin of American Mathematical Society, 74, 683-686.
- [5] Hammersley, J. (1960). Monte Carlo methods for solving multivariate problems, Annals of the New York Academy of Sciences, 86, 844-874.
- [6] Hammersley, J. and Hanscomb, D. (1964). Monte Carlo Methods. Methuen and Company, Ltd., London, England, 1-9, 43-75.
- [7] Kahn, H. (1956). Use of different Monte Carlo sampling techniques, Symposium on Monte Carlo Methods, edited by H. A. Meyer. John Wiley and Sons, Inc., New York, New York, 146-190.
- [8] Meier, R., Newell, W. and Pazer, H. (1969). Simulation in Business and Economics. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 27-144, 252-328.
- [9] Meyer, H. (1956). Symposium on Monte Carlo Methods. John Wiley and Sons, Inc., New York, New York.
- [10] Nielsen, K. (1964). Methods in Numerical Analysis. The Macmillan Co., New York, New York, 129-133.

- [11] Ringer, L. and Suharto, S. (1968). Stratified Monte Carlo sampling for symmetric statistics, Technical Report #9, Grant # DA-ARO-31-124-G793, Institute of Statistics, Texas A&M University.
- [12] Somerville, P. (1968). A technique for the computation of percentage points of a statistic, Technometrics, 12, 373-382.
- [13] Spanier, J. and Gelbard, E. (1969). Monte Carlo Principles and Neutron Transport Problems. Addison-Wesley Publishing Company, Menlo Park, California, 85-138.
- [14] Speed, F. and Smith, W. (1969). Riedwyl's goodness of fit with unknown parameters, Submitted to the Journal of the American Statistical Association.