X-643-70-303
PREPRINT

NASA TM X- 65414

A RANDOM DISTRIBUTION OF RADIAL MOTIONS

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AUGUST 1970

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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By

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ABSTRACT

Two quite unrelated problems motivated the efforts described in this report. One pertains to the question of whether a singularity in space-time can develop from an initially non-singular distribution of matter. The other problem pertains to the initial distribution of comet orbits about the sun from which certain "capture" theories attempt to predict the observed number of short period comets. For both of these problems it is instructive to start with a model consisting of a random distrubution of particles constrained, however, to strictly radial motion in a central gravitational field.

1. INTRODUCTION

Imagine a swarm of particles with random velocities and distances from a suitable origin. For example a swarm of comets about the sun, or a cluster of galaxies under their mutual gravitational attraction. Further suppose the system is in quasi-equilibrium, i.e. only very slowly permit the density of the distribution to change, e. g. because of random close encounters or a weak resisting medium. It is instructive to consider a grossly simplified version of such a swarm of particles by restricting attention to purely radial motions. This model may be motivated by two quite unrelated problems which we may take up in turn.

First, for a swarm of comets it should be noticed that we on Earth only see or observe those that come within one or two Astronomical Units (A.U.) of the Sun. Hence that sub-set of the set of all long period comets contains only those whose orbits are long slender ellipses with eccentricities close to one. Thus we only observe those that from time to time drop almost radially toward the Sun from a great distance. One might at first glance get the impression that such comets are sort of kept out in "deep storage" for centuries at a time only to drop in for short visits. In a sense this is true as we shall show. Further, by varying the randomness of the distribution one might hope to throw some light on an old question about the short period comets usually attributed to Laplace but most recently studied by Everhard[1].

Secondly the model to be discussed in this report may be reviewed as an attempt to start with a simple easily visualized Newtonian picture and then by slowly increasing the density at the center try, so to speak, to creep up upon a gravitational collapse as predicted by general relativity. In the first section of this report then we shall discuss the random distribution of radial motions in Newtonian Mechanics. Then we shall review briefly the classical solution of Oppenheimer and Snyder[2] on gravitational collapse. Finally we shall view collapse from the point of view of a non-uniform density distribution.

2. THE MOTIONS IN NEWTONIAN MECHANICS ABOUT A CENTRAL BODY

Imagine the swarm of particles with random velocities and distances from the origin. We desire their distribution for all subsequent time. The equation of motion for each is given by

$$(2.1) \qquad \frac{d^2r}{dt^2} = -G\frac{M}{r^2}, \qquad (M = \text{total mass}, G \text{ the grav. const.}, r \text{ the distance})$$

if one assumes each can be regarded as moving in a central field and ignore the shielding effect which occurs when the motion of the one particle is within most of the mass of the swarm.

The motion of each particle is next regarded as purely radial. This idealizes the motion of a long slender ellipse. The motion may be described as follows. (These may be verified by differentiating, or by referring to a standard text such as Szebehely[3].)

$$(2.2) \qquad \begin{aligned} \zeta &= (1 + \cos\psi) \\ t' &= (\psi + \sin\psi), \end{aligned}$$

where the new radial variable ζ instead of r is

$$\zeta = r/a \tag{2.3}$$

and a new time t' instead of t is

$$t' = \sqrt{\frac{GM}{a}}\; t \tag{2.4}$$

In these variables the motion is in the sense of increasing ψ. Equation (2.2) furnishes, for $\psi = 0$, $\zeta = 2$, and $t' = 0$. Subsequently ζ decreases to zero as ψ increases to π; for which value $t' = \pi$. The plot of the motion in the (ζ, t')-plane is a cycloid, with its cusp corresponding to infinite speed at the origin (occurring at $t' = \pi$). The particle then rises back to its maximum distance at $\zeta = 2$, or in the original variable, at $r = 2a$. Denote this value by ℓ.

Before leaving these familiar relations, observe that the single particle spends most of its time out near $r = \ell$ and very little near the origin. It will be worth stating this precisely in what follows by defining a density distribution for a single particle whose integral over the whole radial path is the mass m of the particle.

To do this observe either by going back to (2.1) and obtaining the first integral or more directly from (2.2) that

$$\frac{d\zeta}{dt'} = -\frac{\sqrt{2-\zeta}}{\sqrt{\zeta}}\;, \tag{2.5}$$

hence the reciprocal of $\frac{d\zeta}{dt'}$, is proportional to the relative time spent in the interval ζ to $\zeta + d\zeta$. Let us denote the density by σ_ℓ, with

(2.6) $$\int_0^\ell \sigma_\ell(r)\, dr = m, \text{ the particle mass.}$$

Then one has

(2.7) $$\sigma_\ell(r) = \frac{2}{3\pi}\frac{m}{\ell}\ \frac{(r/\ell)^{1/2}}{\sqrt{1-(r/\ell)}}\ .$$

Notice, as stated, that $\sigma_\ell(0) = 0$, whereas $\sigma_\ell(\ell)$ is infinite, so indeed in this sense such a particle is indeed out in "cold storage" and only "drops in" from time to time.

If now we have a swarm of particles n_ℓ in number, each of mass m, all in radial orbits of major axis ℓ, but uniformly distributed in all directions, we may define a density $\rho_\ell(r)$ for all of them defined so that the integral over the whole space out to $r = \ell$ will be $n_\ell m$:

(2.8) $$\rho_\ell(r) = \begin{cases} \dfrac{n_\ell m}{6\pi 2\ell^3} \quad \dfrac{(r/\ell)^{1/2}}{(\frac{r}{\ell})^2\sqrt{1-(r/\ell)}} & , \quad 0 < r < \ell \\ 0 & , \quad r \geq \ell . \end{cases}$$

Notice $\rho_\ell(0)$ is no longer zero because of the large supply from all directions that enter a small neighborhood of the origin.

Let us next make the swarm less artificial by permitting a wide distribution of energies, characterized by ℓ, to be present. Indeed let us introduce a distribution in ℓ, namely $\mu(\ell)$ with n_ℓ now being the number with maximum distances from ℓ to $\ell + d\ell$. Then for the density distribution $\rho(r)$ for the whole cluster, we have

$$\rho(r) = \int_r^\infty \rho_\ell \, d\ell ;$$

where the lower limit must be r, not zero since the function $\rho_\ell(r)$ vanishes for $\ell \leq r$. Thus

$$\rho(r) = \frac{m}{6\pi^2} r^{-3/2} \int_r^\infty \frac{\mu(\ell)\,d\ell}{\ell\sqrt{\ell - r}} . \tag{2.10}$$

Before continuing we should discuss the consistency of this equation. It appears to define $\rho(r)$ for a given distribution of orbit sizes $\mu(\ell)$. However we assumed the central force law (2.1) in the derivation. Thus (2.10) is correct for the density of comets, for example, whose total mass is negligible compared to a central body such as the sun; but (2.10) will not hold for a swarm of particles in their mutual field because of the shielding effect. We shall discuss this latter point in the next section.

To get some feeling for (2.10) and to keep the integrations easy, let us consider a simple power distribution for $\mu(\ell)$ and write

$$\mu(\ell) = c\ell^{1-n} . \tag{2.11}$$

If we let N denote the total number of all energies out to some remote last value of ℓ say L, we may chose c so that

$$\int_0^L \mu(\ell)\,d\ell = N . \tag{2.12}$$

Then

$$(2.13) \qquad c = \frac{(2-n)}{L^{2-n}} N, \qquad n < 2.$$

We chose the restriction $n < 2$ here because any larger value gives such a high central concentration that, as we shall see, the total mass at the origin becomes infinite. To evaluate the integral in (2.10) we set $x = (\ell - r)/r$, whence

$$I = \int_r^\infty \frac{d\ell}{\ell^n(\ell-r)^{1/2}} = r^{\frac{1}{2}-n} \int_0^\infty \frac{dx}{(1+x)^n x^{1/2}} = r^{\frac{1}{2}-n} \frac{\Gamma(\frac{1}{2})\ \Gamma(n-\frac{1}{2})}{\Gamma(n)},$$

or as we prefer to write

$$(2.14) \qquad I_n = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)} \sqrt{\pi}\ r^{\frac{1}{2}-n}.$$

Thus from (2.10) we have

$$(2.15) \qquad \rho(r) = C_n r^{-(n+1)}, \qquad C_n = Nm \frac{2-n}{L^{2-n}} \frac{\sqrt{\pi}}{6\pi^2} \frac{\Gamma(n-1/2)}{\Gamma(n)}.$$

Finally integrating over a sphere of radius r we find for the mass

$$(2.16) \qquad M(r) = \frac{4\pi}{2-n} C_n r^{2-n}.$$

To sum up as follows: For any arbitrary distribution $\mu(\ell)$ of orbit sizes the corresponding density distribution $\rho(r)$ is given by (2.10).

In the special case of all having the same size ℓ the density is given by (2.8). In the special case of $\mu(\ell)$ given by a plausible power law (2.11) the density is given by (2.15) and the mass interior to a sphere of radius r is given by (2.16). Thus n=2 corresponds to the usual infinite mass-at-the-origin, central field approximation. Cases with $n < 2$ correspond to less concentration, and indeed $n = -1$ gives a uniform density.

Example: Suppose an astronomer is aware at any one time of 5 comets in our neighborhood (i.e. a cube 1 A.U. on a side centered at $r = 1$) whose major axes are known to be 1000 A.U. How many are there altogether?
Ans. 10,000,000.

3. THE MUTUAL MOTIONS IN NEWTONIAN MECHANICS.

As mentioned before, to treat the case where there is no large central mass so that particles in the inner regions of the swarm are shielded, we must give up equation (2.1). To correctly account for the shielding even in this setting of pure radial motion is difficult, it involves integrating $r = - k' M(r)/r^2$ for $M(r)$ arbitrary, working through to an equation analogous to (2.16) and then regarding the latter as an integral equation for that $\mu(\ell)$ which furnishes the given $M(r)$. We shall not attempt this in general but consider the problem for $M(r)$ given as a power law in the same spirit as the last section.
Indeed for

$$(3.1) \qquad \rho = C_n r^{-(1+n)},$$

we have

$$(3.2) \qquad M(r) = \int_V \rho r^2 \, dr \, d\Omega = 4\pi \int_0^r \rho r^2 dr$$

$$= 4\pi \int_0^r C_n \, r^{1-n} \, dr = \frac{4\pi \, C_n}{2 - n} \, r^{2-n}, \quad n < 2$$

and hence

$$\ddot{r} = - G \frac{M}{r^2} = - \frac{4\pi \, C_n G}{2 - n} \, r^{-n} \, .$$

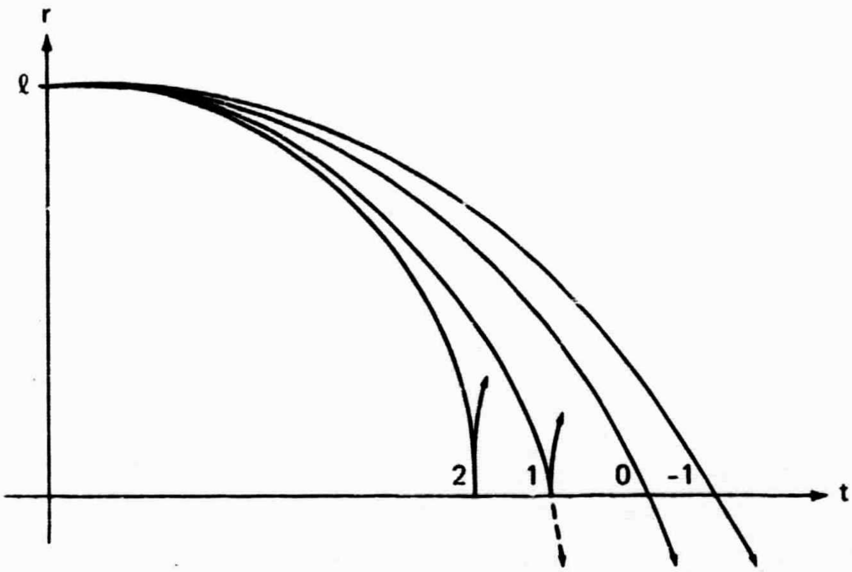
Let $$k = \frac{4\pi \, C_n G}{2 - n} \, , \text{ so}$$

$$(3.3) \qquad \ddot{r} = -k \, r^{-n}$$

$$(3.4) \qquad \frac{dr}{dt} = \begin{cases} - \sqrt{E - \frac{2k}{1-n} \, r^{1-n}} \, , & n \neq 1 \\ - \sqrt{E - 2k \, \ell n \, r} \, , & n = 1. \end{cases}$$

Where E is merely an integration constant.

Finally t may be obtained by quadrature as an incomplete gamma function. However the behavior of the solution is easily found from (3.4). The principle effect of the shielding is to remove the infinite velocity at the origin for $n < 1$. If we restrict attention to bounded orbits and choose E so that $r(0) = \ell$, the motions qualitatively may be sketched as in the Figure 1.



FREE FALL PATHS

Figure 1

Notice that n = 1, the logarithmic case, serves as the dividing line between those orbits that return on themselves and those that cross the origin (The understanding here is that the radial paths are thought of as approximating long slender ellipses in the n= 2 case). The n = - 1 case is that of a uniform dust cloud of constant density.

To continue as described after equation (2.5), we have from (3.4),

$$(3.5) \qquad \sigma_{\ell}(r) = \frac{\alpha}{\sqrt{E - \frac{2k}{1-n}\, r^{1-n}}} \, ,$$

where α is a suitable constant chosen so that the integral of σ_{ℓ} taken over r is the mass of the particle. We eschew the detail here of explicitly expressing α in terms of m and n by suitable gamma functions.

If we pursue the more general case through to the analogue of equation (2.10) we obtain

$$(3.6) \qquad \rho(r) = \frac{\alpha\sqrt{1-n}}{4\pi r^2\sqrt{2k}} \int_r^\infty \frac{\mu(\ell)\, d\ell}{\sqrt{\ell^{1-n}\; r^{1-n}}} ,$$

or introducing x through $\ell = rx$

$$(3.7) \qquad \rho(r) = \frac{\alpha\sqrt{1-n}}{4\pi\sqrt{2k}}\; r^s \int_1^\infty \frac{\mu\,(rx)\, dx}{\sqrt{x^{1-n} - 1}} , \quad s = \frac{n-3}{2} .$$

Now this relation is based on the assumption (3.1), hence we seek a function $\mu(\ell)$ for which this is the case. But from the form of (3.7) it is clear that $\mu(\ell)$ must be a power function in ℓ if ρ is to be a power function in r. Thus we assume

$$(3.8) \qquad \mu = b_n \ell^m ,$$

where m is an as yet unknown power. Absorbing all the constants and equating (3.1) to (3.7):

$$C_n r^{-(1+n)} = C_n' \, r^{(n-3+2m)/2 \; + \; m}$$

hence we have a solution provided

$$(3.9) \qquad C_n' = C_n ,$$

and

$$(4.0) \qquad m = \frac{1}{2} - \frac{3}{2}\, n .$$

Thus we have a family of solutions to (3.7) depending upon n. The functional dependence of the various variables may be summarized in a table as shown in Table 1. The extreme central concentration n = 2 is on the left, the complete lack of concentration is n = -1 on the right.

variable	power of r	n=2	n=1	$n = \frac{1}{3}$	n=0	n= -1
ρ	-1-n	-3	-2	1.33	-1	0
$\ddot{r}$	-n	-2	-1	-0.33	0	1
μ	$\frac{1}{2} - \frac{3n}{2}$	-2.5	-1	0	0.5	2
$E-\dot{r}^2$	1-n	-1	(logr)	0.66	1	2
M	2-n	0	1	1.66	2	3

TABLE 1

The case $n = \frac{1}{3}$ has a rather nice interpretation. Consider a system of particles initially at rest and uniformly distributed and suddenly allowed to fall under their mutual gravitation. Then if we allow their energies to be unchanged, but the phases of their motions to drift incommensurably, the state of motion can shift statistically from n = -1 to n = 1/3. It may be of interest to pursue the question implied here more seriously in view of the recent efforts of Barnes and Whitrow[19] and Penston[20]. However we shall not follow this line of argument further in this report.

4. RELATIVISTIC GRAVITATIONAL CONTRACTION

In their now classic report on gravitational contraction (1939), Oppenheimer and Snyder[2] obtained a solution of the Einstein equations of general relativity that correspond to the fall of a ball of matter of uniform density. Indeed it had previously been shown for sufficiently large mass concentrations that "cold" matter must contract[4]. The implications of this, especially for stellar theory has prompted numerous papers since[5-18]. The literature is in fact much too extensive to review here. The basic conclusion seems inescapable and has led Geroch[10], Hawking[11], and Penrose[12] to propose rigorous mathematical proofs, under fairly broad assumptions, that such collapse must occur.

One feature of the original Oppenheimer-Snyder paper that is bothersome, is the fact that to obtain a first result of this kind they quite naturally considered the simplest case. It turns out that a uniform cloud of particles, c_o-moving, will contract steadily and uniformly down to a point singularity. In spite of general theorems such as that of Penrose it is interesting to look at other specific examples. One recent example due to Israel[13] considers a thin spherical shell of matter that collapsed due to its own gravity. Israel would have it rebound; appearing to go backwards in time after the "bounce". Other examples suggest themselves. Before going into them, we shall review the principle results of Oppenheimer and Snyder here. They assume, for an observer at rest at a great distance, the Schwarzschild metric given in terms of coordinates (r, θ, φ, t) by

$$(4.1)\quad ds^2 = (1 - \frac{r_o}{r})\, dt^2 - (1 - \frac{r_o}{r})^{-1}\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

For the falling matter they introduce co-moving coordinates $(R, \theta, \varphi, \tau)$ with metric

$$(4.2)\qquad ds^2 = d\tau^2 - e^{\overline{\omega}}\, dR^2 - e^{\omega}\,(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where $\overline{\omega}$ and ω are functions to be determined. Since the matter is at rest in these coordinates and any internal stresses and pressures are taken as negligible, all components of the stress tensor vanish except $T_4^{\ 4} = \rho(R)$. Substitution into Einstein's equations then furnish equations for $\overline{\omega}$ and ω; indeed they find

$$(4.3)\qquad e^{\overline{\omega}} = e^{\omega}\, \omega'^2/4f^2(R),$$

where $f(R)$ is arbitrary. A restricted class of solutions is found for $f(R) \equiv 1$ for which they find

$$(4.4)\qquad e^{\omega} = (F\tau + G)^{4/3},$$

with F and G arbitrary functions of R. Since the equations are invariant under a coordinate transformation, such as taking R to be a function of a new variable R^*, only one of F and G is really arbitrary so they set $G = R^{3/2}$. If now the density $\rho(R)$ be given initially as $\rho_o(R)$ they find

$$(4.5)\qquad FF' = 9\pi R^2 \rho_c(R).$$

We have repeated much of the argument here because we wish to use these equations for $\rho_o(R)$ chosen as in Section 3. At this point the easiest case is to take $\rho_o(R)$ = const. However such a distribution cannot extend to all R but must fall off faster at infinity; so they introduce a value R_b such that $\rho_o(R)$ = const. $R \leq R_b$, $\rho_o(R) = 0$ for $R > R_b$. One then wishes to observe the collapse in the (r, θ, φ, t) coordinates. The transformation is obtained by finding $r = r(R,\tau)$, $t = t(R,\tau)$ using (4.3) and (4.4) to transform (4.2) over to (4.1). Skipping the details here, we shall simply give the result (r_o being the Schwarzschild radius corresponding to the total mass.):

$$(4.6) \qquad r = \begin{cases} R\left(1 - \frac{3}{2} r_o^{\frac{1}{2}} R_b^{-3/2} \tau\right)^{2/3} ; & R < R_b \\ R\left(1 - \frac{3}{2} r_o^{\frac{1}{2}} R^{-3/2} \tau\right)^{2/3} ; & R > R_b \end{cases}$$

$$(4.7) \qquad t = \begin{cases} \frac{2}{r_o^{\frac{1}{2}}}\left(R^{3/2} - r^{3/2}\right) - 2(rr_o)^{\frac{1}{2}} + r_o \ln \frac{r^{\frac{1}{2}} + r_o^{\frac{1}{2}}}{r^{\frac{1}{2}} - r_o^{\frac{1}{2}}}, & R > R_b \\ M(y), \quad R < R_b \end{cases}$$

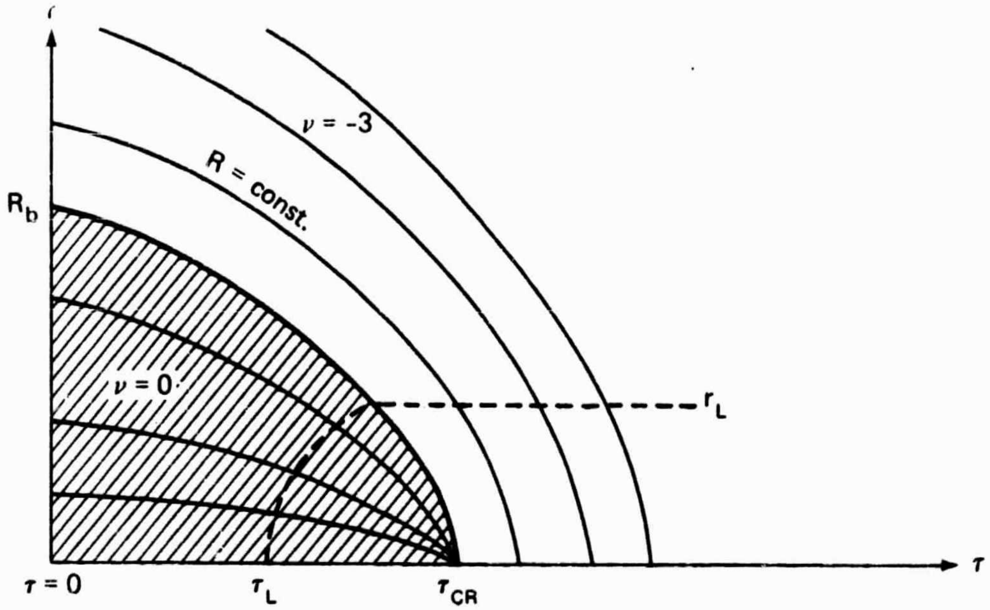
where

$$(4.8) \qquad y = \frac{1}{2}\left[\left(\frac{R}{R_b}\right)^2 - 1\right] + R_b r/r_o R.$$

Here $M(y)$ must be chosen to satisfy continuity at $R = R_b$ which furnishes

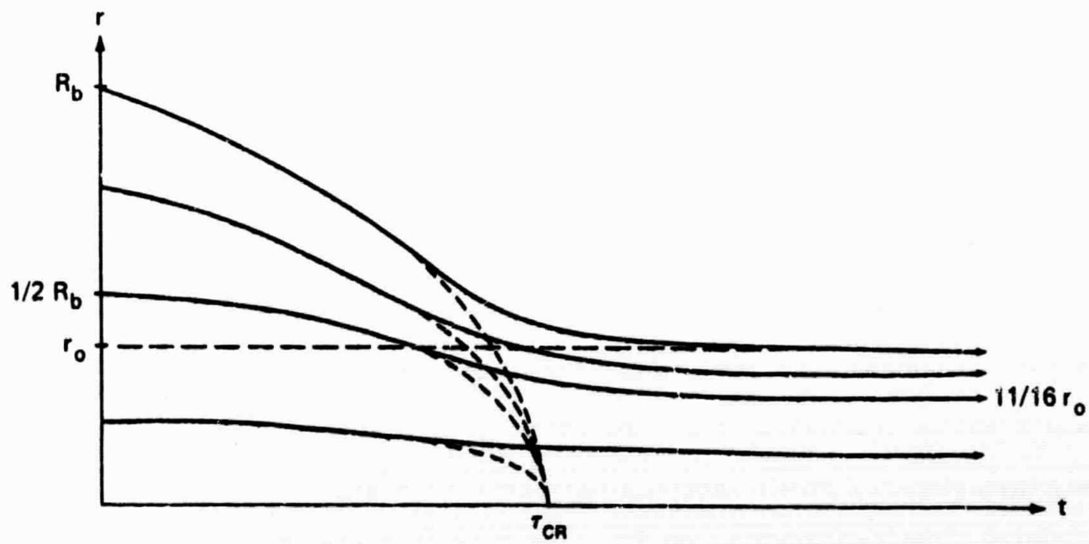
$$(4.9) \qquad M(y) = \frac{2}{3} r_o^{-\frac{1}{2}}\left(R_b^{3/2} - r_o^{3/2} y^{3/2}\right) - 2r_o y^{\frac{1}{2}} + r_o \ln \frac{y^{\frac{1}{2}}+1}{y^{\frac{1}{2}}-1} .$$

The appearance of the solution is easiest to visualize by appropriate graphs. Figure 2, gives the paths R = const. in the mixed (r,τ)-plane.



FREE FALL IN (r, τ)-PLANE

Figure 2



FREE FALL IN (r, t)-PLANE

Figure 3

Figure 3 shows the motion in (r,t). The latter shows the phenomena known as cut off. If we denote the moment of complete collapse by τ_{CR} we see that the falling motions appear to slow down for t in the neighborhood of τ_{CR} and rather quickly approach asymptotic distances which are the Schwarzschild radii for the total mass interior to each. That is $\lim_{t\to\infty} R_b = r_o$, $\lim \frac{1}{2} R_b = \frac{11}{16} r_o$, etc. each finding a limiting position. Since the asymptotic values are in fact approached rather quickly for $t > \tau_{CR}$ one may ask if this is an effect of the assumption that all the particles started out together and hence all converged at the same time. We may also wonder at the consistency of this picture with the Newtonian picture of Section 3 where particles may fall from different initial points at different times.

To answer some of these questions let us first point out an important distinction between the particles falling in Section 3 and the co-moving coordinate R following the motion here. Consider a small volume of space in the neighborhood Y, of a point at distance R initially. The volume Y at any instant contains mass. The momentum components T_{i4} of the stress tensor will be zero even for matter in motion, so long as the net flux into or out of Y is zero. Thus R, the "coordinate-following-the-motion", only has this meaning in the sense that the center of mass of Y has a motion described by $r = r(R;t)$ such as shown in Figures 2 or 3. Thus individual particle motions different from the mean collapse motion can be understood to be encompassed by the relativistic theory. On the other hand the picture in Section 3 suggests that we look at other motions following equation (4.5) with $\rho_o(R)$ chosen from Table 1. Perhaps we can sort of sneak up on the singularity without the common suddeness of $\tau = \tau_{CR}$ shown in Figure 2. We shall indeed take this up in Section 5.

Before leaving the subject of uniform collapse, it is worth one more fact of note. If each region characterized by the coordinate R is furnished with a clock, the clock will appear to stop at a certain proper time τ_L before the critical value of τ. Moreover this occurs asymptotically as R appears to sink down to its ultimate lowest value, say r_L. This defines a curve $r_L = r_L(\tau_L)$ in the (r,τ)-plane beyond which further collapse is unobservable to our distant rest observer. This may be found from (4.7) (corresponding to y=1), with a little algebra, and is shown on Figure 2 as the dashed line.

5. COLLAPSE FOR ALTERNATIVE DENSITY DISTRIBUTIONS

In this section we shall adopt an assumed power law

$$\rho = C_\upsilon R^\upsilon \tag{5.1}$$

where $\upsilon = -(1+n)$ in terms of the exponent in (3.1), and introduce the co-moving coordinate $R = r(0)$ in place of $r(t)$. Following Oppenheimer and Snyder, as described in Section 4, the only departure comes after (4.5). We shall follow their argument for general υ. Actually looking at Figure 2, we see that by taking $\rho_0(R)$ = const. for $R \leq R_b$, $\rho_0(R) = 0$ for $R > R_b$ they have in a sense covered two cases: $\upsilon = 0$ and $\upsilon = -3$. So that for $0 < \upsilon < -3$ we may expect the paths of constant R in the figure to resemble those labeled $\upsilon = -3$ rather than all converging to

τ_{CR} as in the $\upsilon = 0$ case. That this is indeed the more general situation we shall see shortly.

Let us integrate (4.5) then to find

$$F^2 = \frac{18\pi C\upsilon}{3+\upsilon} R^{3+\upsilon} + \text{constant}.$$

Dropping the non-essential constant we may write

$$(5.2) \qquad F = -\gamma R^{\frac{3+\upsilon}{2}}, \qquad \gamma = \sqrt{\frac{18\pi C\upsilon}{3+\upsilon}}$$

hence $e^{\omega/2} = r$ furnishes in place of (4.6)

$$(5.3) \qquad r = R(1-\gamma\tau R^{\upsilon/2})^{2/3}$$

Thus this motion is a solution of Einstein's equations and indeed should describe a possible relativistic behavior of the swarm of particles described in Section 2. Thus even though the motion is a "non-zero-temperature-motion", collapse can still occur. (This is in large part due to the zero angular momentum of pure radial motions.) To see what the motion looks like to our distant observer we need, in addition to the function $r = r(R, \tau)$ given by (5.3), a relation $t = t(R,\tau)$. This is obtainable (still following [2]) by solving

$$\frac{\partial t(R,\tau)}{\partial \tau} \frac{\partial r(R,\tau)}{\partial \tau} = \frac{\partial t/\partial R}{\partial r/\partial R}$$

for $t(R,\tau)$ regarded as a first order partial differential equation, since $\partial r/\partial\tau$ and $\partial r/\partial R$ are obtained from (5.3). Thus one finds

$$(5.4)\qquad \frac{\frac{2}{3}\gamma R^{1+\frac{\upsilon}{2}}}{(1-\gamma\tau R^{\upsilon/2})^{1/3}}\ \frac{\partial t}{\partial\tau} + \frac{(1-\gamma\tau R^{\upsilon/2})^{1/3}}{(1-\gamma\frac{\upsilon+3}{3}R^{\upsilon/2})}\ \frac{\partial t}{\partial R} = 0.$$

If $\varphi = \varphi(R,\tau)$ be any first integral of this equation the general solution is $t = \psi(\varphi)$, for ψ an arbitrary function. Note however that for all $\upsilon < 0$ the density given by (5.1) decreases to zero as $R \to \infty$, so we wish the mapping $(R,\tau) \to (r,t)$ to approach the Schwarzschild field asymptotically for all t. This fixes the function ψ. At this point one could, for given υ, carry out the integration of (5.4) by setting up the equations

$$(5.5)\qquad \frac{dR}{dt} = \frac{-3}{3\gamma}\ R^{-\frac{\upsilon+2}{2}}\ \frac{[1+\gamma\tau R^{\upsilon/2}]^{2/3}}{1+\frac{\upsilon+3}{3}\gamma\tau R^{\upsilon/2}}\ .$$

If $M = \int_0^\infty \rho_0(R)R^2\,dR$ be the total mass of the system, and we denote by r_s the Schwarzschild radius corresponding to M, we may now write the transformation as

$$(5.6)\qquad \begin{aligned} r &= R(1-\gamma\tau R^{\upsilon/2})^{2/3} \\ t &= \frac{-r_s}{\gamma}\left(\frac{2}{3}\varphi^{2/3} + 2\varphi^{1/2} - \ln\frac{\sqrt{\varphi}+1}{\sqrt{\varphi}-1}\right), \end{aligned}$$

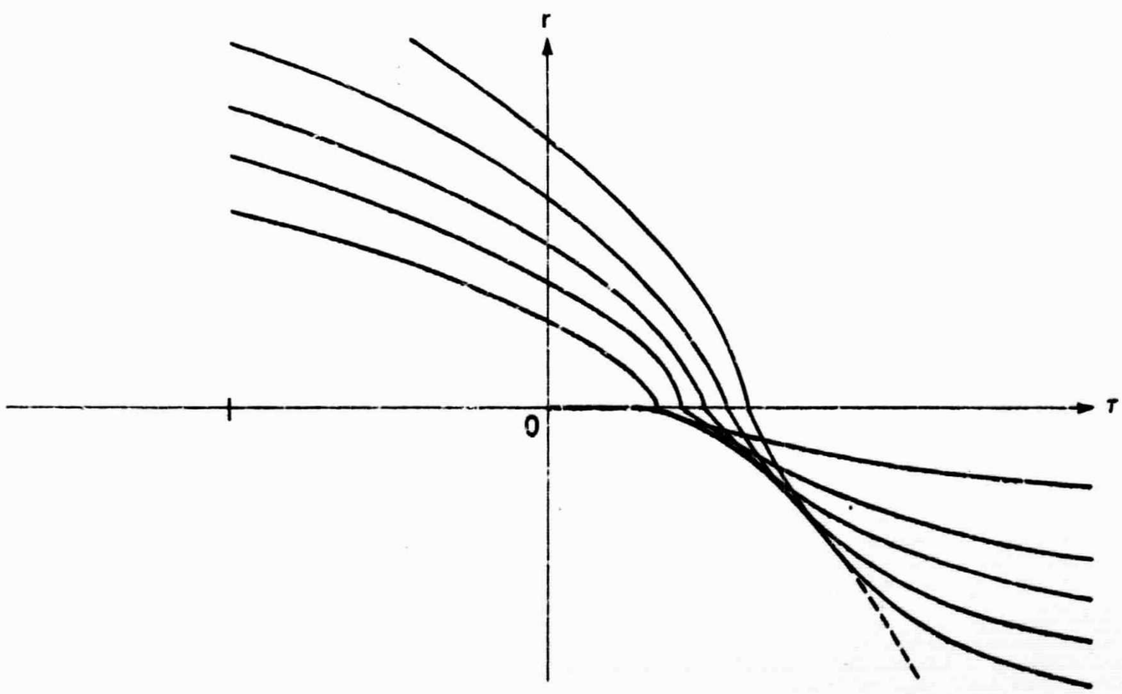
where $\varphi = \varphi(R,\tau)$ is any solution of (5.5).

Of particular interest will be the solution $\varphi(R,\tau) = 1$ which describes the locus in the (R,τ) space of all points that map to $t = \infty$

in the (r,t) space. These are the ultimate-lowest-apparent-radii reached. It is of passing interest to note that (5.5) is separable for two cases, $\gamma = 0$ and $\upsilon = -2$. Even in the $\upsilon = -2$ case, however, a numerical integration is necessary.

Rather than get into a detailed solution of (5.5) we will content ourselves with a few comments on the nature of solutions.

Figure 4 furnishes a sketch of the curves $r=r(\tau)$ described by (5.3) for $\gamma = 1$, $\upsilon = -1$. These have been extended across the r=0 axis in the spirit of a "rebound" from the singularity. It should be noticed that two difficulties occur. For one thing if we compare with the Newtonian figure 1, we see that for all υ, not only $n \geq 0$, one has $dr/d\tau$ infinite.



The Case $\upsilon = -1$

FIGURE 4

This is a reflection of the fact that even though r is only a coordinate variable, none-the-less $r=0$ is the apparent location of a true singularity since the intrinsic curvatures are infinite.

Secondly, as continued across $r=0$, neighboring curves cross forming an envelope so that the coordinate transformation $(R,\tau) \to (r,t)$ is no longer single-valued. This leads to two questions. One: what is the equation of the envelope? Two: Does the mapping ever break down with such a fold before the axis is crossed? The answer to the latter question is no, since for $\upsilon < 0$, $R_2 > R_1$ implies $R_2^{\upsilon/2} < R_1^{\upsilon/2}$ and hence

$$1 - \gamma \tau R_2^{\upsilon/2} > 1 - \gamma \tau R^{\upsilon/2},$$

provided $1 - \gamma \tau R_1^{\upsilon/2} > 0$. Thus $r_2(\tau) > r_1(\tau)$ if $r_2(0) > r_1(0)$ for all τ for which $r_1 > 0$.

That an envelope does in general exist for negative r is found from (5.3) by direct construction, i.e. eliminating R from (5.3) using $\partial r/\partial R = 0$. The equation of the envelope is

$$r = \gamma^{-\frac{2}{\upsilon}} (1 + \tfrac{\upsilon}{3})^{-(\frac{2}{3} + \frac{2}{\upsilon})} (\tfrac{\upsilon}{3})^{2/3} \tau^{-\frac{2}{\upsilon}}.$$

The case $\upsilon = 0$ degenerates to a point at $\tau = \tau_{CR}$, the case $\upsilon = -3$ gives $r = \gamma^{2/3} \tau^{2/3}$. For $\upsilon = -1$ we have r proportional to τ^2 as shown in Figure 4.

The only really basic difference then between the case $\upsilon = 0$ and other power laws is that instead of the whole space seeming to simultaneously settle down to the ultimate state as in Figure 3, one has the inner regions effectively cut off first, followed by the outer regions later, so that one must view a steadily expanding "black hole".

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