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GENERAL RELATIVISTIC PRECESSION OF A GYROSCOPE IN AN INCLINED ORBIT

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The general relativistic precession of a gyroscope in an inclined orbit is calculated. The magnitude of the precession is found to be proportional to the cosine of the inclination angle, apart from periodic terms. It is shown that the geodetic precession cannot be separated from the motional precession no matter what initial orientation is chosen for the gyroscope.

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SUMMARY

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INTRODUCTION

A perfectly spherical gyroscope in orbit around the earth will precess as a result of two relativistic effects. The first and larger effect results from the motion of the gyroscope along a geodesic in a 4-space which is not flat; this is called the geodetic precession. The second effect is associated with the rotation of the earth and is called the motional or Lense-Thirring precession. A polar orbit was originally chosen for the Stanford Gyroscope Relativity Experiment because this orbit will allow these two effects to be separated. Recently, it has been suggested that a preliminary test flight of the Gyro Experiment be performed on Skylab II, for which an inclined orbit is planned. In this paper we calculate the expected gyroscopic precession for such an inclined orbit and show that one cannot separate the geodetic and motional precession.

THE GEODETIC PRECESSION

First, we will consider the geodetic precession, which is generally about two orders of magnitude larger than the motional precession. To obtain this precession we must solve Schiff's equation [1] for the orbit under consideration. This equation will be written as

$$\frac{d\vec{S}}{dt} = \vec{S} \times \vec{L} \quad , \tag{1}$$

$$\vec{L} = -\frac{3}{2}\frac{GM}{c^2 r^3}(\vec{r} \times \vec{v}) \quad , \tag{2}$$

which is a first order equation in the gyroscope spin vector $\vec{S}$. $2GM/c^2$ is the Schwarzschild radius of the earth, and $\vec{r}$ and $\vec{v}$ are the orbital position and velocity vectors. $\vec{L}$ has been defined to be antiparallel to the orbital angular momentum vector.

In the simple case of a circular orbit for which $\vec{L}$ is constant, the vector $\vec{S}$ simply precesses about $\vec{L}$ with constant angular velocity $\omega = |\vec{L}|$ and $\vec{S} \cdot \vec{L}$ remains constant. For the more general case in which $\vec{L}$ is not constant, one must obtain $\vec{r}$ and $\vec{v}$, thus $\vec{L}$, as functions of time from an integration of the orbit equations and then obtain $\vec{S}$ as a function of time by integrating equation (1).

For a circular inclined orbit about the earth, the orbital plane will not remain fixed in space. Because the earth is not perfectly spherical, the orbital plane will precess about the earth's axis in such a way that the orbit inclination angle and radius remain nearly the same. So to good approximation, one can consider that the orbital angular momentum, and thus $\vec{L}$, will remain constant in magnitude and precess about the earth's axis with an angular velocity Ω. Classical perturbation theory [2, equation 11.15.6] gives Ω as

$$\Omega = \frac{3J_2}{2r^2}(\cos i)\ \omega_o \quad , \tag{3}$$

where i is the orbital inclination, ω_o is the orbital angular velocity, and J_2 is the earth's mass quadrupole moment. $\vec{L}$ can then be written in the form

$$\begin{aligned} L_x &= L_1 \sin \Omega t \\ L_y &= L_1 \cos \Omega t \\ L_z &= -L_2 \quad , \end{aligned} \tag{4}$$

where L_1 and L_2 are positive constants. The z-axis has been chosen to correspond to the earth's axis while the x-y plane coincides with the equatorial plane. $\vec{L}$ has been given a negative z-component so that the orbital angular momentum vector has a positive z-component. Initially $\vec{L}$ is in the y-z plane. It precesses in a direction consistent with the fact that the nodes regress for an oblate body.

When $\vec{L}$ is given by equation (4), the following analytic solution to equation (1) can be obtained.

$$S_x = -C_1 \sin\phi \sin\Omega t - C_2 [\sin(\omega t + \psi) \cos\Omega t - \cos(\omega t + \psi) \sin\Omega t \cos\phi]$$

$$S_y = -C_1 \sin\phi \cos\Omega t + C_2 [\sin(\omega t + \psi) \sin\Omega t + \cos(\omega t + \psi) \cos\Omega t \cos\phi]$$

$$S_z = C_2 \sin\phi \cos(\omega t + \psi) + C_1 \cos\phi \quad , \tag{5}$$

where

$$\omega = \left[L^2 + \Omega^2 + 2L_2\Omega\right]^{1/2} \quad , \quad L = \left[L_1^2 + L_2^2\right]^{1/2} \quad ,$$

$$\sin\phi = L_1/\omega \quad , \tag{6}$$

$$\cos\phi = (L_2 + \Omega)/\omega \quad ,$$

and the constants C_1, C_2, and ψ are determined by the initial condition on $\vec{S}$. We will now demonstrate explicitly that this is a solution to equation (1). By differentiating equations (5), we obtain

$$\frac{d\vec{S}}{dt} = \begin{bmatrix} \frac{dS_x}{dt} \\ \frac{dS_y}{dt} \\ \frac{dS_z}{dt} \end{bmatrix}$$

$$= \begin{bmatrix} -\Omega C_1 \sin\phi \cos\Omega t + C_2 [(\Omega - \omega\cos\phi) \sin(\omega t + \psi) \sin\Omega t \\ \qquad - (\omega - \Omega\cos\phi) \cos(\omega t + \psi) \cos\Omega t] \\ \Omega C_1 \sin\phi \sin\Omega t + C_2 [(\Omega - \omega\cos\phi) \sin(\omega t + \psi) \cos\Omega t \\ \qquad + (\omega - \Omega\cos\phi) \cos(\omega t + \psi) \sin\Omega t] \\ - \omega C_2 \sin\phi \sin(\omega t + \psi) \end{bmatrix} .$$

Using equations (4), we have

$$(\vec{S} \times \vec{L}) = \begin{bmatrix} S_y L_z - S_z L_y \\ \\ S_z L_x - S_x L_z \\ \\ S_x L_y - S_y L_x \end{bmatrix} = \begin{bmatrix} S_y[-L_2] - S_z[L_1 \cos \Omega t] \\ \\ S_z[L_1 \sin \Omega t] - S_x[-L_2] \\ \\ S_x[L_1 \cos \Omega t] - S_y[L_1 \sin \Omega t] \end{bmatrix} ,$$

and now substituting from equation (5), this becomes

$$\begin{bmatrix} C_1 [L_2 \sin \phi - L_1 \cos \phi] \cos \Omega t - C_2 L_2 \sin (\omega t + \psi) \sin \Omega t \\ -C_2 [L_2 \cos \phi + L_1 \sin \phi] \cos (\omega t + \psi) \cos \Omega t \\ \\ -C_1 [L_2 \sin \phi - L_1 \cos \phi] \sin \Omega t - C_2 L_2 \sin (\omega t + \psi) \cos \Omega t \\ +C_2 [L_2 \cos \phi + L_1 \sin \phi] \cos (\omega t + \psi) \sin \Omega t \\ \\ -L_1 C_2 \sin (\omega t + \psi) [\cos^2 \Omega t + \sin^2 \Omega t] - L_1 C_2 \cos \phi \cos (\omega t + \psi) \\ [-\sin \Omega t \cos \Omega t + \cos \Omega t \sin \Omega t] \end{bmatrix} .$$

Now from equations (6), we have

$$L_2 \sin \phi - L_1 \cos \phi = L_2 \left(\frac{L_1}{\omega}\right) - L_1 \left(\frac{L_2 + \Omega}{\omega}\right) = -\Omega \left(\frac{L_1}{\omega}\right) = -\Omega \sin \phi$$

$$L_2 \cos \phi + L_1 \sin \phi = L_2 \left(\frac{L_2 + \Omega}{\omega}\right) + L_1 \left(\frac{L_1}{\omega}\right) = \frac{\left(L_1^2 + L_2^2\right)}{\omega} + \Omega \left(\frac{L_2}{\omega}\right)$$

$$= \frac{\left(\omega^2 - \Omega^2 - 2L_2\Omega\right)}{\omega} + \Omega \left(\frac{L_2}{\omega}\right)$$

$$= \omega - \Omega \left(\frac{L_2 + \Omega}{\omega}\right) = \omega - \Omega \cos \phi$$

$$L_2 = -\Omega + (L_2 + \Omega) = -(\Omega - \omega \cos \phi)$$

$$L_1 = \omega \sin \phi \quad ,$$

so that a comparison of the expressions for $d\vec{S}/dt$ and $\vec{S} \times \vec{L}$ shows that they are equal. Hence, equations (5) have been shown to be a solution of equation (1) when $\vec{L}$ is given by equation (4). It can also be shown, using equations (5) that

$$\vec{S} \cdot \vec{S} = S_x^2 + S_y^2 + S_z^2 = C_1^2 + C_2^2$$

so that $\vec{S}$ remains constant in magnitude, as it must.

This solution to equation (1) was obtained by transforming to a rotating coordinate system in which $\vec{L}$ is at rest. In this system, $\vec{S}$ precesses about $\vec{L}$ and one can write a solution to equation (1) in the rotating system immediately. When this solution is transformed to the nonrotating system, equations (5) are obtained. The problem is mathematically identical to the motion of a magnetic dipole in a magnetic field with a constant z-component and a rotating component in the x-y plane. This configuration is often considered in NMR work.

Using standard identities for the products of two trigonometric functions, equations (5) can be written in the form

$$S_x = -C_1 \sin\phi \sin\Omega t - C_2 \frac{(1 + \cos\phi)}{2} \sin[(\omega - \Omega)t + \psi]$$

$$-C_2 \frac{(1 - \cos\phi)}{2} \sin[(\omega + \Omega)t + \psi]$$

$$S_y = -C_1 \sin\phi \cos\Omega t + C_2 \frac{(1 + \cos\phi)}{2} \cos[(\omega - \Omega)t + \psi]$$

$$-C_2 \frac{(1 - \cos\phi)}{2} \cos[(\omega + \Omega)t + \psi]$$

$$S_z = C_1 \cos\phi + C_2 \sin\phi \cos(\omega t + \psi)$$

so that $\vec{S}$ consists of terms with frequency $\omega - \Omega$, Ω, ω, and $\omega + \Omega$. We can simplify this expression by using the fact that $L \ll \Omega$; that is

$$L = \frac{3GM}{2c^2 r} \frac{|\vec{r} \times \vec{v}|}{r^2} = \frac{3GM}{2c^2 R} \left(\frac{R}{r}\right) \omega_o$$

$$= [1.05 \times 10^{-9}] \left(\frac{R}{r}\right) \omega_o = 6.6 \times 10^{-9} \left(\frac{R}{r}\right) \frac{\text{rad}}{\text{rev}}$$

where R is the earth's radius. Now since $J_2 \sim 10^{-3} R^2$, we have from equation (3) that $\Omega \sim 10^{-3}\,\omega_o \sim 5$ degrees/day for a near-earth orbit so that

$$\frac{L}{\Omega} \sim \frac{10^{-9}\,\omega_o}{10^{-3}\,\omega_o} = 10^{-6} \quad .$$

Expanding the first of equations (6) to first order in L/Ω, we have

$$\omega = \Omega \left[1 + \frac{2L_2}{\Omega} + \frac{L^2}{\Omega^2}\right]^{1/2} \simeq \Omega \left[1 + \frac{L_2}{\Omega}\right] \;,$$

so that

$$\omega - \Omega \simeq L_2 \tag{7}$$

and

$$\sin\phi \simeq L_1/\Omega \quad , \tag{8}$$

all valid to first order in L/Ω. By expanding ω to second order in L/Ω and using the expression for $\cos\phi$ in equations (6), we find

$$\cos\phi \simeq 1 - \frac{1}{2}\left(\frac{L_1}{\Omega}\right)^2 \quad , \tag{9}$$

so that $(1 - \cos\phi)$ is second order in (L/Ω).

If we let the initial value of $\vec{S}$ be $\vec{S}^{\,o}$, then at time $t = 0$, equations (5) give

$$\begin{aligned} S_x^{\,o} &= -C_2 \sin\psi \\ S_y^{\,o} &= -C_1 \sin\phi + C_2 \cos\psi \cos\phi \\ S_z &= C_1 \cos\phi + C_2 \cos\psi \sin\phi \quad . \end{aligned} \tag{10}$$

These three equations can be solved for C_1, C_2 and ψ to give

$$\psi = \tan^{-1}\left\{ -S_x^{\,o} \Big/ \left[S_y^{\,o} \cos\phi + S_z^{\,o} \sin\phi \right] \right\}$$

$$C_1 = S_z^{\,o} \cos\phi - S_y^{\,o} \sin\phi \tag{11}$$

$$C_2 = \left[\left(S_x^{\,o}\right)^2 + \left(S_y^{\,o}\right)^2 \cos^2\phi + \left(S_z^{\,o}\right)^2 \sin^2\phi + 2\left(S_y^{\,o}\right)\left(S_z^{\,o}\right) \cos\phi \sin\phi \right]^{1/2} ,$$

and a substitution of equations (10) into equations (11) verifies that this is a solution. So that for any initial $\vec{S}^{\,o}$, a solution can be constructed. At any time t, the angle Θ through which the spin vector $\vec{S}$ has precessed can be obtained from the equation

$$\Theta = \cos^{-1}\left[\frac{\vec{S}\cdot\vec{S}^{\,o}}{S^2}\right] ,$$

where $\vec{S}$ is given by equations (5) and $\vec{S}^{\,o}$ by equations (10). This is not a very convenient way to compute Θ. Since Θ is very small, it is easier to use the following method. Define the vector $\vec{B}$ by the equation

$$\vec{S} = \vec{S}^{\,o} + \vec{B} \ . \tag{12}$$

Then, we have

$$\sin\Theta = \frac{|\vec{S}\times\vec{S}^{\,o}|}{S^2} = \frac{|\vec{B}\times\vec{S}^{\,o}|}{S^2} = \frac{\left[(\vec{B}\times\vec{S}^{\,o}) \cdot (\vec{B}\times\vec{S}^{\,o})\right]^{1/2}}{S^2} \ .$$

Using a standard vector identity this becomes

$$\sin\Theta = \left[S^2B^2 - (\vec{S}^{\,o}\cdot\vec{B})^2\right]^{1/2} \Big/ S^2$$

since

$$(S^o)^2 = (S)^2 \ .$$

Now

$$S^2 = (\vec{S}^{\,o} + \vec{B}) \cdot (\vec{S}^{\,o} + \vec{B}) = S^2 + 2\,B \cdot \vec{S}^{\,o} + B^2$$

so that

$$(\vec{S}^{o} \cdot \vec{B}) = -B^2/2 \tag{13}$$

$$\sin \Theta = \frac{1}{S^2}\left[S^2B^2 + \frac{B^4}{4}\right]^{1/2} = \frac{B}{S}\left[1 + \frac{1}{4}\left(\frac{B}{S}\right)^2\right]^{1/2} . \tag{14}$$

A general formula for $\sin \Theta$ can be calculated using equations (5), (10), (12), and (14) but the result is quite complicated. So we will restrict ourselves to three specific choices of $\vec{S}^{o}$ and work out explicit expressions for Θ in these cases only.

1. $S_y^{o} = S$, $S_x^{o} = 0$, $S_z^{o} = 0$; i. e., $\vec{S}$ is initially perpendicular to the earth's axis and in the plane formed initially by $\vec{L}$ and the earth's axis. From equations (10) we have

$$\psi = 0 \quad , \quad C_1 = -S \sin \phi \quad , \quad C_2 = S \cos \phi$$

so that equations (5) become

$$S_x = S\left[\sin^2 \phi \sin \Omega t - \cos \phi \frac{(1 + \cos \phi)}{2} \sin (\omega - \Omega)t - \cos \phi \frac{(1 - \cos \phi)}{2} \sin (\omega + \Omega)t\right]$$

$$S_y = S\left[\sin^2 \phi \cos \Omega t + \cos \phi \frac{(1 + \cos \phi)}{2} \cos (\omega - \Omega)t - \cos \phi \frac{(1 - \cos \phi)}{2} \cos (\omega + \Omega)t\right]$$

$$S_z = S\,[\cos \phi \sin \phi\, (\cos \omega t - 1)] \quad .$$

Now from equations (8) and (9) we note that $\sin^2 \phi$ and $(1 - \cos \phi)$ are second order in L/Ω. Also, we are interested in times of the order of 1 year so that from equation (7)

$$(\omega - \Omega)t \simeq L_2 t \ll 1 \quad , \tag{15}$$

since L $\sim$ 3.5 $\times$ 10^{-5} rad/year. We can then expand the sin function to obtain

$$S_x \simeq -S[L_2 t] \tag{16}$$

so that S_x is linear in t to first order in L/Ω. Also

$$S_z \simeq S\left[\frac{L_1}{\Omega}\right][\cos \omega t - 1] \tag{17}$$

from equations (8) and (9), so S_z oscillates with frequency $\omega \sim \Omega$ to first order. The precession angle Θ can be calculated as follows: Using equations (12) and (13) we have

$$\frac{B}{S} = \left[\frac{-2(\vec{B}\cdot\vec{S}^{\,o})}{S^2}\right]^{1/2} = \sqrt{2}\left[\frac{-B_y S_y^{\,o}}{S^2}\right]^{1/2}$$

$$= \sqrt{2}\left[1 - \sin^2\phi\cos\Omega t - \cos\phi\,\frac{(1+\cos\phi)}{2}\cos(\omega-\Omega)t\right.$$

$$\left. + \cos\phi\,\frac{(1-\cos\phi)}{2}\cos(\omega+\Omega)t\right]^{1/2} .$$

If we now use equations (8), (9), and (15) to expand this to second order in L/Ω and $(\omega - \Omega)t$, we find that B/S $\ll$ 1 and

$$\Theta = \frac{B}{S} = \left\{[L_2 t]^2 + \left(\frac{L_1}{2}\right)^2\left[\frac{3}{2} - 2\cos\Omega t + \frac{1}{2}\cos(\omega+\Omega)t\right]\right\}^{1/2} \tag{18}$$

so Θ is not strictly linear in t but only becomes so for $t \gg \frac{L_1}{L_2}\left(\frac{1}{\Omega}\right) = \frac{L_1}{L_2}\left(\frac{\tau}{2\pi}\right)$ $\simeq$ 10 days, where $\tau \sim$ 60 days is the precession period of Ω.

2. $S_x^{\,o} = S$, $S_y^{\,o} = 0$, $S_z^{\,o} = 0$; i. e., $\vec{S}$ is initially perpendicular to the earth's axis and to the initial value of $\vec{L}$. From equations (10) we have

$$\psi = -\pi/2,\ C_1 = 0,\ C_2 = S$$

$$S_x = S\left[\frac{(1+\cos\phi)}{2}\cos(\omega-\Omega)t + \frac{(1-\cos\phi)}{2}\cos(\omega+\Omega)t\right]$$

$$S_y = S\left[\frac{(1 + \cos\phi)}{2}\sin(\omega - \Omega)t - \frac{(1 - \cos\phi)}{2}\sin(\omega + \Omega)t\right]$$

$$S_z = S\sin\phi\sin\omega t \quad .$$

Again we see that, since $(1 - \cos\phi)$ is second order in L/Ω, we have after expanding $\sin(\omega - \Omega)t$ to first order that

$$S_y = S\,[L_2 t] \tag{19}$$

and S_z has an oscillatory behavior with frequency $\omega \sim \Omega$. The precession angle is calculated in the same way as is case 1:

$$\frac{B}{S} = \left[\frac{-2(\vec{B}\cdot\vec{S}^{o})}{S^2}\right]^{1/2} = \left[\frac{-2\left(B_x S_x^{\,o}\right)}{S^2}\right]^{1/2}$$

$$= 2\left[1 - \frac{(1 + \cos\phi)}{2}\cos(\omega - \Omega)t + \frac{(1 - \cos\phi)}{2}\cos(\omega + \Omega)t\right]^{1/2} .$$

Expanding these terms to second order in L/Ω and $(\omega - \Omega)t$, we find again that $B/S \ll 1$ and

$$\Theta = \frac{B}{S} = \left\{[L_2 t]^2 + \left(\frac{L_1}{\Omega}\right)^2\left[\frac{1 - \cos(\omega + \Omega)t}{2}\right]\right\}^{1/2} \tag{20}$$

and again we have an oscillatory term in the expression for Θ but the detailed time dependence is different from case 1. For $t \gg \frac{L_1}{L_2}\left(\frac{1}{\Omega}\right)$, Θ again becomes linear in t.

3. $S_z^{\,o} = S$, $S_x^{\,o} = 0$, $S_y^{\,o} = 0$; i. e., $\vec{S}$ is initially parallel to the earth's axis. Equations (10) give $\psi = 0$, $C_1 = S\cos\phi$, $C_2 = S\sin\phi$,

$$S_x = -S\left[\sin\phi\cos\phi\sin\Omega t + \sin\phi\,\frac{(1 + \cos\phi)}{2}\sin(\omega - \Omega)t\right.$$
$$\left. + \sin\phi\,\frac{(1 - \cos\phi)}{2}\sin(\omega + \Omega)t\right]$$

$$S_y = -S\left[\sin\phi\cos\phi\cos\Omega t - \sin\phi\frac{(1+\cos\phi)}{2}\cos(\omega-\Omega)t\right.$$

$$\left. + \sin\phi\frac{(1-\cos\phi)}{2}\cos(\omega+\Omega)t\right]$$

$$S_z = S\,[\cos^2\phi + \sin^2\phi\cos\omega t] \quad .$$

The dominant terms in $\vec{S}$ become, to first order in L/Ω,

$$S_x = -S\left(\frac{L_1}{\Omega}\right)\sin\Omega t \tag{21}$$

$$S_y = S\left(\frac{L_1}{\Omega}\right)[1 - \cos\Omega t] \tag{22}$$

with only terms with frequency Ω. S_z is constant to first order in L/Ω. The expression for the precession angle is

$$\Theta \simeq \frac{B}{S} = \left[\frac{-2\left(B_z \cdot S_z^{\,o}\right)}{S^2}\right]^{1/2}$$

$$= \sqrt{2}\sin\phi\,[1 - \cos\omega t]^{1/2} \tag{23}$$

so that Θ is periodic with frequency $\omega \sim \Omega$.

THE MOTIONAL PRECESSION

The complete version of Schiff's equation, including both geodetic and motional precession, is

$$\frac{d\vec{S}}{dt} = \vec{S} \times \vec{L}' \tag{24}$$

$$\vec{L}' = \vec{L} + \frac{GI}{c^2r^3}\left[\frac{3\vec{r}}{r^2}(\vec{\sigma} \cdot \vec{r}) - \vec{\sigma}\right]$$

$$\equiv \vec{L} + \vec{H} \quad ,$$

where $\vec{\sigma}$ is the earth's angular velocity and I its moment of inertia. Now the first term in H oscillates with period equal to the orbital period so that the exact solution to equation (24) for a precessing circular orbit is much more complicated than the solution to equations (5), which we have already found. However, the effect of $\vec{H}$ can be taken into account by replacing $\vec{H}$ by its average over one orbital period T. The justification for this statement is contained in the following argument:[1] If we integrate equation (24) over an orbital period t and divide by T, we obtain

$$\frac{\vec{S}(t+T) - \vec{S}(t)}{T} = \frac{1}{T}\int_t^{t+T} dt' [\vec{S}(t') \times \vec{L}'(t')] \tag{25}$$

Now $\vec{S}(t')$ can be expanded in a Taylor series about t:

$$\begin{aligned}\vec{S}(t') &= \vec{S}(t) + \frac{d\vec{S}(t)}{dt}(t'-t) + \ldots \\ &= \vec{S}(t) + [\vec{S}(t) \times \vec{L}'(t)](t'-t) + \left\{\begin{matrix}\text{terms of higher} \\ \text{order in } \vec{L}'\end{matrix}\right\}\end{aligned}$$

Defining the average value of $\vec{S}$ over an orbital period as

$$\langle \vec{S}(t) \rangle = \frac{1}{T}\int_t^{t+T} dt'\, \vec{S}(t') \quad ,$$

we see that if we differentiate this equation we obtain the left hand side of equation (25). So that substituting the Taylor series for $\vec{S}(t')$ in the right hand side of equation (25) gives

$$\frac{d\langle \vec{S}(t)\rangle}{dt} = -\frac{1}{T}\int_t^{t+T} dt'\, \vec{L}'(t') \times \{\vec{S}(t) + [\vec{S}(t) \times \vec{L}'(t)](t'-t) + \ldots\} \,.$$

Now if we neglect the second and higher terms in the bracket and we replace S(t) by $\langle S(t)\rangle$ in the bracket, this equation will still be valid to first order in $\vec{L}'$, and it can then be written

$$\frac{d\langle \vec{S}(t)\rangle}{dt} = \langle \vec{S}(t)\rangle \times \langle \vec{L}'(t)\rangle$$

1. A more rigorous argument could probably be given by applying some kind of classical perturbation theory to equation (24).

where

$$\langle L'(t) \rangle = \frac{1}{T} \int_t^{t+T} dt' \vec{L}'(t') \quad .$$

This means that to first order in $\vec{L}'(t)$, the average value of $\vec{S}(t)$ over an orbital period satisfies equation (24) with $\vec{L}'(t)$ replaced by $\langle L'(t) \rangle$ so that we can replace $\vec{H}$ by $\langle \vec{H} \rangle$ in equation (24) if we are interested only in the average value of $\vec{S}$ over an orbital period.

This type of argument is borne out by the solution given in the first section if we consider T to be τ, the period of the precession of $\vec{L}$ rather than the orbital period. The terms with period $\tau_{-} = 2\pi/\Omega$ in equations (5) are seen to contribute nothing to the average value of $\vec{S}$, which is seen to precess about the earth's axis; i. e., about the average value of $\vec{L}$, to first order in $\vec{L}$.

We also note that this argument allows us to drop the restriction to circular orbits in the previous calculation. Since $\vec{r} \times \vec{v}$ is a constant for noncircular orbits, we simply replace $\frac{1}{r^3}$ by $\left\langle \frac{1}{r^3} \right\rangle$, the average of $\frac{1}{r^3}$ over the orbital period, in the expression for $\vec{L}$. This is given by [2, equation 11.15.4]

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3 \left(1 - \epsilon^2\right)^{3/2}} \quad ,$$

where a is the semimajor axis of the orbit and ϵ the eccentricity.

We will now compute $\langle H \rangle$ for a circular orbit with inclination angle i. To do this, we construct a coordinate system such that the orbit is the x'-y' plane and the orbit normal is the z' axis. We will let $\vec{\sigma}$ be in the y'-z' plane, so that it will make an angle i with the z' axis. In this coordinate system the orbit is given by

$$r_x' = -a \sin \alpha t \qquad r_z' = 0$$

$$r_y' = a \cos \alpha t \qquad T = \frac{2\pi}{\alpha}$$

and $\sigma_x' = 0$, $\sigma_y' = \sigma \sin i$, $\sigma_z' = \sigma \cos i$, so that $(\vec{\sigma} \cdot \vec{r}) = a\sigma_y' \cos \alpha t$.

Letting $\langle H_x' \rangle$, $\langle H_y' \rangle$, $\langle H_z' \rangle$ denote the components of $\langle \vec{H} \rangle$ in this coordinate system we have

$$\langle H_x' \rangle = \frac{GI}{c^2a^3}\,\frac{1}{T}\int_o^T [(-3\sin\alpha t)(\sigma_y'\cos\alpha t)]dt$$

$$= \frac{-3GI\sigma_y'}{c^2Ta^3}\left[\frac{\sin^2\alpha\ t}{2\alpha}\Bigg|_o^{\frac{2\pi}{\alpha}}\right] = 0$$

$$\langle H_y' \rangle = \frac{GI}{c^2a^3}\,\frac{1}{T}\int_o^T [(3\cos\alpha t)(\sigma_y'\cos\alpha t) - \sigma_y']dt$$

$$= \frac{GI}{c^2a^3}\,\sigma_y'\,\frac{1}{T}\left[\frac{3}{\alpha}\left(\frac{\alpha t}{3} + \frac{1}{4}\sin 2\alpha t\right)\Bigg|_o^{\frac{2\pi}{\alpha}} - T\right]$$

$$= \frac{GI}{2c^2a^3}\,\sigma_y'$$

$$\langle H_z' \rangle = \frac{GI}{c^2a^3}\,\frac{1}{T}\int_o^T (-\sigma_z')dt = \frac{-IG}{c^2a^3}\,\sigma_z' \ .$$

Performing a rotation about the x'-axis through an angle i we have

$$\langle H_y \rangle = \langle H_y' \rangle \cos i - \langle H_z' \rangle \sin i$$

$$= \frac{3GI\sigma}{2c^2a^3}[\sin i\cos i]$$

$$\langle H_z \rangle = \langle H_y' \rangle \sin i + \langle H_z' \rangle \cos i$$

$$= \frac{GI\sigma}{c^2a^3}\left[\frac{\sin^2 i}{2} - \cos^2 i\right] = \frac{I\sigma}{2c^2a^3}[1 - 3\cos^2 i]$$

$$\langle H_x \rangle = 0.$$

Since the orbital angular momentum, and thus $\vec{L}$, remains in the y-z plane after the rotation, we see that $\langle \vec{H} \rangle$ has components only in the plane which contains $\vec{L}$ and $\vec{\sigma}$. The components of $\langle \vec{H} \rangle$ perpendicular to this plane average to zero, and we expect this to remain true for noncircular orbits. Now as the orbit plane and $\vec{L}$ precess, $\langle \vec{H} \rangle$ will remain in the plane described by $\vec{L}$ and $\vec{\sigma}$, and $\vec{L} + \langle \vec{H} \rangle$ will precess about the earth axis with period $\tau = 2\pi/\Omega$. This means that the solution to equation (24) for $L' = \vec{L} + \langle \vec{H} \rangle$ can be obtained by substituting $\vec{L}'$ for $\vec{L}$ in the solution which we found previously for equation (1). In other words we simply substitute L_1' for L_1 and L_2' for L_2 in equations (5) and all subsequent equations in the first section where

$$L_1' = L_1 + \frac{3GI\sigma}{2c^2r^3} [\sin i \cos i]$$

$$L_2' = L_2 + \frac{GI\sigma}{2c^2r^3} [1 - 3 \cos^2 i] \quad .$$

It is now clear that $\vec{H}$ contributes to the precession no matter what initial orientation for $\vec{S}$ is chosen. If $\vec{S}$ is initially oriented perpendicular to the earth's axis (cases 1 and 2), it will, apart from effects with period τ, precess about the z-axis with angular velocity L_2' as described by equations (16), (17), (18), (19), and (20) with L_1 (L_2) replaced by L_1' (L_2').

Both $\vec{L}$ and $\vec{H}$ will contribute to the precession. If $\vec{S}$ is initially aligned parallel to the earth's axis (case 3), it will exhibit oscillatory behavior with period τ, as described by equations (21), (22), and (23), with L_1 replaced by L_1'. So $\vec{H}$ contributes to the precession in this case also. For a general inclination angle i, $\vec{H}$ will contribute to the precession for any initial orientation of $\vec{S}$, since both L_1' and L_2' contain the effects of $\vec{H}$. (The only exceptions to this statement occur for a polar orbit, an equatorial orbit, and an orbit for which $\cos i = 1/\sqrt{3}$, $i \cong 54$ degrees.) So that in general it is not possible to separate the effects of $\vec{L}$ and $\vec{H}$ and thus verify the existence of the geodetic and Lense-Thirring precessions separately for an inclined orbit.

SKYLAB II ORBIT

The nominal Skylab II orbit will be circular at 235 nautical miles inclined at 35 degrees. We will compute $\vec{S}$ for this case for the three initial orientations discussed previously. For this orbit

$$r = 6.8 \times 10^6 \text{ meters}$$

and Kepler's Law gives for the period τ

$$\tau = 2\pi \left[\frac{r^3}{MG} \right]^{1/2} = 93 \text{ min}$$

$$\omega_o = (2\pi)/T \quad .$$

We have $R/r = 0.94$, so $L = 7.2$ arc sec/year, and $L_2 = L \cos i = 5.9$ arc sec/year.

Now in units of earth radii R

$$J_2 = 1.08 \times 10^{-3}$$

so that equation (3) gives $\Omega = 6.5$ degrees/day, $\tau = 55$ days, and $\left(\frac{L_1}{\Omega}\right) = \left(\frac{L}{\Omega}\right) \sin i = 0.10$ arc sec.

The components of $\vec{S}$ for cases 1, 2, and 3 are plotted in Figures 1 through 7, using equations (16), (17), (18), (19), (20), (21), (22), and (23). For this case, L_2 differs from L_2' by less than 1 percent.

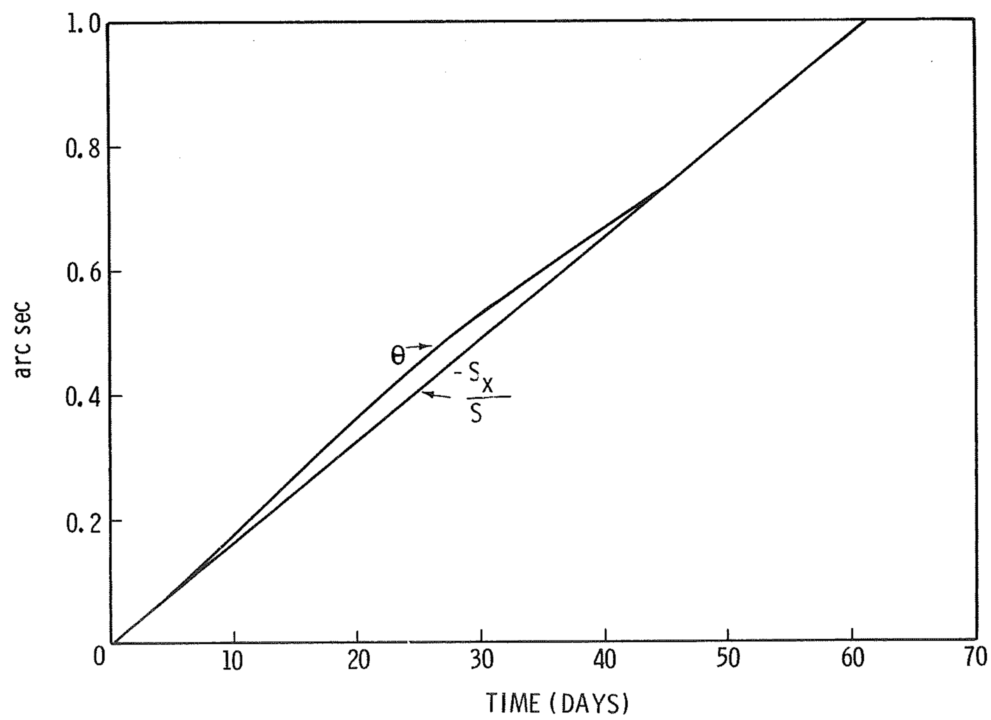

Figure 1. Case 1: $\frac{S_x}{S}$, Θ

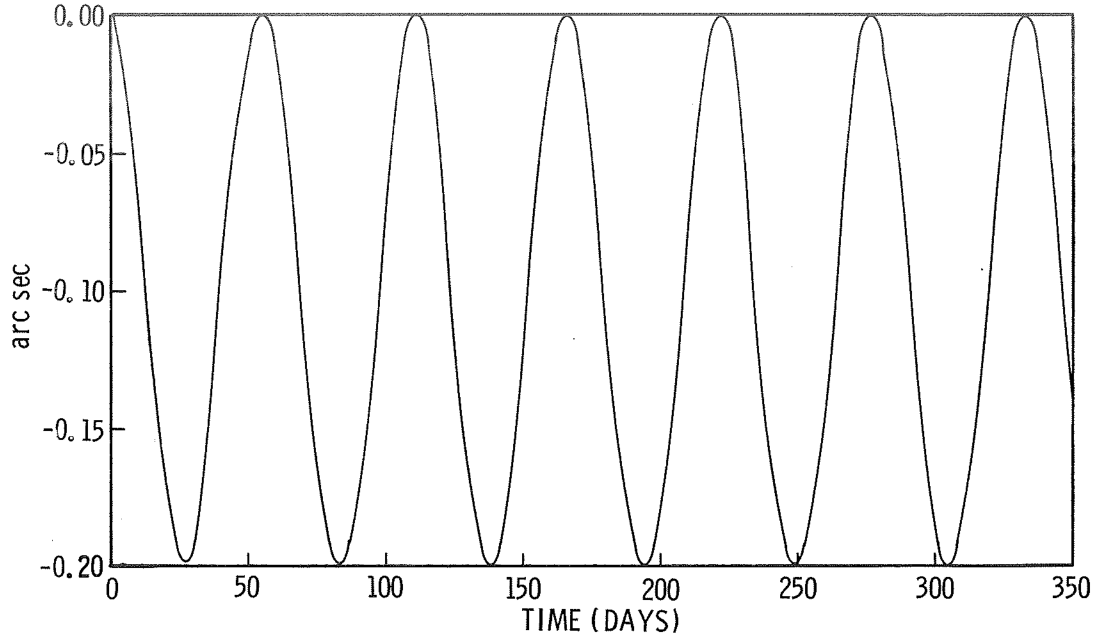

Figure 2. Case 1: $\frac{S_z}{S}$

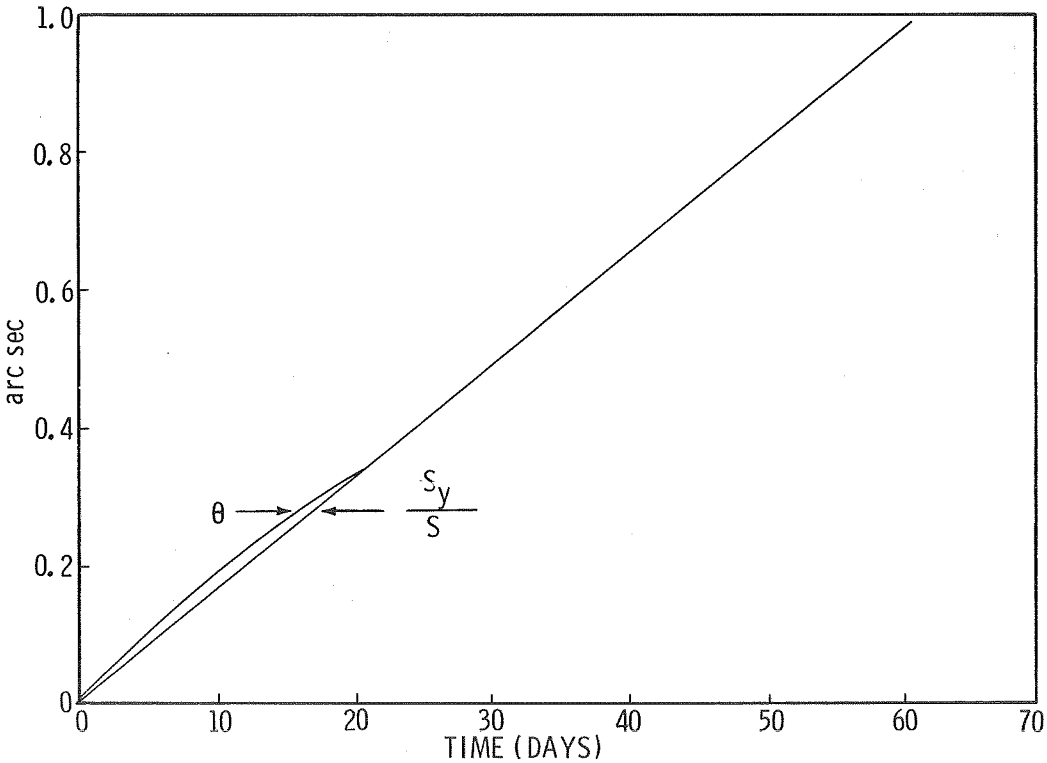

Figure 3. Case 2: $\frac{S_y}{S}$, Θ

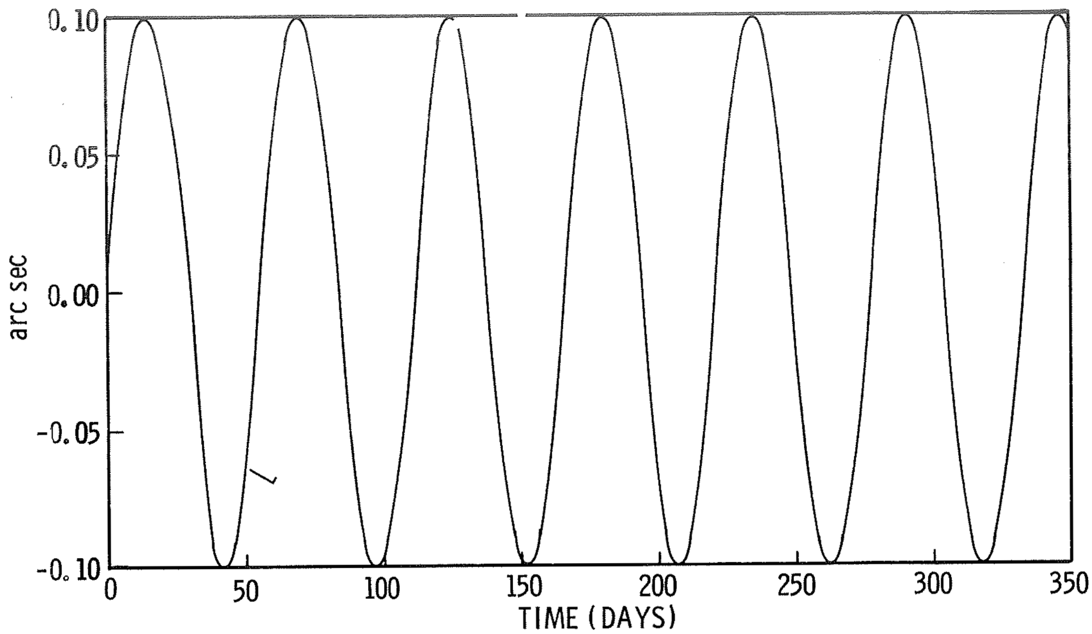

Figure 4. Case 2: $\frac{S_z}{S}$

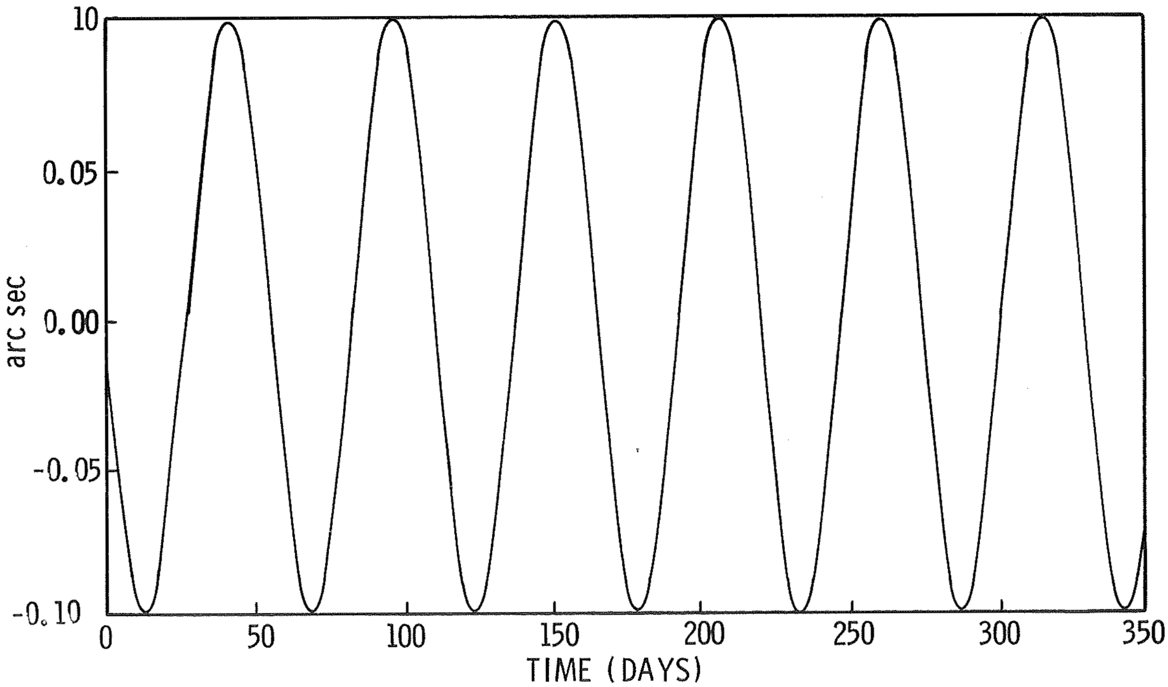

Figure 5. Case 3: $\frac{S_x}{S}$

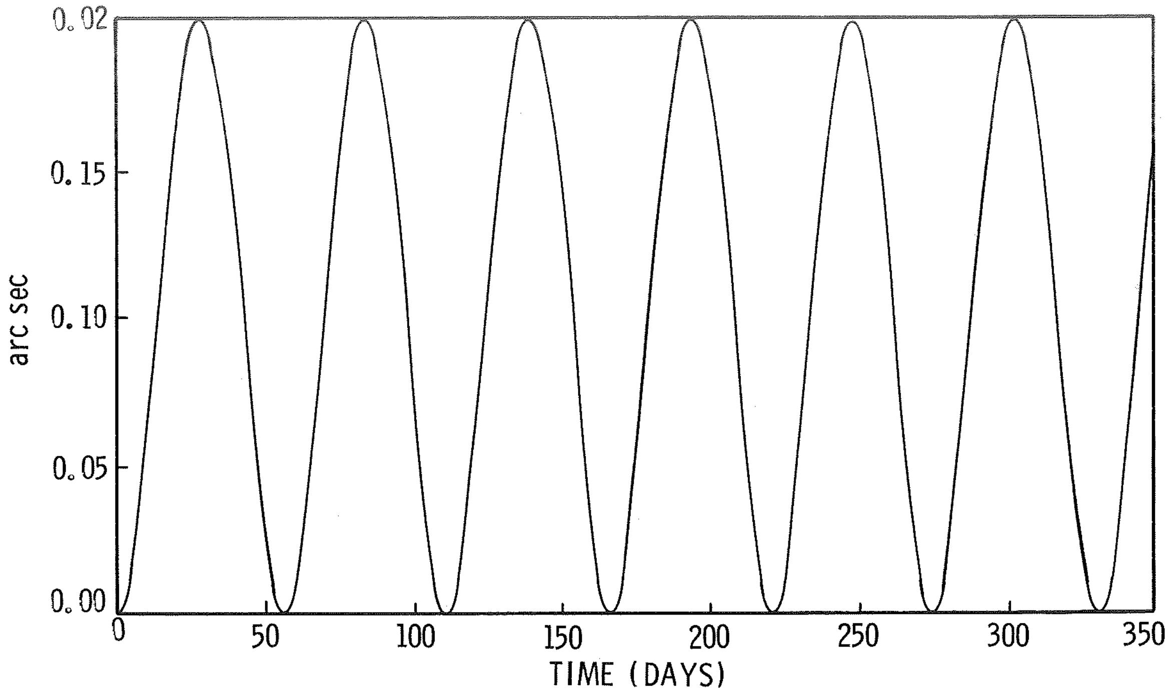

Figure 6. Case 3: $\frac{S_y}{S}$

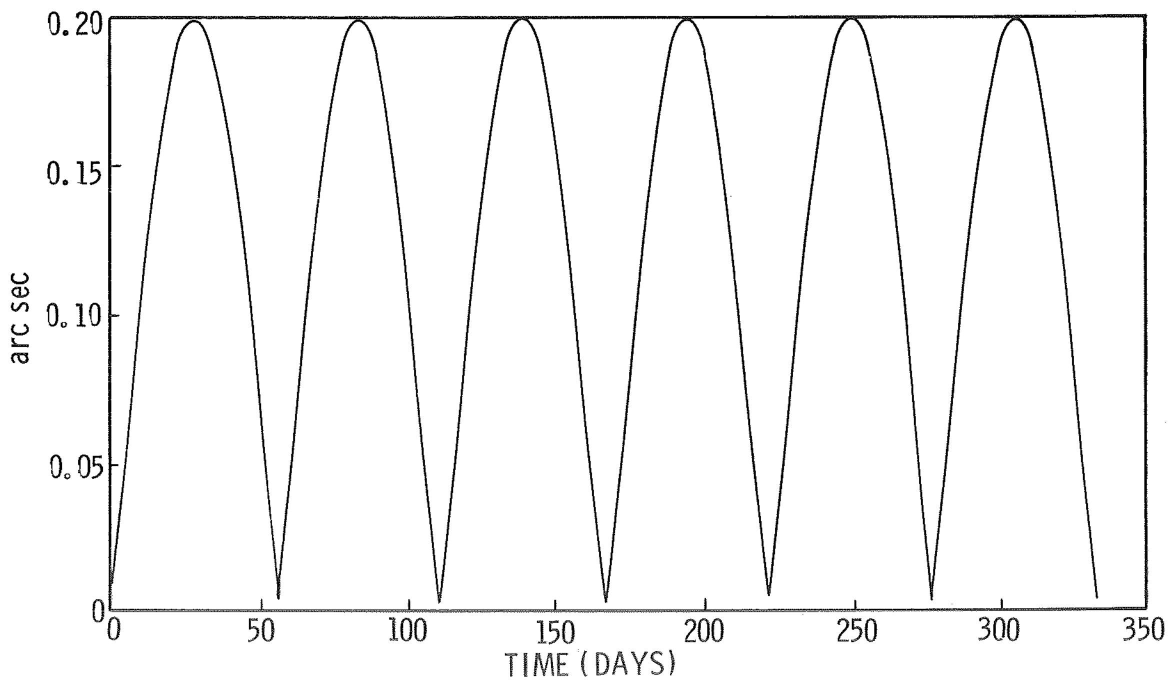

Figure 7. Case 3: Θ

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APPROVAL

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GENERAL RELATIVISTIC PRECESSION OF A GYROSCOPE IN AN INCLINED ORBIT

By Peter Eby

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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