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by

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## ABSTRACT

A number of examples are given of the use of singular perturbation theory in space trajectory optimization. Singular perturbation problems have resulted from the use of four different small parameters. These parameters are 1) the mass of the launch, target, or swingby planet, 2) the reciprocal of the thrust for high thrust rockets, 3) the thrust for low thrust rockets, and 4) the reciprocal of the transfer time.

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## INTRODUCTION

The theory of minimum fuel space trajectories was originally developed by Lawden in the early nineteen fifties. For trajectories in three spatial dimensions the nonlinear equations of state are of seventh order.

$$\ddot{\bar{r}} = \bar{g}(\bar{r}, t) + \frac{\bar{f}}{m} \tag{1}$$

$$\dot{m} = \frac{f}{c} \tag{2}$$

$$0 \leq f \leq f_{max} \tag{3}$$

In these equations $\bar{r}$ is the position vector, $\bar{g}$ is the gravitational acceleration, $\bar{f}$ is the rocket thrust, m is the mass, and c is the constant exhaust velocity of the rocket. Lawden expressed the necessary conditions in terms of the adjoint vector for velocity, $\bar{\lambda}$, which he called the primer vector (Ref. 1). The Hamiltonian for this problem may be written as Eq. 4 where $\sigma$ is the adjoint variable for mass.

$$H = \frac{f}{m}\left(\lambda - \frac{\sigma m}{c}\right) + \bar{\lambda} \cdot \bar{g} - \dot{\bar{\lambda}} \cdot \dot{\bar{r}} \tag{4}$$

In this equation the maximum principle has already been used to optimize the thrust direction by pointing it in the direction of the primer vector. The adjoint equations are given by equations (5) and (6).

$$\ddot{\bar{\lambda}} = \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial \bar{r}} \tag{5}$$

$$\dot{\sigma} = \frac{\lambda f}{m^2} \tag{6}$$

The Hamiltonian is linear in the thrust magnitude, so the optimal magnitude of the thrust depends on the sign of the switching function.

$$\lambda - \frac{\sigma m}{c} < 0 \qquad f = 0 \tag{7}$$

$$\lambda - \frac{\sigma m}{c} = 0 \qquad 0 \leq f \leq f_{max} \tag{8}$$

$$\lambda - \frac{\sigma m}{c} > 0 \qquad f = f_{max} \tag{9}$$

The optimal trajectory will usually consist of a sequence of coasting arcs and maximum thrust arcs. A typical time history of $\lambda$ and $\sigma m/c$ for a fixed time trajectory with three thrusting periods is shown in the upper portion of Fig. 1. During the coasting arcs $\sigma m/c$ will be constant. The primer vector has a stationary maximum during the interior thrusting arc. This will always be true for an interior arc as the primer vector is continuous through several time derivatives (Eq. 5).

## IMPULSIVE TRAJECTORIES

Lawden considered the case where $f_{max}$ is allowed to become infinite, leading to impulsive changes in velocity. Across such a velocity impulse the mass will change by a finite amount given by Eq. 10.

$$\frac{m_2}{m_1} = e^{-\frac{\Delta V}{c}} \tag{10}$$

Lawden derived the necessary conditions for impulsive trajectories by a limiting argument (Ref. 1). It is interesting that he obtained the correct condition even though his work predated the maximum principle. In fact, the impulsive case is of such a nature that the standard maximum principle is not applicable. In recent years Lawden's results have been rigorously verified by Neustadt, Rishel, and Warga (Refs. 2, 3, 4).

If the thrust is allowed to become unbounded $\sigma m/c$ becomes a constant of the motion which may be taken as unity. The thrust is applied only when the primer vector has unit magnitude. The primer vector cannot exceed unity on an optimal trajectory. The Hamiltonian is only defined on the open interval from the initial time to the final time for fixed time problems. A typical primer vector history for a fixed-time three-impulse trajectory is shown in the lower part of Fig. 1.

Although the impulsive case can be treated by the extensions of the maximum principle in Refs. 2-4, Jack Warga has suggested a simple regularizing transformation which allows the ordinary form of the maximum principle to be used. Define a new independent variable $\tau$ and a new control variable $u$ by Eqs. 11 and 12.

$$d\tau = \left(1 + \frac{f}{mu^*}\right)dt \tag{11}$$

$$u = \frac{fu^*}{mu^* + f} \tag{12}$$

When $u$ is equal to its maximum value $u^*$ the thrust is infinite, Eq. 13.

$$\frac{f}{m} = \frac{u}{1 - \frac{u}{u^*}} \tag{13}$$

The Hamiltonian for the new variables is given by Eq. 14.

$$H_\tau = (\lambda - 1)u + \left(1 - \frac{u}{u^*}\right)(\bar{\lambda} \cdot \bar{g} - \dot{\bar{\lambda}} \cdot \dot{\bar{r}}) \tag{14}$$

The conventional form of the maximum principle can be used for this new Hamiltonian. The Hamiltonian is now defined on the closed interval from the initial to the final time.

The extremals neighboring an optimum impulsive trajectory provide an interesting example of a singular perturbation problem. If the nominal extremal has two or more impulses, then neighboring extremals with the same number of impulses will exist (Ref. 5). However, near the end of any extremal only one impulse will remain on the nominal extremal. Neighboring extremals to this single impulse nominal trajectory may require as many as three small midcourse impulses to meet the desired terminal conditions. While the neighboring extremals only involve small variations in the state, they obviously require large variations in the adjoint equations. The primer vector magnitude will be required to go from a single endpoint maximum to as many as three initial and/or interior maxima and an endpoint maximum, all of unit magnitude.

If the nominal extremal has a finite bound on thrust magnitude and a single coasting arc followed by a single maximum thrust arc, then neighboring extremals with the same sequence of arcs will generally exist. However, the region covered by these neighboring extremals may be very small.

## SINGULAR EXTREMALS

The case where the switching function remains identically zero over a finite time interval leads to what are known as singular arcs in both the classical calculus of variations and modern control theory. This usage of the term "singular" may be distinct from its usage in "singular perturbation theory". However, singular extremals do form a manifold of lower dimensionality than do the non-singular extremals so neighboring extremals do involve a singular perturbation.

The theory of singular extremals is developing rapidly at the present time (Refs. 6-9). Ref. 6 shows that junctions between singular and non-singular extremals for bounded thrust rocket trajectories are very complex. The junctions involve a well defined infinite sequence of maximum thrust and coasting arcs whose limit point is the beginning of the intermediate thrust singular arc. What is surprising is that if the bound on thrust is removed then there can be a simple junction between an impulse and a singular arc. Robbin's demonstration of the existance of minimizing singular arcs in an inverse square field is of great theoretical interest and might eventually prove to have practical consequences (Ref. 10).

## LARGE THRUST TRAJECTORIES

Impulsive thrust extremals are much easier to calculate than finite thrust extremals. As a result it is often desirable to calculate finite thrust trajectories as a singular perturbation of impulsive trajectories. Practical theories for this problem have been developed by Robbins, Hazelrigg and Lion, and Andrus, Refs. 11-13. These analyses are usable as long as the thrusting time is not too long. Their range of validity coincides with a large class of practical problems.

## SWINGBY TRAJECTORIES

The masses of the planets are quite small compared to the mass of the sun. However, a close approach to a planet can strongly perturb a Heliocentric trajectory. These close planetary approaches can often decrease the flight time and/or the fuel consumption for interplanetary missions. A typical example is shown in Fig. 2. This is a 3.33 year trajectory to Saturn which passes Jupiter at 5 planetary radii after 1.55 years. Except for guidance corrections, the vehicle is in a free coasting orbit after leaving the Earth. A direct flight to Saturn would require a larger launch energy even if the flight time was greatly increased. The effect of Jupiter in this example is so large that it changes the heliocentric trajectory from an ellipse with an eccentricity of 0.786 to a hyperbola with an eccentricity of 2.10.

An accurate and useful theory has been developed for these problems by Breakwell and Perko (Ref. 14). This theory has been constructed by the method of matched asymptotic expansions. The inner solutions consist of planetcentric hyperbola's which are perturbed by the sun while the outer solutions consist of heliocentric conics perturbed by the planets. The solution contains terms of order $\epsilon^{1/2}$, $\epsilon$, and $\epsilon^{3/2}$ and is in error by terms of order $\epsilon^2$. The small parameter $\epsilon$ is the ratio of the planetary mass to the solar mass.

This theory has been developed for the unpowered, ballistic case. However, it can easily be extended to handle optimum impulsive trajectories. The Breakwell-Perko theory can also handle lunar trajectories although the errors in this case are large enough to be significant.

A different singular perturbation approach to lunar trajectories was developed earlier by Kevorkian and various coworkers (e.g. Ref. 15). This theory uses a near rectilinear earth-moon trajectory as the outer solution. The accuracy of this theory is comparable to the Breakwell-Perko theory. It has recently been shown to yield many of the qualitative properties of earth-moon trajectories (Ref. 16).

## LONG DURATION TRAJECTORIES

The theory of minimum fuel impulsive orbit transfer in an inverse square field has been extensively developed, particularly for the time open case (Ref. 17). For some cases the absolute minimum fuel solution requires infinite time while in other cases the time involved is finite. One case where the optimum transfer time is almost always infinite is transfer from an ellipse to a hyperbola (Ref. 18). In these cases a long transfer time may be used to reduce the fuel consumption close to the absolute minimum. A singular perturbation theory is being developed for these problems (Refs. 19, 20).

Figure 3 illustrates a four impulse transfer from an initial elliptic orbit to a specified hyperbolic asymptote. The first impulse transfers the vehicle from the initial ellipse onto a highly eccentric ellipse which approximates a straight line in the outer region. The second and third impulses are used to change the orientation of this highly eccentric ellipse and to reduce its perigee radius to the minimum allowable. A fourth impulse then transfers the vehicle onto the escape hyperbola.

The solution is developed as an asymptotic series in inverse third powers of the time between the first and fourth impulses. The magnitude of the first and fourth impulses involve terms of order unity and minus two-thirds. The magnitude of the second and third impulses involve terms of order minus one-third and minus two-thirds. The location and directions of the impulses are also developed as asymptotic series.

## LOW THRUST TRAJECTORIES

One class of space propulsion devices is characterized by very small accelerations on the order of one ten-thousandth of the standard acceleration of gravity. For these electric propulsion devices, the thrust may be regarded as a small parameter, at least in the close vicinity of planets.

The method of averaging may be used to treat these problems (Refs. 21-23). A variation of parameters formulation is used with the conventional elliptic orbit elements being the slowly varying parameters. In first approximation the order is reduced because the position in the orbit is averaged out. By going to the improved first approximation (Ref. [illegible]) which considers first order periodic as well as secular terms, the orbital position can be reintroduced into the problem. Fig. 4 illustrates a minimum fuel, fixed time transfer between two coplanar elliptic orbits.

## LOW THRUST ESCAPE TRAJECTORIES

If an electric propulsion system is to be used for interplanetary missions it might first be placed in a low altitude circular orbit and allowed to spiral slowly away from the earth or it might be injected onto a hyperbolic orbit which rapidly takes it into interplanetary space. Both possibilities are illustrated in Fig. 5.

The case of a circular initial orbit has been treated most thoroughly by Breakwell and Rauch (Refs. 24-26). They asymptotically match an analytical inner solution with a numerically determined intermediate solution and an analytical outer solution.

The case of a hyperbolic initial orbit has been treated by Melbourne and Sauer (Ref. 27), and by Edelbaum (Ref. 28). In this case the inner solution is an unpowered hyperbola. By approximating the trajectory as being essentially radial an analytic solution is obtained for the effect of the planet on the heliocentric trajectory.

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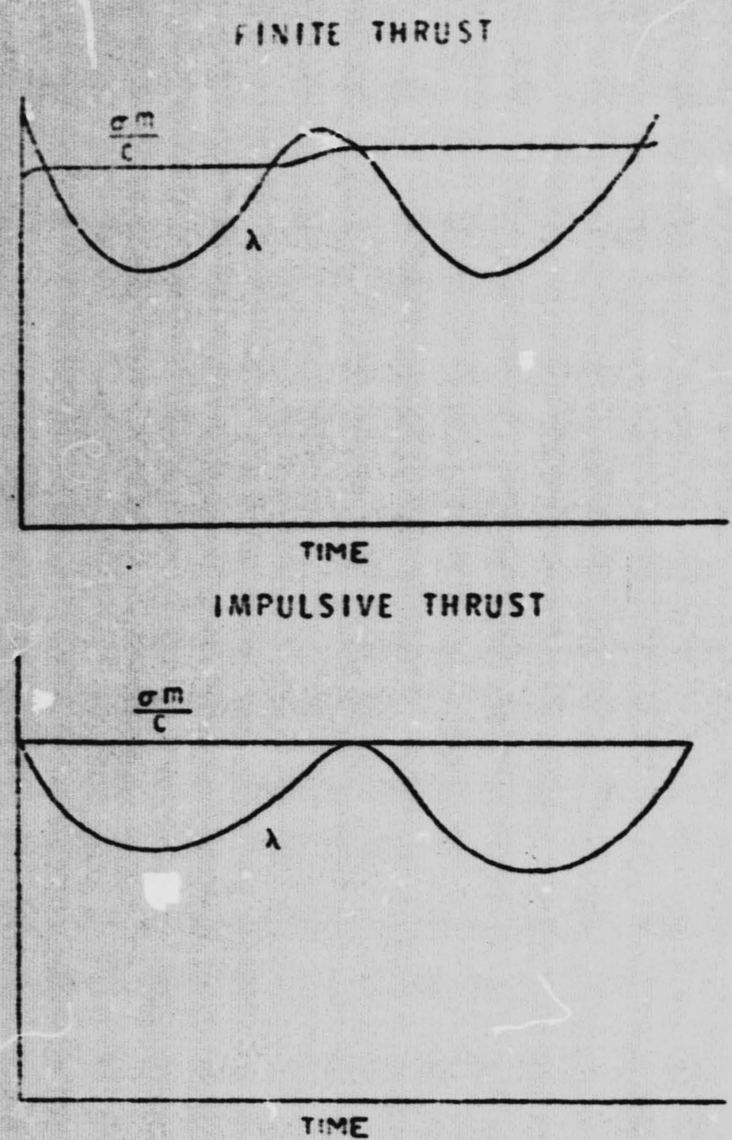


Figure 1

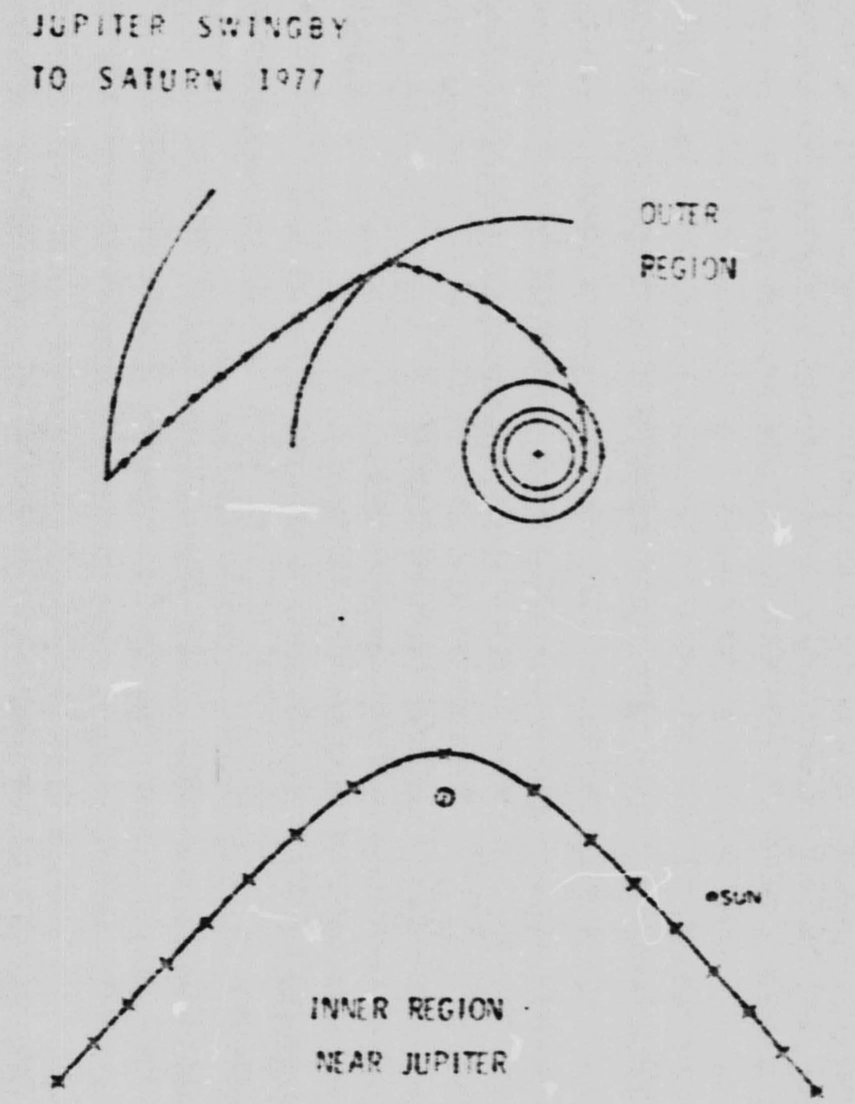


Figure 2

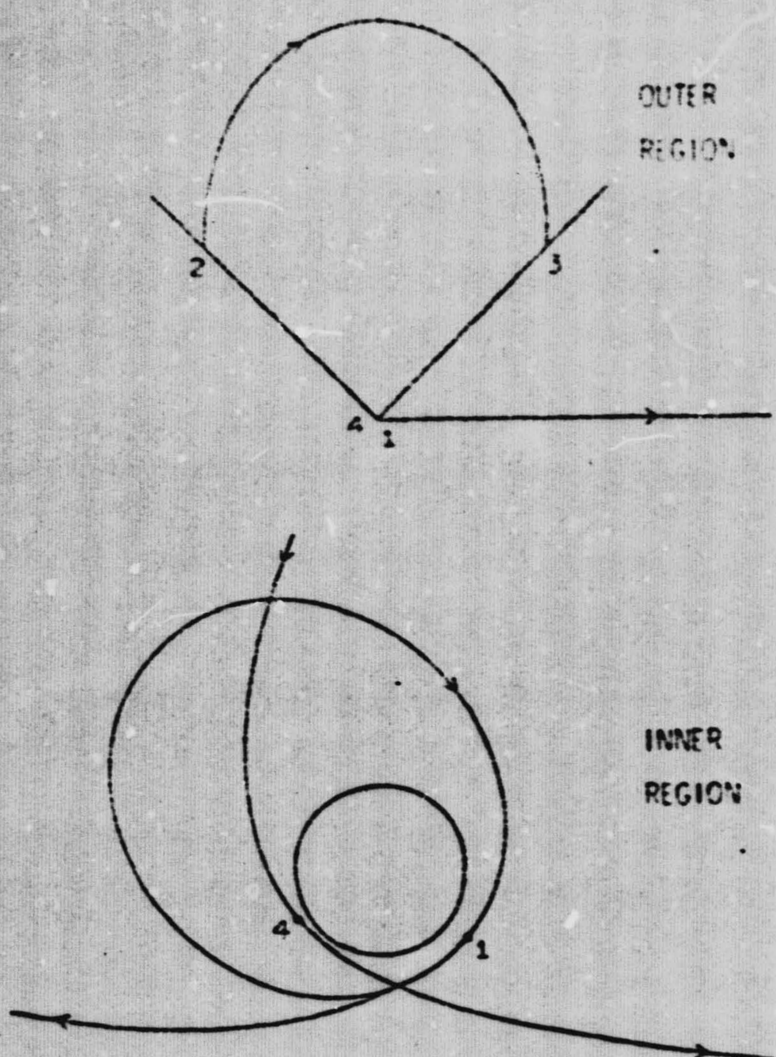


Figure 3 Four Impulse Escape

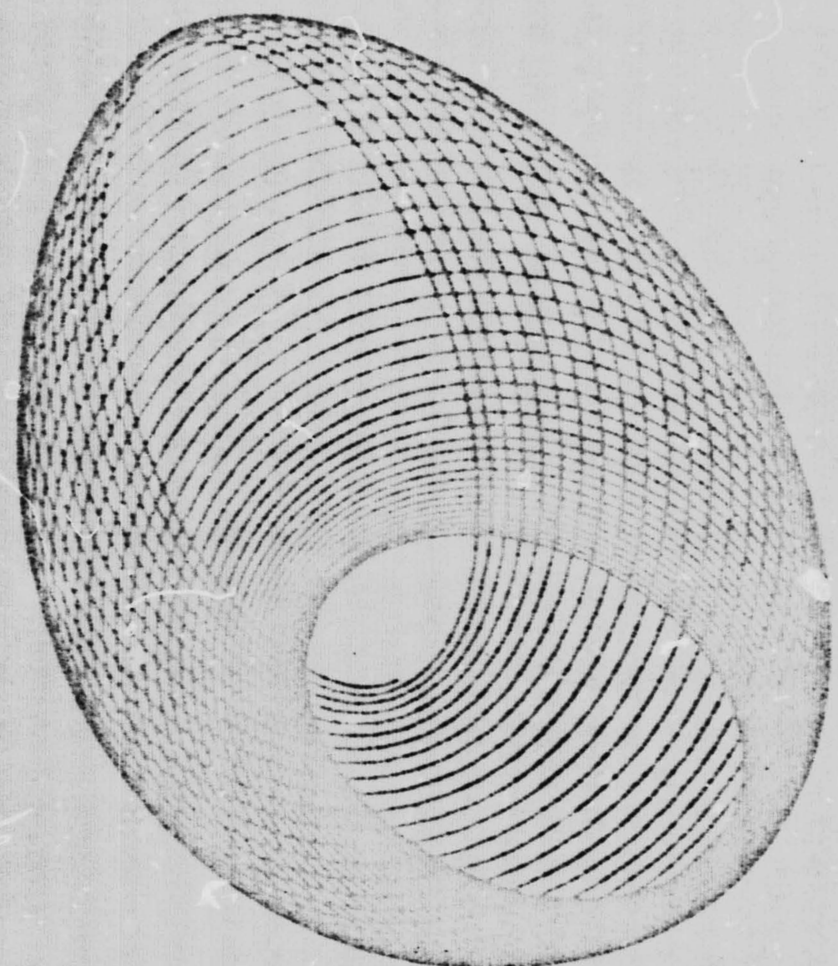

Figure 4 Low Thrust Transfer

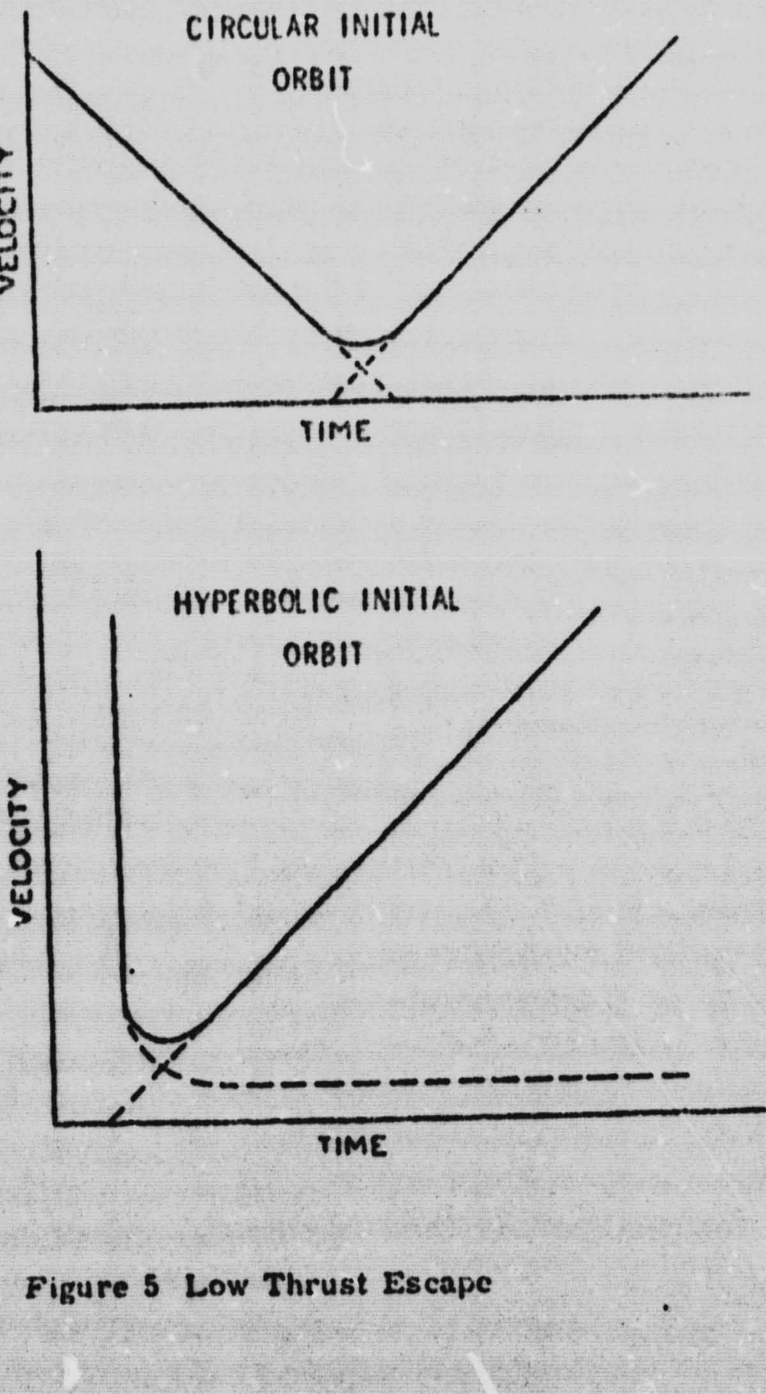


Figure 5 Low Thrust Escape

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