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MATHEMATICAL CHARACTERIZATION OF MECHANICAL BEHAVIOR OF POROUS FRICTIONAL GRANULAR MEDIA

VOLUME I

by

G. Aguirre-Ramirez

Final Technical Report

This research work was supported by the National Aeronautics and Space Administration under Contract NAS8-25102

Research Institute
The University of Alabama in Huntsville
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PREFACE

This report presents the results of studies conducted during the period July 1, 1969 - October 31, 1970, under NASA research contract NAS 8-25102, "Mathematical Characterization of Mechanical Behavior of Porous Friction Granular Media". This study was monitored by Dr. N. C. Costes, the Geotechnical Laboratory of NASA's Marshall Space Flight Center.

The objectives of this project are:

(1) to develop a consistent three-dimensional mathematical theory describing the mechanical behavior of porous, frictional granular media exhibiting a small amount of cohension. -- Volume I of this report by Dr. G. Aquirre-Rameriz.

(2) to solve boundary-value problems related to in-situ measurements performed on the lunar or planetary surface. -- Volume II of this report by Dr. T. J. Chung and Mr. J. K. Lee.

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SECTION I

ON THE SOILD MASS CONTINUUM

1. INTRODUCTION

A general accepted practice for the analysis of the response of a soil mass under external loads is to model the soil mass as a continuum. Under this assumption the distinction between modeling of a soil mass and a strictly solid mass (such as rolled steel, say) as a continuum is made through the constitutive equations.

In this paper the basic equations of continuum mechanics are reassessed as to their applicability to model a soil mass as a continuum. For the purpose of simplicity the discussion is limited to a dry (i.e., unsaturated) soil mass. Under these conditions the effect of interaction in a water soil system need not be considered.

As a starting point the two physical properties that bodies are known to have and which are used in the construction of a model for continua are taken as fundamental. These are [1] (a) that they occupy regions of space, and (b) that they have mass. These two porperties are used together with the porous geometry of a soil mass to show that in order to define the thermodynamic state variable $\bar{\rho}$, the density of solid aggregate, as a field variable one needs to introduce two field variable $\rho(\underset{\sim}{x},t)$, the soil bulk mass density, and $n(\underset{\sim}{x},t)$ the porosity of the soil. By interpreting the mass density appearing in the local balance laws of mass, momentum, and energy of continuum mechanics as the bulk mass density, these balance equations can be taken over unchanged into soil mechanics to locally describe the corresponding balance laws for the soil.

In soild mechanics the two soil geometric variables, surface porosity and volume porosity are taken to be the same. Here it is shown that by defining Terzaghi's effective stress [5] by boundary conditions this assumption can be removed. Researchers in soil mechanics have consistently verified that the mechanical behavior of the effective soil structure is governed by the effective stress (cf. Schofield and Wroth [11]). For this reason the balance of momentum and energy are formulated in terms of effective stress.

The introduction of the porosity as a field variable requires an additional constitutive equation for the rate of change of porosity. The need for this constitutive equation is shown to be given by the Second Law of Thermodynamics. It is shown that porosity influences the free energy of the soil as an internal state variable and since its rate of change does not appear in the Clausius-Duhem inequality a constitutive equation for it must be postulated. As an example a possible set of constitutive equations is examined. The restrictions on the proposed constitutive response functions are also found by a method introduced by Coleman and Gurtin [6] constitutive equations are further linearized.

NOTATION

In this paper direct tensor notation is used. Second order tensors and linear transformations of the three dimensional vector space U into itself are regarded as the same. If $\underset{\sim}{T}$ is a linear transformation, $\underset{\sim}{T}^T$ denotes its transpose, $\underset{\sim}{T}^{-1}$ its inverse, tr $\underset{\sim}{T}$ its trace and det $\underset{\sim}{T}$ its determinant. The gradient with respect to spatial coordinates is denoted by grad and the gradient with respect to material coordinates by ∇.

2. THE SOIL MASS CONTINUUM

It is important to realize at the outset that the two physical properties that bodies are known to have and which are taken in the construction of a model for continua are [1]: (a) that they occupy regions of space, and (b) that they have mass. These two properties will now be examined in relation to a soil mass.

Let R be an arbitrary region of space and consider two bodies B_1 and B_2 with masses M_1 and M_2 respectively. B_1 will be considered to be a strictly solid body (rolled steel, say) but B_2 will definitely be taken as a dry soil. Assuming for the present that B_1 and B_2 occupy equal regions of space R, densities ρ_1 and ρ_2 can be introduced such that

$$M_1 = \int_R \rho_1 dv \tag{2.1}$$

$$M_2 = \int_R \rho_2 dv \tag{2.2}$$

The density ρ_1 for B_1 so introduced is the <u>mass density</u>. However, since B_2 has mass by virtue of its soil skeleton, the density ρ_2 is the <u>bulk mass density</u>. Thus, for the solid continuum and soil mass continuum the definitions (2.1) and (2.2) yield two different types of mass densities.

The question arises as to whether a <u>mass density</u> for the soil body can be introduced as a field variable. The answer to this question is positive provided that another field variable be admitted. To this purpose let V_s denote the volume of the solid aggregate of the soil body B_2 which occupies the region of space R. A density ω can then be introduced such that

$$V_s = \int_R \omega dv. \tag{2.3}$$

The density ω is the additional field variable that may be introduced for the soil mass continuum [2] which, if it is also introduced for the solid continuum, has a constant value of one. The variable ω may be called the solids material volume density. Note that if dv is an element of volume of soil mass then

$$dv_s = \omega dv \tag{2.4}$$

gives the element of solid aggregate volume. Having introduced ω the more familiar variable, porosity, of the soil mass continuum is obtained from

$$n = 1 - \omega. \tag{2.5}$$

The mass density $\overline{\rho}$ for the soil mass continuum is then given by

$$\overline{\rho} = \frac{\rho_2}{\omega}. \tag{2.6}$$

Therefore, the soil mass continuum differs from the solid continuum in the sense that two field variables ρ_2, ω are needed to obtain the mass density of the solid aggregates. This idea will now be put into formal grounds.

In continuum mechanics a body B is considered to be a manifold of particles, denoted by X. The particles, however, are primitive elements in the sense that numbers are primitive elements in analysis [3]. The body manifold is further assumed to be (1) smooth and isomorphic to regions in Euclidean 3-space, and (2) endowed with a non-negative measure M of space which is its mass distribution [3]. In the same sense, a soil body can be considered to be a set of particles, denoted by X. These particles may be called soil particles which, of course, are not to be confused with the physical soil particles. The soil body manifold can also be assumed to be (1) smooth and isomorphic to regions in Euclidean 3-space, and (2) endowed with two non-negative measures of space: (a) M, which is the distribution of mass of the solid aggregate, and (b) V_s, which is the distribution of material volume of the solid aggregate. The measure M is assigned once and for all. This is not so with the measure V_s which can vary in time due to local microscopic deformation of the solid aggreagre.

Let B denote a soil body. B can then be considered a set of particles X, with the above mentioned structure. A configuration of B is then a smooth homeomorphism $\underset{\sim}{\chi}$ of B onto a subset of Euclidean 3-space $\mathcal{E}$:

$$\underset{\sim}{x} = \underset{\sim}{\chi}(X), \; X = \underset{\sim}{\chi}^{-1}(\underset{\sim}{x}). \tag{2.7}$$

Here $\underset{\sim}{\chi}^{-1}$ denotes the inverse of the mapping $\underset{\sim}{\chi}$.

Consider a small element of cross-sectional area ΔA of the soil body in the configuration $\underset{\sim}{\chi}$. To an observer ΔA will then appear to have a "Swiss cheese-like" structure in which part of ΔA will correspond to the solid aggregates and the remaining part will be voids. Assuming that the voids are randomly distributed in their location within ΔA then in the limit as ΔA becomes infinitesimal a distribution $a_{\underset{\sim}{\chi}}(\underset{\sim}{x})$ can be assigned to the ratio of dA_v, the element of area occupied by the voids, to dA, i.e.,

$$dA_v = a_{\underset{\sim}{\chi}}(\underset{\sim}{x})\,dA. \tag{2.8}$$

Equation (2.8) will be used when the soil stresses are discussed.

3. KINEMATICS

A motion of the soil body B is a one-parameter family of configurations $\underset{\sim}{\chi}_t$,

$$\underset{\sim}{x} = \underset{\sim}{\chi}_t(X) = \underset{\sim}{\chi}(X,t). \tag{3.1}$$

where the parameter $t \in (-\infty,\infty)$ is the time. In view of the smoothness assumptions made $\underset{\sim}{\chi}_t$ will be invertible for each $t \in (-\infty,\infty)$.

A reference configuration for B is a fixed configuration $\underset{\sim}{\kappa}$. The place of $X \in B$ in its reference configuration $\underset{\sim}{\kappa}$ is denoted by $\underset{\sim}{X}$,

$$\underset{\sim}{X} = \underset{\sim}{\kappa}(X), \; X = \underset{\sim}{\kappa}^{-1}(\underset{\sim}{X}). \tag{3.2}$$

Insertion of $(3.2)_2$ into $(3.1)_2$ leads to

$$\underset{\sim}{x} = \underset{\sim}{\chi}_{\kappa}(\underset{\sim}{X},t) = \underset{\sim}{\chi}(\bar{\kappa}^{1}(\underset{\sim}{X}),\ t). \tag{3.3}$$

The function $\underset{\sim}{\chi}_{\kappa}$ is the deformation function for B from its reference configuration $\underset{\sim}{\tilde{\kappa}}$.

The velocity and acceleration for X∈B are defined by

$$\dot{\underset{\sim}{x}} = \frac{\partial}{\partial t}\underset{\sim}{\chi}_{\kappa}(\underset{\sim}{X},t)\ ,\quad \ddot{\underset{\sim}{x}} = \frac{\partial^2}{\partial t^2}\underset{\sim}{\chi}_{\kappa}(\underset{\sim}{X},t) \tag{3.4}$$

while the deformation gradient is given by

$$\underset{\sim}{F} = \nabla\, \underset{\sim}{\chi}_{\kappa}(\underset{\sim}{X},t) \tag{3.5}$$

$\underset{\sim}{F}$ is a second order non-singular tensor with the property

$$|\det \underset{\sim}{F}| > 0. \tag{3.6}$$

If $\underset{\sim}{L}$ denotes the gradient of the velocity $\dot{\underset{\sim}{x}}$ then it can be shown that

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} = \text{grad}\ \dot{\underset{\sim}{x}}(\underset{\sim}{x},t). \tag{3.7}$$

The various strain measures used in continuum mechanics are constructed from the deformation gradient $\underset{\sim}{F}$ [4] and these can be used to describe the deformation of the soil mass continuum. However, for the soil mass continuum an additional "strain" measure can be constructed. If dV denotes an element of volume in $\underset{\sim}{\kappa}(B)$ at $\underset{\sim}{X}$ and dv its image under the mapping (3.3) then it is known that [4]

$$dv = |\det \underset{\sim}{F}|dV. \tag{3.8}$$

Letting dV_s be the element of solid aggregate volume in $\underset{\sim}{\kappa}(B)$ and dv_s the corresponding element in $\underset{\sim}{\chi}_{\kappa}(B,t)$ the following quantity can be constructed

$$\Delta = \frac{dv_s}{dv_s} = \frac{\omega|\det \underset{\sim}{F}|}{\omega_o} \tag{3.9}$$

where ω_o is the solid aggregate volume density in $\underset{\sim}{\varkappa}(B)$. Δ is the average expansion (or contraction) of the solid aggregate. The observation is made that if the assumption of incompressibility of the solid aggregate of the microscopic level is introduced, as is common practice in soil mechanics, then

$$\Delta = 1 \tag{3.10}$$

and (3.9) becomes

$$\omega_o = \omega|\det \underset{\sim}{F}|. \tag{3.11}$$

Introducing the change in porosity, α through

$$\alpha = n - n_o \tag{3.12}$$

where n_o is the porosity in $\underset{\sim}{\varkappa}(B)$, Eq. (3.9) can be written in the form

$$\Delta = (1 - \frac{\alpha}{1 - n_o})|\det \underset{\sim}{F}|. \tag{3.13}$$

Whenever the deformation of the soil mass continuum is infinitesimal the linear strain measure $\underset{\sim}{\tilde{E}}$ given by

$$\underset{\sim}{\tilde{E}} = \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T), \tag{3.14}$$

where $\underset{\sim}{H}$ is the displacement gradient, can be used. In this case under the assumption of small displacement gradients, i.e., $\sqrt{\mathrm{tr}\underset{\sim}{H}\underset{\sim}{H}^T} \ll 1$,

$$|\det \underset{\sim}{F}| \approx 1 + \mathrm{tr}\, \underset{\sim}{\tilde{E}}. \tag{3.15}$$

Then assuming α is of the same order as the strains, $\underset{\sim}{\tilde{E}}$, Eq. (3.9) can be written as

$$\Delta - 1 = \mathrm{tr}\, \underset{\sim}{\tilde{E}} - \frac{\alpha}{1 - n_o} \tag{3.16}$$

The term $\operatorname{tr} \tilde{\underset{\sim}{E}}$ is the volumetric strain. When the solid aggregate is assumed to be microscopically incompressible (3.16) becomes

$$\operatorname{tr} \tilde{\underset{\sim}{E}} = \frac{\alpha}{1 - n_o} . \tag{3.17}$$

which is the formula used in soil mechanics to compute infinitesimal volumetric strains.

Quite often, in soil mechanics, the void ratio e is used instead of the porosity. This is defined by

$$e = \frac{dv_v}{dv_s} = \frac{n}{1 - n} \tag{3.18}$$

where dv_v is the element of volume of the voids. Note that if e_o and e denote initial and current void ratios respectively then

$$\frac{\alpha}{1 - n_o} = \frac{\zeta}{1 + e_o} \tag{3.19}$$

where $\zeta = e - e_o$. Equation (3.17) can then be written in the form

$$\operatorname{tr} \tilde{\underset{\sim}{E}} = \frac{\zeta}{1 + e_o} . \tag{3.20}$$

4. THE BALANCE EQUATIONS

The basic balance equations of continuum mechanics are local formulations of the principles of physics of conservation of (a) mass, (b) linear momentum, (c) moment of momentum, (d) energy. These will be taken one at a time.

Balance of Mass - The balance of mass equation is given in one of two forms [4]: the spatial form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \dot{\underset{\sim}{x}} = 0 \tag{4.1}$$

or the material form

$$\rho|\det \underset{\sim}{F}| = \rho_o \tag{4.2}$$

where ρ_o and ρ are the mass densities in the reference and current configurations respectively.

It can be shown that (4.1) and (4.2) can be taken over unchanged as the balance equation for the soil mass continuum provided that the mass density appearing therein be interpreted as the "bulk mass density". However, for the soil mass continuum (3.8) is available. This equation can be used to eliminate $|\det \underset{\sim}{F}|$ from (4.2) and obtain

$$\bar{\rho}\,\Delta = \bar{\rho}_o \tag{4.3}$$

for the balance of mass equation in terms of "mass density" of the solid aggregate.

Balance of Momentum and Moment of Momentum - Consider an element of oriented surface area $d\underset{\sim}{a}$ of the soil mass continuum in its current configuration and denote the area fraction defined by (2.8) by

$$a = a(\underset{\sim}{x},t) = a_{\chi_t}(\underset{\sim}{x}) \tag{4.4}$$

Further let $\underset{\sim}{t}$ be the stress vector acting on $d\underset{\sim}{a}$. The stress vector $\underset{\sim}{t}_s$ acting on the solid aggregate portion of $d\underset{\sim}{a}$ is defined by

$$\underset{\sim}{t}_s = a\underset{\sim}{t}. \tag{4.5}$$

Cauchy's stress hypothesis is invoked and the existence of a stress tensor $\underset{\sim}{T}_s$ is assumed such that

$$\underset{\sim}{t}_s = \underset{\sim}{T}_s\,\underset{\sim}{n} \tag{4.6}$$

where $\underset{\sim}{n}$ is the unit normal vector of orientation of $d\underset{\sim}{a}$. The stress tensor $\underset{\sim}{T}_s$ is called Terzaghi's effective stress. The reason for calling $\underset{\sim}{T}_s$ the effective stress will now be given.

Consider a completely saturated soil and let $\boldsymbol{T}$ be the total stress acting at a point of the soil. Terzaghi's effective stress is then defined by [5]

$$\boldsymbol{T} = \boldsymbol{T}_s + p\boldsymbol{1} \tag{4.7}$$

where p is the pore water pressure which here is taken as positive in tension. Now the portion $d\boldsymbol{a}_f$ of the oriented element of surface $d\boldsymbol{a}$ occupied by the fluid is given by

$$d\boldsymbol{a}_f = (1-\alpha)d\boldsymbol{a} \tag{4.8}$$

and the portion of the surface vector $\boldsymbol{t}$ acting on the fluid is given by

$$\boldsymbol{t}_f = (1-\alpha)\boldsymbol{t}. \tag{4.9}$$

Note that

$$\boldsymbol{t}_s + \boldsymbol{t}_f = \boldsymbol{t}. \tag{4.10}$$

Cauchy's stress hypothesis is invoked and the existence of a total stress tensor $\boldsymbol{T}$ and partial stress tensor $\boldsymbol{T}_f$ is assumed such that

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n},\ \boldsymbol{t}_f = \boldsymbol{T}_f\boldsymbol{n}. \tag{4.11}$$

Substitution of (4.6), (4.11) into (4.10) leads to

$$(\boldsymbol{T}_s + \boldsymbol{T}_f - \boldsymbol{T})\boldsymbol{n} = \boldsymbol{0}. \tag{4.12}$$

If (4.12) is to hold for arbitrary $\boldsymbol{n}$ it follows that

$$\boldsymbol{T} = \boldsymbol{T}_s + \boldsymbol{T}_f. \tag{4.13}$$

Equation (4.7) follows from (4.13) by setting $\boldsymbol{T}_f = p\boldsymbol{1}$. Thus the reason for calling $\boldsymbol{T}_s$ the effective stress.

Having defined the soil stress, the balance of momentum and moment of momentum for the soil mass continuum is then postulated to be given by

$$\rho\ddot{\underset{\sim}{x}} = \operatorname{div} \underset{\sim}{T}_s + \rho\underset{\sim}{b} \tag{4.14}$$

$$\underset{\sim}{T}_s = \underset{\sim}{T}_s^{\,T} \tag{4.15}$$

wherein $\underset{\sim}{b}$ is the specific body force density, i.e., the body force per unit solid aggregate mass.

Balance of Energy - Let ϵ denote the specific internal energy of the soil body, i.e., the internal energy per unit solid aggregate mass, r the energy source due to external radiation, and $\underset{\sim}{q}$ the heat flux. In a manner similar to that used to define the effective stress, an effective heat flux $\underset{\sim}{q}_s$ can be defined by

$$\underset{\sim}{q}_s = \mathcal{A}\underset{\sim}{q}. \tag{4.16}$$

The balance of energy for the soil mass continuum is then postulated to be given by

$$\rho\dot{\epsilon} = \operatorname{tr} \underset{\sim}{T}_s \underset{\sim}{L} - \operatorname{div} \underset{\sim}{q}_s + \rho r. \tag{4.17}$$

5. THE CLAUSIUS-DUHEM INEQUALITY

Let $\theta = \Theta(\underset{\sim}{x},t)$ be the temperature of the soil mass continuum which is assumed to be positive and let η be the specific entropy. Then regarding $\underset{\sim}{q}_s/\theta$ as the flux of entropy due to heat flow and r/θ the supply of entropy from radiation, the specific rate γ of entropy production for the soil mass continuum is postulated to be given by [4]

$$\rho\gamma = \rho\dot{\eta} - \left(\frac{\rho r}{\theta} - \operatorname{div} \frac{1}{\theta} \underset{\sim}{q}_s\right). \tag{5.1}$$

The Clausius-Duhem inequality is the assertion that the rate of entropy production is not negative, i.e.,

$$\gamma \geq 0. \tag{5.2}$$

Equation (5.1) may be combined with (4.17) so as to obtain

$$\gamma = \dot{\eta} - \frac{\dot{\varepsilon}}{\theta} + \frac{1}{\theta}\,\mathrm{tr}\,\underset{\sim}{S}_s\dot{\underset{\sim}{F}} - \frac{1}{\rho\theta^2}\,\underset{\sim}{q}_s\cdot\underset{\sim}{g}\;. \tag{5.3}$$

where (3.7) has been used and

$$\underset{\sim}{g} = \mathrm{grad}\;\theta,\quad \underset{\sim}{S}_s = \frac{1}{\rho}\,\underset{\sim}{T}_s(\underset{\sim}{F}^T)^{-1} \tag{5.4}$$

The specific free energy ψ may be introduced through the definition

$$\psi = \varepsilon - \theta\eta. \tag{5.5}$$

Under this definition (5.4) may be written in the form

$$\theta\gamma = -\dot{\psi} - \eta\dot{\theta} + \mathrm{tr}\,\underset{\sim}{S}_s\dot{\underset{\sim}{F}} - \frac{1}{\rho\theta}\,\underset{\sim}{q}_s\cdot\underset{\sim}{g}. \tag{5.6}$$

The Clausius-Duhem inequality (5.2) will be used to find restrictions on constitutive equations.

6. CONSTITUTIVE EQUATIONS

It is well known that the deformation of a soil mass continuum is accompanied by dissipative effects which are in addition to heat conduction. Therefore the constitutive equations for the soil body must be such as to show this feature. In continuum mechanics there are various ways of accounting for dissipative effects which (in addition to heat conduction) accompany deformation. One of these is to postulate the existence of internal state variables which influence the free energy and whose rate of change is governed by differential equations in which the strain appears. These have been studied in detail by Coleman and Gurtin [6] for single continua and by Bowen [7,8] for mixtures of continua. It will be shown below that by considering the porosity of the soil mass continuum as an internal state variable the dissipative effects can be accounted for.

It is important to realize at the outset that a thermodynamic state variable for a soil mass continuum is the solid aggregate mass density and not the bulk mass density. The solid aggregate mass density is given by $\bar{\rho}$.

The balance laws suggest that constitutive equations are needed for ψ, η, $\underset{\sim}{T}_s$, and $\underset{\sim}{q}_s$. The soil mass behavior which will be studied here is one which is characterized by four response functions $\bar{\psi}$, $\bar{\eta}$, $\bar{\underset{\sim}{T}}$, $\bar{\underset{\sim}{q}}$ which give ψ, η, $\underset{\sim}{T}_s$, and $\underset{\sim}{q}_s$ when $\bar{\rho}$, $\underset{\sim}{F}$, $\dot{\underset{\sim}{F}}$, θ, $\underset{\sim}{g}$ are known, i.e.,

$$(\psi, \eta, \underset{\sim}{T}_s, \underset{\sim}{q}_s) = \bar{f}(\bar{\rho}, \underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}) \tag{6.1}$$

where $\bar{f}$ stands for any $\bar{\psi}, \ldots, \bar{\underset{\sim}{q}}$. The reason for including $\dot{\underset{\sim}{F}}$ as an independent variable will be explained below.

Now through (2.5), (4.2), equation (2.6) may be written in the form

$$\bar{\rho} = \frac{\rho_o}{(1-n)\,|\det \underset{\sim}{F}|} \tag{6.2}$$

In view of this equation the existence of functions $\hat{f}$ such that

$$(\psi, \eta, \underset{\sim}{T}_s, \underset{\sim}{q}_s) = \hat{f}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \tag{6.3}$$

can be reasoned. This demonstrates how the porosity enters as an independent variable. Since the rate of change $\dot{n}$ of porosity does not appear in the Clausius-Duhem inequality a constitutive equation for $\dot{n}$ must be postulated. Thus to (6.3) must be added

$$\dot{n} = \hat{n}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n). \tag{6.4}$$

Therefore the porosity influences the free energy and its rate of change is postulated by a differential equation in which the strain appears through $\underset{\sim}{F}$. Of course the assumption has to be made that n, $\underset{\sim}{F}$, $\dot{\underset{\sim}{F}}$, θ, $\underset{\sim}{g}$ as functions of $\underset{\sim}{X}$ and t are smooth enough to insure the existence of a unique solution $n = n(\underset{\sim}{X}, t)$ of (6.4) for all t in some interval $[t_o, t_o + \tau]$ with $n(\underset{\sim}{X}, t_o) = n_o(\underset{\sim}{X})$.

The inclusion of $\dot{\underset{\sim}{F}}$ as an independent variable will now be explained. It can be shown that whenever the solid aggregate is incompressible the porosity is governed by the differential equation

$$\dot{n} = (1-n) \text{ tr } \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1} \tag{6.5}$$

Equation (6.5) is a special case of (6.4) with

$$\partial_\theta \hat{\eta} = 0, \quad \partial_{\underset{\sim}{g}} \hat{n} = \underset{\sim}{0}$$

so that

$$\hat{n}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, n) = (1-n) \text{ tr } \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1}.$$

Thus the reason for including $\dot{\underset{\sim}{F}}$ as an independent variable.

It is also assumed that the effective stress $\underset{\sim}{T}_s$ is the sum of a non-dissipative part $\underset{\sim}{T}_o$ and a dissipative part $\underset{\sim}{T}_D$, i.e.,

$$\underset{\sim}{T}_s = \underset{\sim}{T}_o + \underset{\sim}{T}_D \tag{6.6}$$

such that

$$\begin{aligned} \underset{\sim}{T}_o &= \hat{\underset{\sim}{T}}_o(\underset{\sim}{F}, \theta, n) \\ \underset{\sim}{T}_D &= \hat{\underset{\sim}{T}}_D(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n). \end{aligned} \tag{6.7}$$

Therefore

$$\hat{\underset{\sim}{T}}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) = \hat{\underset{\sim}{T}}_o(\underset{\sim}{F}, \theta, n) + \hat{\underset{\sim}{T}}_D(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n). \tag{6.8}$$

In Section 7 it will be shown that

$$\hat{\underset{\sim}{T}}_D(\underset{\sim}{F}, \underset{\sim}{0}, \theta, \underset{\sim}{0}, n) = \underset{\sim}{0} \quad \text{whenever } \dot{n} = 0. \tag{6.9}$$

The constitutive response functions $\hat{f}$ and $\hat{n}$ have to satisfy the Clausius-Duhem inequality (5.6). The restrictions on the response functions $\hat{f}$ and $\hat{n}$ by the Clausius-Duhem inequality can be found by the method used by Coleman and Gurtin [6] and Bowen [7,8]. The main results of this exercise are:

(I) The response functions $\hat{\psi}$ and $\hat{\eta}$ are independent of $\dot{\underset{\sim}{F}}$ and $\underset{\sim}{g}$, i.e.,

$$(\psi, \eta) = \hat{h}(\underset{\sim}{F}, \theta, n) \tag{6.10}$$

where $\hat{h}$ stands for either $\hat{\psi}$ or $\hat{\eta}$.

(II) $\hat{\psi}$ determines $\hat{\eta}$ through the entropy relation

$$\eta = -\partial_\theta \hat{\psi}(\underset{\sim}{F}, \theta, n). \tag{6.11}$$

(III) $\hat{\psi}$ determines $\hat{\underset{\sim}{T}}_D$ through a stress relation

$$\underset{\sim}{T}_o = \rho \partial_{\underset{\sim}{F}} \hat{\psi}(\underset{\sim}{F}, \theta, n)\underset{\sim}{F}^T. \tag{6.12}$$

(IV) $\hat{\psi}$, $\hat{n}$, $\hat{\underset{\sim}{T}}_D$, and $\hat{\underset{\sim}{q}}$ obey the general dissipation inequality

$$\begin{aligned} \operatorname{tr} \hat{\underset{\sim}{S}}_D(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n)\dot{\underset{\sim}{F}} - \partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\hat{n}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \\ - \frac{1}{\rho\theta} \hat{\underset{\sim}{q}}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \cdot \underset{\sim}{g} \geq 0. \end{aligned} \tag{6.13}$$

where

$$\hat{\underset{\sim}{S}}_D(\quad) = \frac{1}{\rho} \hat{\underset{\sim}{T}}_D(\quad)(\underset{\sim}{F}^T)^{-1}. \tag{6.14}$$

Equations (6.10) through (6.13) are necessary and sufficient conditions that the Clausius-Duhem inequality be satisfied by the constitutive response functions. It is possible to extract additional information from the general dissipation inequality (6.13). This information will be examined in the next section.

7. EQUILIBRIUM STATES

The additional information that can be extracted from the general dissipation inequality (6.13) is obtained for certain values of the independent variables. It has been found convenient to name equilibrium the state in which these values occur.

The general dissipation inequality (6.13) implies that when $\underset{\sim}{g} = \underset{\sim}{0}$, the mechanical dissipation inequality

$$-\operatorname{tr} \hat{\underset{\sim}{S}}_D(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{0}, n)\dot{\underset{\sim}{F}} + \partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\hat{n}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{0}, n) \leq 0 \tag{7.1}$$

holds, and when $(\dot{\underset{\sim}{F}}, \partial_n \hat{\psi} \hat{n}) = (\underset{\sim}{0}, 0)$ the heat conduction inequality

$$\hat{\underset{\sim}{q}}(\underset{\sim}{F}, \underset{\sim}{0}, \theta, \underset{\sim}{g}, n) \cdot \underset{\sim}{g} \leq 0 \tag{7.2}$$

holds. Also when $(\dot{\underset{\sim}{F}}, \underset{\sim}{g}) = (\underset{\sim}{0}, \underset{\sim}{0})$ the internal dissipation inequality

$$\partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\hat{n}(\underset{\sim}{F}, \underset{\sim}{0}, \theta, \underset{\sim}{0}, n) \le 0 \tag{7.3}$$

holds.

It is convenient to call a triplet $(\underset{\sim}{F}^*, \theta^*, n^*)$ with

$$\hat{n}(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*) = 0 \tag{7.4}$$

an equilibrium state for the soil material point X. Note that if

$$\mathcal{l}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) = \operatorname{tr} \hat{\underset{\sim}{S}}_D(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n)\dot{\underset{\sim}{F}} - \partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\, \hat{n}\, (\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n)$$

$$- \frac{1}{\rho\theta} \hat{\underset{\sim}{q}}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \cdot \underset{\sim}{g} \tag{7.5}$$

then (6.13) can be written as

$$\mathcal{l}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \ge 0. \tag{7.6}$$

Clearly

$$\mathcal{l}(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*) = 0 \tag{7.7}$$

Therefore as a function of $(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n)$, $\mathcal{l}$ is a minimum at the equilibrium state $(\underset{\sim}{F}^*, \theta^*, n^*)$. Consequently

$$\frac{d}{d\lambda} \mathcal{l}(\underset{\sim}{F}^* + \lambda\underset{\sim}{A}, \lambda\underset{\sim}{B}, \theta^* + \lambda a, \lambda\underset{\sim}{a}, n^* + \lambda d)\Big|_{\lambda=0} = 0 \tag{7.8}$$

for all scalars a, d, all vectors $\underset{\sim}{a}$, and all second order tensors $\underset{\sim}{A}$, $\underset{\sim}{B}$ in the domain of $\mathcal{l}$.

For a function G( ) evaluated at $(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*)$ the following notation is used

$$G^+ = G(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*).$$

Equations (7.5) and (7.8) imply that

$$\operatorname{tr}(\hat{\underset{\sim}{S}}_D^+ - \partial_n \hat{\psi}^+ \, \partial_{\underset{\sim}{F}} \hat{n}^{+T})\underset{\sim}{B} - \partial_n \hat{\psi}^+ \{\operatorname{tr} \partial_{\underset{\sim}{F}} \hat{n}^{+T} \underset{\sim}{A} + \partial_\theta \hat{n}^+ \, a + \partial_n \hat{n}^+ d\}$$

$$-(\partial_n \hat{\psi}^+ \, \partial_{\underset{\sim}{g}} \hat{n}^+ + \frac{1}{\rho^+ \theta^*} \hat{\underset{\sim}{q}}^+) \cdot \underset{\sim}{a} = 0 \tag{7.9}$$

From the second term in this equation and the arbitrariness of $\underset{\sim}{A}$, a, and d the following is concluded: either

$$\partial_{\underset{\sim}{F}} \hat{n}^+ = \underset{\sim}{0} \,, \quad \partial_\theta \hat{n}^+ = 0 \,, \quad \partial_n \hat{n}^+ = 0 \tag{7.10}$$

or

$$\partial_n \hat{\psi}^+ = 0. \tag{7.11}$$

For obvious reasons (7.10) must be discarded. Therefore in view of (7.11) and the arbitrariness of $\underset{\sim}{B}$ and $\underset{\sim}{a}$ the following additional information is extracted from (7.9)

$$\hat{\underset{\sim}{S}}_D(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*) = \underset{\sim}{0} \tag{7.12}$$

$$\hat{\underset{\sim}{q}}(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, n^*) = \underset{\sim}{0} \tag{7.13}$$

Thus at the equilibrium state the dissipative stress and the heat flux vanish. Also (7.11) reads

$$\partial_n \hat{\psi}(\underset{\sim}{F}^*, \theta^*, n^*) = 0. \tag{7.14}$$

Equation (7.14) is called the equation of internal equilibrium.†

---

†Coleman and Gurtin [6] have derived an equation identical to (7.14) in their study of constitutive equations for which the independent variables are $(\underset{\sim}{F}, \theta, \underset{\sim}{g}, \underset{\sim}{\alpha})$, $\underset{\sim}{\alpha}$ being an internal state N-vector.

It is assumed that corresponding to each strain-temperature pair $(\underset{\sim}{F}^*, \theta^*)$ there is exactly one porosity n^* such that (7.4) holds. This correspondence is given by the function

$$n^* = \alpha^*(\underset{\sim}{F}^*, \theta^*) \tag{7.15}$$

which is called the equilibrium response function for n. Equilibrium response functions $\hat{\psi}^*$, η^*, $\hat{\underset{\sim}{T}}^*_o$ may then be constructed through

$$(\psi^*, \eta^*, \underset{\sim}{T}^*_o) = \hat{m}^*(\underset{\sim}{F}^*, \theta^*) \stackrel{\text{def}}{=\!=} \hat{m}(\underset{\sim}{F}^*, \theta^*, \alpha(\underset{\sim}{F}^*, \theta^*)) \tag{7.16}$$

where $\hat{m}^*$ stands for any of ψ^*, η^*, or $\hat{\underset{\sim}{T}}^*_D$ and $\hat{m}$ for any of $\hat{\psi}$, $\hat{\eta}$, or $\hat{\underset{\sim}{T}}_o$. Considering that

$$\partial_\theta \hat{\psi}^*(\underset{\sim}{F}, \theta) = \partial_\theta \hat{\psi}(\underset{\sim}{F}, \theta, n) + \partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\partial_\theta \alpha(\underset{\sim}{F}, \theta)$$

$$\partial_{\underset{\sim}{F}} \hat{\psi}^*(\underset{\sim}{F}, \theta) = \partial_{\underset{\sim}{F}} \hat{\psi}(\underset{\sim}{F}, \theta, n) + \partial_n \hat{\psi}(\underset{\sim}{F}, \theta, n)\partial_{\underset{\sim}{F}} \alpha(\underset{\sim}{F}, \theta)$$

It follows through the internal equilibrium equation that

$$\begin{aligned} \partial_{\theta*} \hat{\psi}^*(\underset{\sim}{F}^*, \theta^*) &= \partial_\theta \hat{\psi}(\underset{\sim}{F}^*, \theta^*, n^*) \\ \partial_{\underset{\sim}{F}*} \hat{\psi}^*(\underset{\sim}{F}^*, \theta^*) &= \partial_{\underset{\sim}{F}} \hat{\psi}(\underset{\sim}{F}^*, \theta^*, n^*). \end{aligned} \tag{7.17}$$

Therefore in view of (6.11) and (6.12)

$$\begin{aligned} \eta^* &= -\partial_{\theta*} \hat{\psi}^*(\underset{\sim}{F}^*, \theta^*) \\ \underset{\sim}{S}^*_o &= \partial_{\underset{\sim}{F}*} \hat{\psi}^*(\underset{\sim}{F}^*, \theta^*) \end{aligned} \tag{7.18}$$

where

$$\underset{\sim}{S}^*_o = \frac{1}{\rho^*} \underset{\sim}{T}^*_o (\underset{\sim}{F}^{*T})^{-1}. \tag{7.19}$$

Equation (7.18) defines the equilibrium entropy and stress relations.

8. MATERIAL FRAME-INDIFFERENCE

The constitutive equations are further restricted by the axiom of material-frame indifference. This axiom states that the constitutive response functions must be form invariant under a change of frame. The change of frame is characterized by a time-dependent orthogonal tensor $\underset{\sim}{Q}(t)$ (cf. Truesdell and Noll [9]).

Under a change of frame scalars θ, ϵ, η, and ψ are unaffected. However $\underset{\sim}{F}$, $\dot{\underset{\sim}{F}}$, $\underset{\sim}{g}$, $\underset{\sim}{q}_s$, and $\underset{\sim}{T}_s$ transform as follows

$$
\begin{aligned}
&\underset{\sim}{F} \to \underset{\sim}{Q}\,\underset{\sim}{F} \\
&\dot{\underset{\sim}{F}} \to \underset{\sim}{Q}\,\dot{\underset{\sim}{F}} + \dot{\underset{\sim}{Q}}\,\underset{\sim}{F} \\
&\underset{\sim}{g} \to \underset{\sim}{Q}\,\underset{\sim}{g} \qquad\qquad (8.1)\\
&\underset{\sim}{q}_s \to \underset{\sim}{Q}\,\underset{\sim}{q}_s \\
&\underset{\sim}{T}_s \to \underset{\sim}{Q}\,\underset{\sim}{T}_s\,\underset{\sim}{Q}^T .
\end{aligned}
$$

The manner in which the porosity n transforms needs to be specified. It is postulated that n is unaffected under a change of frame. Therefore

$$n \to n, \quad \dot{n} \to \dot{n} \qquad (8.2)$$

under a change of frame.

In view of (3.7) dependence on $\dot{\underset{\sim}{F}}$ can be indicated through dependence on $\underset{\sim}{L}$. Moreover $\underset{\sim}{L}$ can be written as the sum of its symmetric part $\underset{\sim}{D}$ and skew symmetric part $\underset{\sim}{W}$,

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W} , \qquad (8.3)$$

where $2\underset{\sim}{D} = \underset{\sim}{L} + \underset{\sim}{L}^T$, $2\underset{\sim}{W} = \underset{\sim}{L} - \underset{\sim}{L}^T$. Under a change of frame

$$
\begin{aligned}
&\underset{\sim}{D} \to \underset{\sim}{Q}\,\underset{\sim}{D}\,\underset{\sim}{Q}^T \\
&\underset{\sim}{W} \to \underset{\sim}{Q}\,\underset{\sim}{W}\,\underset{\sim}{Q}^T + \dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T .
\end{aligned} \qquad (8.4)
$$

Note that from the orthogonality of $\underset{\sim}{Q}$, it follows that $\dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T = -\underset{\sim}{Q}\,\dot{\underset{\sim}{Q}}^T$. The fact that $\underset{\sim}{W}$ does not transform as a tensor under a change of frame can be used to show that the constitutive equations should be independent of $\underset{\sim}{W}$. Therefore

$$(\underset{\sim}{T}_s, \underset{\sim}{q}_s, n) = \hat{a}(\underset{\sim}{F}, \dot{\underset{\sim}{D}}, \theta, \underset{\sim}{g}, n) \tag{8.5}$$

where $\hat{a}$ stands for either $\hat{\underset{\sim}{T}}$, $\hat{\underset{\sim}{q}}$, or $\hat{n}$.

Necessary and sufficient conditions that the constitutive equations (6.10) and (8.3) satisfy the axiom of material frame indifference are the following

$$\bar{a}(\underset{\sim}{F}, \theta, n) = \bar{a}(\underset{\sim}{Q}\,\underset{\sim}{F}, \theta, n)$$

$$\begin{aligned}
\underset{\sim}{Q}\,\hat{\underset{\sim}{T}}\,(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n)\underset{\sim}{Q}^T &= \hat{\underset{\sim}{T}}(\underset{\sim}{Q}\underset{\sim}{F}, \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\underset{\sim}{g}, n) \\
\underset{\sim}{Q}\,\hat{\underset{\sim}{q}}\,(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \hat{\underset{\sim}{q}}(\underset{\sim}{Q}\underset{\sim}{F}, \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\underset{\sim}{g}, n) \\
\hat{n}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \hat{n}(\underset{\sim}{Q}\,\underset{\sim}{F}, \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\underset{\sim}{g}, n)
\end{aligned} \tag{8.6}$$

where $\bar{a}$ stands for either $\hat{\psi}$ or $\hat{\eta}$. Choosing $\underset{\sim}{Q} = -\underset{\sim}{1}$, $(8.6)_{2-4}$ becomes

$$\begin{aligned}
\hat{\underset{\sim}{T}}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \hat{\underset{\sim}{T}}(-\underset{\sim}{F}, \underset{\sim}{D}, \theta, -\underset{\sim}{g}, n) \\
-\hat{\underset{\sim}{q}}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \hat{q}(-\underset{\sim}{F}, \underset{\sim}{D}, \theta, -\underset{\sim}{g}, n) \\
\hat{n}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \hat{n}(-\underset{\sim}{F}, \underset{\sim}{D}, \theta, -\underset{\sim}{g}, n).
\end{aligned} \tag{8.7}$$

Thus $\hat{\underset{\sim}{T}}$, $\hat{n}$ are even functions of $\underset{\sim}{F}$ and $\underset{\sim}{g}$ and $\hat{\underset{\sim}{q}}$ is an odd function of $\underset{\sim}{F}$ and $\underset{\sim}{g}$.

Using standard arguments (cf. Truesdell and Noll [9]) it can be shown that a set of reduced forms of the response functions which are frame-indifferent are

$$\begin{aligned}
\bar{a}(\underset{\sim}{F}, \theta, n) &= a^{+}(\underset{\sim}{C}, \theta, n) \\
\hat{\underset{\sim}{T}}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \underset{\sim}{F}\, \underset{\sim}{T}^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n)\underset{\sim}{F}^{T} \\
\hat{\underset{\sim}{q}}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= \underset{\sim}{F}\, \underset{\sim}{q}^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n) \\
\hat{n}(\underset{\sim}{F}, \underset{\sim}{D}, \theta, \underset{\sim}{g}, n) &= n^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n)
\end{aligned} \tag{8.8}$$

where

$$\underset{\sim}{C} = \underset{\sim}{F}^{T}\underset{\sim}{F} \tag{8.9}$$

is the right Cauchy-Green tensor. To arrive at (8.8) the identity $\dot{\underset{\sim}{C}} = 2\underset{\sim}{F}^{T}\underset{\sim}{D}\underset{\sim}{F}$ has been used.

Considering that

$$\partial_{\underset{\sim}{F}}\hat{\psi}(\underset{\sim}{F}, \theta, n) = 2\underset{\sim}{F}\, \partial_{\underset{\sim}{C}}\psi^{+}(\underset{\sim}{C}, \theta, n)$$

equation (6.12) can be written as

$$\underset{\sim}{T}_{o} = 2\rho\underset{\sim}{F}\partial_{\underset{\sim}{C}}\psi^{+}(\underset{\sim}{C}, \theta, n)\underset{\sim}{F}^{T}. \tag{8.10}$$

Also

$$\begin{aligned}
\eta &= -\partial_{\theta}\psi^{+}(\underset{\sim}{C}, \theta, n) \\
\underset{\sim}{T}_{s} &= 2\rho\underset{\sim}{F}\partial_{\underset{\sim}{C}}\psi^{+}(\underset{\sim}{C}, \theta, n)\underset{\sim}{F}^{T} + \underset{\sim}{F}\, \underset{\sim}{T}^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n)\underset{\sim}{F}^{T} \\
\underset{\sim}{q}_{s} &= \underset{\sim}{F}\, \underset{\sim}{q}^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n) \\
\dot{n} &= n^{+}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{F}^{T}\underset{\sim}{g}, n)
\end{aligned} \tag{8.11}$$

and the equilibrium entropy and stress relations (7.18) become

$$\begin{aligned}
\eta^{*} &= -\partial_{\theta *}\psi^{*+}(\underset{\sim}{C}^{*}, \theta^{*}) \\
\underset{\sim}{T}_{o}^{*} &= 2\rho^{*}\underset{\sim}{F}^{*}\partial_{\underset{\sim}{C}*}\psi^{*+}(\underset{\sim}{C}^{*}, \theta^{*})\underset{\sim}{F}^{*T}
\end{aligned} \tag{8.12}$$

9. MATERIAL SYMMETRY

The constitutive response functions for the soil mass continuum are further restricted by material symmetries. Recall Noll's [10] definition of the isotropy group $\mathscr{G}$ of a material response function $\hat{F}$: the local isotropy group $\mathscr{G}$ of a material is the set of mass density-preserving changes of local reference configuration which leaves the response function $\hat{F}$ unaltered.

The above definition was arrived at through the recognition that a change from a given reference configuration $\underset{\sim}{\kappa}_1$ to another reference configuration $\underset{\sim}{\kappa}_2$ which is indistinguishable from $\underset{\sim}{\kappa}_1$ by relating the values F of $\hat{F}$ to deformation must be obtained by a mapping from $\underset{\sim}{\kappa}_1$ to $\underset{\sim}{\kappa}_2$ such that

$$\rho_{\underset{\sim}{\kappa}_1} = \rho_{\underset{\sim}{\kappa}_2}. \tag{9.1}$$

For the soil mass continuum it appears more natural to base the definition of the isotropy group $\mathscr{G}$ of the soil for a constitutive response $\hat{F}$ on the solid aggregate mass density $\bar{\rho}$ and porosity n. Thus (9.1) is replaced by

$$\bar{\rho}_{\underset{\sim}{\kappa}_1} = \bar{\rho}_{\underset{\sim}{\kappa}_2}, \quad n_{\underset{\sim}{\kappa}_1} = n_{\underset{\sim}{\kappa}_2} \tag{9.2}$$

or in view of (2.5) and (2.6) by the equivalent statement

$$\rho_{\underset{\sim}{\kappa}_1} = \rho_{\underset{\sim}{\kappa}_2}. \tag{9.3}$$

Therefore the definition of the isotropy group for a response function $\hat{F}$ of the soil mass continuum is essentially the same as Noll's. Thus the isotropy group $\mathscr{G}$ for the soil mass continuum is the set of all unimodular tensors $\underset{\sim}{H}$ such that the following identities hold:

$$\begin{aligned}
\hat{a}(\underset{\sim}{F}, \theta, n) &= \hat{a}(\underset{\sim}{F}\underset{\sim}{H}, \theta, n) \\
\hat{\underset{\sim}{T}}(\underset{\sim}{F}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{g}, n) &= \hat{\underset{\sim}{T}}(\underset{\sim}{F}\underset{\sim}{H}, \underset{\sim}{H}^T\dot{\underset{\sim}{C}}\underset{\sim}{H}, \theta, \underset{\sim}{g}, n) \\
\hat{\underset{\sim}{q}}(\underset{\sim}{F}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{g}, n) &= \hat{\underset{\sim}{q}}(\underset{\sim}{F}\underset{\sim}{H}, \underset{\sim}{H}^T\dot{\underset{\sim}{C}}\underset{\sim}{H}, \theta, \underset{\sim}{g}, n) \\
\hat{n}(\underset{\sim}{F}, \dot{\underset{\sim}{C}}, \theta, \underset{\sim}{g}, n) &= \hat{n}(\underset{\sim}{F}\underset{\sim}{H}, \underset{\sim}{H}^T\dot{\underset{\sim}{C}}\underset{\sim}{H}, \theta, \underset{\sim}{g}, n)
\end{aligned} \tag{9.4}$$

where $\hat{a}$ stands for either $\hat{\psi}$ or $\hat{\eta}$.

It can be reasoned that if a solid is thought of as a body which has some preferred configuration from which any change of shape will change some of its properties then a soil mass is a solid. The ideas set forth by Noll [10] can then be carried over unchanged, i.e., since the soil mass continuum is a solid its isotropy group $\mathscr{G}$ is a subgroup of the orthogonal group $\mathscr{O}(\mathcal{E})$ of linear transformations of Euclidean 3-space onto itself,

$$\mathscr{G} \subset \mathscr{O}(\mathcal{E}) \tag{9.5}$$

When the soil mass-continuum is isotropic the isotropy group $\mathscr{G}$ is equal to the orthogonal group $\mathscr{O}(\mathcal{E})$. It can then be shown that the constitutive equations for ψ, n, $\underset{\sim}{T}_s$, $\underset{\sim}{q}_s$, and $\dot{n}$ for an isotropic soil mass which are frame indifferent are

$$\begin{aligned}
\psi &= \psi^+(\underset{\sim}{B}, \theta, n) \\
\eta &= \eta^+(\underset{\sim}{B}, \theta, n) = -\partial_\theta \eta^+(\underset{\sim}{B}, \theta, n) \\
\underset{\sim}{T}_s &= 2\rho\partial_{\underset{\sim}{B}}\psi^+(\underset{\sim}{B}, \theta, n)\underset{\sim}{B} + \underset{\sim}{T}^{\neq}(\underset{\sim}{B}, \dot{\underset{\sim}{B}}, \theta, \underset{\sim}{g}, n) \\
\underset{\sim}{q}_s &= \underset{\sim}{q}^{\neq}(\underset{\sim}{B}, \dot{\underset{\sim}{B}}, \theta, \underset{\sim}{g}, n) \\
\dot{n} &= n^{\neq}(\underset{\sim}{B}, \dot{\underset{\sim}{B}}, \theta, \underset{\sim}{g}, n)
\end{aligned} \tag{9.6}$$

where $\underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{F}^T$ is the right Cauchy-Green deformation tensor and all the response functions are isotropic tensor functions. Representation theorems for the response functions are available in [9].

10. ELASTIC BEHAVIOR

In solid mechanics elastic deformations are recoverable deformations and the thermodynamic process under which these take place is a reversible process. It follows that elastic behavior occurs under thermodynamic equilibrium. Therefore for a soil mass continuum the constitutive equations for elastic behavior are

$$\begin{aligned} \psi &= \psi^*(\underset{\sim}{F}, \theta) \\ \eta &= -\partial_\theta \psi^*(\underset{\sim}{F}, \theta) \\ \underset{\sim}{T}_s &= \rho \partial_{\underset{\sim}{F}} \psi^*(\underset{\sim}{F}, \theta)\underset{\sim}{F}^T \\ \underset{\sim}{q}_s &= \underset{\sim}{0} \end{aligned} \tag{10.1}$$

and

$$n = \alpha(\underset{\sim}{F}, \theta) \tag{10.2}$$

wherein all deformations are recoverable and the temperature field is homogeneous. Note that (10.2) can be written as

$$\lambda(\underset{\sim}{F}, \theta, n) = 0. \tag{10.3}$$

Upon reduction of (10.1), (10.3) for material frame-indifference the following is obtained

$$\begin{aligned} \psi &= \psi^+(\underset{\sim}{C}, \theta) \\ \eta &= -\partial_\theta \psi^+(\underset{\sim}{C}, \theta) \\ \underset{\sim}{T}_s &= 2\rho\, \underset{\sim}{F}\, \partial_{\underset{\sim}{C}} \psi^+(\underset{\sim}{C}, \theta)\underset{\sim}{F}^T \\ \underset{\sim}{q}_s &= \underset{\sim}{0} \\ \lambda(\underset{\sim}{C}, \theta, n) &= 0. \end{aligned} \tag{10.4}$$

For an isotropic soil mass (10.4) reduces to

$$
\begin{aligned}
\psi &= \psi^{+}(\underset{\sim}{B},\ \theta) \\
\eta &= -\partial_{\theta}\psi^{+}(\underset{\sim}{B},\ \theta) \\
\underset{\sim}{T}_{s} &= 2\rho\,\partial_{\underset{\sim}{B}}\psi^{+}(\underset{\sim}{B},\ \theta)\underset{\sim}{B} \\
\underset{\sim}{q}_{s} &= \underset{\sim}{0} \\
\lambda(\underset{\sim}{B},\ \theta,\ n) &= 0.
\end{aligned}
\tag{10.5}
$$

Inspection of $(10.4)_{1-4}$ indicates that the elastic behavior of a soil mass is described by constitutive equations for ψ, η, $\underset{\sim}{T}_{s}$, $\underset{\sim}{q}_{s}$ which are of the same form as the equations for a hyperelastic solid under isothermal conditions. However this is only possible whenever $(10.4)_{5}$ is satisfied. Consequently elastic behavior of the soil mass implies the existence λ of $(\underset{\sim}{C},\ \theta,\ n)$ such that $(10.5)_{5}$ is satisfied.

11. LINEARIZATION

In this section the constitutive equations postulated in the previous sections will be linearized by considering small departures from thermodynamic equilibrium. To this end it is assumed that the soil mass in its reference configuration is in a state of thermodynamic equilibrium.

Introduce the Green-St. Venant strain tensor $\underset{\sim}{E}$ which is related to $\underset{\sim}{C}$ through

$$
2\underset{\sim}{E} = \underset{\sim}{C} - \underset{\sim}{1}. \tag{11.1}
$$

Consequently

$$
\psi = \hat{\psi}^{+}(\underset{\sim}{E},\ \theta,\ n) = \psi^{+}(\underset{\sim}{1} + 2\underset{\sim}{E},\ \theta,\ n). \tag{11.2}
$$

Thus (8.11), (8.12) can be written as

$$\eta = -\partial_\theta \hat{\psi}^+(\underset{\sim}{E},\ \theta,\ n)$$

$$\underset{\sim}{T}_s = \rho\ \underset{\sim}{F}\ \partial_{\underset{\sim}{E}} \hat{\psi}^+(\underset{\sim}{E},\ \theta,\ n)\underset{\sim}{F}^T + \underset{\sim}{F}\ \hat{\underset{\sim}{T}}_D{}^+(\underset{\sim}{E},\ \dot{\underset{\sim}{E}},\ \theta,\ \underset{\sim}{F}^T\underset{\sim}{g},\ n)\underset{\sim}{F}^T$$

$$\underset{\sim}{q}_s = \underset{\sim}{F}\ \hat{\underset{\sim}{q}}^+(\underset{\sim}{E},\ \dot{\underset{\sim}{E}},\ \theta,\ \underset{\sim}{F}^T\underset{\sim}{g},\ n) \tag{11.3}$$

$$\dot{n} = \hat{n}^+(\underset{\sim}{E},\ \dot{\underset{\sim}{E}},\ \theta,\ \underset{\sim}{F}^T\underset{\sim}{g},\ n)$$

where $2\dot{\underset{\sim}{E}} = \dot{\underset{\sim}{C}}$ has been used.

Let $\underset{\sim}{u}$ be the displacement vector field of the soil mass. The displacement gradient is then given by

$$\underset{\sim}{H} = \nabla\underset{\sim}{u}(\underset{\sim}{X},t). \tag{11.4}$$

$\underset{\sim}{H}$ is related to the deformation gradient through

$$\underset{\sim}{F} = \underset{\sim}{1} + \underset{\sim}{H} \tag{11.5}$$

The departures from thermodynamic equilibrium can be measured by the quantity δ defined by

$$\delta^2 = (\theta-\theta_o)^2 + (n-n_o)^2 + \underset{\sim}{g}\cdot\underset{\sim}{g} + \mathrm{tr}\ \underset{\sim}{H}\ \underset{\sim}{H}^T + \mathrm{tr}\ \dot{\underset{\sim}{H}}\ \dot{\underset{\sim}{H}}^T, \tag{11.6}$$

where θ_o is the temperature in the reference state. The departure from thermodynamic equilibrium is said to be small if $\delta < 1$. A quantity of order δ^{m} is any scalar, vector, or tensor, denoted by $0(\delta^{m})$, with the property that there exists a real number N such that

$$||\ 0(\delta^{m})\ || = N\delta^{m} \tag{11.7}$$

as $\delta \rightarrow 0$. Observe that $0(\delta^{m_1})0(\delta^{m_2}) = 0(\delta^{m_1+m_2})$.

Under the assumption of $\delta < 1$ it follows that

$$\theta - \theta_o = O(\delta)$$
$$n - n_o = O(\delta)$$
$$\underset{\sim}{H} = O(\delta) \qquad (11.8)$$
$$\dot{\underset{\sim}{H}} = O(\delta)$$
$$\underset{\sim}{g} = O(\delta).$$

Also

$$\underset{\sim}{E} = \underset{\sim}{\tilde{E}} + O(\delta^2)$$
$$\dot{\underset{\sim}{E}} = \dot{\underset{\sim}{\tilde{E}}} + O(\delta^2)$$
$$\underset{\sim}{F}^T\underset{\sim}{g} = \underset{\sim}{g} + O(\delta^2) \qquad (11.9)$$
$$|\det \underset{\sim}{F}| = 1 + \operatorname{tr} \underset{\sim}{\tilde{E}} + O(\delta^2)$$
$$\rho = \rho_o(1 - \operatorname{tr} \underset{\sim}{\tilde{E}}) + O(\delta^2) = \rho_o + O(\delta),$$

where $\underset{\sim}{\tilde{E}}$ is the linear strain tensor given by (3.14).

Consider the ordered sextuple

$$(\underset{\sim}{E},\ \dot{\underset{\sim}{E}},\ \theta,\ \underset{\sim}{F}^T\underset{\sim}{g},\ n)$$

Its value in the reference configuration is

$$(\underset{\sim}{0},\ \underset{\sim}{0},\ \theta_o,\ \underset{\sim}{0},\ n_o).$$

Therefore $\hat{\psi}^+$ can be expanded about $\underset{\sim}{E} = \underset{\sim}{0}$, $\theta = \theta_o$, $n = n_o$ to yield an expression of the form

$$\hat{\psi}^+(\underset{\sim}{0} + \underset{\sim}{\tilde{E}},\ \theta_o + \bar{\theta},\ n_o + \alpha) = \psi_o + a_1\bar{\theta} + \tfrac{1}{2}a_2\theta^2 + a_3\alpha\,\bar{\theta} + \tfrac{1}{2}a_4\alpha^2 + b_1[\underset{\sim}{\tilde{E}}]\,\bar{\theta}$$
$$+ b_2[\underset{\sim}{\tilde{E}}]\alpha + \operatorname{tr} \underset{\sim}{\mathcal{L}}\,\underset{\sim}{\tilde{E}} + \tfrac{1}{2}\, b_3[\underset{\sim}{\tilde{E}},\ \underset{\sim}{\tilde{E}}] + O(\delta^3) \qquad (11.10)$$

where

$$\bar{\theta} = \theta - \theta_o \ , \quad \alpha = n - n_o \tag{11.11}$$

and ψ_o, a_1, a_2, a_3 are constants, $\underset{\sim}{\mathcal{L}}$ is a second order tensor, $b_1[\cdot]$, $b_2[\cdot]$ are linear and $b_3[\cdot,\cdot]$ bilinear functions of $\underset{\sim}{\tilde{E}}$. The constant ψ_o is the value of the free energy in the reference configuration. A term of the form $a_5\alpha$ would ordinarily appear in (11.10) but because of the internal equilibrium equation (7.14), $a_5 = 0$. Thus to within $O(\delta^3)$

$$\psi = \psi_o + a_1\bar{\theta} + \tfrac{1}{2}a_2\bar{\theta}^2 + a_3\alpha\bar{\theta} + \tfrac{1}{2}a_4\alpha^2 + b_1[\underset{\sim}{\tilde{E}}]\,\bar{\theta} + b_2[\underset{\sim}{\tilde{E}}]\,\alpha + \mathrm{tr}\,\underset{\sim}{\mathcal{L}}\,\underset{\sim}{\tilde{E}} + \tfrac{1}{2}b_3[\underset{\sim}{\tilde{E}},\underset{\sim}{\tilde{E}}]. \tag{11.12}$$

It follows from (11.12) that approximate expressions for η, $\underset{\sim}{T}_o$ for small departures from equilibrium are

$$\eta = -a_1 - a_2\bar{\theta} - a_3\alpha - b_1[\underset{\sim}{\tilde{E}}]$$

$$\underset{\sim}{T}_o = \rho_o \frac{\partial b_1}{\partial \underset{\sim}{\tilde{E}}}[\underset{\sim}{\tilde{E}}]\,\bar{\theta} + \rho_o \frac{\partial b_2}{\partial \underset{\sim}{\tilde{E}}}[\underset{\sim}{\tilde{E}}]\,\alpha + \rho_o\underset{\sim}{\mathcal{L}} + \tfrac{1}{2}\rho_o \frac{\partial b_3}{\partial \underset{\sim}{\tilde{E}}}[\underset{\sim}{\tilde{E}},\underset{\sim}{\tilde{E}}]. \tag{11.13}$$

Also the linear approximations for small departures from equilibrium of $\underset{\sim}{T}_D$, $\underset{\sim}{q}_s$, $\dot{n}$ are

$$\begin{aligned} \underset{\sim}{T}_D &= (b_1\bar{\theta} + b_2\alpha)\underset{\sim}{1} + \underset{\sim}{M}_1[\underset{\sim}{\tilde{E}}] + \underset{\sim}{M}_2[\dot{\underset{\sim}{\tilde{E}}}] \\ \underset{\sim}{q}_s &= \bar{\theta}\,\underset{\sim}{a}_1 + \alpha\,\underset{\sim}{a}_2 + \underset{\sim}{K}_1[\underset{\sim}{\tilde{E}}] + \underset{\sim}{K}_2[\underset{\sim}{g}] \\ \dot{n} &= c_1\bar{\theta} + c_2\alpha + N_1[\underset{\sim}{\tilde{E}}] + N_2[\dot{\underset{\sim}{\tilde{E}}}] \end{aligned} \tag{11.14}$$

where b_1, b_2, c_1, c_2 are constants, $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$ are constant vectors, $\underset{\sim}{M}_1[\cdot]$, $\underset{\sim}{M}_2[\cdot]$ are symmetric linear tensor functions, $\underset{\sim}{K}_1[\cdot]$, $\underset{\sim}{K}_2[\cdot]$ are linear vector functions, and $N_1[\cdot]$, $N_2[\cdot]$ are linear scalar functions. In writing (11.4) use was made of the fact that $\underset{\sim}{T}_D$, $\dot{n}$ are even functions of $\underset{\sim}{g}$ and $\underset{\sim}{q}_s$ is an even function of $\dot{\tilde{\underset{\sim}{E}}}$.

The case of an isotropic soil mass is now considered. In this case $b_1[\cdot]$, $b_2[\cdot]$, $\underset{\sim}{\mathcal{L}}$, $b_3[\cdot,\cdot]$ in (11.12) have the following representation:

$$
\begin{aligned}
& b_\alpha[\tilde{\underset{\sim}{E}}] = d_\alpha \mathrm{tr}\, \tilde{\underset{\sim}{E}},\ \alpha = 1,2 \\
& \underset{\sim}{\mathcal{L}} = d_3\, \underset{\sim}{1} \qquad (11.15) \\
& b_3[\tilde{\underset{\sim}{E}},\tilde{\underset{\sim}{E}}] = d_4 \mathrm{tr}\, \tilde{\underset{\sim}{E}}^2 + d_5(\mathrm{tr}\, \tilde{\underset{\sim}{E}})^2
\end{aligned}
$$

Also since there are no isotropic tensors of odd rank, $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, and $\underset{\sim}{K}_1[\cdot]$ must drop out of $(11.4)_2$, and $\underset{\sim}{K}_2[\cdot]$, $\underset{\sim}{M}_1[\cdot]$, $\underset{\sim}{M}_2[\cdot]$, $N_1[\cdot]$, $N_2[\cdot]$ have the following representation:

$$
\begin{aligned}
& \underset{\sim}{K}_2[\underset{\sim}{g}] = K\, \underset{\sim}{g} \\
& \underset{\sim}{M}_1[\tilde{\underset{\sim}{E}}] = e_1(\mathrm{tr}\, \tilde{\underset{\sim}{E}})\underset{\sim}{1} + e_2\, \tilde{\underset{\sim}{E}} \\
& \underset{\sim}{M}_2[\dot{\tilde{\underset{\sim}{E}}}] = e_3(\mathrm{tr}\, \dot{\tilde{\underset{\sim}{E}}})\underset{\sim}{1} + e_4\, \dot{\tilde{\underset{\sim}{E}}} \qquad (11.16) \\
& N_1[\tilde{\underset{\sim}{E}}] = c_3 \mathrm{tr}\, \tilde{\underset{\sim}{E}} \\
& N_2[\dot{\tilde{\underset{\sim}{E}}}] = c_4 \mathrm{tr}\, \dot{\tilde{\underset{\sim}{E}}}.
\end{aligned}
$$

In (11.15), (11.16), d_1, d_2, d_3, d_4, d_5, K, e_1, e_2, e_3, e_4, c_3, and c_4 are material constants.

With the representations (11.15), (11.16) the linear constitutive equations (11.12), (11.13), and (11.14) become

$$
\begin{aligned}
\psi = \psi_o + a_1\overline{\theta} + \tfrac{1}{2}a_2\overline{\theta}^2 + a_3\alpha\,\overline{\theta} + d_1\overline{\theta}\ \mathrm{tr}\, \tilde{\underset{\sim}{E}} + d_2\alpha\ \mathrm{tr}\, \tilde{\underset{\sim}{E}} + d_3 \mathrm{tr}\, \tilde{\underset{\sim}{E}} + \tfrac{1}{2}d_4 \mathrm{tr}\, \tilde{\underset{\sim}{E}}^2 \\
+ \tfrac{1}{2}d_5(\mathrm{tr}\, \tilde{\underset{\sim}{E}})^2 \qquad (11.17)
\end{aligned}
$$

[Equation (11.17) continued on next page]

$$\eta = -a_1 - a_2\bar{\theta} - a_3\alpha - d_1 \mathrm{tr}\,\tilde{\underset{\sim}{E}}$$

$$\underset{\sim s}{T} = p\underset{\sim}{1} + (f_1\bar{\theta} + f_2\alpha + \lambda \mathrm{tr}\,\tilde{\underset{\sim}{E}} + e_3 \mathrm{tr}\,\dot{\tilde{\underset{\sim}{E}}})\underset{\sim}{1} + 2\mu\,\tilde{\underset{\sim}{E}} + e_4\,\dot{\tilde{\underset{\sim}{E}}}$$

$$\underset{\sim s}{q} = K\,\underset{\sim}{g}$$

$$\dot{n} = c_1\dot{\bar{\theta}} + c_2\dot{\alpha} + c_3 \mathrm{tr}\,\tilde{\underset{\sim}{E}} + c_4 \mathrm{tr}\,\dot{\tilde{\underset{\sim}{E}}}$$

where

$$p = \rho_o d_3 \ , \ f_1 = \rho_o d_1 + b_1 \ , \quad f_2 = \rho_o d_2 + b_2$$

$$\lambda = \rho_o d_4 + e_1 \ , \quad 2\mu = \rho_o d_5 + e_2. \qquad (11.18)$$

The general dissipation inequality (6.13) will give some inequalities for the constants b_1, b_2, c_1, c_2 and the constants appearing in (11.16). However since the constitutive equations examined here are merely for illustrative purposes and have no bearing on true soil behavior these inequalities will not be found.

12. CONCLUSIONS

In this study the basic equations of continuum mechanics for non-polar continuum have been reassessed as to their applicability to model a dry soil mass as a contimuum. It was found that the physical properties of a body which are taken in the construction of a continuum model, namely that of occupying a region of space and having mass, allow for the introduction of two densities, the bulk mass density and the solid aggregate volume density. In this respect the soil mass continuum differs from the strictly solid continuum. The introduction of the solid aggregate volume density brings the porosity of the soil mass into play as an additional field variable. It must be pointed out that in applied soil mechanics the solid aggregate of a soil is assumed incompressible for computational convenience, for in this case the volumetric strain is computed through a phase diagram as

$$\frac{\Delta v}{v} = \frac{\Delta e}{1 + e_o}, \tag{12.1}$$

e being the void ratio. Now it can be shown that in the limit the left hand side of (12.1) becomes tr $\tilde{\underset{\sim}{E}}$. Thus

$$\text{tr}\ \tilde{\underset{\sim}{E}} = \frac{\Delta e}{1 + e_o} \tag{12.2}$$

which was found earlier (see Eq. (3.20)). Now it is realized that soil is extremely difficult to sample and that volumetric measurements are hard to make. Nevertheless the assumption leading to (12.2) is dictated by this physical handicap and it is not in general a property of the soil.

The equations used in continuum mechanics to depict the balance of mass, linear momentum, and energy also hold for the soil mass continuum provided that the mass density appearing therein be interpreted as the bulk mass density. In addition, through the balance of mass equation in terms of bulk mass density, a balance of mass equation in terms of bulk mass density, a balance of mass equation in terms of solid aggregate mass density was found.

The introduction of the porosity as a field variable introduces complications since in general there is no equation relating porosity to deformation. This indicates that a constitutive equation for the porosity is needed. As an example of how a constitutive equation for the porosity may be introduced, a set of constitutive equations for a special kind of soil mass was studied. It must be pointed out that the constitutive equations studied here may not describe true soil behavior under load. However, the work presented here does yield results which are very important, namely that under isothermal conditions,

(1) The constitutive equation for the effective stress should be of the form

$$\underset{\sim}{T}_s = \underset{\sim}{\mathcal{F}}(\underset{\sim}{F}, n) \tag{12.3}$$

where $\underset{\sim}{\mathcal{F}}$ indicates a general functional relationship.

(2) A constitutive equation has to be postulated for the rate of change of porosity.

(3) Elastic behavior of the soil implies the existence of a function λ such that

$$\lambda(\underset{\sim}{F},n) = 0 \tag{12.4}$$

for all elastic deformations.

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SECTION II

A THEORY OF SOIL PLASTICITY

1. INTRODUCTION

A valid solution to a problem of the mechanical response of a soil mass to applied load must satisfy the basic balance equations of continuum mechanics. However, in order to obtain valid solutions, the constitutive relation between stress and strain of the soil must be established: Without knowledge of the stress-strain relation or its equivalent a so-called solution is merely a guess.

Partly because of the difficulty in obtaining self-consistent constitutive relations, problems in soil mechanics are treated in several and unrelated ways. For example, when no failure of the soil is involved stresses at points in a soil mass under a footing, or behind a retaining wall are computed using linear elasticity. Problems of bearing capacity, stability of slopes, failure of retaining walls are now being considered in the realm of plasticity, while settlement and consolidation problems are treated as essentially viscoelastic.

In this paper our primary concern is with the establishing of a self-consistent phenomenological theory for the mechanical behavior of granular media which shows stress-strain behavior similar to that of an elastic-work hardening plastic metal.

Drucker and Prager [8] suggested that the Mohr-Coulomb failure criteria for soils could serve as a yield function with which one could associate a flow rule and to treat the soil mass as a perfectly plastic material. Even though several important and interesting results may be obtained by considering the soil mass to a perfectly plastic body with a Mohr-Coulomb yield surface, predictions of volume changes, under this idealization, were higher than those found by experiments. Based on observations made on pressure-volume change curves Drucker, Gibson, and Henkel [7] subsequently explained that soil could be treated as a work hardening material. Henkel [9], however, concluded that much of the available experimental information for soils lay outside the scope of a useful theory of plasticity. Nevertheless Roscoe and co-workers at Cambridge have indicated that certain soils can be described remarkably well by a simple isotropic work-hardening idealization. The work at Cambridge is discussed in considerable detail by Roscoe and Burland [5].

In the work described above a soil mass differs from a strictly solid mass by the constitutive relations describing the mechanical behavior. Aguirre-Ramirez [1] questioned the applicability of the basic balance equations of continuum mechanics to model a dry soil mass without investigation. He found that the two basic properties that bodies are known to have and which are used in the construction to model a body as a continuum, namely that (1) they have mass and (2) occupy regions of space, lead to the introduction of two densities one of which is the soil bulk mass density and the other can be related to the porosity of the soil mass. The basic local balance equations of a continuum can then be used for the physical-mathematical description of a dry soil mass and the processes occuring in it provided the mass density appearing therein be interpreted as the bulk mass density. The porosity, however, appears as an additional field variable that plays the role of a hidden variable. In soil mechanics porosity changes are related to bulk volume changes by an assumption which will be discussed in the text.

In this paper we establish a self-consistent phenomenological theory for the mechanical behavior of granular media which shows stress-strain behavior of the type discussed by Drucker, Gibson, and Henkel [7]. This is done by extending the ideas presented in [1]. The main results of this paper are presented in Sections 4 and 5 we have felt compelled to include in Section 3 those aspects of soil mass behavior that we have used as a guide in arriving at them. In Section 4 we present a theory of plasticity for soils. The theory is a phenomenological theory in which statements are made directly put into mathematical form and studied as such. Using the theory developed in Section 4 we construct constitutive relations for soils in a triaxial compression condition. This is done in Section 5. We find that the Cambridge triaxial compression theory developed by Roscoe and co-workers [5] comes out as a special case of the theory constructed in Section 5. This is very promising because the Cambridge triaxial compression theory of Roscoe and co-workers [5] has been found to give reasonable agreement with experimental results.

We must remark that by considering the soil mass to be dry we have disregarded the influence of the pore pressure on its behavior. This corresponds to a soil in what is called a drained condition.

NOTATION

In this paper direct tensor notation is used except in Section 5. For the most part vectors in the three-dimensional inner product vector space U and points in Euclidean 3-space ε are indicated by bold faced Latin minuscules: x,...,u. Linear transformations from U into U are indicated by boldfaced Latin majuscules T,...,N. Second order tensors and linear transformations are regarded as the same. If T is a linear transformation, T^T indicates its transpose, T^{-1} its inverse, tr T its trace, and det T its determinant. The gradient with respect to spatial coordinates is denoted by grad and the gradient with respect to material coordinates by ∇.

2. PRELIMINARIES

We consider a dry soil mass body B which occupies a region R in Euclidean 3-space in a reference configuration and denote by $\underset{\sim}{X}$ the position in R of the particle $X \in B$. We further suppose B to occupy the region R_t at time t and denote by $\underset{\sim}{x}$ the position in R_t of the particle $X \in B$. The motion of B from R to R_t is given by

$$\underset{\sim}{x} = \chi(\underset{\sim}{X}, t). \tag{2.1}$$

Let $n_o = n_o(\underset{\sim}{X})$ denote the porosity of B in R. According to the ideas set forth by Aguirre-Ramirez [1] in order to describe the deformed state of B at time t we have to set alongside (2.1),

$$n = \hat{n}(\underset{\sim}{X}, t) \tag{2.2}$$

where n is the porosity of B at time t. The function $\hat{n}$ is such that

$$n_o = n_o(\underset{\sim}{X}) = \hat{n}(\underset{\sim}{X}, t_o) \tag{2.3}$$

where t_o is the reference time.

The gradient of $\underset{\sim}{\chi}$,

$$\underset{\sim}{F} = \nabla \underset{\sim}{\chi}(\underset{\sim}{X}, t) \tag{2.4}$$

is called the deformation gradient. $\underset{\sim}{F}$ is a second order non-singular tensor with the property

$$|\det \underset{\sim}{F}| > 0. \tag{2.5}$$

We let $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{X}, t)$ be the displacement vector from R to R_t and $\underset{\sim}{H}$ be its gradient,

$$\underset{\sim}{H} = \nabla \underset{\sim}{u}(\underset{\sim}{X}, t). \tag{2.6}$$

The deformation gradient $\underset{\sim}{F}$ is related to $\underset{\sim}{H}$ by

$$\underset{\sim}{F} = \underset{\sim}{1} + \underset{\sim}{H}. \tag{2.7}$$

We denote by dV_s and dv_s the element of solid aggregate of the soil body in R and R_t respectively. The mean solid aggregate dilation or expansion Δ is then defined by [1]

$$\Delta = \frac{dv_s}{dV_s}. \tag{2.8}$$

It can be shown that Δ is given by [1]

$$\Delta = \frac{(1-n)}{(1-n_o)} \left|\det \underset{\sim}{F}\right| \tag{2.9}$$

and is an additional strain measure that is characteristic of the soil mass body.

The quantity $e = e(\underset{\sim}{X},t)$ defined by

$$e = \frac{n\left|\det \underset{\sim}{F}\right|}{(1-n_o)} \tag{2.10}$$

is the void ratio of the soil mass defined as "the ratio of the element of void volume at time t to the element of solid aggregate of the soil mass in the reference configuration". Note that

$$\Delta = \frac{(1-n)}{n} e. \tag{2.11}$$

In what follows the word "specific" shall mean per unit mass of solid aggregate. Let ρ^o, ρ be the soil bulk mass density and ρ_s^o, ρ_s the solid aggregate mass density in R and R_t respectively. The differential equations governing the deformation and motion of the soil mass body are given by [1]

(i) Balance of mass

$$\rho\left|\det \underset{\sim}{F}\right| = \rho^o \quad \text{or} \quad \rho_s \Delta = \rho_s^o. \tag{2.12}$$

(ii) Balance of linear and moment of momentum

$$\operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{x}} \tag{2.13}$$

$$\underset{\sim}{T} = \underset{\sim}{T}^T \tag{2.14}$$

(iii) Balance of energy

$$\rho\dot{\epsilon} = \text{tr}\, \underset{\sim}{T}\, \underset{\sim}{L} - \text{div}\, \underset{\sim}{q} + \rho r. \tag{2.15}$$

In (2.13) and (2.14), $\underset{\sim}{T}$ is Terzaghi's [2] effective stress which was defined in [1] in such a way so as to obtain (2.13) from a global balance law, $\underset{\sim}{b}$ the specific body force density, $\ddot{\underset{\sim}{x}}$ the acceleration, ϵ the specific internal energy density, $\underset{\sim}{q}$ the effective heat flux vector, r the specific heat source density, and $\underset{\sim}{L}$ the velocity gradient which is related to $\underset{\sim}{F}$ by

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1} = \text{grad}\, \dot{\underset{\sim}{x}}\, (\underset{\sim}{x}, t) \tag{2.16}$$

Alongside (2.12), (2.13), and (2.15) constitutive equations are needed for the soil mass. The work presented in [1] indicates that these constitutive equations should be

$$\begin{aligned} \psi &= \Psi(\underset{\sim}{F}, \theta, n) \\ (\underset{\sim}{T}, \underset{\sim}{q}) &= \mathfrak{F}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \\ \dot{n} &= \Xi(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \theta, \underset{\sim}{g}, n) \end{aligned} \tag{2.17}$$

where $\Psi, \mathfrak{F}, \Xi$ indicate a general functional relationship, ψ is the specific free energy density, θ the temperature and

$$\underset{\sim}{g} = \text{grad}\, \theta \tag{2.18}$$

Also in [1] a triplet $(\underset{\sim}{F}^*, \theta^*, n^*)$ with

$$\Xi(\underset{\sim}{F}^*, \underset{\sim}{0}, \theta^*, \underset{\sim}{0}, \underset{\sim}{n}^*) = 0 \tag{2.19}$$

was called a thermodynamic equalibrium state for the material point X of the soil mass.

If the stress is written as the sum of a non-dissipative part $\underset{\sim}{T}_o$ and a dissipative part $\underset{\sim}{T}_D$,

$$\underset{\sim}{T} = \underset{\sim}{T}_o + \underset{\sim}{T}_D \tag{2.20}$$

with constitutive equations

$$\underset{\sim}{T}_o = \underset{\sim}{\mathcal{L}}_o(\underset{\sim}{F},\theta,n) \tag{2.21}$$

$$\underset{\sim}{T}_D = \underset{\sim}{\mathcal{L}}_D(\underset{\sim}{F},\dot{\underset{\sim}{F}},\theta,\underset{\sim}{g},n) \tag{2.22}$$

where $\underset{\sim}{\mathcal{L}}_o$, $\underset{\sim}{\mathcal{L}}_D$ indicate a general functional relationship, then $\underset{\sim}{\mathcal{L}}_D$ must be such that

$$\underset{\sim}{\mathcal{L}}_D(\underset{\sim}{F}^*,\underset{\sim}{0},\theta^*,\underset{\sim}{0},n^*) = \underset{\sim}{0}. \tag{2.23}$$

Also

$$\underset{\sim}{q} = \underset{\sim}{\mathcal{K}}(\underset{\sim}{F}^*,\underset{\sim}{0},\theta^*,\underset{\sim}{0},n^*) = \underset{\sim}{0}, \tag{2.24}$$

i.e., at equilibrium the effective heat flux vanishes. In (2.24), $\underset{\sim}{\mathcal{K}}$ indicates a general functional relationship.

In continuum mechanics elastic deformations are recoverable deformations and the thermodynamic process under which these take place is a reversible process. Therefore one may reason that elastic behavior occurs under thermodunamic equilibrium. Suppose the soil mass is responding elastically with respect to some configuration[+] R_o at time t. Then according to (2.19) all deformations $\underset{\sim}{F}$, all porosities n, and all temperatures θ are such that

$$\hat{\Xi}(\underset{\sim}{F},\theta,n) = \Xi(\underset{\sim}{F},\underset{\sim}{0},\theta,\underset{\sim}{0},n) = 0 \tag{2.25}$$

and since $\underset{\sim}{g} = \underset{\sim}{0}$, the temperature field θ is homogeneous. The effective stress, free energy, and heat flux is given by

$$\begin{aligned} \underset{\sim}{T} &= \hat{\underset{\sim}{T}}(\underset{\sim}{F},\theta,n) \\ \psi &= \hat{\psi}(\underset{\sim}{F},\theta,n) \\ \underset{\sim}{q} &= \underset{\sim}{0} \end{aligned} \tag{2.26}$$

where $\hat{\underset{\sim}{T}}, \hat{\psi}$ are ordinary functions.

[+]Here a configuration and the region the soil mass B occupies in $\mathcal{E}$ in that configuration are taken to mean the same. No confusion need arise since B is isomorphic to regions in $\mathcal{E}$.

To demonstrate that the above ideas of elastic behavior of the soil mass are equivalent to the elastic behavior (constitutive equation wise) of a simple continuum it suffices to indicate that the constitutive equations for an elastic soil mass are (2.25) and (2.26). Then under sufficient smoothness assumptions on $\hat{\Xi}$ the following equation

$$n = f(\underset{\sim}{F}, \theta) \tag{2.27}$$

for the porosity may be obtained from (2.25). It follows that

$$\begin{aligned} \underset{\sim}{T} &= \underset{\sim}{T}^{+}(\underset{\sim}{F}, \theta) = \hat{\underset{\sim}{T}}(\underset{\sim}{F}, \theta, f(\underset{\sim}{F}, \theta)) \\ \psi &= \psi^{+}(\underset{\sim}{F}, \theta) = \hat{\psi}(\underset{\sim}{F}, \theta, f(\underset{\sim}{F}, \theta)) \end{aligned} \tag{2.28}$$

which are the constitutive equations for an elastic simple continuum.

3. SOIL BEHAVIOR

In this section we shall discuss some aspects of soil behavior which we shall use as a guide in the next section. Before doing this, however, let us indicate that most experimental data on the response of a soil mass under load is for infinitesimal deformations under isothermal conditions. Also in the reduction of soil test data the following formula is used to compute volumetric strains

$$\frac{\Delta v}{v} = - \frac{\Delta n}{1-n_o} \tag{3.1}$$

where $\Delta v, \Delta n$ are the change in sample volume and porosity respectively. It can be shown that under infinitesimal deformations (3.1) follows from (2.9) under the assumption of incompressibility of the solid aggregate of the soil. In compressibility of the solid aggregate is really not a general property of the soil. The reason for using (3.1) is that volumetric measurements on a soil sample are extermely difficult to make. Measurements of porosity changes on the other hand are simpler to make since these can be related to the amount of fluid expelled from the pores during the test.

Most soils show a phenomena that is generally not shown by metallic solids undergoing infinitesimal deformations. This is the phenomena of dilatancy, i.e., bulk volume changes in a state of apparent pure shear.

Current methods of testing soils have been, almost always, restricted to the conventional triaxial compression test, the oedometer and to a far less extent, the direct shear test. The reader is referred to Lamb and Whitman [2] for a discussion of these tests. In the direct shear test the sample is assumed to be subjected to plane strain. In the triaxial compression test and oedometer the sample is assumed to be in a stress state in which the intermediate and minor physical components of the stress tensor are equal.

The triaxial test is essentially a cylindrical sample first put under an equal all around pressure, called the confining pressure (denoted by σ in Fig. 1), and then adding increments of load in the direction of the axis of the cylinder (Fig 1). As mentioned above the stress state of the sample during the test is assumed to be such that

$$\sigma_1 = \sigma_2 = \sigma \ , \quad \sigma_3 \tag{3.2}$$

are the only non-vanishing physical components of the stress tensor in a polar coordinate system. In soil mechanics it is a practice to use the generalized stress parameters p and q defined by

$$p = \frac{1}{3}(\sigma_3 + 2\sigma) \;, \quad q = \sigma_3 - \sigma \qquad (3.3)$$

as an appropriate set of independent parameters that can be used in analyzing data obtained in the triaxial test.

A typical stress-strain curve for a dry soil mass obtained in a vacuum triaxial test is shown in Fig. 2. This curve was obtained at the Geotechnical Laboratory of NASA's Marshall Space Flight Center at Huntsville for a lunar soil simulant material. For the purpose of discussion of this curve and for the remainder of the paper compressive stresses will be taken as positive. Examination of the curve (Fig. 2) indicates that the stress-strain behavior of this particular soil mass is similar to the behavior of an elastic-strain hardening plastic material. We also note that the response to a decrease in stress is an elastic recovery.

The confined compression (oedometer) test is a cylindrical sample subjected to axial load but prevented from horizontal movement. Because of this last constraint lateral stresses develop which in general are not measured. As mentioned above accurate measurements of volume changes in dry soil are not easy to make. In the oedometer test, however, because of the no lateral movement constraint, the axial strain is exactly equal to the volumetric strain. The parameters used to analyze data from this test are generally the vertical effective stress denoted by p and the porosity n.

A typical p-n curve obtained in the confined compression test is shown in Fig. 3. The significant features of this curve are the non-linear relationship between p and n and the elastic response to a decrease in stress. We note that this curve also shows behavior which is similar to the behavior of an elastic-strain hardening material.

The similarity of soil stress-strain curves of the type shown in Figs. 2 and 3 to that of an elastic-strain hardening plastic material has led some researchers to suggest that the Mohr-Coulomb[+] criteria could serve as a

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[+]In the next section we shall discuss the Mohr-Coulomb failure criteria.

yield function with which one could properly associate a flow rule [7]. While this is a valid assumption it is not a reasonable one, because if the Mohr-Coulomb envelope is used as a yield surface then yielding does not occur until failure takes place. As shown by the curve (Fig. 2) soils yield long before they fail. The use of the Mohr-Coulomb failure criteria as a yield surface gives erroneous predictions of high rates of change of volume during shear distortion. This is very unfortunate because research workers who reject these predictions will have the tendency to discount the usefulness of the theory of plasticity to model some aspects of the stress-strain behavior of soils.

4. SOIL PLASTICITY

In this section we shall formulate a theory of plasticity for soils. While the considerations of Section 2 hold for finite deformations, the discussion in this and the following sections is limited to small strains. Thus the strain tensor that we shall use will be the linear strain tensor,

$$\underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{H} + \underset{\sim}{H}^T). \tag{4.1}$$

Also we shall assume that the change in porosity

$$\zeta = n - n_o \tag{4.2}$$

is of the same order of magnitude as the strains.

In a soil we may define yield as a permanent irrecoverable deformation. We may write the strain $\underset{\sim}{E}$ as the sum of an elastic or recoverable part $\underset{\sim}{E}_r$ and a plastic or irrecoverable part $\underset{\sim}{E}_p$,

$$\underset{\sim}{E} = \underset{\sim}{E}_r + \underset{\sim}{E}_p. \tag{4.3}$$

In addition we shall assume porosity to be given as the sum of an elastic or recoverable part n_r and a plastic or irrecoverable part n_p,

$$n = n_r + n_p. \tag{4.4}$$

In view of this assumed resolution we shall have for the change in porosity

$$\zeta = \zeta_r + n_p \tag{4.5}$$

where $\zeta_r = n_r - n_o$. The quantity ν given by

$$\nu = \Delta - 1 \tag{4.6}$$

where Δ is given by (2.9) is the solid aggregate dilatation. For small strains ν is given by

$$\nu = \frac{\zeta}{1-n_o} + \mathrm{tr}\,\underset{\sim}{E}. \tag{4.7}$$

Under the assumed resolution (4.3), (4.5), ν is given by

$$\nu = \left(\frac{\zeta_r}{1-n_o} + \mathrm{tr}\, \underset{\sim}{E}_r\right) + \left(\frac{n_p}{1-n_o} + \mathrm{tr}\, \underset{\sim}{E}_p\right). \tag{4.8}$$

Thus the solid aggregate dilatation is the sum of two parts, a recoverable part

$$\nu_r = \frac{\zeta_r}{1-n_o} + \mathrm{tr}\, \underset{\sim}{E}_r \tag{4.9}$$

and an irrecoverable part

$$\nu_p = \frac{n_p}{1-n_o} + \mathrm{tr}\, \underset{\sim}{E}_p. \tag{4.10}$$

The plastic deformation of the soil is therefore described by the pair $(\underset{\sim}{E}_p, n_p)$.

A fundamental assumption of metal plasticity is that the elastic strains may be computed through the elastic constitutive equations for the stress. We carry this assumption into soil plasticity. Thus the elastic strains and elastic porosity may be computed through the soil elastic constitutive equation for an isothermal process,

$$\underset{\sim}{T} = \hat{\underset{\sim}{T}}(\underset{\sim}{E}_r, \zeta_r) \tag{4.11}$$

$$f(\underset{\sim}{E}_r, \zeta_r) = 0 \tag{4.12}$$

where we have introduced the change in recoverable porosity. We assume (4.11) to be invertible in $\underset{\sim}{E}_r$,

$$\underset{\sim}{E}_r = \hat{\underset{\sim}{E}}(\underset{\sim}{T}, \zeta_r). \tag{4.13}$$

This allows us to write (4.12) in the form

$$\hat{f}(\underset{\sim}{T}, \zeta_r) = 0. \tag{4.14}$$

We find it convenient to decompose the stress tensor into the sum of the mean pressure p and deviatoric stress $\underset{\sim}{\mathcal{T}}$, i.e.,

$$\underset{\sim}{T} = p\,\underset{\sim}{1} + \underset{\sim}{\mathcal{T}} \tag{4.15}$$

$$p = \frac{1}{3}\,\mathrm{tr}\,\underset{\sim}{T},\ \mathrm{tr}\,\underset{\sim}{\mathcal{T}} = 0. \tag{4.16}$$

In view of this assumed resolution we may write (4.14) as

$$g(p,\underset{\sim}{\mathcal{T}},\zeta_r) = 0 \tag{4.17}$$

Equation (4.17) defines a surface Ω in ζ-$\underset{\sim}{T}$ space. We call this surface Ω, the elastic surface. Note that we may solve (4.17) for ζ_r so as to obtain

$$\zeta_r = g^*(p,\underset{\sim}{\mathcal{T}}). \tag{4.18}$$

The curve in ζ-p space defined by

$$\zeta_r^* = \hat{\zeta}_r(p) = g^*(p,\underset{\sim}{0}) \tag{4.19}$$

is called the <u>elastic swelling curve</u> of the consolidation curve of the soil.

We consider a curve in ζ-p space given by

$$\psi(p,\zeta) = 0 \tag{4.20}$$

with ψ fixed and unique. For a given soil there exists such a curve which is called the <u>virgin consolidation curve</u>. The intersection of the elastic swelling curve with the virgin consolidation curve defines a point in ζ-p space which is a <u>yield point</u> for the soil. We denote the mean pressure corresponding to such a point by p_o. Now (4.20) may be solved for the change in porosity ζ,

$$\zeta = \hat{\zeta}(p). \tag{4.21}$$

From (4.19) and (4.21) we find

$$n_p^o = \hat{\zeta}(p_o) - \hat{\zeta}_r(p_o) \tag{4.22}$$

which indicates that with the pressure p_o at yield we can associate a plastic porosity n_p^o. We shall use this result below.

One of the main ingredients of metal plasticity is that of a yield surface. We shall now demonstrate how to construct a yield surface for a soil. Our construction of a yield surface for the soil is wholly dependent on the hypothesis that "plastic porosity is a unique function of stress," i.e.,

$$n_p = N(\underset{\sim}{T}). \tag{4.23}$$

We further assume (4.23) to be the solution for n_p of the equation

$$G(\underset{\sim}{T}, n_p) = 0 \tag{4.24}$$

where G is unique. In plastic pososity-stress space (4.24) defines a six-dimensional hypersurface Σ that is called the state boundary surface. A soil particle will be said to be in a plastic state if the value of the stress and plastic porosity are such that (4.24) is satisfied.

We consider a curve on Σ. The projection of this curve on stress space is a five-dimensional hypersurface Ψ. There are curves on Σ which have the unique feature that the value of plastic porosity is the same all along the curve. Let n_p^* be the fixed value of the plastic porosity along one of these curves and consider the set B of all $\underset{\sim}{T}$ such that

$$G(\underset{\sim}{T}, n_p^*) = 0. \tag{4.25}$$

We call B a yield domain. The projection of this equi-plastic porosity curve on stress space is a five-dimensional hypersurface whose equation is given by

$$A(\underset{\sim}{T}, \varkappa) = 0 \ , \quad \underset{\sim}{T} \in B \tag{4.26}$$

where the parameter $\varkappa$ depends upon the value n_p^* of plastic porosity, i.e.,

$$\varkappa = \hat{\varkappa}(n_p^*). \tag{4.27}$$

Our concept of a yield surface Ψ, for the soil is given by (4.26). Note that along the yield surface the plastic porosity has a constant value n_p^*, i.e., for all states of stress which locate points on Ψ the value of the plastic porosity is the same. In particular we can find a yield surface such that the value of plastic porosity associated with it is given by (4.22), i.e., $n_p^* = n_p^o$. We call such yield surfaces "volumetric yield surfaces" and denote them by Ψ_v. Thus a volumetric yeild surface is characterized by

$$\varkappa = \hat{\varkappa}(n_p^o). \tag{4.28}$$

Also since with each n_p^o we can associate a mean pressure p_o we can also characterize volumetric yield surfaces by

$$\varkappa = \varkappa^+(p_o) \tag{4.29}$$

If we consider another equi-plastic porosity curve in Σ its projection in stress space is given by an equation of the form (4.26) but with a different yield domain and, of course, a different value of the parameter corresponding to a different value of plastic porosity. In addition, since for all states of stress on a given yield surface, the plastic porosity is constant, it follows that the change in porosity ζ is also constant. We have envisioned soil as a work-hardening material. Thus in order for plastic deformation to occur, the stress point must move outside the yield curve, i.e., the initial yield point must be exceeded. A new yield curve is then established which, depending upon the shape of the state boundary surface Σ, may or may not resemble the old yield curve. We shall also show below that the yield surface for soils is not a closed surface.

With our concept of a yield surface the loading and unloading criteria are respectively given by

$$\begin{aligned} &\operatorname{tr} \partial_{\underset{\sim}{T}} A(\underset{\sim}{T},\varkappa)\dot{\underset{\sim}{T}} < 0, \ A(\underset{\sim}{T},\varkappa) = 0 \\ &\operatorname{tr} \partial_{\underset{\sim}{T}} A(\underset{\sim}{T},\varkappa)\dot{\underset{\sim}{T}} > 0, \ A(\underset{\sim}{T},\varkappa) = 0 \end{aligned} \tag{4.30}$$

while the neutral loading is given by

$$\operatorname{tr} \partial_{\underset{\sim}{T}} A(\underset{\sim}{T},\varkappa)\dot{\underset{\sim}{T}} = 0, \quad A(\underset{\sim}{T},\varkappa) = 0. \tag{4.31}$$

An important ingredient of plasticity is that of Drucker's postulate of stability of material [3]. This postulate is used to classify a material as a work-hardening material. According to this postulate, if an external agency applies a small surface fraction which alters the stress at each point by $\dot{\underset{\sim}{T}}$, then upon gradual application and removal of this surface fraction

$$\operatorname{tr} \dot{\underset{\sim}{T}} \dot{\underset{\sim}{E}}_p \geq 0 \tag{4.32}$$

if the material is work-hardening. Important consequences of Drucker's postulate are [3]:

(i) The yield surface and all subsequent loading surfaces must be convex.

(ii) The plastic strain increment vector must be normal to the loading surface at a regular point, and it must lie between adjancent normals to the loading surface at a corner of the surface.

The normality condition (ii) implies that at smooth points on the yield surface

$$\dot{\underset{\sim}{E}}_p = \dot{\lambda} \, \underset{\sim}{D}(\underset{\sim}{T},\varkappa) \tag{4.33}$$

where λ is a function of the deformation history and is such that (4.33) is homogeneous in time and

$$\underset{\sim}{D}(\underset{\sim}{T},\varkappa) = \partial_{\underset{\sim}{T}} A(\underset{\sim}{T},\varkappa). \tag{4.34}$$

We can also write the yield surface in the form

$$\hat{A}(p,\underset{\sim}{\mathcal{T}},\varkappa) = 0 \tag{4.35}$$

The normality condition then leads to

$$\dot{\underset{\sim}{e}}_p = \dot{\lambda}\, \hat{\underset{\sim}{D}}(p, \underset{\sim}{\mathcal{T}}, \kappa)$$
$$\dot{\varphi}_p = \dot{\lambda}\, \alpha(p, \underset{\sim}{\mathcal{T}}, \kappa) \tag{4.36}$$

where

$$\hat{\underset{\sim}{D}}(p, \underset{\sim}{\mathcal{T}}, \kappa) = \partial_{\underset{\sim}{\mathcal{T}}} \hat{A}(p, \underset{\sim}{\mathcal{T}}, \kappa)$$
$$\alpha(p, \underset{\sim}{\mathcal{T}}, \kappa) = \partial_p \hat{A}(p, \underset{\sim}{\mathcal{T}}, \kappa) \tag{4.37}$$

and $\underset{\sim}{e}_p$ is the plastic strain deviator, φ_p the plastic volumetric strain given by

$$\underset{\sim}{e}_p = \underset{\sim}{E}_p - \frac{1}{3}\varphi_p \underset{\sim}{1}$$
$$\varphi_p = \mathrm{tr}\, \underset{\sim}{E}_p, \quad \mathrm{tr}\, \underset{\sim}{e}_p = 0. \tag{4.38}$$

We note that the dependence of $\hat{A}$ on $\underset{\sim}{\mathcal{T}}$ must be such that

$$\mathrm{tr}\, \hat{\underset{\sim}{D}}(p, \underset{\sim}{\mathcal{T}}, \kappa) = 0. \tag{4.39}$$

We shall now indicate how the Mohr-Coulomb failure criteria may be used together with our concept of the yield surface. The Mohr-Coulomb failure criteria states that the magnitude of the shearing stress τ on any section through a mass of an isotropic cohesive soil must not be greater than an amount which depends linearly upon the normal stress σ acting on the section. This condition is expressed as

$$\tau \leq c + \sigma \tan \Phi \tag{4.40}$$

where c is the cohesive and Φ the angle of friction of the soil. Failure can occur when the equality sign in (4.40) holds for some section through the soil. Shield [4] has constructed a surface in principal stress space corresponding to

$$\tau = c + \sigma \tan \Phi \tag{4.41}$$

Letting $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses with $\sigma_1 \geq \sigma_2 \geq \sigma_3$, Shield [4] finds that the surface is a right hexagonal pyramid equally inclined to the $\sigma_1, \sigma_2, \sigma_3$ axes and with vertex at the point $\sigma_1 = \sigma_2 = \sigma_3 = -c \cot \Phi$ (Fig. 4). When the stress point of the soil is on this surface, the soil mass is said to be on a failure condition. The significant feature of the failure surface is that it is not a closed surface in stress space but rather an open surface. Consequently it divides stress space into two regions I and II. Stress points in region I are such that the inequality

$$< c + \sigma \tan \Phi \tag{4.42}$$

is violated and consequently such states of stress are not possible for the soil since by definition the soil has failed. On the other hand, points in the region II are such that the inequality (4.42) holds and consequently equilibrium of the soil mass is possible. However, since soils yield before they fail, all points in region II in the vicinity of the failure surface also lie in some yield surface. For a soil which has work-hardened isotropically two possibilities arise: (1) Ψ is tangent to the failure surface, or (2) Ψ traverses the failure surface. If the possibility (1) prevails the plastic increment vector at the tangent point will be normal to both surfaces. Considering that the use of normality of the plastic strain increment vector to the failure surface gives erroneous predictions of high rates of change of volume during shear distortion we disregard the possibility (1) above. Therefore the failure surface is traversed by the yield surface. This indicates that stress points can only lie on a portion of the yield surface, that portion which lies in region II. Consequently, the yield surface is not a closed surface.

We now assume the soil to be isotropic. We further assume $\underset{\sim}{E}_r$ to be linear in $\underset{\sim}{T}$. Therefore (4.13) has the representation

$$\underset{\sim}{E}_r = (a_1(\zeta_r)\operatorname{tr} \underset{\sim}{T})1 + a_2(\zeta_r)\underset{\sim}{T} \tag{4.43}$$

where a_1 and a_2 are functions of the change in recoverable porosity. Letting

$$J_2 = \operatorname{tr} \underset{\sim}{T}^2, \quad J_3 = \operatorname{tr} \underset{\sim}{T}^3, \tag{4.44}$$

(4.18) reduces to

$$\zeta_r = g^*(p,\ J_2,\ J_3) \tag{4.45}$$

Also we assume the soil to be isotropic work-hardening material and use the form (4.35) of the yield surface. Thus

$$\hat{A}(p, \underset{\sim}{T},\ \varkappa) = \hat{A}(p,\ J_2,\ J_3,\ \varkappa) \tag{4.46}$$

From this follows

$$\underset{\sim}{\hat{D}}(p, \underset{\sim}{T}, \varkappa) = 2\partial_{J_2}\hat{A}(p, J_2, J_3, \varkappa)\underset{\sim}{T} + 3\partial_{J_3}\hat{A}(p, J_2, J_3, \varkappa)\underset{\sim}{T}^2 \tag{4.47}$$

and consequently

$$\mathrm{tr}\ \underset{\sim}{\hat{D}}(p, \underset{\sim}{T}, \varkappa) = 3\partial_{J_3}\hat{A}(p, J_2, J_3, \varkappa)J_2.$$

In view of (4.39) and the fact that $J_2 \neq 0$, this last equation leads to

$$\partial_{J_3}\hat{A}(p, J_2, J_3, \varkappa) = 0 \tag{4.48}$$

or equivalently that $\hat{A}$ must be independent of J_3. Consequently (4.35) reduces to

$$\hat{A}(p,\ J_2,\ \varkappa) = 0. \tag{4.49}$$

Considering that

$$J_2 = I_2 - 3p^2\ ,\quad I_2 = \mathrm{tr}\ \underset{\sim}{T}^2 \tag{4.50}$$

we can write (4.49) in the form

$$\hat{A}(p, J_2, \varkappa) = A(p, I_2, \varkappa) = 0 \tag{4.51}$$

or using principal components of stress

$$A(p, I_2, \varkappa) = A^*(\sigma_1, \sigma_2, \sigma_3, \varkappa) = 0. \tag{4.52}$$

We indicated at the beginning of the last section that (3.1) follows under the assumption of incompressibility of the solid aggregate. A formula similar to (3.1) may also be derived by recognizing that fundamentally there are two mechanisms that contribute to the deformation of the soil: distortion of individual particles and relative motion between particles as the result of sliding or rolling. If we assume that plastic deformations are mainly due to the relative motion of the soil particles due to sliding and rolling then a reasonable assumption we may make is that during this motion the soil particles are essentially incompressible. Mathematically this assumption is stated as

$$\nu_p = 0 \tag{4.53}$$

where ν_p is the irrecoverable part of the solid aggregate dilatation given by (4.10). In view of the ass mption (4.53) we obtain from (4.10)

$$\varphi_p = - \frac{n_p}{1-n_o} \tag{4.54}$$

which is a formula similar to (3.1).

Now considering that

$$\dot{\hat{A}} = \alpha\dot{p} + \mathrm{tr}\, \hat{\underset{\sim}{D}}\, \dot{\underset{\sim}{\tau}} + \partial_\varkappa \hat{A}\ \partial_{n_p} \hat{\varkappa} \dot{n}_p = 0$$

where α and $\hat{\underset{\sim}{D}}$ are given by (4.37), we can write (4.54) in the form

$$\begin{aligned} \dot{\underset{\sim}{e}}_p &= \Lambda(\alpha\dot{p} + \mathrm{tr}\, \hat{\underset{\sim}{D}}\, \dot{\underset{\sim}{\tau}})\hat{\underset{\sim}{D}} \\ \dot{\varphi}_p &= \Lambda(\alpha\dot{p} + \mathrm{tr}\, \hat{\underset{\sim}{D}}\, \dot{\underset{\sim}{\tau}})\alpha \end{aligned} \tag{4.55}$$

where

$$\Lambda = \frac{1}{(1-n_o)\partial_\varkappa \hat{A}\ \partial_{n_p} \hat{\varkappa}\alpha}. \tag{4.56}$$

Thus under the assumption (4.53) the flow rule is given by (4.55).

5. STRESS-STRAIN THEORY FOR TRIAXIAL COMPRESSION CONDITION

In this section we shall construct from the theory presented in the previous section stress-strain relations for the independent interpretation of triaxial compression condition. With reference to Fig. 4 we assume, that in a triaxial compression condition, the non-vanishing physical components of the stress tensor to be

$$\sigma_1,\ \sigma_2,\ \sigma_3$$

We further assume a homogeneous stress field in which case it follows from equilibrium that

$$\sigma_1 = \sigma_2. \tag{5.1}$$

Under these assumptions the mean pressure and physical components of the deviatoric stress tensor q_i $(i = 1,2,3)$ are

$$\begin{aligned} p = \frac{1}{3}(\sigma_3 + 2\sigma_2)\ , \quad q_3 &= \frac{2}{3}(\sigma_3 - \sigma_1) \\ q_1 = q_2 &= -\tfrac{1}{2}q_3. \end{aligned} \tag{5.2}$$

If we assume an isotropic material then the principal directions of the stress tensor and strain tensor coincide. The non-vanishing physical components of the strain tensor are then given by

$$\varepsilon_1 = \varepsilon_2\ ,\quad \varepsilon_3$$

and the physical components of the strain deviator by

$$\begin{aligned} e_3 &= \varepsilon_3 - \frac{1}{3}\varphi \\ e_1 = e_2 &= -\tfrac{1}{2}e_3 \end{aligned} \tag{5.3}$$

where φ is the volumetric strain.

We can write the stress power of the soil mass in the form,

$$D = \mathrm{tr}\ \underset{\sim}{T}\ \dot{\underset{\sim}{E}} = \mathrm{tr}\ \underset{\sim}{\mathcal{T}}\ \dot{\underset{\sim}{e}} + p\dot{\varphi}. \tag{5.4}$$

Using the principal components of $\underset{\sim}{\mathcal{T}}$ and e we bring (5.4) into the form

$$D = q\dot{e} + p\dot{\varphi} \tag{5.5}$$

where

$$q = \frac{3}{2}q_3 = \sigma_3 - \sigma_1 \ , \quad e = e_3. \tag{5.6}$$

The form (5.5) suggests that in a triaxial compression condition the generalized stress and strain parameters be (q,p), (e,φ). Accordingly we decompose (e,φ) into the sum of recoverable and irrecoverable parts,

$$\begin{aligned} e &= e_r + e_p \\ \varphi &= \varphi_r + \varphi_p \end{aligned} \tag{5.7}$$

and construct a yield function in q-p space.

In two dimensional stress space the yield surface reduces to a curve and the Mohr-Coulomb failure surface to two straight lines meeting at a point on the p-axis. In Fig. 5 we show these two lines for a cohesionless soil $(c = 0)$. The angles Φ_1, Φ_2 are defined by

$$\tan \Phi_1 = \frac{6 \sin \Phi}{3 - \sin \Phi} \ , \quad \tan \Phi_2 = \frac{6 \sin \Phi}{3 + \sin \Phi} \tag{5.8}$$

where Φ is the angle of internal friction of the soil. We note that the two Coulomb lines OC_1, OC_2 divide q-p space into two regions I and II. In Fig. 5 the current yield locus Ψ is the curve F_1F_2 which traverses the p-axis at p_o. Recall that along the yield curve the value of the change in porosity is constant. This value of the change in porosity can be obtained from the soil virgin consolidation line which is obtained through another experiment.

Without loss of generality we take $\Phi_1 = \Phi_2$ and introduce the dimensionless quantities

$$M = \tan \Phi_1$$

$$N = \frac{p}{p_o} \tag{5.9}$$

$$\eta = \frac{q}{p} .$$

The value of p,q at the point F_1 is then given by

$$\bar{q} = M\bar{p} \tag{5.10}$$

where a bar over a quantity indicates that these are the values of q,p when the soil is in a failure condition. We also introduce the dimensionless quantity N_f defined by

$$N_f = \frac{\bar{p}}{p_o} \tag{5.11}$$

We note that whereas M is a constant for the soil, N_f is constant only for a particular yield. However for lack of experimental evidence to use as a guide we shall consider N_f to be a constant for the soil. This assumption in itself suggests experimentation.

We shall assume the current and subsequent yield locii to be symmetrical about the p-axis and to be segments of ellipses which pass through the origin. It can then be shown that the equation of the ellipse is

$$N = \frac{M^2}{M^2 + K\eta^2} \tag{5.12}$$

where

$$K = \frac{1 - N_f}{N_f}. \tag{5.13}$$

Thus the current yield locus is given by (5.12). In (5.12), p_o appears as a parameter that we may use as a strain-hardening parameter. This being the case the yield locus will be a volumetric yield locus. The dependence of p_o on n_p has to be determined in order to obtain a flow rule associated with (5.12). To this purpose we assume that the virgin consolidation line of the soil when plotted on ζ- ln p space is given by Terzaghi's well known equation

$$\zeta = -\lambda_o \ln(\frac{p_o}{p_i}) \tag{5.14}$$

where λ_o is a soil constant and p_i is the consolidation pressure of the soil in its reference state. We also assume the elastic swelling curves to be straight and parallel lines of slope κ when plotted on ζln p space. We can then show that the construction depicted by (4.22) leads to the following relation

$$p_o = Y \exp(\frac{-n_p}{\gamma}) \tag{5.15}$$

where Y is the initial yield pressure under confined consolidation and

$$\gamma = \lambda_o - \kappa . \tag{5.16}$$

In order to construct a flow rule associated with (5.12) we assume that (4.54) holds. We can then show that the coefficient of proportionality $\dot{\lambda}$ in (4.36) is given by

$$\dot{\lambda} = \frac{\gamma p_o}{(1-n_o)} (\frac{\dot{p}}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2})(\frac{M^2 + K\eta^2}{M^2 - K\eta^2}) . \tag{5.17}$$

The plastic strain rates can then be shown to be given by

$$\begin{aligned} \dot{e}_p &= \frac{\gamma}{(1-n_o)} (\frac{2K\eta}{M^2 - K\eta^2})(\frac{\dot{p}}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2}) \\ \dot{\varphi}_p &= \frac{\gamma}{(1-n_o)} (\frac{\dot{p}}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2}) . \end{aligned} \tag{5.18}$$

Having obtained expressions for the plastic strain rates the next order of business is to obtain expressions for the recoverable strain rates. It can easily be shown that (4.43) and (4.45) reduce to

$$e_r = a_3(\zeta_r)q$$

$$\varphi_r = 3a_4(\zeta_r)p \qquad (5.19)$$

$$\zeta_r = g^*(p,q)$$

where

$$a_3(\zeta_r) = \frac{2}{3}a_2(\zeta_r)\ , \quad a_4(\zeta_r) = 3a_1(\zeta_r) + a_2(\zeta_r) \qquad (5.20)$$

For lack of experimental information regarding the dependence of ζ_r on q we assume

$$\partial_q g^*(p,q) = 0\ , \qquad (5.21)$$

i.e., g^* is independent of q. It follows that

$$\zeta_r = \hat{g}(p). \qquad (5.22)$$

This being the case, we have already assumed the form of (5.22). Equation (5.27) describes the elastic swelling curve. Consequently

$$\dot{\zeta}_r = -\kappa\frac{\dot{p}}{p}\ . \qquad (5.23)$$

Using $(5.19)_2$, (5.22), and (5.23) we find

$$\dot{\varphi}_r = \kappa\psi(\zeta_r,p)\frac{\dot{p}}{p} \qquad (5.24)$$

where

$$\psi(\zeta_r,p) = 3p(a_4(\zeta_r) - \partial_{\zeta_r}a_4(\zeta_r)). \qquad (5.25)$$

If we assume that there is no recoverable energy during shear distortion then

$$\dot{e}_r = 0. \tag{5.26}$$

The final stress-strain relations for triaxial compression condition are obtained by combining (5.18), (5.24), (5.26), and (5.7),

$$\begin{aligned} \dot{e} = \dot{e}_r &= \frac{\gamma}{(1-n_o)}\left(\frac{2K\eta}{M^2 - K\eta^2}\right)\left(\frac{\dot{p}}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2}\right) \\ \dot{\varphi} = \dot{\varphi}_r + \dot{\varphi}_p &= \frac{1}{(1-n_o)}\left(\frac{2\gamma K\eta\dot{\eta}}{M^2 + K\eta^2} + \lambda^*(\zeta_r, p)\frac{\dot{p}}{p}\right) \end{aligned} \tag{5.27}$$

where

$$\lambda^*(\zeta_r, p) = \gamma + \kappa(1-n_o)\psi(\zeta_r, p). \tag{5.28}$$

Let us now consider the following form of the material function $a_4(\zeta_r)$ appearing in $(5.19)_2$,

$$a_4(\zeta_r) = \frac{\kappa}{3R(1-n_o)(1-\kappa)}\left(\exp\zeta_r - \exp\left(\frac{\zeta_r}{\kappa}\right)\right) \tag{5.29}$$

where R is the residual pressure, i.e., the pressure experience by the soil when it has been held at rest in its reference configuration at all times. The assumed equation of the elastic swelling curve is

$$p = R\exp\left(\frac{-\zeta_r}{\kappa}\right). \tag{5.30}$$

Substitution of (5.29) and (5.30) into (5.19) yields

$$\varphi_r = \frac{\kappa}{(1-n_o)(1-\kappa)}\left(\exp\left(\frac{-\zeta_r(1-\kappa)}{\kappa}\right) - 1\right). \tag{5.31}$$

If we expand the exponential function into a power series we can write (5.31) in the form

$$\varphi_r = \frac{\zeta_r}{(1-n_o)} + O(\zeta_r^{\,2}).$$

But since we have assumed φ_r and ζ_r to be of the same order of magnitude, it follows that

$$\varphi_r = - \frac{\zeta_r}{(1-n_o)} \tag{5.32}$$

which is an equation of the form (3.1). Therefore if we assume (5.29) we may replace $(5.19)_2$ by (5.30) which may be written in terms of φ_r as

$$\varphi_r = \frac{\kappa}{(1-n_o)} \ln \left(\frac{p}{R}\right). \tag{5.33}$$

Under the assumed form (5.29) for $a_4(\zeta_r)$, the function $\lambda^*(\zeta_r,p)$ given by (5.28) reduces to

$$\lambda^*(\zeta_r,p) = \lambda_o. \tag{5.34}$$

In this case equation $(5.27)_2$ reduces to

$$\dot{\varphi} = \frac{1}{(1-n_o)} \left(\frac{2\gamma K\eta\dot{\eta}}{M^2 + K\eta^2} + \lambda_o \frac{\dot{p}}{p}\right) \tag{5.35}$$

while $(5.27)_1$ remains the same.

Let us assume the particular value of ½ for the coefficient N_f given by (5.11). In this case K = 1 and the yield locus and stress-strain relations $(5.27)_1$ and (5.35) reduce to

$$N = \frac{M^2}{M^2 + \eta^2}$$

$$\dot{e} = \frac{\gamma}{(1-n_o)} \left(\frac{2\eta}{M^2 - \eta^2}\right)\left(\frac{\dot{p}}{p} + \frac{2\eta\dot{\eta}}{M^2 + \eta^2}\right) \tag{5.36}$$

$$\dot{\varphi} = \frac{1}{(1-n_o)} \left(\frac{2\gamma\eta\dot{\eta}}{M^2 + \eta^2} + \lambda_o \frac{\dot{p}}{p}\right) .$$

If we assume the line OC_1 in Fig. 5 to be critical state line+ instead of a

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+See Schoffield and Wroth [6] for a thorough treatment of the critical state concept in soil mechanics.

Coulomb line then (5.36) is identical to the Cambridge triaxial compression theory presented by Roscoe and Burland [5] for "wet" clay. We must point out that Roscoe and co-workers at Cambridge have, for the past decade, concentrated considerable effort to arrive at self-consistent constitutive relations for soils. The reader is referred to the recent article by Roscoe and Burland [5] for an account of the work at Cambridge.

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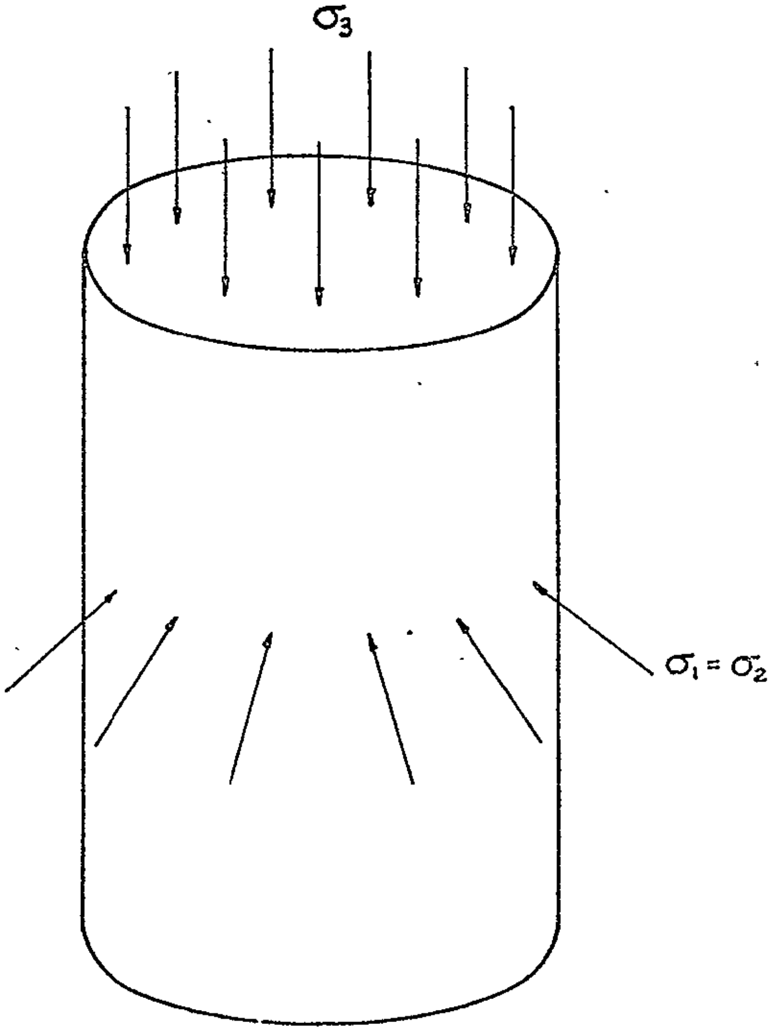

Fig. 1 Triaxial Test $\sigma_1 = \sigma_2$

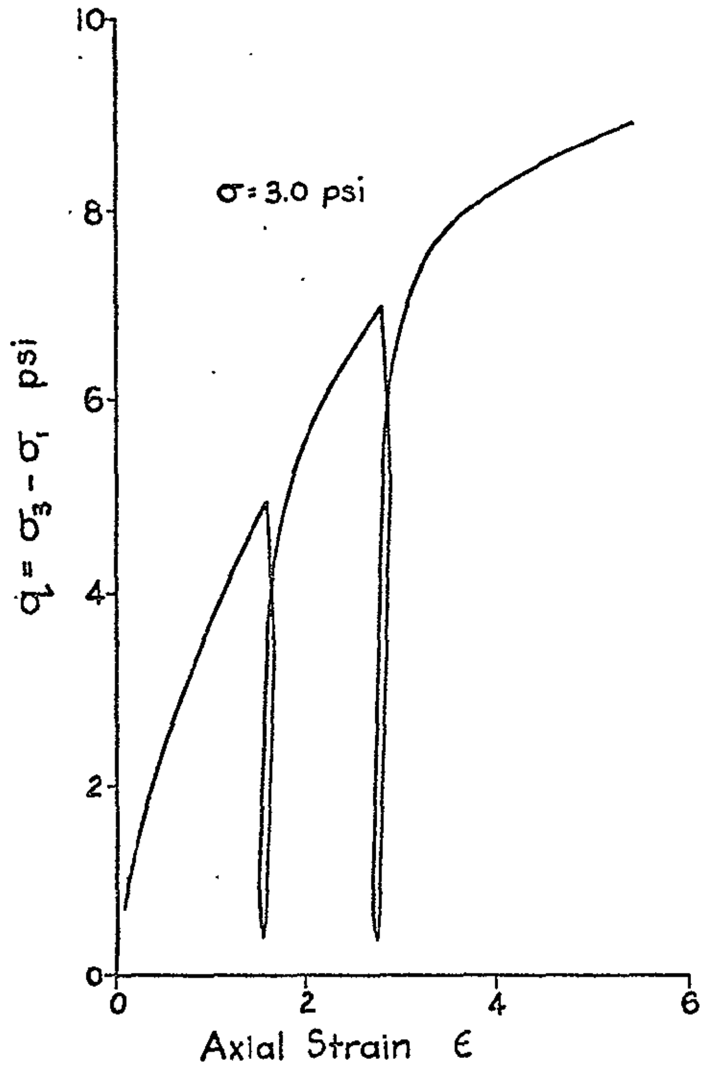

Fig. 2 Triaxial Stress - Strain Curve

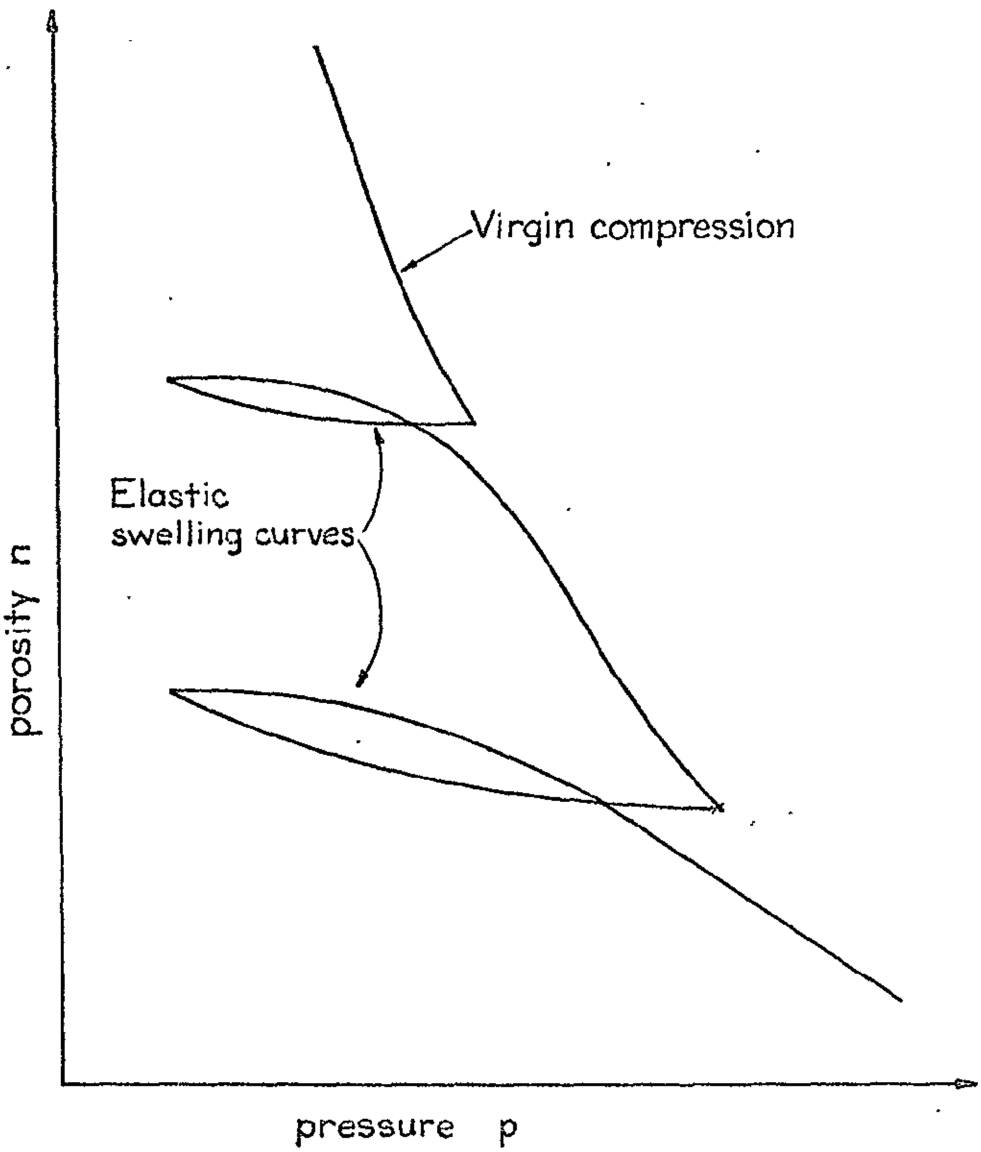

Fig. 3 Hydrostatic Pressure - Porosity

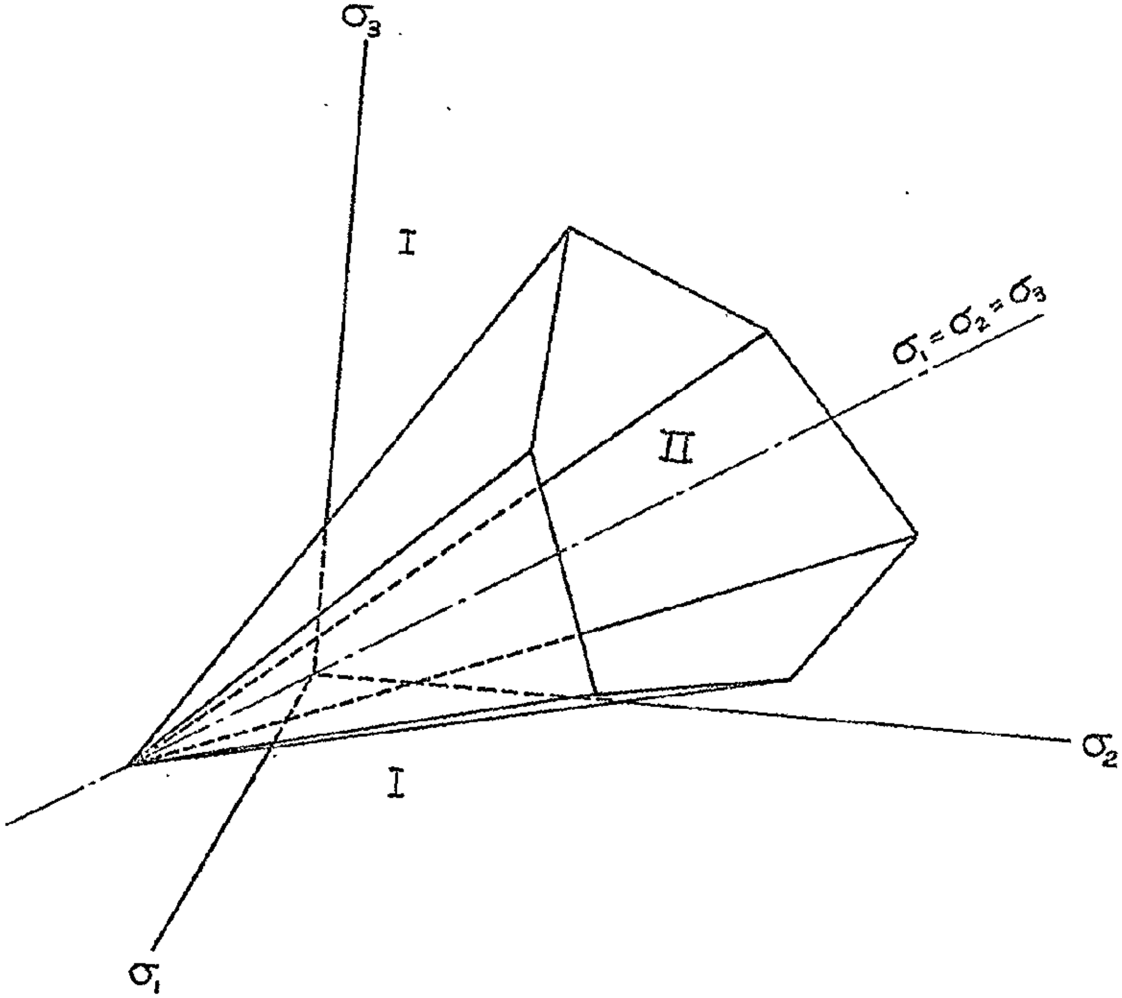

Fig. 4 Mohr - Coulomb Failure Surface In Principal Stress Space

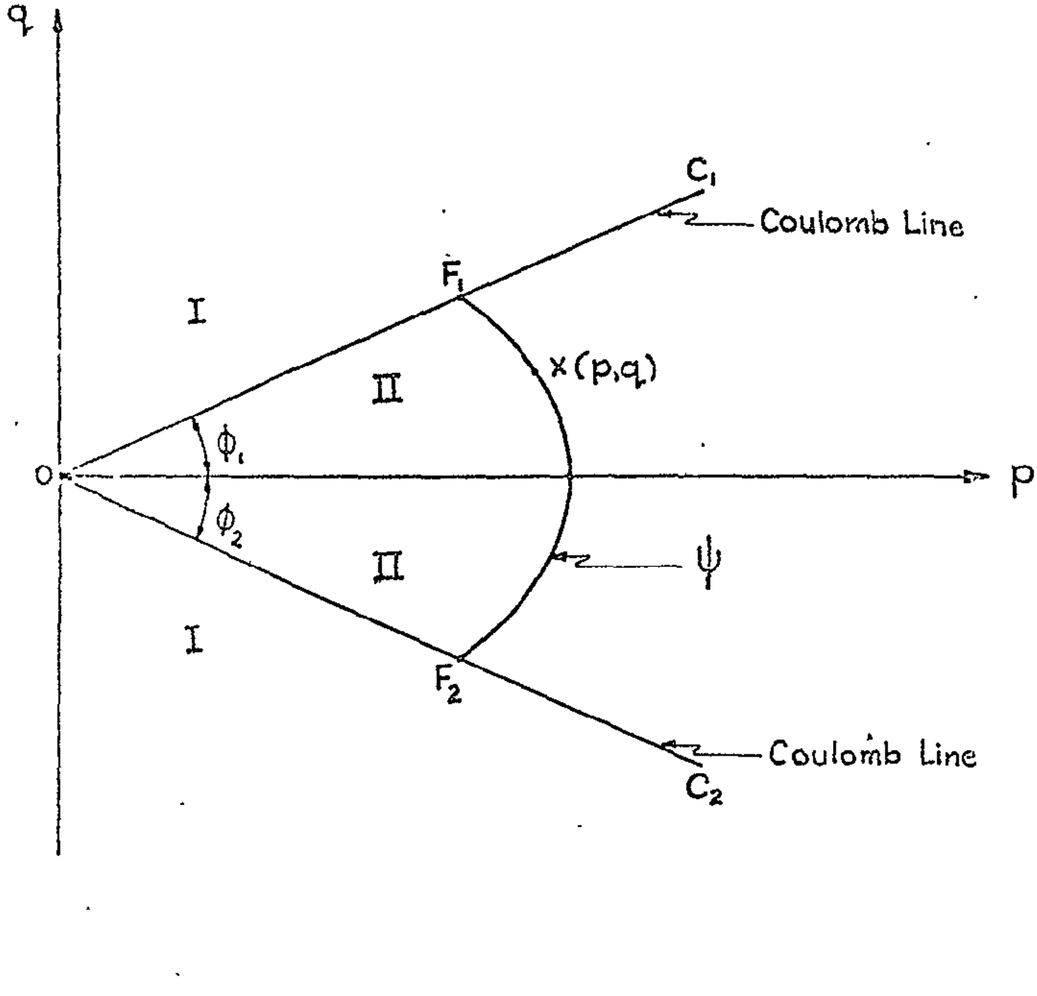

Fig 5 Triaxial Compression Yield Curve

SECTION III

ON A YIELD SURFACE FOR SOILS

1. INTRODUCTION

In 1952, Drucker and Präger [3] introduced an idealization for the phenomenological behavior of soils under load. In this idealization the soil mass is treated as a perfectly plastic material with the Mohr-Coulomb failure criteria for soils as a yield function with which a flow rule can be associated. Volume changes predicted under this idealization, however, were higher than those found by experiments. Based on observations made on pressure-volume change curves Drucker, Gibson, and Henkel [4] explained how soils could be treated as a work-hardening material. Roscoe and co-workers at Cambridge have indicated that certain soils can be described remarkably well by a simple isotropic work-hardening idealization. The work at Cambridge is discussed in considerable detail by Roscoe and Burland [2].

Aguirre-Ramirez and Costes [1] presented a self-consistent phenomenological theory for the mechanical behavior of granular media which shows stress-strain behavior of the type discussed by Drucker, Gibson, and Henkel [4]. As an example a yield surface and associated flow rule, for triaxial compression conditions, was constructed in [1]. It was also shown in [1] that the Cambridge triaxial compression theory developed by Roscoe and co-workers [2] comes out as a special case of this theory.

In this paper we generalize, to complex stress fields, the special triaxial compression theory presented in [1]. We have been encouraged to do so in view of the fact that the Cambridge triaxial compression theory of Roscoe and co-workers has been found to give reasonable agreement with experimental results. Accordingly in Section 2 we briefly review the theory of soil plasticity presented in [1]. In Section 3 the special triaxial compression theory of [1] is also reviewed. This theory is generalized to three-dimensional complex stress field in Section 4. Recognizing that many soil mechanic problems can be idealized to plane-strain situations we present in Section 5 a theory for plane-strain. In Section 6 we compare both the three-dimensional and plane-strain theories to those of Roscoe and Burland [2] and find perfect agreement.

2. PRELIMINARIES

We consider the particles of a dry soil mass continuum to be referred to a fixed rectangular Cartesian system x_i $(i=1,2,3)$ and let $\underset{\sim}{u}$ denote the displacement field. The linear strain tensor γ_{ij} is then given by

$$\gamma_{ij} = 1/2(u_{i,j} + u_{j,i}) \tag{2.1}$$

where we have denoted partial differentiation with respect to x_i by $(\ )_{,i}$. We also let n, n_0 denote the current and initial porosity of the soil mass continuum respectively and define the change in porosity ζ by

$$\zeta = n - n_0. \tag{2.2}$$

We further assume ζ to be of the same order of magnitude as the linear strain tensor γ_{ij}.

Earlier we have defined yield in a soil to be a permanent irrecoverable deformation [1]. The strain may then be written as the sum of an elastic or recoverable part γ'_{ij} and a plastic or irrecoverable part γ''_{ij},

$$\gamma_{ij} = \gamma'_{ij} + \gamma''_{ij}. \tag{2.3}$$

In addition we shall assume porosity to be given as the sum of an elastic or recoverable part n' and a plastic or irrecoverable part n'',

$$n = n' + n'' \tag{2.4}$$

In view of this assumed resolution we shall have for the change in porosity

$$\zeta = \zeta' + n'' \tag{2.5}$$

where $\zeta' = n' - n_0$.

According to the ideas set forth in [1] the irrecoverable deformation of the soil is described by the pair (γ_{ij}'', n''). The theory of soil plasticity presented in [1] is wholly dependent on the hypothesis that there exists a unique function F of plastic porosity n" and Terzaghi's effective stress σ_{ij} such that

$$F(\sigma_{ij}, n'') = 0. \tag{2.6}$$

In plastic porosity-stress space (2.6) defines a six-dimensional hypersurface Σ which was called in [1], the state boundary surface. A soil particle was then said to be in a plastic state if the value of the stress and plastic porosity are such that (2.6) is satisfied.

A yield surface Ω for the soil was defined in [1] to be the projection on stress space of curves on Σ along which the plastic porosity has the constant value $\bar{n}''$. This is given by

$$A(\sigma_{ij}, k) = 0 \tag{2.7}$$

where

$$k = k(\bar{n}'') \tag{2.8}$$

is the strain-hardening parameter. The flow rule associated with (2.7) is given by

$$\dot{\gamma}''_{mr} = \dot{\lambda}\, D_{mr}(\sigma_{ij}, k) \tag{2.9}$$

where $\dot{\lambda}$ is a function of the deformation history and is such that (2.9) is homogeneous in time. Also in (2.9)

$$D_{mr}(\sigma_{ij}, k) = \frac{\partial A}{\partial \sigma_{mr}}(\sigma_{ij}, k). \tag{2.10}$$

We often find it convenient to decompose σ_{ij} into the sum of its deviatoric part τ_{ij} and mean pressure p,

$$\sigma_{ij} = p\delta_{ij} + \tau_{ij} \tag{2.11}$$

where δ_{ij} is the Kronecker's delta and

$$p = 1/3\ \sigma_{mm}, \qquad \tau_{mm} = 0. \tag{2.12}$$

Under the resolution (2.11) of the stress we may write (2.7) in the form

$$\hat{A}(p, \tau_{ij}, k) = 0. \tag{2.13}$$

The flow rule (2.9) then reads

$$\begin{aligned} \dot{\varepsilon}''_{mr} &= \dot{\lambda}\, \hat{D}_{mr}(p, \tau_{ij}, k) \\ \dot{\gamma}''_{mr} &= \dot{\lambda}\, \alpha(p, \ {}_{ij}, k) \end{aligned} \tag{2.14}$$

where ε''_{mr} is the plastic strain deviator and

$$\begin{aligned} \hat{D}_{mr}(p, \tau_{ij}, k) &= \frac{\partial \hat{A}}{\partial \tau_{mr}}(p, \tau_{ij}, k) \\ \alpha(p, \tau_{ij}, k) &= \frac{\partial A}{\partial p}(p, \tau_{ij}, k) \end{aligned} \tag{2.15}$$

Fundamentally there are two mechanisms that contribute to the deformation of the soil: distortion of individual particles and relative motion between particles as the result of sliding or rolling. We assume that soil irrecoverable deformations are mainly due to the relative motion of the soil particles due to sliding and rolling and that during this motion the soil particles are essentially incompressible. It can then be shown that under this assumption [1]

$$\gamma''_{mm} = - \frac{n''}{(1-n_0)} . \tag{2.16}$$

The flow rule (2.14) may now be written as

$$\begin{aligned} \dot{\varepsilon}''_{mr} &= \Lambda(\hat{D}_{ij} \ \dot{\tau}_{ji} + a\dot{p})\hat{D}_{mr} \\ \gamma''_{mm} &= \Lambda(\hat{D}_{ij} \ \dot{\tau}_{ji} + a\mathring{p})a \end{aligned} \tag{2.17}$$

where

$$\Lambda = [(1-n_0)a \frac{\partial A}{\partial k} \frac{\partial k}{\partial n''}]^{-1} . \tag{2.18}$$

The recoverable strains γ'_{mr} and porosity n' may be computed through the soil elastic constitutive equations [1]

$$\begin{aligned} \gamma'_{mr} &= \gamma'_{mr}(\sigma_{ij}, \zeta') \\ \zeta' &= \zeta'(\sigma_{ij}) . \end{aligned} \tag{2.19}$$

For an isotorpic soil for which the strains are linear in σ_{ij}, $(2.19)_1$ has the representation

$$\gamma'_{mr} = a_1(\zeta')\sigma_{ii} \ \zeta_{mr} + a_2(\zeta')\sigma_{mr} \tag{2.20}$$

where a_1, a_2 are material functions of recoverable porosity ζ'. We may also write (2.20) in the form

$$\varepsilon'_{mr} = \alpha^2(\zeta')\tau_{mr} \tag{2.21}$$

$$\gamma'_{mm} = 3\alpha_3(\zeta')p$$

where

$$\alpha_3(\zeta') = 3\alpha_1(\zeta') + \alpha_2(\zeta'). \tag{2.22}$$

The following form of (2.19) was assumed in [1]

$$\zeta' = -\kappa \ln(\frac{P}{\kappa}) \tag{2.23}$$

where κ is a soil constant and R is the residual pressure, i.e., the pressure experienced by the soil when it has been held at rest in its reference configuration at all times. We note that in view of (2.23) we may write $(2.21)_2$ in the form

$$\gamma'_{mm} = 3R\alpha_3(\zeta') \exp(-\frac{\zeta'}{\kappa}) \tag{2.24}$$

which relates volumetric strains to recoverable change in porosity. In [1] it was shown that under a suitable chosen function α_3 the relation (2.24) can be reduced to

$$\gamma'_{mm} = -\frac{\zeta'}{(1-n_0)}. \tag{2.25}$$

3. TRIAXIAL THEORY

In this section and for the duration of the paper we shall consider the soil mass to be isotropic. We shall also use the convention that compressive stresses will be taken as positive.

We consider the family of deformations γ_{mr} such that when referred to a suitable set of orthogonal axes

$$\gamma_{mr} = 0, \quad m \neq r$$
$$\gamma_{11} = \gamma_{22}. \tag{3.1}$$

Such deformations are called triaxial deformations. Since we have assumed the soil to be isotropic it follows that the principal directions of the strain and stress tensor coincide. This being the case then it follows that under triaxial deformations the components of the stress tensor take the particular form

$$\sigma_{mr} = 0, \quad m \neq r$$
$$\sigma_{11} = \sigma_{22}. \tag{3.2}$$

It can be shown [1] that under triaxial deformations a suitable set of generalized stress and strain parameters are (q,p), (ε,θ) where

$$q = \sigma_{33} - \sigma_{11}, \quad p = 1/3(\sigma_{33} + 2\sigma_{11})$$
$$\varepsilon = 2/3(\gamma_{33} - \gamma_{11}), \quad \theta = \gamma_{mm} = \gamma_{33} + 2\gamma_{11}. \tag{3.3}$$

In terms of these generalized parameters, the stress power is given by

$$D = q \dot{\varepsilon} + p\dot{\theta}. \tag{3.4}$$

We consider the following invariant of the deviatoric stress tensor

$$J = \tau_{mr} \tau_{rm} \tag{3.5}$$

It can be shown that in a triaxial stress field J is given by

$$J = 2/3(\sigma_{33} - \sigma_{11})^2 = 2/3\ q^2. \tag{3.6}$$

The octahedral shear stress τ_o is related to J through

$$\tau_0 = \frac{\sqrt{J}}{3}. \tag{3.7}$$

In a triaxial stress field (3.7) reduces to

$$\tau_0 = \sqrt{2/9}\ q. \tag{3.8}$$

Equations (3.6), (3.8) give meaning to the stress parameter q in terms of invariants of the stress tensor.

The strain measures ε, θ may be decomposed into the sum of recoverable and irrecoverable parts

$$\varepsilon = \varepsilon' + \varepsilon'' \quad , \quad \theta = \theta' + \theta''. \tag{3.9}$$

In [1] a one-parameter family of yield curves was constructed in q-p space for a cohesionless soil with an angle of internal friction ϕ at failure. A member of this family is shown in Fig. 1 and is given by

$$\frac{p}{p_0} = \frac{M}{M^2 + K\eta^2} \tag{3.10}$$

where $\eta = q/p$ and

$$M = \frac{6 \sin\phi}{3 - \sin\phi} \quad , \quad K = \frac{p_0 - p_u}{p_u} \tag{3.11}$$

Here p_u is the value of the mean pressure at failure at a change in porosity ζ and p_0 represents the pressure corresponding to ζ on the virgin isotropic compression curve of the soil. The parameter of the family was taken in [1] to be p_0 and is given by

$$p_0 = Y \exp\left(\frac{-n''}{\beta}\right) \tag{3.12}$$

where Y is the initial yield pressure under confined consolidation and

$$\beta = \lambda_0 - \kappa \tag{3.13}$$

λ_0 being a soil constant.

Under an assumed constant value of K the following flow rule was established in [1],

$$\dot{\varepsilon} = \frac{\beta}{(1-n_0)} \left(\frac{2K\eta}{M^2 - K\eta^2}\right) \left(\frac{\dot{p}}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2}\right) \tag{3.14}$$

$$\theta = \frac{\beta}{(1-n_0)} \left(\frac{p}{p} + \frac{2K\eta\dot{\eta}}{M^2 + K\eta^2}\right) \; .$$

In [1] we indicated that the yield curve given by (3.10) and associated flow rule (3.14) reduce to the "Cambridge" triaxial compression theory presented by Roscoe and Burland [2] if we interpret the line OC in Fig. 1 as a critical state line instead of a Coulomb line and if we take K = 1.

4. THREE DIMENSIONAL THEORY

The yield curve given by (3.10) and associated flow rule (3.14) is in terms of triaxial compression stress parameters q-p and as it stands it is only good for analysis of triaxial deformations under a triaxial stress field. In general, however, it is desirable to obtain a yield surface and associated flow rule to analyze deformations under complex stress fields. Now for an isotropic work hardening material (2.13) reduces to [1]

$$\hat{A}(p, J, k) = 0. \tag{4.1}$$

To obtain the yield curve (3.10) in terms of stress invariants we use (3.6) to arrive at

$$\frac{p}{p_0} = \frac{\Phi^2}{\Phi^2 + K\xi^2} \tag{4.2}$$

where

$$\Phi = \sqrt{2/3}\ M \ , \quad \xi^2 = \frac{J}{p^2} . \tag{4.3}$$

We see that (4.2) is of the form (4.1) with the strain hardening parameter k identified with p_0. We can also write (4.2) in the form

$$\tau_0 = f(p;\ p_0) \tag{4.4}$$

where τ_0 is the octahedral shear stress and

$$f(p;\ p_0) = \frac{\Phi}{\sqrt{3K}}\ p\left(\frac{p_0}{p} - 1\right)^{1/2} \tag{4.5}$$

We also note that by using (2.16) and (3.12) we may write (4.2) in the form

$$\theta'' = \frac{\beta}{(1-n_0)} \ln \left[\frac{p(\Phi^2+K\xi^2)}{Y\Phi^2}\right] \tag{4.6}$$

It can easily be shown that the flow vector associated with the yield surface (4.2) is given by

$$\dot{\gamma}''_{mr} = \frac{\beta}{3(1-n_0)p(\Phi^2-K\xi^2)} (pg(\xi)\delta_{mr} + 6K\sigma_{mr})\left(\frac{\dot{p}}{p} + \frac{2K\xi\dot{\xi}}{\Phi^2+K\xi^2}\right) \tag{4.7}$$

where

$$g(\xi) = \Phi^2 - K(\xi^2 + 6). \tag{4.8}$$

Let us assume that there is no recoverable energy during shear distortion, then

$$\gamma'_{mr} = 1/2\ \theta'\ \delta_{mr}. \tag{4.9}$$

Also combining (2.24) and (2.26) we obtain

$$\theta' = \frac{\kappa}{(1-n_0)} \ln \left(\frac{p}{R}\right). \tag{4.10}$$

Therefore

$$\theta = \frac{1}{(1-n_0)} \left[\ln\left(\frac{p(\Phi^2+K\xi^2)}{Y\Phi^2}\right) + K\ln\left(\frac{p}{R}\right)\right] \tag{4.11}$$

and

$$\dot{\gamma}_{mr} = \frac{1}{3p(\Phi^2-K\xi^2)}(pg(\xi)\delta_{mr} + 6K\sigma_{mr})\dot{\theta}'' + \frac{\kappa}{3(1-n_0)} \frac{\dot{p}}{p} \delta_{mr} \tag{4.12}$$

Here

$$\dot{\theta}'' = \frac{\beta}{(1-n_0)} \left(\frac{\dot{p}}{p} + \frac{2K\xi\dot{\xi}}{\Phi^2+K\xi^2}\right). \tag{4.13}$$

To the above equations we add the equilibrium equations

$$\sigma_{ij,j} + X_i = 0 \tag{4.14}$$

where X_i is the body force, and the strain rate-velocity relations

$$\dot{\gamma}_{mr} = 1/2\,(v_{m,r} + v_{r,m}) \tag{4.15}$$

where $v_r = \dot{u}_r$ are the components of the velocity vector. Equations (4.6), (4.12), (4.14) and (4.15) form a system of sixteen equations for the sixteen unknowns $\dot{\gamma}_{mr}$, θ'', σ_{mr} and v_r.

5. PLANE STRAIN

In this section and for the duration of the paper Greek indices will have the range from 1 to 2. We further consider the case in which the soil constant κ is negligible in comparison with λ_0. Under this assumption the elastic response is negligible and we may set

$$\gamma'_{mr} = 0. \tag{5.1}$$

Consequently

$$\gamma_{mr} = \gamma''_{mr} \tag{5.2}$$

and we may drop the double primes to identify the irreversible strains.

A state of plane strain is characterized by the assumption

$$\begin{aligned} u_\alpha &= u_\alpha(x_1, x_2) \\ u_3 &= 0 \end{aligned} \tag{5.3}$$

It follows that for plane strain

$$\gamma_{m3} = 0 \quad , \quad \dot{\gamma}_{m3} = 0 \tag{5.4}$$

and therefore

$$\theta = \gamma_{\alpha\alpha} \quad , \quad \dot{\theta} = \dot{\gamma}_{\alpha\alpha}. \tag{5.5}$$

Now in view of (5.1) the constitutive relation (4.12) reduces to

$$\dot{\gamma}_{mr} = \frac{1}{3p(\Phi^2-K\xi^2)} (pg(\xi)\delta_{mr} + 6K\sigma_{mr})\dot{\theta}''. \tag{5.6}$$

Also considering (5.4) we obtain from (5.6) the pair of equations

$$\sigma_{13} = \sigma_{23} = 0 \tag{5.7}$$

$$pg(\xi) + 6K\sigma_{33} = 0. \tag{5.8}$$

Equation (5.8) is a quadratic equation in σ_{33} which may be solved to obtain σ_{33} as a function of $\sigma_{\alpha\beta}$. However, since $\sigma_{\alpha\beta}$, σ_{33} must also satisfy (4.2) we may combine (4.2) and (5.8) so as to obtain

$$\sigma_{33} = \frac{3p\phi^2 - 2(\phi^2-3K)\sigma}{2(\Phi^2+6K)} \tag{5.9}$$

where

$$\sigma = \sigma_{\alpha\alpha}. \tag{5.10}$$

With σ_{33} given by (5.9) the mean pressure is given by

$$p = \frac{6K\sigma+p_0\,\phi^2}{2(\Phi^2+6K)}. \tag{5.11}$$

Using (5.9) and (5.11) we reduce (4.2) to

$$C_1 I_2 + C_2\sigma^2 + C_3 p_0\sigma = 1/4\, p_0{}^2\Phi^4 \tag{5.12}$$

where

$$\begin{aligned} C_1 &= K(\Phi^2 + 6K) \\ C_2 &= \frac{(\phi^2-3K)(\phi^2-3K+9K^2)}{(\Phi^2+6K)} \\ C_3 &= -3\Phi^2\left(1 + \frac{(1-K)(\phi^2-3K)}{\Phi^2 + 6K}\right) \end{aligned} \tag{5.13}$$

and

$$I_2 = \sigma_{\alpha\beta}\,\sigma_{\beta\alpha}. \tag{5.14}$$

Equation (5.12) describes the yield surface in plane-strain. Now from (5.12) we find

$$\overset{\circ}{\theta} = \frac{2\lambda_0}{(1-n_0)(p_0{}^2\phi^4-2C_3\sigma p_0)}\,(2C_1\sigma_{\alpha\beta}+(2C_2\sigma+C_3p_0)\delta_{\alpha\beta})\,\dot{\sigma}_{\alpha\beta} \tag{5.15}$$

where we have used (2.16), (3.12), and (5.1). We can then show that with the yield locus (5.14), the associated flow rule is

$$\overset{\circ}{\gamma}_{\alpha\beta} = \frac{1}{(C_4\sigma+2C_3p_0)}\,(2C_1\sigma_{\alpha\beta}+(2C_2\sigma+C_3p_0)\delta_{\alpha\beta})\,\dot{\theta} \tag{5.16}$$

where

$$C_4 = 2C_1 + 4C_2. \tag{5.17}$$

To the above equations we add the strain rate-velocity and equilibrium relations

$$\dot{\gamma}_{\alpha\beta} = 1/2\,(v_{\alpha,\beta} + v_{\beta,\alpha}) \tag{5.18}$$

$$\sigma_{\alpha\beta,\beta} + X_2 = 0.$$

Equations (5.12), (5.16), and (5.18) form a system of nine equations for the nine unknowns p_0, $\sigma_{\alpha\beta}$, $\dot{\gamma}_{\alpha\beta}$, and v_α.

6. COMPARISON WITH THE "CAMBRIDGE THEORY"

In this section we shall compare the three-dimensional theory of Section 4 and the plane strain theory of the previous section with the corresponding theories presented by Roscoe and Burland [2].

In order to make the comparison we must point out that Roscoe and Burland [2] use only principal components of the stress and strain tensor. Using principal components of stress the invariant J is given by

$$J = 1/3[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\alpha_1)^2] \tag{6.1}$$

and is related to the stress parameter r used in [2] by

$$r = \sqrt{J} \; . \tag{6.2}$$

Therefore from $(4.3)_2$, $\xi = r/p$ and for K = 1, (4.2) reduces to

$$\frac{p}{p_0} = \frac{\Phi^2}{\Phi^2 + \xi^2} \tag{6.3}$$

which is the equation for the yield surface of the three-dimensional theory given in [2].

Now combining (4.10) and 4.13) and setting K = 1 we obtain

$$\dot{\theta} = \frac{1}{(1-n_0)}\left[\frac{2\beta\xi\dot{\xi}}{\Phi^2 + \xi^2} + \kappa\frac{\dot{p}}{p}\right] \tag{6.4}$$

which is the equation given in [2] for the volumetric strain increment.

We consider the invariant $\dot{\varepsilon}$ of the increment of irrecoverable deviatoric strain given by

$$(\mathring{\varepsilon})^2 = \dot{\varepsilon}''_{mr}\,\dot{\varepsilon}''_{rm}\,.$$

Since we have assumed $\varepsilon'_{mr} = 0$ we may write this as

$$(\dot{\varepsilon})^2 = \mathring{\varepsilon}_{mr}\,\mathring{\varepsilon}_{rm}\,. \tag{6.5}$$

Using principal components of total strain, $\mathring{\varepsilon}$ is given by

$$\dot{\varepsilon} = \frac{1}{\sqrt{3}}\,[(\dot{\gamma}_1-\mathring{\gamma}_2)^2 + (\mathring{\gamma}_2-\dot{\gamma}_3) + (\mathring{\gamma}_3-\dot{\gamma}_1)\,]^{1/2}. \tag{6.6}$$

Roscoe and Burland [2] use $\mathring{\varepsilon}$ as a strain-increment parameter.

Now from (4.7) we can show for $K = 1$

$$\dot{\varepsilon}_{mr} = \frac{2\tau_{mr}}{p(\Phi^2-\xi^2)}\,\dot{\theta}'', \tag{6.7}$$

and from this equation we find

$$\mathring{\varepsilon} = \frac{2\xi}{(\Phi^2-\xi^2)}\,\dot{\theta}''. \tag{6.8}$$

Substituting (4.13) into (6.3) and setting $K = 1$ we find

$$\dot{\varepsilon} = \frac{\beta}{(1-n_0)}\left(\frac{2\xi}{\Phi^2-\xi^2}\right)\left(\frac{\dot{p}}{p} + \frac{2\xi\dot{\xi}}{\Phi^2+\xi^2}\right). \tag{6.9}$$

Equation (6.9) is the equation given in [2] for the strain-increment parameter $\dot{\varepsilon}$.

We observe that we may combine (6.7) and (6.8) so as to obtain

$$\mathring{\varepsilon}_{mr} = \tau_{mr}\,\frac{\dot{\varepsilon}}{r}. \tag{6.10}$$

Since $\dot{\varepsilon}'_{mr} = 0$ it follows that

$$\dot{\gamma}_{mr} = \tau_{mr}\,\frac{\dot{\varepsilon}}{r} + 1/3\,\mathring{\theta}\,\delta_{mr}. \tag{6.11}$$

Using principal components of stress and strain (6.11) is equivalent to the three equations

$$\dot{\gamma}_1 = 1/3\left[(2\sigma_1-\sigma_2-\sigma_3)\,\frac{\dot{\varepsilon}}{r} + \dot{\theta}\right]$$
$$\dot{\gamma}_2 = 1/3\left[(2\sigma_2-\sigma_3-\sigma_1)\,\frac{\dot{\varepsilon}}{r} + \dot{\theta}\right] \qquad (6.12)$$
$$\dot{\gamma}_3 = 1/3\left[(2\sigma_3-\sigma_1-\sigma_2)\,\frac{\dot{\varepsilon}}{r} + \dot{\theta}\right]$$

which are the equations given in [2] for the three principal strain-increments. Thus, for K = 1, the three dimensional theory given in Section 4 is in complete agreement with the three-dimensional theory presented by Roscoe and Burland [2] for "wet" clay.

We shall now compare the plane strain theory of the previous section with that given in [2]. The plane strain theory given in [2] is derived under the assumption $\kappa = 0$. This assumption was also made in the prévious section. For K = 1, (5.9) reduces to

$$\sigma_{33} = \frac{3p_0\phi^2 - 2(\phi^2-3)\sigma}{2(\phi^2+6)} \qquad (6.13)$$

which is the equation given in [2] for the determination of σ_{33}. Also for K = 1, (5.12) reduces to

$$(\phi^2+6)I_2 + (\phi^2-3)\sigma^2 - 3\phi^2 p_0\sigma = 1/4\ p_0{}^2\phi^4. \qquad (6.14)$$

Using principal components of stress we reduce (6.14) into the form

$$(2\phi^2+3)(\phi_1{}^2+\sigma_2{}^2) - 3\phi^2 p_0(\sigma_1+\sigma_2) + 2(\phi^2-3)\sigma_1\sigma_2 = 1/4\ p_0{}^2\phi^4 \qquad (6.15)$$

which is the yield curve for plane strain in σ_1-σ_2 space given in [2].

Roscoe and Burland [2] introduce stress parameters t and τ defined by

$$t = 1/2(\sigma_1+\sigma_2) \quad , \quad \tau = 1/2(\sigma_1-\sigma_2) . \tag{6.16}$$

Using experimental observations as a guide Roscoe and Burland [2] introduce the following approximation to (6.15),

$$\omega = \Omega(\frac{p_0}{t} - 1)^{1/2} \tag{6.17}$$

where

$$\omega = \frac{\tau}{t} \quad , \quad \Omega = \frac{1}{\sqrt{2}}\Phi = \frac{1}{\sqrt{3}} M. \tag{6.18}$$

The incremental stress-strain relations, based on (6.17), given in [2] are

$$\mathring{\theta} = \frac{\lambda_0}{(1-n\ )}(\frac{2\omega\mathring{\omega}}{\Omega^2+\omega^2} \quad \frac{\dot{t}}{t}) \tag{6.19}$$

$$\dot{\gamma} = \frac{2\omega}{(\Omega^2=\omega^2)} \mathring{\theta}$$

where

$$\gamma = \gamma_1-\gamma_2 . \tag{6.20}$$

In order to compare our results with those in [2] we find it convenient to introduce Mohr's circle variables t, τ, and ψ through

$$t = 1/2(\sigma_{11} + \sigma_{22})$$

$$\tau = [1/4(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}\sigma_{12}]^{1/2} \tag{6.21}$$

$$\tan 2\psi = \frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}} .$$

In terms of stress parameters t and τ the yield locus given by

(5.12) reduces to

$$2C_1\tau^2 + 2(C_1+2C_2)t^2 + 2C_3p_0t = 1/4\ p_0{}^2\Phi^4 \qquad (6.22)$$

where C_i ($i=1,2,3$) are given by (5.13). For $K = 1$, (6.22) reduces to

$$6\Phi^2t^2 + 2(\Phi^2+6)\tau^2 - 6\Phi^2p_0t = 1/4\ p_0{}^2\Phi^4 \qquad (6.23)$$

which is the equation for the yield locus given in [2] in terms of stress parameters t and τ.

Equation (6.23) may be put into the form

$$(1+\frac{\Omega^2}{3})\ \frac{\omega^2}{\Omega^2} - \frac{p_0}{t} + (1 - \frac{\Omega^2 p_0{}^2}{12t^2}) = 0. \qquad (6.24)$$

The expression (6.17) emerges from (6.24) under the following two conditions: (i) $\Omega^2/3 << 1$ and (ii) $\Omega^2p_0{}^2/12t^2 << 1$. It can be shown that under these conditions

$$p(\Phi^2 - \xi^2) \simeq 2t(\Omega^2 - \omega^2). \qquad (6.25)$$

In view of (6.25) we obtain from (5.6) and (5.8) with $K = 1$, the expression

$$\mathring{\gamma}_{\alpha\beta} = \frac{1}{t(\Omega^2-\omega^2)}\ (\sigma_{\alpha\beta} - \sigma_{33}\delta_{\alpha\beta})\dot{\theta}. \qquad (6.26)$$

Using this expression we construct

$$(\mathring{\gamma}_{11} - \dot{\gamma}_{22}) = \frac{(\sigma_{11}-\sigma_{22})}{t(\Omega^2-\omega^2)}\ \dot{\theta}. \qquad (6.27)$$

If we use principal components of stress and strain rate (6.27) will reduce to

$$\dot{\gamma} = \frac{2\omega}{(\Omega^2-\omega^2)}\,\dot{\theta}. \tag{6.28}$$

Now from the equation obtained from (6.24) under the assumptions (i) and (ii) mentioned above we find

$$\frac{\dot{p}_0}{t} \quad \frac{(\Omega^2+\omega^2)}{\Omega^2}\left(\frac{\dot{t}}{t} + \frac{2\omega\dot{\omega}}{\Omega^2+\omega^2}\right). \tag{6.29}$$

However, since

$$\dot{p}_0 = \frac{(1-n_0)}{\lambda_0}\,p_0\,\dot{\theta}.$$

we obtain through (6.29)

$$\dot{\theta} = \frac{\lambda_0}{(1-n_0)}\left(\frac{2\omega\dot{\omega}}{\Omega^2+\omega^2} \quad \frac{\dot{t}}{t}\right) \tag{6.30}$$

Equations (6.28) and (6.30) are in complete agreement with (6.19). Thus for K=1, the approximate plane strain theory extracted from the plane strain theory presented in the previous section is in complete agreement with that of Roscoe and Burland [2].

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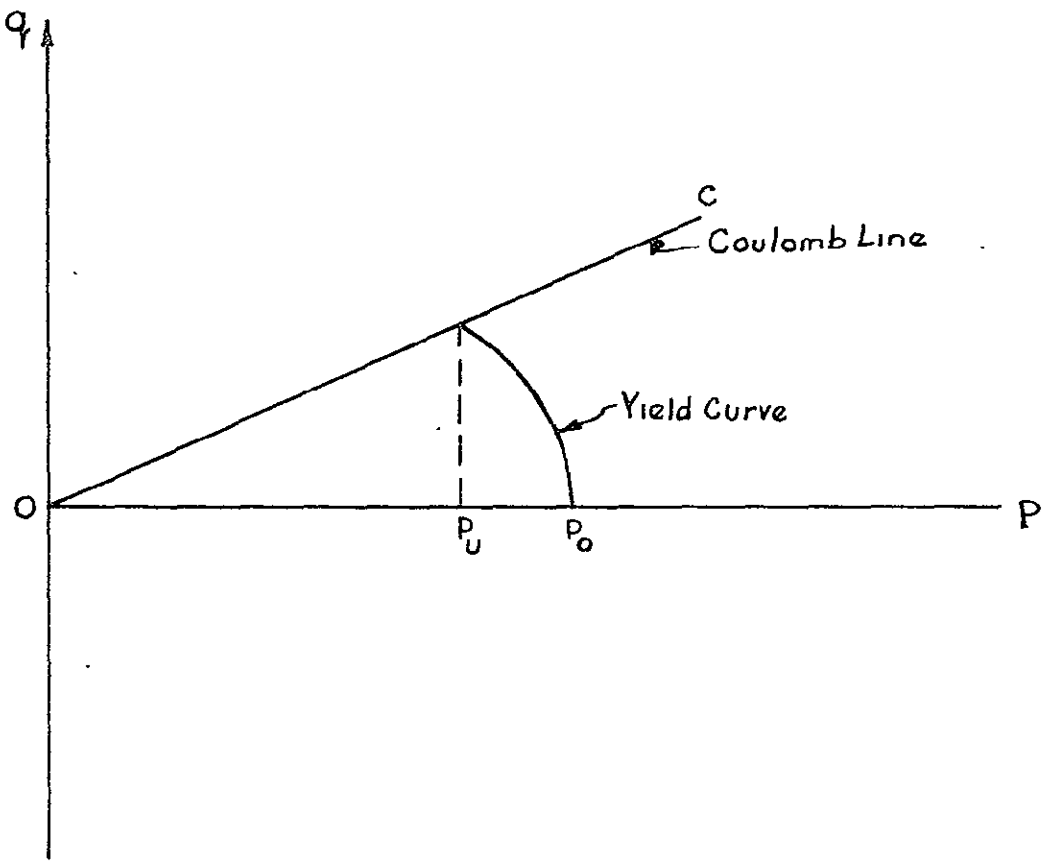

Figure 1. Triaxial Compression Yield Curve

SECTION IV

AXISYMMETRIC PLASTIC FLOW

1. INTRODUCTION

The present investigation in theoretical soil plasticity is concerned with ideal soils whose postulated mechanical behavior is an approximation to that of a wide class of natural soils. It is the object of this investigation to provide a theoretical analysis, valid under certain mathematical and physical assumptions, that has applications to a fairly wide class of problems that concerns the general situation of quasi-static axially symmetric plastic flow.

2. PRELIMINARIES

We consider the particles of a dry soil mass continuum to be referred to a fixed rectangular Cartesian system $x_i (i = 1,2,3)$ and let $\underset{\sim}{u}$ denote the displacement vector field. The components of the linear strain tensor γ_{ij} are then given by

$$2\gamma_{ij} = \partial_j u_i + \partial_i u_j \tag{2.1}$$

We also let n, n_0 denote the current and initial porosity of the soil mass continuum respectively and define the change in porosity ξ by

$$\xi = n - n_0 \tag{2.2}$$

which we assume to be of the same order of magnitude as the linear strain tensor γ_{ij}.

In an earlier work we defined yield in a soil as a permanent irrecoverable deformation [1]. The strain may then be written as the sum of an elastic or recoverable part γ_{ij}^{e} and a plastic or irrecoverable part γ_{ij}^{p},

$$\gamma_{ij} = \gamma_{ij}^{e} + \gamma_{ij}^{p}. \tag{2.3}$$

We shall also assume porosity to be given as the sum of an elastic part n^{e} and a plastic part n^{p},

$$n = n^{e} + n^{p}. \tag{2.4}$$

In view of this assumed resolution we can write

$$\xi = \xi^{e} + n^{p} \tag{2.5}$$

where $\xi^{e} = n^{e} - n_{0}$.

The theory presented in [1] is based upon the hypothesis that the irrecoverable deformation of the soil is described by the pair (γ_{ij}^{p}, n^{p}). Also in [1] we hypothesized the existence of a unique function F of plastic porosity n^{p} and Terzaghi's effective stress such that

$$F(\sigma_{ij}, n^{p}) = 0. \tag{2.6}$$

In plastic porosity-stress space (2.6) defines a six-dimensional hypersurface Σ which was called in [1], the state boundary surface. A soil particle is then said to be in a plastic state if the value of the stress and plastic porosity at the particle are such that (2.6) is satisfied.

In the theory of soil plasticity presented in [1] a yield surface for the soil was defined to be the projection on stress space of curves on Σ along which the plastic porosity has the constant value $\overline{n}^p$. This is given by

$$A(\sigma_{ij}, \varkappa) = 0 \tag{2.7}$$

where

$$\varkappa = \varkappa(\overline{n}^p) \tag{2.8}$$

is the strain hardening parameter. For a properly chosen yield surface we can associate a flow rule by invoking Drucker's postulates [1,2].

We assume that soil irrecoverable deformations are mainly due to the relative motion of the soil solid particles and that during this motion the soil solid particles are essentially incompressible. It can then be shown that under this assumption

$$\theta^p = -\frac{n^p}{(1-n_0)} \tag{2.9}$$

where $\theta^p = \gamma^p_{mm}$ is the plastic volumetric strain. We note that in view of (2.4) it follows from (2.8)

$$\varkappa = \hat{\varkappa}(\theta^p), \tag{2.10}$$

i.e., the strain hardening parameter is a function of the irrecoverable volumetric strains.

Until further notice we adopt the convention to consider compressive stresses as positive. We further consider the stress field for which

$$\sigma_{mr} = 0, \quad m \neq r \tag{2.11}$$

$$\sigma_{11} = \sigma_{22}$$

Such a stress field is called a triaxial stress field. In a triaxial stress field it is convenient to use the set of generalized stress parameters (q,p) where

$$q = \sigma_{33} - \sigma_{11}$$
$$p = \frac{1}{3}(\sigma_{33} + 2\sigma_{11}). \tag{2.12}$$

The soil mass that we consider is one that obeys the Coulomb theory of internal friction according to which the strength of the soil is limited by its ability to resist shearing stresses. In the two dimensional q-p space the Coulomb failure surface reduces to two straight lines F_1 and F_2 (Figure 1) meeting at a common point on the p-axis and whose equations are

$$q = M(p + c\ \cot\phi) \quad (F_1)$$
$$q = -M(p + c\ \cot\phi) \quad (F_2) \tag{2.13}$$

where c and ϕ are respectively the unit cohesion and angle of internal friction and

$$M = \frac{6\ \sin\phi}{3 - \sin\phi}\ , \quad M^1 = \frac{6\ \sin\phi}{3 + \sin\phi} \tag{2.14}$$

The lines F_1 and F_2 (Fig. 1) are respectively the compression and tension Coulomb failure lines. We introduce the parameter N defined by

$$N = N(p) = M(1 + \frac{c}{p}\ \cot\phi) \tag{2.15}$$

We shall use this parameter below.

In [1] a one-parameter family of yield curves was constructed in q-p space for a cohesionless soil (c=0) under the assumption that $M = M^1$.

We carry this assumption into the present study and consider the following curve in q-p space (Fig. 1),

$$\frac{p}{p_o} = \frac{N_u^2}{N_u^2 + K\eta^2} \tag{2.16}$$

where $\eta = q/p$ and

$$\begin{aligned} N_u &= N(P_u) \\ K &= \frac{p_o - p_u}{p_u} \end{aligned} \tag{2.17}$$

Here p_u is the value of the mean pressure at failure at a change in porosity ξ and p_o represents the pressure corresponding to ξ on the virgin isotropic compression curve of the soil. Equation (2.16) defines a one-parameter family of curves. For the parameter of this family we take p_o which we assume to be given by

$$p_o = \gamma \exp\left(-\frac{n^p}{\beta}\right) \tag{2.18}$$

where γ is the initial yield pressure under isotropic compression and β is a soil constant. We can also write (2.18) in the form

$$p_o = \gamma \exp(\lambda\theta^p) \tag{2.19}$$

where $\lambda = (1-n_o)/\beta$.

The yield locus given by (2.16) is in terms of triaxial stress field parameters (q,p) and as it stands it is only good for analysis under this particular stress field. Let us introduce the invariant

$$J = \tau_{rm}\,\tau_{rm} \tag{2.20}$$

where τ_{rm} is the stress deviator. We also introduce the parameter ξ through

$$\xi^2 = \frac{J}{p^2} \tag{2.21}$$

where now $3p = \sigma_{mm}$. We can then show that the expression

$$\frac{p}{p_o} = \frac{\Phi^2}{\Phi^2 + K\xi^2} \tag{2.22}$$

will reduce to (2.16) in a triaxial stress field. Here

$$\Phi = \sqrt{\frac{2}{3}}\, N_u \tag{2.23}$$

Combining (2.22) and (2.19) we obtain

$$\theta^p = \frac{\beta}{(1-n_o)} \ln\left[\frac{p(\Phi^2 + K\xi^2)}{\gamma\Phi^2} \right] \tag{2.24}$$

The requirement that in stress space the strain-rate vector be normal to the yield surface leads to the following

$$\dot{e}^p_{rm} = \frac{2K}{p(\Phi^2 - K\xi^2)} \tau_{rm}\, \dot{\theta}^p, \tag{2.25}$$

where e^p_{rm} is the plastic strain deviator. Also from (2.24) we obtain

$$\dot{\theta}^p = \frac{\beta}{(1-n_o)} \left[\frac{2K\xi\dot{\xi}}{\Phi^2 - K\xi^2} + \frac{\dot{p}}{p} \right] \tag{2.26}$$

where we have assumed K to be constant.

We consider the stress states σ_{ij} for which $p = p_u$. We call such states, limiting stress states. For σ_{ij} a limiting stress state (2.22) reduces to

$$J = p^2\Phi^2 = \frac{2}{3} M^2(p + c \cot\phi)^2. \tag{2.27}$$

Also in this case (2.25), (2.26) reduce to

$$\dot{e}^p_{rm} = \frac{2K}{(1-K)} \frac{p}{J} \tau_{rm} \dot{\theta}^p$$
$$\dot{\theta}^p = \frac{\beta}{(1-n_o)} \left(\frac{2K}{(1+K)} \frac{\dot{\xi}}{\xi} + \frac{\dot{p}}{p}\right) \tag{2.28}$$

We recall that the theory of earth pressure is based upon the concept of states of limiting equilibrium satisfying Coulomb's law of failure [3]. The defect of the theory, however, lies in its development without constitutive relations. Thus in the theory a stress field can be found, in principle, without explicit knowledge of an acceptable velocity field. The need for the necessity of a compatible velocity field with a limiting stress field satisfying Coulomb's law of failure led to the suggestion made by Drucker and Prager [4] of using Coulomb's failure criteria as a yield criteria and to treat the soil mass as a perfectly plastic material. Prediction of volume changes under the idealization of a flow vector normal to the Coulomb failure surface, however, were higher than those found by experiments. Below we shall state our concept of associating a flow rule compatible with limiting stress fields.

Even though Shield [5] has shown that the interpretation of the Coulomb law leads to only one failure surface for three-dimensional stress fields we find that (2.27) gives a convenient valid generalization of the Coulomb law to three dimensions. Equation (2.27) defines a surface in stress space that we call the limiting surface. The set of stress points that lie on the limiting surface are <u>not</u> on one yield surface. However, each limiting stress point does lie on the curve defined by the intersection of some yield surface with the limiting surface. Hence it is correct to associate a flow vector with each of the limiting stress points on the limiting surface. Here is the main difference between our use of the Coulomb surface as a limiting surface rather than a yield surface. The strain-rate is then not normal to the limiting surface.

3. EQUATIONS FOR AXIALLY SYMMETRIC DEFORMATIONS

In a cylindrical coordinate system (r,θ,z) we denote by $(\sigma_r, \sigma_\theta, \tau_{\theta z}, \tau_{rz}, \tau_{r\theta})$ the components of the stress tensor, $(\dot{\epsilon}_r, \dot{\epsilon}_\theta, \dot{\epsilon}_z, \dot{\gamma}_{\theta z}, \dot{\gamma}_{rz}, \dot{\gamma}_{r\theta})$ the components of the strain-rate tensor and (u,v,w) the components of the velocity. We are interested in axially symmetric deformations in which the z-axis is taken as the axis of symmetry. Under the assumption of axial symmetry the shear stresses $\tau_{\theta z}$, $\tau_{r\theta}$, shear strain-rates $\dot{\gamma}_{\theta z}$, $\dot{\gamma}_{r\theta}$ and circumferential velocity v all vanish identically and the remaining stresses, strain-rate components and velocities are only functions of (r,z,t).

The strain-rate-velocity relations for axial symmetry are

$$\dot{\epsilon}_r = \frac{\partial u}{\partial r}$$

$$\dot{\epsilon}_\theta = \frac{u}{r}$$

$$\dot{\epsilon}_z = \frac{\partial w}{\partial z}$$

$$\dot{\gamma}_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

Also for quasi-static conditions the stress components satisfy the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$(3.2)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + \rho g = 0$$

assume the soil mass to be in a limiting state of equilibrium so that in effect the stress field must satisfy (2.27) in addition to (3.2).

In view of (3.6) we may reduce (2.27) into

$$s = f(\sigma) = a\ \sigma \sin\phi + a\ c \cos\phi$$

or $\qquad$ (3.8)

$$s = g(p) = b\ p \sin\phi + b\ c \cos\phi$$

where

$$a = \frac{3}{3 - (1 + 3\alpha)\sin\phi}\ ,\quad b = \frac{a}{1 + \alpha\ a \sin\phi} \tag{3.9}$$

We can write σ_r, σ_z, τ_{rz} in terms of the two independent variables s and ψ

$$\begin{aligned}\sigma_r &= \frac{1}{a}(s \csc\phi - ac \cot\phi) + s \cos 2\psi\\ \sigma_z &= \frac{1}{a}(s \csc\phi - ac \cot\phi) - s \cos 2\psi\\ \tau_{rz} &= s \sin 2\psi.\end{aligned} \tag{3.10}$$

Also we can show that under the Haar-von Karman hypothesis

$$\sigma_r - \sigma_\theta = s(\cos 2\psi - 3\alpha) \tag{3.11}$$

Substitution of (3.10), (3.11) into (3.2) leads to the two differential equations for the two unknowns s and ψ

$$\begin{aligned}&(\csc\phi + a \cos 2\psi)\frac{\partial s}{\partial r} + a \sin 2\psi \frac{\partial s}{\partial z}\\ &\quad - 2as\left[\sin 2\psi \frac{\partial\psi}{\partial r} - \cos 2\psi \frac{\partial\psi}{\partial z} - \frac{1}{2r}(\cos 2\psi - 3\alpha)\right] = 0\\ &a \sin 2\psi \frac{\partial s}{\partial r} + (\csc\phi - a \cos 2\psi)\frac{\partial s}{\partial z}\end{aligned} \tag{3.12}$$

$$+ 2as \left[\cos 2\psi \frac{\partial\psi}{\partial r} + \sin 2\psi \frac{\partial\psi}{\partial z} + \frac{1}{2r} \sin 2\psi \right] + pag = 0$$

Now from (2.28), (3.6), and (3.10), we obtain

$$\dot{\gamma}_{rz} = \frac{3K}{4(1 - K)} \frac{p}{s} \sin 2\psi \, \dot{\theta}$$

$$\dot{\theta} = \frac{\beta}{(1-n_0)(1+K)} \left[2K \frac{\dot{s}}{s} + (1 - K) \frac{\dot{p}}{p} \right] \tag{3.13}$$

where we have dropped the identification of the plastic strains since we are neglecting elastic strains.

The volumetric strain-rate $\dot{\theta}$ is given by

$$\theta = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} \tag{3.14}$$

Combining (3.13), (3.14), and $(3.1)_4$ we obtain

$$\sin 2\psi \frac{\partial u}{\partial r} - \varkappa \frac{\partial u}{\partial r} + \sin 2\psi \frac{\partial w}{\partial z} + \sin 2\psi \frac{u}{r} = 0 \tag{3.15}$$

where

$$\varkappa = \varkappa(s) = \frac{2(1 - K)b \sin\phi}{3K} \frac{s}{(s - bc \cos\phi)} \tag{3.16}$$

Equations (3.12), (3.15) together with the equation of istropy $(3.3)_2$

$$\tan 2\psi \frac{\partial u}{\partial r} - \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} - \tan 2\psi \frac{\partial w}{\partial z} = 0 \tag{3.17}$$

form a system of four equations for the four unknowns s, ψ, u, and w.

4. STRESS AND VELOCITY CHARACTERISTICS

We examine the case for which $3\alpha = -1$, i.e., $\sigma_3 = \sigma_2$. For this case,

$$a = 1$$
$$b = \frac{3}{3 - \sin\phi} \tag{4.1}$$

Using known methods we may obtain the equations for the characteristics of the system of equations governing s and ψ. The slopes of the characteristics in the z-r plane are given by

$$\left(\frac{dz}{dr}\right)_{1,2} = \frac{\sin 2\psi \pm \cos\phi}{\cos 2\psi + \sin\phi}. \tag{4.2}$$

We see that the characteristics will be real and non-zero. It follows that the system will be hyperbolic. It is convenient to introduce the angle φ through

$$\varphi = \frac{\pi}{4} - \frac{\phi}{2} \tag{4.3}$$

and to name the characteristic with slope $\tan(\psi - \varphi)$ an α-line, and that with slope $\tan(\psi + \varphi)$ a β-line, thus

$$\frac{dz}{dr}_{\alpha} = \tan(\psi - \varphi)$$
$$\frac{dz}{dr}_{\beta} = \tan(\psi + \varphi). \tag{4.4}$$

The α and β characteristic directions are illustrated graphically in Fig. 3.

The equations along the characteristics can be shown to be

$$\cot\phi ds - 2sd\psi - (\rho g \cos(\psi+\varphi) - \frac{2s}{r} \sin\varphi \cos\psi)dS_{\alpha} = 0 \text{ on } \alpha\text{-line}$$
$$\cot\phi ds + 2sd\psi + (\rho g \cos(\psi-\varphi) + \frac{2s}{r} \sin\varphi \cos\psi)dS_{\beta} = 0 \text{ on } \beta\text{-line} \tag{4.5}$$

where S_α and S_β are arc lengths measured along the α and β lines respectively. Introducing the quantity χ defined by

$$\chi = \cot\phi \ \ln \frac{s}{s_o} \tag{4.6}$$

where s_o is some reference stress, we may write (4.5) in the form

$$\begin{aligned} &d\chi - 2d\psi + \frac{1}{r}(\cos\phi \ dr - (1 - \sin\phi)dz) \\ &\quad - \frac{\rho g}{s_o}(\sin\phi \ dr - \cos\phi \ dz) \exp(-\chi \tan\phi) = 0 \text{ an } \alpha\text{-line} \\ &d\chi + 2d\psi + \frac{1}{r}(\cos\phi \ dr - (1 - \sin\phi)dz) \\ &\quad + \frac{\rho g}{s_o}(\sin\phi \ dr + \cos\phi \ dz) \exp(-\chi \tan\phi) = 0 \text{ on } \beta\text{-line.} \end{aligned} \tag{4.7}$$

We consider the case when ψ has a constant value ψ_o along a stress α-characteristic (say). In this case (4.7), becomes

$$\frac{d\chi}{dr} - B \exp(-\chi \tan\phi) + \frac{A}{r} = 0 \tag{4.8}$$

where

$$\begin{aligned} A &= \cos\phi - (1 - \sin\phi) \tan(\psi_o - \varphi) \\ B &= (\sin\phi - \cos\phi \tan(\psi_o - \varphi)) \frac{\rho g}{s_o} \end{aligned} \tag{4.9}$$

The general solution of (4.8) can be shown to be

$$s = \frac{KS_o}{{}_rA \tan\phi} + \frac{Bs_o \tan\phi}{(1 + A \tan\phi)} r \tag{4.10}$$

where K is a constant of integration. At a later time we shall refer to (4.10).

The slopes of the characteristics associated with the system (3.15), (3.17) are given by

$$\frac{dz}{dr}_{1,2} = \frac{\sin 2\psi \pm \sqrt{1-\kappa^2}}{\cos 2\psi - \kappa} \quad (4.11)$$

where now in view of $(4.1)_2$

$$\kappa = \frac{2(1-K)}{3K} \sin\phi \frac{s}{p} . \quad (4.12)$$

We assume $\kappa < 1$. This being the case the system (3.15), (3.17) will be hyperbolic with characteristic directions given by (4.11). By defining the angle η through

$$- \sin 2\eta = \kappa \quad (4.13)$$

we may write (4.11) in the form

$$\left(\frac{dz}{dr}\right)_1 = \tan(\psi - \hat{\varphi})$$
$$\left(\frac{dz}{dr}\right)_2 = \tan(\psi + \hat{\varphi}) \quad (4.14)$$

where

$$\hat{\varphi} = \frac{\pi}{4} - \eta \quad (4.15)$$

Comparing (4.15) with (4.3) we see that in general the velocity characteristics do not coincide with the stress characteristics. The equations along the velocity characteristics are

$$du + \left(\frac{dz}{dr}\right)_{1,2} dw - (\cos 2\psi - \sin 2\eta)^{-1} \left(\sin 2\psi - \left(\frac{dz}{dr}\right)_{1,2} \cos 2\psi\right) \frac{u}{r} dz = 0 \quad (4.16)$$

5. INDENTATION OF SEMI-INFINITE SOIL MASS BY A LUBRICATED CIRCULAR RIGID CONE

As an application of the theory presented we consider a problem that is related to the cone-penetrometer test used in soil mechanics. This problem is associated with the incipient plastic flow in a semi-infinite region of soil due to load applied through a lubricated circular rigid cone.

We define the origin of cylindrical coordinates as shown in Fig. 4 and we shall suppose the soil to occupy the semi-infinite region $z \geq 0$. In addition to the load applied to the soil through the circular cone we take into account the normal stress t (Fig. 4) which may correspond to atmosphere pressure or an equivalent surcharge.

The boundary conditions on ψ, s, and velocity components for the problem of interest are

$$\left.\begin{array}{l} \psi = \delta \\ w = \text{const.} \\ u = \text{const.} \end{array}\right. \quad \text{on } z = (R-r)\cot\delta \tag{5.1}$$

$$\left.\begin{array}{l} \psi = 0 \\ s = \dfrac{t \sin\phi + c \cos\phi}{1 - \sin\phi} \end{array}\right. \quad \text{on } r = R,\ z = 0 \tag{5.2}$$

The boundary conditions (5.1), (5.2) together with the governing equations (3.12), (3.15), and (3.17) define a boundary value problem. We note that the velocity boundary conditions are not sufficient to completely determine the velocity field anywhere in the soil mass. This indicates that we may impose further arbitrary conditions on the solution to find an acceptable velocity field. However we should not expect this solution to be unique. Cox, Eason, and Hopkins [6] discuss the uniqueness of a similar problem.

Following Cox, Eason, and Hopkins [6] we expect the stress characteristic net to exhibit the geometrical features depicted schematically in Fig. 5. The unknown of interest is the limiting mean pressure at the face of the cone. To find this we need only to consider that part of the stress field bounded by OA, AB and BCDO. Lines such as PS will be α-lines while those such as PQR will be β-lines. In particular ODCB is the β-line through the apex of the cone O. We shall discuss this β-line below.

A solution for the stress field can be constructed following arguments similar to those used by Cox, Eason, and Hopkins [6]. Thus from a knowledge of s and ψ on AB, s and ψ can be determined on ABC by using (4.4) and (4.5). At $r = R$, $z = 0$ a singularity is introduced at which s and ψ will be multivalued. This fact together with the known values of s and ψ can be used to determine s and ψ on ACD. Finally the now known values of s and ψ on AD together with the known values of ψ on OA determine s and ψ on ADO. From this solution the limiting stresses at the face of the cone can be determined. If we find a velocity field compatible with this stress field then the solution will given an upper bound for the limiting stress field. If this stress field can be extended in such a way so as to satisfy the conditions of limiting equilibrium then the solution is said to be complete and will be a lower bound for the limiting stress field [6].

Now by considering the singular point A as a β-line of zero length we find the conditions at point A,

$$\left.\begin{aligned} &0 \le \psi \le \delta \\ &s = \frac{t \sin\phi + c \cos\phi}{1 - \sin\phi} \end{aligned}\right\} \quad \text{at } r = R,\ z = 0. \tag{5.3}$$

In order to extend the solution into the rigid region we must know the location of the boundary between the rigid and deforming regions. Since at this boundary the velocity components or their spatial derivatives must have some discontinuities it follows that the boundary between rigid and deforming regions must be a velocity characteristic line. Considering that in the theory presented here the velocity and stress characteristics do not coincide we reason that the stress β-characteristic curve ACDO is not the boundary between the rigid and deforming region.

Let us name the first velocity characteristic an α'-line and the second velocity characteristic a β'-line. In Fig. 4, the curve B'O depicts the velocity β'-line through the apex of the cone. Here we have assumed that

$$\eta < \frac{\phi}{2} . \tag{5.4}$$

Now the velocity field must accomodate the incipient motion of the rigid cone. For this reason AO must lie within the deforming region. The simplest configuration that can occur is when the β'-line B'O is the boundary between the deforming and rigid region. We shall return to this question below.

The stress field and stress characteristic net in the region BAOD can be determined by numerically integrating (4.4) and (4.7). Using pgR as the reference stress s_0 indicated in (4.6) we replace (5.2) and (5.3) by

$$
\left.
\begin{array}{ll}
\psi = 0 & \\
x = \cot\phi \ \ln\left(\dfrac{t^* \sin\phi + c^* \cos\phi}{1 - \sin\phi}\right) & \text{on } r > R,\ z = 0 \\
x = \cot\phi \ \ln\left(\dfrac{t^* \sin\phi + c^* \cos\phi}{1 - \sin\phi}\right) & \text{at } r = R,\ z = 0 \\
0 \le \psi \le \delta &
\end{array}
\right\} \tag{5.5}
$$

Here $t^* = t/pgR$, $c^* = c/pgR$. The numerical integration is based upon the approximation of (4.4) and (4.7) by finite difference equations [7].

It remains to determine the velocity of β'-line B'O which we have assumed to be the boundary between the deforming and rigid region. Since in addition to ψ, X is now known on AO we can determine the slope of B'O at O. Suppose we extend the stress α-lines beyond BCDO as straight lines with slopes obtained by using the calculated values of ψ on BCDO. Then along each of these stress α-lines (4.10) holds with the arbitrary constant K evaluated for each α-line by values on BCDO. With reference to Figure 6 we can determine the coordinates (r_T, z_T) of the point T of the intersection of the straight stress α-line with the velocity β'-line as follows: We assume values of the slopes known at S and M. Then the first approximation (r_1, z_1) to (r_T, z_T) is obtained from

$$
\begin{aligned}
z_1 - z_S &= (r_1 - r_S)\tan(\psi_S - \varphi) \\
z_1 - z_M &= (r_1 - r_2)\tan\psi^*_M
\end{aligned} \tag{5.6}
$$

where ψ^*_M is known with the calculated value. With this calculated value of r_1 we obtain through (4.10), (4.12) the first approximation $\varkappa_1$ to $\varkappa_T$. With this value $\varkappa$, we can compute the first approximation ψ^*_1 to ψ^*_T. We can then use the average value of ψ^*_1 and ψ^*_M to obtain second approximations. Using this iteration procedure we can determine the location of the β'-line B'O and the value of S at each point of intersection of the straight stress α-line with the β'-line B'O.

Once the stress distribution along B'O is known we can determine the total vertical load P, exerted by the indenter, by simple statics. The ultimate bearing capacity qu is then obtained from

$$qu = \frac{P}{\pi R^2} \qquad (5.7)$$

6. CONCLUSIONS

We have presented an approach of obtaining ultimate loads that differs from the traditional approach used in soil mechanics. The traditional approach for obtaining ultimate loads for soil mechanics problems is to exhibit an equilibrated limiting stress field, solve for the load equilibrating the stress field, and then simply term this load a "failure" or "ultimate" load. The approach presented here, however, demands more of a load before it is termed an ultimate load. The ultimate load must, in addition to equilibrating a limiting stress field, be associated with a deforming solution in a theory involving material deformation.

The problem formulated in Section 5 can be tied to the cone penetrometer test. The amount that the cone has penetrated is equal to the amount h_s that the shaft has penetrated plus the height $h_c = R \tan\delta$

of the cone. The distance h_s is reflected in the equivalent surcharge $t = \rho g h_s$ which is a boundary value. We can then associate with an ultimate load P a depth $h = h_c + h_s$. Thus we can plot a cone load P vs. depth h curve. This, of course, is for a given soil with cohesion C, angle of internal friction ϕ and bulk mass density ρ.

We introduce the quantities q_u*, $h*$, and $c*$ defined by

$$q_u* = \frac{q_u}{\rho g R}$$
$$h* = \frac{h}{R} \qquad (6.1)$$
$$c* = \frac{c}{\rho g R}$$

Then for a given soil with cohesion c, angle of internal friction ϕ, bulk mass density ρ and given fixed cone geometry R and δ we can find an ultimate bearing capacity q_u at depth h. We can then generate the curve

$$q_u* = f\ (h*;\ c*,\ \phi) \qquad (6.2)$$

wherein ϕ and $c*$ appear as parameters.

The solutions for q_u* most widely used in soil mechanics are of the form

$$q_u* = c*H_c + h*H_q \qquad (6.3)$$

where H_c, H_q are bearing capacity factors which depend on ϕ alone. It is possible to obtain from (6.2) values of N_c, N_q such that the form (6.3) holds. However by doing so we would have to accept the obvious consequences of the form (6.3), viz., linearity in c^* and h^*.

The use of the cone penetrometer as a useful test of obtaining *in situ* properties of soils is highly dependent on being able to solve the following problem: given the response curve q_u^* vs. h^* determine the parameters c^* and ϕ that correspond to this curve. In this section we have presented a theory to generate q_u^* vs. h^* curves for different values of c^* and ϕ. It remains to look into a systematic and logical manner of using response q_u^* vs. h^* curves to obtain values of c^* and ϕ.

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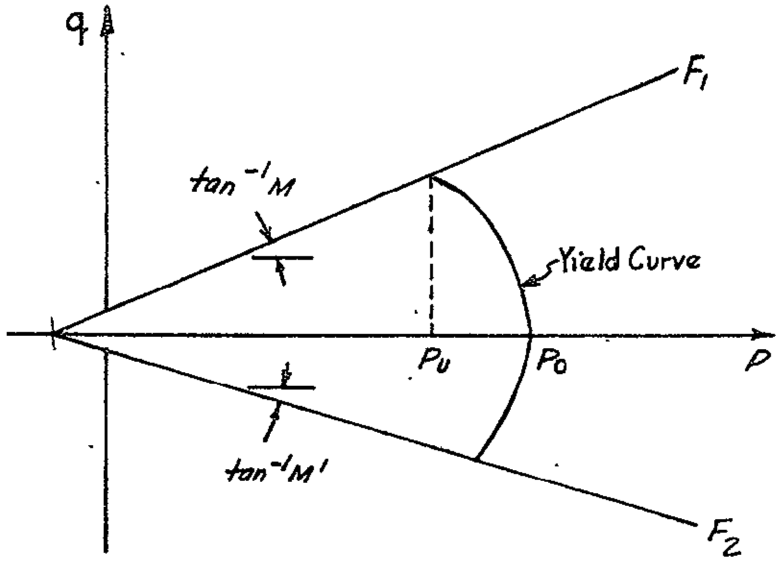

Figure 1. Triaxial Compression Yield Curve

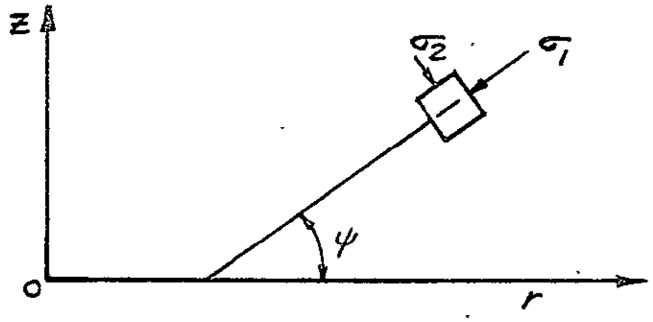

Figure 2. Directions of Principal Stresses

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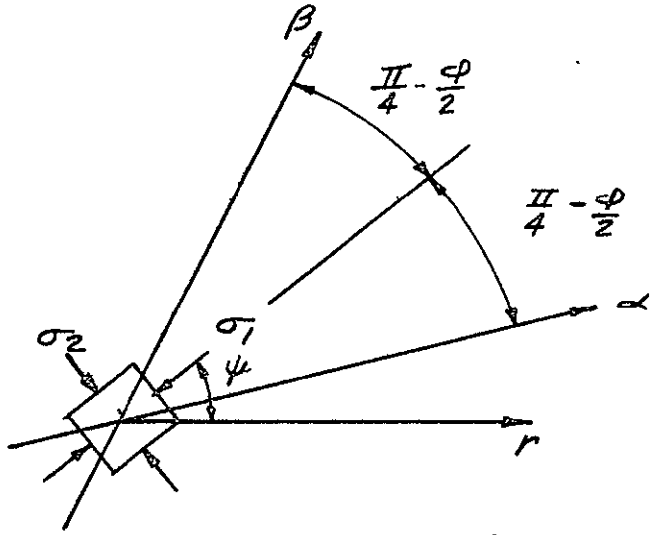

Figure 3. Stress Characteristics Directions

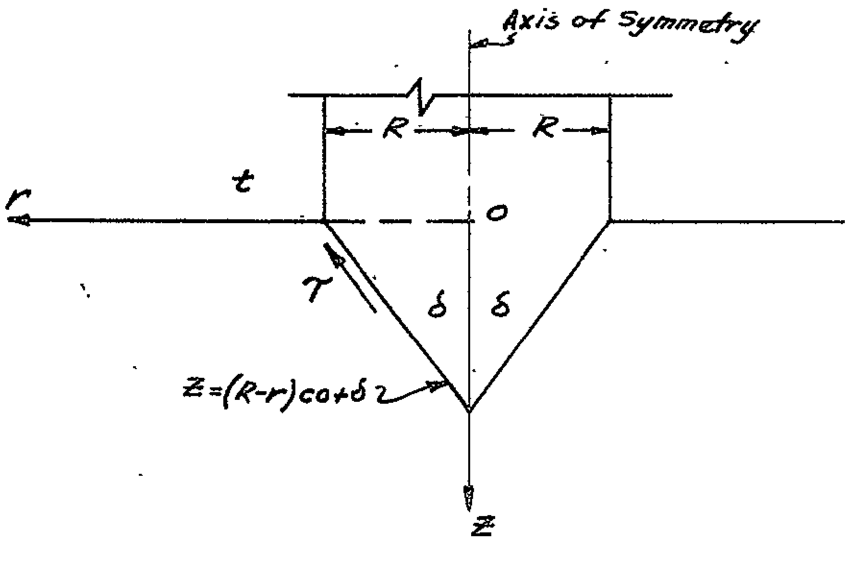

Figure 4. Lubricated Circular Rigid Cone

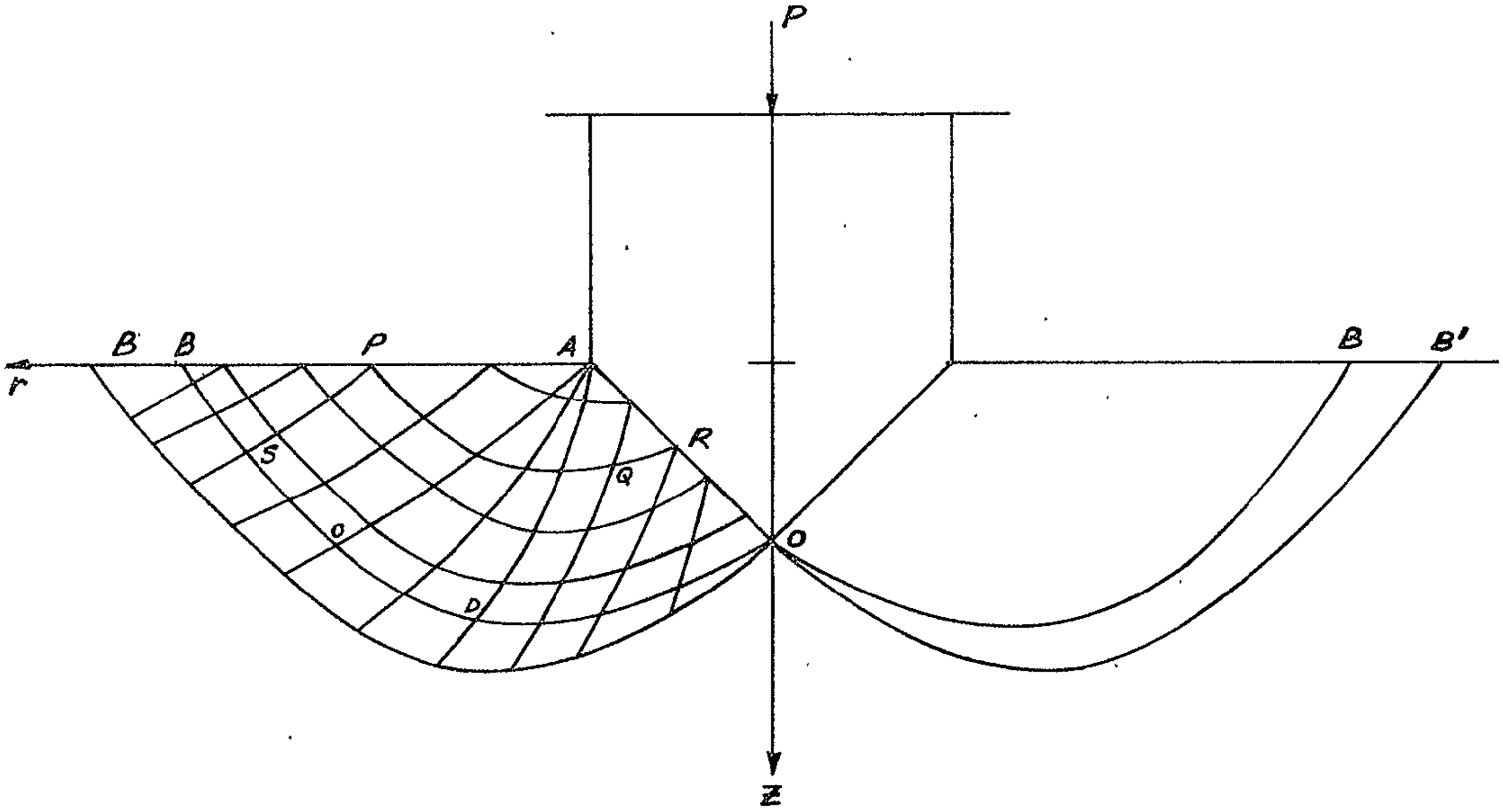

Figure 5. Schematic Diagram of Characteristics

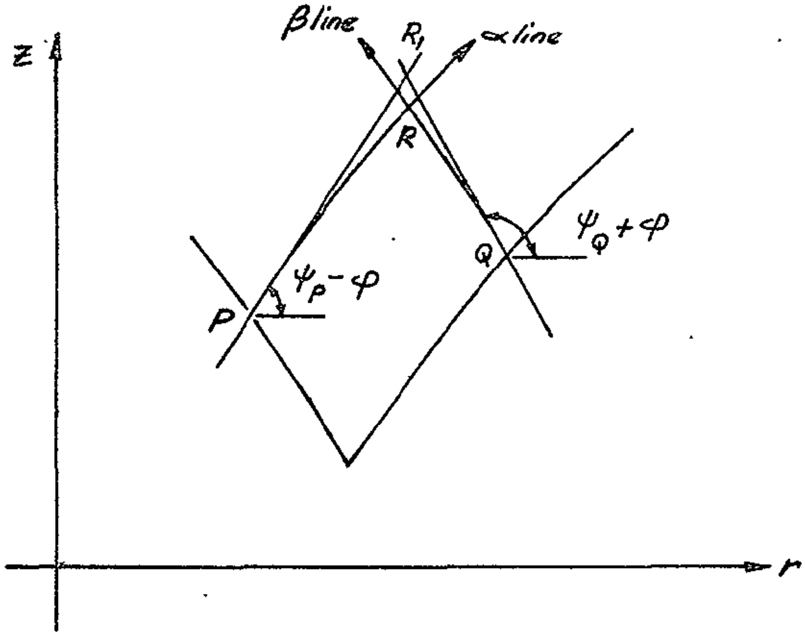

Figure 6. Intersection of Stress Characteristics Through Neighboring Points P & Q

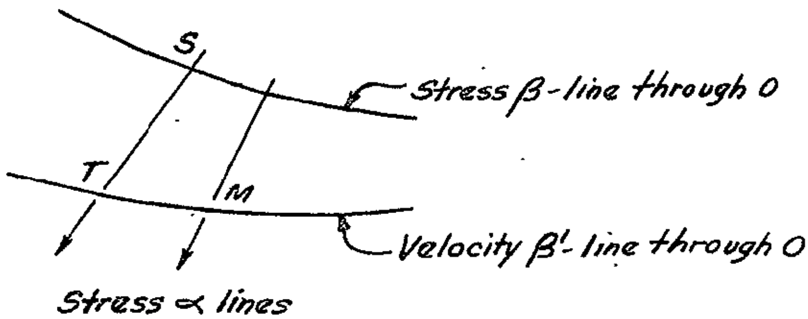

Figure 7. Intersection of Stress α-Lines with Stress β-Line & Velocity β'-Line Through O