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STABLE ADAPTIVE CONTROL WITH GAIN CONSTRAINTS

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I. Introduction

The application of Liapunov Theory to the design of stable model tracking adaptive systems requires use of state variables in formulating the control law [1-3], and for the important class of problems to be considered the state variables must be in normal (phase variable) form [4]. Hence complexity of instrumentation and the presence of measurement noise are likely to be the factors which most severly limit the practicality of the design. Significant results which offer means of improving this situation have been reported for the case in which derivatives of the forcing function appear in the differential equation of the plant [1,5]. It has been shown that the highest-order state which must be measured can be reduced in proportion to the number of input derivatives.

The problem treated here has been motivated by the desire to reduce further the noise entering the system, and at the same time to find possibilities for simplifying the complexity of the adaptive controller.

The essential idea is to consider the problem in which some of the compensating gains are fixed at nominal values, as would be justified if the associated plant parameters are known with sufficient accuracy. Hence the adaptive control is applied only to those terms in which the parameters are known with poor accuracy. Using this approach the tracking error will not in general be asymptotically stable, and a criterion is needed to establish that the error will in fact be satisfactorily bounded. The object of this paper is to derive a test for such a bound. The result is a sufficient condition for Lagrange Stability which, though conservative, offers a basis for design.

Consideration is given to the effect of measurement noise upon the solution,

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and results of simulation are presented, illustrating an application of the method.

II. Description of the Adaptive System

The time-invariant linear plant is defined according to the state equation

$$\dot{\underline{x}} = A_p\underline{x} + \underline{b}_p u + \underline{d} + \underline{c}_p r \tag{2.1}$$

wherein the state vector $\underline{x} = [x_i]$ is of dimension n. u and r are scalar control and reference inputs respectively, with $\underline{b}_p = [b_i]$, $\underline{c}_p = [c_i]$, and $\underline{d} = [d_i]$ is an unknown bounded disturbance. In assuming that $\underline{x}$ is in normal form, only the coefficients of the last row of A_p are subject to uncertainty. Specifically, if $A_p = [a_{ij}]$, then some or all of the parameters a_{ni}, $i = 1, \ldots, n$ may be unknown, the other coefficients being zero except for the terms on the super diagonal which, for the normal form, are defined as unity.

As discussed in [4], if the elements of $\underline{b}_p$ are not presumed to be known exactly, then the design requires, for the case treated here in which u is a scalar, that $b_i \equiv 0$ for $i \neq n$. This is compatible with the normal form, and is in keeping with the development in [5] which allows for the presence of derivatives of the forcing function in the differential equation of the plant. The case in which there are derivatives of the input is treated in Section V.

The stable time-invariant model, which is assumed to have the same mathematical structure as the plant, is defined by

$$\dot{\underline{y}} = A_m\underline{y} + \underline{b}_m r \tag{2.2}$$

where

$\underline{y} = [y_i]$, $\underline{b}_m = [\beta_i]$, $A_m = [\alpha_{ij}]$, and r is the scalar input.

The design objective is to adapt one or more of the compensating gains (to be defined) so as to realize an asymptotic bound on the norm of the error, $\|\underline{e}\|$, where $\underline{e} = \underline{y}-\underline{x}$. The design approach is based on [2,5] except that here we assume some ignorance of <u>all</u> plant parameters, even though some of the compensating gains may be nonadaptive. Recognizing that a reduction of the number of adaptive gains certainly simplifies the requirements on instrumentation, the purpose of this study is to examine the consequences of such a simplification upon system performance. In past works the presence of measurement noise has been ignored, and the parameters associated with the nonadaptive gain terms have been assumed to be known exactly.

III. Summary Statement of the Control Law with $\underline{n} = \underline{0}$, $\underline{d} = \underline{0}$

In order to simplify the notation, the results which serve as background material to the paper will first be stated subject to the assumptions that there are no derivatives of the input, and that there are no measurement noise and disturbance present. Thus, summarizing the results of previous work, and making the assumption that the control of the plant can be accomplished through u only, that is to say, that plant parameters cannot be adjusted directly, we shall write the differential equation for the tracking error as

$$\dot{\underline{e}} = A_m \underline{e} + \underline{f} \tag{3.1}$$

with

$$\underline{f} = A\underline{x} + (\underline{\beta}_m - \underline{c}_p)\, r - \underline{b}_p u \tag{3.2}$$

wherein $A = A_m - A_p$ with elements $[\delta_{ij}]$, $(\underline{\beta}_m - \underline{c}_p) = [\delta_i]$, and $\underline{f} = [f_i]$. The object being to guarantee stability of the equilibrium at $\underline{e} = \underline{0}$, a Liapunov function is defined by

$$V = \underline{e}^T P \underline{e} + \underline{\phi}^T \underline{\phi} \tag{3.3}$$

in which $P = [p_{ij}]$ and $\underline{\phi} = [\phi_i]$ is a vector to be defined, with dimension n + 1.

The time derivative of V following the motion can be shown to be given by

$$\dot{V} = A_m^T P + PA_m + 2\,(\gamma f_n + \underline{\phi}\,\dot{\underline{\phi}}) \tag{3.4}$$

wherein

$$\gamma = \sum_{i=1}^{n} p_{in}\, e_i, \tag{3.5}$$

and f_n is the last element of $\underline{f}$. In deriving (3.4) it is helpful to recognize that, for the normal form, $\underline{e}^T P \underline{f} = \gamma f_n$, since f_n is the only element of $\underline{f}$ that can be non-zero, as given by

$$f_n = \sum_{i=1}^{n} \delta_{ni} \; x_i + \delta_n \; r - b_n u. \tag{3.6}$$

If it is now required in (3.4) that $(\gamma f_n + \underline{\phi} \; \dot{\underline{\phi}}) \equiv 0$, then with any positive definite symmetric Q, and P satisfying the equation

$$-Q = A_m^T \; P + PA_m \tag{3.7}$$

V is a Liapunov function.

This result is obtained by satisfying the following relationships:

$$\begin{aligned} \phi_i &= \lambda_i \; (\delta_{ni} - b_n \; k_i), \; i = 1, \ldots, n \\ \phi_{n+1} &= \lambda_r \; (\delta_n - b_n \; k_r) \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} \dot{k}_i &= x_i \; \gamma/\lambda_i^2 \; b_n, \; i = 1, \; \ldots, \; n \\ \dot{k}_r &= r \; \gamma/\lambda_r^2 \; b_n \end{aligned} \tag{3.9}$$

wherein λ_i, λ_r are real and non zero, and

$$u = \sum_{i=1}^{n} k_i \; x_i + k_r \; r. \tag{3.10}$$

It is important to note that V as defined in (3.3) is positive definite in the space defined by $\underline{e}$, $\underline{\phi}$. Thus $\dot{V}$ in (3.4) is negative semidefinite, and the equilibrium at $\underline{e} = \underline{0}$, $\underline{\phi} = 0$ is stable. However since $\dot{V}$ is negative definite in the $\underline{e}$ space, the equilibrium at $\underline{e} = \underline{0}$ is asymptotically stable.

It is of further interest to note that, by the well known corollary to the Liapunov Theorem on asymptotic stability [6], the equilibrium at $\underline{e} = \underline{0}$, $\underline{\phi} = 0$ is in some cases asymptotically stable as well. It must merely be shown that the solution at $\underline{e} = 0$, $\underline{\phi} \neq 0$ cannot be an equilibrium. For the conditions that r(t) contains all frequencies, and that the plant and model have the same structure, it follows that there is only one unique set of gains $[k, k_r]$ for which $\underline{e}$ is identically zero, i.e. with $\underline{\phi} = \underline{0}$. Hence there can be no solution for $\dot{\underline{e}} = \underline{0}$, $\dot{\underline{\phi}} = \underline{0}$ at other than $\underline{e} = \underline{0}$, $\underline{\phi} = \underline{0}$, and the corollary applies.

IV. Stability Criterion with Gain Constraints and Disturbance

Assume now that one or more of the gain terms in $[\underline{k}, k_r]$ is constant. Then $\underline{\phi}$ in (3.3) is constructed to have elements corresponding to each of the adjustable gains in $[\underline{k}, k_r]$, as defined in (3.7). Since it is always possible to redefine the coefficients in A_p and $\underline{c}_p$ of (2.1) so as to include the constant gain terms, the statement that the gain terms in $[\underline{k}, k_r]$ which are constant are also zero valued suffers no loss of generality. If then we let m elements of $\underline{k}$, as well as k_r, be identically zero, (3.4) becomes equal to

$$\dot{V} = -\underline{e}^T Q \underline{e} + 2\gamma[\sum^m \delta_{ni} x_i + \delta_n r - d_n]. \tag{4.1}$$

Here $\sum^m$ signifies the sum of m terms, but not necessarily in a sequence of successive integers. Clearly asymptotic stability in $\underline{e}$ is no longer assured by (4.1). However if a spherical region R_e in $\underline{e}$ can be found outside which $\dot{V}$ is negative, then the motion in $\underline{e}$ will be bounded [7].

To find such a region, if it exists, we replace x_i by $y_i - e_i$, and let Q be the identity matrix. Then substituting the expression for γ from (3.5), (4.1) becomes

$$\dot{V} = -\underline{e}^T\underline{e} + 2\sum_{i=1}^{n} p_{in} e_i (\sum^m \delta_{ni}(y_i - e_i) + \delta_n r - d_n). \tag{4.2}$$

It is readily shown that $\dot{V} < 0$ if

$$\underline{e}^T\underline{e} > 2\sum_{i=1}^{n} p_{in}|e_i|(\sum^m|\delta_{ni}||e_i| + \sum^m|\delta_{ni}||y_i| + |\delta_n||r| + |d_n|). \tag{4.3}$$

Denoting p_{max} and $|\delta|_{max}$ as the maximum values of p_{in} and $|\delta_{ni}|$ respectively, a stronger condition than (4.3) is given by

$$\underline{e}^T\underline{e} > 2p_{max}\sum^m|e_i|\ [|\delta|_{max}(\sum^m|e_i| + \sum^m|y_i|) + |\delta_n||r| + |d_n|]. \tag{4.4}$$

Using inequalities [8]

$$\sum_{i=1}^{n} |e_i| \leq (n \sum_{i=1}^{n} e_i^2)^{1/2}$$

$$\sum^{m} |e_i| \leq (m \sum^{m} e_i^2)^{1/2} \leq \sqrt{m} \, (\sum_{i}^{n} e_i^2)^{1/2} \tag{4.5}$$

a still stronger condition than (4.4), with $R_e \equiv (e^T e)^{1/2}$, becomes

$$R_e^2 > 2p_{max} \sqrt{n} \, [|\delta|_{max}(\sqrt{m} \, R_e + m \, |y|_{max}) + |\delta_n| |r|_{max} + |d_n|_{max}], \tag{4.6}$$

where $|y|_{max}$, $|r|_{max}$, and $|d_n|_{max}$ denote $\overset{max}{t}$ of the set $|y_i|$, $|r|$, and $|d_n|$, respectively. If

$$(1 - 2p_{max} \, |\delta|_{max} \sqrt{nm}) > 0, \tag{4.7}$$

it follows that $\dot{V} < 0$ outside the region

$$R_e = \frac{2\sqrt{n} \, p_{max}(|\delta|_{max} \, m \, |y|_{max} + |\delta_n| \, |r|_{max} + |d_n|_{max}).}{1 - 2p_{max} \, |\delta|_{max} \sqrt{nm}} \tag{4.8}$$

The condition (4.7) is a test for the existence of a region $R_e < \infty$, and is a sufficiency condition for stability (Lagrange) if, as has been assumed, $e^T Pe$ in (3.3) is positive definite. Thus using the notion that the actual ultimate bound for $\underline{e}$ must be determined by a contour of $\underline{e}^T P\underline{e}$ = constant circumscribing the sphere of radius R_e, where P is defined by (3.7), it is readily shown that $\underline{e}$ must ultimately be within a sphere of radius R_e' where

$$R_e' = \left(\frac{\lambda_{max}}{\lambda_{min}}\right)^{1/2} R_e, \tag{4.9}$$

and λ_{max}, λ_{min} are the max, min eigenvalues, respectively, of the P matrix. Since P is positive definite, $\lambda_{max}/\lambda_{min}$ is a finite positive real number, and R_e' is finite if R_e is finite.

If R_e is zero, the system is asymptotically stable in $\underline{e}$ space.

V. Stability Criterion where there are Input Derivatives

In [5] the adaptive control law is derived for the general case in which there may be up to n-1 derivatives of the control input appearing in the differential equation of the plant. The essential result in [5] is to show that a reduced-order error equation similar to (3.1) can be expressed as

$$\underline{\dot{e}} = A_m \underline{e} + \underline{b}\,[\Sigma \alpha_i z_i - b_{op} u] \tag{5.1}$$

where $\underline{e}^T = (e, \dot{e}, \ldots, e^{(n-p-1)})$ is a reduced-order state vector, n being the dimension of the plant state vector $\underline{x}$, and p being the highest order of the input derivative. In (5.1) $\underline{b}^T = (0, 0, \ldots, 1)$; and b_{op}, $[\alpha_i]$, are unknown constants comprised of model and plant parameters. The terms $[z_i]$ represent the inputs to the compensating gains. Whereas in the case treated previously the inputs to these gains are composed of plant state variables and the reference input, some of the z_i's are derived by processing these signals through a low-pass filter, as a consequence of the technique for reducing the order of the state equation (5.1).

In order to find an error bound in this case when certain of the gains $[k_i]$ are held constant, attention is directed to the expression for $\dot{V}$ which evolves in a form similar to (4.1), namely

$$\dot{V} = -\underline{e}^T\underline{e} + 2\sum_{i=1}^{n-p} p_{in} e_i (\sum^{m} \alpha_i z_i - d_n) \tag{5.2}$$

where m is the number of gain terms held constant (at zero). A difficulty now arises in paralleling the step in going from (4.1) to (4.2) in which $y_i - e_i$ was substituted for x_i. Consider a term $z_i = \mathcal{L}^{-1}[H(s)X_i(s)] = \mathcal{L}^{-1}[H(s)(Y_i(s)-E_i(s))]$, and let

$$\zeta_i = \mathcal{L}^{-1}[H(s)E_i(s)] \tag{5.3}$$

where H(s) describes the low pass filter mentioned above. It will be seen that, in order to arrive at an expression such as (4.6), it is necessary to derive an inequality relationship from (5.3) on the assumption that a bound R_e' exists.

Since by (4.9) R_e' can be expressed in terms of R_e, the inequality relationship for satisfying the condition $\mathring{V} < 0$ can then be formulated in terms of R_e, and the test for the existence of R_e can be applied. If R_e exists, then the assumption that R_e' exists is valid.

From (5.3) it follows with $h(t) = \mathcal{L}^{-1}H(s)$ that

$$\xi_i(t) = \int_0^\infty e_i(t-\lambda)\, h\,(\lambda)\, d\lambda. \tag{5.4}$$

If we assume that $|e_i| \leq R_e'$, then from (5.4) it follows

$$|\xi_i(t)| \leq R_e' \int_0^\infty |h(\lambda)|\, d\,\lambda. \tag{5.5}$$

By the condition imposed in [5] that H(s) is a stable filter, there is a finite number N such that

$$|\xi_i(t)| \leq R_e'\, N.$$

Now by (4.9)

$$|\xi_i(t)| \leq \left(\frac{\lambda_{max}}{\lambda_{min}}\right)^{1/2} N\, R_e.$$

From this point, the derivation of a relationship similar to (4.6) follows without difficulty.

It has been the purpose of this section to show that the results contained in Section V can be extended to the general case.

VI. An Error Bound due to Measurement Noise

If the measured state vector of the plant is defined as $\underline{w} = \underline{x} + \underline{n}$, where $\underline{n}$ is the measurement noise, it follows that (2.1) becomes

$$\dot{\underline{w}} = A_p\underline{w} + \underline{b}_p u + \underline{h} \tag{6.1}$$

with $\underline{h} = \underline{d} + \underline{c}_p r + \dot{\underline{n}} - A_p\underline{n}$. If a pseudo tracking error is now defined by $\underline{\varepsilon} = \underline{y} - \underline{w}$ where $\underline{y}$ is the output of the model (2.2), then the state equation for the pseudo error is given by

$$\underline{\dot{\varepsilon}} = A_m \underline{\varepsilon} + \underline{f}' \tag{6.2}$$

where

$$\underline{f}' = A\underline{w} + \underline{\beta}_m r - \underline{h} - \underline{b}_p u.$$

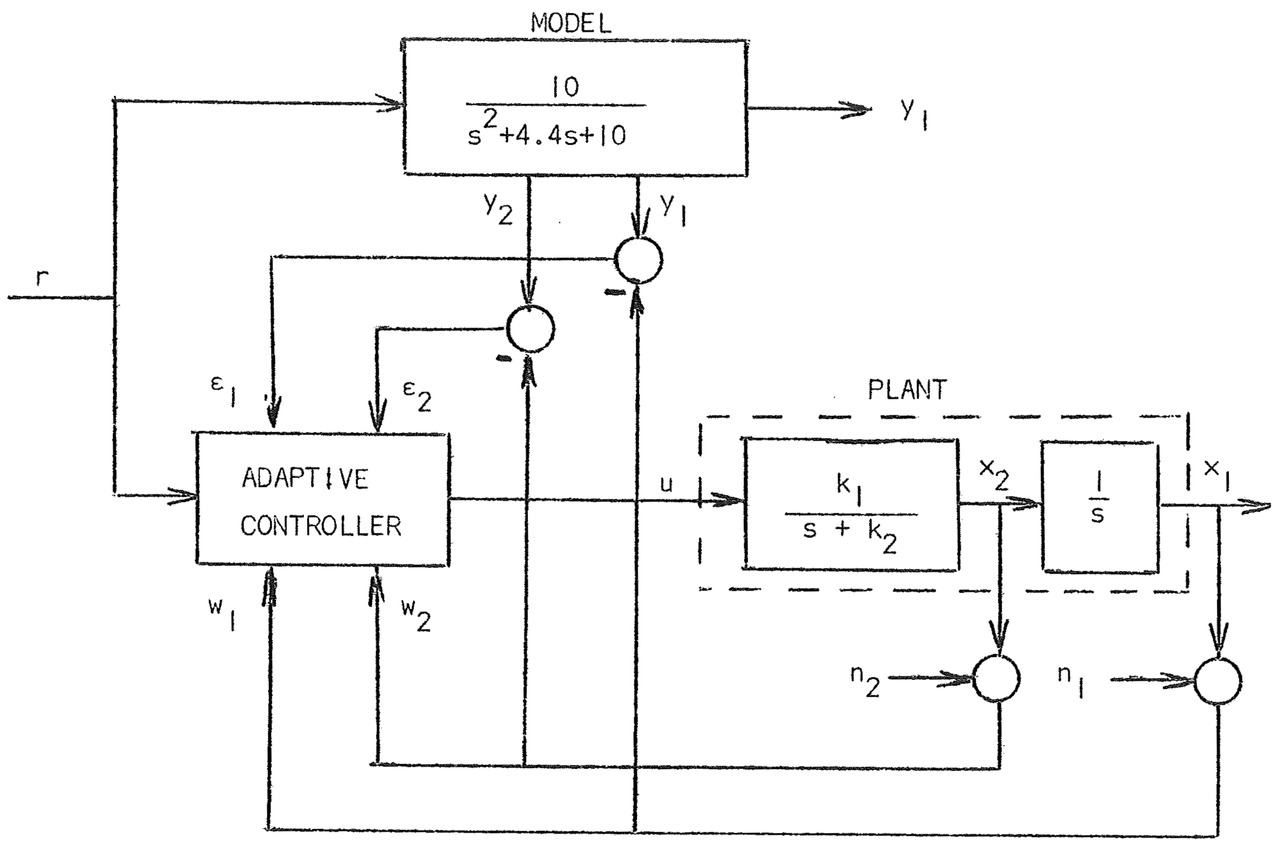
By having assumed that the system equation is in normal form, it is reasonable to consider the elements of the noise vector to be defined as $\underline{n}^T = [n_1, \dot{n}_1, \ldots, n^{(n-1)}]$. Hence it follows that $\varepsilon^T = [\varepsilon_1, \dot{\varepsilon}_1, \ldots, \varepsilon^{(n-1)}]$. Thus (6.4) is in normal form, and $\underline{f}'$ can be shown to have only one non-zero component, namely f_n'. This allows the analysis of Section III and IV to be applied directly to the present problem wherein the bound on $\underline{\varepsilon}$ is sought about the equilibrium $\underline{\varepsilon} = \underline{0}$. Since $\underline{e} = \underline{n} + \underline{\varepsilon}$, a bound on $\underline{e}$ can be found if bounds on $\underline{n}$ and $\underline{\varepsilon}$ are known.

Analysis shows that if the system is stable in $\underline{e}$ space without noise, then the error is bounded when noise is present, if the terms $\underline{n}$, $\underline{\dot{n}}$ appearing in $\underline{h}$ are bounded. This result is not very satisfying, however, because the dependence of the bound on $\underline{\dot{n}}$ was caused by the necessity for restructuring the system equation to be in the form (6.1) so that the noise enters the system in the form of a disturbance. Although the solution is found by this artifice, the result is highly artificial. For this reason it is important to determine the system performance by simulation if a reasonable evaluation of the effects attributable to noise is to be had.

VII Example

The system shown in Figure 1 is used to illustrate the concepts which have been introduced, taking into account the presence of measurement noise. Results obtained by simulation are compared with the calculated error bound.

The assumption is made that k_1 and k_2 are not known exactly. We assume however that k_2 is known to within 20% of the nominal value whereas k_1 may differ widely from nominal. Hence, although the adaption with respect to



Adaptive System Used in Example

Figure 1

$[w_1, r]$ is required, the need for an adaptive loop with respect to w_2 is questionable.

For this example, the input was taken to be a five-volt square wave having a period of two seconds. A twenty percent offset was made in k_2 ($k_2 = 5.28$), compared to its nominal value of 4.4. The value of k_1 was not important since the w_1 loop was always operative. A value of $k_1 = 5$ was used throughout.

In the absence of measurement noise, and with the w_2 loop inoperative, the bound R_e' can be found as follows:

Let $Q = I$. Then using (3.7), with A_m as specified by the model, it follows that

$$P = \begin{bmatrix} 1.47 & .05 \\ .05 & .125 \end{bmatrix}.$$

Substituting numbers into (4.8) according to

$$n = 2,\ m = 1,\ p_{max} = 0.125,\ \delta_{max} = 0.84,\ y_{max} = |y_2|_{max} = 10,\ \delta_n = 0,$$

it follows that $R_e = 4.4$ volts. If the eigenvalues of the P matrix are computed, the desired result in (4.9) is found to be $R_e' = 15$ volts.

By simulation the actual error bound was determined to be 1.2 volts. Hence, as is to be expected, the calculated bound is conservative.

Adding measurement noise to the system, the simulation was used to determine at what noise level, if any, the inclusion of the adaptive loop involving w_2 would in fact degrade system performance. For this purpose, the noise n_1 was chosen to be a random signal with a low frequency power spectrum spanning the bandwidth of the system. With $n_2 = \dot{n}_1$, this resulted in n_2 having peak values eighty times the peak values of n_1.

It was found that, with the w_2 adaptive loop operating, a noise n_1 having peak amplitudes of 50 millivolts produced an error signal (ε_1) with the same peak amplitude (0.35 volts) as that which was obtained with the w_2 adaptive loop inoperative. Hence the noise inserted through the adaptive loop involving w_2 nullified the advantage offered by that adaptive gain. The fact that the noise signal had such a pronounced effect is attributed to the relatively large noise level associated with n_2.

VIII Conclusions

The presence of measurement noise is shown to be an important factor in the design of adaptive systems. For those cases in which certain plant parameters are known with fair accuracy, it is found that better system performance may result by eliminating certain adaptive loops. In this case stability of the adaptive system is no longer asymptotic, and stability must be expressed in terms of an error bound. The expression for an ultimate bound on the tracking error is derived in terms of known bounds on certain of the plant parameter deviations. However, the result is conservative, so that particularly when noise is considered it is advisable to resort to simulation methods.

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