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ON THE EFFECTS OF MONOENERGETIC ELECTRONS IN A STEADY-STATE, LOW PRESSURE PLASMA

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16. Abstract

This paper describes an expansion of the Tonks-Langmuir static free-fall theory of a one-dimensional, beam-generated plasma in a plane diode configuration. An electron density consisting of a superposition of cathode-emitted, collisionally scattered monoenergetic (primary) electrons upon a Maxwellian distribution is considered. Ion production due to both species of electrons is included. Cases are treated in which the primary electron space charge contributes to plasma neutralization as well as ion production. In such cases the potential distribution, ion current density and mean ion energy in the plasma differ from the beam-tree case. Special cases of the derived plasma equation are solved and discussed.

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ON THE EFFECTS OF MONOENERGETIC ELECTRONS IN A STEADY-STATE, LOW PRESSURE PLASMA

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SUMMARY

This paper describes a steady-state free-fall theory of a one-dimensional, low-pressure, beam-generated plasma in a plane diode configuration. The motivation for this work derives from the fact that the low pressure plasma discharge encountered in electron-bombardment thrusters is believed to develop an electron density composed of both monoenergetic and Maxwellian electrons. It is of interest to determine the effects of such a distribution on plasma potential, plasma-sheath boundaries, ion current density, and mean ion energy. For this reason the earlier theory of the positive column introduced by Tonks and Langmuir was expanded to include an electron density consisting of a superposition of cathode emitted, collisionally scattered monoenergetic (primary) electrons upon a Maxwellian distribution. Cases are considered in which the primary electron space charge contributes to plasma neutralization as well as ion production. In such cases, the potential distribution, ion current density, and mean ion energy in the plasma differ from the primary-free case.

The presence of a monoenergetic electron space charge density made the usual Tonks-Langmuir plasma-sheath boundary condition inapplicable. (The Tonks-Langmuir condition is that the reciprocal of the spatial variation of potential is zero at the sheath.) Instead, the Bohm sheath criterion was used to replace the Tonks-Langmuir boundary condition by one that depends on primary electron collision phenomena. Plasma potential depended on the primary electron density, ionization frequency, and to a lesser extent, beam scattering phenomena.

Approximate models of the plasma discharge are considered in which the primary electrons contribute to ionization but not to the space charge density. With the appropriate choice of parameters, potential profiles obtained from the approximate models coincide with profiles from a more complete model over a wide range. The solutions based on the approximate models are more easily obtained and can be used to investigate appropriate regions of low pressure plasmas.

INTRODUCTION

This paper describes a steady-state theory of a beam-generated plasma in which beam electron scattering by collisions is included. The motivation for this work derives from the fact that the low-pressure plasma discharge encountered in electron-bombardment thrusters is believed to develop an electron density composed of both monoenergetic primary electrons and Maxwellian electrons (ref. 1). It is of interest to determine the effects of such a distribution on plasma phenomena.

The analysis considers a one-dimensional configuration associated with a plane parallel diode system. The diode consists of a plasma discharge region between a cathode emitter and an anode. In previous work on beam-generated plasma in a plane symmetric configuration (ref. 2), the electron beam flowed parallel to and between two grounded plane electrodes. These electrons were presumed to suffer no scattering collisions and were the principle source of ionization. The present study considers effects of an electron distribution consisting of cathode-emitted monoenergetic primary electrons superimposed upon a Maxwellian electron distribution. The primaries are allowed to suffer nonionizing as well as ionizing collisions. The details of such collisions need not be known explicitly.

For the case in which primaries contribute to the space charge density in the plasma, a low pressure approximation is used. In essence, this means that the primary electron mean free path λ is of the order of or greater than system dimensions. The primary electrons in the plasma are assumed to follow an exponential spatial distribution $\sim e^{-z/\lambda}$ in their flow from cathode to anode. Ion generation is taken to be proportional to contributions from both primary and Maxwellian electrons.

The theory is based upon an extension of the one-dimensional free-fall theory of Tonks and Langmuir (ref. 3). Starting with the one-dimensional Poisson equation, an integral equation is derived in reference 3, which is presumed to be valid in both the plasma and collisionless sheath regions of the discharge. This equation is the Tonks-Langmuir plasma-sheath equation. Solutions to this equation yield the plasma potential distribution. The original theory, in which second order terms could be neglected, resulted in solutions for the potential and current distributions in plane, cylindrical, and spherically symmetric low-pressure plasma arcs. Interest in this theory was increased by the work of Harrison and Thompson (ref. 4) on a low pressure, plane symmetric discharge. Their paper stimulated further work by several authors (refs. 5 to 9). Recent efforts extend the work of Dunn and Self (ref. 2) to a cylindrically symmetric beam-generated plasma (ref. 10) and provide experimental verification of the collisionless sheath theory (ref. 11).

The present calculations yield information on the effects of different modes of ion generation on the plasma potential distribution and ion current density in the plasma due to the presence of monoenergetic electrons. In addition, the mean ion energy and Bohm sheath criterion are investigated in the context of the theory. The presentation begins with a description of the physical system to be considered. The analysis incorporates an electron distribution consisting of primary beam electrons plus Maxwellian electrons. In the limit of vanishingly small ratio of Debye length to characteristic plasma length, this equation reduces to the so-called plasma equation in which charge neutrality holds. Solutions to this plasma equation are investigated.

PHYSICAL SYSTEM

A schematic diagram of the diode configuration to be studied is given in figure 1. In the figure the plasma potential at potential maximum is assumed zero. Electrons emitted from the cathode gain kinetic energy in the cathode sheath and arrive at the potential maximum with an energy V_c, where V_c is the cathode potential. The dashed line represents the potential variation in the sheath regions. As a result of ion production and randomizing processes, the two kinds of electrons presumed to exist in the plasma discharge region are cathode emitted monoenergetic electrons (primaries) and Maxwellian electrons at uniform temperature T. Primaries suffer collisions (ionization, excitation, scattering, etc.) when they traverse the discharge. Once a primary electron suffers an energy absorbing or scattering collision, it is assumed to be removed from the beam, so that the primaries follow an exponentially decaying spatial distribution.

The plasma discharge occupies the region shown on figure 1. The region considered herein for analysis is between the potential maximum and the anode sheath. (Except for the absence of a magnetic field, this approximates the cathode-anode region in the ion thruster.) In this region the potential is decreasing monotonically towards the anode. Potential defined in this manner is thus negative. A low pressure plasma is assumed, so that plasma ions have an effective collisional mean free path greatly in excess of the cathode-anode spacing. Ion motion in the plasma is determined solely by the electric fields developed from slight deviations from charge neutrality. These fields are assumed to be so small that they provide a negligible perturbation on the primary electron energy. (Conditions for plasma shielding by space charge sheaths have been given by Friedman and Levi (ref. 12).) The anode acts as a charged particle sink, so that no reflection of charged particles back into the discharge plasma occurs.

In this study only the steady-state properties of such discharges are considered. In particular, only the plasma approximation in which the Debye length λ_D is much less than the characteristic plasma length L (proportional to the electron mean free path for ionization) is solved.

THEORY

The starting point for the calculation is the Poisson equation

$$\frac{d^2V}{dz^2} = -\frac{e}{\epsilon_o}(n_i - n_e) \tag{1}$$

(All symbols defined in appendix A. All quantities are in SI units.) Following Tonks and Langmuir, an expression for the ion density at z $(z \geq 0)$ is obtained as follows: Consider all positions $z' < z$ at which ions are created by electron bombardment. It is desired to obtain the ion density at z resulting from ion production at all positions z'. This density is

$$n_i(z) = \int_0^z N_i(z')[v(z, z')]^{-1}\, dz' \tag{2}$$

where $N_i(z')$ is the volume ion production rate, and

$$v(z, z') = \left\{\frac{2}{M} e\left[V(z') - V(z)\right]\right\}^{1/2} \tag{3}$$

is the speed of ions produced at z' arriving at z. The electron density is assumed to be a superposition of Maxwellian and monoenergetic (primary) components (analogous to ref. 2),

$$n_e(z) = n_m(0)\exp\left[\frac{eV(z)}{kT}\right] + n_p(0)\exp(-z/\lambda) \tag{4}$$

where λ is the total collisional mean free path for primary electrons, T is the Maxwellian electron temperature, $n_m(0)$ is the Maxwellian electron density at $z = 0$, and $n_p(0)$ is the primary electron density at $z = 0$. Also note that $V(0) = 0$ by definition. The ion generation is given by

$$N_i(z') = \nu_m n_e^*(z') \tag{5}$$

where

$$n_e^*(z) = n_m(0)\left\{\xi\varphi\exp\left(-\frac{z}{\lambda}\right) + \exp\left[\frac{eV(z)}{kT}\right]\right\} \tag{6}$$

The parameter ν_m is the ionization frequency for Maxwellian electrons, $\xi = \nu_p/\nu_m$ is the ratio of primary electron ionization frequency to Maxwellian electron ionization frequency, and φ is the electron density ratio $n_p(0)/n_m(0)$. For mercury ξ, the ionization frequency ratio, is of the order of 10 to 10^3, but in cesium, ξ is of the order of 10 to 10^2 (ref. 13).

The introduction of the dimensionless variables

$$\eta = -\frac{eV}{kT}$$

$$s = \frac{z}{L} \tag{7}$$

$$s' = \frac{z'}{L}$$

into the Poisson equation (1) results in

$$0 = \left(\frac{\lambda_D}{L}\right)^2 \frac{d^2\eta}{ds^2} + e^{-\eta} + \varphi e^{-\rho s} - \nu_m\left(\frac{M}{2kT}\right)^{1/2} L \int_0^s \frac{n_e^*(s')ds'}{\sqrt{\eta(s) - \eta(s')}} \tag{8}$$

Here λ_D is the Debye length $\left[\epsilon_0 kT/e^2 n_m(0)\right]^{1/2}$, L is a characteristic length to be defined, and ρ is the mean free path ratio L/λ. Since L in equation (8) is an adjustable constant, it is convenient to define it so as to render the coefficient of the integral equal to unity. Thus

$$\frac{1}{L} = \nu_m\left(\frac{M}{2kT}\right)^{1/2} \tag{9}$$

or, alternatively, L can be defined as the reduced mean free path for ionization by Maxwellian electrons

$$L = \lambda_{im}\left(\frac{m}{M}\right)^{1/2} \tag{10}$$

In the form of equation (8), the Poisson equation is the complete Tonks-Langmuir plasma-sheath equation valid throughout the plasma and sheath regions of the discharge. A general closed form solution to this integro-differential equation is not attainable. Numerical solutions for the case $\varphi = 0$ have been given by Self (ref. 6). A general simplification to be used in this study is that within the main body of plasma, $\lambda_D \ll L$ in equation (8). That is, in the plasma discharge region of figure 1, the Debye length is presumed to be much less than the characteristic length L. This leads to the plasma condition

$$n_i \approx n_e \tag{11}$$

and is equivalent to the statement that plasma potential gradient changes little over a Debye length. The complete equation (8) thus reduces to the integral equation

$$e^{-\eta} + \varphi e^{-\rho s} - \int_0^s g(s')[\eta(s) - \eta(s')]^{-1/2}\, ds' = 0 \tag{12}$$

where $g(s)$ is defined as $n_e^*(s)/n_m(0)$. Equation (12) states that both Maxwellian and primary electrons contribute to space charge neutralization as well as ion production in the plasma. Solutions to the plasma equation (12) for the case $\varphi = 0$ have been given in references 2 and 3. Solutions for the case $\rho = 0$, $g(s) = 1$ have been given in reference 2. For the problem considered herein equation (12) was solved on a digital computer in the low pressure approximation $\rho s << 1$. (J. S. Sovey and R. W. Palmer assisted in performing some of the numerical calculations.) In this general low pressure case, the primary and Maxwellian electrons contribute to both the ionization and space charge. In a later section, the limiting case in which the primaries do not contribute to space charge are considered.

GENERAL LOW PRESSURE SOLUTION

Equation (12) can be solved in the low pressure approximation $\rho s << 1$ to yield the plasma potential distribution $\eta = \eta(s)$ (appendix B). A series solution of the form

$$s = \sum_{\mu=1}^{\infty} a_\mu x^\mu \tag{13}$$

was used, where $x = \sqrt{\eta}$. Recurrence relations for the coefficients are

$$a_{2\mu+1} = \left[\frac{\pi}{2}\rho\xi\varphi \sum_{q=0}^{2\mu-1}(q+1)a_{q+1}a_{2\mu-q} + \rho^2\varphi \sum_{q=0}^{2(\mu-1)}\sum_{\nu=0}^{\mu-1}(q+1)B_{\mu-1,\nu}a_{q+1}a_{2\mu-q-1} + \frac{(-1)^\mu 2^\mu}{(2\mu-1)!!} - \frac{\pi}{2}\sum_{\nu=1}^{\mu} C_{\mu\nu}a_{2\mu+1-2\nu} - 2\rho\varphi\sum_{\nu=0}^{\mu-1} B_{\mu-1,\nu}\mu a_{2\mu}\right]\cdot \frac{2}{\pi(2\mu+1)(\xi\varphi+1)} \tag{14}$$

where

$$\mu = 1, 2, \; . \; . \; .$$

with

$$a_1 = \frac{\frac{2}{\pi}(1 + \varphi)}{(\xi\varphi + 1)} \tag{15}$$

and

$$a_{2\mu+2} = \left[\rho^2\varphi \sum_{q=0}^{2\mu-1} (q + 1)A_\mu a_{q+1} a_{2\mu-q} + \rho\xi\varphi \sum_{q=0}^{2\mu} (q + 1)a_{q+1} a_{2\mu+1-q} \right.$$

$$\left. -\rho\varphi A_\mu (2\mu + 1)a_{2\mu+1} - \sum_{\nu=1}^{\mu} D_{\mu\nu} a_{2\mu+2-2\nu}\right] \frac{1}{2(\mu + 1)(\xi\varphi + 1)} \tag{16}$$

where

$$\mu = 1, 2, \; . \; . \; .$$

with

$$a_2 = \frac{\rho\varphi\left(\xi a_1^2 - a_1\right)}{2(\xi\varphi + 1)} \tag{17}$$

In the relations (14) and (16), we have

$$\left.\begin{aligned} A_\mu &= \frac{1}{2^{2\mu}} \frac{(2\mu)!}{(\mu!)^2} \\ B_{\mu\nu} &= \frac{(-1)^\mu}{2^{2\mu}} \left[\frac{(-1)^\nu (2\mu + 1)!}{(2\mu + 1 - 2\nu)(2\mu + 1 - \nu)!\nu!}\right] \\ C_{\mu\nu} &= \frac{(-1)^\nu}{\nu!} (2\mu + 1 - 2\nu) \\ D_{\mu\nu} &= \frac{2(-1)^\nu}{\nu!} (\mu + 1 - \nu) \end{aligned}\right\} \tag{18}$$

Forty terms of the series (13) were computed and found to converge rapidly for $\eta \leq 1.2$ independent of the values of ξ, φ, and ρ used. Several solutions are shown graphically in figure 2 for various values of the parameters. An interesting feature is noted from the curves of figure 2. The Tonks-Langmuir (T-L) boundary condition (which defines the limit of validity of the plasma equation)

$$\frac{ds}{d\eta} = 0 \tag{19}$$

is not satisfied for any $\varphi \neq 0$. In fact, for very small φ ($\varphi < 10^{-3}$) and $\xi\varphi < 10^{-1}$, the solutions approach a limiting curve which closely approximates the $\varphi = 0$ solution for $\eta \lesssim \frac{1}{2}$. The lack of agreement for $\eta > \frac{1}{2}$ implies that the inclusion of a monoenergetic electron space charge density precludes locating the sheath boundary by solving the Tonks-Langmuir plasma equation. The reason for this behavior can be explained as follows. Transforming equation (12) by means of the Schlömilch inversion (appendix B) yields

$$\frac{d\eta}{ds} = \frac{\pi\sqrt{\eta}\,(\xi\varphi e^{-\rho s} + e^{-\eta})}{1 + \varphi - 2\sqrt{\eta}\,e^{-\eta}D(\sqrt{\eta}) - \rho\varphi\mathscr{I}(\eta)} \tag{20}$$

where $D(\sqrt{\eta})$ is the Dawson function. For $\eta = 0.854$, where

$$1 - 2\sqrt{\eta}\,\epsilon^{-\eta}D(\sqrt{\eta}) = 0 \tag{21}$$

equation (20) reduces to

$$\frac{d\eta}{ds} = \frac{\pi\sqrt{\eta}\,(\xi\varphi e^{-\rho s} + e^{-\eta})}{\varphi[1 - \rho\mathscr{I}(\eta)]} \tag{22}$$

Because the denominator of equation (22) does not vanish at $\eta = 0.854$ for nonvanishing φ, the Tonks-Langmuir boundary condition (19) cannot be satisfied.

This is true even if the denominator of equation (22) vanishes for some $\eta > 0.854$ satisfying $\mathscr{I}(\eta) = (1/\rho)$, since then the factor $1 - 2\sqrt{\eta}\,e^{-\eta}D(\sqrt{\eta})$ would not vanish. Thus, as $\eta \rightarrow 0.854$ in the limit $\xi\varphi \rightarrow 0$, the solutions depend on the values of ξ, φ, and ρ and on the nature of the approximations used to evaluate $\mathscr{I}(\eta)$.

In general, the solutions were nearly independent of ρ. Thus, in the plasma, the potential distribution is essentially dependent only on the relative number of primaries entering the discharge and their ionization rate, while depending only slightly on primary electron scattering phenomena.

SPECIAL CASES - NO PRIMARY ELECTRON CONTRIBUTION TO SPACE CHARGE

Both Primaries and Maxwellians Produce Ionization, $\varphi \ll 1$

As shown in appendix B, equation (12) can be transformed by means of the Schlömilch inversion (ref. 14) into a form amenable to computation. In cases for which the primary contribution to space charge neutralization is negligible (i.e., $\varphi \ll 1$), the Schlömilch transformation yields the differential equation

$$\frac{d\eta}{ds} = \frac{\pi\sqrt{\eta}\,(\xi\varphi e^{-\rho s} + e^{-\eta})}{1 - 2\sqrt{\eta}\,e^{-\eta}D(\sqrt{\eta})} \tag{23}$$

The condition (19) is satisfied when the denominator of (23) vanishes. This leads to the usual Tonks-Langmuir value of η as determined by Harrison and Thompson (ref. 4). We thus note that the absence of primary electron space charge allows the Tonks-Langmuir sheath boundary condition to be satisfied independent of the nature of ion generation. Numerical solutions to (23) are plotted in figure 3. They are functions of the product $\xi\varphi$ and, to a lesser extent, of ρ. In fact, at a value of $\xi\varphi \sim 6$, the solutions were found to be independent of ρ over two orders of magnitude variation.

These approximate solutions can be compared to actual solutions to the complete plasma equation (12) in the limits,

$$\xi\varphi \sim 1$$

$$\varphi \ll 1$$

Such a comparison is given in figure 4 in which solutions to equation (23) for $\xi\varphi = 1$ and 10 are compared with the appropriate solutions of equation (12) in which $\varphi = 0.01$ and 0.001. It is apparent that a solution to equation (23) is a better approximation to the actual solution if, in the latter case, $\varphi = 0.001$. Again, the approximate potential profile diverges from the actual profile as the Tonks-Langmuir boundary value of η is approached.

Only Primaries Produce Ionization, $\varphi \ll 1$, $\xi\varphi \gg 1$

Another physically interesting situation results when primaries are assumed to be the only contributors to ionization. The Maxwellian plasma electrons account for the space charge neutralization of the ions. Letting

$$\eta = \beta\sigma \tag{24}$$

where

$$\left.\begin{aligned} \sigma &= -\frac{V}{V_c} \\ \beta &= \frac{eV_c}{kT} \end{aligned}\right\} \tag{25}$$

equation (23) becomes (using $\xi \varphi e^{-\rho s} >> e^{-\eta}$)

$$\frac{d\sigma}{ds} = \frac{\pi\left(\frac{\sigma}{\beta}\right)^{1/2} \xi \varphi e^{-\rho s}}{1 - 2\sqrt{\beta\sigma}\, e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right)} \tag{26}$$

Now $s = z/L$, where $L = \nu_m (2kT/M)^{1/2}$. Introducing a new length $L^* = \nu_p (2eV_c/M)^{1/2}$, a new dimensionless position s^* is defined such that $s = (L^*/L)s^*$. Noting that $\rho = (L/L^*)\rho^*$ and remembering that $\xi = \nu_p/\nu_m$, equation (26) becomes

$$\frac{d\sigma}{ds^*} = \frac{\pi\sqrt{\sigma}\ \varphi e^{-\rho^* s^*}}{1 - 2\sqrt{\beta\sigma}\, e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right)} \tag{27}$$

with solution

$$s^* = \frac{1}{\rho^*} \ln\left[\frac{1}{1 - \frac{2\rho^*}{\pi\sqrt{\beta}\,\varphi}\, e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right)}\right] \tag{28}$$

Again the Tonks-Langmuir sheath boundary condition (19) is satisfied at $\beta\sigma = 0.854$, and the solutions depend only slightly on ρ^*. Since equation (28) describes the potential profile of an approximate model of the plasma discharge in which $\xi\varphi >> 1$ and $\varphi << 1$, the validity of the model can be estimated by comparing these profiles with profiles obtained from solving equation (12). Results of such a comparison are shown in figure 5. Here solutions to equation (27) for values $\varphi = 0.1$ and 0.01 and $L^*/L = 10^{-3}$ are compared with actual potential profiles in which $\xi\varphi = 10$, 100, and 1,000 and $\varphi = 10^{-1}$, 10^{-2}, and 10^{-3}. Agreement between the approximate and exact curves was best for $\varphi = 10^{-3}$, since here the condition $\varphi << 1$ is best satisfied. A solution of equation (23) for $\xi\varphi = 10$ and $\rho = 1$ is also given in figure 5. Comparison of this solution with the curve solution to equation (27), shows that solutions to the two approximate models of the discharge differ for $\xi\varphi = 10$ by about 6.4 percent at the Tonks-Langmuir boundary, $\eta = 0.854$. For values of η less than the Tonks-Langmuir boundary value, the difference is less.

Condition for Existence of Solutions, $\varphi \ll 1$, $\xi\varphi \gg 1$

An interesting feature of equation (28) is that real, stable solutions require that

$$1 - \frac{2\rho^*}{\pi\varphi\sqrt{\beta}} e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right) > 0 \tag{29}$$

This situation leads to conditions on the plasma parameters. From (26) one obtains the condition

$$\beta^{1/2} > \frac{2\rho^*}{\pi\varphi} e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right) \tag{30}$$

which holds for all $\beta\sigma$. In particular, condition (30) must hold for the largest value of $2/\pi\ e^{-\beta\sigma} D\left(\sqrt{\beta\sigma}\right) = 1/2.92$. Thus, when substituting for β and ρ^*, equation (30) becomes

$$\left(\frac{V_c}{T}\right)^{1/2} > \left(\frac{m}{M}\right)^{1/2} \frac{\lambda_{ip}}{\lambda} \frac{1}{2.92\ \varphi} \tag{31}$$

where T is in electron volts. The total primary electron mean free path can be written as

$$\frac{1}{\lambda} = \frac{1}{\lambda_{ip}} + \frac{1}{\lambda'} \tag{32}$$

where $(\lambda')^{-1}$ represents the sum of the reciprocals of mean free paths for all primary collisions other than ionizing. Substitution of (29) into (28) yields

$$\lambda_{ip} < \left[2.92\ \varphi\left(\frac{V_c}{T}\right)^{1/2}\left(\frac{M}{m}\right)^{1/2} - 1\right]\lambda' \tag{33}$$

Relation (33) is essentially two conditions. First, it represents a condition on the ionization mean free path when primary electron collisions other than ionizing collisions occur. Second, physically useful solutions require that

$$\varphi > \frac{1}{2.92}\left(\frac{mT}{MV_c}\right)^{1/2} \tag{34}$$

or, in terms of ion atomic mass A,

$$\varphi > \frac{1}{125.3}\left(\frac{T}{V_c A}\right)^{1/2} \tag{35}$$

Condition (35) holds even if no primary electron collisions other than ionizing collisions occur. Thus, for example, if a mercury plasma is used, relation (35) results in

$$\varphi > 5.65\times10^{-4}\left(\frac{T}{V_c}\right)^{1/2} \tag{36}$$

Relation (33) is thus a lower bound on the allowed value of the primary electron fraction in order that the approximate model of the plasma discharge described by equation (27) be valid. Physically, this means that strict charge neutrality in the plasma places a lower limit on the primary electron fraction required for ion production. An upper bound on φ can be estimated from figure 5. This comparison between the approximate solution given by equation (28) and the actual potential profiles yields, when combined with relation (35),

$$\frac{1}{125.3}\left(\frac{T}{V_c A}\right)^{1/2} < \varphi \lesssim 0.01 \tag{37}$$

The lower bound in relation (37) is the condition for the existence of real solutions to this approximate model. The upper bound represents the limit on φ below which the approximate solution and the exact solution are in close agreement.

ION CURRENT DENSITY

The ion current density in the plasma can be calculated from the one-dimensional continuity equation

$$\frac{dJ_+}{dz} = \nu n_e^* e \tag{38}$$

where ν is the ionization rate and n_e^* is given by equation (6). Equation (38) states that the ion current flux is balanced by the rate of ion production. Introducing the dimensionless variable

$$s = \frac{z}{L}$$

where

$$L = \nu^{-1}\left(\frac{2kT}{M}\right)^{1/2}$$

yields for (38)

$$\frac{dJ_+}{ds} = \left(\frac{2kT}{M}\right)^{1/2} en_e^* \tag{39}$$

Defining the dimensionless current density

$$I_+ = \frac{1}{e}\left(\frac{M}{2kT}\right)^{1/2} \frac{J_+}{n_m(0)} \tag{40}$$

and integrating (39), one obtains

$$I_+ = \int_0^{\eta} \frac{n_e^*(\eta')}{n_m(0)} \frac{ds'}{d\eta'} d\eta' \tag{41}$$

For those cases in which the primary electron space charge contribution can be neglected, $\varphi \ll 1$, equation (41) yields the same dependence of ion flux on plasma potential, independent of ion generation mechanisms.

$$I_+ = \frac{2}{\pi} e^{-\eta} D(\sqrt{\eta}) \tag{42}$$

At the Tonks-Langmuir boundary $\eta_0 = 0.854$, equation (39) results in $I_+ = 0.344$. This lack of dependence of I_+ on the mode of ion generation is due to the nature of the plasma approximation. The electron density sets the ion density so as to preserve charge neutrality in the plasma. Thus, as pointed out by Auer (ref. 7), the electron distribution in the plasma sets the ion current to the sheath. To demonstrate the role of the electron distribution, the ion flux resulting from inclusion of primary electrons in the space charge electron density can be calculated numerically from the relation

$$I_+ = \frac{2\varphi}{\pi}\eta^{1/2} + \frac{2}{\pi}e^{-\eta}D(\sqrt{\eta}) - \frac{\rho\varphi}{\pi}\int_0^{\eta} \mathscr{I}(\eta')d\eta' \tag{43}$$

where

$$\mathscr{I}(\eta') = \int_0^{\eta'} \frac{e^{-\rho s''}}{\sqrt{\eta' - \eta''}} \frac{ds''}{d\eta''} d\eta'' \tag{44}$$

Using a series approximation for $\mathscr{I}(\eta)$, the resulting current density profiles are shown in figure 6. The major difference between equations (42) and (43) was found to be the first term in (43). Variations in the parameters ξ and ρ had only a slight effect on the current density. Inclusion of the first and third terms of equation (43), which are due to nonvanishing φ, increased the current level above that which would be obtained if $\varphi = 0$. However, as in the case of the $\varphi = 0$ curve, a maximum occurs at some value of η. In the case of the $\varphi = 0$ solution, this maximum occurs at the Tonks-Langmuir boundary value of $\eta = 0.854$. If we postulate that the maximum ion current is delivered to the sheath boundary, curves of $I_+(\eta)$ would yield the values of η at such boundaries, and thus serve to spatially locate the plasma-sheath boundary. This postulate is justified physically since the solution for the ion current density including the sheath would have no maximum. These maxima in $I_+(\eta)$ were found to shift to values of $\eta > 0.854$ with increasing φ. Thus at $\varphi = 10^{-3}$ and $\xi = 10^2$, the maximum occurred at $\eta \approx 0.854$, but at $\varphi = 0.1$ and $\xi = 10^2$ the maximum was at $\eta \approx 1.0$.

BOHM SHEATH CRITERION

The stable sheath criterion as originally derived by Bohm (ref. 15) is a statement on the lower limit of the ion energy delivered from the plasma to a sheath boundary.

$$M\left(v_i^2\right) \geq kT \tag{45}$$

where T is the electron temperature in the plasma. The criterion assumes that the electron space charge density is completely thermalized at temperature T. Following Harrison and Thompson (ref. 4), the Poisson equation in the sheath can be written

$$\frac{\epsilon_0 kT}{e^2}\frac{\partial^2\eta}{\partial z^2} = \int f(v)dv - n_e \tag{46}$$

where

$$n_+ = \int f(v)dv$$

and

$$n_e = n_p(0)e^{-\rho s} + n_m(0)e^{-\eta}$$

Assuming no ion generation or ion momentum transfer collisions in the sheath region, ion current continuity requires that

$$vf(v)dv = v_b f(v_b)dv_b \tag{47}$$

where the subscript b denotes sheath boundary. Writing $v = v_s \eta^{1/2}$, where $v_s = (2kT/M)^{1/2}$, the ion density in the sheath region is

$$n_+ = \int \frac{v_b f(v_b)dv_b}{v_s \eta^{1/2}} \tag{48}$$

Let

$$F(\eta) = \frac{1}{v_s} \int v_b \eta^{-1/2} f(v_b)dv_b - n_p(0)e^{-\rho s} - n_m(0)e^{-\eta} \tag{49}$$

Expanding $F(\eta)$ about $\eta = \eta_b$ yields for (46), to first order,

$$\frac{\epsilon_o kT}{e^2} \frac{\partial^2 \Delta\eta}{\partial z^2} = \left\{n_e\left(1 - \frac{kT}{M}\left\langle\frac{1}{v_b^2}\right\rangle\right) + n_p\left[\rho\left(\frac{ds}{d\eta}\right)_b - 1\right]\right\}\Delta\eta \tag{50}$$

where $F(\eta_b) = (n_i - n_e)_b = 0$ in the plasma approximation. Nonoscillatory solutions to equation (50) require that

$$n_e\left(1 - \frac{kT}{M}\langle v_b^{-2}\rangle\right) \geq n_p\left[1 - \rho\left(\frac{ds}{d\eta}\right)_b\right]$$

or

$$M\langle v_b^{-2}\rangle^{-1} \geq kT\left\{1 + \left(\frac{n_p}{n_e}\right)_b\left[\rho\left(\frac{ds}{d\eta}\right)_b - 1\right]\right\}^{-1} \tag{51}$$

where $n_p = n_p(0)e^{-\rho s}$ is the primary electron density and n_e is the total electron or plasma density. Assuming that at the sheath edge

$$\langle v_b^{-2} \rangle^{-1} \rightarrow \langle v_b^2 \rangle$$

one obtains the stable sheath criterion

$$M\langle v_b^2 \rangle \geq kT\left\{1 + \left(\frac{n_p}{n_e}\right)_b \left[\rho\left(\frac{ds}{d\eta}\right)_b - 1\right]\right\}^{-1} \tag{52}$$

If no primary electrons reach the sheath ($n_p = 0$), then one obtains the usual Bohm criterion (45).

It is also of interest to note that in the limit $\rho \rightarrow 0$, the criterion (49) becomes

$$M\langle v_b^2 \rangle > \left(\frac{n_{pl}}{n_m}\right)_b kT \tag{53}$$

where $n_m = n_m(0)e^{-\eta}$ and n_{pl} is the plasma density. This result is identical to an expression used recently by Masek (ref. 16) to describe conditions at the anode sheath in the discharge chamber of bombardment thrusters.

Relation (52) yields a condition that replaces the Tonks-Langmuir boundary condition given by equation (19). If primary electrons are presumed to arrive at the sheath boundary, then in order that the Bohm sheath criterion, given by inequality (45), be satisfied, one has the condition (from eq. (52))

$$\frac{ds}{d\eta} \leq \frac{1}{\rho} \tag{54}$$

at the plasma-sheath boundary. Condition (53) states that the potential gradient at the sheath boundary depends on primary electron collision phenomena. It is a less restrictive condition than the Tonks-Langmuir condition (eq. (19)) in the sense that the precise location of the sheath boundary is not defined. In the limit of very large ρ, relation (54) approaches the Tonks-Langmuir condition.

THE MEAN ION ENERGY

The mean ion energy distribution resulting from a monoenergetic electron density distribution in the plasma can be calculated from the ion distribution function. For the case $\varphi = 0$, the mean square ion speed is given by (ref. 4)

$$\langle v_i^2 \rangle = \frac{kT}{M} (e^{\eta} - 1) \tag{55}$$

This relation is, in general, valid provided that the spatial variation of electron density is given by $n_m(0)e^{-\eta}$. For the case in which n_p, the primary density, does not vanish, the plasma equation (12) can be written

$$n_m(0)e^{-\eta} + n_p(0)e^{-\rho s} - \left(\frac{M}{2kT}\right)^{1/2} L \int_0^{\eta} g(\eta')(\eta - \eta')^{-1/2} \frac{ds'}{d\eta'} d\eta' = 0 \tag{56}$$

where

$$g(\eta') = \nu_m n_m(0)[\xi \varphi e^{-\rho s} + e^{-\eta}]$$

The ion distribution function is given by

$$f(v)dv = L\left(\frac{M}{2kT}\right)^{1/2} g(\eta')(\eta - \eta')^{-1/2} \frac{ds'}{d\eta'} d\eta' \tag{57}$$

A Schlömilch transformation of equation (56) allows equation (57) to be written as

$$f(v)dv = \frac{n_m(0)}{\pi}\left(\frac{1 + \varphi}{\sqrt{(\eta - \eta')\eta'}} - \frac{2e^{-\eta'}D(\sqrt{\eta'})}{\sqrt{\eta - \eta'}} - \rho\varphi \frac{\mathscr{I}(\eta')}{\sqrt{\eta - \eta'}}\right)d\eta' \tag{58}$$

Noting that

$$\int f(v)dv = n_i$$

one obtains

$$n_i = n_m(0)(1 + \varphi) - \frac{n_m(0)}{\pi}\left[2\int_0^{\eta} \frac{e^{-\eta'}D(\sqrt{\eta'})}{\sqrt{\eta - \eta'}} + \rho\varphi \int_0^{\eta} \frac{\mathscr{I}(\eta')d\eta'}{\sqrt{\eta - \eta'}}\right] \tag{59}$$

The mean square ion speed is given by

$$\langle v_i^2 \rangle = \frac{1}{n_i} \int v^2 f(v)dv \tag{60}$$

where

$$v^2 = \left(\frac{2kT}{M}\right)(\eta - \eta') \tag{61}$$

Substituting (58) and (61) into (60) yields the mean square speed as a function of η,

$$\langle v_i^2 \rangle = \frac{n_m(0)}{\pi n_i}\left(\frac{2kT}{M}\right)\left[\int_0^\eta \frac{e^{-\eta'}D(\sqrt{\eta'})d\eta'}{\sqrt{\eta - \eta'}} + \frac{\pi}{2}\eta\varphi - \rho\varphi \int_0^\eta \sqrt{\eta - \eta'}\,\mathscr{I}(\eta')d\eta'\right] \tag{62}$$

In appendix C, it is shown that the last integrations in equations (59) and (62) can be reduced, so that equation (59) becomes

$$n_i = n_m(0)(1 + \varphi) - \frac{n_m(0)}{\pi}\left[2\int_0^\eta \frac{e^{-\eta'}D(\sqrt{\eta'})d\eta'}{\sqrt{\eta - \eta'}} + \pi\varphi(1 - e^{-\rho s})\right] \tag{63}$$

and equation (62) becomes

$$\langle v_i^2 \rangle = \frac{n_m(0)}{\pi n_i}\left(\frac{2kT}{M}\right)\left[\int_0^\eta \frac{e^{-\eta'}D(\sqrt{\eta'})d\eta'}{\sqrt{\eta - \eta'}} + \frac{\pi}{2}\varphi \int_0^\eta e^{-\rho s'}\,d\eta'\right] \tag{64}$$

Using equation (63) to eliminate the first integral in equation (62) and noting that $n_i = n_m(0)(\varphi e^{-\rho s} + e^{-\eta})$ yields, finally,

$$\langle v_i^2 \rangle = \frac{kT}{M}\left[\frac{1 + \varphi\left\{1 + \int_0^\eta e^{-\rho s'}\,d\eta' - (1 - e^{-\rho s})\right\}}{\varphi e^{-\rho s} + e^{-\eta}} - 1\right] \tag{65}$$

For purposes of calculation, this result can be simplified in the low-pressure approximation $\rho s << 1$ by using the substitution $e^{-\rho s} \approx 1$, so that equation (65) becomes

$$\langle v_i^2 \rangle \approx \frac{kT}{M}\left[\frac{1 + \varphi(1 + \eta)}{\varphi e^\eta + 1}e^\eta - 1\right] \tag{66}$$

In the limit $\varphi \rightarrow 0$, equations (65) and (66) reduce to equation (55). Equation (66) is plotted in figure 7 for several values of φ, including $\varphi = 0$. It is seen that the mean square ion speed (proportional to mean ion energy) decreases with increasing φ. In the limit of very large φ, equation (66) yields the result that

$$\frac{M}{kT} \langle v_i^2 \rangle \approx \eta \tag{67}$$

Thus, for large φ the mean ion energy $\langle E_i \rangle = \frac{1}{2} M \langle v_i^2 \rangle$ is given by

$$\langle E_i \rangle \approx -\frac{1}{2} eV_{plasma} \tag{68}$$

where V_{plasma}, the plasma potential, is negative. Physically, relation (68) means that the average ion energy at any point in the plasma is given by the average potential between $z = 0$ and that point. A large primary electron fraction means that ion production is more uniformly distributed throughout the plasma than for the $\varphi = 0$ case. Because ions are produced with a negligibly small initial kinetic energy, the mean ion energy at any point in the discharge would be less than that for the $\varphi = 0$ case, in which ion production is greatest near the $\eta = 0$ point (where $n_e = n_m(0)$), and ions gain kinetic as they fall through potential η.

Also evident from figure 7 is that the Bohm criterion (relation (45)) is satisfied at values of η increasing with increasing φ. For very large values of φ, relation (45) requires that, at a plasma-sheath boundary, $\eta \geq 1$. Thus comparing such calculations of the mean ion energy with calculations of the maximum ion current, I_+, could serve to yield the plasma-sheath boundary value of η for values of $\varphi > 0$.

PLASMA DENSITY PROFILES

Often it is the plasma electron density that is experimentally determined from diagnostic techniques such as plasma oscillation measurements (refs. 17 and 18). In the present problem, density profiles can be obtained from equation (4) and the curves of figure 2. Figure 8 gives the plasma density (normalized to the Maxwellian electron density at zero potential) as a function of η and s for two values of φ. The value of s is normalized by the K-factor, where $K = 100$ for $\varphi = 0.01$ and $K = 10$ for $\varphi = 10^{-3}$. The curves show an expected decline in plasma density with increasing η and s.

CONCLUSIONS

The results of including primary electrons in the plasma electron density in a one-dimensional, low pressure plasma are as follows:

1. The potential distribution was found to depend on ion generation mechanisms in the plasma, the ratio of primary electron to Maxwellian electron densities at the potential maximum, and the ratio of primary to Maxwellian electron ionization frequencies.

2. With the inclusion of primary electron space-charge density, the Tonks-Langmuir boundary condition was not satisfied for any value of the primary electron to Maxwellian electron ratio. An examination of the Bohm sheath criterion led to the replacement of the Tonks-Langmuir boundary condition by one relating the plasma potential gradient to the primary electron collision mean free path.

3. The effect of primary electron scattering was much less pronounced and often negligible over a two order of magnitude variation in the ratio of characteristic plasma length (proportional to ionization mean free path) to beam electron total collision mean free path.

4. Two approximate models of the plasma were considered to correspond to the condition of negligible primary electron fraction. The plasma potential solutions obtained for these models coincided with the exact solution except at values of potential near the Tonks-Langmuir boundary value. For the case in which the product of the two ratios of primary electron density to Maxwellian electron density and primary to Maxwellian electron ionization frequencies was very large, upper and lower bounds in the beam fraction were obtained. The lower bound is essentially an existence condition, whereas the upper bound was obtained by matching the approximate solution to the exact solution.

5. The ion current density distribution in the plasma was also affected by the presence of primary electron space charge. The deviation from the case of no primary electrons depended primarily on the primary fraction. In the absence of a primary electron space charge contribution the variation of ion current density with plasma potential was independent of ion generation mechanisms. The variations of the ion current density with plasma potential exhibited maxima for different values of primary electron fraction. By analogy to the zero beam solution, these maxima could serve to define a plasma-sheath boundary, assuming that the maximum ion current is delivered to the boundary.

6. The mean ion energy in the plasma was found to be modified by the presence of primary electrons such that the mean ion energy at a given value of potential was less than the case of zero primary electrons. This result could be understood from the fact that the presence of primary electrons yields a more uniform ion production throughout the plasma.

It is of interest to conjecture on the applicability of the preceding analysis to the discharge plasma of bombardment thrusters. It is apparent that primary electrons in the discharge play an important role in determining plasma behavior. For example, energy losses due to ions dissipating their energy in wall collisions could be minimized in the presence of a high primary fraction, because the mean ion energy would be lower (see (6)) and ion production would be more uniformly distributed throughout the discharge. It is obvious, however, that the analysis is a gross simplification of the actual discharge environment of such thrusters. If possible, a more refined analysis should include the effect of a magnetic field on the electron distribution.

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120-26.

APPENDIX A

SYMBOLS

A	ion mass, atomic mass units
$A_{\mu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}$	coefficients defined by relations (18)
a	coefficient in series expansion, equation (13)
$D(x)$	the Dawson function, $\int_0^x e^{x^2}\,dx$
d	modified Debye length (eq. (21))
E_i	ion kinetic energy
e	electron charge
$g(s)$	the ratio $n_e^*(s)/n_m(0)$
I_+	normalized ion current density (eq. (36))
$\mathcal{J}(\eta)$	integral defined by eq. (37)
J_+	ion current density
K	normalization factor in fig. 8
k	Boltzmann constant
L	plasma characteristic length, $\nu_m\left(\frac{2kT}{M}\right)^{1/2}$
L^*	plasma characteristic length, $\nu_p\left(\frac{2eV_c}{M}\right)^{1/2}$
M	ion mass
m	electron mass
N_i	volume ion production rate
n_e	electron density
n_e^*	defined by eq. (6)
n_i	ion density
$n_m(0)$	Maxwellian electron density at $z = 0$
n_p	primary electron density

n_{pl}	total electron or plasma density
$n_p(0)$	primary electron density at $z = 0$
s, s'	dimensionless position variables defined in eq. (7)
s^*	dimensionless position variable defined in eq. (28)
T	plasma electron temperature
V	plasma potential
V_c	cathode sheath drop, V
$\langle v_i^2 \rangle$	mean square ion speed
z, z'	position variables
β	eV_c/kT
ϵ_o	free space permittivity
η	dimensionless potential variable defined in eq. (7)
η_o	value of η at Tonks-Langmuir boundary
θ	angle parameter used in Schlömilch transformation of equation (12)
λ	total primary electron collisional mean free path
λ_D	Debye length, $\left[\epsilon_o kT/\epsilon^2 n_m(0)\right]^{1/2}$
λ_{im}	Maxwell electron ionization mean free path
λ_{ip}	primary electron ionization mean free path in eq. (21)
μ, ν	summation indices
$\nu_{m,p}$	Maxwellian, primary electron ionization rates, sec^{-1}
ξ	the ratio ν_p/ν_m
ρ, ρ^*	the ratio L/λ and L^*/λ
σ	dimensionless variable, $-V/V_c$
φ	the ratio $n_p(0)/n_m(0)$

APPENDIX B

SERIES SOLUTION TO THE PLASMA EQUATION (12)

The equation to be solved is

$$0 = \int_0^{\eta} \frac{\xi \varphi e^{-\rho s'} + e^{-\eta}}{\sqrt{\eta(s) - \eta'(s')}} ds' - e^{-\eta} - \varphi e^{-\rho s} \tag{B1}$$

Let

$$\eta = x^2, \ \eta' = x^2 \sin^2\theta = x'^2$$

Then

$$\sqrt{\eta - \eta'} = x \cos \theta = \frac{dx'}{d\theta}$$

and equation (B1) transforms to

$$0 = \int_0^{\pi/2} \left(\xi \varphi e^{-\rho s'} + e^{-x'^2}\right) \frac{ds'}{dx'} d\theta - e^{-x^2} - \varphi e^{-\rho s(x)} \tag{B2}$$

The Schlömilch inversion formulas (ref. 14) can now be applied.

$$f(x) = e^{-x^2} + \varphi e^{-\rho s} \tag{B3}$$

$$\frac{df(x)}{dx} = -2xe^{-x^2} - \rho \varphi \frac{ds}{dx} e^{-\rho s} \tag{B4}$$

$$h(x') = \frac{\pi}{2} \left(\xi \varphi e^{-\rho s'} + e^{-x'^2}\right) \frac{ds'}{dx'} \tag{B5}$$

We note that

$$f(x) = \frac{2}{\pi} \int_0^{\pi/2} h(x')d\theta \tag{B6}$$

where

$$h(x) = f(0) + x \int_0^{\pi/2} f'(x \sin \theta) d\theta \tag{B7}$$

where the prime on f denotes differentiation with respect to its argument. Substituting (B3) to (B5) into (B7) yields

$$\frac{\pi}{2}\left(\xi \varphi e^{-\rho s} + e^{-x^2}\right)\frac{ds}{dx} = 1 + \varphi - 2x \int_0^{\pi/2} e^{-x^2 \sin^2\theta}\, x \sin \theta \, d\theta$$

$$- \rho \varphi x \int_0^{\pi/2} e^{-\rho s'} \frac{ds'}{dx'} d\theta \tag{B8}$$

Using the identity $\sin^2 = 1 - \cos^2$ allows equation (B8) to be written as

$$\rho \varphi x \int_0^{\pi/2} e^{-\rho s'} \frac{ds'}{dx'} d\theta = 1 + \varphi - 2xe^{-x^2} D(x) - \frac{\pi}{2}\left(\xi \varphi e^{-\rho s} + e^{-x^2}\right)\frac{ds}{dx} \tag{B9}$$

Equation (B9) is thus the transformed plasma equation (12) of the text.

To equation (B9) the series

$$s = \sum_{n=1}^{\infty} a_n x^n \tag{B10}$$

is applied. The factor $e^{-\rho s}$ requires an approximation. For low pressure plasmas $\rho s \ll 1$, so that $e^{-\rho s} \approx 1 - \rho s$. Then

$$e^{-\rho s} \frac{ds}{dx} \approx \left(1 - \rho \sum_{k=1}^{\infty} a_k x^k\right)\left[\sum_{q=0}^{\infty} (q + 1)a_{q+1} x^q\right] \tag{B11}$$

and

$$\int_0^{\pi/2} e^{-\rho s'} \frac{ds'}{dx'} d\theta \approx \sum_{q=0}^{\infty} (q + 1)a_{q+1} x^q \int_0^{\pi/2} \sin^q\theta \, d\theta$$

$$- \rho \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} (q + 1)a_k a_{q+1} x^k x^q \int_0^{\pi/2} \sin^{k+q}\theta \, d\theta \tag{B12}$$

In order to evaluate the integrals on the right side of equation (B12), the sums need to be separated into even and odd powers of x.

$$\Sigma_1 = \sum_{\mu=0}^{\infty} (2\mu + 1)a_{2\mu+1}x^{2\mu} \int_0^{\pi/2} \sin^{2\mu}\theta \, d\theta$$

$$+ 2 \sum_{\mu=0}^{\infty} (\mu + 1)a_{2\mu+2}x^{2\mu+2} \int_0^{\pi/2} \sin^{2\mu+1}\theta \, d\theta \tag{B13}$$

From tables (ref. 19)

$$\int_0^{\pi/2} \sin^{2\mu}\theta \, d\theta = \frac{1}{2^{2\mu}} \binom{2\mu}{\mu} \frac{\pi}{2} \tag{B14}$$

and

$$\int_0^{\pi/2} \sin^{2\mu+1}\theta \, d\theta = \frac{1}{2^{2\mu}} (-1)^{\mu} \sum_{\nu=0}^{\mu} \frac{(-1)^{\nu}}{2\mu + 1 - 2\nu} \binom{2\mu + 1}{\nu} \tag{B15}$$

so that

$$\Sigma_1 = \sum_{\mu=0}^{\infty} \frac{(2\mu + 1)}{2^{2\mu}} a_{2\mu+1} \left[\frac{\pi}{2} \binom{2\mu}{\mu}\right] x^{2\mu}$$

$$+ 2 \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{2^{2\mu}} (\mu + 1)a_{2\mu+2} \left[\sum_{\nu=0}^{\mu} (-1)^{\nu} \frac{(2\mu + 1)!}{(2\mu + 1 - 2\nu)(2\mu + 1 - \nu)!\nu!}\right] x^{2\mu+1} \tag{B16}$$

For the second sum,

$$\Sigma_2 = \rho \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} (q + 1)a_k a_{q+1} x^{k+q} \int_0^{\pi/2} \sin^{k+q}\theta \, d\theta \tag{B17}$$

One must account separately for $k + q$ even and odd.

$$\Sigma_2 = \rho \sum_{\mu=1}^{\infty} \sum_{q=0}^{2\mu-1} (q + 1)a_{q+1}a_{2\mu-q}x^{2\mu} \int_0^{\pi/2} \sin^{2\mu}\theta \, d\theta$$

$$+ \rho \sum_{\mu=0}^{\infty} \sum_{q=0}^{2\mu} (q + 1)a_{q+1}a_{2\mu+1-q}x^{2\mu+1} \int_0^{\pi/2} \sin^{2\mu+1}\theta \, d\theta \qquad \text{(B18)}$$

Substituting equation (B14) and (B15) into (B18) yields the series form for Σ_2. Thus, the integral (eq. (B12)) can be written

$$\int_0^{\pi/2} e^{-\rho s'} \frac{ds'}{dx'} d\theta \approx \frac{\pi}{2} \sum_{\mu=0}^{\infty} (2\mu + 1)A_\mu a_{2\mu+1}x^{2\mu} + 2 \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} B_{\mu\nu}(\mu + 1)a_{2\mu+2}x^{2\mu+1}$$

$$- \frac{\pi}{2} \rho \sum_{\mu=1}^{\infty} \sum_{q=0}^{2\mu-1} (q + 1)A_\mu a_{q+1}a_{2\mu-q}x^{2\mu}$$

$$- \rho \sum_{\mu=0}^{\infty} \sum_{q=0}^{2\mu} \sum_{\nu=0}^{\mu} (q + 1)B_{\mu\nu}a_{q+1}a_{2\mu+1-q}x^{2\mu+1} \qquad \text{(B19)}$$

where

$$A_\mu = \frac{1}{2^{2\mu}} \frac{(2\mu)!}{(\mu!)^2} \qquad \text{(B20)}$$

and

$$B_{\mu\nu} = \frac{(-1)^\mu}{2^{2\mu}} \left[\frac{(-1)^\nu(2\mu + 1)!}{(2\mu + 1 - 2\nu)(2\mu + 1 - \nu)!\nu!}\right] \qquad \text{(B21)}$$

The series expansions of other terms in equation (B9) are

$$e^{-\rho s} \frac{ds}{dx} \approx \sum_{\mu=0}^{\infty} (2\mu + 1)a_{2\mu+1}x^{2\mu} + 2 \sum_{\mu=0}^{\infty} (\mu + 1)a_{2\mu+2}x^{2\mu+1}$$

$$- \rho \sum_{\mu=1}^{\infty} \sum_{q=0}^{2\mu-1} (q + 1)a_{q+1}a_{2\mu-q}x^{2\mu} - \rho \sum_{\mu=0}^{\infty} \sum_{q=0}^{2\mu} (q + 1)a_{q+1}a_{2\mu+1-q}x^{2\mu+1} \qquad \text{(B22)}$$

$$2x\epsilon^{-x^2} D(x) = - \sum_{\mu=1}^{\infty} \frac{(-1)^{\mu} 2^{\mu} x^{2\mu}}{(2\mu - 1)!!} \tag{B23}$$

$$e^{-x^2} \frac{ds}{dx} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} C_{\mu\nu} a_{2\mu+1-2\nu} x^{2\mu} + \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} D_{\mu\nu} a_{2\mu+2-2\nu} x^{2\mu+1} \tag{B24}$$

where

$$C_{\mu\nu} = \frac{(-1)^{\nu}}{\nu!} (2\mu + 1 - 2\nu) \tag{B25}$$

and

$$D_{\mu\nu} = 2 \frac{(-1)^{\nu}}{\nu!} (\mu + 1 - \nu) \tag{B26}$$

Now, we substitute equations (B19) and (B22) to (B24) into (B9) to obtain the series representation of the plasma equation.

$$\begin{aligned} &\frac{\pi}{2} \rho\varphi \sum_{\mu=0}^{\infty} A_{\mu} (2\mu + 1) a_{2\mu+1} x^{2\mu+1} + 2\rho\varphi \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} B_{\mu\nu} (\mu + 1) a_{2\mu+2} x^{2\mu+2} \\ &\quad - \frac{\pi}{2} \rho^2 \varphi \sum_{\mu=1}^{\infty} \sum_{q=0}^{2\mu-1} (q + 1) A_{\mu} a_{q+1} a_{2\mu-q} x^{2\mu+1} - \rho^2 \varphi \sum_{\mu=0}^{\infty} \sum_{q=0}^{2\mu} \sum_{\nu=0}^{\mu} (q + 1) B_{\mu\nu} a_{q+1} a_{2\mu+1-q} x^{2\mu+2} \\ &\quad - \sum_{\mu=1}^{\infty} \frac{(-1)^{\mu} 2^{\mu} x^{2\mu}}{(2\mu - 1)!!} + \frac{\pi}{2} \xi\varphi \sum_{\mu=0}^{\infty} (2\mu + 1) a_{2\mu+1} x^{2\mu} + \pi\xi\varphi \sum_{\mu=0}^{\infty} (\mu + 1) a_{2\mu+2} x^{2\mu+1} \\ &\quad - \frac{\pi}{2} \rho\xi\varphi \sum_{\mu=1}^{\infty} \sum_{q=0}^{2\mu-1} (q + 1) a_{q+1} a_{2\mu-q} x^{2\mu} - \frac{\pi}{2} \rho\xi\varphi \sum_{\mu=0}^{\infty} \sum_{q=0}^{2\mu} (q + 1) a_{q+1} a_{2\mu+1-q} x^{2\mu+1} \\ &\quad + \frac{\pi}{2} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} C_{\mu\nu} a_{2\mu+1-2\nu} x^{2\mu} + \frac{\pi}{2} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} D_{\mu\nu} a_{2\mu+2-2\nu} x^{2\mu+1} - 1 - \varphi = 0 \end{aligned} \tag{B27}$$

The expansion (eq. (B27)) is then used to obtain the recurrence relations in the coefficients a_n of the series (eq. (B10)).

For $\mu = 0$, one obtains

$$\frac{\pi}{2} \xi \varphi a_1 + \frac{\pi}{2} C_{00} a_1 = 1 + \varphi \tag{B28}$$

and since $C_{00} = 1$

$$a_1 = \frac{\frac{2}{\pi}(1 + \varphi)}{\xi \varphi + 1} \tag{B29}$$

Also, from the coefficient of x^1,

$$\frac{\pi \rho \varphi}{2} A_0 a_1 + \pi \xi \varphi a_2 - \frac{\pi}{2} \rho \xi \varphi a_1^2 + \frac{\pi}{2} D_{00} a_2 = 0 \tag{B30}$$

so that

$$a_2 = \frac{\rho \varphi (\xi a_1^2 - a_1)}{2(\xi \varphi + 1)} \tag{B31}$$

For $\mu > 0$, one obtains

$$a_{2\mu+2} = \frac{1}{2(\mu + 1)(\xi \varphi + 1)} \left[\rho^2 \varphi \sum_{q=0}^{2\mu-1} (q + 1) A_\mu a_{q+1} a_{2\mu-q} + \rho \xi \varphi \sum_{q=0}^{2\mu} (q + 1) a_{q+1} a_{2\mu+1-q} \right.$$

$$\left. - \rho \varphi A_\mu (2\mu + 1) a_{2\mu+1} - \sum_{\nu=1}^{\mu} D_{\mu\nu} a_{2\mu+2-2\nu} \right] \tag{B32}$$

For the coefficients of odd powers of x, and from even powers of x

$$a_{2\mu+1} = \left[\frac{\pi}{2} \rho \xi \varphi \sum_{q=0}^{2\mu-1} (q + 1) a_{q+1} a_{2\mu-q} + \rho^2 \varphi \sum_{q=0}^{2(\mu-1)} \sum_{\nu=0}^{\mu-1} (q + 1) B_{\mu-1,\nu} a_{q+1} a_{2\mu-q-1} \right.$$

$$\left. + \frac{(-1)^\mu 2^\mu}{(2\mu - 1)!!} - \frac{\pi}{2} \sum_{\nu=1}^{\mu} C_{\mu\nu} a_{2\mu+1-2\nu} - 2\rho\varphi \sum_{\nu=0}^{\mu-1} \mu B_{\mu-1,\nu} a_{2\mu} \right] \frac{2}{\pi(2\mu + 1)(\xi \varphi + 1)} \tag{B33}$$

Using equations (B29) and (B31) to (B33) the series (B10) was computed for 40 terms. The parameter ξ was varied from 0.1 to 100, φ was varied from 0.00016 to 0.5, and ρ was varied from 0.01 to 1.0. The convergence of the series was quite rapid for $\mu \geq 10$ and $\eta \lesssim 1.21$.

APPENDIX C

REDUCTION OF CERTAIN INTEGRALS

In the calculation of mean ion energy in the plasma, two integrals occur that require special treatment. Since the procedure for reducing these integrals is identical, a detailed treatment will be given only for one of the integrals.

Consider the integral occurring in equation (56)

$$\int_0^{\eta} \frac{\mathscr{f}(\eta')d\eta'}{\sqrt{\eta - \eta'}} = \int_0^{\eta} \int_0^{\eta'} \frac{f(\eta'')d\eta''\,d\eta'}{\sqrt{\eta - \eta'}\sqrt{\eta' - \eta''}} \tag{C1}$$

where

$$f(\eta) = e^{-\rho s} \frac{ds}{d\eta} \tag{C2}$$

In order to reverse the order of integration, we introduce the step function

$$\Delta(\eta' - \eta'') = \begin{cases} 0 & \text{for} \quad \eta'' > \eta' \\ 1 & \text{for} \quad \eta'' < \eta' \end{cases} \tag{C3}$$

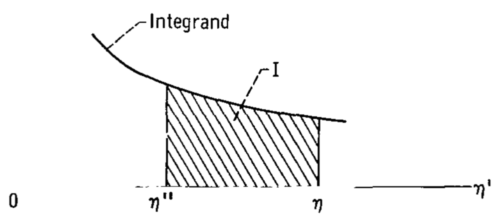
to eliminate the η' upper limit in equation (C1). Thus

$$\int_0^{\eta} \int_0^{\infty} \frac{\Delta(\eta' - \eta'')f(\eta'')d\eta''\,d\eta'}{\sqrt{\eta - \eta'}\sqrt{\eta' - \eta''}} = \int_0^{\eta} \frac{\mathscr{f}(\eta')d\eta'}{\sqrt{\eta - \eta'}} \tag{C4}$$

Let

$$I = \int_0^{\eta} \frac{\Delta(\eta' - \eta'')d\eta'}{\sqrt{\eta - \eta'}\sqrt{\eta' - \eta''}} \tag{C5}$$

The following sketch can be used to evaluate equation (C5).



For $\eta'' < \eta$, equation (C5) yields

$$I = 2 \tan^{-1} \sqrt{\frac{\eta - \eta'}{\eta' - \eta''}} \Bigg|_{\eta''}^{\eta} = -\pi$$

For $\eta'' > \eta$, it is apparent that $I = 0$. Thus, equation (C4) becomes

$$\int_0^{\eta} \frac{\mathscr{f}(\eta')d\eta'}{\sqrt{\eta - \eta'}} = -\pi \int_0^{\eta} f(\eta'')d\eta'' \tag{C6}$$

From equation (C2), we can write

$$f(\eta) = -\frac{1}{\rho}\frac{d}{d\eta}\left(e^{-\rho s}\right) \tag{C7}$$

so that

$$\int_0^{\eta} \frac{\mathscr{f}(\eta')d\eta'}{\sqrt{\eta - \eta'}} = \frac{\pi}{\rho}\left(1 - e^{-\rho s}\right) \tag{C8}$$

The second integral of interest is, from equation (60),

$$\int_0^{\eta} \sqrt{\eta - \eta'}\, \mathscr{f}(\eta')d\eta' = \int_0^{\eta}\int_0^{\eta'} \left(\frac{\eta - \eta'}{\eta' - \eta''}\right)^{1/2} f(\eta'')d\eta''\, d\eta' \tag{C9}$$

Again, using the step function (C3) to replace the η' upper limit by ∞ allows reversal of the order of integration. In this case

$$I = \int_0^{\eta} \Delta(\eta' - \eta'')\left(\frac{\eta - \eta'}{\eta' - \eta''}\right)^{1/2} d\eta' \tag{C10}$$

so that for $\eta'' < \eta$

$$I = (\eta - \eta'')\frac{\pi}{2} \tag{C11}$$

and, again, $I = 0$ for $\eta'' > \eta$. In this way the integral (eq. (C9)) becomes

$$\int_0^{\eta} \sqrt{\eta - \eta'}\; \mathcal{f}(\eta')d\eta' = \frac{\pi}{2}\int_0^{\eta} (\eta - \eta'')f(\eta'')d\eta'' \tag{C12}$$

Integration by parts of the right side of equation (C12) yields

$$\int_0^{\eta} \sqrt{\eta - \eta'}\; \mathcal{f}(\eta')d\eta' = \frac{\pi}{2\rho}\left[\eta - \int_0^{\eta} e^{-\rho s'}\, d\eta'\right] \tag{C13}$$

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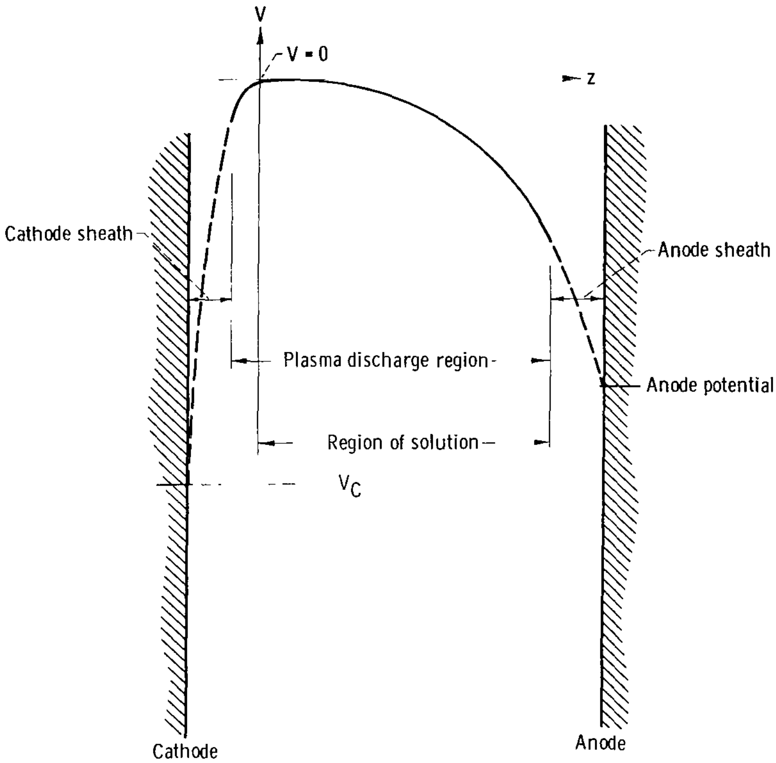
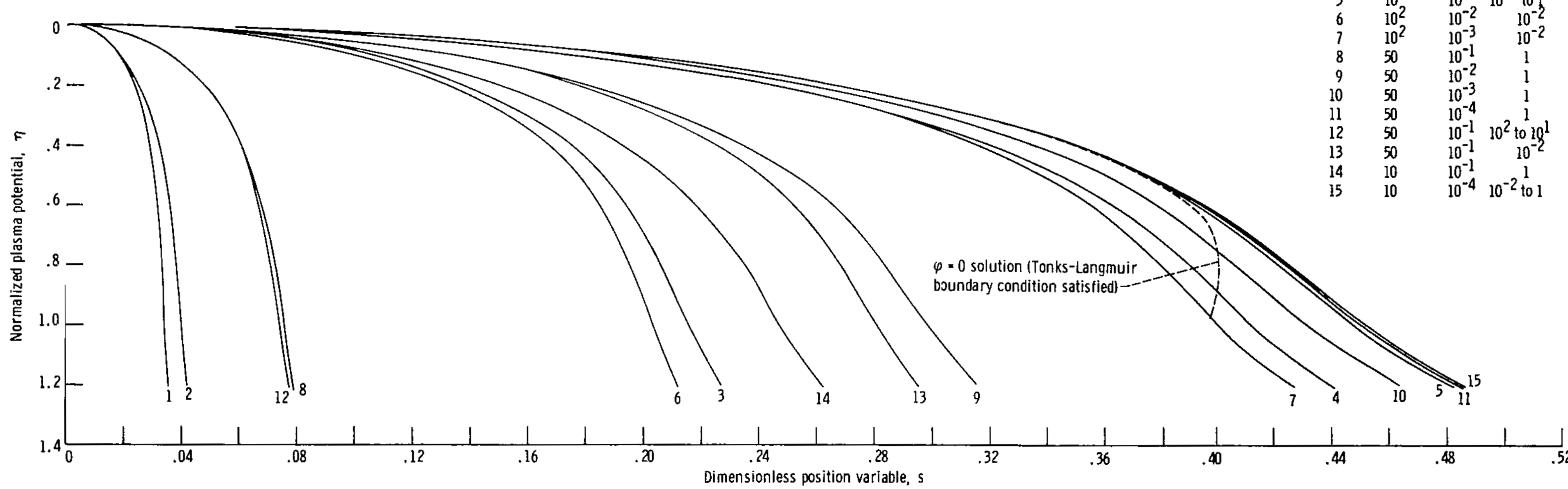


Figure 1. - Diode configuration and plasma potential distribution.



Curve	Ionization frequency ratio,	Electron density ratio, φ	Mean free path ratio, ρ
1	10^4	10^{-3}	1
2	10^2	10^{-1}	1
3	10^2	10^{-1}	1
4	10^2	10^{-3}	1
5	10^2	10^{-4}	10^{-2} to 1
6	10^2	10^{-2}	10^{-2}
7	10^2	10^{-3}	10^{-2}
8	50	10^{-1}	1
9	50	10^{-2}	1
10	50	10^{-3}	1
11	50	10^{-4}	1
12	50	10^{-1}	10^2 to 10^1
13	50	10^{-1}	10^{-2}
14	10	10^{-1}	1
15	10	10^{-4}	10^{-2} to 1

Figure 2. - Plasma potential profiles. Solutions to exact plasma equation (12).

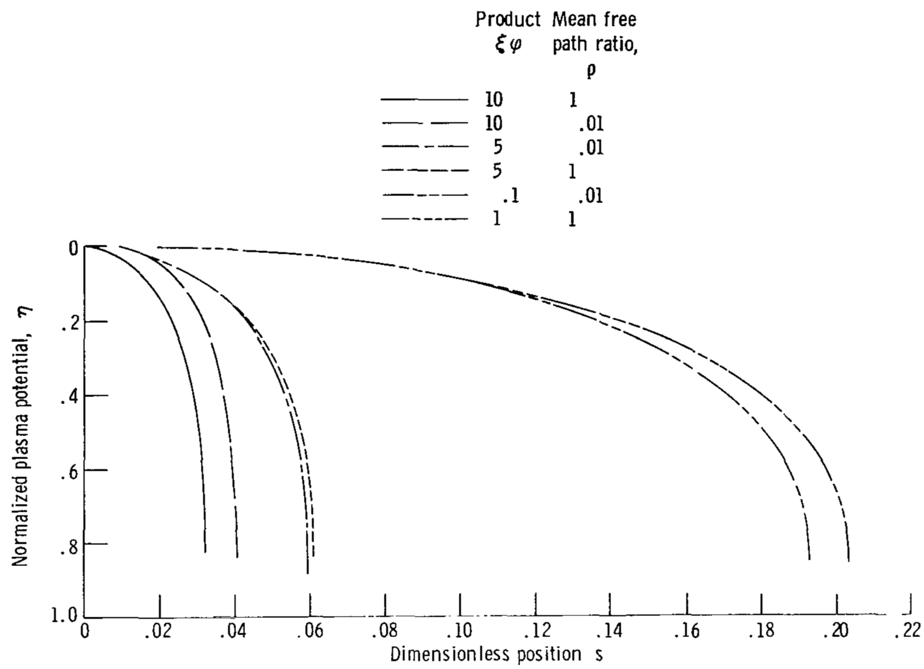


Figure 3. - Solutions to equation (23).

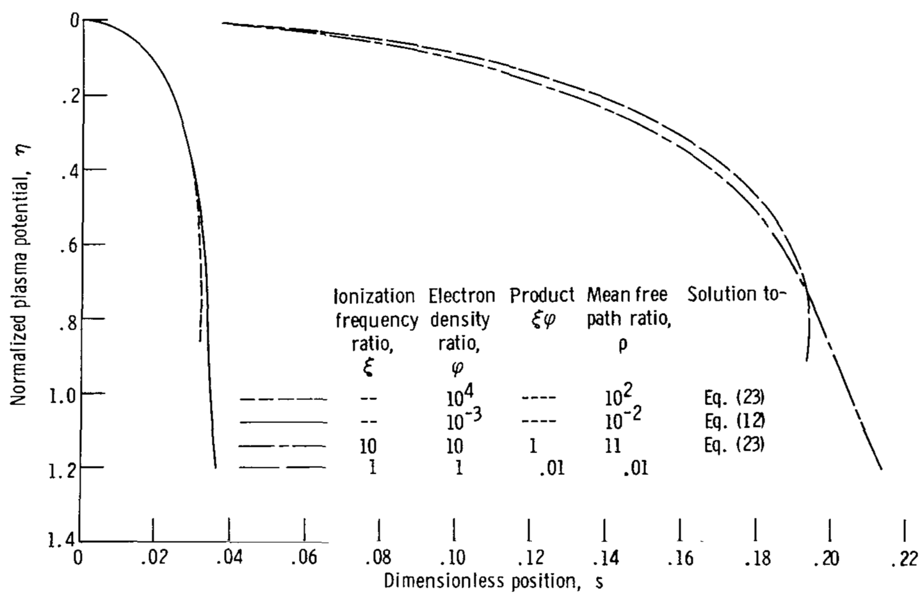


Figure 4. - Comparison of exact solutions to approximate solutions $\varphi \ll 1$.

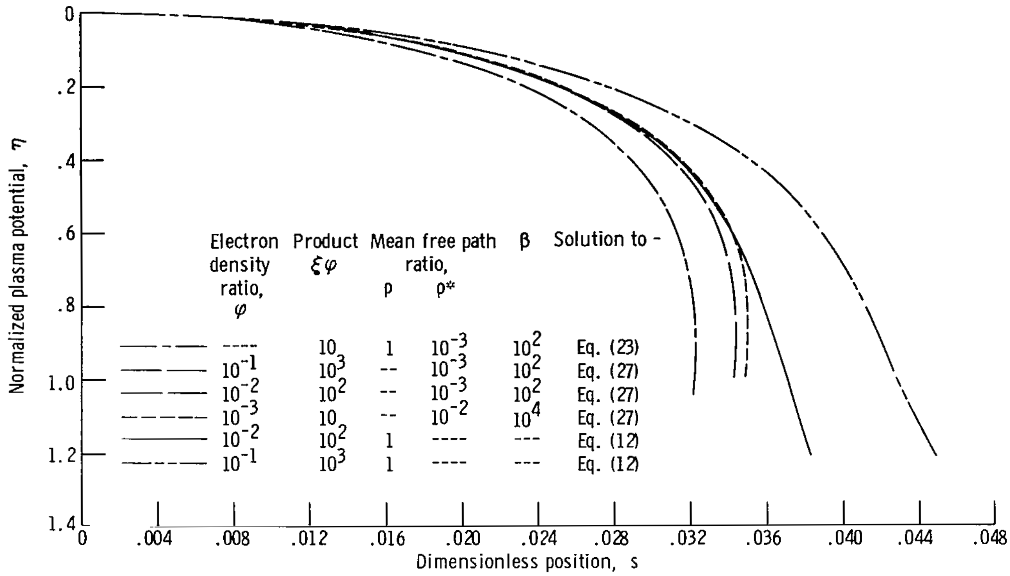


Figure 5. - Comparison of exact solutions with approximate solutions, $\varphi \ll 1$. $\beta = eV_C\ kT$; $\rho^* = \sqrt{\beta/\xi}\rho$.

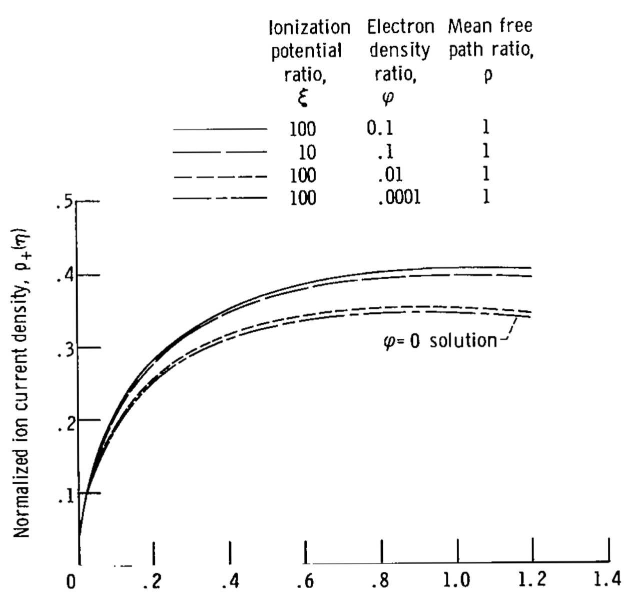


Figure 6. - Normalized ion current density.

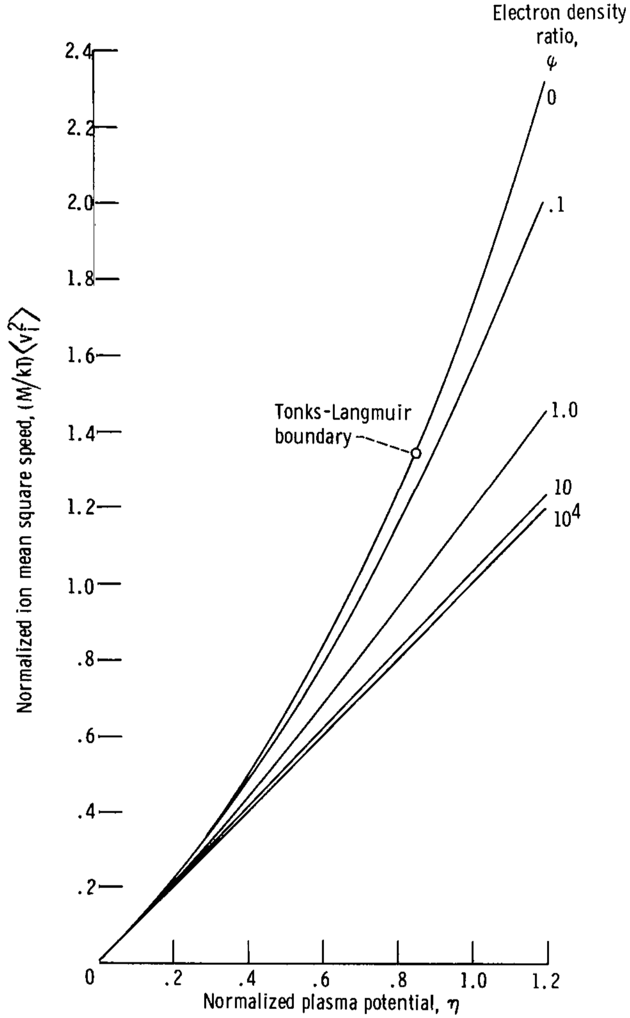


Figure 7. - Ion mean square speed against normalized plasma potential.

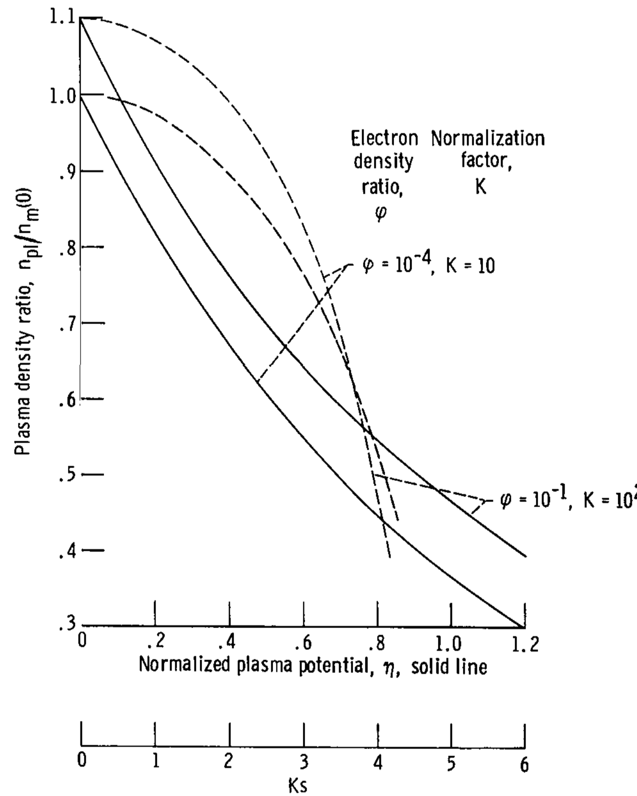


Figure 8. - Normalized plasma density, dashed line.

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