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FULLY IONIZED QUASI-ONE DIMENSIONAL MAGNETIC NOZZLE FLOW

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Abstract

To investigate magnetic nozzle processes, a theoretical quasi-one dimensional analysis was employed. Included in the analysis are the effects of unequal electron and ion temperatures and electron thermal conductivity. The acceleration of the plasma is controlled by an imposed magnetic nozzle. The parameters that control the flow are the nozzle shape, the ratio of kinetic pressure to magnetic pressure, and the ratio of the nozzle throat radius to the collisional mean free path. Results show that higher nozzle exit velocities than a corresponding adiabatic flow are achieved. Also, the plasma temperature remains higher throughout the nozzle than a corresponding adiabatic flow.

Introduction

Plasma devices such as the MPD arc[1] and the Q machine wind tunnel[2] produce high speed flows in a diverging magnetic field. One possible explanation for the acceleration mechanism in these devices is that the diverging magnetic field acts as a nozzle. Axial acceleration occurs as a result of both the magnetic pressure exerted by the nozzle and the conversion of thermal motion into axially directed motion. To investigate magnetic nozzle processes, a theoretical quasi-one dimensional analysis was undertaken.

The recent analysis of Walker and Seikel[3] considers this same problem in a different manner. Their analysis considers the flow near the axis of the nozzle including the electron thermal conductivity. The main assumptions of their analysis are that the ion temperature is negligible and that the Hall parameter (electron cyclotron frequency/electron-ion collision frequency) is a constant, at least near the nozzle's axis.

The geometry of the problem considered in this study is shown in Fig. 1. An imposed axially symmetric magnetic field controls the flow. Any change in the magnetic field from induced currents is neglected.

One fairly simple method of analyzing this plasma flow is by a quasi-one-dimensional analysis similar to those applied in ordinary gasdynamics. In the case of a fully-ionized plasma, however, we must consider two species (ions and electrons) rather than a single species. Also, since electrons are much lighter than ions, the electrons tend to have larger random energies and therefore a higher temperature than the ions. The small electron mass also results in large values of the thermal conductivity for the electrons. Therefore, in the analysis herein unequal electron and ion temperatures as well as electron thermal conductivity are included.

Quasi-One Dimensional Equations

The quasi-one dimensional equations are derived in appendix A by integrating the various equations over the control volume shown in Fig. 1 (this is a segment of the nozzle). Also, the following assumptions are made.

(1) Plasma properties are uniform across the nozzle
(2) No current in x direction, $J_x = 0$
(3) Quasi-neutrality, $n_e = n_i$
(4) Heat flux in r and θ directions is negligible
(5) Viscous forces are negligible
(6) No energy added to flow
(7) Ratio of electron to ion mass, $m_e/m_i = \epsilon^2$, negligible compared to 1

It should be noted that 1 is the basic assumption for any quasi-one dimensional analysis.

Under the above assumptions the following equations are derived in appendix A.

$$\overline{V}_e = \overline{V}_i = \overline{V} \qquad \text{equal x-components of electron and ion velocities} \tag{A4}$$

$$\overline{A\rho V} = (\overline{A\rho V})_{\overline{x}=0} \qquad \text{plasma continuity} \tag{A5}$$

or,

$$\overline{AnV} = (\overline{AnV})_{\overline{x}=0}$$

$$\overline{\rho}\overline{V}\frac{d\overline{V}}{d\overline{x}} + \frac{d\overline{p}}{d\overline{x}} + \frac{1}{2\mu_o}\frac{d\overline{B}_r^2}{d\overline{x}} = 0 \qquad \text{Plasma momentum} \tag{A17}$$

$$\overline{\rho}\overline{V}\frac{d}{d\overline{x}}\left(\frac{\overline{V}^2}{2} + \frac{5}{2}\frac{\overline{p}}{\overline{\rho}}\right) + \frac{d\overline{q}_x}{d\overline{x}} = 0 \qquad \text{Plasma energy} \tag{A25}$$

$$\frac{k\overline{T}_e}{\overline{V}}\frac{d\overline{V}}{d\overline{x}} + \frac{d}{d\overline{x}}\ \frac{3}{2}k\overline{T} \quad + \frac{1}{\overline{n}\overline{V}}\frac{d\overline{q}_x}{d\overline{x}} = -\frac{3\epsilon^2}{\overline{V}}\overline{\nu}_{ei}(k\overline{T}_e - k\overline{T}_i)$$

$$\frac{k\overline{T}_e}{\overline{A}}\frac{d\overline{A}}{d\overline{x}} \qquad \text{electron energy} \tag{A31}$$

$$\overline{V}\frac{dq_x}{d\overline{x}} + \overline{q}_x\left(\frac{\overline{V}}{\overline{A}}\frac{d\overline{A}}{d\overline{x}} + \frac{16}{5}\frac{d\overline{V}}{d\overline{x}}\right) + \frac{5}{2}\frac{k\overline{n}\overline{T}_e^2}{m_e}\frac{d\overline{T}_e}{d\overline{x}}$$

$$= 1.866\,\overline{\nu}_{ei}\overline{q}_x \qquad \text{electron heat flux} \tag{A34}$$

The bars appearing in the above equations are used to denote dimensional variable quantities. The subscripts e and i are used to denote electrons and ions, respectively. All quantities are defined in the usual manner; $\overline{V}$ is velocity, $\overline{T}$ is temperature, $\overline{\rho}$ is mass density, $\overline{n}$ is number density, $\overline{p}$ is pressure, $\overline{B}$ is magnetic field strength, $\overline{q}$ is heat flux, and $\overline{A}$ is the cross sectional area of the nozzle at position $\overline{x}$. The quantity $\overline{\nu}_{ei}$ is the electron-ion collision frequency for transfer of momentum given by Equa. (A28) in the appendix. In appendix A, Equa. (A17) is derived without assuming that B_x and B_r are uniform across the nozzle. However, in the above equations it has been assumed that B_x and B_r are uniform across the nozzle.

In addition to the above equations, we need expressions for the total pressure $\overline{p}$, and $\overline{B}_r$. For the total pressure we have

$$\overline{p} = \overline{n}_e k\overline{T}_e + \overline{n}_i k\overline{T}_i = \overline{n}k(\overline{T}_i + \overline{T}_e) \qquad (1)$$

Define the temperature,

$$\overline{T}_o \equiv \overline{T}_e + \overline{T}_i \qquad (2)$$

therefore

$$\overline{p} = \overline{n}k\overline{T}_o \qquad (3)$$

The magnetic field is assumed to determine the nozzle shape,

$$\frac{d\overline{R}}{d\overline{x}} = \frac{\overline{B}_r}{\overline{B}_x} \qquad (4)$$

where $\overline{R}$ is the nozzle radius at position $\overline{x}$. Also, from $\nabla \cdot \vec{B} = 0$ and assuming $\overline{B}_x$ is uniform across the nozzle (see equation immediately preceeding Eq. (A16)),

$$\overline{B}_x\overline{A} = (\overline{B}_x\overline{A})_{\overline{x}=0} \qquad (5)$$

We now nondimensionalize the equations using the following definitions.

$$n = \rho = \frac{\overline{\rho}}{\overline{\rho}_*},\ V = \frac{\overline{V}}{\overline{V}_*},\ X = \frac{\overline{X}}{l_*},\ T_e = \frac{\overline{T}_e}{\overline{T}_*},\ T_o = \frac{\overline{T}_o}{\overline{T}_*},$$

$$\nu = \frac{\overline{\nu}_{ei}}{(\overline{\nu}_{ei})_*},\ A = \frac{\overline{A}}{\overline{A}_*} = \frac{\overline{A}}{\pi l_*^2},\ q = \frac{\overline{q}_x}{\frac{1}{2} m_e \overline{n}_*\left(\frac{2k\overline{T}_*}{m_e}\right)^{3/2}},$$

$$B_r = \frac{\overline{B}_r}{(\overline{B}_x)_*} \qquad (6)$$

The asterisk subscript is used to denote conditions that exist at the nozzle throat. Unbarred quantities are dimensionless, $\overline{T}_* = (\overline{T}_o)_{x=0}$ and l_* is the throat radius. Also, ν_{ei*} has been defined as a function of $\overline{T}_*$ rather than $\overline{T}_{e*} = (\overline{T}_e)_{x=0}$.

If Eqs. (A5) and (3) are used to eliminate $\overline{\rho}$, $\overline{n}$, and $\overline{p}$, Eqs. (A17), (A25), (A31), and (A34) become, respectively, the following dimensionless equations

$$\left(\frac{5}{3} M_*^2 - \frac{T_o}{V^2}\right)\frac{dV}{dx} + \frac{1}{V}\frac{dT_o}{dx} = G(x) + \frac{T_o}{AV}\frac{dA}{dx} \qquad (7)$$

$$\frac{5}{3} M_*^2 V \frac{dV}{dx} + \frac{5}{2}\frac{dT_o}{dx} + \sqrt{\frac{6}{5}}\frac{A}{\epsilon M_*}\frac{dq}{dx} = 0 \qquad (8)$$

$$\frac{T_e}{V}\frac{dV}{dx} + \frac{3}{2}\frac{dT_e}{dx} + \sqrt{\frac{6}{5}}\frac{A}{\epsilon M_*}\frac{dq}{dx} = 3\left(\frac{l_*}{\lambda_{ei*}}\right)\frac{\epsilon}{M_*}\frac{\nu}{V}(T_o - 2T_e) - \frac{T_e}{A}\frac{dA}{dx} \qquad (9)$$

$$\frac{16}{5}\frac{q}{V}\frac{dV}{dx} + \sqrt{\frac{6}{5}}\frac{5T_e}{4\epsilon M_* AV^2}\frac{dT_e}{dx} + \frac{dq}{dx} = -1.866\left(\frac{l_*}{\lambda_{ei_*}}\right)\frac{\nu q}{\epsilon M_* V} - \frac{q}{A}\frac{dA}{dx} \qquad (10a)$$

Appearing in Eqs. (7) to (10) are the throat Mach number M_*, the ratio of the throat radius to the throat mean free path l_*/λ_{ei_*}, the magnetic pressure term $G(x)$, and the square root of the mass ratio ϵ.

$$M_*^2 = \frac{m_i \overline{V}_*^2}{\frac{5}{3} k\overline{T}_*} \qquad (11)$$

$$\lambda_{ei*} = \frac{\sqrt{\frac{5}{3}\frac{k\overline{T}_*}{m_e}}}{\overline{\nu}_{ei_*}}$$

$$G(x) = -\frac{A}{\beta_*}\frac{dB_r^2}{dx} = \frac{1}{4\beta_* A^2}\frac{dA}{dx}\left[\frac{3}{A}\left(\frac{dA}{dx}\right)^2 - 2\frac{d^2A}{dx^2}\right] \qquad (13)$$

$$\epsilon = \sqrt{\frac{m_e}{m_i}} \qquad (14)$$

The term β_* is the ratio of kinetic pressure to magnetic pressure at the throat.

$$\beta_* = \frac{\overline{n}_* k\overline{T}_*}{\frac{1}{2\mu_o}\overline{B}_{x_*}^2} \qquad (15)$$

(The approximation $\overline{\rho} \approx m_i\overline{n}$ (eq. (A6)) is used throughout the paper.) From Eqs. (7) to (10) we see that the solution will depend on the parameters M_*, β_*, (l_*/λ_{ei_*}), and ϵ. Also, we must know the area, A, as a function of x. As we shall see later, M_* will be determined by the value of $(\overline{T}_e)_{x=0}$. The parameter β_* determines the relative energy density of the magnetic field compared to the thermal energy density. Large β_* implies small magnetic fields. The parameter l_*/λ_{ei_*} is a measure of the "collisionness" of the flow. A large value of this parameter means there are many collisions occurring. For the basic equations from which the quasi-one dimensional equations are derived to be valid, the flow must be collisional. Therefore, $l_*/\lambda_{ei_*} \geq 1$. The mass ratio ϵ merely depends on the gas considered.

In Eq. (10a) the dT_e/dx and νq terms have a coefficient proportional to $1/\epsilon M_*$. Since this quantity is of the order of 10^2, the other terms in Eq. (10a) will be small compared to the dT_e/dx and νq terms. As a result the following approximate heat flux equation was used.

$$\sqrt{\frac{6}{5}}\frac{5T_e}{4AV}\frac{dT_e}{dx} = -1.866\left(\frac{l_*}{\lambda_{ei_*}}\right)\nu q \qquad (10b)$$

If Eqs. (7) to (10) are solved for the various derivatives the following results are obtained.

$$\frac{dV}{dx} = V\left[\frac{\frac{1}{A}\frac{dA}{dx}\left(\frac{5}{2}T_o - T_e\right) + \frac{5}{2}GV + \theta}{D}\right] \quad (16)$$

$$\frac{dT_o}{dx} = -\frac{\frac{1}{A}\frac{dA}{dx}\left[\frac{5}{3}M_*^2V_*^2(T_o-T_e)\right] + GV\left[\frac{5}{3}M_*^2V^2-T_e\right] + \left[\frac{5}{3}M_*^2V^2-T_o\right]\theta}{D} \quad (17)$$

$$\frac{dT_e}{dx} = -1.244\sqrt{\frac{6}{5}}\left(\frac{l_*}{\lambda_{ei_*}}\right)\frac{q}{T_e^{5/2}} \quad (18)$$

$$\frac{dq}{dx} = -\frac{5}{2}\sqrt{\frac{5}{6}}\frac{\epsilon M_*}{A}\left[\frac{\frac{1}{A}\frac{dA}{dx}M_*^2V_*^2T_e + GVT_e - (M_*^2V^2-T_o)\theta}{D}\right] \quad (19)$$

Appearing in Eqs. (16) to (19) are the quantities,

$$D = \frac{5}{2}\left[M_*^2V^2 - \left(T_o - \frac{2}{5}T_e\right)\right] \quad (20)$$

$$\theta = \left(\frac{l_*}{\lambda_{ei_*}}\right)T_e^{-3/2}\left[\frac{55.98}{25}\sqrt{\frac{5}{6}}\frac{q}{T_e} + \frac{3\epsilon}{AV^2M_*}(T_o-2T_e)\right] \quad (21)$$

In obtaining these expressions we have used the results of Eq. (A28) in the appendix.

$$\nu = \frac{\bar{\nu}_{ei}}{\bar{\nu}_{ei_*}} = \frac{n}{T_e^{3/2}}\ \frac{\ln \Lambda_{ei}}{\ln \Lambda_{ei_*}} \quad (22a)$$

Since $\ln \Lambda_{ei}$ is a slowly varying function we have used the approximation,

$$\nu = \frac{n}{T_e^{3/2}} = \frac{1}{AVT_e^{3/2}} \quad (22b)$$

The continuity equation has been used to replace n in Eq. (22b).

Critical Point

The system of Eqs. (16) to (19) has a critical point when $D = 0$ or when,

$$M^2 = \frac{m_i\bar{V}^2}{\frac{5}{3}k\bar{T}_o} = \frac{M_e^2}{M_e^2 + \frac{2}{5}\epsilon^2} \quad (23)$$

where

$$M_e^2 = \frac{m_e\bar{V}^2}{\frac{5}{3}k\bar{T}_e} \quad (24)$$

Equation (23) shows that the critical point occurs at a subsonic Mach number. In order for the flow to pass smoothly through this singular point, the numerators in Eqs. (16), (17), and (19) must go to zero when $D = 0$. The first two terms in the numerators of Eqs. (16), (17), and (19) are proportional to dA/dx. Therefore, when $dA/dx = 0$ these terms will vanish. The last term in the numerators of Eqs. (16), (17), and (19) will vanish when $\theta = 0$. As a result, by requiring that $da/dx = \theta = 0$ when $D = 0$ the solution will be continuous at the critical point. From the requirement $\theta = 0$, we obtain a condition on the heat flux at the critical point (see eq. (21)).

$$q_* = \frac{25}{18.66}\sqrt{\frac{6}{5}}\frac{\epsilon T_{e_*}}{M_*A_*V_*^2}(2T_{e_*} - T_{o_*}) \quad (25a)$$

The reason for the asterisk will be explained shortly.

For any value of T_{e_*} that gives a physically meaningful solution $D < 0$ just upstream of the critical point (see eq. (20)). Therefore, in order for the flow to be accelerating the numerator of Eq. (16) must be negative in magnitude. For the values of β_* and l_*/λ_{ei_*} considered here, the predominate term in the numerator of Eq. (16) is the first one, which is multiplied by $(1/A)(dA/dx)$. Therefore, in order for the numerator to be negative, $dA/dx < 0$. Similarly, just downstream of the critical point $D > 0$. In this case dA/dx must be positive. Since dA/dx goes from negative to positive in passing through the critical point we have established the critical point as that corresponding to the minimum area of the nozzle (i.e., the throat). It is for this reason that the asterisk subscript is used on the quantities in Eq. (25a). Also, since we have nondimensionalized in terms of the throat properties, $T_{o_*} = 1$, $V_* = 1$, $A_* = 1$ and Eq. (25a) becomes the following.

$$q_* = \frac{25}{18.66}\sqrt{\frac{6}{5}}\frac{\epsilon T_{e_*}}{M_*}(2T_{e_*} - 1) \quad (25b)$$

Using Eq. (20) at the critical point we obtain an expression for M_*.

$$M_*^2 = 1 - \frac{2}{5}T_{e_*} \quad (26)$$

It should be pointed out that the condition $dA/dx = \theta = 0$ is not the only one that will make the numerators of Eqs. (16), (17), and (19) vanish. However, based on ordinary gasdynamic flows, it appears physically meaningful that the critical point should occur at the throat. Also, the requirement $\theta = 0$ establishes a boundary condition on the heat flux in terms of T_{e_*}.

Nozzle Area Ratio

So far it has been established that the equations describing the flow have a critical point. Further, this critical point has been fixed at the nozzle throat and yielded a boundary condition on the heat flux. This information was obtained without specifying the cross-sectional area variation of the nozzle. To integrate the equations, however, we must specify how the nozzle area varies with $\bar{x}$.

The nozzle radius dependence on $\bar{x}$ was assumed to be the following.

$$\bar{R} = a\bar{x}^p + l_* \tag{27}$$

where l_* is the throat radius and a and p are constants. Using Eq. (27) the area can be calculated. In dimensionless form it is the following.

$$A = (A_s x^p + 1)^2 \tag{28}$$

where,

$$A_s = a l_*^{p-1} \tag{29}$$

From Eq. (28) the following is obtained,

$$\frac{1}{A}\frac{dA}{dx} = 2pA_s\left(\frac{x^{p-1}}{A_s x^p + 1}\right) \tag{30}$$

Also, from Eqs. (4), (5), and (27) to (29),

$$B_r = \frac{pA_s x^{p-1}}{(A_s x^p + 1)^2} \tag{31}$$

and using Eq. (13),

$$G = \frac{2p^2(p-1)A_s^2}{\beta_*}\left[\frac{p+1}{p-1}A_s x^p - 1\right]\frac{x^{2p-3}}{(A_s x^p + 1)^3} \tag{32}$$

Dependence of Solution on Electron Temperature

Considering Eqs. (16) to (19) and (25) to (32), we see that the solution will depend on the following quantities; ϵ, β_*, (l_*/λ_{ei_*}), A_s, p, and T_{e_*}. The first three of these parameters have already been discussed. The quantities A_s and p merely determine how fast the area changes with x.

Since T_{e_*} is a boundary condition on the electron temperature it might at first seem that solutions should be obtainable for all values of T_{e_*} from 0 to 1. However, it was found that for given values of the other parameters a physically meaningful solution was obtained for only one value of T_{e_*}. By a physically meaningful solution is meant one in which $T_e \leq T_o$ for large values of x.

Asymptotic Solution

For very large x the expected results are T_o, T_e, $q \to 0$ and $V \to V_\infty$, where V_∞ is some asymptotic limit. However, Eqs. (16) to (19) will not produce such results. This can be seen by considering Eq. (19) for large x. In this case

$$\frac{dq}{dx} \approx \frac{\frac{5}{2}\sqrt{\frac{5}{6}}\frac{\epsilon M_*}{A}M_*^2V^2\left(\frac{l_*}{\lambda_{ei_*}}\right)\frac{55.98}{25}\sqrt{\frac{5}{6}}\frac{q}{T_e^{5/2}}}{\frac{5}{2}M_*^2V^2}$$

$$= 1.866\frac{\epsilon M_*}{A}\frac{q}{T_e^{5/2}}\left(\frac{l_*}{\lambda_{ei_*}}\right) = -\frac{3}{2}\sqrt{\frac{5}{6}}\frac{\epsilon M_*}{A}\frac{dT_e}{dx}$$

where Eq. (18) has been used. From this expression we see that q will be increasing while T_e is decreasing for large x. Such a result is not physically meaningful since both T_e and q should approach zero as x approaches infinity. Therefore, the integration of Eqs. (16) to (19) was cut off when q started to increase. This occurred for values of the area ratio, A of the order of 100. A discussion of how this result influences the conclusions of the analysis is presented in the RESULTS section.

Adiabatic and Isentropic Solutions

It is of interest to compare the solution of Eqs. (16) to (19) with the adiabatic and isentropic flows for a similar nozzle. To obtain the adiabatic solution we merely let $q = 0$ in Eqs. (7) to (10). In this case Eq. (10) is not used and Eq. (9) (electron energy) is decoupled from Eqs. (7) and (8). Equation (8) can be integrated to obtain

$$T_o = \frac{M_*^2}{3}(1 - V^2) + 1 \tag{33a}$$

We also obtain from Eqs. (7) to (9) the following results for dV/dx and dT_e/dx;

$$\frac{dV}{dx} = V\left[\frac{GV + \frac{T_o}{A}\frac{dA}{dx}}{M_*^2V^2 - T_o}\right] \tag{34a}$$

$$\frac{dT_e}{dx} = \frac{2\epsilon}{M_*}\left(\frac{l_*}{\lambda_{ei_*}}\right)\frac{T_e^{-3/2}}{AV^2}(T_o - 2T_e) - \frac{2}{3}T_e\left[\frac{GV + M_*^2V^2\frac{1}{A}\frac{dA}{dx}}{M_*^2V^2 - T_o}\right] \tag{35a}$$

This system of equations has a critical point when,

$$M_*^2V^2 - T_o = 0$$

or when

$$M^2 = \frac{m_i\bar{V}^2}{\frac{5}{3}k\bar{T}_o} = 1 \tag{36}$$

This is the ordinary gasdynamic situation where the critical point occurs at $M = 1$. Similar to the case for $q \neq 0$, the critical point must occur at the nozzle throat in order for the flow to pass smoothly through this point. Therefore, since M_* is the throat Mach number, $M_* = 1$ and Eqs. (33a) to (35a) become, respectively

$$T_o = \frac{1}{3}(1 - V^2) + 1 \quad (33b)$$

$$\frac{dV}{dx} = V\left[\frac{GV + \frac{T_o}{A}\frac{dA}{dx}}{V^2 - T_o}\right] \quad (34b)$$

$$\frac{dT_e}{dx} = 2\epsilon\left(\frac{l_*}{\lambda_{ei_*}}\right)\frac{T_e^{-3/2}}{AV^2}(T_o - 2T_e) - \frac{2}{3}T_e\left[\frac{GV + V^2\frac{1}{A}\frac{dA}{dx}}{V^2 - T_o}\right] \quad (35b)$$

Equation (33b) yields the maximum velocity ratio that can be attained in an adiabatic flow for a monatomic gas. To reach the maximum velocity, the flow is expanded until $T_o \rightarrow 0$. Therefore, from Eq. (33b)

$$(V_{max})_{ad} = 2 \quad (37)$$

For isentropic flow, $q = 0$, and no current may flow in the plasma. The magnetic field term G results from the $\vec{J} \times \vec{B}$ term in the plasma momentum equation. Therefore, since $J \rightarrow 0$, $G \rightarrow 0$ as well. Under these conditions the usual one-dimensional isentropic flow equations for a monatomic gas ($\gamma = 5/3$)) are obtained.[4] The temperature and velocity are related by Eq. (33b) and the flow Mach number is given as a function of the area ratio A by the following expression.

$$M^4 + 6M^2 - 16\,AM + 9 = 0 \quad (38)$$

Since $M_* = 1$,

$$V = M\sqrt{T_o} \quad (39)$$

Using Eq. (39) in (33b) we obtain T_o as a function of M.

$$T_o = \frac{4}{3 + M^2} \quad (40)$$

Equation (38) can be solved for M(A) and then used in Eqs. (39) and (40) to obtain V and T_o as functions of the area ratio A.

Results for the adiabatic (eqs. (33b) to (35b)) and isentropic (eqs. (38) to (40)) cases will be compared to the complete solution in the following section.

Results

Equations (16) to (19) were integrated using the Runge-Kutta method. The integration was started near the throat of the nozzle (x = 0). Since the throat is at the critical point it is not possible to begin the integration at exactly x = 0. However, it was found that different starting points very close to x = 0 did not greatly effect the solution obtained. For all the cases presented here the following starting values were used; $x_* = 0.001$, $V_* = 1.0001$, and $T_{o*} = 1.0$. An iterative process was necessary to establish the proper value of the initial electron temperature T_{e*}. The critical heat flux q_*, and Mach number M_*, were calculated using Eqs. (25b) and (26), respectively. Solutions were obtained for various values of the parameters β_* and (l_*/λ_{ei_*}) in argon ($\epsilon = 3.71\times10^{-3}$).

In Fig. 2 the complete solution is compared to the adiabatic and isentropic solutions for $A_s = 0.02$, p = 2, $\beta_* = 5$, and $(l_*/\lambda_{ei_*}) = 5$. The velocity and temperature profiles for the adiabatic and isentropic cases are nearly the same. That is why they are shown together in Fig. 2. There are two significant things to notice from these results. First of all, velocities higher than the adiabatic limit can be attained when $q \neq 0$. Secondly, the electron temperature, and likewise the total temperature, remains relatively high throughout the nozzle. The cause of this temperature result is the large thermal conductivity for the electrons.

Changing β_* while keeping the other parameters constant results in only a slight change in the velocity and heat flux profiles. Increasing β_* produces a small increase in the velocity. The temperature profiles show a greater change with β_*. Figure 3 illustrates this effect. Larger values of β_* result in higher temperatures throughout the nozzle. Such a result is expected since large values of β_* mean that the thermal energy density is large compared to the magnetic energy density.

The effect of the collision parameter l_*/λ_{ei_*}, is more pronounced than β_*. Increasing the value of l_*/λ_{ei_*} produces smaller values for V, T_o, T_e, and q. Figure 4 shows this effect on V, and T_o. Large values of l_*/λ_{ei_*} imply a more collisional flow. As the flow becomes more collisional the electron thermal conductivity is decreased. Therefore, the flow approaches the adiabatic limit and the resulting lower values for V, T_o, T_e, and q.

In Ref. 5 the plasma properties were measured in the magnetic nozzle of a low power MPD arc. It is difficult to compare these experimental results with the theory since the values of β_* and l_*/λ_{ei_*} appropriate to the experiment are not known. However, there is qualitative agreement between the theory and experiment for the electron temperature. The experimental electron temperature decreases much slower than that in an adiabatic expansion which is in agreement with the theory.

It has already been pointed out that the asymptotic behavior of the heat flux equation is not physically meaningful. This raises two questions. Is the heat flux equation correct? If it is incorrect, how are the results effected. The first question can not be answered without experimentally testing the electron heat flux equation used in this analysis (eq. (A34)). However, a qualitative answer can be given for the second question. As long as electron heat flux, q, exists, no matter what equation we use to describe it, the conclusions about higher temperatures and velocities than a corresponding adiabatic flow ($q = 0$) will still be valid. This is so because the presence of heat flux gives the electrons another mechanism for transporting their energy down the nozzle and therefore producing higher temperatures. This additional transport of thermal energy down the nozzle eventually ends up being converted into directed energy ($V^2/2$). As a result higher velocities than the adiabatic case ($q = 0$) will always be attained.

Conclusion

Two significant results were obtained from the quasi-one dimensional analysis of a magnetic nozzle. First of all, higher velocities than the adiabatic limit can be attained in the nozzle. Secondly, the electron temperature and therefore the total temperature remains high throughout the nozzle. Both of these results occur because of the inclusion of the electron thermal conductivity. The experimental electron temperature measurements of Ref. 5 indicate that the temperature remains high throughout the nozzle. This result agrees qualitatively with the theoretical result.

Appendix A

Quasi-One Dimensional Equations

The steady-state quasi-one dimensional equations are derived by integrating the plasma equations over a control volume (which is a segment of the nozzle) such as that shown in Fig. 1. For example, consider the steady-state continuity equation for the ions or electrons.

$$\frac{\partial}{\partial x_i}(n_s u_{s_i}) = 0 \qquad \text{(A1)}$$

The subscript s denotes either electrons or ions. Also, cartesian tensor notation is used.

Integrating Eq. (A1) over the control volume yields

$$\int_{\Upsilon} \frac{\partial}{\partial x_i}(n_s u_{s_i}) d\Upsilon = 0$$

Now apply the divergence theorem and assume $n_s u_{s_i}$ is uniform across the nozzle.

$$\oint_S n_s u_{s_i} h_i dS = A n_s V_s - (A n_s V_s)_{x=0} = 0 \quad \text{(A2)}$$

In obtaining Eq. (A2) the condition $u_{s_i} h_i = 0$ on the nozzle's lateral surface, was used, where h is the unit vector perpendicular to the control volume surface S. The quantity V_s is the x component of the velocity of species $s (V_s > 0)$.

Equation (A2) holds for both the ions and electrons. Using Eq. (A2) for the ions and electrons the following expression for the current density in the x direction is obtained.

$$J_x = e(n_i V_i - n_e V_e) = \frac{1}{A}(A n_i V_i - A n_e V_e)_{x=0} \text{(A3)}$$

where e is the magnitude of the electron charge.

Assuming $J_x = 0$ and charge neutrality ($n_i = n_e = n$), Eq. (A3) yields,

$$V_e = V_i = V \qquad \text{(A4)}$$

where V is the x component of the plasma mean velocity. Also, Eq. (A2) for the ions and electrons together with the result (eq. (A4)) can be used to obtain the continuity equation for the plasma.

$$A\rho V = (A\rho V)_{x=0} \qquad \text{(A5)}$$

where

$$\rho = \rho_e + \rho_i = m_i n_i + m_e n_e = n(m_i + m_e)$$

and since $m_e \ll m_i$,

$$\rho \approx \rho_i = m_i n \qquad \text{(A6)}$$

The steady-state plasma momentum equation is, (in MKS units) neglecting the gravitational term, the following (eq. 17.14 of ref. 6).

$$\rho U_i \frac{\partial U_j}{\partial x_i} + \frac{\partial P_{ij}}{\partial x_i} + \frac{\partial p}{\partial x_j} - J_l B_m \epsilon_{lmj} - \sigma E_j = 0 \qquad \text{(A7)}$$

Where, U_i is the plasma velocity, P_{ij} is the stress tensor, p is the scalar pressure, J_l is the total current density, B_m is the magnetic field strength, ϵ_{lmj} is the alternating tensor (or Levi-cevita density), σ is the net charge density, and E_j is the electric field. Since $n_e \approx n_i$, $\sigma = 0$. Also, neglecting the stress tensor the following is obtained from Eq. (A7).

$$\rho U_i \frac{\partial U_j}{\partial x_i} + \frac{\partial p}{\partial x_j} = J_l B_m \epsilon_{lmj} \qquad \text{(A8)}$$

Using the steady state plasma continuity equation (eq. 17.9 of ref. 6)

$$\frac{\partial(\rho U_i)}{\partial x_i} = 0 \qquad \text{(A9)}$$

in the first term of Eq. (A8) and Maxwell's equation,

$$\frac{\partial B_j}{\partial x_i}\epsilon_{ijl} = \mu_o J_l \qquad \text{(A10)}$$

for J_l, the following result is obtained,

$$\frac{\partial}{\partial x_i}\left[\rho U_i U_j - \frac{1}{\mu_o} B_i B_j\right] + \frac{\partial}{\partial x_j}\left[p + \frac{B^2}{2\mu_o}\right] = 0 \quad \text{(A11)}$$

where the fact that $\partial B_i/\partial x_i = 0$ has been used.

If Eq. (A11) is now integrated over the control volume shown in Fig. 1 and the divergence theorem applied, the following is obtained.

$$\oint_S \left[\rho U_i U_j - \frac{1}{\mu_o} B_i B_j\right] h_i dS + \oint_S \left[p + \frac{B^2}{2\mu_o}\right] h_j dS = 0$$

Since the nozzle surface S' is assumed to be determined by the magnetic field, $B_i h_i = 0$ and $U_i h_i = 0$ on S'. Therefore,

$$\int_0^{A(x)} \left[\rho V U_j - \frac{B_x B_j}{\mu_o} + \left(p + \frac{B^2}{2\mu_o}\right)\delta_{xj}\right] dA'$$

$$- \int_0^{A_o = A(x=0)} \left[\rho V U_j - \frac{B_x B_j}{\mu_o} + \left(p + \frac{B^2}{2\mu_o}\right)\delta_{xj}\right] dA'$$

$$+ \oint_{S'} \left(p + \frac{B^2}{2\mu_o}\right) h_j dS' = 0 \qquad (A12)$$

The quantity $\delta_{xj} = 0$ for $x \neq j$ and $\delta_{xj} = 1$ if $x = j$.

$$h_x dS' = -\frac{dA}{dx'} dx' \qquad (A13)$$

Using Eq. (A13) and the assumption that the plasma properties are uniform across the nozzle gives the following for the x component of Eq. (A12)

$$A\rho V^2 + pA - \left[A\rho V^2 + pA\right]_{x=0} - \int_0^x p\left(\frac{dA}{dx'}\right) dx'$$

$$+ \frac{1}{2\mu_o} \int_0^{A(x)} (B_r^2 - B_x^2) dA'$$

$$- \frac{1}{2\mu_o} \int_0^{A_o} (B_r^2 - B_x^2) dA_o'$$

where the fact that $B_\theta = 0$ has been used.

Now take the x derivative, using Leibnitz's rule to differentiate the integral terms.

$$\rho V \frac{dV}{dx} + \frac{dp}{dx} + \frac{1}{2\mu_o A} \int_0^{A(x)} \frac{\partial B_r^2}{\partial x} dA'$$

$$- \frac{1}{2\mu_o A} \left[\int_0^{A(x)} \frac{\partial B_x^2}{\partial x} dA' + 2B_x^2\Big|_{r=R} \frac{dA}{dx}\right] = 0 \qquad (A14)$$

where the result $A\rho V$ = constant (eq. (A5)) has been used. The last term in this expression can be rewritten using Maxwell's equation

$$\nabla \cdot \vec{B} = \frac{\partial B_i}{\partial x_i} = 0 \qquad (A15)$$

Integrating Eq. (A15) over the volume using the divergence theorem yields the following

$$\int_\gamma \frac{\partial B_i}{\partial x_i} d\gamma = \oint B_i h_i dS = 0$$

Since $B_i h_i = 0$ on the nozzle surface this becomes

$$\int_0^{A(x)} B_x dA' - \int_0^{A_o} B_x dA_o' = 0$$

Taking the x derivative yields

$$B_x\Big|_{r=R} \frac{dA}{dx} = -\int_0^{A(x)} \frac{\partial B_x}{\partial x} dA' \qquad (A16)$$

Substituting Eq. (A16) in (A14) results in the plasma momentum equation.

$$\rho V \frac{dV}{dx} + \frac{dp}{dx} + \frac{1}{2\mu_o A}\left[\int_0^{A(x)} \frac{\partial B_r^2}{\partial x} dA'\right.$$

$$\left. - 2\int_0^{A(x)} \left(B_x - B_x\Big|_{r=R}\right) \frac{\partial B_x}{\partial x} dA'\right] = 0 \qquad (A17)$$

It should be noted that the plasma properties have been assumed constant across the nozzle in obtaining Eq. (A17). However, the magnetic field has not been assumed uniform across the nozzle. If we assume that B_x is uniform so that $B_x = B_x\big|_{r=R}$, then the B_x term in Eq. (A17) vanishes.

The steady-state plasma energy equation is the following (eq. 17.20, ref. 6)

$$U_i \frac{\partial\left(\frac{3}{2} p\right)}{\partial x_i} + \frac{5}{2} p \frac{\partial U_i}{\partial x_i} + P_{ij} \frac{\partial U_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}$$

$$= J_i(E_i + U_l B_m \epsilon_{lmi}) \qquad (A18)$$

Where q_i is the total heat flux defined with respect to the plasma flow velocity. If the momentum equation (eq. (A7)) is multiplied by U_j and the result added to Eq. (A18), the following is obtained.

$$\rho U_i \frac{\partial\left(\frac{U^2}{2}\right)}{\partial x_i} + \frac{5}{2} \frac{\partial}{\partial x_i} (p U_i) + \frac{\partial}{\partial x_i} (P_{ij} U_j)$$

$$+ \frac{\partial q_i}{\partial x_i} = J_i E_i \qquad (A19)$$

Now use the steady-state continuity equation,

$$\frac{\partial(\rho U_i)}{\partial x_i} = 0 \qquad (A20)$$

to rewrite the first term in Eq. (A19),

$$\frac{\partial}{\partial x_i}\left[\rho U_i\left(\frac{U^2}{2} + \frac{5}{2} \frac{p}{\rho}\right)\right] + \frac{\partial}{\partial x_i} (P_{ij} U_j) + \frac{\partial q_i}{\partial x_i} = J_i E_i \qquad (A21)$$

Using Maxwell's equation,

$$(\nabla \times \vec{B})_i = \frac{\partial B_m}{\partial x_l} \epsilon_{lmi} = \mu_o J_i \qquad \text{(A22)}$$

the term $J_i E_i$ can be written as

$$J_i E_i = - \frac{\partial}{\partial x_l}\left(\frac{E_i B_m}{\mu_o}\right) \epsilon_{lim}$$

since $\nabla \times \vec{E} = 0$ in steady state conditions. Therefore, Eq. (A21) becomes the following.

$$\frac{\partial}{\partial x_i}\left[\rho U_i\left(\frac{U^2}{2} + \frac{5}{2}\frac{p}{\rho}\right)\right] + \frac{\partial(P_{ij}U_j)}{\partial x_i} + \frac{\partial q_i}{\partial x_i}$$

$$+ \frac{\partial}{\partial x_l}\left(\frac{E_i B_m}{\mu_o}\right)\epsilon_{lim} = 0 \qquad \text{(A23)}$$

If we now neglect the viscous term and the Poynting vector term $E_i B_m$ and integrate the result over the control volume in Fig. 1 using the divergence theorem, the following is obtained,

$$\int_0^{A(x)} \left[\rho V\left(\frac{U^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + q_x\right] dA' -$$

$$- \int_0^{A_o} \left[\rho V\left(\frac{U^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + q_x\right] dA'_o$$

$$+ \oint_{S'} \left[\rho U_i\left(\frac{V^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + q_i\right] h_i dS' = 0 \qquad \text{(A24)}$$

Now assume the plasma properties are uniform across the nozzle and that q_r and q_θ are negligible. Then, using the fact that $V_i h_i = 0$ on S' Eq. (A24) becomes

$$\left[\rho V\left(\frac{V^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + q_x\right] A - \left[\rho V\left(\frac{V^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + q_x\right] A_o$$

$$- \int_0^x q_x \frac{dA}{dx'} dx' = 0$$

where U_θ and U_r have been assumed small compared to $U_x = V$ (i.e., $U^2 \approx V^2$). Now take the x derivative and use the continuity equation (eq. (A5)).

$$\rho V \frac{d}{dx}\left(\frac{V^2}{2} + \frac{5}{2}\frac{p}{\rho}\right) + \frac{dq_x}{dx} = 0 \qquad \text{(A25)}$$

Equation (A25) is the steady-state plasma energy equation. In obtaining Eq. (A25) the electromagnetic energy transfer given by the Poynting vector has been ignored.

To complete the set of quasi-one dimensional equations we need the steady state electron energy equation and the heat flux equation. The heat flux q_x that appears in Eq. (A25) is the sum of the electron and ion heat fluxes. However, since q_{ion}/q_e is of the order $(m_e/m_{ion})^{1/2}$ (ref. 6, p. 62, p. 155), the ion heat flux q_{ion}, will be neglected. Therefore, $q_x = q_{e_x}$.

The steady-state electron energy equation is the following (ref. 6, eq. (12.4a))

$$u_{e_i} \frac{\partial}{\partial x_i}\left(\frac{3}{2} p_e\right) + \frac{5}{2} p_e \frac{\partial u_{e_i}}{\partial x_i} + P_{e_{ij}} \frac{\partial u_{e_j}}{\partial x_i} + \frac{\partial q_{e_i}}{\partial x_i}$$

$$= \frac{3 m_e m_{ion}}{(m_e + m_{ion})^2} n_e \nu_{e_i} k(T_i - T_e) \qquad \text{(A26)}$$

The x component of the steady-state electron heat flux equation is also obtained from Ref. 6 (eq. 12.9)

$$u_{e_i} \frac{\partial q_{e_x}}{\partial x_i} + \frac{7}{5} q_{e_x} \frac{\partial u_{e_i}}{\partial x_i} + \frac{7}{5} q_{e_i} \frac{\partial u_{e_x}}{\partial x_i} + \frac{2}{5} q_{e_i} \frac{\partial u_{e_i}}{\partial x}$$

$$+ p_e \frac{\partial}{\partial x}\left(\frac{5}{2}\frac{kT_e}{m_e}\right) + P_{e_{xi}} \frac{\partial}{\partial x_i}\left(\frac{5}{2}\frac{kT_e}{m_e}\right) + p_e \frac{\partial}{\partial x_i}\left(\frac{P_{e_{ix}}}{m_e n_e}\right)$$

$$+ \frac{e}{m_e} q_{e_l} B_m \epsilon_{lmx} = -1.866\, \nu_{ei} q_{e_x} \qquad \text{(A27)}$$

In obtaining the right hand sides of Eqs. (A26) and (A27) results of Ref. 6 given on pages 63 and 150-152 were used (the diffusion velocities are assumed to be small). The quantity ν_{ei} is the electron-ion collision frequency for transfer or momentum given in MKS units by the following expression (ref. 7, p. 252),

$$\nu_{ei} = 3.62\text{x}10^{-6}\, n_i \frac{\ln \Lambda_{ei}}{T_e^{3/2}} \qquad \text{(A28)}$$

$$\Lambda_{ei} = 1.23\text{x}10^{7} \sqrt{\frac{T_e^3}{n_e}} \qquad \text{(A29)}$$

It should be noted that the heat flux vector used in Eq. (A25) is defined in terms of the mean gas velocity U_i while the heat flux vector used in Eqs. (A26) and (A27) is defined in terms of the species mean velocity u_{e_i}. However, in the problem being considered $u_{e_x} = u_{ion_x} = u_x$ so that the q_x of Eq. (A25) is equivalent to q_{e_x} of Eqs. (A26) and (A27).

Now convert Eq. (A26) into the quasi-one dimensional form. First of all, neglect the viscous term, $P_{e_{ij}}$. Also since $(m_e/m_{ion}) \ll 1$, $m_e m_{ion}/(m_e + m_{ion})^2 \approx m_e/m_{ion}$. Therefore, Eq. (A26) can be written as the following.

$$\frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{e_i}p_e\right) + \frac{\partial q_{e_i}}{\partial x_i} - u_{e_i}\frac{\partial p_e}{\partial x_i}$$

$$= -3\epsilon^2 n_e \nu_{ei} k(T_e - T_i) \quad (A30)$$

Where, $\epsilon^2 = (m_e/m_{ion})$. Integrating Eq. (A30) over the control volume in Fig. 1 and using the divergence theorem results in the following.

$$\int_0^{A(x)} \left[\frac{5}{2} p_e V + q_x\right] dA' - \int_0^{A_o} \left[\frac{5}{2} p_e V + q_x\right]_{x=0} dA_o'$$

$$+ \oint_{S'} \left[\frac{5}{2} p_e u_{e_i} + q_{e_i}\right] h_i dS'$$

$$= \int_\gamma \left[u_{e_i}\frac{\partial p_e}{\partial x_i} - 3\epsilon^2 n_e \nu_{ei} k(T_e - T_i)\right] d\gamma$$

Using the relations $d\gamma = A\,dx'$, $h_x dS' = -(dA/dx')dx'$, $u_{e_i}h_i = 0$ on the surface S' and assuming that q_r and q_θ are negligible the following is obtained

$$\int_0^{A(x)} \left[\frac{5}{2} p_e V + q_x\right] dA' - \int_0^{A_o} \left[\frac{5}{2} p_e V + q_x\right] dA_o'$$

$$- \int_0^x q_x \frac{dA}{dx'} dx'$$

$$= \int_0^x \left[u_{e_i}\frac{\partial p_e}{\partial x_i'} - 3\epsilon^2 n_e \nu_{e_i} k(T_e - T_i)\right] A\,dx'$$

Now use the assumption that the plasma properties are uniform across the nozzle.

$$A\left[\frac{5}{2} p_e V + q_x\right] - A_o\frac{5}{2}\left[p_e V + q_x\right] - \int_0^x q_x \frac{dA}{dx'} dx'$$

$$= \int_0^x \left[V\frac{dp_e}{dx'} - 3\epsilon^2 n_e \nu_{e_i} k(T_e - T_i)\right] A\,dx'$$

where u_{e_r} and u_{e_θ} have been neglected in comparison to $u_{e_x} = V$. Taking the x derivative and using $p_e = n_e kT_e$ and Eq. (A2) yields the following.

$$n_e V \frac{d}{dx}\left(\frac{3}{2} kT_e\right) + \frac{dq_x}{dx} - VkT_e\frac{dn_e}{dx} = -3\epsilon^2 n_e \nu_{ei} k(T_e - T_i)$$

Using the electron continuity equation to replace dn_e/dx results in the following.

$$\frac{kT_e}{V}\frac{dV}{dx} + \frac{d}{dx}\left(\frac{3}{2}kT_e\right) + \frac{1}{nV}\frac{dq_x}{dx}$$

$$= -\frac{3\epsilon^2}{V}\nu_{e_i}k(T_e - T_i) - \frac{kT_e}{A}\frac{dA}{dx} \quad (A31)$$

Equation (A31) is the quasi-one dimensional electron energy equation.

Now consider the quasi-one dimensional form of the electron heat flux equation. Neglect the stress tensor terms in Eq. (A27) and rewrite as follows.

$$\frac{\partial}{\partial x_i}(u_{e_i}q_{e_x}) + \frac{2}{5}\left[q_{e_x}\frac{\partial u_{e_i}}{\partial x_i} + q_{e_i}\frac{\partial u_{e_i}}{\partial x}\right] + \frac{7}{5}q_{e_i}\frac{\partial u_{e_x}}{\partial x_i}$$

$$+ p_e\frac{\partial}{\partial x}\left(\frac{5}{2}\frac{kT_e}{m_e}\right) + \frac{e}{m_e}q_{e_l}B_m\epsilon_{lmx}$$

$$= -1.866\,\nu_{e_i}q_{e_x}$$

Integrating over the control volume in Fig. 1 and applying the divergence theorem yields the following.

$$\int_0^{A(x)} q_{e_x}V\,dA' - \int_0^{A_o}(q_{e_x}V)_{x=0}\,dA_o'$$

$$+ \oint_{S'} q_{e_x}u_{e_i}h_i dS' + \frac{2}{5}\int_\gamma \left[q_{e_x}\frac{\partial u_{e_i}}{\partial x_i} + q_{e_i}\frac{\partial u_{e_i}}{\partial x}\right] d\gamma$$

$$+ \frac{7}{5}\int_\gamma q_{e_i}\frac{\partial u_{e_x}}{\partial x_i} d\gamma + \int_\gamma p_e\frac{\partial}{\partial x}\left(\frac{5}{2}\frac{kT_e}{m_e}\right) d\gamma$$

$$+ \frac{e}{m_e}\int_\gamma q_{e_l}B_m\epsilon_{lm_x}d\gamma = -1.866\int_\gamma \nu_{e_i}q_{e_x}d\gamma \quad (A32)$$

Take the x component of Eq. (A32). Using the approximations that q_{er} and q_{e_θ} are negligible and that the plasma properties are uniform across the nozzle yields

$$q_x AV - (q_x AV)_{x=0} + \frac{11}{5}\int_0^x q_x\frac{dV}{dx'}A\,dx'$$

$$+ \int_0^x p_e\frac{d}{dx'}\left(\frac{5}{2}\frac{kT_e}{m_e}\right)A\,dx' = -1.866\int_0^x \nu_{ei}q_x A\,dx'$$

$$(A33)$$

where the subscript "e" on q_x has been dropped since $q_{e_x} \approx q_x$. In obtaining Eq. (A33) the results $d\gamma = A\,dx'$ and $u_{e_i}h_i = 0$ on S' have been used. Taking the x derivative results in the quasi-one dimensional electron heat flux equation.

$$V \frac{dq_x}{dx} + q_x \left[\frac{V}{A} \frac{dA}{dx} + \frac{16}{5} \frac{dV}{dx} \right] + \frac{5}{2} \frac{k^2 n T_e}{m_e} \frac{dT_e}{dx}$$

$$= -1.866\ \nu_{ei} q_x \qquad (A34)$$

The electron pressure, p_e, has been replaced by $n_e kT_e = nkT_e$ in Eq. (A34).

The complete set of quasi-one dimensional equations for ρ, V, T_i, T_e, and q_x are given by Eqs. (A5), (A17), (A25), (A31), and (A34).

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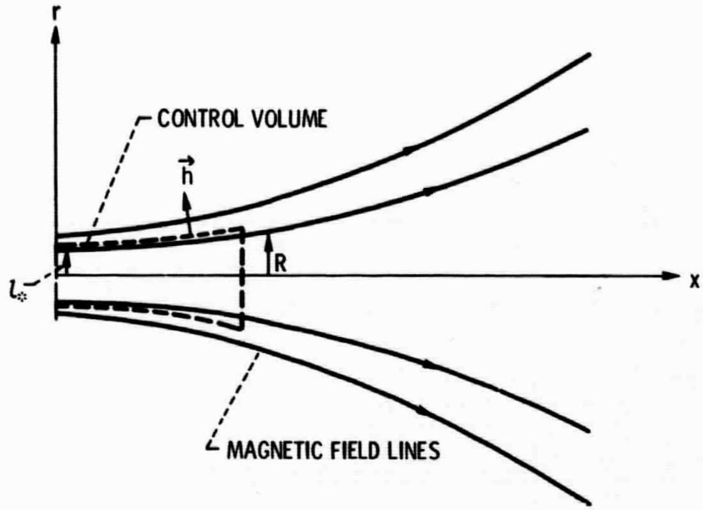
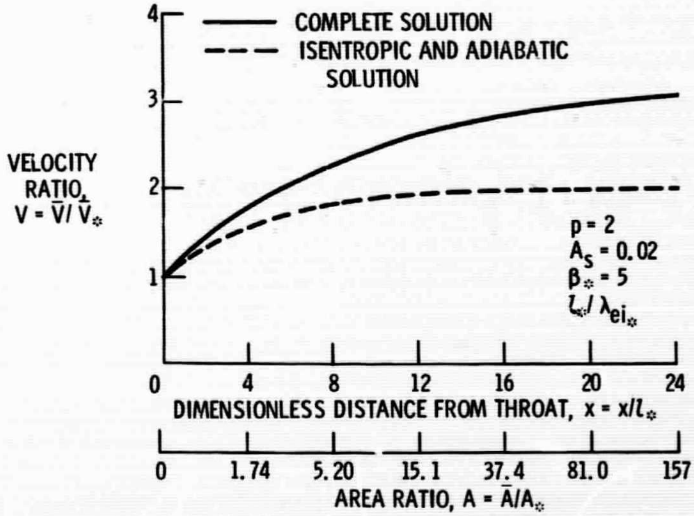


Figure 1. - Magnetic nozzle.



(A) VELOCITY PROFILE.

Figure 2. - Comparison of solution with adiabatic and isentropic solutions. Magnetic nozzle parameters, $p = 2$ and $A_s = 0.02$, ratio kinetic to magnetic pressure $\beta_* = 5$, collision parameter $l_*/\lambda_{ei_*} = 5$.

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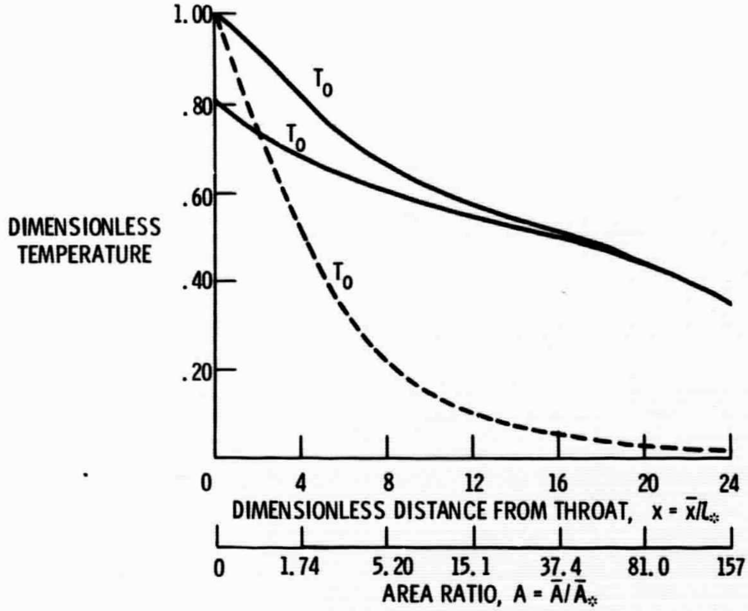
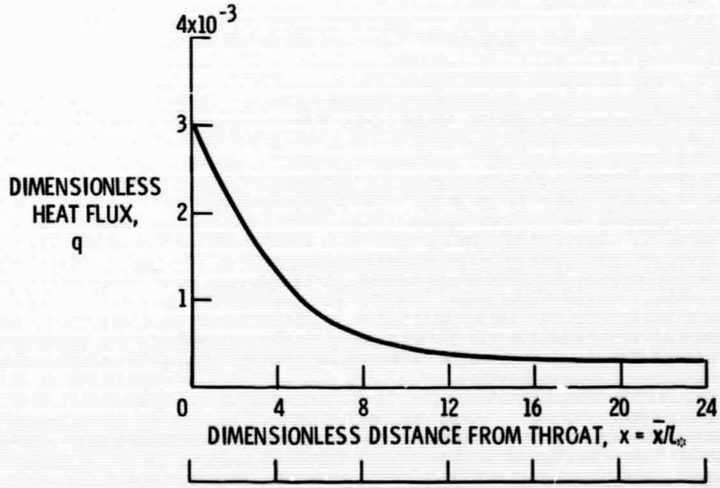


(B) TEMPERATURE PROFILES.

Figure 2. - Continued.



(C) HEAT FLUX PROFILE.

Figure 2. - Concluded.

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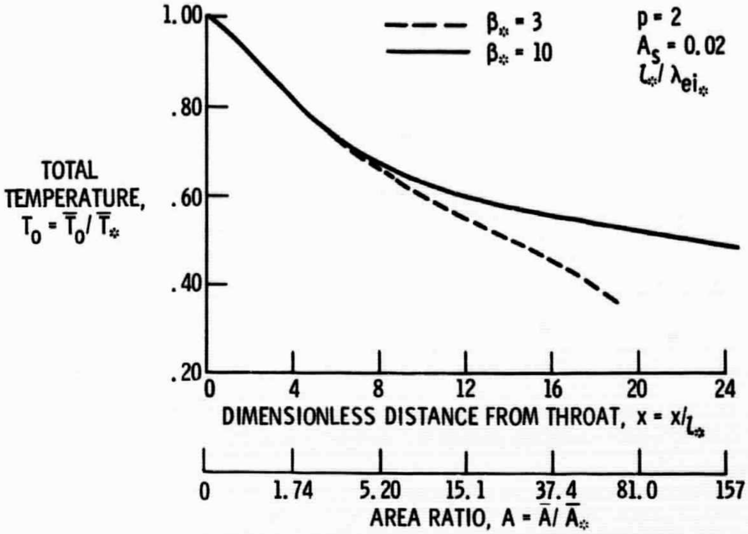
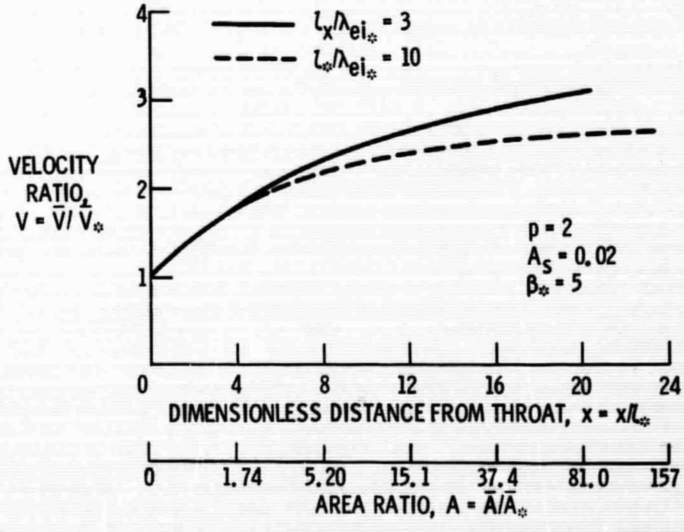
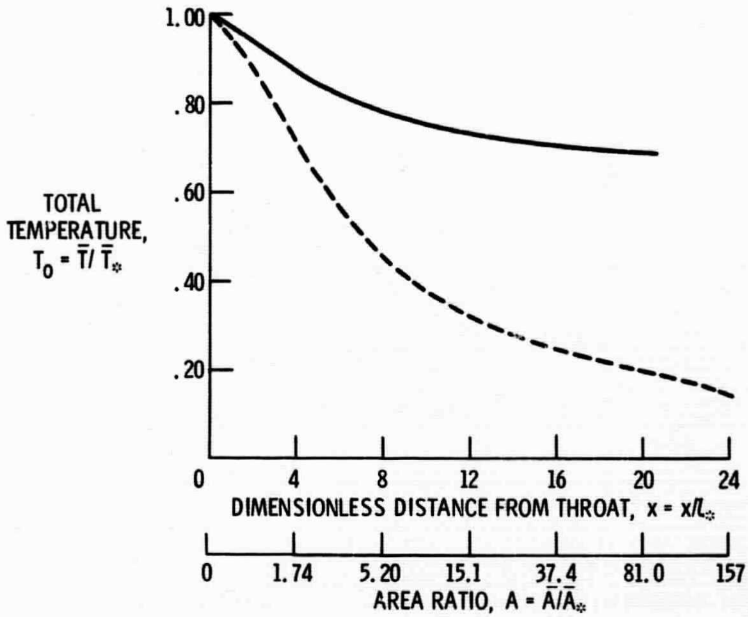


Figure 3. - Effect of kinetic to magnetic pressure ratio, β_*, on total temperature. Magnetic nozzle parameter p = 2 and A_s = 0.02, collision parameter l_*/λ_{ei_*} = 5.



(A) VELOCITY PROFILE.

Figure 4. - Effect of collision parameter on velocity and total temperature. Magnetic nozzle parameter p = 2 and A_s = 0.02, ratio of kinetic to magnetic pressure β_* = 5.



(B) TEMPERATURE PROFILE.

Figure 4. - Concluded.