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COMBINED CONDUCTION, CONVECTION AND RADIATION EFFECTS IN INTERNAL FLOWS - PARTICIPATING GASES

by

Richard S. Thorsen
Department of Mechanical Engineering

Prepared for the
Office of University Affairs
National Aeronautics and Space Administration
Under Research Grant NGL-33-016-067.

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New York University
School of Engineering and Science
University Heights, New York, N.Y. 10453

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ABSTRACT

A general formulation is presented for determining the gas and surface temperature distributions for laminar flow in a circular tube. The analysis allows for arbitrary wall heat generation and radiation effects in the gas and at the tube surface. Though optically thin radiation is considered for purposes of obtaining numerical results with water vapor as the fluid, the analytical development is applicable to more general radiation situations, any gas for which the relevant properties are known, and to fluids with heat sources due to effects other than radiation.

Numerical results are presented for a variety of parameters, selected to demonstrate important qualitative trends. By comparing the results with results obtained for non-participating gases (surface radiation effects considered) it is found that for a prescribed wall heat generation the non-participating gas solution represents a poor estimate of the wall temperature distribution as well as the gas bulk temperature variation. It is also demonstrated that introducing a so called "quasi-one dimensional" approximation for the radiation term in the energy equation simplifies the computational procedure without sacrificing accuracy.

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NOMENCLATURE

A_w Cross sectional area of tube wall

B Radiosity

C_p Specific heat at constant pressure

d Tube inside diameter

H Irradiation

I Spectral intensity

k Thermal conductivity of gas

k_w Thermal conductivity of wall

$\vec{l}$ Unit vector denoting pencil of radiation (fig. 2)

L Tube length

$\vec{n}$ Unit normal vector

N Defined by eq. (54)

P Defined by eq. (52)

q'' Wall heat generation per unit inside area

$\vec{q}^R$ Radiation heat flux vector

r Radial coordinate

r_o Tube inside radius, d/2

R General three-dimensional region

S Surface of R or coordinate directed opposite to $\vec{l}$ (fig.2)

T Temperature

u Axial velocity

$\bar{u}$ Mass averaged velocity

x Axial coordinate

Greek

α Thermal diffusivity

δ Dirac delta function

θ Solution to Reduced Problem - also polar angle

ϕ Angle

κ_ν Monochromatic (mass) absorption coefficient

κ_P Planck mean absorption coefficient

ν Frequency

ξ, ρ Dummy variables ; also ρ = density

σ Stefan-Boltzmann constant

τ Defined by eq. (53)

τ_ν Monochromatic optical thickness, eq. (21)

ψ T/T_i

ω Solid angle

<u>Subscripts</u>

e Pertains to tube exit

i Pertains to tube inlet

w Pertains to tube wall

I. INTRODUCTION

General

Continuing efforts to develop high temperature engineering systems has necessitated a careful study of the role of thermal radiation as a mechanism for energy transport. In many of these systems, including high temperature heat exchangers and nuclear reactors, gases are passed through tubes, channels and annuli and the combined modes of conduction, convection and radiation energy transfer must be considered.

In addition to the usual complications present in conduction-convection problems (temperature dependence of properties, entrance region effects and selection of an adequate turbulence model if the flow is not laminar) severe additional difficulties arise when thermal radiation is of importance. Basically, this is due to the absence of a simple phenomenological law, analogous to the Fourier conduction law, to express the divergence of the radiation heat flux vector in terms of temperature in the energy equation. In a rigorous formulation of the radiation effect the divergence of the radiation heat flux vector is expressed in terms of the spectral intensity which in turn must satisfy the transfer equation. The formal solution of the transfer equation then relates the intensity to the gas and surface temperature distribution provided a knowledge of the spectral absorption coefficient is available.

As discussed in greater detail below, implementation of this scheme for relating the divergence of the radiation heat flux vector to the temperature field depends strongly on a suitable absorption coefficient model, geometry and boundary conditions.

A considerable amount of recent research [1]* has been directed toward establishing suitable absorption coefficient models and some advances

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* Numbers in brackets [ ] designate references in the Bibliography.

have been made in treating radiating gases in geometries other than the infinite parallel plate channel (which is the simplest geometry to treat).[2,3,4,5] However, except for the investigation of Perlmutter, Siegel and Keshock[6,7,8,9] and Thorsen[10] (which are limited to radiatively non-participating gases) Landram, et al [11] (which is limited to optically thin radiation and neglects the possibility of axial radiation) and Cess[12] (which also neglects the possibility of axial radiation) the influence of non-isothermal boundaries has not been considered. Yet in many engineering applications, including high temperature heat exchangers and gas cooled nuclear reactors, the boundary temperature is unknown a-priori, though it is of critical importance from a structural design viewpoint.

The importance of these applications and the conspicuous lack of satisfactory analytic or experimental work dealing with internal flow of radiating gases subject to non-uniform boundary temperatures has motivated the present study. It is also worth mentioning that not only are isothermal walls infrequently encountered in practical high temperature applications, they are also difficult to achieve in laboratory investigations which seek experimental verification of radiating gas flow analyses. On the other hand, controlled wall heat generation boundary conditions can be achieved by electric current or electric arcs, for example. The results of this work, therefore, not only have direct engineering applicability, but can also serve as a base for future experimental research.

Review of Previous Work

To evaluate the specific applicability of the previous analyses dealing with combined conduction, convection and radiation heat transfer and their limitations, a review of past investigations is presented.

Viskanta[13,14] considered hydrodynamically fully-developed laminar flow and slug flow between two parallel, isothermal plates. Assuming a gray gas and expanding the non-linear radiation terms in a Taylor series the governing non-linear integro-differential equation was reduced to a non-linear ordinary differential equation. Digital computer solutions were then obtained for the temperature field and wall heat flux as a function of optical thickness. Einstein [5] considered slug flow in a circular, isothermal tube. Again, a gray gas was assumed. Hottel's zonal method was used to evaluate the radiation term in the energy equation.

It is to be emphasized that the analyses of both Viskanta and Einstein are restricted to isothermal boundaries and gray gases. Because of the severe dependence of the spectral absorption coefficient on wave number, a gray gas appproximation is not realistic and can lead to severe errors in temperature distribtution and heat flux calculations. Quantitiative support demonstrating the inadequancy of the gray gas approximation has been given by deSoto and Edwards [2] and deSoto [3]. These authors considered laminar flow of CO_2 in isothermal tubes. However, Edwards' exponential band model [15,16] was introduced to account for non-gray gas behavior. It was demonstrated that errors arising from the gray gas approximation could be on the order of 50%-100% in the estimate of the wall radiation heat flux. Since the radiation heat flux was of the same order of magnitude as the convection heat flux for the parameters considered, this represents a sizeable error.

In reference [2] deSoto and Edwards uncoupled the radiation effects from the conduction and convection effects by assuming the gas temperature field a priori. The radiation heat flux vector , $\vec{q}^R$, was obtained by numerically integrating the formal solution to the transfer equation. An important contribution of this investigation was its justification for neglecting axial radiation under isothermal boundary conditions. This was done by computing the radiation heat flux at the wall on the basis of two different a-priori temperature fields. First, the classical Graetz distribution [17] was used to describe the axial and radial gas temperature variation. Then it was assumed that the radial Graetz profile existing at the axial location at which $\vec{q}^R$ was to be calculated, prevailed throughout the entire tube. Since there was no axial temperature variation in the gas or on the tube wall for the second case, net axial radiation was precluded. Numerical results for the radiation contribution to the wall heat flux were in excellent agreement for the two cases, thereby justifying the neglect of axial radiation for the problem considered in reference [2]. The validity of this approximation was further confirmed by deSoto [3] for the coupled conduction, convection and radiation problem. In addition de Soto's results demonstrate that neglect of coupling effects between the three modes of energy transfer can lead to errors in the temperature profiles and wall heat fluxes on the order of twenty-five to fifty percent.

Further work involving isothermal walls was successfully undertaken by Nichols [4] who considered turbulent, participating gas flow in a concentric tube annulus. The results are, however, limited to situations where radiation effects are inherently small compared to turbulent convection.

Analyses undertaken for prescribed heat generation in channel and tube walls were reported in the previously mentioned papers by Perlmutter, Siegel and Keshock, [6,7,8,9] Thorsen [10,31] and Liu and Thorsen [32]. All of these analyses are restricted to non-participating gases and an a priori knowledge of the convection heat transfer coefficient is necessary to apply the methods of references [6,7,8,9]. In references [10,31,32] it is shown that such an a priori assumption of the convection coefficient is unsound because of the coupling, through the non-linear boundary conditions, of the gas conduction and convection processes with the wall radiation. For suitable parameters, it is shown in reference [10] that the convection coefficient can actually become locally negative as the tube exit is approached. Since this could never be the case for a prescribed uniform wall temperature boundary condition it would seem that caution should be exercised in extrapolating uniform temperature boundary condition results to situations where the boundaries will be at non-uniform temperatures.

Recently Landram, Greif and Habib [11] analyzed hydrodynamically established turbulent flow in a circular tube subjected to uniform wall heat flux. Only the optically thin limit was considered and axial radiation was neglected. Furthermore, the thermal entrance problem was neglected. Finally, Cess and Tiwari [12] have considered laminar thermally developed flow in an infinite parallel plate channel. Axial radiation was neglected and a one-dimensional approximation [18] was used for the radiation term in the energy equation. Tien's and Lowder's [19] correlation for the total band absorptance was used and the effect of pressure and path length were studied. Since the primary purpose of their investigation was to establish a satisfactory method of handling non-gray gases for a complete range of radiation path lengths their simplifying assumptions are justified. It is, however, noted that neither the work of Landram, et al [11] or Cess and Tiwari [12] give results consistent with those of Thorsen [10] if the absorption coefficient is allowed to approach zero. This is because, for uniform wall heat generation the local wall heat flux to the gas (due to conduction and radiation) will not be uniform since the non-isothermal tube (or channel) wall elements can radiate to each other and the tube (or channel) ends. The importance of this effect would appear to depend on the optical thickness of the gas.

Present Work

In the present investigation a general approach is presented for treating combined mode heat transfer. Specific application is made to a circular tube of finite length with uniform wall heat generation. However, the basic features of the solution technique are applicable to other geometries and arbitrarily prescribed wall heat generations or temperature distributions. Simplifying assumptions, made to reduce the complexity of the calculations required, do not preclude the applicability of the basic techniques to less restrictive situations. Furthermore, a basis for most of the simplifying assumptions can be found in the results of this and other authors. Thus, for example, results are obtained only for hydrodynamically developed flows in tubes having black surfaces. Both of these restrictions shall be qualitatively justified.

The question of the importance of axial radiation is studied, the importance of gas participation is demonstrated and both wall and gas temperature distributions for a variety of flow and thermal conditions are established when the gas is taken to be optically thin water vapor.

II. ANALYSIS

The Gas Temperature Field

The problem to be considered is depicted in Fig. 1. Gas is assumed to enter the passage with a fully developed velocity profile and a uniform temperature T_i. Neglecting compressibility effects and viscous dissipation and further assuming steady flow, it is necessary to consider only the energy equation for the gas in the form [18]

$$u \frac{\partial T}{\partial x} = \nabla \cdot \alpha \nabla T - \nabla \cdot \vec{q}^R / \rho C_p \; . \qquad (1)$$

For turbulent flow, α in equation (1) must be replaced by $(\alpha + \epsilon_H)$. The divergence of the radiation heat flux vector appearing in equation (1) cannot be related to the local gas temperature by a simple phenomenological law as in the case of thermal conduction. However, as discussed in detail below, the radiation term in equation (1) can be related to integrals of the spectral intensity, which in turn must satisfy the Radiative Transfer equation. The solution to this equation depends, in a non-linear manner, on the temperature field through the Planck function, the surface temperatures and the gas absorption coefficient. For the present discussion, however, it is sufficient to recognize that the divergence of the radiation heat flux vector depends in a complicated, non-linear manner on the entire temperature field. Then in anticipation of an iterative scheme for solving equation (1) the radiation term shall be treated as a known function of the space coordinates. Thus, if T represents the nth iterated temperature at a point and $T^-(x,r)$ represents the $(n-1)$st iterated temperature field, equation (1) can be written as

$$u(r) \frac{\partial T}{\partial x} = \nabla \cdot \alpha \nabla T + g\{T^-(x,r)\} \qquad (2)$$

where, for compactness,

$$g\{T^{-}(x,r)\} = -\nabla\cdot\vec{q}^{R}/\rho c_p \tag{3}$$

has been introduced. It is to be emphasized that g is considered to be a known function of position in equation (2) through the (n-1)st iterated temperature field.

As a final restriction, axial conduction in the gas will be neglected. Then, if x is interpreted as a time-like variable, equation (2) is analogous to the transient heat conduction equation with spatially dependent properties (density and thermal conductivity) and heat source. However, before pursuing this analogy it will be necessary to specify initial and boundary conditions. The initial (inlet) condition, as already mentioned, is

$$T(o,r) = T_i \tag{4}$$

whereas the boundary condition at the tube surface will be taken as

$$T(x,r_o) = T_w(x) \tag{5}$$

In equation (5) $T_w(x)$ can be arbitrary. For the problem at hand, the wall heat generation and not the wall temperature (or heat flux) is prescribed. $T_w(x)$ will actually be determined from the boundary condition derived below. However, for the present, only the gas temperature field in terms of an arbitrary surface temperature is desired. Continuing, symmetry at the tube centerline dictates that

$$(\partial T/\partial r)_{r=o} = 0 \tag{6}$$

Returning to the analogy between equation (2) and the conduction equation it is recognized that if u and α were constant the solution techniques for determining $T(x,r)$ could be found in several sources, e.g., [23,24]. Though this is not the case when u and α depend on r, it will now be shown that the Green's function approach presented in these references can be modified and extended to the problem at hand.

A Green's function, $G(x,r \,\|\, \xi,\rho)$ is defined to satisfy

$$u(\rho)\,\frac{\partial G}{\partial \xi} + \nabla\cdot\alpha\,(\rho)\,\nabla\, G = \underline{\delta}\ ,\quad x > \xi \tag{7a}$$

$$G = 0\ ,\quad x \leqq \xi \tag{7b}$$

$$G = 0 \qquad \text{at } \rho = r_o \tag{7c}$$

$$\frac{\partial G}{\partial \rho} = 0 \qquad \text{at } \rho = 0\ . \tag{7d}$$

Paralleling the procedure for constant coefficients, the expression

$$\begin{aligned}&\int_o^x\int_R G(\nabla\cdot\alpha\nabla T - u\,\frac{\partial T}{\partial \xi})\,dR\,d\xi = \\ &\int_o^x\int_R G(\nabla\cdot\alpha\nabla T)\,dR\,d\xi - \int_R\int_o^x Gu\,\frac{\partial T}{\partial \xi}\,d\xi\,dR\end{aligned} \tag{8}$$

is considered. In order to continue the Green's function formulation, an extended form of Green's identity must be derived. This is done in a manner similar to the derivation of Green's identity for the case of $\alpha = 1$, presented in [21]. The generalized result is

$$\int_R \left[v\ \nabla\cdot\alpha\nabla w - w\ \nabla\cdot\alpha\nabla\, v\right] dR = \oint_S \alpha\left[v\,\frac{\partial w}{\partial n} - w\,\frac{\partial v}{\partial n}\right]\,d\,S \tag{9}$$

where v and w are arbitrary scalar functions and n is the outward directed normal on S . Applying equation (9) to (8) then results in

$$\begin{aligned}&\int_o^x\int_R G(\nabla\cdot\alpha\nabla T - u\,\frac{\partial T}{\partial \xi})\,dR\,d\xi = \int_o^x\int_R T(\nabla\cdot\alpha\nabla G + u\,\frac{\partial G}{\partial \xi})\,dR\,d\xi \\ &- \int_R \Big[uGT\Big]_{\xi=o}^{\xi=x}\,dR + \int_o^x\oint_S a\left[G\,\frac{\partial T}{\partial n} - T\,\frac{\partial G}{\partial n}\right]dS\,d\xi\end{aligned} \tag{10}$$

Introducing equations (2), (4), (5), (6) and (7a-7d) results in the solution for the temperature field, viz.,

$$T = -\int_0^x \int_R G g \, dR \, d\xi - \int_R u [G]_{\xi=0} \, T_i \, dR + \int_0^x \oint_S \alpha T \frac{\partial G}{\partial n} \, dS \, d\xi \, . \tag{11}$$

Details of all the manipulations leading to equation (11) along with the derivation of equation (9) are presented in Appendix A of Ref. [33].

It is noted that neither the restriction of tangential symmetry nor uniform inlet temperature is required to derive equation (11). Furthermore, it is applicable to channels and annuli as well as tubes provided axial conduction is neglected. However, for the problem under consideration, i.e., a circular tube, equation (11) can be written as

$$\begin{aligned} T(x,r) = &- 2\pi \int_0^x \int_0^{r_o} G(x,r \mid \xi,\rho) \, g(\xi,\rho) \rho \, d\rho \, d\xi \\ &- 2\pi \, T_i \int_0^{r_o} u(\rho) G(x,r \mid 0,\rho) \, \rho \, d\rho + 2\pi \, r_o \alpha(r_o) \int_0^x T_w(\xi) \left(\frac{\partial G}{\partial \rho}\right)_{\rho=r_o} d\xi \end{aligned} \tag{12}$$

Since the eddy diffusivity at the wall will be zero even for turbulent flow, $\alpha(r_o)$ appearing in (13) will be equal to the molecular thermal diffusivity.

The formalism leading to equation (12) would be of little value unless the Green's function could be determined. Using generalized transform theory it can be shown, as is done in [24] for constant coefficients, that the form of G is

$$G(x,r \mid \xi,\rho) = \sum_{k=0}^{\infty} \frac{1}{2\pi} A_k \, e^{-\beta_k^2 (x-\xi)} \, R_k(r) \, R_k(\rho) \tag{13}$$

However, the eigenvalue problem for β_k and R_k requires a second order ordinary differential equation with variable coefficients to be solved.

These variable coefficients which arise from $u(r)$ and $\alpha(r)$ make this a formidable problem and asymptotic methods are generally required for solution. Fortunately, an alternate approach, which exploits solutions to related problems in the heat transfer literature, can be applied. A reduced problem is introduced:

$$u(r)\,\frac{\partial T}{\partial x} = \nabla\cdot\alpha\,\nabla\,T \tag{14a}$$

$$T_i(r) = 1 \tag{14b}$$

$$T_w(x) = 0 \tag{14c}$$

For laminar flow in a circular tube this is referred to as the "Graetz Problem" [17]. However, since turbulent flow is also to be considered and because the procedure presented here is applicable to channels and annuli, (14a - 14c) shall be consistently referred to as the Reduced Problem. The solution to the Reduced Problem is

$$\theta(x,r) = \sum_{k=o}^{\infty} B_k\, e^{-\beta_k^2 x}\, R_k\,(r)\ . \tag{15}$$

B_k, β_k and R_k have been established for a variety of flow situations including flows in tubes [17], channels [17] and annuli [25,26] . To obtain the A_k's in (13) the solution given by equation (12) is applied to the Reduced Problem resulting in

$$\theta(x,r) = -\,2\pi \int_o^{r_o} u(\rho)\ G(x,r\,|\,o,\rho)\ \rho d\rho\ . \tag{16}$$

When equation (13) is substituted into equation (16) and the results compared termwise with equation (15) the desired A_k's can be determined. The integrals of the eigenfunctions which arise in this procedure are evaluated in Appendix A where the coefficients, A_k , are evaluated for laminar flow in a circular tube.

To strengthen the reader's confidence in equation (12), it will be put in an alternate form. Integrating the last term in (12) by parts and employing equations (13), (15) and (16) leads

$$T(x,r) = T_i + \int_0^x [1-\theta(x-\xi,r)]\ dT_w - 2\pi \int_0^x \int_0^{r_o} gG\rho d\rho d\xi \qquad (17)$$

The complete details leading to equation (17) are presented for reference in Appendix B.

In the absence of gas radiation or internal heat generation ($g = 0$) equation (17) is identical to equation (33) of [17]. The present approach, however, has the added generality of including heat generation or, as explained earlier, radiation in the gas.

It is clear from equation (17) that if g were known and the surface temperature variation were prescribed, $T(x,r)$ could be computed. However, when the wall heat generation is prescribed these quantities are determined by coupling between equation (17) and the energy equation for the tube surface.

Energy Equation for the Tube Wall

Assuming the outer surface of the tube to be perfectly insulated and the wall to be sufficiently thin to permit neglecting radial temperature gradients the first law of thermodynamics for a differential ring at x (see Fig. 1) leads to

$$\left(\frac{k_w A_w}{\pi d}\right) \frac{d^2T_w}{dx^2} + q''(x) + H(x) = B(x) + k \left(\frac{\partial T}{\partial r}\right)_{r=r_o} \qquad (18)$$

In the last equation $H(x)$ and $B(x)$ represent the local irradiation and radiosity of the tube inside surface at x and can be expressed as integrals of the spectral intensity, which generally depends on the entire temperature field. Equations (17) and (18) represent a coupled integro-differential equation system which will generally require an iterative scheme for solution. It is to be emphasized that when the wall temperature is known equation (17) completely describes the gas temperature field for a given absorption coefficient model. However, when the surface heat generation is prescribed then equation (18) is also required.

With the restrictions of (a) tangential symmetry, (b) hydrodynamically developed flow (c) uniform inlet temperature and (d) negligibly small axial gas conduction equations (17) and (18) represent an equation system for the gas and wall temperature distributions provided suitable end conditions are imposed on the wall temperature. (These conditions will be discussed below). In order to apply this rather general formulation the details of the gas/wall radiation processes must be specified so that the radiation source term, g, the irradiation H and radiosity B can be prescribed in terms of the temperature field.

The Radiation Field

Knowledge of the local monochromatic intensity, I_ν, is sufficient to determine g, H and B in equations (17) and (18). The intensity must satisfy the equation of radiative transfer,

$$\vec{\ell} \cdot \nabla I_\nu = \rho \kappa_\nu (I_{b\nu} - I_\nu) \quad . \tag{19}$$

The formal solution to eq. (19) is

$$I_\nu = I_\nu(S_W) e^{-\tau_\nu(S_W)} + \int_0^{\tau_\nu(S_W)} I_{b\nu} e^{-\tau_\nu(S)} \, d\tau_\nu \tag{20}$$

where

$$\tau_\nu(S) = \int_0^S \rho\, \kappa_\nu dS \tag{21}$$

and in Fig. 2 the definitions of $\vec{\ell}$ and s are clarified.

Following the development of ref. [20] , the source term, g, is given by

$$g(x,r) = -\frac{\nabla \cdot \vec{q}^R}{\rho C_p} = \frac{1}{\rho C_p}\int_{\nu=0}^{\infty}\int_{\omega=4\pi} \rho\, \kappa_\nu (I_\nu - I_{b\nu})\ d\omega d\nu \tag{22}$$

The definitions of H and B respectively result in

$$H(x) = \left[-\int_{\nu=0}^{\infty}\int_{\substack{\omega=2\pi \\ \vec{\ell}\cdot\vec{n}<0}} I_\nu\, \vec{\ell}\cdot\vec{n}\ d\omega d\nu\right]_{S\to S_W} \tag{23}$$

and

$$B(x) = \left[\int_{\nu=0}^{\infty}\int_{\substack{\omega=2\pi \\ \vec{\ell}\cdot\vec{n}>0}} I_\nu\, \vec{\ell}\cdot\vec{n}\ d\omega d\nu\right]_{S\to S_W} \tag{24}$$

Clearly the evaluation of g, H and B is a rather formidable task and requires complete knowledge of the spectral absorption coefficient, κ_ν, which depends strongly on ν and on temperature.

Before proceeding with simplifications which will facilitate the evaluation of g, H and B some qualitative observations are in order. The results of ref. [10] for non-participating gases indicate that the wall and gas temperature distributions are relatively insensitive to surface absorptivity (except near the tube ends). Thus the wall is taken to be black and $I_\nu(S_W)$ in eq. (20) becomes $I_{b\nu}(T_W)$. The first term in eq. (20) represents energy radiated by the wall that is partially attenuated by gas absorption before it reaches point (x, r). In general this term depends on the wall temperature and gas temperature along s, through κ_ν. The second term in eq. (20) represents energy emitted by the gas along s that is then partially attenuated before reaching (x,r). Again this term depends on the gas temperature along s. When I_ν from eq. (20) is substituted into eq. (22) it becomes clear that g can be looked upon as a known function of the temperature field, thus confirming the validity of the iterative scheme alluded to in eq. (2).

Optically Thin Radiation

The essential mathematical features of combined conduction, convection and radiation can be preserved while achieving computational simplifications by considering the case of optically thin radiation. Furthermore, since axial radiation effects are known to be small in the gas (see Introduction), it is only the non-isothermal wall of the present investigation that can give rise to axial radiation effects. In the case of optically thin radiation the wall contribution to the radiation will be most pronounced and therefore represents a good case for establishing whether or not axial radiation is ever important.

The optically thin case is achieved by assuming that for all frequencies and all directions $\tau_\nu << 1$. Recognize that

$$e^{-\tau_\nu} = 1 - \tau_\nu + O(\tau_\nu^2) \tag{25}$$

Substituting eq. (20) into eq. (22) the expression for g, correct to order τ_ν, is

$$g(x,r) = \frac{1}{\rho C_p}\left\{\int_{\nu=0}^{\infty}\int_{\omega=4\pi}\rho\kappa_\nu I_\nu(S_W)\, d\omega d\nu - 4\rho\kappa_p\sigma T^4\right\} \tag{26}$$

It is observed that the source term behaves as if all the radiant energy arriving at (x,r) came directly from the solid boundaries, unattenuated by interlayer absorption and that the gas does not radiate to itself.

When the order of integration in eq. (26) is reversed and the black wall assumption introduced there results

$$g(x,r) = \frac{1}{\rho C_p}\left\{\int_{\omega=4\pi}\left[\rho \kappa_{PM}\,(T,T_w)\sigma T^4(S_w)/\pi\right] d\omega - 4\rho\kappa_p\sigma T^4\right\} \tag{27}$$

In eq. (27) $\kappa_{PM}(T,T_w)$ is the modified Planck mean absorption coefficient (see ref. [18]). Cess and Mighdoll [30] have shown that the modified Planck mean absorption coefficient can be very accurately approximated by

$$\kappa_{PM}\,(T,T_w) = \kappa_p\,(T_w)\cdot\frac{T_w}{T} \tag{28}$$

Equation (27) can thus be further simplified to read

$$g(x,r) = \frac{1}{\rho C_p}\left\{\int_{\omega=4\pi}\left[\rho\,\kappa_p(T_w)\sigma T^5(S_w)/\pi T(x,r)\right]\,d\omega - 4\rho\kappa_p\sigma T^4\right\} \tag{29}$$

Physically, there are three distinct contributions to the integral in eq. (26). These arise from the tube wall and the two tube ends. The tube ends shall be taken as black surfaces at the inlet and exit gas bulk temperatures respectively. In an approximate manner these inlet and exit conditions correspond to inlet and exit headers. Furthermore, it is noted that the exit gas bulk temperature is a priori unknown since not all of the energy generated in the wall actually enters the gas while it is passing through the tube; some energy is radiated out the ends of the tube and can result in pre- or post-heating of the gas. This, however, is of no concern in the analysis. The details of evaluating the integral in eq. (29) are quite substantial and are presented in Appendix C. Only the result is presented here, viz.,

$$\begin{aligned} g(x^+,r^+) = \frac{2\sigma}{\pi C_p T}&\left\{\int_0^{L^+} \kappa_p(T_w)\,T_w^5\left[\frac{4-2A}{(A-B)\sqrt{A+B}}\,E(k) - \frac{2}{\sqrt{A+B}}\,K(k)\right] d\xi\right. \\ &\left. + \pi\kappa_p(T_i)T_i^5\,F(x^+,r^+) + \pi\kappa_p(T_e)T_e^5\,F(L^+-x^+,r^+)\right\} - \frac{4\sigma\kappa_p(T)}{C_p}\,T^4 \end{aligned} \tag{30}$$

In eq. (30) $L^+ = L/d$, $x^+ = x/d$ and $r^+ = r/r_o$. The superscript "+" will be dropped in subsequent equations, where it shall be understood that the dimensionless space variables are being used. A , B and k are defined in Appendix C and shall not be repeated here. They depend only on location. K and E are the complete elliptic integrals of the first and second kind respectively. $F(x,r)$ is the shape factor between a differential area element, with normal parallel to the tube axis, and a circular area, of diameter d , normal to the tube axis and located at the tube inlet.

The mathematical description of the gas temperature field is now completed by substituting eq. (30) into eq. (17) and introducing the Green's function given by eq. (13) and the Graetz solution given by eq. (15). The coefficients A_n and C_n are given in Appendix A and ref. [17] respectively. It now remains to establish the expressions for $H(x)$ and $B(x)$ appearing in the energy equation for the tube wall, eq. (18).

The general expression for the radiosity is given by eq. (24). For the case of a black surface, however, this reduces to

$$B(x) = \sigma T_w^4(x) \tag{31}$$

Evaluation of the local irradiation, $H(x)$, is not near so trivial. In fact, it will be evaluated by two different methods utilizing two different assumptions.

<u>Evaluation of $H(x)$, Method 1.</u> The first method for evaluating the irradiation at the tube wall is based on an oversimplified interpretation of eq. (26). This interpretation, however, simplifies the evaluation of $H(x)$ while still permitting an estimation of the importance of axial radiation effects on the gas and wall temperature solutions. Physically, the interpretation of the solution, eq. (20), to the Transfer Equation reveals that the intensity at a point P , having direction $\vec{\ell}$ (see fig. 2), is equal to

the intensity at S_w , i.e., at the wall, attenuated by interlayer gas absorption plus a contribution (the integral in eq. (20)) due to gas emission which is then also attenuated. Now, comparing eqs. (22) and (26) implies that $I_\nu \approx I(S_w)$, i.e., at any point the intensity is just equal to the intensity associated with the radiation leaving the wall. Thus, for purposes of evaluating $H(x)$ it would appear reasonable to treat the intervening gas as non-participating. Then, as shown in eq. (20) of ref. [33]

$$H(x) = \sigma T_i^4 F^+(x) + \sigma T_e^4 F^+(L-x) + \int_0^L \sigma T_w^4(\xi) K(x,\xi)\, d\xi \tag{32}$$

Expressions for $F^+(x)$, the shape factor between a ring at x (see Fig. 1) and a circular disk at the tube entrance and for $K(x,\xi)$, the shape factor between differential rings at x and ξ , are given by eqs. (35) and (36) of ref. [33].

Together with auxilliary equations for G , θ , g , B and H [eqs. (13), (15), (30), (31) and (32) respectively], equations (17) and (18) represent a coupled integro-differential equation system for the entire temperature field. The numerical iterative scheme used to solve these equations is described below. In the graphical results, discussed below, solutions to this equation system are shown as broken lines.

Considerable simplification in the evaluation of eq. (29) could have been achieved by making what shall be called the "quasi-one-dimensional" assumption. This assumption consists of replacing $T(S_w)$ in eq. (29) by $T_w(x)$, x being the axial position at which $g(x,r)$ is to be evaluated. It then clearly follows from eq. (C-1) that

$$\begin{aligned} g(x,r) = \frac{2\sigma}{\pi C_p T} \Big\{ &\pi[2-F(x,r) - F(L-x,r)]\, K_p(T_w) T_w^5 \\ &+ \pi F(x,r) K_p(T_i) T_i^5 + \pi F(L-x,r) K_p(T_e) T_e^5 \Big\} - \frac{4\sigma K_p(T)}{C_p} T^4 \end{aligned} \tag{33}$$

where x and r are dimensionless and F is defined in Appendix C. Results obtained for $T_w(x)$ and $T(x,r)$ using eq. (33) rather than eq. (30) for $g(x,r)$ reveal that the error introduced by the quasi-one-dimensional assumption is undetectable in the graphs presented, the error always being less than 1%.

Evaluation of $H(x)$, Method 2. Evaluation of $H(x)$ by Method 1, though simpler than Method 2, has been presented only to establish the accuracy of the quasi-one-dimensional approximation for g. However, it introduces a fundamental error into the formulation, i.e., the solution obtained is in violation of the First Law of Thermodynamics. This is evident from an examination of results (not presented here). It was found that the sum of the energy radiated out the tube ends and the energy added to the gas was greater than the energy generated in the tube wall. The magnitude of this error depended upon the tube diameter, length to diameter ratio and level of wall heat generation. For the cases studied it varied from 1% to 10% of the wall heat generation.

The physical explanation for this error is as follows. A non-zero radiation source term, g, implies that the gas is absorbing radiant energy. Thus, when a wall element radiates, its energy cannot reach the other wall elements and tube ends unattenuated. Though eq. (26) for g is correct to order τ_ν, it cannot be concluded that $I_\nu = I_\nu(S_w)$ as on page 18. This point is also made in chapter 7 of ref. [18]. The expression for I_ν, correct to order τ_ν, must be obtained independent of eq. (26). Equation (20), correct to order τ_ν, becomes

$$I_\nu = I_\nu(S_w)\ [1 - \tau_\nu(S_w)] + \int_0^{S_w} \rho\, \kappa_\nu\, I_{b\nu}\, dS \tag{34}$$

Employing eqs. (23) and (34) results in $H(x)$ being composed of three terms,

$$H(x) = I_1(x) - I_2(x) + I_3(x) \tag{35}$$

where

$$I_1(x) = \int_\nu \int_{\omega=2\pi} I_\nu(S_w) \cos\theta d\omega d\nu \tag{36}$$

$$I_2(x) = \int_\nu \int_{\omega=2\pi} \rho \bar{\kappa}_\nu S_w I_\nu(S_w) \cos\theta \, d\omega d\nu \tag{37}$$

$$I_3(x) = \int_\nu \int_{\omega=2\pi} \int_0^{S_w} \rho \kappa_\nu I_{b\nu} dS \cos\theta \, d\omega d\nu \tag{38}$$

Here θ is the angle between S_w and the normal vector of the receiving wall element. In eq. (37) $\bar{\kappa}_\nu$ is defined by the relation

$$\bar{\kappa}_\nu = \frac{1}{\rho S_w} \int_0^{S_w} \rho \kappa_\nu dS \tag{39}$$

and will be further discussed below.

$I_1(x)$ represents radiation leaving a wall element and arriving unattenuated at a receiving element. Evaluation of eq. (36) would therefore result in the right hand side of eq. (32), i.e.,

$$I_1(x) = \sigma T_i^4 F^+(x) + \sigma T_e^4 F^+(L-x) + \int_0^L \sigma T_w^4(\xi) \, K(x,\xi) \, d\xi \tag{40}$$

The quasi-one-dimensional assumption is now employed in the evaluation of $I_2(x)$. In the quasi-one-dimensional case the radial temperature distribution at x ($0 \leq r \leq 1$, dimensionless r), is assumed to exist at all axial positions. Thus, eq. (37) becomes

$$I_2(x) = \int_\nu \rho \bar{\kappa}_\nu I_\nu [T_w(x)] \int_{\omega=2\pi} S_w \cos\theta \, d\omega d\nu \tag{41}$$

The inner integral evaluates as πd . Thus, applying the approximation given by eq. (28) for the modified Planck mean absorption coefficient, results in

$$I_2(x) = \frac{\rho \kappa_P(T_W)\sigma\, T_W^5}{T_m} \cdot d \tag{42}$$

T_m in eq. (42) is the temperature at which $\overline{\kappa}_\nu$ is to be evaluated. It is clear from the definition of $\overline{\kappa}_\nu$ that determination of T_m is no trivial matter. For the case of the symmetric parallel plate channel, S.T. Liu* has evaluated the appropriate mean temperature, T_m . His result, applied to the tube, becomes

$$T_m(x) = \frac{1}{\int_0^1 \frac{1}{T(x,r)}\, dr} \tag{43}$$

It is interesting to note that $I_2(x)$ is directly proportional to d , a point that will be discussed further when the results are presented.

Performing the frequency integration in eq. (38) results in

$$I_3(x) \int_{\omega=2\pi} \int_0^{S_W} \frac{\rho \kappa_P(T)\sigma T^4}{\pi} \cos\theta \; dS \; d\omega \tag{44}$$

$I_3(x)$ has also been evaluated for the parallel plate channel. However,

$$\kappa_P(T)T^4 << \frac{\kappa_P(T_W)T_W^5}{T_m} \tag{45}$$

except near the wall, and the contribution to the inner integral in eq. (44) from gas near the wall will be small compared to I_2 since only a small fraction of S_W contributes to the integral in eq. (44) whereas eq. (41) has S_W as a factor. This qualitative argument has been confirmed by Liu's numerical results. Thus $H(x)$ can reasonably be approximated by

* This result, not yet published, was obtained by S.T. Liu at New York University in connection with work on his Ph.D. dissertation. It will be incorporated into his dissertation which is expected to be available by February 1971.

$$H(x) = \sigma T_i^4 F^+(x) + \sigma T_e^4 F^+(L-x) + \int_0^L \sigma T_w^4 K(x,\xi)d\xi - \frac{\rho \kappa_p(T_w)\sigma T_w^5}{T_m} \cdot d \quad (46)$$

It is noted that if a hot gas were being cooled this argument would not apply since the inequality expressed in (45) would be reversed.

The term in eq. (18) accounting for conduction to the gas at the tube wall can be determined by evaluating the radial derivative of eq. (17) at the tube wall, i.e.,

$$k \left.\frac{\partial T}{\partial r}\right)_{r=r_o} = - k \int_0^x \theta_r(x-\xi, r_o)\, dT_w - 2\pi k \int_0^x \int_0^{r_o} g(\xi,\rho) \left.\frac{\partial G}{\partial r}\right|_{r=r_o} \rho\, d\rho\, d\xi \quad (47)$$

Substituting equations (31), (46) and (47) into eq. (18) results in

$$\left(\frac{k_w A_w}{\pi d}\right) \frac{d^2 T_w}{dx^2} + q''(x) + \sigma T_i^4 F^+(x) + \sigma T_e^4 F^+(L-x) + \int_0^L \sigma T_w^4 K(x,\xi)d\xi =$$

$$\sigma T_w^4 + \frac{\rho \kappa_p(T_w)\sigma T_w^5}{T_m} d - k \int_0^x \theta_r(x-\xi, 1)\, dT_w - 2\pi k \int_0^x \int_0^1 g(\xi,\rho) \left.\frac{\partial G}{\partial r}\right|_{r=1} \rho\, d\rho\, d\xi$$

(48)

In equation (48) dimensionless space variables are being used. G, g, T_m, F^+ and K have been previously defined and discussed and

$$\theta_r(x-\xi, 1) = \sum_{n=0}^{\infty} C_n R_n'(1) \exp\left[\frac{-2\lambda_n^2}{Pe}(x-\xi)\right] . \quad (49)$$

The complete analytic formulation of combined conduction, convection and optically thin radiation for laminar tube flow with prescribed wall heat generation is contained in eqs. (17) and (48) along with the auxiliary equations for G, θ, g, T_m and geometric factors. The details leading to determination of $T(x,r)$ are described in the next chapter. Readers not interested in these details can skip to Chapter IV where the results are presented and discussed.

III. MATHEMATICAL AND COMPUTATIONAL DETAILS

The solution of equations (17) and (47) is a formidable computational task. However, by careful examination of these two equations, it will be shown that much of the technique established for the non-participating gas case can be used for the participating gas case. The first order of business, however, will be to put eqs. (17) and (47) into dimensionless form. Introducing

$$\psi = T/T_i \tag{50}$$

$$q''_{Av} = \frac{1}{L}\int_0^L q''(x)\,dx \tag{51}$$

$$P = \left(\frac{k_w A_w}{q''_{Av}\, d^3}\right)\left(\frac{q''_{Av}}{\sigma}\right)^{1/4} \tag{52}$$

$$\mathcal{T} = T_i(\sigma/q''_{Av})^{1/4} \tag{53}$$

$$N = q''_{Av}\, d/kT_i \tag{54}$$

$$\bar{g} = \frac{k}{\bar{u}q''_{Av}}\, g \tag{55}$$

$$\bar{G} = (\alpha r_o Pe)\, G = (2\bar{u}\, r_o^2)\, G \tag{56}$$

$$f(x) = q''(x)/q''_{Av} \tag{57}$$

results in

$$\psi(x,r) = 1 + \int_0^x [1-\Theta(x-\xi,r)]d\psi_w - (2\pi)\left(\frac{N}{2}\right)\int_0^x\int_0^1 \bar{g}(\xi,\rho)\bar{G}(x,r|\xi,\rho)\rho\, d\rho\, d\xi \tag{58}$$

and

$$P\tau \frac{d^2\psi_w}{dx^2} + \frac{2}{N}\int_0^x \Theta_r(x-\xi,1)\,\frac{d\psi_w}{d\xi}\,d\xi = \tau^4\left[1-\rho\kappa_P(T_w)\cdot\left(\frac{\psi_w}{\psi_m}\right)d\right]\psi_w^4 - \tau^4\int_0^L \psi_w^4\,K(x,\xi)d\xi - S(x) \tag{59}$$

where

$$S(x) = f(x) + \tau^4\left[F^+(x) + \psi_e^4\,F^+(L-x)\right] + 2\pi\int_0^x\int_0^1 \bar{g}(\xi,\rho)\left.\frac{\partial\bar{G}}{\partial r}\right|_{r=1}\rho\,d\rho\,d\xi \tag{60}$$

When equations (59) and (60) are compared with eq. (30) of ref. [33] or eqs. (26) and (28) of ref. [31] for the case of a black surface, it is noted that the difference between the wall energy equation for the present case and the non-participating case is contained in the factor in brackets multiplying $\tau^4\psi_w^4$ in eq. (59) and the integral in eq. (60). In terms of the computational algorithm for solving the wall energy equation these changes are relatively minor.

In broad terms the solution technique proceeds as follows:

Step 1. Initial values for $\psi(x,r)$ and $\psi_w(x)$, referred to as $\psi^-(x,r)$ and $\psi_w^-(x)$, are estimated. $\psi_w^-(x)$ is determined from the non-participating gas solution (see ref. [33]). Then $\psi^-(x,r)$ is determined from eq. (58) with $g = 0$.

Step 2. Based on $\psi^-(x,r)$ $g(x,r)$ is determined.

Step 3. ψ_e is evaluated from the definition of bulk temperature, i.e.,

$$\psi_e = 2\int_0^1 \psi(L,r)[1-r^2]\,r\,dr \tag{61}$$

Step 4. The integral in eq. (60), which shall be referred to as $V(x)$, is evaluated based on $g(x,r)$ from Step 2 and ψ_e

from Step 3. Also, the last term in eq. (58), referred to henceforth as $W(x,r)$, is evaluated based on Steps 2 and 3.

Step 5. A new ψ_W is computed using substantially the same numerical iterative scheme described in Appendix D of ref. [33].

Step 6. A new $\psi(x,r)$ is determined from eq. (58) and compared with $\psi^-(x,r)$ obtained in Step 1. The answer was accepted when $|\psi(x,r) - \psi^-(x,r)| < .01$ at every point. (A finite difference grid of 21 axial points including the ends and 11 radial points was used.) If $|\psi(x,r) - \psi^-(x,r)| \geq .01$ at any one of the 231 points then $\psi(x,r)$ and $\psi(x,1)$ replaced $\psi^-(x,r)$ and $\psi_W^-(x)$ in Step 1 and Steps 2 through 6 repeated.

The radiation source term, $g(x,r)$, is evaluated in Appendix C and the final expression is given by eq. (30). It was treated as a subroutine in the computer program where Simpson's rule was used for evaluation of the integral in eq. (30). $V(x)$ and $W(x,r)$, which require additional integration for their evaluation merit further discussion. Introducing the Greens function given by eq. (13) with coefficients given in Appendix A, and recognizing that $\beta_k^2 = 2\lambda_k^2/Pe$ results in

$$V(x) = 2.84606 \sum_{n=0}^{\infty} (-1)^n \lambda_n^{4/3} \int_0^x \exp\left[\frac{-2\lambda_n^2}{Pe}(x-\xi)\right] \cdot \int_0^1 \bar{g}(\xi,\rho)\, R_n(\rho)\, \rho\, d\rho\, d\xi \tag{62}$$

and

$$W(x,r) = -3.999 \left(\frac{N}{2}\right) \sum_{n=0}^{\infty} \lambda_n R_n(r) \int_0^x \exp\left[\frac{-2\lambda_n^2}{Pe}(x-\xi)\right] \cdot \int_0^1 \bar{g}(\xi,\rho) R_n(\rho)\, \rho\, d\rho\, d\xi \tag{63}$$

Here, use has been made of the relation between A_n and C_n in Appendix A, along with the numerical values of C_n and $R'_n(1)$ given in reg. [17], to arrive at

$$\overline{A}_n \, R'_n(1) = (-1)^n \, 2.84606 \, \lambda_n^{4/3} \tag{64}$$

and

$$\overline{A}_n = -\,3.999 \, \lambda_n \tag{65}$$

where

$$\overline{A}_n = 2\overline{u} \, r_o^2 \, A_n \tag{66}$$

(see eq. (56)).

It is noted that in both eq. (62) and eq. (63) the term

$$\int_0^1 \overline{g}\,(\xi,\rho)\, R_n(\rho)\, \rho \, d\rho \, d\xi \tag{67}$$

has to be evaluated and that evaluation of $V(x)$ and $W(x,r)$ are of roughly equal complexity.

Infinite sums appear in several terms of the governing equations. $V(x)$, $W(x,r)$ and the integrals involving θ and θ_r in eqs. (58) and (59) all contain infinite sums. The first six terms were found to give sufficient accuracy.

Numerical Constants. The constants λ_n , C_n and the eigenfunctions R_n are now presented for completeness. λ_n and C_n are given in ref. [17] as

$$\lambda_o = 2.704 \tag{68}$$

$$\lambda_1 = 6.679 \tag{69}$$

$$\lambda_n = 4n + \frac{8}{3} \,, \quad n > 1 \tag{70}$$

$$C_n = (-1)^n \; 2.84606 \, \lambda_n^{-2/3} \tag{71}$$

$$C_n R'_n(1) = 2.02552 \, \lambda_n^{-1/3} \tag{72}$$

Numerical values were obtained for $R_n(r)$ for $n = 0,1,\ldots,5$. at $r = 0,.1,\ldots,1$ by solving the eigenfunction problem for R_n .[35] Because these values were not available in the literature they are presented in Table I .

TABLE I

Eigenfunctions for Circular Tube - Laminar Flow

r	R_o	R_1	R_2	R_3	R_4	R_5
0	1	1	1	1	1	1
.1	.9818	.8922	.7358	.5313	.3025	.0750
.2	.9289	.6060	.1532	-.2327	-.4026	-.3213
.3	.8454	.2360	-.3147	-.3594	.0002	.2897
.4	.7380	-.1073	-.3924	.0674	.2992	-.0474
.5	.6144	-.3406	-.1435	.3151	-.0793	-.2055
.6	.4829	-.4321	.1686	.1149	-.2554	.1973
.7	.3508	-.3991	.3312	-.1955	.0356	.1041
.8	.2240	-.2874	.3036	-.2924	.2591	-.2087
.9	.1064	-.1451	.1643	-.1785	.1887	-.1955
1.0	-.0003	-.0046	.0021	-.0013	.0008	-.0006

IV. RESULTS

Examination of eqs. (58) and (59) reveals that a large number of parameters enter the solution to the problem considered in this report. Since it is not the purpose of this report to present a complete parametric study of the factors influencing the quantitative aspects of the solution, the influence of some factors shall be ignored. In particular, conduction effects in the tube wall shall be neglected. This corresponds to setting $P = 0$. It is shown in refs. [31, 33], for the non-participating case, that over a practical range of values the influence of P is small and does not alter the qualitative aspects of the solution. It should, however, be noted that where P is non-zero it is necessary to prescribe two boundary conditions at the tube ends. In refs. [31, 33] either the conditions

$$\frac{d\,Tw}{dx} = 0 \quad , \quad x = 0, L$$

or

$$Tw\,(0) = Ti \quad \text{and} \quad \frac{d\,Tw}{dx} = 0 \quad \text{at } x = L$$

were used. In the present work, with $P = 0$, it is necessary only to prescribe $T_w\,(0) = 1$.

It was also shown that the influence of the Peclet number, Pe, on the qualitative aspects of the results was also small. Therefore in the present case a single Peclet number of 1740 was considered. This corresponds to a Reynolds number of approximately 2000 when the properties of water vapor are evaluated at $1400^{o}F$. It was arbitrarily decided to evaluate all gas properties at $1400^{o}F$ and one atmosphere. The selection of $1400^{o}F$ was based on preliminary numerical results indicating an average gas bulk temperature of approximately $1400^{o}F$ for the range of parameters studied. One exception to this was Planck mean absorption coefficient which was treated as a temperature dependent quantity. The data for the Planck absorption coefficient was taken from fig. 3 of ref. [36] .

As previously mentioned the tube wall emissivity was taken as unity and hydrodynamically developed flow was considered. Some justification for this last assumption comes from the results for non-participating gases given in refs. [31, 33] . There, all other things being equal, the results for the following two cases are compared. In one case there is an abrupt entrance into the tube and it is assumed that the gas immediately becomes hydrodynamically developed as in the present case. In the second case a long unheated hydrodynamic entrance section is provided prior to the tube section having wall heat generation. Of course in this case the gas will indeed have a fully developed velocity profile at the start of the heating section. When the resultant wall temperature distributions for the two cases are compared it is found that within a few diameters downstream of the entrance to the heated section of tube the results are almost identical. It can therefore be concluded that the assumption of hydrodynamically developed flow at the tube entrance does not deteriorate the validity of the present solution for the tube wall temperatures.

The results to be presented and discussed below are based on a uniform inlet temperature of $1000^{o}R$. This temperature was selected because it is the lowest value for which Planck absorption coefficient data was readily available. Higher values of T_i were not investigated because it was desired to have resultant wall temperatures within the realm of practicality. $q''(x)$ was also selected to yield reasonable wall temperatures. In ref.[33] uniform, sinusoidal and step-uniform wall heat generation were studied. In the present work the thrust of attention is focused on the effect of gas participation. Therefore, there appeared to be no _a priori_ reason for considering anything other than uniform heat generation. In eq. (60) $f(x)$ was therefore taken as unity. Merely by prescribing suitable values for $f(x)$, other than uniform heat generation could be considered. In the results presented in this report numerical values for q'' were 40,000 and 60,000 $BTU/hr\text{-}ft^{2}\text{-}{}^{o}F$.

The two remaining quantities to be prescribed are L/d and d. The numerical formulation was such that for a given axial increment in the finite difference scheme, the storage space required was proportional to $(L/d)^2$. Reasonable values of L/d were considered but results are presented only for $L/d = 20$. Representative cases for $L/d = 40$ were investigated, but complete production runs providing results suitable for complete comparison with $L/d = 20$ were not sought. It was found that while L/d certainly influenced the numerical results, just as for the non-participating gas case (see refs. [31, 33]), no interesting qualitative differences, dependent on L/d, were exhibited. As mentioned below eq. (43), and shown in eq. (46), it is reasonable to expect the solution to depend explicitly on d. Three values of d were considered (1", 1/2" and 1/4").

In the graphical (figs. 4-7) and tabular (Tables 3 and 4) results to follow the appropriate parameters are given in Table 2.

TABLE II

Numerical Values Associated with Prescribed Constants Leading to Results

	q'' (BTU/hr-ft^2-^{o}F)	d (inches)
Case 1	40,000	1
Case 2	40,000	1/2
Case 3	60,000	1/4

In each case $T_i = 1000^oR$, $L/d = 20$, $Pe = 1740$, $\epsilon = 1$ and $P = 0$ and the gas is water vapor with properties other than the absorption coefficient evaluated at 1860^oR.

The results based on the numerical constants discussed above and in Table II are presented in Figs. 4 - 7. These results were obtained on a Univac 1108 computer. In the figures the dashed curves correspond to the non-participating gas solution obtained as described in ref. [33]. The broken curves represent the optically thin solution with H evaluted by method 1, i.e., eq. (32) and the solid curves represent the solution obtained using method 2, eq. (46).

It is noted in Figs. 4 - 6 that the wall temperature distribution for the optically thin case behaves qualitatively like that of the non-participating gas, i.e., it rises rapidly in the entrance region, reaches a maximum at an interior point and then declines as the tube exit is approached. The decline is due to end "losses", i.e., a net radiant energy exchange from the tube wall to the exit header. In Fig. 4 the gas bulk temperature variation is also shown. As expected the gas temperature is considerably higher in the non-participating case.

When Figs. 4 - 6 are compared the effect of d is revealed. It is noted that the solid and broken curves becomes progressively closer to each other as d decreases. This merely confirms, as previously indicated, that the difference between method 1 and method 2 for evaluating H(x) depends on d as can be seen by comparing eqs. (32) and (46).

The accuracy of the quasi-one dimensional approximation is shown in Table III where method 1 was used to evaluate H(x). Results for T_w/T_i when g is evaluated using eq. (30) (general) and eq. (33) (quasi-one dimensional) reveal that the difference occurs in the third decimal place, i.e., the two wall temperature distributions are within $10^{o}R$ of each other. Hence, the quasi-one dimensional approximation appears to be quite satisfactory.

In Fig. 7 the radial gas temperature variation for case 1 is shown for three axial positions. Of interest is the reversal in slope as the wall is approached when $x/d = 20$. This result can be explained by noting that the heat conducted to the gas depends on the slope of the wall temperature (see eq. (47)).

Though not shown in the graphs because of its limited applicability, a Nusselt number can be defined and evaluated. Let $q(x)$ represent the heat transferred to the gas at x (note: $q(x) \neq q''(x)$). $q(x)$ can be evaluted from the relation

$$q(x) = \frac{\dot{m}\ Cp}{\pi\ d} \frac{dT_b}{dx} \tag{73}$$

where $\dot{m}$ is the mass flow rate and T_b is the local bulk temperature. The a heat transfer coefficient, h, can be defined as

$$h(x) = \frac{q(x)}{T_w(x) - T_b(x)} \tag{74}$$

Defining a Nusselt number by

$$Nu = hd/k \tag{75}$$

then results in

$$Nu = \frac{Pe}{4(\psi_w - \psi_b)} \frac{d\,\psi_b}{dx} \tag{76}$$

Whereas in the case of a non-participating gas it can be expected that Nu is bounded by the Nusselt number results for the isothermal wall and uniform heat flux wall in the absence of <u>all</u> radiation effects (see refs. [31,33]), no such statement can be made for the present case since two parallel mechanisms, conduction and radiation, exist for transferring energy to the gas.

TABLE III

Effect of Quasi One-Dimensional Approximation on Wall Temperature Distribution. H Evaluated from Eq. (32) (CASE 1)

x/d	T_w/T_i (g From Eq. (30))	T_w/T_i (g From Eq. (33))
0	1.000	1.00
2	4.256	4.256
4	4.958	4.959
6	5.325	5.328
8	5.537	5.541
10	5.651	5.657
12	5.689	5.695
14	5.645	5.651
16	5.483	5.490
18	5.101	5.107
20	4.098	4.104

V. CONCLUSIONS

A general mathematical approach has been presented for handling combined conduction, convection and radiation gas flow problems. This technique which exploits the Greens function method for parabolic equations results in equations (17) and (18) which are not restricted to optically thin gases. To arrive at eq. (17) it was necessary to

(1) Neglect axial conduction effects in the gas

(2) Consider hydrodynamically developed flow

(3) limit consideration to flows having uniform inlet temperatures.

It has been shown (see ref. [33]) that the error introduced by the second restriction is small. The third restriction is not serious since eq. (11) instead of eq. (17) could have been employed. To obtain g, B and H a radiation model must be introduced and in the present work the optically thin gas has been considered. It is noted that if the wall temperature, rather than heat generation, is prescribed eq. (18) is not necessary and the difficulty in evaluating $H(x)$ (see pages 17 - 22) will not be encountered.

It can also be concluded from the present work that the quasi one-dimensional approximation for $g(x,r)$ leads to computational simplifications without sacrificing accuracy.

Lastly it is noted that the mathematical technique presented here is applicable to turbulent flow and non-circular passages provided the solution to the reduced problem is known. Furthermore the Greens function approach is applicable to combined conduction-radiation problems in radiating media other than gases. In the transient heating of a plastic slab, for example, the governing equation is the Heat Equation with a spatially dependent source term. The time variable would then correspond to x in the present work.

Recommendations

Several unanswered questions regarding problems of the class considered still remain. Some of these represent details in the writer's view; however, others are more fundamental. In the first category are:

(1) the effect of non-black surfaces

(2) establishment of correct T_m for tube problems

(3) evaluation of I_3 for tube problems

(4) consideration of participating gases that are not optically thin.

A more fundamental question is the importance of the axial conduction effect. Though this effect is likely to be overshadowed by radiation effects, it is important tc note that the solution technique presented here would not be applicable because the governing gas equation would become elliptic.

BIBLIOGRAPHY

1. C.L. Tien, "Thermal Radiation Properties of Gases," Advances in Heat Transfer, V, Academic Press, New York, 1968, pp. 253-324

2. S. deSoto and D.K. Edwards, "Radiative Emission and Absorption in Non-Isothermal, Non-Gray Gases in Tubes," Proceedings of the 1965 Heat Transfer and Fluid Mechanics Institute, Stanford University Press, Palo Alto California, 358-372 (1965).

3. S. deSoto, "Coupled Radiation, Conduction and Convection in Entrance Region Flow," International Journal of Heat and Mass Transfer, vol. 11, 39-53 (1968).

4. L.D. Nichols, "Temperature Profile in the Entrance Region of an Annular Passage Considering the Effects of Turbulent Convection and Radiation," International Journal of Heat and Mass Transfer, vol. 8, 589-607 (1965).

5. T.H. Einstein, "Radiant Heat Transfer to Absorbing Gases Enclosed in a Circular Pipe with Conduction, Gas Flow, and Internal Heat Generation," NASA TR R-156 (1963).

6. M. Perlmutter and R. Siegel, "Heat Transfer by Combined Forced Convection and Thermal Radiation in a Heated Tube," Journal of Heat Transfer, Trans. ASME, Series C, vol. 84, 301-311 (1962).

7. R. Siegel and M. Perlmutter, "Convective and Radiant Heat Transfer for Flow of a Transparent Gas in a Tube with a Gray Wall." International Journal of Heat and Mass Transfer, vol. 5, 639-660 (1962).

8. R. Siegel and E.G. Keshock, "Wall Temperatures in a Tube with Forced Convection, Internal Radiation Exchange and Axial Wall Heat Conduction," NASA TN D-2116, March 1964.

9. E.G. Keshock and R. Siegel, "Combined Radiation and Convection in a Asymmetrically Heated Parallel Plate Flow Channel," Journal of Heat Transfer, Trans. ASME, Series C, vol. 86, 341-350 (1964).

10. R.S. Thorsen, "Heat Transfer in a Tube with Forced Convection, Internal Radiation Exchange, Axial Wall Heat Conduction and Arbitrary Wall Heat Generation", International Journal of Heat and Mass Transfer, vol. 12, 1182 (1969).

11. C.S. Landram, R. Greif and I.S. Habib, "Heat Transfer in Turbulent Pipe Flow with Optically Thin Radiation", A.S.M.E. Paper No. 68-WA/HT-17. To be published in the Journal of Heat Transfer, Trans. A.S.M.E. Series C.

12. R.D. Cess and S.N. Tiwari, State University of New York at Stony Brook, College of Engineering Report, No. 90, 1967.

13. R. Viskanta, "Interaction of Heat Transfer by Conduction, Convection, and Radiation in a Radiating Fluid," Journal of Heat Transfer, Trans. ASME, Series C, vol. 85, 318-328 (1963).

14. R. Viskanta, "Heat Transfer in a Radiating Fluid with Slug Flow in a Parallel Plate Channel", Applied Scientific Research, Sec. A, vol. 13, 291-311 (1964).

15. D.K. Edwards and W.A. Menard, "Comparison of Models for Correlation of Total Band Absorption", Applied Optics, vol. 3, 621-625 (1964).

16. D.K. Edwards and W.A. Menard, "Correlations for Absorption by Methane and Carbon Dioxide Gases", Applied Optics, vol. 3, 847-852 (1964).

17. J. Sellars, M. Tribus and J. Klein, "Heat Transfer to Laminar Flow in a Round Tube or Flat Conduit - The Graetz Problem Extended," Trans. ASME, vol. 78, 441-448 (1956).

18. E.M. Sparrow and R.D. Cess, Radiation Heat Transfer, Brooks/Cole Publishing Co., Belmont, Cal. (1966).

19. C.L. Tien and J.E. Lowder, "A Correlation for Total Band Absorptance for Radiating Gases," International Journal of Heat and Mass Transfer, vol. 9, 698-701 (1966).

20. W.G. Vincenti, and C.H. Kruger, Jr. Introduction to Physical Gas Dynamics, John Wiley and Sons, Inc., New York, N. Y. Ch. XI (1965).

21. G.E. Hay, Vector and Tensor Analysis, Dover Publ., Inc., New York, pp. 145-146 (1953).

22. W.M. Kays, Convective Heat and Mass Transfer, McGraw-Hill Book Co. New York (1966).

23. H.S. Carslaw and J.C. Jaeger, Conduction of Heat in Solids, 2nd Ed., Oxford University Press, London, Ch. XIV (1959).

24. M.N. Özisik, Boundary Value Problems of Heat Conduction, International Textbook Co., Scranton, Penn., Ch. 1 and 5 (1968).

25. W.C. Reynolds, R.E. Lundberg and P.A. McCuen, "Heat Transfer in Annular Passages. General Formulation of the Problem for Arbitrarily Prescribed Wall Temperature or Heat Fluxes," International Journal of Heat and Mass Transfer, vol. 6, 483-493 (1963).

26. R.E. Lundberg, P.A. McCuen and W.C. Reynolds, "Heat Transfer in Annular Passages. Hydrodynamically Developed Laminar Flow with Arbitrarily Prescribed Wall Temperatures or Heat Fluxes," International Journal of Heat and Mass Transfer, vol. 6, 495-529 (1963).

27. R.D. Cess, P. Mighdoll and S.N. Tiwari, "Infrared Radiative Heat Transfer in Non-Gray Gases," International Journal of Heat and Mass Transfer, vol. 10, 1521-1532(1967).

28. R.D. Cess and S.N. Tiwari, "The Large Path Length Limit for Infrared Gaseous Radiation," Applied Scientific Research, vol. 19, 439-449 (1968).

29. R.D. Cess and S.N. Tiwari, "The Interaction of Thermal Conduction and Infrared Gaseous Radiation," Applied Scientific Research, vol. 20, 25-39 (1969).

30. R.D. Cess and P. Mighdoll, "Modified Planck Mean Coefficients for Optically Thin Gaseous Radiation," International Journal of Heat and Mass Transfer, vol. 10, 1291-1292 (1967).

31. R.S. Thorsen and D. Kanchanagom, "The Influence of Internal Radiation Exchange, Arbitrary Wall Heat Generation and Wall Heat Conduction on Heat Transfer in Laminar and Turbulent Tube Flows," Heat Transfer 1970, Preprints of Papers Presented at the Fourth International Heat Transfer Conference, Paris - Versailles, vol. III, R2.8 (1970).

32. S.T. Liu and R.S. Thorsen, "Combined Forced Convection and Radiation Heat Transfer in Asymmetrically Heated Parallel Plate Channels," Proceedings of the 1970 Heat Transfer and Fluid Mechanics Institute, June 10-12, 1970, Monterey, Cal., 32-44 (1970).

33. R.S. Thorsen, "Combined Conduction, Convection and Radiation Effects in Internal Flows - Nonparticipating Gases," New York University, Department of Mechanical Engineering Report No. F-69-4, September 1969.

34. I. S. Gradshleyn and I. M. Ryzhik, Tables of Integrals Series and Products, Academic Press, New York (1965).

35. M. Jakob, Heat Transfer, John Wiley and Sons, Inc., New York, N.Y. vol. 1, 1949.

36. M. M. Abu-Romia and C. L. Tien, "Appropriate Mean Absorption Coefficients for Infrared Radiation of Gases," J. Heat Transfer, Trans. ASME, Series C, vol. 89, pp. 321-327 (1967).

APPENDIX A. GREENS FUNCTION FOR LAMINAR TUBE FLOW

For laminar flow in a circular tube, the governing equation (14a) can be written as [17],

$$\frac{\partial T}{\partial x^+} = \frac{1}{1-r^{+2}} \cdot \frac{1}{r^+} \frac{\partial}{\partial r^+} \left(r^+ \frac{\partial T}{\partial r^+}\right) \tag{A-1}$$

where $r^+ = r/r_o$ and $x^+ = \frac{x}{P_e r_o}$.

The solution to the reduced problem can be written in two ways, viz.,

$$\theta = \sum_{n=0}^{\infty} C_n e^{-\lambda_n^2 x^+} R_n(r^+) \tag{A-2}$$

or

$$\theta = - 2\bar{u} r_o^2 \int_0^1 (1-\rho^2)\, G(x^+, r^+ | o, \rho)\, 2\pi\rho d\rho \tag{A-3}$$

Substituting equation (13) into (A-3) results in

$$\sum_{n=0}^{\infty} C_n e^{-\lambda_n^2 x^+} R_n(r^+) = - (2\bar{u}\, r_o^2) \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 x^+} R_n(r^+) \int_0^1 (1-\rho^2)\, R_n(\rho)\, d\rho \tag{A-4}$$

However, the separation equation for R_n is (see [17])

$$r^+ R_n'' + R_n' + \lambda_n^2 r^+ (1-r^{+2})\, R_n = 0 \tag{A-5}$$

Rearranging (A-5) leads to

$$- (1 - r^{+2})\, r^+ R_n = \frac{1}{\lambda_n^2} \frac{d}{dr^+} (r^+ R_n') \tag{A-6}$$

Substituting (A-6) into (A-4) and performing the indicated integration leads to

$$\sum_{n=0}^{\infty} C_n e^{-\lambda_n^2 x^+} R_n(r^+) = (2\bar{u}\, r_o^2) \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 x^+} R_n(r^+) \frac{R_n'(1)}{\lambda_n^2} \tag{A-7}$$

Thus

$$A_n = \lambda_n^2 C_n / [2\bar{u}\, r_o^2 R_n'(1)] \tag{A-8}$$

C_n and $R_n'(1)$ are given in reference [17]. A_n is thus evaluated as

$$A_n = -\frac{4 \times 6^{2/3} \times 3^{5/6} \Gamma(2/3) \Gamma(4/3)}{2^{2/3}\, (2\bar{u}\, r_o^2)} \lambda_n = \frac{-4\lambda_n}{\alpha\, r_o Pe} \tag{A-9}$$

APPENDIX B. DERIVATION OF EQUATION (17)

Since the integral containing g remains unchanged in going from equation (12) to equation (17), it is necessary only to prove the equivalence of the last two terms of equation (12) and the first two terms on the right hand side of (17). To this end consider

$$u(r)\,\frac{\partial T}{\partial x} = \frac{\alpha}{r}\,\frac{\partial}{\partial r}\, r\,\frac{\partial T}{\partial r} \tag{B-1}$$

$$T(o,r) = T_i \tag{B-2}$$

$$T(x,r_o) = T_w(x) \tag{B-3}$$

with

$$u(r) = 2\bar{u}\; r_o^2\left[1 - \left(\frac{r}{r_o}\right)^2\right] \tag{B-4}$$

The Green's function solution to this problem (eq.12)) is written as

$$T(x,r) = -2\pi\; T_i \int_o^{r_o} u(\rho)\; G(x,r|o,\rho)\rho d\rho + 2\pi\; r_o\; \alpha \int_o^{x} T_w(\xi)\; \left(\frac{\partial G}{\partial \rho}\right)_{\rho=r_o} d\,\xi \tag{B-5}$$

The dimensionless variables $r^+ = r/r_o$ and $x^+ = x/Per_o$ are now introduced, resulting in

$$T(x^+,r^+)/T_i = -\left[(2\pi)(2\bar{u}\; r_o^2)\right]\cdot\left\{\int_o^{1} (1-\rho^2)\; G(x^+,r^+|o,\rho)\rho d\rho - \int_o^{x^+} \left[T_w(\xi)/T_i\right]\left(\frac{\partial G}{\partial \rho}\right)_{\rho=1} d\xi\right\} \tag{B-6}$$

Clearly, the first term on the right hand side of (B-6) is the solution to the Graetz or Reduced Problem. Thus

$$T(x^+,r^+) = \theta(x^+,r^+) + 2\pi(2\bar{u}\ r_o^2)\int_o^{x^+} \left[T_w(\xi)/T_i\right]\left(\frac{\partial G}{\partial \rho^+}\right)_{\rho^+=1} d\xi \tag{B-7}$$

It will now be proven that

$$\left(\frac{\partial G}{\partial \rho^+}\right)_{\rho^+=1} = \left(\frac{1}{2\pi}\right)\left(\frac{1}{2\bar{u}\ r_o^2}\right)\frac{\partial \theta(x^+-\xi^+,r^+)}{\partial \xi^+} \tag{B-8}$$

A new function ϕ is introduced which satisfies eq. (B-1) together with $T_i=0$ and $T_w = 1$. Clearly $\theta = 1 - \phi$. From eq. (12)

$$\phi = 2\pi\ \alpha r_o \int_o^{x} \left(\frac{\partial G}{\partial \rho}\right)_{\rho=r_o} d\xi = (2\pi)(2\bar{u}\ r_o^2) \int_o^{x^+} \left(\frac{\partial G}{\partial \rho^+}\right)_{\rho^+=1} d\xi \tag{B-9}$$

Now define $f(x^+,\xi^+,r^+)$ such that

$$\left(\frac{\partial G}{\partial \rho^+}\right)_{\rho^+=1} = -\left(\frac{1}{2\pi}\right)\left(\frac{1}{2\bar{u}\ r_o^2}\right)\frac{\partial \phi(x^+-\xi^+,r^+)}{\partial \xi^+} + f(x^+,\xi^+,r^+)\ . \tag{B-10}$$

It will be shown that f is equal to zero. Integrating (B-10) with respect to ξ^+ between $\xi^+ = 0$ and $\xi^+ = x^+$ results in

$$\int_o^{x^+} \left(\frac{\partial G}{\partial \rho^+}\right)_{\rho^+=1} d\xi = -\left(\frac{1}{2\pi}\right)\left(\frac{1}{2\bar{u}\ r_o^2}\right)\left[\phi(o,r^+) - \phi(x^+,r^+)\right] + \int_o^{x^+} f(x^+,\xi^+,r^+)\ d\xi^+ \tag{B-11}$$

From (B-9) and the fact that $\phi(o,r^+) = 0$ it follows that

$$\int_o^{x^+} f(x^+,\xi^+,r^+)\ d\xi^+ = 0 \tag{B-12}$$

Since this is true for all x^+ , it follows that $f(x^+,\xi^+,r^+)$ is identically zero and that

$$-\left(\frac{1}{2\pi}\right)\left(\frac{1}{2\bar{u}\ r_o^2}\right)\ \frac{\partial\phi(x^+-\xi^+,r^+)}{\partial\xi^+} = \left(\frac{1}{2\pi}\right)\left(\frac{1}{2\bar{u}\ r_o^2}\right)\ \frac{\partial\theta(x^+-\xi^+,r^+)}{\partial\xi^+} = \left(\frac{\partial G}{\partial\rho^+}\right)_{\rho^+=1} \quad \text{(B-13)}$$

Substituting eq. (B-8) into eq. (B-7) results in

$$T(x,r)/T_i = \theta(x,r) + \frac{1}{2\pi}\int_o^x \left[T_w(\xi)/T_i\right]\ \frac{\partial\theta(x-\xi,r)}{\partial\xi}\ d\xi \quad \text{(B-14)}$$

The superscript "+" has been dropped for convenience but the independent variables in (B-14) and below are still dimensionless. Letting $T_i = 1$ (or alternately considering T to be T/T_i) and integrating (B-14) by parts results in

$$T(x,r) = \theta(x,r) + T_w(x)\cdot\theta(o,r) - T_w(o)\cdot\theta(x,r) - \int_o^x \theta(x-\xi,r)\ \frac{dT_w}{d\xi}\ d\xi \quad \text{(B-15)}$$

But $\theta(o,r) = 1$. Hence (B-15) can be written as

$$T(x,r) = \theta(x,r) + T_w(o) - T_w(o)\ \theta(x,r) + \int_o^x [1-\theta(x-\xi,r)]\ \frac{dT_w}{d\xi}\ d\xi \quad \text{(B-16)}$$

Furthermore,

$$\theta(x,r) + T_w(o) - T_w(o)\cdot\theta(x,r) = 1 + [1-\theta(x,r)]\cdot[T_w(o)-1\} \ . \quad \text{(B-17)}$$

Then, from the definition of the Stieltjes integral

$$T(x,r) = 1 + \int_o^x [1 - \theta(x-\xi,r)]\ \ dT_w \quad \text{(B-18)}$$

Q.E.D.

APPENDIX C Evaluation of Integral in Equation 29.

First it is recognized that $1/[\pi T\ (x,r)]$ can be taken outside the integral sign in eq. (29). Furthermore, because of the assumed isothermal nature of the tube ends, the integral is considered as three terms, i.e., referring to the integral in eq. (29) as $J\ (x,r)$,

$$J(x,r)=\frac{1}{\pi T}\left\{\int_{\substack{\omega=\\(4\pi-\omega_i-\omega_e)}}[\rho K_p(T_w)\sigma T^5(S_w)]d\omega+\rho K_p(T_i)\sigma T_i^5\omega_i+\rho K_p(T_e)\sigma T_e^5\omega_e\right. \quad \text{(C-1)}$$

Here ω_i and ω_e are the solid angles, with vertices at P, subtended by areas A_i and A_e respectively (see fig. 3). The evaluation of ω_i and ω_e is quite simple. Consider a differential area element, dA, located at point P with normals n_i and n_e parallel to the tube axis. Further, imagine a sphere of unit radius with center at P. The solid angle ω_i intercepts an area on this unit sphere which shall be called a_i (now shown in fig. 3). From the definition of a solid angle

$$\omega_i = a_i \quad \text{(C-2)}$$

The shape factor between dA and A_i, denoted as $F_{dA\to A_i}$ is

$$F_{dA\to A_i} = a_i/2\pi \quad \text{(C-3)}$$

thus, from (C-2) and (C-3)

$$\omega_i = 2\pi\ F_{dA\to A_i} \quad \text{(C-4)}$$

Similarly

$$\omega_e = 2\pi\ F_{dA\to A_e} \quad \text{(C-5)}$$

Fortunately these shape factors have been evaluated. Thus $F_{dA\to A_i}$, which shall be called $F(x,r)$, is (see ref [18])

$$F(x,r) = \frac{1}{2} - \frac{1}{2}\left[\frac{r^2 - 1 + 4x^2}{\sqrt{r^4+1+16x^2+8x^2(r^2+1) - 2r^2}}\right] \quad \text{(C-6)}$$

and

$$F_{dA \to A_e} = F\,(L-x,\ r) \qquad \text{(C-7)}$$

Equations (C-6) and (C-7) are written in terms of dimensionless variables, i. e., $x = (\text{dimensional } x)/d$ and $r = (\text{dimensional } r)/r_o$.

The integral appearing in (C-1) is now evaluated. First it is noted that circumferential symmetry exists in the problem. In addition it follows from the solid angle definition that

$$d\omega = \frac{\cos\,\Theta\ dA_T}{S_w^{\ 2}} \qquad \text{(C-8)}$$

where θ, dA_T and S_W are shown in fig. 3.

$$dA_T = r_o\, d\phi\ d\,\xi \qquad \text{(C-9)}$$

where ϕ is the azmuthal angle, varying from 0 to 2π. It is also a relatively simple matter to show that

$$S_w^{\ 2} = [(x-\xi)^2 + r^2 + r_o^{\ 2} - 2r\, r_o\, \sin\,\phi\,] \qquad \text{(C-10)}$$

and

$$\cos\,\theta = \frac{r_o - r\,\sin\,\phi}{\sqrt{(x-\xi)^2 + r^2 + r_o^2 - 2rr_o\,\sin\,\phi}} \qquad \text{(C-11)}$$

where dimensional variables are being used.

Introducing dimensionless coordinates, denoting the integral in (C-1) as $I(x,r)$ and using equations (C-8) through (C-11) results in

$$I(x,r) = 2\int_o^L \rho\, K_p(T_w)\sigma T_w^5(\xi) \int_o^{2\pi} \frac{(1-\sin\,\phi)d\phi d\xi}{[4(x-\xi)^2 + r^2 + 1 - 2r\,\sin\,\phi]^{3/2}} \qquad \text{(C-12)}$$

The inner integral in (C-12) will now be evaluated. Denoting this inner integral as $II\,(x,\,\xi,\,r)$,

$$II(x,\xi,r) = \int_o^{2\pi} \frac{d\,\phi}{[A - B\,\sin\phi\,]^{3/2}} - r\int_o^{2\pi} \frac{\sin\phi\ d\phi}{[A - B\,\sin\phi\,]^{3/2}} \qquad \text{(C-13)}$$

Using the periodicity of the sine function (C-13) can be written as

$$\mathrm{II}(x,\xi,r) = 2\int_{-\pi/2}^{\pi/2} \frac{d\,\phi}{[A + B\sin\phi]^{3/2}} + 2r\int_{-\pi/2}^{\pi/2} \frac{\sin\phi\; d\phi}{[A + B\sin\phi]^{3/2}} \qquad \text{(C-14)}$$

In (C-13) and (C-14)

$$A = 4\,(x - \xi)^2 + r^2 + 1 \qquad \text{(C-15)}$$

$$B = 2r \qquad \text{(C-16)}$$

The two integrals in (C-14) are quite difficult to evaluate and all of the algebra will not be presented. However, employing the substitution

$$\alpha = \arcsin\sqrt{\frac{1 - \sin\theta}{2}}\,, \qquad \text{(C-17)}$$

considerable patience and formulas 2.575.1 and 2.584.40 of ref. [34] finally results in

$$\mathrm{II}(x,\xi,r) = \frac{(4-2A)E(k)}{(A-B)\sqrt{A+B}} + \frac{2\,K(k)}{B\sqrt{A+B}} \qquad \text{(C-18)}$$

$K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively. The argument k is defined by

$$k = \sqrt{2B/\,(A + B)} \qquad \text{(C-19)}$$

Combining (C-18), (C-12), (C-5), (C-4) and (C-1) with eq (29) then results in eq. (30).

FIGURES

Interpretation of Curves in Figures 4 - 7

----------- Non-participating Gas Solution

Optically Thin Gas Solution
H evaluated from eq. (32)

Optically Thin Gas Solution
H evaluated from Eq. (46)

	L/d	Pe	T_i (^oR)	q''_{Av} (BTU/hr-ft^2)	d (inches)
Fig. 4	20	1740	1000	40,000	1
Fig. 5	20	1740	1000	40,000	1/2
Fig. 6	20	1740	1000	60,000	1/4
Fig. 7	20	1740	1000	40,000	1

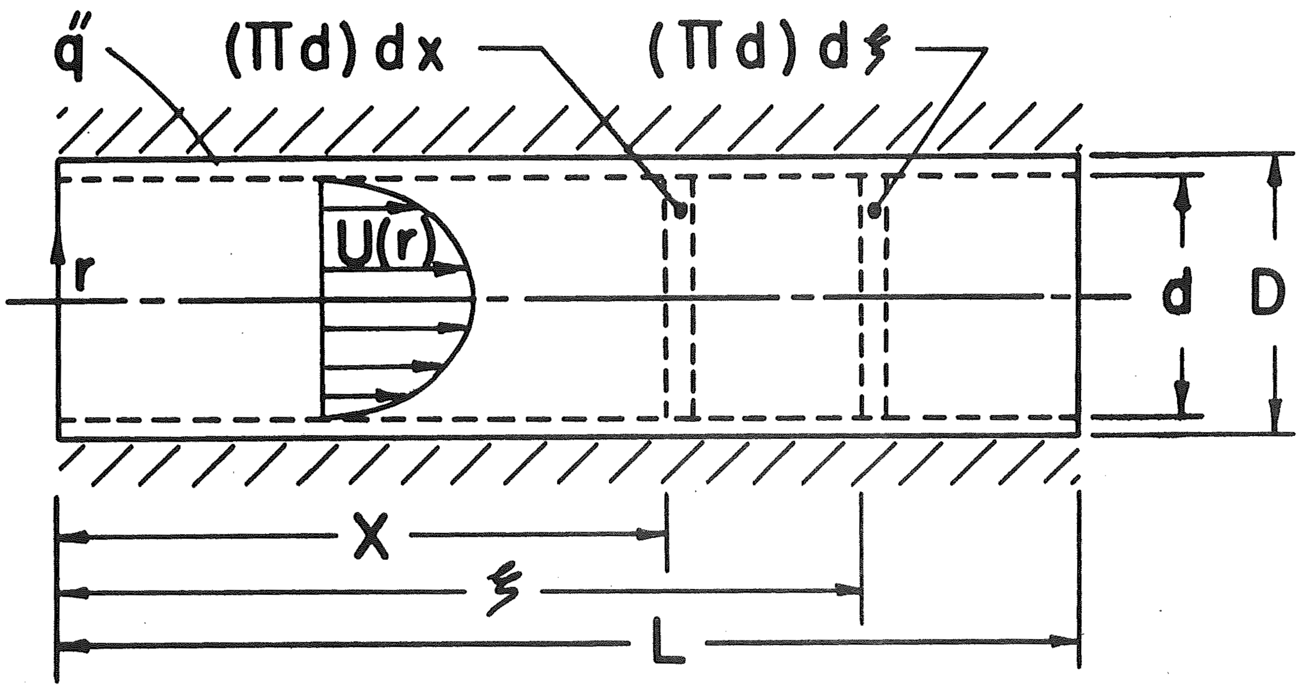

Figure 1. Diagram of Problem Under Consideration

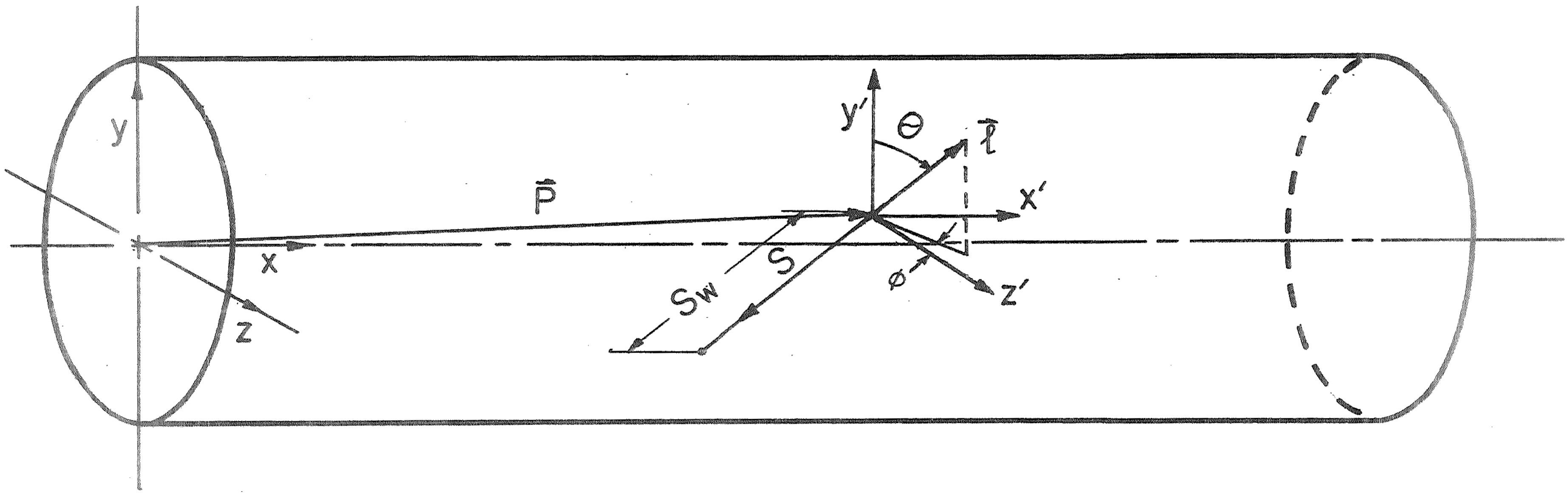

Figure 2. Geometry Associated with Spectral Intensity.

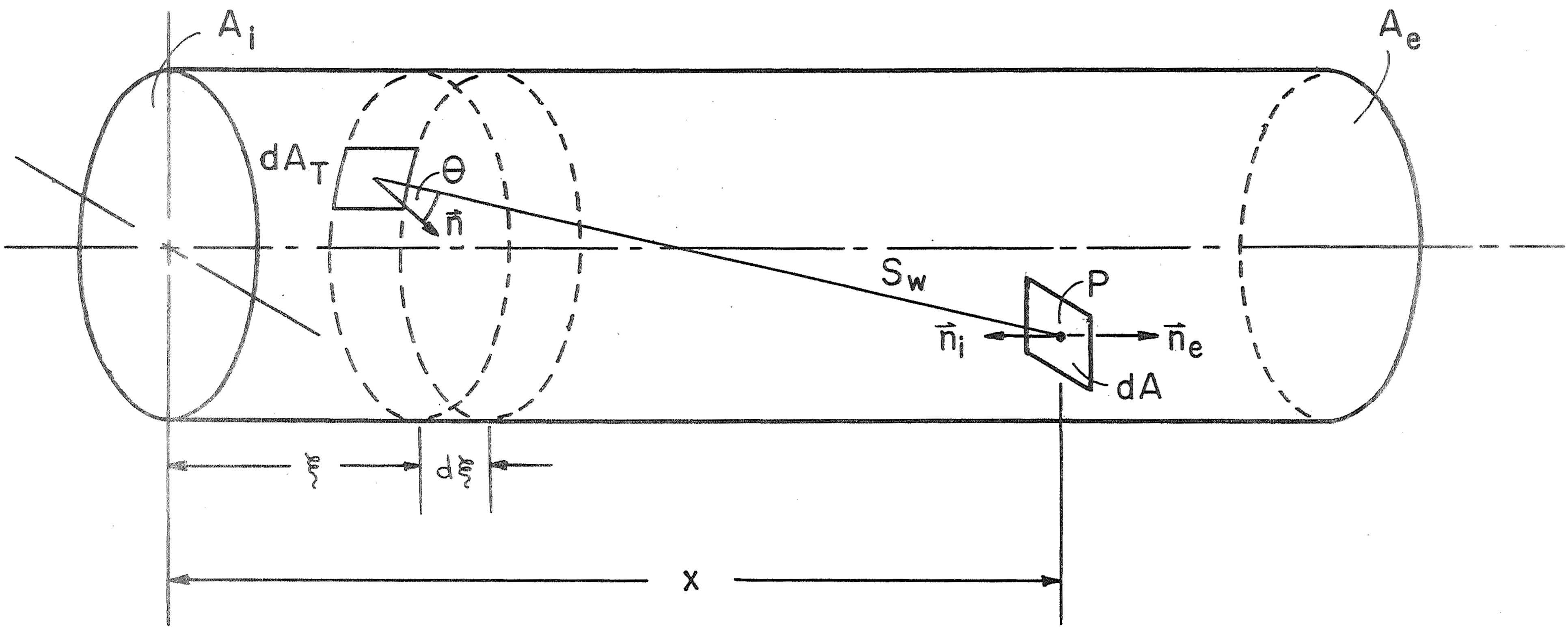

Figure 3. Geometry for Evaluation of Source Term, g .

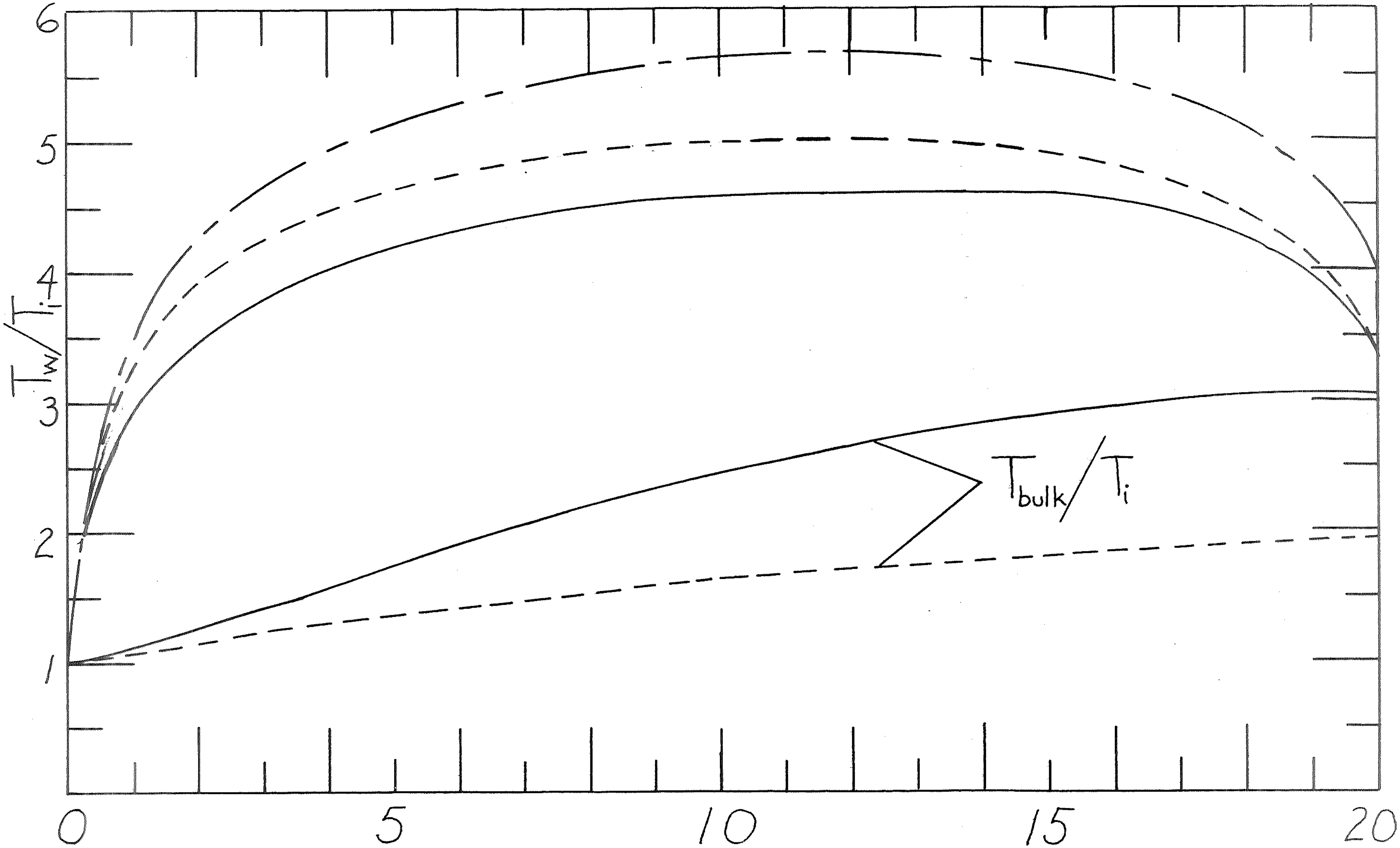

Figure 4.

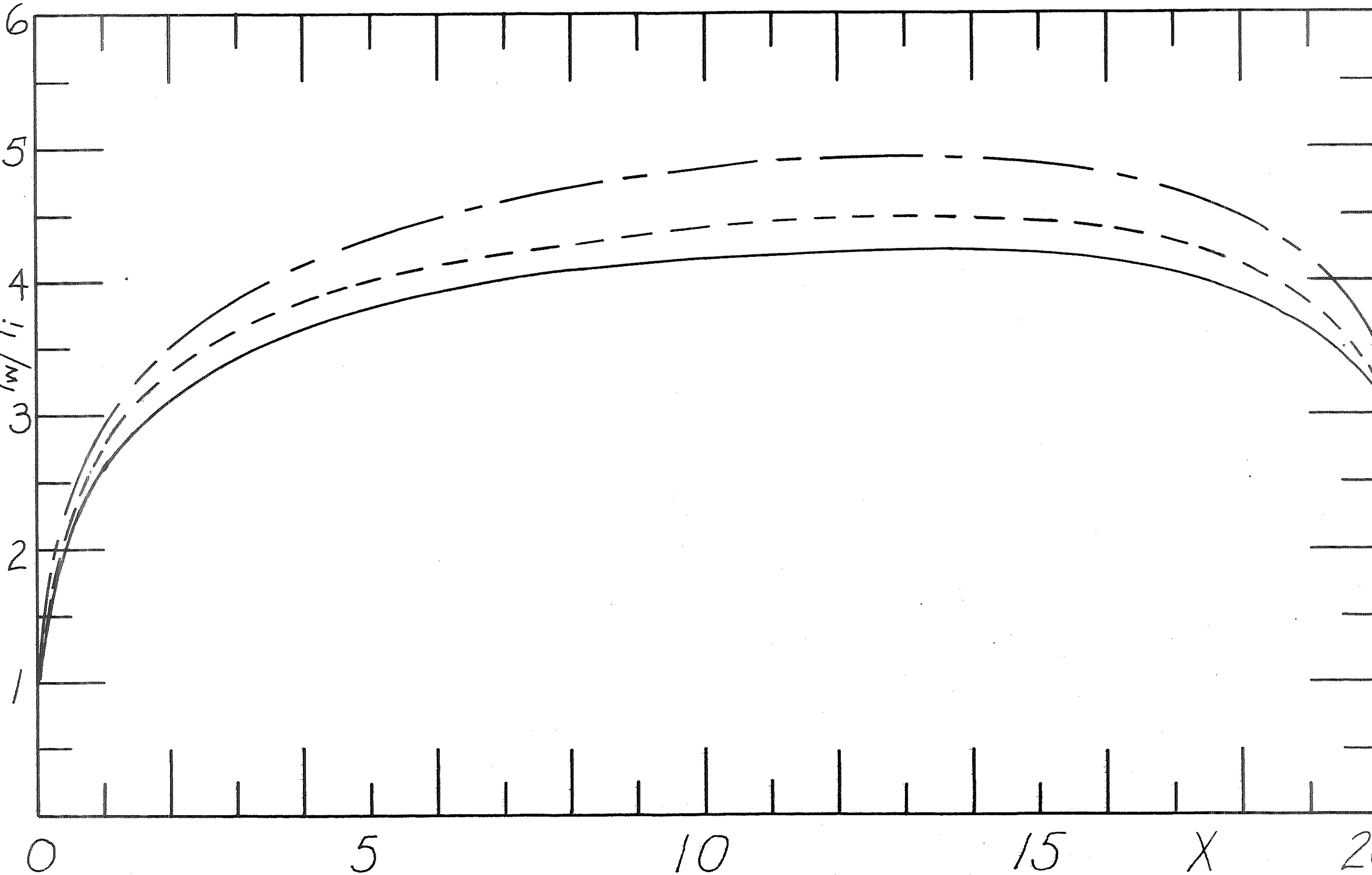

Figure 5.

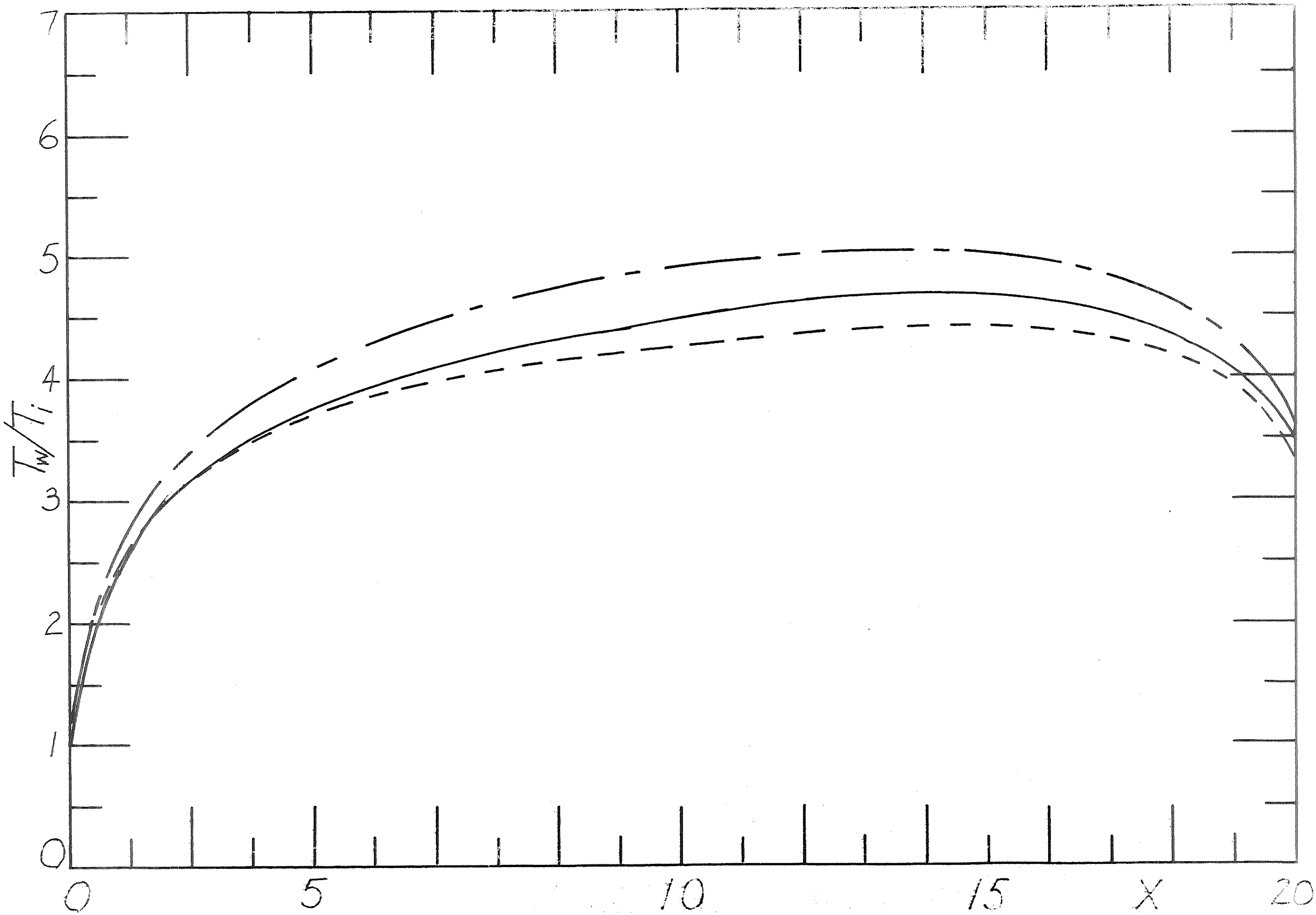

Figure 5.

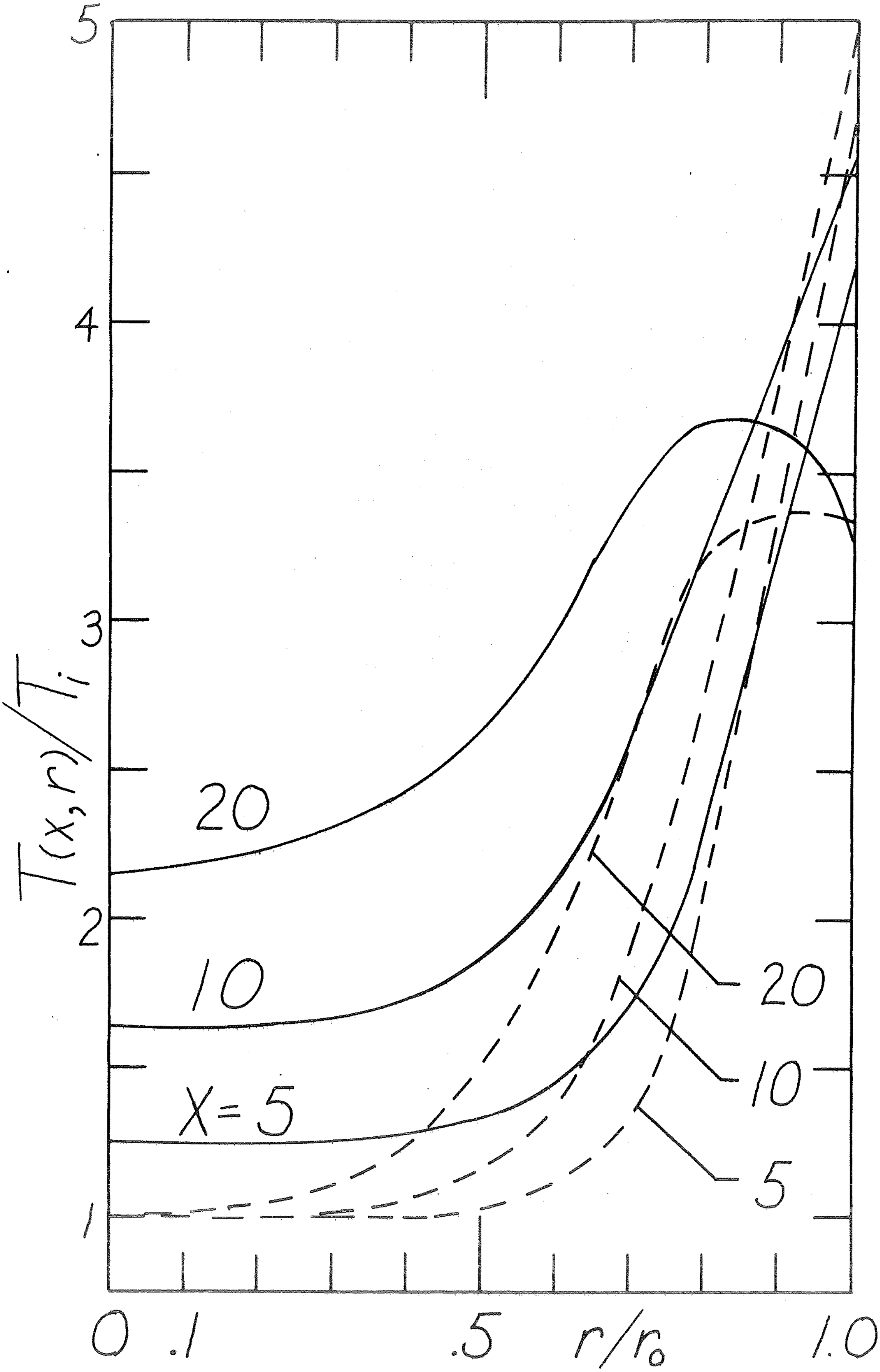

Figure 7.