THEORY OF A GENERAL CLASS OF DISSIPATIVE PROCESSES

by

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1. Introduction.

The object of this paper is to develop a theory of periodic processes of sufficient generality that it can be applied to systems defined by partial differential equations (distributed parameter systems), functional differential equations of retarded and neutral type (hereditary systems), systems arising in the theory of elasticity, etc. A large number of examples of autonomous processes (dynamical systems) and more complete references can be found in the paper [1] by Hale. There the principal objective was to obtain a generalized invariance principle and to exploit this invariance to obtain a general stability theory. The results in [1] were extended in [2] by Slemrod to periodic dynamical systems and Dafermos in [3] gave an invariance principle for compact processes which include periodic processes. Recent applications of this stability theory can be found in [4]-[7].

The purpose here is to develop in the spirit of the work above a general and meaningful theory of dissipative periodic systems. More specifically, we study the iterates of the period map T associated with a class of dissipative periodic processes, prove that large iterates of T always have fixed points, characterize and prove the existence and stability of the maximal compact invariant set of T. Nonlinear ordinary differential equations which are periodic and dissipative were studied by Levinson [8] in 1944 and more general results can be found in [9], [10], [11]. This paper also includes all of the results stated in [12]. For ordinary differential equations, the period map T is topological and the space is locally compact. However, for the applications we have in mind, the mapping may not be topological and the state spaces are not locally compact. Because of this and because we wish a unified theory with a wide range of applications, the identification of suitable hypotheses and their theory is by no means a trivial exercise.

Section 2 is devoted to the definition and examples of processes. Sections 3 and 4 contain the basic theory for a special class of dissipative processes. Some indication is given in Section 3 of how the theory relates to retarded functional differential equations, neutral functional differential equations and partial differential equations. Applications of the theory and sufficient conditions for dissipativeness in terms of Liapunov functions will be discussed in a later paper.

<u>2. Processes.</u> Let R denote the real numbers, R^+ the nonnegative reals and let X be a Banach space with norm $|\cdot|$. Consider a mapping $u: R \times X \times R^+ \to X$ and define a mapping $U(\sigma,\tau): X \to X$ for each $\sigma \in R$, $\tau \in R^+$ by

$$U(\sigma,\tau)x = u(\sigma,x,\tau).$$

It is convenient to interpret $U(\sigma,\tau)x$ as the state of the system at time $\sigma+\tau$ if initially the state of the system at time σ was x. A <u>process</u> on a Banach space X is a mapping $u: R \times X \times R^+ \to X$ with the following properties:

(2.1) u is continuous

(2.2) $U(\sigma,0) = I$, the identity

(2.3) $U(\sigma+s,\tau)U(\sigma,s) = U(\sigma,s+\tau)$

Thus a process is essentially what was called in [2] a "generalized non-autonomous dynamical system" and differs by the continuity condition on u from what was called a process in [3]. The (<u>positive</u>) <u>motion</u> or <u>orbit</u> through (σ,x) is $\bigcup_{\tau\geq 0}U(\sigma,\tau)x$. A motion is said to be <u>periodic</u> of period $\alpha > 0$ if $U(\sigma,\tau+\alpha) = U(\sigma,\tau)$ for all $\tau \in R^+$.

A process is said to be <u>periodic</u> of period $\omega > 0$ if $U(\sigma+\omega,\tau) = U(\sigma,\tau)$ for all $\sigma \in R$, $\tau \in R^+$. A process is said to be <u>autonomous</u> (or the process is a <u>dynamical system</u>) if $U(\sigma,\tau) = U(0,\tau)$ for all $\sigma \in R$, $\tau \in R^+$.

Let us now give some examples of processes.

<u>Example 2.1.</u> <u>Ordinary differential equations.</u> Suppose $f: R \times E^n \to E^n$ is continuous and for any $\sigma \in R$, $\xi \in E^n$, the solution $\varphi(t,\sigma,\xi)$, $\varphi(\sigma,\sigma,\xi) = \xi$, of the equation

$$\dot{x} = f(t,x)$$

exists for all $t \geq 0$, is unique and depends continuously upon t,σ,ξ. Uniqueness of the solution implies $\varphi(t+\tau,\sigma,\xi) = \varphi(t+\tau,t,\varphi(t,\sigma,\xi))$. Therefore, if we let $u(\sigma,\xi,\tau) = \varphi(\sigma+\tau,\sigma,\xi)$, then u is a process on E^n.

<u>Example 2.2.</u> <u>Retarded functional differential equations.</u> Let $r \geq 0$ be given, $C = C([-r,0],E^n)$ be the space of continuous functions mapping $[-r,0]$ into E^n with the topology of uniform convergence. For any continuous function x defined on $[-r,A)$, $A > 0$, and any $t \in [0,A)$, let x_t in C be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Suppose $f: R \times C \to E^n$ is continuous. A function $x = x(\sigma,\varphi)$ defined and continuous on $[\sigma-r,\sigma+A)$, $A > 0$, is said to be a solution of the retarded functional differential equation

$$\dot{x}(t) = f(t,x_t) \tag{2.4}$$

with initial value φ at σ if $x_\sigma = \varphi$ and $x(t)$ satisfies (2.4) for $t \in (\sigma,\sigma+A)$. For any $(\sigma,\varphi) \in R \times C$, suppose a solution $x(\sigma,\varphi)$ exists on $[-r,\infty)$, is unique and $x(\sigma,\varphi)(t)$ depends continuously on (σ,φ,t). If $u(\sigma,\varphi,\tau) = x_{\sigma+\tau}(\sigma,\varphi)$, then u is a process on C. ▲

brief survey of the history of functional differential equations is given in [13]. For sufficient conditions for existence uniqueness, continuity and continuation to the right, see [1], [4], [14], [15] or [16]. These conditions are quite similar to the ones for ordinary differential equations.

If we assume further that $f(t+\omega,\varphi) = f(t,\varphi)$ for all $(t,\varphi) \in R \times C$ and f takes bounded sets of $R \times C$ into bounded sets of E^n, then every bounded orbit of (2.4) is precompact in C. In fact, even more is true: <u>for any bounded set $B \subset C$, there is a compact set B^* in C such that $x_t \in B$ for $\sigma \leq t \leq \sigma+s$, $s \geq r$ implies $x_t \in B^*$ for $\sigma+r \leq t \leq \sigma+s$.</u> This is clear because if $|x(t)| < b$ for $t \in [\sigma,\sigma+s]$, then $|\dot{x}_t| < d$ for $t \in [\sigma+r,\sigma+s]$ and some constant d. This smoothing of the initial data was exploited by Hale in [1] although he did not use and did not need a smoothing property as strong as this one.

<u>Example 2.3.</u> <u>Functional differential equations of neutral type.</u> A general definition of equations of neutral type as well as basic theorems of existence, uniqueness, continuous dependence and continuation may be found in [16] (see also [4]). For the purposes of this paper, we are content to illustrate them in a special case.

With the notation as in Example 2.2, let $D, f\colon R \times C \to E^n$ be continuous,

$$(2.5) \qquad D(t,\varphi) = \varphi(0) + B_1(t)\varphi(-r_1) + \dots + B_k(t)\varphi(-r_k)$$

where $0 < r_j \leq r$ (if $B_j \not\equiv 0$), $B_j(t)$ are uniformly continuous and bounded for $t \in R$. A function $x = x(\sigma,\varphi)$ defined and continuous on $[\sigma-r,\sigma+A)$,

$A > 0$ is said to be a solution of the neutral functional differential equation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{2.6}$$

with initial value φ at σ if $x_\sigma = \varphi$, $D(t,x_t)$ is continuously differentiable on $(\sigma,\sigma+A)$ and satisfies (2.6) on $(\sigma,\sigma+A)$. We assume that for any $(\sigma,\varphi) \in R \times C$, a solution $x(\sigma,\varphi)$ exists on $[\sigma-r,\infty)$, is unique and $x(\sigma,\varphi)(t)$ is continuous in (σ,φ,t). The function $u(\sigma,\varphi,\tau) = x_{\sigma+\tau}(\sigma,\varphi)$ is a process on C.

In general, the solutions of a neutral functional differential equation are no smoother than the initial data. In particular, a bounded orbit in C may not belong to a precompact set in C. To overcome this difficulty, further conditions were imposed in [4] on the operator D. More specifically, the operator D is said to be stable if there is a constant N such that for every function h continuous on R, the solution $x(\sigma,\varphi,h)$, $x_\sigma(\sigma,\varphi,h) = \varphi$ of the equation

$$D(t,x_t) = D(\sigma,\varphi) + h(t)-h(\sigma)$$

satisfies

$$|x_t(\sigma,\varphi,h)| \le N[|\varphi| + \sup_{\sigma\le u\le t} |h(u) - h(\sigma)|], \quad t \ge \sigma.$$

It is shown in [4] that the operator D is stable if and only if the solutions of the homogeneous equation $D(t,x_t) = 0$ are uniformly asymptotically stable.

Now suppose $D(t,\varphi)$, $f(t,\varphi)$ are ω-periodic in t, D is stable and f takes bounded sets into bounded sets. Then it is not difficult to show that an orbit of (2.6) bounded in C is precompact in C. In fact, if x is the solution corresponding to the bounded orbit, then

$$D(t+\tau,x_{t+\tau}) = D(\sigma+\tau,x_{\sigma+\tau}) + \int_{\sigma+\tau}^{t+\tau} f(s,x_s)ds$$

$$D(t,x_t) = D(\sigma,x_\sigma) + \int_\sigma^t f(s,x_s)ds$$

and

$$D(t+\tau,x_{t+\tau}-x_t) = -D(t+\tau,x_t) + D(t,x_t) + D(\sigma+\tau,x_{\sigma+\tau}) - D(\sigma,x_\sigma) + \left(\int_t^{t+\tau} - \int_\sigma^{\sigma+\tau}\right)f(s,x_s)ds$$

Using the definition of a stable operator D, the uniform continuity of $D(t,\varphi)$ in t and the continuity of $x_{\sigma+\tau}$ in τ, one obtains the result.

Actually the same proof gives the following result:

Remark 2.1. <u>If K is a compact subset of C and $\cup_{\varphi\in K}\gamma^+(\sigma,\varphi)$ is bounded, then $\cup_{\varphi\in K}\gamma^+(\sigma,\varphi)$ is precompact</u>. This latter property is not necessarily true if K is only a bounded set (compare with Example 2.2)

Under the same hypotheses as above, the solution operator $T(t,\sigma)\varphi \overset{def}{=} x_t(\sigma,\varphi)$ has the following interesting property:

<u>Remark 2.2. There is a linear operator $T_1(t,\sigma)$ and constants $K > 0$, $\alpha > 0$ such that</u>

$$|T_1(t,\sigma)| \leq Ke^{-\alpha(t-\sigma)}, \quad t \geq \sigma,$$

<u>and a nonlinear operator $T_2(t,\sigma)$ such that for any bounded set $B \subset C$, there is a compact set $B^* \subset C$ such that $T(t,\sigma)\varphi \in B$ for $\sigma \leq t \leq \sigma+s$, $s \geq r$, implies $T_2(t,\sigma)\varphi \in B^*$ for $\sigma+r \leq t \leq \sigma+s$ and</u>

$$T(t,\sigma) = T_1(t,\sigma) + T_2(t,\sigma)$$

A special case of this result was proved by Hale in [17]. The same proof gives the more general result stated here.

<u>Example 2.4. Partial differential equations.</u> Certain types of parabolic and hyperbolic partial differential equations have been shown to define processes on appropriate Sobolev spaces (see [18], [19], [20]). In the parabolic case, the solution is generally smoother than the initial data and a bounded orbit is precompact. In the hyperbolic case, this smoothing effect does not take place. On the other hand, if we know that a hyperbolic equation defines a process on two Sobolev spaces $\mathscr{B}$, $\mathscr{L}$ with $\mathscr{B} \subset \mathscr{L}$ algebraically and topologically and the injection map completely continuous, then a bounded orbit in $\mathscr{B}$ will be precompact in $\mathscr{L}$. This property has been used effectively in the analysis of the asymptotic behavior of the solutions of partial differential equations

(see [1], [21], [22]) and we will use it again for periodic processes.

Our objective in this paper is to study the existence of periodic solutions and asymptotic behavior of periodic processes. For a periodic process and any fixed $t \in R$, there is associated a continuous mapping $T: X \to X$ defined by

$$Tx = U(\sigma,\omega)x.$$

If T^n is the nth iterate of T, it follows from (2.3) that $T^n = U(\sigma,n\omega)$. Since for a periodic process $U(\sigma,\tau+k\omega) = U(\sigma+k\omega,\tau)U(\sigma,k\omega) = U(\sigma,\tau)U(\sigma,k\omega)$, it follows that the fixed points of T^k correspond to periodic motions of period $k\omega$ of the periodic process.

With this motivation, we now turn our attention to the study of <u>discrete dynamical systems</u>; namely, the iterates of a continuous mapping $T: X \to X$, where X is a Banach space. The (<u>positive</u>) <u>motion</u> or <u>orbit</u> $\gamma^+(x)$ through $x \in X$ is the sequence $T^n x$, $n = 0,1,2,\ldots$. A point y is said to be a <u>limit point</u> of the motion through x if there exists a subsequence n_k of integers such that $n_k \to \infty$ and $T^{n_k}x \to y$ as $k \to \infty$. The <u>limit set</u> $L(x)$ is the set of all limit points of $T^n(x)$. Note that

$$L(x) = \bigcap_{j=0}^{\infty} \mathrm{Cl} \bigcup_{n=j}^{\infty} T^n(x), \tag{2.7}$$

where Cl denotes closure.

A set $M \subset X$ is said to be <u>positively invariant</u> if $T(M) \subset M$ and <u>negatively invariant</u> if $M \subset T(M)$. It is said to be <u>invariant</u> if $T(M) = M$; that is, if it is both positively and negatively invariant. Negative invariance and the axiom of choice implies the existence on M of a right inverse T^{-1} to T. Hence we have T^n defined for all integers n (when n is negative $T^n = (T^{-1})^{-n}$) with the property that $T^{k-j} = T^k T^{-j}$ for all nonnegative integers k, j. Thus negative invariance implies the existence of an extension over all integers of each positive motion through a point of M and the negative extension is contained in M. Although the following lemma is essentially contained in [1], [2], [3], the proof for the case of discrete motions is especially simple and is included. The same proof yields the more general result, Lemma 4.1.

<u>Lemma 2.1</u>. If $\gamma^+(x)$ is precompact, then the limit set $L(x)$ is nonempty, compact and invariant.

<u>Proof</u>: $L(x)$ is the intersection (2.7) of a descending sequence of nonempty compact sets and is therefore nonempty and compact. The continuity of T implies $L(x)$ is positively invariant. Let $y \in L(x)$. Then there is a sequence of integers n_j such that $n_j \to \infty$, $T^{n_j}x \to y$ as $j \to \infty$. By the precompactness assumption, we can select a subsequence (which we again label as n_j) such that $T^{n_j - 1} x \to z$ as $j \to \infty$. Now $z \in L(x)$ and $Tz = \lim_{j \to \infty} T^{n_j} x = y$ by the continuity of T. Hence $L(x) \subset T(L(x))$ which shows $L(x)$ is invariant.

We now wish to impose on the operator T "smoothing" properties which will permit T to be the period map associated with any of the periodic processes mentioned in the above examples. Also, one expects that real processes will be dissipative for large displacements and the notion of dissipativeness is naturally associated with boundedness. With applications in mind, we develop a theory of dissipative processes based on boundedness and the smoothing property alluded to above.

3. Dissipative systems. Suppose $\mathscr{B}, \mathscr{C}$ are Banach spaces, $\mathscr{B} \subset \mathscr{C}$ algebraically and topologically and the injection map I taking $\mathscr{B}$ into $\mathscr{C}$ is continuous. Suppose T defines a discrete dynamical system on both $\mathscr{B}$ and $\mathscr{C}$. The symbol $\mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A)$ denotes the ε-neighborhood in $\mathscr{B}$ of a subset A of $\mathscr{B}$. The symbol $Cl_{\mathscr{C}}A$ denotes the closure in $\mathscr{C}$ of a subset A of $\mathscr{C}$. If B is a subset of $\mathscr{B}$, then $\gamma^+(B) = \cup_{x \in B} \gamma^+(x)$. At times, we need the following hypotheses on T.

H_1) A dissipative property. There exists a bounded set $B \subset \mathscr{B}$ such that for each bounded set $M \subset \mathscr{B}$ and $x \in \mathscr{C}$, there is a neighborhood $O_x^{\mathscr{C}} \subset \mathscr{C}$ of x and an integer $N(x,M)$ such that $T^n[O_x^{\mathscr{C}} \cap M] \subset B$ for all $n \geq N(x,M)$.

H_2) A smoothness property. For each bounded set $B \subset \mathscr{B}$, there exists a bounded set $A \subset \mathscr{B}$ with $B^* \overset{def}{=} Cl_{\mathscr{C}}A$ compact in $\mathscr{C}$ such that, for every $\varepsilon > 0$, there is an integer $n_0(\varepsilon,B)$ with the property that $T^n x \in B$ for $0 \leq n \leq N$, $N \geq n_0(\varepsilon,B)$, implies $T^n x \in \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A)$ for $n_0(\varepsilon,B) \leq n \leq N$.

H_3) A fixed point property. There is an integer k_0 such that for every closed bounded convex set $B \subset \mathscr{B}$ and every integer $k \geq k_0$, if $T^n Cl_{\mathscr{C}}B$ is bounded for $0 \leq n \leq k$ and $T^k: Cl_{\mathscr{C}}B \to Cl_{\mathscr{C}}B$, then T^k has a fixed point in $Cl_{\mathscr{C}}B$.

H_4) A smoothness property. For any bounded set $B \subset \mathscr{B}$ with $Cl_{\mathscr{C}} B$ compact in $\mathscr{C}$, if $\gamma^+(B)$ is bounded in $\mathscr{B}$ then $\gamma^+(Cl_{\mathscr{C}}B)$ is precompact in $\mathscr{C}$.

Before discussing some of the properties of discrete dynamical systems described by a system satisfying property $H_1) - H_4)$, let us discuss these hypotheses in connection with the examples mentioned above.

Let us consider first the case where $\mathscr{B} = \mathscr{L} = X$. Hypotheses $H_1) - H_4)$ becomes

$H_1')$ There is a bounded set $B \subset X$ such that for any $x \in X$, there is a neighborhood O_x of x and an integer $N(x)$ such that $T^n O_x \subset B$ for $n \geq N(x)$.

$H_2')$ For any bounded $B \subset X$, there is a compact $B^* \subset X$ such that for any $\varepsilon > 0$, there is an $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in \mathfrak{N}_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

$H_3')$ There is an integer k_0 such that for every closed bounded convex set $B \subset X$ and every integer $k \geq k_0$, if $T^n B$ is bounded for $0 \leq n \leq k$ and $T^k: B \to B$, then T^k has a fixed point in B.

$H_4')$ For any compact set $B \subset X$, $\gamma^+(B)$ bounded implies $\gamma^+(B)$ precompact.

In the previous section, it was shown that the operator T associated with retarded functional differential equations satisfied the following conditions

$H_2'')$ There is a nonnegative integer n_0, such that for any bounded set $B \subset X$, there is a compact set B^* in X such that $T^n x \in B$ for $0 \leq n \leq N$, $N \geq n_0$, implies $T^n x \in B^*$ for $n_0 \leq n \leq N$.

Any operator that satisfies $H_2'')$ automatically satisfies $H_2')$, $H_3')$, $H_4')$. For instance, H_3' follows from the Schauder fixed point theorem, since with $k_0 = n_0$ it follows that $T^{k_0}(B)$ is contained in a compact set of X. For operators satisfying $H_2'')$, one can establish the results to follow under an hypothesis of dissipativeness weaker than $H_1')$; in fact, we will show that it is enough to assume

$H_1'')$ There is a bounded set B in X with the property that given $x \in X$ there is a positive integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$.

Theorem 3.1. If T satisfies $H_1'')$, $H_2'')$, then T satisfies $H_1') - H_4')$. Furthermore, there is a compact set K in X with the property that given a compact set H in X, there is a positive integer $N(H)$ and an open neighborhood H_0 of H such that $T^n(H_0) \subset K$ for all $n \geq n(H)$.

Proof: From the above remarks, it is only necessary to show that T satisfying $H_1'')$, $H_2'')$ implies T satisfies the last property stated in the theorem. We may always assume B in $H_1'')$ is open. Let B* be the correspond compact set in $H_2'')$ and n_0 the integer in $H_2'')$. By the continuity of T, there is an open neighborhood O_x of x such that $T^n(O_x) \subset B$ for $N(x) \leq n \leq N(x)+n_0$. Therefore, $T^{n(x)}O_x \subset B^*$, where $n(x) = N(x)+n_0$. Suppose H is an arbitrary compact set in X. The neighborhoods O_x, $x \in H$ form a covering of H. Selecting from this covering a finite covering, we see there is an integer $n(H)$ such that for each O_x of this finite covering there is an $i = i(x)$ such that $0 \leq i \leq n(H)$ and $T^i(O_x) \subset B^*$. Hence all we need to prove the theorem is to show that there is an integer $N(B^*)$ and a compact set K such that $T^n(B^*) \subset K$

for $n \geq N(B^*)$. Let $x \in B^*$ and let $n \geq n(B^*)$ be any positive integer. There is then a least integer j, $0 \leq j \leq n$, such that $T^{n-j}x \in B^*$ and $T^{n-k} \notin B^*$ for $0 \leq k < j$. It follows by what was shown above that $0 \leq j \leq n(B^*)$. Hence $T^n x$ is contained in the union K of B^*, $T(B^*),\ldots,T^{n(B^*)}(B^*)$, which is compact. This completes the proof.

Remark 3.1. From the above proof, if T satisfies $H_2'')$, then T satisfies $H_1'')$ if there is a bounded set B in X with the property that given $x \in X$ there is a positive integer $N(x)$ such that $T^n(x) \in B$ for $N(x) \leq n \leq N(x)+n_0$. It is only necessary to require that $T^n(x)$ remain in B long enough to "smooth".

When T satisfies $H_1'')$, $H_2'')$, then it follows from Theorem 3.1 that, for some integer k_1, T^n has a fixed point for each $n \geq k_1$ (Corollary 3.2 below). If, in addition, T maps bounded sets into bounded sets, we can prove a bit more and include a result of Yoshizawa for retarded functional differential equations whose solutions are uniformly bounded and uniformly ultimately bounded (see Yoshizawa [23] or [24]). The integer n_0 of this corollary is the n_0 of $H_2')$. For ordinary differential equations $n_0 = 1$ and for retarded functional differential equations $n_0 = 1$ if $\omega \geq r$ (ω is the period and r is the retardation).

Corollary 3.1. If T satisfies $H_1'')$, $H_2'')$ and maps bounded sets into bounded sets, then T^n has a fixed point for each $n \geq n_0$.

Proof. With $n \geq n_0$ we know that T^n is completely continuous. Since the closed convex hull of a compact set is closed, we may assume that the compact set K in Theorem 3.1 is convex. Let $f = T^n$, select j sufficiently large that $f^j(K) \subset K$, let G be the union of $K, f(K), \ldots, f^j(K)$, and let S_1 be an open ball containing G. Then $f^k(K) \subset S_1$ for all $k = 0, 1, \ldots,$ and for m sufficiently large $f^m(S_1) \subset K$ by Theorem 3.1 and the fact that f is completely continuous. It then follows from Browder's extension in [25] of the Schauder fixed point theorem that f has a fixed point.

For the neutral equation in Example 2.3, Remark 2.2. implies that $H_2')$ is satisfied by the corresponding operator T. Also, the same type of argument shows that, on a set B satisfying $H_3')$, the operator T^{k_0} is the sum of a contraction and a completely continuous operator and thus, has the fixed point property (see [16] and [17]). Hypothesis $H_4')$ is the same as Remark 2.1. Thus, we see the significance of our hypotheses for functional differential equations of both retarded and neutral type.

The case $\mathscr{B} \subset \mathscr{L}$, $\mathscr{B} \neq \mathscr{L}$, was introduced to treat partial differential equations and especially hyperbolic equations. The spaces $\mathscr{B}, \mathscr{L}$ are usually chosen so that the injection map $I: \mathscr{B} \to \mathscr{L}$ is completely continuous. For such a siutation hypothesis $H_2)$ is satisfied by any continuous T since we can take $A = B$, $n_0(\varepsilon, B) = 1$. Similarly $H_3)$ and $H_4)$ are satisfied by any continuous T. Therefore, when the injection map is completely continuous, the only hypotheses that will ever be made on T are continuity and $H_1)$.

The following result is the more general analog of Theorem 3.1.

<u>Theorem 3.2</u>. If T satisfied $H_1)$, $H_2)$, then there is a bounded set $A \subset \mathscr{B}$ such that $K \overset{\text{def}}{=} \mathrm{Cl}_{\mathscr{L}} A$ is compact and, given any compact set

$H \subset \mathscr{C}$, there is a $\mathscr{C}$-neighborhood H_0 of H in $\mathscr{C}$ such that for any $\varepsilon > 0$ and any bounded set $M \subset \mathscr{B}$, $T^n[H_0 \cap Cl_{\mathscr{C}} M]$ is bounded in $\mathscr{C}$ for each $n \geq 0$ and there is an integer $n_1(H,M,\varepsilon)$ with

$$T^n[H_0 \cap M] \subset \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A), \; n \geq n_1(H,M,\varepsilon).$$

Proof: Let B be as in $H_1)$, A and B^* as in $H_2)$. Then $K = B^*$ is compact. Let H be an arbitrary compact set. Since T is locally dissipative, for any $x \in H$ and any bounded $M \subset \mathscr{B}$, there is a $\mathscr{C}$-neighborhood O_x of x and an integer $N(x,M)$ such that $T^n[O_x \cap M] \subset B$ for $n \geq N(x,M)$. Selecting from this covering of H a finite covering, we see there is an integer $N(H,M)$ and a $\mathscr{C}$-neighborhood $H_1 \subset \mathscr{C}$ of H such that $T^n[H_1 \cap M] \subset B$ for $n \geq N(H,M)$. Since H is compact and T is continuous, one can choose a finite covering of H in such a way that it yields a $\mathscr{C}$-neighborhood H_0 of H with $H_0 \subset H_1$ and $T^n[H_0]$ bounded in C for $0 \leq n \leq N(H,M)$. Since $T^n[H_0 \cap M] \subset B$ for $n \geq n(H,M)$, this implies $T^n[H_0 \cap Cl_{\mathscr{C}} M]$ is bounded for $n \geq 0$. Using $H_2)$, for any $\varepsilon > 0$, it follows that

$$T^n[H_0 \cap M] \subset \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A) \quad \text{for} \quad n \geq N(H,M) + n_0(\varepsilon,B).$$

If we suppress the dependence on B and let $n_1(H,M,\varepsilon) = N(H,M) + n_0(\varepsilon,B)$, then the theorem is proved.

Corollary 3.2. If T satisfies $H_1)$, $H_2)$, $H_3)$, then there is an integer k_1 such that T^n has a fixed point in $\mathscr{C}$ for each $n \geq k_1$.

Proof: Take K,A as in Theorem 3.2. Since the closed convex hull of a compact set in a Banach space is compact, we may assume K,A convex. Choose $\varepsilon > 0$ so small that $S \overset{\text{def}}{=} Cl_{\mathscr{C}}(\mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A)) \subset K_0$. For $M = \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A)$, Theorem 3.2 implies there is an integer $n_1(A,\varepsilon)$ such that $T^n(\mathfrak{N}_\varepsilon^{\mathcal{B}}(A)) \subset \mathfrak{N}_\varepsilon^{\mathcal{B}}(A)$ for $n \geq n_1(A,\varepsilon)$. Therefore, the continuity of T^n implies

$$T^n(\mathrm{Cl}_{\mathcal{C}}\mathfrak{N}_\varepsilon^{\mathcal{B}}(A)) \subset \mathrm{Cl}_{\mathcal{C}} T^n(\mathfrak{N}_\varepsilon^{\mathcal{B}}(A)) \subset \mathrm{Cl}_{\mathcal{C}}\mathfrak{N}_\varepsilon^{\mathcal{B}}(A)$$

for $n \geq n_1(A,\varepsilon)$, or T^n takes the closed bounded convex set S into itself for $n \geq n_1(A,\varepsilon)$. Let $k_1 = \max\{k_0, n_1(A,\varepsilon)\}$ where k_0 is the integer in H_3). From Theorem 3.2, $T^n(S)$ is bounded for all $n \geq 1$. Therefore, H_3) implies T^n has a fixed point in S for each $n \geq k_1$.

4. The limit set J. We wish now to show that if T satisfied hypotheses H_1), H_2) and H_4), then there is a compact invariant set J that is globally asymptotically stable. The set J is the natural generalization of the maximal compact invariant set introduced in [8], and when $\mathscr{B} = \mathscr{G}$, the set J is the maximal compact invariant set of [12].

Let $A \subset \mathscr{B}$ be the bounded set of Theorem 3.1. Then $K = Cl_{\mathscr{G}} A$ is compact. Let

$$(4.1) \qquad J = \bigcap_{n=0}^{\infty} T^n(K).$$

Of course, since A is not unique, K is not unique, but we can prove that J is independent of A. Observe first that if $J(A)$ is the set defined by (4.1), then $J(A) \subset T^j(J(A))$ for all $j \geq 1$. If A_1 is any other bounded set satisfying the same conditions as A of Theorem 3.2 and $K_1 = Cl_{\mathscr{G}} A_1$, then for any $\varepsilon > 0$ there is an $n_1(K, K_1, \varepsilon)$ such that $T^n(A) \subset \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A_1)$, $T^n(A_1) \subset \mathfrak{N}_{\varepsilon}^{\mathscr{B}}(A)$ for $n \geq n_1(K, K_1, \varepsilon)$. Thus, for any positive sequence $\varepsilon_j \to 0$ as $j \to \infty$, there is a sequence $n_j \to \infty$ as $j \to \infty$ such that $T^{n_j}(A) \subset \mathfrak{N}_{\varepsilon_j}^{\mathscr{B}}(A_1)$, $T^{n_j}(A_1) \subset \mathfrak{N}_{\varepsilon_j}^{\mathscr{B}}(A)$, $j = 1, 2, \ldots$. Consequently, there are positive $\alpha_j \to 0$ as $j \to \infty$ such that

$$T^{n_j}(K) = T^{n_j}(Cl_{\mathscr{G}} A) \subset Cl_{\mathscr{G}} T^{n_j}(A) \subset Cl_{\mathscr{G}} \mathfrak{N}_{\varepsilon_j}^{\mathscr{B}}(A_1)$$

$$\subset \mathfrak{N}_{\alpha_j}^{\mathscr{G}}(K_1), \quad j = 1, 2, \ldots$$

$$T^{n_j}(K_1) \subset \mathfrak{N}_{\alpha_j}^{\mathscr{G}}(K), \quad j = 1, 2, \ldots \ .$$

Thus, $J(K) \subset K_1$, $J(K_1) \subset K$. Since $J(K) \subset T^j(J(K))$ for all $j \geq 1$ and any K, this proves that J is independent of K.

For any bounded set B in $\mathscr{B}$ with $H \overset{\text{def}}{=} Cl_{\mathscr{L}} B$ compact, define L(H), the limit set of the motion through H, by

$$L(H) = \bigcap_{j=0}^{\infty} Cl_{\mathscr{L}} \bigcup_{n=j}^{\infty} T^n(H).$$

Then $y \in L(H)$ means there exist sequences of integers n_j and elements $y_j \in H$ such that $n_j \to \infty$ and $T^{n_j} y_j \to y$ as $j \to \infty$. When H is a single point this is the usual limit set L(x). Just as for Lemma 2.1, we obtain

<u>Lemma 4.1.</u> If $\bigcup_{n=j}^{\infty} T^n(H)$ is precompact in $\mathscr{L}$ for j sufficiently large, then L(H) is a nonempty compact invariant set of $\mathscr{L}$.

<u>Definition 4.1.</u> A set M in $\mathscr{L}$ is said to be a <u>global</u> <u>attractor</u> $(\mathscr{B}, \mathscr{L})$ if for each $x \in \mathscr{B}$, $T^n x \to M$ in $\mathscr{L}$ as $n \to \infty$.

<u>Theorem 4.1.</u> Suppose T satisfies $H_1)$, $H_2)$, $H_4)$, A is the bounded set in $\mathscr{B}$ of Theorem 3.1 and $K = Cl_{\mathscr{L}} A$. Then $J = L(K)$, J is a nonempty compact invariant set and J is a global attractor $(\mathscr{B}, \mathscr{L})$. If H is any other compact invariant set in $\mathscr{L}$, $H = Cl_{\mathscr{L}} M$ for some bounded $M \subset \mathscr{B}$, then $H \subset J$.

<u>Proof:</u> Since T is locally dissipative and K is compact, there is an integer N(A) such that $T^n(A) = T^n[K \cap A] \subset B$ for $n \geq N(A)$. Thus, $\gamma^+(T^{N(A)}A)$ is bounded in $\mathscr{B}$. Hypothesis $H_4)$ implies $\gamma^+(T^{N(A)} Cl_{\mathscr{L}} A)$

is precompact in $\mathcal{L}$. Lemma 4.1 implies that $L(K)$ is a nonempty, compact invariant set of $\mathcal{L}$.

Clearly $J \subset L(K)$. To prove the converse, suppose $y \in L(K)$ and $T^{n_i}x_i \to y$ as $i \to \infty$ where $n_j \to \infty$ as $j \to \infty$ and each $x_i \in K$. Since $\gamma^+(T^{N(A)}K)$ is precompact in $\mathcal{L}$, for any integer j we can find a subsequence of the $T^{n_i - j}x_i$ (which we label the same as before) and a $y^j \in Cl_{\mathcal{L}}\gamma^+(K)$ such that $T^{n_i - j}x_i \to y^j$ as $i \to \infty$. But then $T^j y^j = y$ and thus $y \in J$. Therefore $J = L(K)$.

Now suppose that H is any compact invariant set in $\mathcal{L}$ with $H = Cl_{\mathcal{L}}M$ and M bounded in $\mathcal{B}$. Then $H = T^n(H)$ for every n. Theorem 3.2 implies, for every $\varepsilon > 0$, there is an $n_1(H,M,\varepsilon) \geq 0$ such that $T^n(H_0 \cap M) \subset \mathfrak{N}^{\mathcal{B}}_{\varepsilon}(A)$ for $n \geq n_1(H,M,\varepsilon)$. Taking the closure in $\mathcal{L}$, we obtain, for $n \geq n_1(H,M,\varepsilon)$,

$$H = T^n(H) \subset T^n\{Cl_{\mathcal{L}}(H_0 \cap M)\} \subset Cl_{\mathcal{L}}T^n\{H_0 \cap M\}$$
$$\subset Cl_{\mathcal{L}}\mathfrak{N}^{\mathcal{B}}_{\varepsilon}(A) \subset \mathfrak{N}^{\mathcal{L}}_{\alpha(\varepsilon)}(K)$$

for some positive $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. This implies $H \subset K$.

Now suppose x is an arbitrary element of $\mathcal{B}$. Then $Cl_{\mathcal{L}}\{x\} = \{x\}$. Theorem 3.2 implies $\gamma^+\{x\}$ is bounded in $\mathcal{B}$ and H_4) implies $\gamma^+\{x\}$ is precompact in $\mathcal{L}$. Thus, $L\{x\}$ is nonempty, compact and invariant, and $L\{x\} \subset J$. This proves J is a global attractor $(\mathcal{B},\mathcal{L})$ and completes the proof of the theorem.

<u>Definition 4.2</u>. Suppose $J \subset \mathscr{S}$ is an invariant set. We say J is <u>stable</u> $(\mathscr{B},\mathscr{S})$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every bounded set M in $\mathscr{S}$, there is an integer $n_0(M,\delta,\varepsilon)$ such that $y \in \mathfrak{N}^{\mathscr{S}}_{\delta}(J) \cap M$ implies $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$ for $n \geq n_0(M,\delta,\varepsilon)$.

<u>Definition 4.3</u>. Suppose $J \subset \mathscr{B}$ is an invariant set. We say J is <u>weakly stable</u> $(\mathscr{B},\mathscr{S})$ if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in \mathfrak{N}^{\mathscr{B}}_{\delta}(J)$, then $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$ for $n \geq 0$.

If $\mathscr{B} = \mathscr{S}$, these two definitions are equivalent.

<u>Lemma 4.2</u>. If $J \subset \mathscr{B}$ is invariant and J is compact in $\mathscr{S}$, then J stable $(\mathscr{B},\mathscr{S})$ implies J is weakly stable $(\mathscr{B},\mathscr{S})$.

<u>Proof:</u> If $J \subset \mathscr{B}$ is stable $(\mathscr{B},\mathscr{S})$, then for any $\varepsilon > 0$, there is a $\delta > 0$ such that for every $\delta_1 > 0$, there is an integer $n_0(\delta_1,\delta,\varepsilon)$ such that $x \in \mathfrak{N}^{\mathscr{S}}_{\delta}(J) \cap \mathfrak{N}^{\mathscr{B}}_{\delta_1}(J)$ implies $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$ for $n \geq n_0(\delta_1,\delta,\varepsilon)$. If J is compact, then there is a $\delta_2 > 0$ such that $x \in \mathfrak{N}^{\mathscr{S}}_{\delta_2}(J)$ implies $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$ for $0 \leq n \leq n_0(\delta_1,\delta,\varepsilon)$. If $\delta_2 < \delta$, then $x \in \mathfrak{N}^{\mathscr{S}}_{\delta_2}(J) \cap \mathfrak{N}^{\mathscr{B}}_{\delta_1}(J)$ implies $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$ for $n \geq 0$. If we choose $\delta_1' < \delta_1$ so that $\mathfrak{N}^{\mathscr{B}}_{\delta_1'}(J) \subset \mathfrak{N}^{\mathscr{S}}_{\delta_2}(J)$, then $x \in \mathfrak{N}^{\mathscr{B}}_{\delta_1'}(J)$ implies $T^n x \in \mathfrak{N}^{\mathscr{S}}_{\varepsilon}(J)$; that is, J is weakly stable $(\mathscr{B},\mathscr{S})$.

The argument used in the proof of the following result is similar to one used by LaSalle in [26].

<u>Theorem 4.2</u>. If T satisfies H_1), H_2), H_4) and J is as defined in (4.1), then J is stable $(\mathscr{B},\mathscr{S})$ and if $J \subset \mathscr{B}$, then J is stable $(\mathscr{B},\mathscr{S})$.

Proof: Assume J is not stable $(\mathscr{B},\mathscr{L})$. Then for some $\varepsilon > 0$ (which may be chosen as small as desired), there are sequences of integers n_j and $y_j \in \mathfrak{N}_\delta^{\mathscr{L}}(J) \cap \mathscr{B}$ such that $n_j \to \infty$, $y_j \to I$ as as $j \to \infty$, $T^n y_j \in \mathfrak{N}_\varepsilon^{\mathscr{L}}(J)$, $0 \le n \le n_j$, and $T^{n_j+1} y_j$ is not in $\mathfrak{N}_\varepsilon^{\mathscr{L}}(J)$. The y_j considered as elements of $\mathscr{B}$ may or may not be unbounded. Since J is compact, we may assume there is a $y \in J$ such that $y_j \to y$ as $j \to \infty$. The set $H = \{y_j, j = 1,2,\dots,y\}$ is compact in $\mathscr{L}$.

If H is bounded in $\mathscr{B}$, then it follows from Theorem 3.2 that for any $\eta > 0$ there is an integer $n^* = n^*(H,\eta)$ such that $T^n(H) \subset \mathfrak{N}_\eta^{\mathscr{B}}(A)$ for $n \ge n^*$. Thus, $\gamma^+(T^{n^*}(H))$ is bounded in $\mathscr{B}$. Hypothesis H_4) implies $\gamma^+_{\mathscr{L}}(T^{n^*}(H))$ is precompact. Therefore, Lemma 4.1 implies $\gamma \overset{\text{def}}{=} L(H)$ is nonempty, compact and invariant. Furthermore, γ is the closure in $\mathscr{L}$ of a bounded set in $\mathscr{B}$. Thus, Theorem 4.1 implies $\gamma \subset J$. Also, since $\gamma^+(T^{n^*}(H))$ is precompact in $\mathscr{L}$ we may assume (by choosing a subsequence if necessary) that $T^{n_j} y_j \to z \in \mathscr{L}$ as $j \to \infty$. Then $z \in \gamma \subset J$. But this choice of the n_j and y_j implies that $Tz \notin \mathfrak{N}_\varepsilon^{\mathscr{L}}(J)$ and therefore $Tz \notin J$. Since J is invariant, this is a contradiction and the proof of the theorem is complete for the case in which H is bounded in $\mathscr{B}$.

If H is unbounded in $\mathscr{B}$, suppose the y_j are ordered in such a way that $\|y_j\|_{\mathscr{B}} \to \infty$ as $j \to \infty$. From Theorem 3.2 for each integer j and real $\eta > 0$, there is an integer $N(j,\eta)$ such that $T^n y_j \in \mathfrak{N}_\eta^{\mathscr{B}}(A)$ for $n \ge N(j,\eta)$. Now we may assume the n_j so chosen that $n_j \ge N(j,\eta)$. Therefore, $T^{n_j} y_j \in \mathfrak{N}_\eta^{\mathscr{B}}(A)$ for $j = 1,2,\dots$. Let $H^* = \{T^{n_1} y_1, T^{n_2} y_2, \dots\}$. Then $\gamma^+_{\mathscr{B}}(H^*)$ is bounded and hypothesis H_4) implies $\gamma^+(H^*)$ is precompact in $\mathscr{L}$

Therefore, there is a subsequence of the $T^{n_j}y_j$ if necessary and a z such that $T^{n_j}y_j \to z$ as $j \to \infty$. Since Tz is not in $\mathfrak{N}_{\varepsilon}^{\mathscr{C}}(J)$, z is not in J. But using the same argument as before $z \in L(H) \subset J$. This contradiction implies J is stable $(\mathscr{B}, \mathscr{L})$. The last part of the theorem follows from Lemma 4.2.

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