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**DIFFRACTION OF A PULSE BY A
THREE-DIMENSIONAL CORNER**

by Lu Ting and Fanny Kung

Prepared by
NEW YORK UNIVERSITY
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16. Abstract For the diffraction of a pulse by a three-dimensional corner, the solution is conical in three variables $\zeta = r/(Ct)$, θ and φ . The three-dimensional effect is confined inside the unit sphere $\zeta = 1$, with the vertex of the corner as the center. The boundary data on the unit sphere is provided by the appropriate solutions for the diffraction of a pulse by a two-dimensional wedge. The solution exterior to the corner and inside the unit sphere is constructed by the separation of the variable ζ from θ and φ . The associated eigenvalue problem is subjected to the differential equation in potential theory with an irregular boundary in $\theta - \eta$ plane. A systematic procedure is presented such that the eigenvalue problem is reduced to that of a system of linear algebraic equations. Numerical results for the eigenvalues and functions are obtained and applied to construct the conical solution for the diffraction of a plane pulse. For the diffraction of a general incident wave by corners or edges, two theorems are presented so that the value at the vertex of the corner or along the edges can be determined without the construction of the three-dimensional non-conical diffraction solutions. Relevant numerical programs for the analysis are presented in the appendix.					
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DIFFRACTION OF A PULSE BY A THREE-DIMENSIONAL CORNER

by Lu Ting and Fanny Kung
New York University, New York, N.Y.

SUMMARY

For the diffraction of a pulse by a three-dimensional corner, e.g., the corner of a cube, the solution is conical in three variables $\zeta = r/(Ct)$, θ and φ . The three-dimensional effect is confined inside the unit sphere $\zeta = 1$, or the sonic sphere $r = Ct$ with the vertex of the corner as the center. The boundary data on the unit sphere is provided by the appropriate solutions for the diffraction of a pulse by a two-dimensional wedge. The solution exterior to the corner and inside the unit sphere is constructed by the separation of the variable ζ from θ and φ . The associated eigenvalue problem is subjected to the same differential equation in potential theory for the spherical angle variables θ and φ , but with an irregular boundary in $\theta - \varphi$ plane. A systematic procedure is presented such that the eigenvalue problem is reduced to that of a system of linear algebraic equations. Numerical results for the eigenvalues and functions are obtained and are applied to construct the conical solution for the diffraction of a plane pulse. For the diffraction of a general incident wave by corners or edges, the solutions are no longer conical. Two theorems are presented so that the value at the vertex of the corner or along the edges can be determined without the construction of the three-dimensional non-conical diffraction solutions. Relevant numerical programs for the analysis are presented in the appendix.

1. Introduction

The problem of diffraction and reflection of acoustic waves or electromagnetic waves by wedges, corners and other two-dimensional or axially symmetric obstacles has received extensive investigations. A survey of these investigations can be found in Reference 1.

The present investigation is motivated by the study of the effect of sonic boom on structures. The pressure wave created by a supersonic airplane is three-dimensional in nature. However, the radius of curvature of the wave front is usually much larger than the length scale of a structure. Therefore, the incident waves can be approximated by progressing plane waves with the wave form usually in the shape of the letter N and are referred to as N-waves.

For two-dimensional structures in the shape of a rectangular block, the diffraction of a plane pulse by the first corner is given explicitly by the two-dimensional conical solution of Keller and Blank [2]. The solution for each subsequent diffraction by the next corners can be obtained by the use of Green's function for a wedge [1, pp. 108-115]. For right angled corners, the diffraction solutions can be obtained by the solution of an Abel type integral equation [3]. By means of the integral of Duhamel, the solution for the diffraction of a plane pulse by a two-dimensional structure was employed to construct the solution for the diffraction of an N-wave by the same structure [4].

For the three-dimensional problem of the diffraction of sonic boom by structures, the first step is the construction of the solution for the diffraction of a plane pulse by a corner of a structure. The formulation of the differential equation and the appropriate boundary conditions for the problem in three-dimensional conical variables is presented in the next section.

2. Formulation

For the acoustic disturbance pressure p , the governing differential equation is the simple wave equation,

$$p_{xx} + p_{yy} + p_{zz} - c^{-2} p_{tt} = 0 \quad (1)$$

in the region outside a trihedron simulating the corner of a structure. As shown in Fig. 1, two edges, OA and OB, of the trihedron are in the x-y plane and are bisected by the negative x-axis with half angle α and the third edge OD, which is the negative z-axis with the vertex O as the origin. Let $t=0$ be the instant when the pulse front hits the vertex. The boundary condition on the three faces of the trihedron is

$$\partial p / \partial n = 0 \quad (2)$$

The three-dimensional disturbance due to the vertex is confined inside the sonic sphere $r = Ct$ where $r = (x^2 + y^2 + z^2)^{1/2}$. Outside the sonic sphere, the pressure distribution is given either by the regular reflection of the plane pulse from the surfaces of the trihedron or by the diffraction of the pulse by the edges. When the pulse front is parallel to the edge, the diffracted wave is given by a two-dimensional unsteady conical solution. When the pulse front is not parallel to the edge, the diffracted wave is given by a steady three-dimensional conical solution. For both cases, solutions are given in two conical variables in [2] by means of Busemann's conical flow method.

Due to the absence of a time scale and a length scale, the disturbance pressure p nondimensionalized by the strength of the incident pulse should be a function of the three-dimensional conical variables, $x/(Ct)$, $y/(Ct)$ and $z/(Ct)$ or in terms of the spherical coordinates by $\zeta = r/(Ct)$, θ and φ . The simple wave equation for $p(\zeta, \theta, \varphi)$ becomes,

$$\zeta^2 (1 - \zeta^2) \frac{\partial^2 p}{\partial \zeta^2} + 2\zeta (1 - \zeta^2) \frac{\partial p}{\partial \zeta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial p}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 p}{\partial \varphi^2} = 0 \quad (3)$$

inside the unit sphere, $\zeta = 1$, and exterior to the trihedron. The boundary conditions are:

$$\partial p / \partial \theta = 0 \quad \text{on surface OAB, } \theta = \pi/2, \quad -(\pi - \alpha) < \varphi < \pi - \alpha \quad (4)$$

$$\partial p / \partial \varphi = 0 \quad \text{on surface OAD, } \varphi = \pi - \alpha, \quad \pi - \alpha, \pi/2 < \theta < \pi \quad (5a)$$

$$\partial p / \partial \varphi = 0 \quad \text{on surface OBD, } \varphi = -\pi + \alpha, \quad \pi/2 < \theta < \pi \quad (5b)$$

and $p = F(\theta, \varphi)$ on unit sphere $\zeta = 1$ outside of the trihedron.

The jump across the sonic sphere is inversely proportional to the square root of the area of the ray tube, $(dS)^{1/2}$. Since all the rays reaching the sphere come from the origin where $dS_0 = 0$, the jump across the sonic sphere is zero. The pressure is continuous across the sphere $\zeta = 1$; and $F(\theta, \varphi)$ is defined by the solutions outside the sonic sphere in two conical variables given by [2].

To construct the solution by the method of separation of variables, the usual trial substitution $p(\zeta, \theta, \varphi) = Z(\zeta) G(\mu, \varphi)$ is introduced where $\mu = \cos\theta$ and eq. (3) becomes

$$\zeta^2 (1-\zeta^2) Z''(\zeta) + 2(1-\zeta^2) Z'(\zeta) - \lambda(\lambda+1) Z(\zeta) = 0 \quad (6)$$

for $1 > \zeta \geq 0$, and

$$\frac{\partial}{\partial \mu} \left[(1-\mu^2) \frac{\partial G}{\partial \mu} \right] + \lambda(\lambda+1) G + \frac{1}{1-\mu^2} \frac{\partial^2 G}{\partial \varphi^2} = 0 \quad (7)$$

for the domain in $\mu - \varphi$ plane with $|\varphi| < \pi - \alpha$ when $0 \geq \mu \geq -1$, and with $|\varphi| \leq \pi$ when $1 \geq \mu > 0$. Since the pressure p and also G are single valued, G should be periodic in φ when $1 \geq \mu > 0$, i.e.,

$$G(\mu, \varphi + 2\pi) = G(\mu, \varphi) \quad (8)$$

The range of φ is therefore restricted from $-\pi$ to π and the two ends are connected by the periodicity condition.

The replacement of the variable θ by μ and the use of the constant of separation $\lambda(\lambda+1)$ are motivated by the intention of representing $G(\mu, \theta)$ by the spherical harmonics.

The boundary conditions on the surfaces of the trihedron, eq. (4), become

$$\partial G / \partial \mu = 0 \quad \text{along } \mu = 0 \text{ with } \pi - \alpha < |\varphi| \leq \pi \quad (9a)$$

$$\partial G / \partial \varphi = 0 \quad \text{along } \varphi = \pm(\pi - \alpha) \text{ with } -1 < \mu < 0 \quad (9b)$$

The condition that p is bounded in particular at the two poles, $\theta = 0$ and $\theta = \pi$, yields the condition,

$$|G| < \infty \text{ at } \mu = \pm 1 \quad (9c)$$

The periodicity condition, eq. (8), supplies the condition along the remaining boundaries,

$$G(\mu, -\pi) = G(\mu, \pi) \text{ and}$$

$$G_\varphi(\mu, -\pi) = G_\varphi(\mu, \pi) \text{ for } 1 \geq \mu > 0 \quad (9d)$$

The differential equation, (7), and the boundary conditions, (9a-d), define an eigenvalue problem. The determination of the eigenvalues, λ 's, and the associated eigenfunctions, $G_\lambda(\mu, \varphi)$, are described in the next section.

3. The Eigenvalue Problem

For the eigenvalue problem formulated in the preceding section, two edges of the trihedron, OA and OB, are assumed to be normal to the third edge. This restriction is imposed due to two considerations: 1) the surfaces of the corner can be defined by surfaces of constant θ or constant φ 's and 2) the solutions outside the sonic sphere for the diffraction problem can be constructed as solutions of two-dimensional problems [2]. For the solution of the eigenvalue problem itself the second consideration is irrelevant. The eigenvalue problem in this section will, therefore, be formulated for a wider class of corners as shown in Fig. 2. The surface OAB is a conical surface with $\theta = \beta$ and the two surfaces OAD and OBD are planes with $\varphi = \pi - \alpha$ and $\varphi = -\pi + \alpha$ respectively. The boundary conditions for the solution of eq. (7) are

$$\partial G / \partial \mu = 0 \quad \text{along } \mu = \mu_0 = \cos \beta \text{ with } \pi - \alpha < |\varphi| \leq \pi \quad (10a)$$

$$\partial G / \partial \varphi = 0 \quad \text{along } \varphi = \pm(\pi - \alpha) \text{ with } -1 \leq \mu < \mu_0 \quad (10b)$$

$$|G| < \infty \text{ at } \mu = \pm 1 \quad (10c)$$

$$G(\mu, \pi) = G(\mu, -\pi), \quad G_\varphi(\mu, \pi) = G_\varphi(\mu, -\pi) \text{ for } \mu_0 < \mu \leq 1 \quad (10d)$$

For the special case of $\beta = \pi/2$, or $\mu_0 = 0$, the boundary conditions of eqs. (10a - d) and the domain in the $\mu - \varphi$ plane (Fig. 2) reduce to those of eqs. (9a - d).

The eigenvalue problem is now defined by the differential eq. (7) and the boundary conditions of eqs. (10a - d). In order to reduce the problem to that for a set of algebraic equations, two representations of the eigenfunction $G_\lambda(\mu, \varphi)$ associated with the eigenvalue λ will be sought: one for the region, R^+ , with $\mu > \mu_0$ and the other for R^- with $\mu < \mu_0$ (Fig. 2). These two solutions and their normal derivatives will be matched across the dividing line $\mu = \mu_0$ for $|\varphi| < \pi - \alpha$.

For the upper region R^+ , the eigenfunction $G_\lambda^+(\mu, \varphi)$ which is periodic in φ on account of eq. (10d) or eq. (8) can be represented by the Fourier series in φ with period of 2π ,

$$G_\lambda^+(\mu, \varphi) = \sum_{m=0,1,\dots} A_m p_\lambda^{-m}(\mu) \cos m\varphi + \sum_{m=1,2,\dots} B_m p_\lambda^{-m}(\mu) \sin m\varphi \quad (11)$$

For each m , eq. (7) yields the Legendre equation for $p_\lambda^{-m}(\mu)$

$$\frac{d}{d\mu} \left[(1-\mu^2) \frac{d}{d\mu} p_\lambda^{-m}(\mu) \right] + \left[\lambda(\lambda+1) - \frac{m^2}{1-\mu^2} \right] p_\lambda^{-m}(\mu) = 0 \quad (12)$$

Since eq. (11) is defined for $1 \geq \mu \geq \mu_0 > -1$, p_λ^{-m} should be finite at

$\mu = 1$. P_{λ}^{-m} is identified as the generalized Legendre function [5] and defined by

$$P_{\lambda}^{-m}(\mu) = \left(\frac{1-\mu}{1+\mu}\right)^{m/2} F\left\{-\lambda, \lambda+1, 1+m; (1-\mu)/2\right\} \quad (13)$$

where F denotes the Gaussian hypergeometric function. The factor $1/\Gamma(1+m)$ is omitted on the right side of eq. (13) since it is automatically absorbed into coefficients A_m and B_m with a net saving of programming and computing time.

For the lower region R^- , i.e., $-1 \leq \mu < \mu_0$ the eigenfunction G_{λ}^- , which fulfills the boundary conditions of eqs. (10b and c), can be expressed as a cosine series in $(\varphi - \pi + \alpha)$ with half period of $2(\pi - \alpha)$ i.e.,

$$G_{\lambda}^-(\mu, \varphi) = \sum_{n=0,1,2,\dots} E_n P_{\lambda}^{-(n\pi/\Phi)}(-\mu) \cos \left[\frac{n\pi}{\Phi} \left(\varphi + \frac{\Phi}{2} \right) \right] \quad (14)$$

where $\Phi = 2(\pi - \alpha)$. Similarly for each n , eq. (7) yields the Legendre equation for the generalized Legendre function $P_{\lambda}^{-n\pi/\Phi}(-\mu)$. It is finite for $-1 \leq \mu \leq \mu_0 < 1$.

Both expressions (12) and (14) fulfill the differential eq. (7) and the boundary conditions (10b-d) appropriate for regions R^+ and R^- respectively.

Across the dividing line of R^+ and R^- , where $\mu = \mu_0$ and $|\varphi| < \pi - \alpha$, the matching conditions are the continuity of the eigenfunction G_{λ} and its normal derivative $\partial G_{\lambda} / \partial \mu$. The continuity of the second derivatives are then assured since both G_{λ}^+ and G_{λ}^- fulfill the same differential eq. (7).

The matching conditions and the remaining boundary condition, eq. (10a), can be written as

$$G_{\lambda}^+(\mu_0^+, \varphi) = G_{\lambda}^-(\mu_0^-, \varphi) \quad \text{for } |\varphi| \leq \pi - \alpha \quad (15)$$

$$\partial G_{\lambda}^+(\mu_0^+, \varphi) / \partial \mu = \begin{cases} \partial G_{\lambda}^-(\mu_0^-, \varphi) / \partial \mu & \text{for } |\varphi| < \pi - \alpha \\ 0 & \text{for } \pi - \alpha < |\varphi| \leq \pi \end{cases} \quad (16)$$

Equations (15) and (16) can be reduced to a system of linear homogeneous algebraic equations for the unknown constants A_m , B_m and E_n by Fourier analysis or by fulfilling eqs. (15,16) at a number of φ 's in the appropriate intervals. Before carrying out the reduction, it should be noted that the differential eq. (7) and the boundary conditions (10a-d) are symmetric with the plane $\varphi = 0$. The eigenfunction can, therefore, be expressed either as even functions or odd functions of φ . Indeed, in the expression (11) for

G_{λ}^{+} the cosine terms represent the even solution and the sine terms represent the odd solution. In the expression (14) for G_{λ}^{-} , the terms with even and odd values of n are the even and odd solutions of φ respectively. They can be rewritten as

$$G_{\lambda}^{-}(\mu, \varphi) = \sum_{j=0,1,\dots} C_j P_{\lambda}^{-\nu_j} (-\mu) \cos \nu_j \varphi + \sum_{j=1,2,\dots} D_j P_{\lambda}^{-\bar{\nu}_j} (-\mu) \sin \bar{\nu}_j \varphi \quad (17)$$

where

$$\nu_j = 2j\pi/\Phi, \quad \bar{\nu}_j = (2j-1)\pi/\Phi, \quad C_j = E_{2j} (-1)^j \text{ and } D_j = E_{2j-1} (-1)^j.$$

It is now obvious that the cosine terms in eq. (17) are even and the sine terms are odd in φ . Since the odd solution and the even solution are orthogonal to each other, the conditions of eqs. (15) and (16) for G_{λ}^{+} and G_{λ}^{-} should be fulfilled by their even terms and also by their odd terms respectively. The eigenvalues and the associated even eigenfunction can be determined separately from the eigenvalues and the associated odd eigenfunction. By this uncoupling, the rank of the characteristic determinant is reduced by a half, and a double root of the characteristic equation of the mixed solutions may now be split to one single root for the even and one for the odd solutions respectively. In actual numerical computations, it is much easier to locate the eigenvalue and construct the eigenfunction for a single root than for a double root of the characteristic equation.

The solution of the eigenvalue problem defined by eqs. (15) and (16) can be carried out by collocation method, i.e., to truncate the series of eqs. (11) and (17) and to impose eqs. (15) and (16) at a finite number of value of φ 's in the appropriate intervals. In this method, it is necessary to uncouple the solutions first and then impose eqs. (15) and (16) to the even and to the odd solutions separately.

When the solution of the eigenvalue problem is to be carried out by Fourier analysis, the uncoupling of the even and odd solutions will come about automatically as follows:

Equation (16) holds for the interval $-\pi < \varphi < \pi$, it will be multiplied by $\cos n\varphi$ and will be integrated over the interval. Equation (16) then becomes the algebraic equations for the coefficients of the even terms,

$$A_n \dot{P}_{\lambda}^{-n}(\mu_0) [1 + \delta_{on}] + \sum_{j=0,1,2} C_j \dot{P}_{\lambda}^{-\nu_j} (-\mu_0) I_{jn} = 0 \quad \text{for } n=0,1,\dots \quad (18)$$

where

$$\begin{aligned}
I_{jn} &= \frac{1}{\pi} \int_{-\Phi/2}^{\Phi/2} \cos v_j \varphi \cos n \varphi d\varphi \\
&= \begin{cases} \Phi[1+\delta_{on}]/(2\pi) & , v_j = n \\ -2(-1)^j \sin(n\Phi/2)/[\pi(v_j^2 - n^2)] & , v_j \neq n \end{cases} \quad (19)
\end{aligned}$$

and (·) means the first derivative with respect to its argument.

Similarly eq. (15), which holds for the interval $|\varphi| \leq \pi - \alpha/2 = \Phi/2$, will be multiplied by $\cos[2k\pi(\varphi+\Phi/2)/\Phi]$ or $\cos v_k \varphi$ with $v_k = 2k\pi/\Phi$ and will be integrated over the interval $-\Phi/2 < \varphi < \Phi/2$. The result is

$$\begin{aligned}
\sum_{m=0,1,2,\dots} A_m P_{\lambda}^{-m}(\mu_o) I_{km} - C_k P_{\lambda}^{-v_k}(-\mu_o) \Phi[1+\delta_{ok}]/(2\pi) &= 0 \\
\text{for } k=0,1,2,\dots & \quad (20)
\end{aligned}$$

Since $\cos n\varphi$ and $\cos v_k \varphi$ are even functions of φ , eqs. (18) and (20) contain only the coefficients A_m and C_j associated with the even parts in G_{λ}^{+} and G_{λ}^{-} respectively. Similarly the algebraic equations for the coefficients B_m and D_j associated with the odd parts are obtained from eqs. (15) and (16). Equation (16) will be multiplied by $\sin n\varphi$ and integrated over the interval $-\pi < \varphi < \pi$ and eq. (15) will be multiplied by $\sin \bar{v}_k \varphi$ and integrated over the interval $-\Phi/2 < \varphi < \Phi/2$ with $\bar{v}_k = (2k-1)\pi/\Phi$. The results are,

$$\begin{aligned}
B_n \dot{P}_{\lambda}^{-n}(\mu_o) + \sum_{j=1,2,\dots} D_j \dot{P}_{\lambda}^{-\bar{v}_j}(-\mu_o) L_{jn} &= 0 \\
\text{for } n = 1,2,\dots & \quad (21)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1,2,\dots} B_m P_{\lambda}^{-m}(\mu_o) L_{km} - D_k P_{\lambda}^{-\bar{v}_k}(-\mu_o) \Phi/(2\pi) &= 0 \\
\text{for } k = 1,2,\dots & \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
L_{jn} &= \frac{1}{\pi} \int_{-\Phi/2}^{\Phi/2} \sin \bar{v}_j \varphi \sin n \varphi d\varphi \\
&= \begin{cases} \Phi/(2\pi) & , v_j = n \\ -2n (-1)^j \cos \frac{n\Phi}{2} / [\pi(\bar{v}_j^2 - n^2)] & , \bar{v}_j \neq n \end{cases} \quad (23)
\end{aligned}$$

Equations (18) and (20) define the eigenvalue problem for the even solution, while eqs. (21) and (22) for the odd solution. The numerical method for the solution of the problem will be described in detail in the next section.

4. Numerical Solutions of the Eigenvalue Problem

To obtain the numerical solutions, the infinite cosine series in eq. (11), the even part of G_{λ}^{+} , will be truncated with the maximum of m equal to M_c . Similarly the cosine series in eq. (17), the even part of G_{λ}^{-} , will be truncated with the maximum of j equal to J_c . Likewise, M_c and J_c will be the maxima of n and k , respectively, so that eqs. (18) and (20) yield $M_c + J_c + 2$ linear homogeneous equations for the $M_c + J_c + 2$ constants $A_0, A_1, \dots, A_{M_c}$ and $C_0, C_1, \dots, C_{J_c}$. These solutions are nontrivial if λ is a root of the characteristic equation,

$$\Delta_c(\lambda) = | a_{hi} | = 0 \quad (24)$$

$$\begin{aligned} \text{where} \quad a_{hi} &= \dot{P}_{\lambda}^{-i+1}(\mu_0) [1 + \delta_{li}] \delta_{hi} \quad \text{for } h, i=1, 2, \dots, M_c+1, \\ a_{hi} &= \dot{P}_{\lambda}^{-\nu_j}(-\mu_0) I_{ji} \quad \text{for } h=1, 2, \dots, M_c+1, \\ &\quad i=M_c+2, \dots, M_c+J_c+2, \text{ and } j=i-M_c-2 \\ a_{hi} &= \dot{P}_{\lambda}^{-i+1}(\mu_0) I_{ki} \quad \text{for } h=M_c+2, \dots, M_c+J_c+2, \\ &\quad i=1, 2, \dots, M_c+1, \text{ and } k=h-M_c-2 \end{aligned}$$

$$\begin{aligned} \text{and} \quad a_{hi} &= -\dot{P}_{\lambda}^{-\nu_k}(-\mu_0) \Phi [1 + \delta_{ok}] \delta_{hi} / (2\pi) \\ \text{for} \quad h, i &= M_c+2, \dots, M_c+J_c+2, \text{ with } k=i-M_c-2. \end{aligned}$$

For given values of M_c and J_c , the roots of $\Delta_c(\lambda^*) = 0$ in a given interval of λ can be located by numerical evaluation of $\Delta_c(\lambda)$ as a function of λ . When the derivative of $\Delta_c(\lambda)$ at a root is non-zero, the coefficients A_m, C_j are proportional to the cofactors of the determinant, i.e.,

$$\begin{aligned} A_m &= N^{-1} A_{1, M+1} \quad \text{for } M=0, 1, 2, \dots, M_c \\ C_j &= N^{-1} A_{1, M_c+1+j} \quad \text{for } j=0, 1, 2, \dots, J_c \end{aligned} \quad (25)$$

where A_{hi} is the cofactor of the element a_{hi} of the determinant with $\lambda = \lambda^*$. The constant N is defined by the normalization condition,

$$\int_{\mu_0}^1 d\mu \int_0^\pi d\varphi \left[G_{\lambda^*}^+(\mu, \theta) \right]^2 + \int_{-1}^{\mu_0} d\mu \int_0^{\bar{\varphi}/2} d\varphi \left[G_{\lambda^*}^-(\mu, \theta) \right]^2 = 1 \quad (26)$$

The equation for N is

$$N^2 = \frac{\pi}{2} \int_{\mu_0}^1 d\mu \sum_{m=0}^{M_c} \left[P_{\lambda^*}^m(\mu) A_{1,m+1} \right]^2 (1 + \delta_{0m}) \\ + \frac{\bar{\varphi}}{4} \int_{-\mu_0}^1 d\mu \sum_{j=0}^{J_c} \left[P_{\lambda^*}^{-vj}(\mu) A_{1,M_c+1+j} \right]^2 (1 + \delta_{0j}) \quad (27)$$

A numerical program is described in the appendix so that for a given pair of M_c and J_c , the determinant $\Delta_c(\lambda)$ can be evaluated as a function of λ . The search for the roots can be restricted to $\lambda > -\frac{1}{2}$ because of the symmetry of the function $P_{\lambda}^{-vj}(\mu)$ about $\lambda = -\frac{1}{2}$, i.e., $P_{\lambda}^{-vj}(\mu) = P_{-\lambda-1}^{-vj}(\mu)$. For the given range of λ , say $-\frac{1}{2} \leq \lambda \leq 3$, the eigenvalues for the even solutions are located and the first derivative of $\Delta_c(\lambda)$ at each root λ^* are computed. In the examples considered in this paper, the derivative does not vanish at λ^* , therefore, the numerical program in the appendix carries out the determinations of the coefficients A_m and C_j in the eigenfunctions by means of eqs. (25) and (27).

Similarly for the odd solutions, the series in eqs. (11) and (17) will be truncated with the maxima of m and j equal to M_s and J_s respectively. There will be M_s equations for eq. (21) and M_j equations for eq. (22). The roots of the characteristic equation yield the eigenvalues for the odd solutions. The ratios of the cofactors and the normalization condition of eq. (26) define the coefficients $B_1 \dots B_{M_s}$ and $D_1 \dots D_{J_s}$ in the eigenfunctions.

It should be pointed out here that the convergence of the eigenvalues and eigenfunctions of the truncated problem has been assumed and the equivalence between the matching conditions of eqs. (15) and (16) and matching of the Fourier coefficients is also assumed. Some confidence in these assumptions will be offered by the following numerical results.

For the special case of a two-dimensional corner, namely $\alpha = \pi/2$, $\beta = \pi/2$ in Fig. 2, the eigenvalue problem can be solved exactly by choosing the edge of the corner as the z axis (Fig. 3). In spherical coordinates $\bar{\theta}$ and $\bar{\varphi}$, the two sides are $\bar{\varphi} = 0$ and $\bar{\varphi} = 3\pi/2$. The eigenfunction can be written as

$$\bar{G}_{\lambda}(\bar{\theta}, \bar{\varphi}) = P_{\lambda}^{2j/3}(\bar{\mu}) \cos[(2j/3)\bar{\varphi}] \quad (28)$$

where $\bar{\mu} = \cos \bar{\theta}$ and $j = 0, 1, 2, \dots$. The eigenvalue λ is defined by the condition that $P_{\lambda}^{-2j/3}$ should be finite at $\bar{\theta} = \pi$ or $\bar{\mu} = -1$. Instead of eq. (13), an equivalent expression for $P_{\lambda}^{-\nu}(\bar{\mu})$ is [5].

$$P_{\lambda}^{-\nu}(\bar{\mu}) = 2^{-\nu} (1 - \bar{\mu}^2)^{\nu/2} F(\nu - \lambda, \nu + \lambda + 1, \nu + 1; X) \quad (29)$$

where $\nu = 2j/3$ and $X = (1 - \bar{\mu})/2$

The power series representation for the hypergeometric function is divergent at $X=1$ or $\bar{\mu} = -1$ unless the series has only finite number of terms, say $k + 1$ terms. The condition for λ is

$$\nu - \lambda + k = 0 \quad \text{or} \quad \lambda_{jk} = k + 2j/3 \quad (30)$$

For $j = 0, 1, 2, 3, \dots$, the eigenvalues* are $\lambda_{0,k} = 0, 1, 2, 3, \dots$,
 $\lambda_{1,k} = 2/3, 5/3, 8/3, \dots$, $\lambda_{2,k} = 4/3, 7/3, \dots$, $\lambda_{3,k} = 2, 3, \dots$,
 $\lambda_{4,k} = 8/3, \dots$

The numerical program in the appendix developed for a three-dimensional corner will now be tested by setting $\beta = \pi/2$ and $\alpha = \pi/2$. The eigenvalues between -0.5 and 3.0 for the even and the odd solutions respectively are shown in Table I for various combinations of M_c , J_c , M_s and J_s . The numerical results for each combination yield the same number of eigenvalues in the same interval of λ as the exact solution. For the combination of $M_c=8$, $J_c=4$, $M_s=9$, $J_s=5$, the eigenvalues given by the numerical program are in the agreement with the corresponding exact values within 0.2%. This is a confirmation of the procedures developed in this paper and the numerical program for the eigenvalue problem.

A few words should now be said about how to pick the numbers M_c and J_c (also M_s and J_s), i.e., how many terms in the series representations of G^{+c} and G^{-} should be employed.

Since the eigenvalues of the truncated problem are assumed to converge as M and J increase simultaneously, their limits should be independent of the differences between M and J where M and J stand for M_c , J_c , or M_s , J_s . This fact is also confirmed by all of our numerical results. Nevertheless, it is relevant to point out the implications of the difference between M and J , i.e., the difference between the number of terms in the series solution for G^{+} and that for G^{-} . The conditions of eqs. (15) and (16) call for the matching of G^{+} and G^{-} and of their normal derivatives G_{μ}^{+} and G_{μ}^{-} .

* Note that $G(\theta, \mu) = \text{constant}$ is an eigensolution, therefore, $\lambda=0$ is an eigenvalue for any combinations of β and α .

across $\mu = \mu_0$ with $|\varphi| < \pi - \alpha$ and the vanishing of G_μ^+ at $\mu = \mu_0$ with $\pi - \alpha < |\varphi| \leq \pi$. Intuitively, the two matching conditions would require the same number of terms in both series while the latter condition on G_μ^+ is handled by the extra M-J terms in G^+ .

The qualitative relationship between M and J can be established clearly if eqs. (15) and (16) are fulfilled by the collocation method, i.e., eq. (15) and (16) will be enforced at the grid points uniformly distributed in the intervals for φ . The grid sizes in these intervals are $2(\pi - \alpha)/J$ and $(2\alpha)/(M - J)$ respectively. The condition for these two grid sizes to be equal or almost the same is

$$\pi/M \sim (\pi - \alpha)/J \quad (31)$$

It means that for given J, the integer M should be so chosen that the ratio M/J will be the nearest rational number to $\pi/(\pi - \alpha)$.

For the special corner considered in the diffraction problem, it is the corner of a cube with $\alpha = \pi/4$, $\beta = \pi/2$. Some of the eigenvalues can be found by inspection, namely, all the even integers. The first few exact eigensolutions are grouped as even and odd solutions of φ and listed in Table II. In the truncation of the series for G^+ and G^- , it is, desirable that these exact eigensolutions with $\lambda \leq M$ should be reproduced. This implies that the factor of φ in the last term of the truncated series of G^+ should be equal to or be the nearest one to that of G^- . For the even solutions, the condition is

$$M_c \approx (4/3) J_c \quad (32a)$$

and for the odd solution it is

$$M_s \approx (2/3) (2J_s - 1) \quad (32b)$$

Condition (32a) is identical to eq. (31) from collocation method while condition (32b) will be equivalent to (31) if the grid points in the collocation method are shifted by a half of the grid size.

Listed in Table I are the numerical results for the eigenvalues corresponding to various combinations of M_c , J_c and M_s , J_s . It is clear that for the same total number of terms $M + J$, the combination of M and J which fulfill eqs. (32a and b) are in better agreement with the exact solution than those with M equal to J.

With the relationship between M and J established, the next step is to choose the integer J for the truncation of the series. To give a measure of the accuracy of the eigenvalue $\lambda_{(J)}$ and the corresponding eigenfunctions $G_{(J)}^+$ and $G_{(J)}^-$, the following norms are introduced

$$I_1 = |\lambda_{(J_1)} - \lambda_{(J_2)}| \quad (33a)$$

$$I_2 = \int_0^{\bar{\Phi}/2} \left\{ G_{(J_1)}^+ + G_{(J_1)}^- - \left[G_{(J_2)}^+ + G_{(J_2)}^- \right] \right\}_{\mu=\mu_0}^2 d\varphi \bigg/ \int_0^{\bar{\Phi}/2} \left\{ G_{(J_1)}^+ + G_{(J_1)}^- \right\}_{\mu=\mu_0}^2 d\varphi \quad (33b)$$

$$I_3 = 4 \int_0^{\bar{\Phi}/2} \left[G^+ - G^- \right]_{\mu=\mu_0}^2 d\varphi \bigg/ \int_0^{\bar{\Phi}/2} \left[G^+ + G^- \right]_{\mu=\mu_0}^2 d\varphi \quad (33c)$$

$$I_4 = 4 \left\{ \int_0^{\bar{\Phi}/2} \left[\frac{\partial G^+}{\partial \mu} - \frac{\partial G^-}{\partial \mu} \right]_{\mu=\mu_0}^2 d\varphi + \int_{\bar{\Phi}/2}^{\pi} \left[\frac{\partial G^+}{\partial \mu} \right]_{\mu=\mu_0}^2 d\varphi \right\} \bigg/ \int_0^{\bar{\Phi}/2} \left[\frac{\partial G^+}{\partial \mu} + \frac{\partial G^-}{\partial \mu} \right]_{\mu=\mu_0}^2 d\varphi \quad (33d)$$

I_1 and I_2 measure the convergence of the eigenvalues and the eigenfunctions as the number of the term increases. I_3 and I_4 measure the degree of accuracy of the approximations to the matching conditions and the boundary condition at $\mu = \mu_0$, i.e., eqs. (15) and (16).

It should be pointed out here that although the eigenfunctions are continuous, its derivatives may not exist along the edges. The singularity along the edge $\theta = \pi$ is built in by the representation for $G^-(\mu, \varphi)$. The singularity along the edges, $\theta = \beta$ and $\varphi = \pi - \alpha$ or $\varphi = -\pi + \alpha$ can be ascertained by investigating the behavior of the solutions of the differential equations for $G(\mu, \varphi)$, eq. (7). In the neighborhood of an edge, i.e., $|\tilde{\mu}| \ll 1$ and $|\tilde{\varphi}| \ll 1$ with $\tilde{\mu} = \mu_0 - \mu$, $\tilde{\varphi} = \pi - \alpha - \varphi$ eq. (7) can be approximated by the Laplace equation

$$(1 - \mu_0^2)^2 \partial^2 G / \partial \tilde{\mu}^2 + \partial^2 G / \partial \tilde{\varphi}^2 = 0$$

in the first three quadrants of the $\tilde{\mu} - \tilde{\varphi}$ plane subjected to the boundary condition that the normal derivative of G should vanish along the positive $\tilde{\mu}$ -axis and along the negative $\tilde{\varphi}$ -axis. The potential solution G near the origin $\tilde{\mu} = \tilde{\varphi} = 0$ should behave as the real part of $[\tilde{\mu}(1 - \mu_0^2)^{-1} + i\tilde{\varphi}]^{2/3}$ [6]. The same result can be obtained by observing the three-dimensional corner directly: the surfaces of $\varphi = \pi - \alpha$ and $\theta = \beta$ intersect at right angle and in the neighborhood of the edge away from the vertex, the solution behaves as that of a two-dimensional convex right corner.

Along the interface of G^+ and G^- , i.e., $\mu = \mu_0$ and $\tilde{\varphi} > 0$, the normal derivative of G behaves as the real part of $(\tilde{\varphi}i)^{-1/3}$ i.e.,

$$G_{\mu}(\mu = \mu_0, \varphi = \pi - \alpha - 0^+) \sim (\pi - \alpha - \varphi)^{-1/3}.$$

G_μ is singular near the edge but its product with cosine or sine functions of φ is integrable and itself are square integrable. Therefore, eqs. (18) and (21) are meaningful and the norm I_4 exists.

Table III shows the eigenvalues between $-1/2$ and 3 for the corner of a cube, i.e., $2\alpha = \pi/2$ and $\beta = \pi/2$. The series representations are truncated with $M_c = 8$ and $J_c = 6$ for the even solutions and with $M_s = 10$ and $J_s = 8$ for the odd solutions. The coefficients in the series representations of the eigenfunctions are also listed. The norms, I_3 and I_4 , are computed for each eigenvalue. It is found that all the I_3 's are less than 1% and all the I_4 's are less than 10%. The matching of the normal derivative of G^+ and G^- as measured by I_4 is less satisfactory as expected due to the singularity at the edge and due to the fact that the exact value of $\partial G/\partial \mu$, i.e. zero, has been used in the denominator for I_4 in the interval $\pi/2 < \varphi < \pi$.

Calculations have been performed for various values of J_c and J_s . It is found that when $J_{1c} = 6$ and $J_{2c} = 9$ and also $J_{1s} = 8$ and $J_{2s} = 11$, all the I_1 's and I_2 's corresponding to the eigenvalues between $-1/2$ and 3 are less than .1%.

Table IV compares the eigenvalues for the corner of a cube with those for a circular cone with the same solid angle, i.e., with half angle, $\Theta = \arccos(3/4)$.

For the circular cone, the eigenfunctions are $P_\lambda^{-n}(\mu) \exp(in\varphi)$. The eigenvalues λ 's are defined by the condition of $dP_\lambda^{-n}(\mu)/d\mu = 0$ at $\mu = -\cos\Theta$. For each integer $n \geq 1$, the eigenvalue λ has a multiplicity of 2 and only for $n = 0$ the eigenvalue λ has a multiplicity of unity.

Table IV shows that the eigenvalues for the three-dimensional corner can be approximated by those of a circular cone with the same solid angle. It also shows that when an eigenvalue for the circular cone has a multiplicity of two, i.e., for $n \geq 1$, the corresponding values are found for the corner with even and odd eigensolutions and when the multiplicity is unity, i.e., $n=0$, the corresponding value for the corner has only an even eigensolution. It should be noted that when a corner degenerates to a plane, it is then identical with the equivalent circular cone of solid angle 2π .

The results in Table IV not only serves as a indirect confirmation of the numerical program but also suggests the use of an equivalent circular cone for a quick approximate determination of the eigenvalues of a three-dimensional corner.

5. Construction of the Conical Solution

In 2 the general formulation of the conical problem, the method of separation of variables was introduced. The disturbance pressure $p(\zeta, \theta, \varphi)$ can be written as

$$p(\zeta, \theta, \varphi) = \sum_{\lambda} K_{\lambda} Z_{\lambda}(\zeta) G_{\lambda}(\mu, \varphi) \quad (34)$$

The eigenvalues λ 's and the eigenfunction G_{λ} are determined in 4. The function $Z_{\lambda}(\zeta)$ is a solution of the differential eq. (6), i.e.,

$$\zeta^2 (1 - \zeta^2) Z''(\zeta) + 2(1 - \zeta^2) \zeta Z'(\zeta) - \lambda(\lambda + 1) Z(\zeta) = 0 \quad (35)$$

The boundary conditions are $Z(0) < \infty$ and $Z(1) = 1$. The condition on the unit sphere $\zeta = 1$ permits the determination of the coefficients K_{λ} from the boundary data on the unit sphere, eq. (5), and the eigenfunction $G_{\lambda}(\mu, \varphi)$ independent of $Z_{\lambda}(\zeta)$. The equation for the coefficient K_{λ} is

$$\begin{aligned} K_{\lambda} = & \frac{1}{2} \int_0^{\mu_0} d\mu \int_{-\pi}^{\pi} d\varphi F(\mu, \varphi) G_{\lambda}^{+}(\mu, \varphi) \\ & + \frac{1}{2} \int_{\mu_0}^{\pi} d\mu \int_{-\pi+\alpha}^{\pi-\alpha} d\varphi F(\mu, \varphi) G_{\lambda}^{-}(\mu, \varphi) \end{aligned} \quad (36)$$

The factor 1/2 is due to the normalization condition of eq. (12b) for G_{λ} in which the integration with respect to φ is carried over only half the interval. In the numerical determination of K_{λ} the double integral can be reduced to a line integral by splitting the boundary data to even and odd solutions and then to a cosine and a sine series in φ respectively. For the even solution the results are:

$$\begin{aligned} K_{\lambda} = & \int_{\mu_0}^1 \sum_{m=0,1}^M A_m P_{\lambda}^{-m}(\mu) f_m^{+}(\mu) d\mu \\ & + \int_{-1}^{\mu_0} \sum_{j=0,1}^J C_j P_{\lambda}^{-vj}(-\mu) f_j^{-}(\mu) d\mu \end{aligned} \quad (37)$$

where

$$f_m^+(\mu) = \frac{1}{2} \int_0^\pi [F(\mu, \varphi) + F(\mu, -\varphi)] \cos m\varphi \, d\varphi,$$

$$f_j^-(\mu) = \frac{1}{2} \int_0^{\Phi/2} [F(\mu, \varphi) + F(\mu, -\varphi)] \cos v_j \varphi \, d\varphi,$$

and $v_j = 2j\pi/\Phi.$

Corresponding to the even boundary data, eq. (37) holds for all the eigenvalues of the even solutions and A_m and C_j are the coefficients defined by eq. (25) for each λ .

Similarly the coefficients K_λ for odd eigensolutions are obtained as follows:

$$K_\lambda = \int_{\mu_0}^1 \sum_{m=1,2}^M B_m P_\lambda^{-m}(\mu) f_m^+(\mu) \, d\mu$$

$$+ \int_{-1}^{\mu_0} \sum_{j=1,2}^J D_j P_\lambda^{-v_j}(-\mu) f_j^-(\mu) \, d\mu \quad (38)$$

where

$$f_m^+ = \int_0^\pi \frac{1}{2} [F(\mu, \varphi) - F(\mu, -\varphi)] \sin m\varphi \, d\varphi$$

$$f_j^- = \int_0^{\Phi/2} \frac{1}{2} [F(\mu, \varphi) - F(\mu, -\varphi)] \sin v_j \varphi \, d\varphi$$

and $v_j = (2j-1)\pi/\Phi.$

For the determination of the pressure distribution inside the unit sphere, it is necessary to construct the solution $Z_\lambda(\zeta)$ of the differential eq. (35), subjected to the boundary conditions at $\zeta=0$ and $\zeta=1$. The solution is obtained numerically by the "shooting method". For Z_λ to be finite at $\zeta=0$ the solution can be represented by the power series,

$$Z(\zeta) = a_0 \zeta^\lambda \left[1 + \frac{\lambda(\lambda+1)}{4\lambda+6} \zeta^2 + \dots \right] \quad (39)$$

By setting $a_0=1$, the differential equation can be integrated numerically

from a small ζ_0 (say 0.001) with initial data $\tilde{Z}_\lambda(\zeta_0) = \zeta_0^\lambda \left[1 + \lambda(\lambda+1)\zeta_0^2 / (4\lambda+6) \right]$ and $\tilde{Z}'_\lambda(\zeta_0) = \lambda \zeta_0^{\lambda-1} \left[1 + (\lambda+1)(\lambda+2) \zeta_0^2 / (4\lambda+6) \right]$.

The numerical integration for $\tilde{Z}_\lambda(\zeta)$ is continued until ζ is close to 1 (say 0.99). The value of $\tilde{Z}_\lambda(1)$ is obtained by patching the numerical solution with the series solution of ζ^{-1} , i.e.,

$$\tilde{Z}_\lambda(\zeta) = \tilde{Z}_\lambda(1) \left\{ 1 - \frac{\lambda(\lambda+1)}{4} (1-\zeta) + \frac{\lambda^2(\lambda+1)^2}{16} (1-\zeta)^2 + a_1 (1-\zeta)^3 \log(1-\zeta) + \dots \right\} \quad (40)$$

Since the problem is linear, the correct solution is

$$Z_\lambda(\zeta) = \tilde{Z}_\lambda(\zeta) / \tilde{Z}_\lambda(1)$$

The appropriate value for a_0 or $\lim_{\zeta \rightarrow 0} Z \zeta^{-\lambda}$ is $1/\tilde{Z}_\lambda(1)$. The numerical program in the appendix computes a_0 automatically and generates the function $Z_\lambda(\zeta)$. Thus for a given boundary data $F(\mu, \varphi)$ on the unit sphere, the conical solution of eq. (3) is defined.

6. A Numerical Example

Figure 4 shows the incidence of a plane pulse normal to a face say OAD of the corner of a cube. In order to demonstrate the use of both the even and the odd eigensolutions, one of the two edges parallel to the plane pulse is chosen as the negative z-axis as shown in the figure. After the incidence, $t > 0$, part of the plane pulse is reflected by the face OAD of the cube with double the intensity. The remaining part advancing over the corner is undisturbed. The diffraction by either one of the edges OA and OD, is given by a two dimensional conical solution. It is confined in a circular cylindrical surface with radius Ct and the edge as the axis provided it is outside the domain of influence of the vertex.

Figure 5 shows the incident plane pulse, the reflected pulse and the sonic circle in a cross-section normal to the edge OA at a distance greater than Ct from the vertex. In terms of polar coordinates ρ and τ , the boundary condition on the sonic circle $\rho = Ct$ for the pressure variation from that behind the incident pulse is

$$p_c = 1 \quad \pi < \tau < 3\pi/2$$

$$p_c = 0 \quad 0 < \tau < \pi$$

The strength of the incident pulse is chosen as unity. For the two-dimensional conical solution, the Busemann conical transformation is introduced [2],

$$\rho^* = \bar{\rho} / \left\{ 1 + [1 - \bar{\rho}^2]^{1/2} \right\}, \quad \bar{\rho} = \rho / (Ct) \text{ and } \tau^* = \tau \quad (41)$$

In the new variables, the disturbance pressure becomes a potential solution and the region inside the circle $\rho^* < 1$ with $0 < \tau^* < 3\pi/2$ is mapped to a half circle with $\tilde{\rho} \exp(i\tilde{\tau}) = [\rho^* \exp(i\tau^*)]^{2/3}$. The solution in terms of $\tilde{\rho}, \tilde{\tau}$ is [2]

$$p_c(\rho, \tau) = \frac{1}{\pi} \arctan \frac{(1 - \tilde{\rho}^2) \sqrt{3/2}}{(1 + \tilde{\rho}^2)/2 + 2\tilde{\rho} \cos \tilde{\tau}} \quad (42)$$

where the arctangent lies in the first and the second quadrants. The circular cylinder with axis OA and radius Ct intersects the sonic sphere about the vertex along a greater circle and covers the spherical surface on the right side of plane OBD. Similarly, the diffraction by the other edge, OD, created another two-dimensional conical solution given by eq. (42) with ρ the distance from the edge OD and τ the angle measured from face OBD. Again, the circular cylinder with axis OD and radius Ct intersects the sonic sphere about the vertex along another great circle and covers the part of the spherical surface below plane OAB, i.e., $\mu < 0$.

The surface of the sonic sphere can be divided into four regions:

- 1) not covered by either one of these circular cylinders
- 2) covered only by the cylinder with axis OA

- 3) covered only by the cylinder with axis OD and
- 4) covered by both cylinders.

With the plane $\varphi=0$ defined as the plane bisecting the exterior angle of the two vertical surface OAD and OBD, the four regions and the boundary data are defined as

$$\begin{aligned} \text{i) Region I} \quad 1 > \mu > 0; \quad -3\pi/4 < \varphi < \pi/4 \\ F(\mu, \varphi) &= 0 \end{aligned} \quad (43a)$$

$$\begin{aligned} \text{ii) Region II} \quad 1 > \mu > 0; \quad \pi/4 < \varphi \leq \pi \text{ and } -\pi \leq \varphi < -3\pi/4 \text{ (or } \pi \leq \varphi < 5\pi/4) \\ F(\mu, \varphi) &= p_c(\bar{\rho}_1, \tau_1) \end{aligned} \quad (43b)$$

$$\begin{aligned} \text{iii) Region III} \quad 0 > \mu > -1; \quad -3\pi/4 < \varphi < \pi/4 \\ F(\mu, \varphi) &= p_c(\bar{\rho}_2, \tau_2) \end{aligned} \quad (43c)$$

$$\begin{aligned} \text{iv) Region IV} \quad 0 > \mu > -1; \quad \pi/4 < \varphi < 3\pi/4 \\ F(\mu, \varphi) &= p_c(\bar{\rho}_1, \tau_1) + p_c(\bar{\rho}_2, \tau_2) - 1 \end{aligned} \quad (43d)$$

where

$$\bar{\rho}_1 = [\mu^2 + (1-\mu^2) \sin^2(\varphi - 3\pi/4)]^{\frac{1}{2}}$$

$$\tau_1 = \begin{cases} \sin^{-1} |\mu/\bar{\rho}_1| & \text{for } \sin(\varphi - 3\pi/4) \geq 0, \mu \geq 0 \\ \pi - \sin^{-1} |\mu/\bar{\rho}_1| & \text{for } \sin(\varphi - 3\pi/4) < 0, \mu \geq 0 \\ \pi + \sin^{-1} |\mu/\bar{\rho}_1| & \text{for } \mu < 0 \end{cases}$$

and

$$\bar{\rho}_2 = (1-\mu^2)^{\frac{1}{2}}$$

$$\tau_2 = \varphi + 3\pi/4$$

With the boundary data $F(\mu, \varphi)$ defined by eq. (43) the coefficients K_λ in eq. (34) for the pressure variations inside the sonic circle are computed for all the eigenvalues between $-\frac{1}{2}$ and 3. The pressure distribution on the three faces of the corner are shown in Figs. 6 & 7. The pressure distributions for $r/(Ct) < 1$ are obtained from the present analysis. They match with the corresponding two-dimensional values across the sonic sphere with a discontinuity in slope. They confirm, with less than 1% variation, the symmetry of the solution with respect to the plane bisecting the faces OAB and OAD and the value of 8/7 at the vertex as predicted by Theorem I in 8. When the calculation is performed with only eigenvalues less or equal to 2, the results differ from those in Figs. 6, 7 within 6%.

7. General Incident Wave

The preceding conical solutions for an incident plane pulse can be employed to construct diffraction solutions for an incident wave of more general type by superposition of plane pulses. In particular, when the incident wave is a plane wave of the type

$$p_i = \psi(\eta) \quad (44)$$

$$\text{with} \quad \eta = Ct - (n_1 x + n_2 y + n_3 z) \quad (45)$$

where the wave form ψ , is an arbitrary function of its phase η and n_1, n_2 and n_3 are the direction cosines of the normal to the plane wave.

The incident wave due to the sonic boom can be locally represented as a plane wave since the length of a structure is usually much smaller than the radius of curvature of the wave front. The wave form ψ is in general a sequence of a weak shock wave and an expansive wave or a compression wave as shown in Fig. 8.

If the wave form is a Heaviside function, the diffraction due to the three-dimensional corner is given by the preceding conical solution and will be designated as $p^*(r/(Ct), \theta, \varphi)$. The solution of the diffracted wave corresponding to a general wave form of a sonic boom is

$$p(r, \theta, \varphi, t) = \sum_i \left\{ (\Delta p)_i p^* \left(\frac{r}{Ct - \eta_i}, \theta, \varphi \right) + \int_{\eta_i}^{\eta_{i+1}} p^* \left(\frac{r}{Ct - \eta}, \theta, \varphi \right) \frac{d\psi}{d\eta} d\eta \right\} \quad (46)$$

where η_i is the phase of the i -th shock wave with strength $(\Delta p)_i = \psi(\eta_i + 0) - \psi(\eta_i - 0)$. Note that η increases in the direction opposite to the normal of the plane wave.

Another method for the construction of nonconical diffraction solutions should be pointed out here. When the boundary data on the sonic sphere $\zeta = r/(Ct) = 1$ around the vertex of a three-dimensional corner, which is now a function of t in addition to the spherical angles θ and φ , can be written as a power series with respect to t , i.e.,

$$F(t, \theta, \varphi) = \sum_Y t^Y F_Y(\theta, \varphi) \text{ for } t > 0 \quad (47)$$

where the summation is carried over a sequence of positive numbers, $\gamma \geq 0$. The solution inside the sonic sphere can likewise be written as

$$p(t, r, \theta, \varphi) = \sum_Y t^Y p^{(\gamma)}(\zeta, \theta, \varphi) \quad (48)$$

The wave equation yields the governing equation for $p^{(\gamma)}(\zeta, \theta, \varphi)$

$$\begin{aligned} \zeta^2 (1-\zeta^2) \frac{\partial^2 p^{(\gamma)}}{\partial \zeta^2} + 2\zeta[1+(\gamma-1)\zeta^2] \frac{\partial p^{(\gamma)}}{\partial \zeta} - \gamma(\gamma-1)\zeta^2 p^{(\gamma)} \\ + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial p^{(\gamma)}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 p^{(\gamma)}}{\partial \varphi^2} = 0 \end{aligned} \quad (49)$$

$p^{(\gamma)}$ is subjected to the boundary condition on the sphere $\zeta=1$,

$$p^{(\gamma)}(1, \theta, \varphi) = F_{\gamma}(\theta, \varphi)$$

and the boundary conditions on the surfaces of the corner remain the same as eqs. (3,4).

Since the differential operators with respect to θ and φ are independent of γ , the corresponding eigenvalue problem in θ - φ plane after the separation of the variable ζ is the same as that for $\gamma=0$, i.e., the conical problem.

The solution $p^{(\gamma)}$ can therefore be written as,

$$p^{(\gamma)}(\zeta, \theta, \varphi) = \sum_{\lambda} K_{\lambda}^{(\gamma)} Z_{\lambda}^{(\gamma)}(\zeta) G_{\lambda}(\mu, \varphi) \quad (50)$$

where $\mu = \cos \theta$. The eigenvalues λ 's and the eigenfunctions $G_{\lambda}(\mu, \varphi)$ are identical with those obtained for $\gamma = 0$ in 3,4. With the boundary condition $Z_{\lambda}^{(\gamma)}(1) = 0$ imposed on $Z_{\lambda}^{(\gamma)}$, the constants $K_{\lambda}^{(\gamma)}$ are also related to the boundary data $F_{\gamma}(\mu, \theta)$ by the same set of equations, eqs. (37) and (38), and can be determined by the same numerical program for $\gamma = 0$. The appearance of γ occurs only in the differential equation for $Z_{\lambda}^{(\gamma)}(\zeta)$, which is

$$\zeta^2 (1-\zeta^2) \frac{d^2 Z}{d\zeta^2} + 2 [1+(\gamma-1)\zeta^2] \frac{dZ}{d\zeta} - [\gamma(\gamma-1)\zeta^2 + \lambda(\lambda+1)] Z = 0 \quad (51)$$

The boundary conditions are the same, i.e., $Z(1) = 1$ and $Z(0)$ is finite. $Z_{\lambda}^{(\gamma)}(\zeta)$ can be determined by the same procedure described in 5 for $Z_{\lambda}^{(0)}(\zeta)$.

Now it is clear that the extension of the proceeding conical solutions to the diffraction of a general incident wave by a three-dimensional corner can be carried out provided that the boundary data $F(t, \theta, \varphi)$ on the sonic sphere can be found. This suggests that the next step is to solve the diffraction problem of a three-dimensional wave by a two-dimensional corner.

For a two dimensional corner, the three dimensional diffracted waves can be constructed by superposition of solutions for incident plane waves with different wave forms and directions of incidence or by the method of separation of variables in cylindrical coordinates and time. These schemes in general are tedious for the actual construction of the solution. By making use of the special geometry of the 90° convex corner, an integral

equation can be set up and solved by an iteration scheme similar to the procedure for a two dimensional incident wave [3]. This method of approach is presently under investigation.

In the next section, two theorems are presented so that the value along the edge or the vertex of a corner can be related directly to an incident wave without the construction of the diffraction solution. The incident wave can be completely arbitrary and hence can also be a diffracted wave from an adjacent corner.

8. "Mean-Value" Theorems for Diffraction by a Conical Surface and Applications

In a previous article [7], a mean value theorem for the diffraction of an incident wave by a cone was derived. It states that the value at the vertex of the cone is equal to $4\pi/(4\pi-\Omega)$ times the value of the incident wave at the vertex as if the value is averaged locally over the exterior solid angle of the cone, $4\pi-\Omega$ instead of over the whole space, 4π . This relationship was obtained under the restriction that the incident wave does not hit any portion of the conical surface prior to its encounter with the vertex of the cone. In this section, the derivation of the theorem is partially repeated with modifications so that the theorem is valid under a weaker restriction, that is, the incident wave may hit the conical surface within a finite time interval T ahead of the epoch of its incidence on the vertex. In case such a finite time interval T cannot be found, a second theorem is introduced. Extension of the results to solutions of inhomogeneous wave equations is also presented in this section.

Let $t=0$ be the instant when an acoustic incident wave $\varphi^{(i)}$ hits the vertex of the cone located at the origin. Let T be the finite time interval such that for $t \leq -T$ the incident wave does not hit the conical surface. Let $D(t)$ denote the domain outside of which the incident wave vanishes at the instant t , e.g., $\varphi^{(i)}(x,y,z,T)=0$ for (x,y,z) not in $D(-T)$. Domain $D(t \leq -T)$ does not intersect the cone G with solid angle Ω , as shown in Fig. 9. In the absence of the cone, the incident wave $\varphi^{(i)}$ at the origin can be related to the initial data at an instant $t=t_0 < -T$ by the Poisson Formula [8],

$$\varphi^{(i)}(0,0,0,t) = \frac{1}{4\pi C} \left[\frac{\partial}{\partial t} \iint_S \frac{f}{r} dS + \iint_S \frac{g}{r} dS \right]$$

where r is distance from the origin,

$$f(x,y,z) = \varphi^{(i)}(x,y,z,-t_0),$$

$$g(x,y,z) = \varphi_t^{(i)}(x,y,z,-t_0)$$

and S is the sphere with radius $R=C(t+t_0) > C(t+T)$ and with its center at the origin. The Poisson formula can be identified as a special form of the Kirchhoff's formula [8],

$$\varphi^{(i)}(0,0,0,t) = - \frac{1}{4\pi} \iint_S \left\{ \left[\varphi^{(i)} \right] \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \left[\frac{\partial \varphi^{(i)}}{\partial n} \right] - \frac{1}{Cr} \frac{\partial r}{\partial n} \left[\frac{\partial \varphi^{(i)}}{\partial t} \right] \right\} dS \quad (52)$$

where $\partial/\partial n$ means differentiation along the outward normal to the surface S and $[\varphi]$ is the retarded value of φ . On the sphere S with $R=a(t+t_0)$, it follows $[\varphi^{(i)}(x,y,z,t)] = \varphi^{(i)}(x,y,z,-t_0)$. Similarly, the retarded values of

$\varphi_n^{(i)}$ and $\varphi_t^{(i)}$ are the corresponding values at $t = -t_0$. All of them vanish outside the domain $D(-t_0)$. Let S_c denote the part of the sphere outside of the cone G , and S^* denote the part inside the domain $D(-t_0)$. Since $D(-t_0)$ with $-t_0 < -T$ does not intersect G , S_c contains S^* . The integrand is non-zero only inside S^* , therefore, the domain of the surface integration can be reduced from S to S^* or to S_c .

To obtain the resultant wave φ at the vertex of the cone, the passages in the derivation of the Kirchhoff formula [7,9] from the Green's theorem will be repeated. The Green's theorem is now applied to the volume bounded by the two concentric spheres S^* and S_σ with radii $r=R$ and $r=\sigma$ respectively and the conical surface ∂G yields

$$\iint_{S^*+S_\sigma+\partial G^*} \left\{ \left(\frac{[\varphi]}{r^2} + \frac{1}{rC} \left[\frac{\partial \varphi}{\partial t} \right] \right) \frac{\partial r}{\partial n} + \frac{1}{r} \left[\frac{\partial \varphi}{\partial n} \right] \right\} dS = 0 \quad (53)$$

where ∂G^* is the part of ∂G lying between S^* and S_σ .

On the surface of the cone ∂G , $\partial r / \partial n = 0$. With the boundary condition of $\partial \varphi / \partial n = 0$, the integral over ∂G^* vanishes.

As $\sigma \rightarrow 0$, the integral over S_σ approaches $-(4\pi - \Omega) \varphi(0,0,0,t)$. Equation (53) reduces to

$$(4\pi - \Omega) \varphi(0,0,0,t) = - \iint_{S^*} \left\{ [\varphi] \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \left[\frac{\partial \varphi}{\partial n} \right] - \frac{1}{Cr} \frac{\partial r}{\partial n} \left[\frac{\partial \varphi}{\partial t} \right] \right\} dS \quad (54)$$

On the spherical surface S^* , $[\varphi] = \varphi(x,y,z,-t_0) = \varphi^{(i)}(x,y,z,-t_0) = [\varphi^{(i)}]$ and similar relationship holds for the t - and n - derivatives. Eqs. (52) and (54) yield:

$$\varphi(0,0,0,t) = \frac{4\pi}{4\pi - \Omega} \varphi^{(i)}(0,0,0,t) \text{ for } t > -T \quad (55)$$

This result can be stated as follows:

Theorem I

The resultant value at the vertex of a cone with solid angle Ω is equal to $4\pi / (4\pi - \Omega)$ times the incident wave at the vertex, if a finite T can be found such that the incident wave hits the conical surface within the time interval T prior to its encounter with the vertex of the cone.

In a practical problem, the conical surface which forms a part of the surface of an obstacle, has a finite length, say L . In applying the theorem, it is essential that the part of the obstacle inside the sphere S with radius $C(t + T)$ is conical, i.e.,

$$C(t + T) < L \quad (56)$$

It defines an upper bound for T . On the other hand, for a given T with $CT < L$, the inequality (56) defines an upper bound for t for which eq. (55) holds.

When an incident wave is diffracted first by a part of the surface of the obstacle other than the conical surface of finite length L , condition (56) cannot be fulfilled. Even for a cone of infinite length, the incident pulse may be in contact with a part of the conical surface all the time, consequently, the finite time interval T assumed in Theorem I does not exist. For both examples, the value at the vertex of the cone cannot be related directly to the incident wave by Theorem I. A theorem relating the value at the vertex of one cone to that of another cone will be presented.

Figure 10 shows the relative orientations of two cones G_1 and G_2 , with solid angles Ω_1 and Ω_2 . Their vertices coincide at the origin and their boundaries ∂G_1 and ∂G_2 have a common region Γ , i.e.,

$$\Gamma = \partial G_1 \cap \partial G_2 \quad (57)$$

Let $\varphi_1(x, y, z, t)$ denote the resultant of an incident wave $\varphi^{(i)}$ and its reflected and diffracted waves by cone G_1 alone with $\partial\varphi_1/\partial n = 0$ on the surface of the cone ∂G_1 . $D_1(t)$ denotes the domain outside of which $\varphi_1 = 0$ at the instant t . At $t=0$, the origin lies on the boundary of $D_1(t)$ and $D_1(t)$ does not contain the origin for $t < 0$.

It will be assumed that there exists a finite time interval T such that the domain $D(t \leq -T)$ is in partial contact with the cone G_1 only over the surface Γ , i.e.,

$$(\partial D(t) \cap \partial G_1) \subset (\partial G_1 \cap \partial G_2) = \Gamma \quad \text{for } t \leq -T \quad (58)$$

The next step is to examine the solution φ_2 which is the resultant of the same incident wave $\varphi^{(i)}$ and its reflected and diffracted waves by cone G_2 alone. The initial data for φ_2 are prescribed by $\varphi_1(x, y, z, -t_0)$ and $\varphi_{1,t}(x, y, z, -t_0)$ for $-t_0 < -T$. If cone G_2 does not intersect the domain $D_1(t)$ for $t < -T$, it is evident from conditions (58) that

$$\varphi_2(x, y, z, t) = \varphi_1(x, y, z, t)$$

$$\text{and} \quad D_2(t) \equiv D_1(t) \quad \text{for } -T \geq t \geq -t_0 \quad (59)$$

The derivation of eq. (54) from the Green's theorem will be repeated for the volume bounded by two concentric spheres S and S_σ and by the conic surface ∂G_1 or ∂G_2 respectively. The two spheres, S and S_σ are centered at the origin and their radii are $C(t+t_0)$ and σ . The following result equivalent to eq. (54) is obtained:

$$(4\pi - \Omega_j) \varphi_j(0,0,0,t) = - \iint_{S_j^*} \left\{ [\varphi_j] \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \left[\frac{\partial \varphi_j}{\partial n} \right] - \frac{1}{rC} \frac{\partial r}{\partial n} \left[\frac{\partial \varphi_j}{\partial t} \right] \right\} dS$$

for $j = 1, 2 \quad t \geq -T$ (60)

On the sphere S with radius $C(t+t_0)$, it is evident $[\varphi_j] = \varphi_j(x,y,z,-t_0)$. Equation (59) shows that $[\varphi_j]$, $[\partial \varphi_j / \partial n]$, $[\partial \varphi_j / \partial t]$ and $D_j(-t_0)$ are invariant with respect to j . S_j^* is defined as the part of the spherical surface S lying in $D_j(-t_0)$ and is also independent of j . With the right side of Eq. (60) invariant with respect to j , it yields

$$(4\pi - \Omega_1) \varphi_1(0,0,0,t) = (4\pi - \Omega_2) \varphi_2(0,0,0,t) \text{ for } t \geq -T \quad (61)$$

So far the problem was formulated as propagation and diffraction of waves. It can also be formulated as an initial and boundary value problem and restated as follows:

Theorem II

For two conical surfaces ∂G_1 and ∂G_2 with the same location for the vertices and with the same initial data $\varphi=f(x,y,z)$ and $\varphi_t=g(x,y,z)$ at the instant $t=-t_0 < 0$, the resultant disturbance at the vertex for each cone alone is inversely proportional to the exterior solid angle of the cone provided that the support D for f and g does not intersect either one of the cones, and that the part of the boundary ∂D in common with one of the conical surface is also with the other, i.e.,

$$(\partial D \cap \partial G_1) = (\partial D \cap \partial G_2) \subset (\partial G_1 \cap \partial G_2) \quad (62)$$

It is obvious that when the wave equation has an inhomogeneous term $h(x,y,z,t)$ with support $E(t)$ similar conclusions can be obtained:

Corollary Theorem I and II will also be valid for solutions of the inhomogeneous wave equation $\Delta \varphi - C^{-2} \varphi_{tt} = h(x,y,z,t)$ provided that the support $E(t)$ of h does not intersect the cone or both cones respectively.

An interesting application for Theorem I and the Corollary will be the initial and boundary value problem for an inhomogeneous wave equation in the interior of a conical surface ∂G_c with solid angle Ω_c (or the exterior of a cone G with solid angle $\Omega=4\pi - \Omega_c > 2\pi$). The mathematical problem is:

D. E. $\Delta \varphi - C^{-2} \varphi_{tt} = h(x,y,z,t)$ inside the cone G_c for $t > 0$,

B. C. $\varphi_n = 0$ on ∂G_c ,

I. C. $\varphi=f(x,y,z)$, $\varphi_t = g(x,y,z)$ at $t=0$.

From the definition of the problem both the support E of h and the support

D of f and g lie in G_c and, therefore, do not intersect G_c . Theorem I and the corollary state that:

The value at the vertex, $\varphi(o,o,o,t)$ is related to the solution, $\varphi_{3D}(o,o,o,t)$, of the initial value problem for the inhomogeneous wave equation in three-dimensional space with the removal of the conical surface by an amplification factor equal to the local enlargement in solid angle, i.e., $4\pi/\Omega_c$. This statement is expressed by the following equation,

$$\begin{aligned}\varphi(o,o,o,t) &= \frac{4\pi}{\Omega_c} \left\{ \varphi_{3D}(o,o,o,t) \right\} \\ &= \frac{4\pi}{\Omega_c} \left\{ \frac{1}{4\pi C} \left[\frac{\partial}{\partial t} \iint_{S_{Ct}^*} \frac{f}{r} dS + \iint_{S_{Ct}^*} \frac{g}{r} dS \right] \right. \\ &\quad \left. + \frac{1}{4\pi} \int_0^{Ct} dr \iint_{S_r^*} \frac{[h]}{r} dS \right\} \quad (63)\end{aligned}$$

where S_{Ct}^* is the part of the spherical surface inside the support D of f and g and S_r^* is the part of the spherical surface inside the support E of h at the retarded instant $t-r/C$. The spheres are centered at the origin and the subscript denotes the radius.

It is of interest to note that the value at the vertex of the cone is independent of the geometry of the cone G_c nor does it depend on the distribution of f, g, and h and their supports D and E with respect to the spherical angles θ and φ so long as the integrals of f, g and h over the spherical surface are invariant as functions of r and t.

Another example is the diffraction problem being analyzed in this paper. As a matter of fact, the investigations in this section are motivated by the desire to determine the values along an edge or a corner of a rectangular block or structure due to the arrival of a conical solution originated from an adjacent edge or corner without the construction of the non-conical solution.

Figure 11 shows an incident plane pulse of strength ϵ impinging on a block with normal parallel to an edge OB. After the instant $t = 0$, the pulse is reflected by the face OAD and diffracted around the edges OA, OD and the corner O. Let OA, OB and OD to be designated as the x-, y-, z-, axis respectively. From Theorem I, the value at the corner O of solid angle $\pi/2$ is

$$p_1(o,o,o,t) = \epsilon [4\pi / (4\pi - \pi/2)] = (8/7) \epsilon \text{ for } W/C > t > 0. \quad (64)$$

This is in agreement with the numerical result in 6 and is valid until the arrival at $t = W/C$ of the diffracted wave from the adjacent corners.

For the convenience of description, it is assumed that the height OD is equal to the width $W(=OA)$ and is related to the length $L(=OB)$ by the inequality

$$2L > W > L \quad (65)$$

For a point $(x_0, 0, 0)$ on the edge of OA with solid angle π , Theorem I gives

$$p_1(x_0, 0, 0, t) = \epsilon [4\pi / (4\pi - \pi)] = (4/3) \epsilon \text{ for } x_0/C > t > 0 \quad (66)$$

Due to symmetry, x_0 can be assumed to be less than $W/2$ and the upper limit for t is the epoch for the arrival of the diffracted wave from the vertex O.

For the disturbance pressure elsewhere such as on the surfaces of the block it is necessary to construct the diffracted waves. Behind the incident front the disturbance pressure is $p_1 = \epsilon$ outside the envelopes of sonic spheres along the edges OA and OD. Inside the sonic sphere around the vertex O the pressure is given by the three-dimensional conical solution constructed in 7, i.e., $p_1 = \epsilon + \epsilon p_{3c}(\bar{x}, \bar{y}, \bar{z})$ with $\bar{x}, \bar{y}, \bar{z}$ equal to $x/(Ct)$, $y/(Ct)$, $z/(Ct)$ respectively. Outside the sonic sphere but inside one of the cylindrical envelopes, the solutions are conical in two dimensions i.e., $p_1 = \epsilon + \epsilon p_{2c}(\bar{x}, \bar{y})$ and $p_1 = \epsilon + \epsilon p_{2c}(\bar{z}, \bar{y})$ for edges OA and OD respectively. Of course there are also diffracted waves from vertices A, D, E, and edges AD and DE.

When $t > L/C$, the wave front advances over the edges BA', BD' and the vertex B. If the edges BA', BD' were absent and the planes OBA and OBD were extended beyond BA' and BD', point B can be considered as the vertex of a cone G_1 , with solid angle $\Omega_1 = \pi$ and the solution p_1 at B is valid for $t > L/C$ (Fig. 12). In the presence of the edges BA' and BD', point B can be considered as the vertex of a cone G_2 , with solid angle $\Omega_2 = \pi/2$. Cone G_1 and cone G_2 have faces OBAB' and ODAD' in common and time interval T can be equated to zero. Theorem II yields the resultant value at B for cone G_2 , i.e., including the diffraction of p_1 by corner B,

$$\begin{aligned} p_2(0, L, 0, t) &= [(4\pi - \pi) / (4\pi - \pi/2)] p_1(0, L, 0, t) \\ &= (6/7) \epsilon [1 + p_{3c}(0, L/Ct, 0)] \text{ for } L/C > t - L/C > 0 \end{aligned} \quad (67)$$

The upper limit of $t = 2L/C$ is due to the finite length L of both cone G_1 and cone G_2 .

In case of $(W^2 + L^2)^{1/2} < 2L$, the conical solutions originated at the vertices A and D arrive at corner B before $t = 2L/C$, additional terms should be added for $t > (W^2 + L^2)^{1/2}/C$. They can again be defined by Theorem II. The incoming wave from corner A is $\epsilon p_{3c}(\bar{x}', \bar{y}, \bar{z})$ with $\bar{x}' = (W - x)/(Ct)$.

Point B can be considered as the vertex of cone G_2 defined as before and as the vertex of cone G_1 with the extension of face OABA' over the edges OB and BA'. The solid angle for G_1 is 2π and the ratio of solid angles exterior to G_1 and G_2 is $(4\pi - 2\pi) / (4\pi - \pi/2) = 4/7$. Theorem II yields the

extra term $(4/7) \epsilon p_{3c}(W/(Ct), L/(Ct), 0)$. Similarly a term $(4/7) \epsilon p_{3c}(0, L/(Ct), W/(Ct))$ should be added to Eq. (67) to take into account the incoming wave from vertex D for $t > (W^2 + L^2)^{1/2}/C$.

The disturbance pressure at a point $P(x_0, L, 0)$ on the edge BA' can also be related to the diffracted waves from edge OA and from vertices O and A by means of Theorem II. Point p can be considered as the vertex of cone G_2 with solid angle $\Omega_2 = \pi$ (Fig. 11). When the face $OABA'$ is extended over the edge BA' , point P becomes the vertex of cone G_1 with solid angle $\Omega_1 = 2\pi$ (Fig. 12). The resultant pressure at point P due to diffraction is equal to $2/3$ of the value of the incoming wave alone. The factor $2/3$ is the ratio of exterior solid angles of G_1 and G_2 . This result is valid until the arrival of the non-conical diffracted wave from vertex B' or A' .

It has been demonstrated that the theorems are useful to extend the knowledge of conical solutions to the adjacent edges and vertices for limited time intervals. To obtain the pressure distribution on the surfaces of the rectangular block or structure, it is still necessary to construct three-dimensional non-conical solutions for diffractions by edges and corners.

Conclusion

In this paper the following results are obtained:

- 1) the conical solution for the diffraction of a plane acoustic pulse by a three-dimensional corner of a cube is obtained by separation of variables.
- 2) the solutions of the eigenvalue problem in spherical angles for the conical problem remain the same for the generalized conical solutions described in 7 and also for potential or unsteady heat conduction problems, so long as the boundary conditions in θ, φ plane are the same.
- 3) the technique for the solution of the eigenvalue problem is applicable to a more general shape of boundary on the unit sphere formed by two great circles of given longitudes and a horizontal circle of given latitude. The boundary conditions can be generalized to the type $a\psi + b \frac{\partial \psi}{\partial n} = 0$ with both a and b as non-negative constants.
- 4) the numerical results suggest that the eigenvalues for corners can be approximated, one by one, by the eigenvalues for circular cones of the same solid angle.
- 5) "mean-value" theorems in 8 are derived for solutions of wave equations so that the resultant wave at the vertex of a cone can be related to the incident wave or the value at the vertex of a different cone. These theorems are useful to extend the knowledge of the conical solutions to the adjacent corners or edges.
- 6) applications of the Theorems in 8 yield interesting and simple results: for an initial value problem of an inhomogeneous wave equation inside a conical surface with vanishing normal derivative on the surface, the value at the vertex of the cone can be related directly to the same initial value problem in three-dimensional space. The value at the vertex depends only on the value of the solid angle of the cone but not on the geometry of the cone. It depends on the radial distribution of the initial data and the inhomogeneous term but not on their distribution with respect to the spherical angles.

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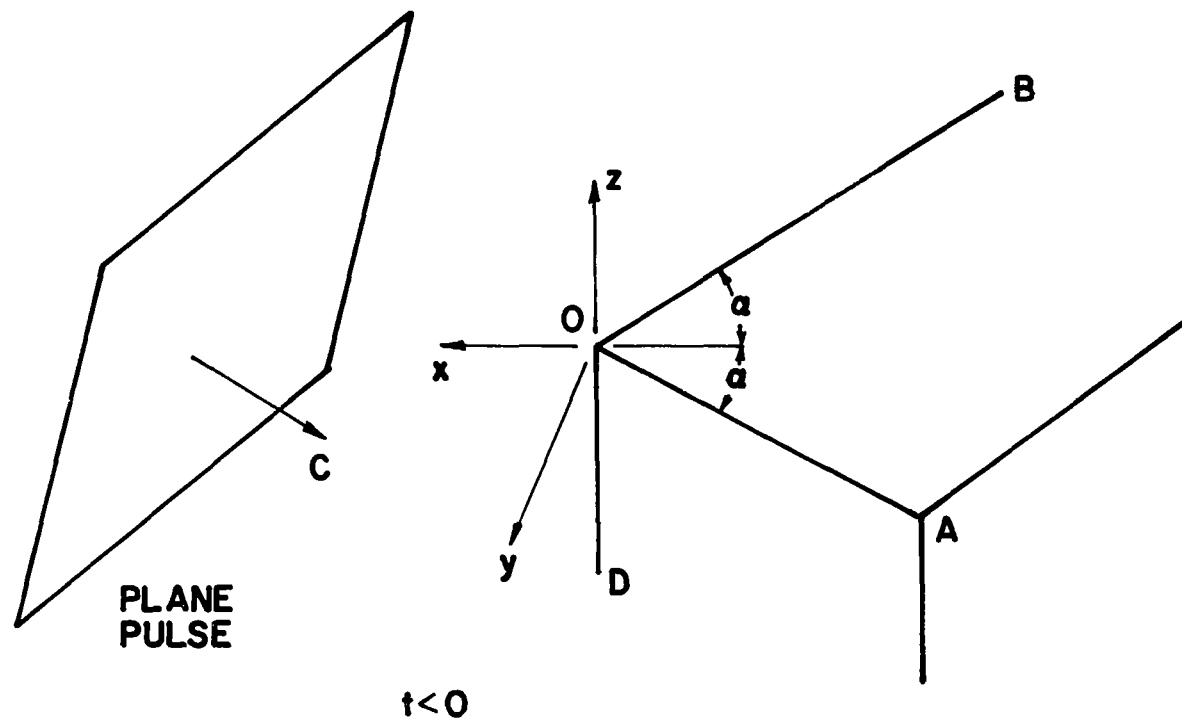


Fig. 1 Incidence of a plane pulse on a corner

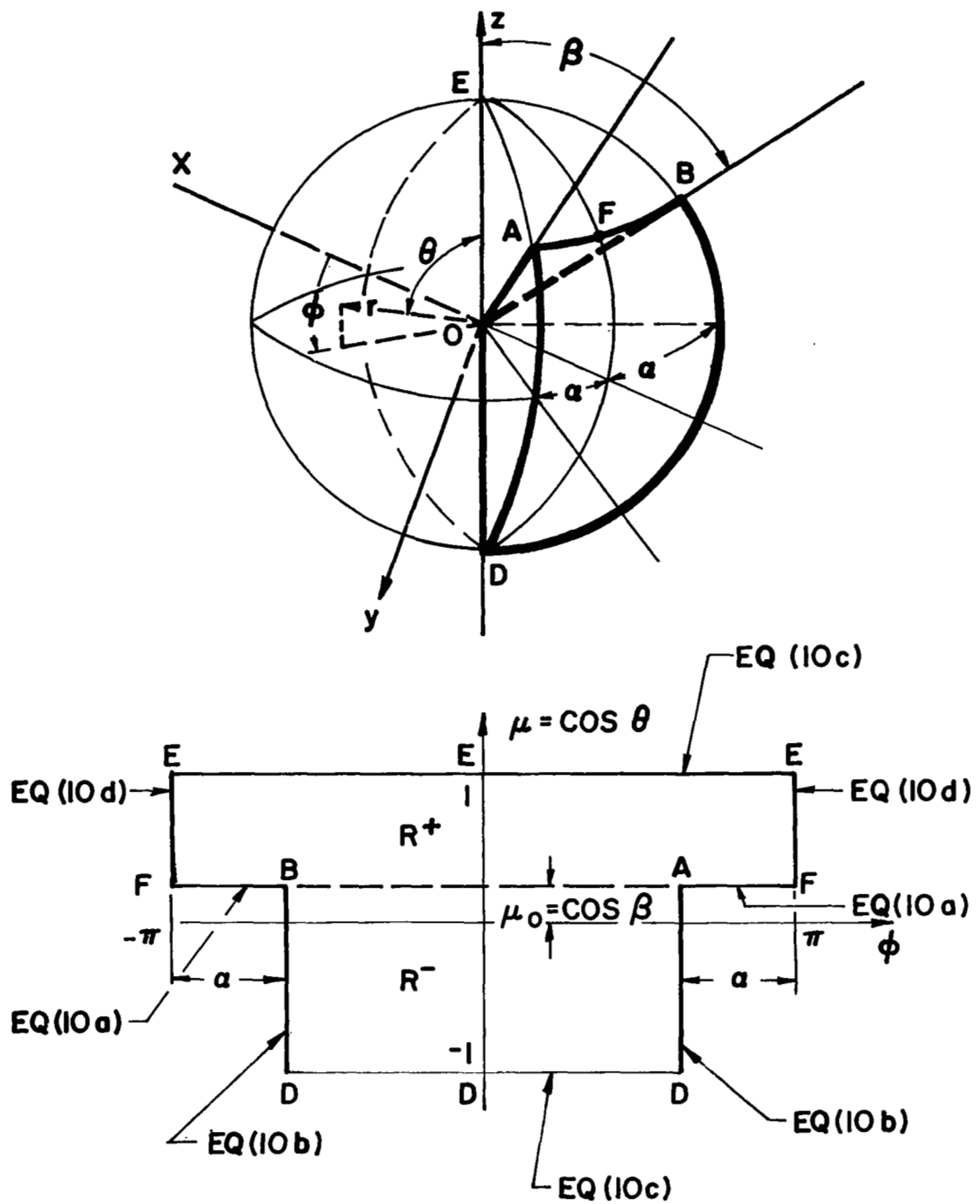


Fig. 2 Eigenvalue problem for the corner

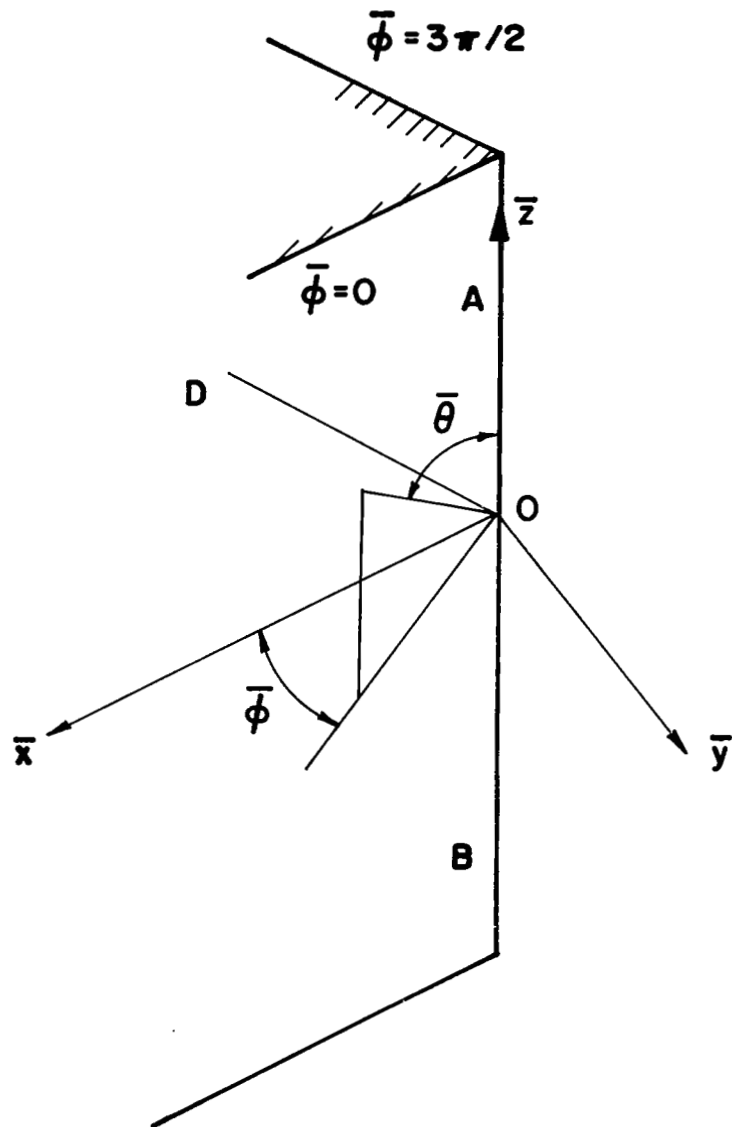


Fig. 3 A two-dimensional corner

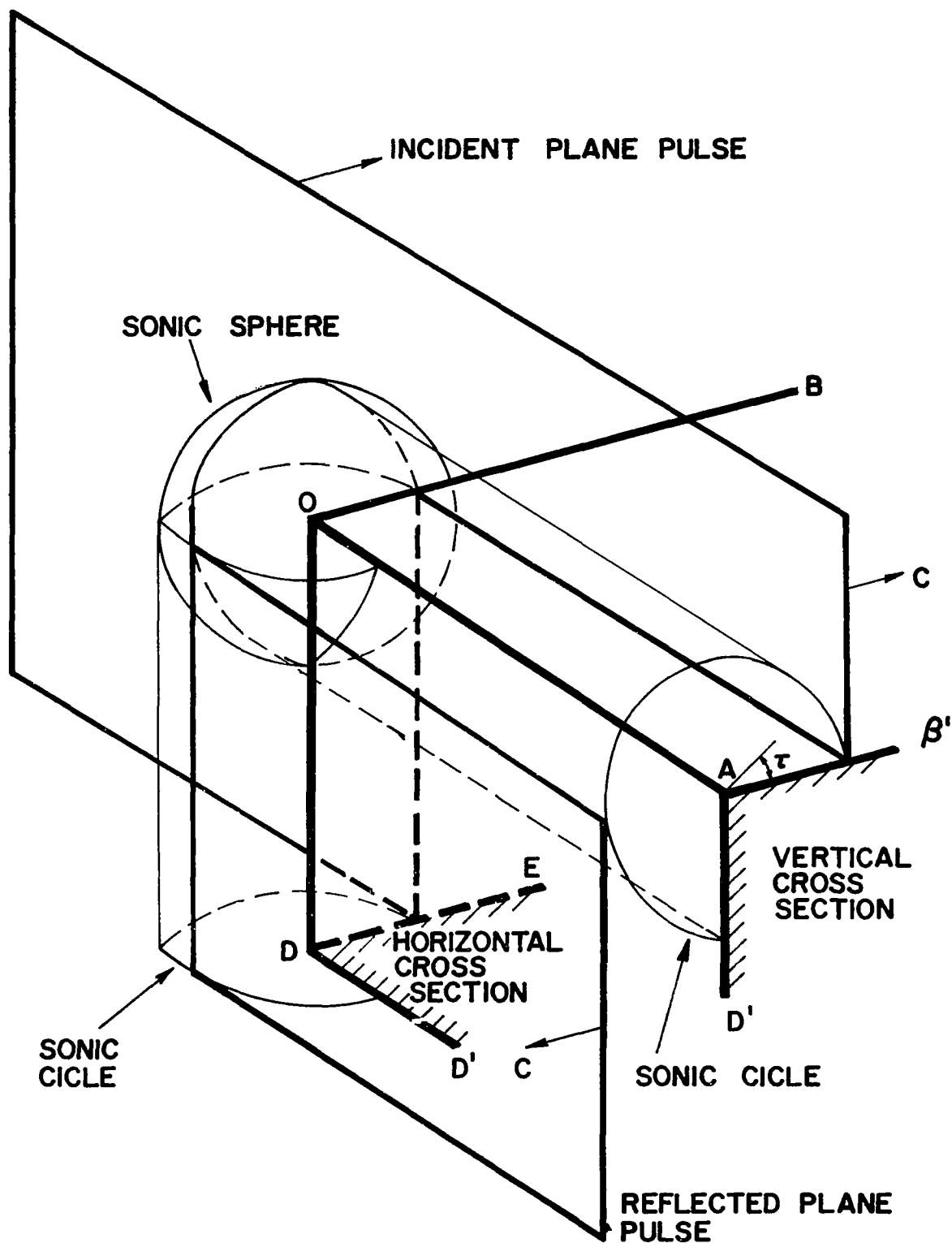
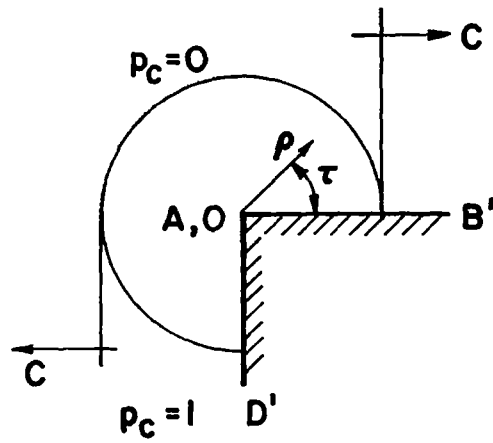
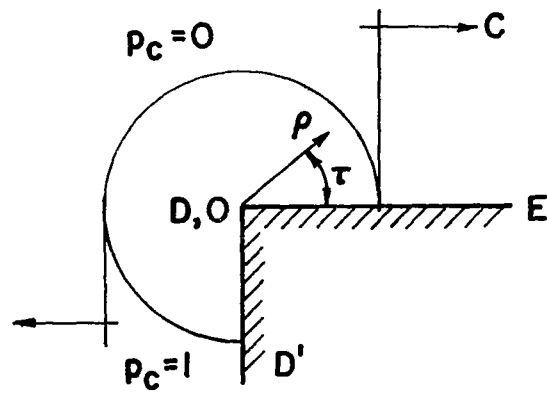


Fig. 4 Plane pulse incidence head on to face OAD, the reflected plane pulse and the diffracted fronts around edges OA, OD, and vertex O



SECTION NORMAL TO EDGE OA



SECTION NORMAL TO EDGE OD

Fig. 5 Two-dimensional conical solutions around edges OA and OD

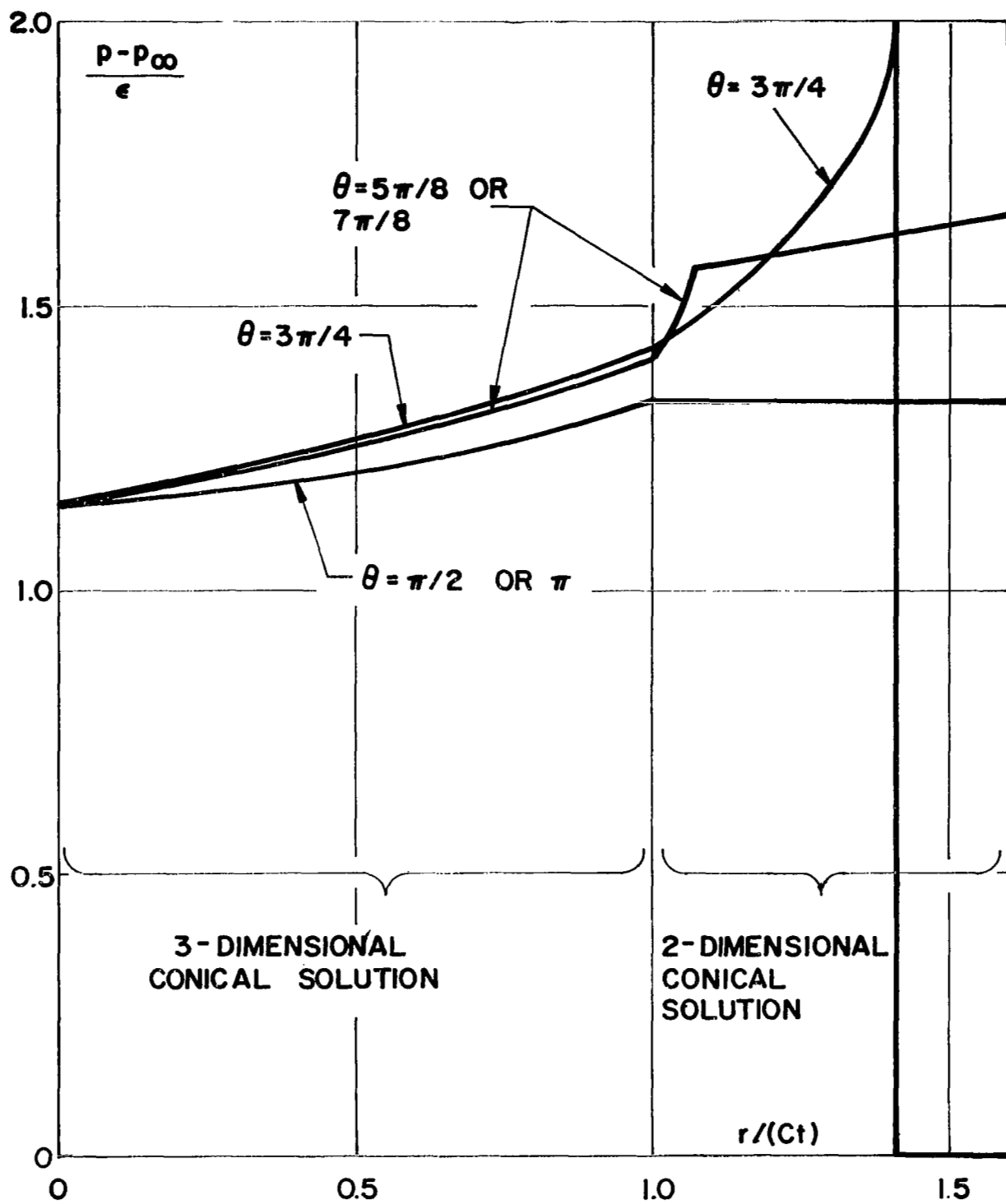


Fig. 6 Over pressure along constant θ lines face OAD ($\psi=3\pi/4$)

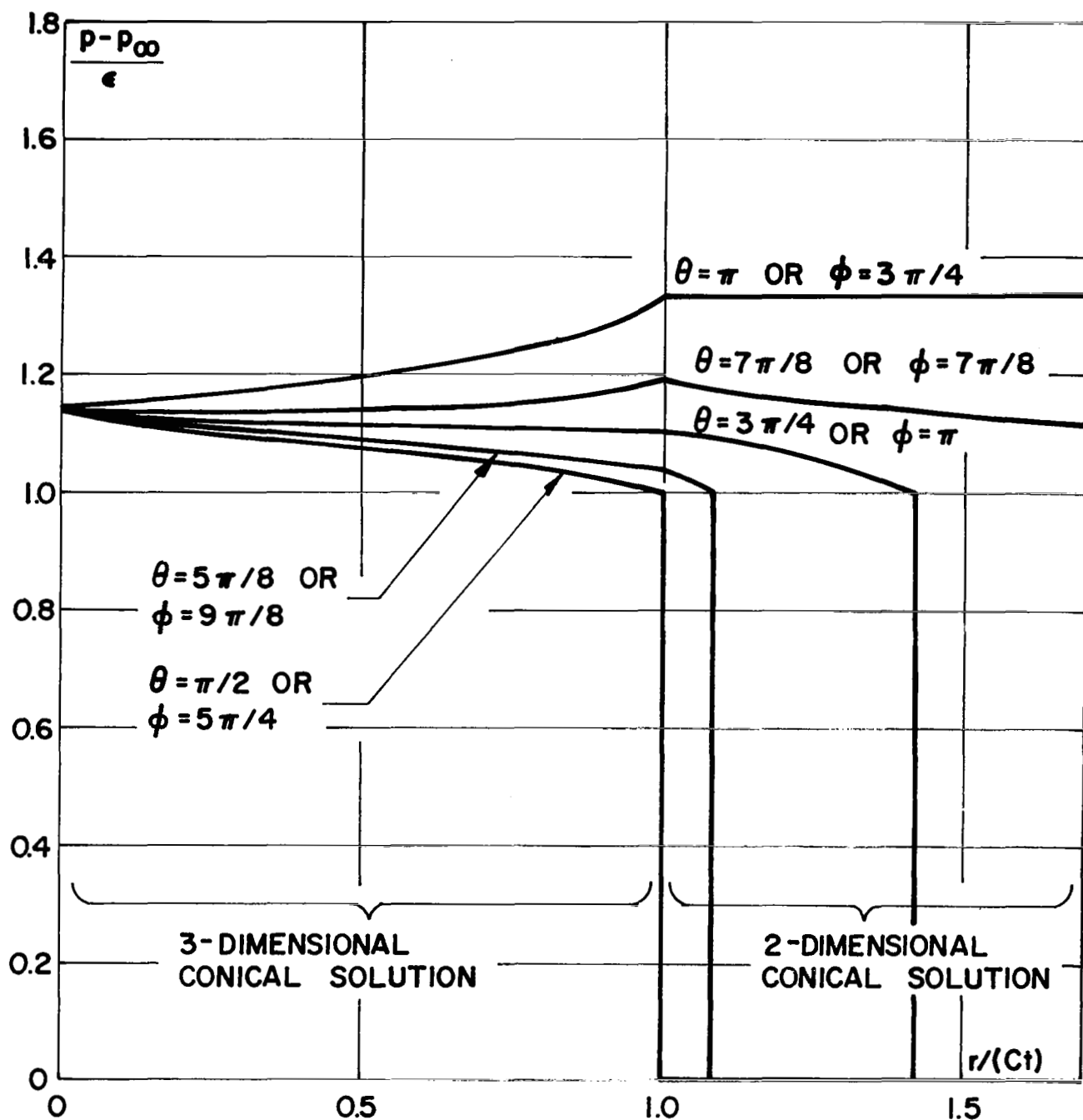


Fig. 7 Over pressure along constant ϕ lines on face OAB ($\theta = \pi/2$) and along constant θ lines on face OBD ($\phi = 3\pi/4$)

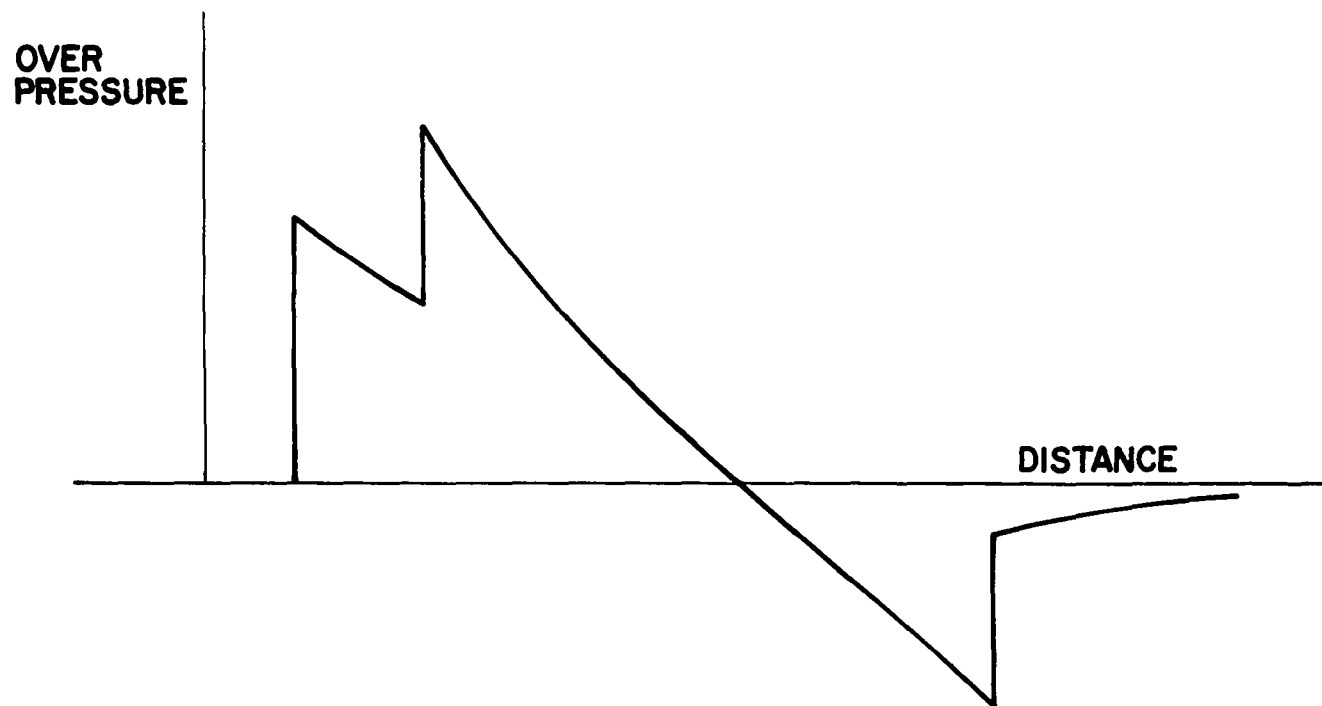


Fig. 8 Typical sonic boom signature

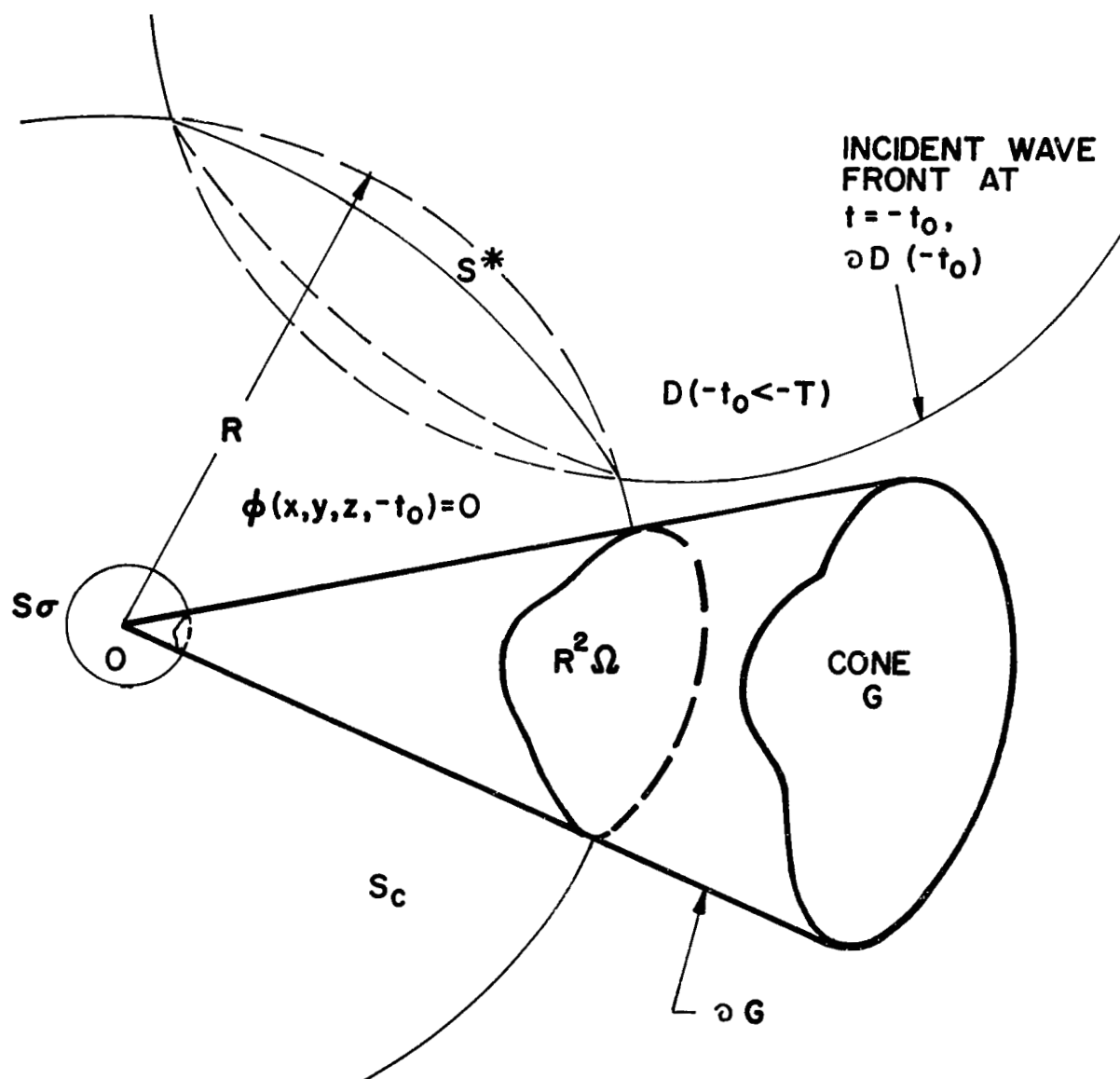


Fig. 9 Diffraction by a cone

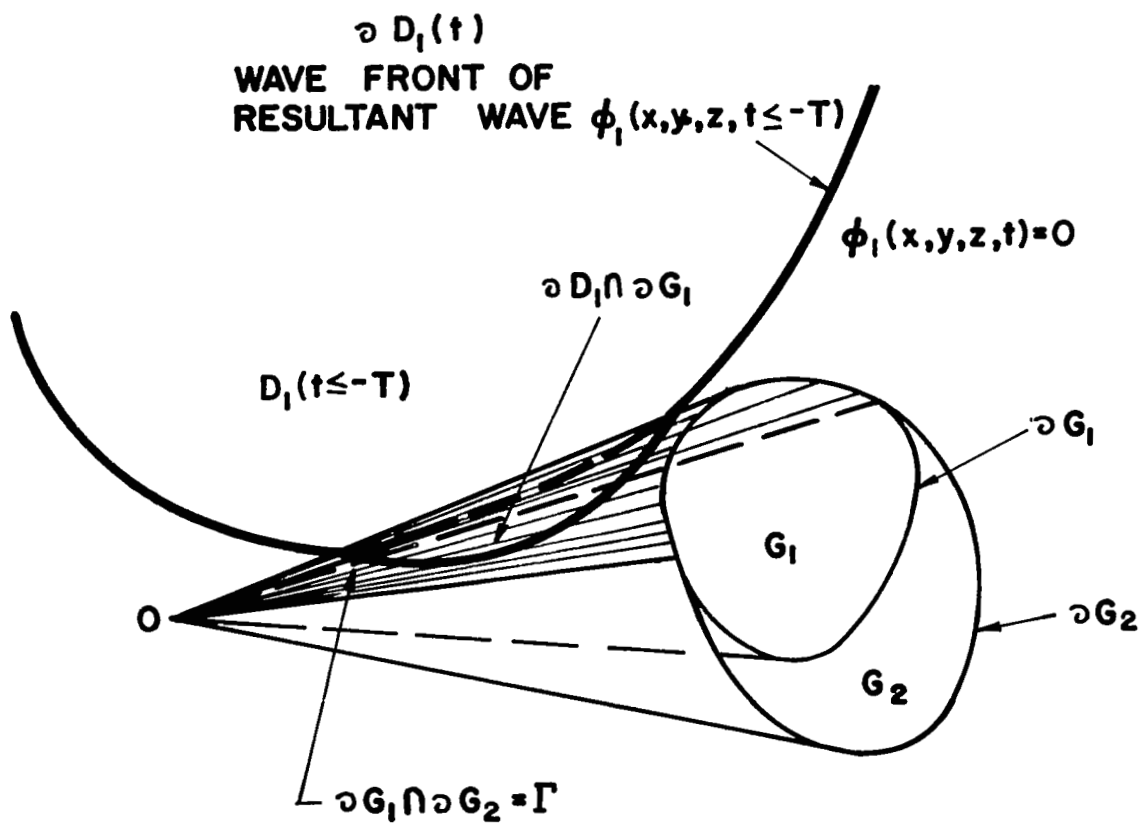


Fig. 10 Comparison of two cones

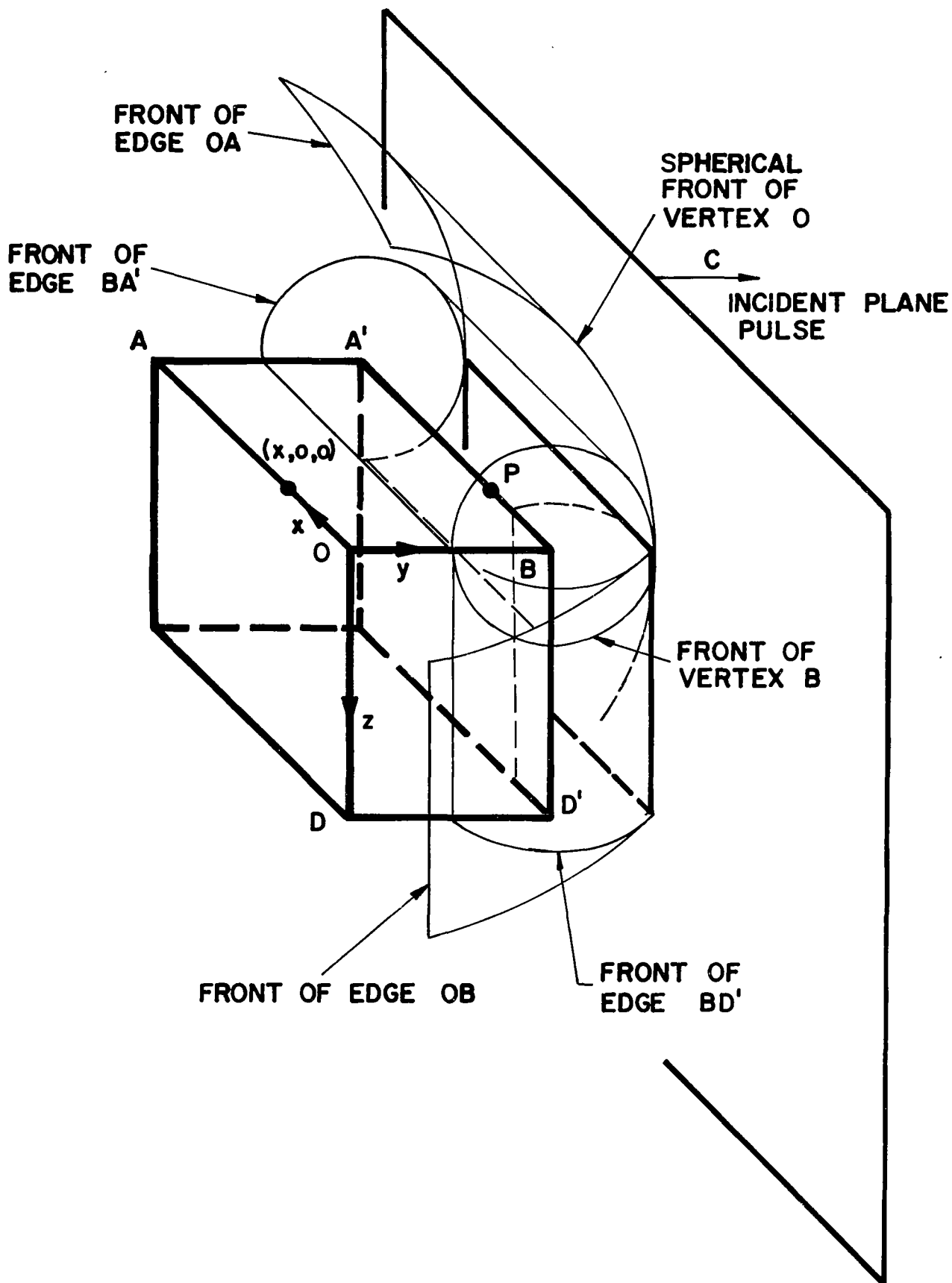


Fig. 11 Incident of a plane pulse on a block at instant $t > L/C$ ($OB=L$, $OA=OD=W$; also shown are the diffracted wave fronts around edges OA, OD, BA', BD' and around vertices O and B).

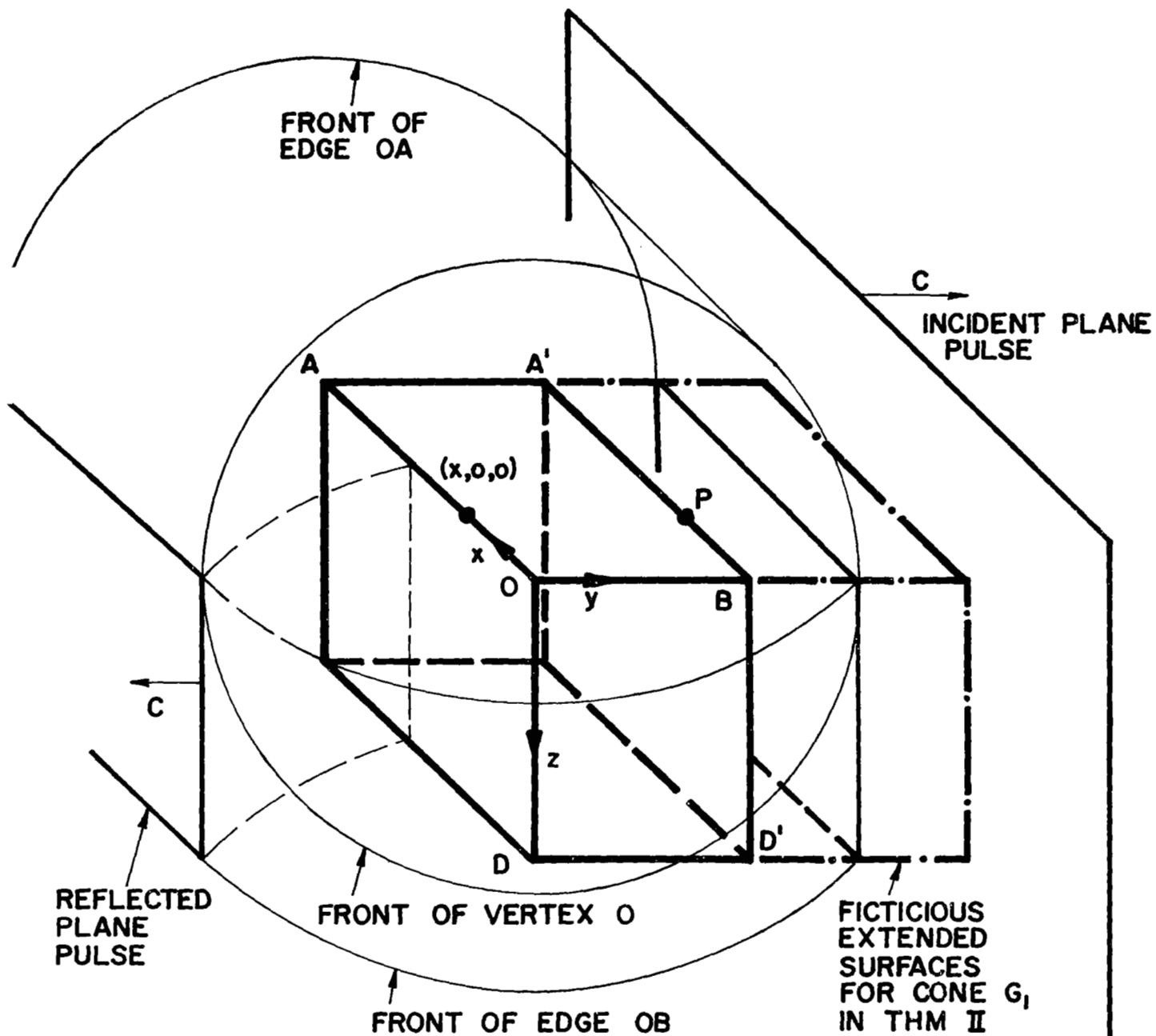


Fig. 12 Incident of a plane pulse on a block with extended fictitious surfaces at instant $t > L/C$ (also shown are the diffracted wave fronts around edges OA, OD and vertex O)

APPENDIX

NUMERICAL PROGRAMS

Four complete sets of listing programs for CDC-6600 are given in the appendix. They are called first program (A), first program (B), second program and third program. Their input and output formats and the operating instructions are presented as follows:

First Program (A): determination of eigenvalues and eigenfunctions for odd function.

First Program (B): determination of eigenvalues and eigenfunctions for even function.

Input Definition

IIPP . a control constant
 To calculate determinant and search for eigenvalues;
 let IIPP equal to any negative or zero integer.
 To calculate determinant and coefficients of eigen-
 function; let IIPP equal to any positive integer.

NMAX : for odd function $NMAX=M_s$; for even function $NMAX=M_c$

LMAX : for odd function $LMAX=J_s$; for even function $LMAX=J_c$

XLAM : λ
 If $IIPP \leq 1$; XLAM is only a searching eigenvalue. More
 accurate value for the roots of $\Delta(\lambda)=0$ can be obtained
 by interpolation. Be careful not to overlook multiple
 root across which $\Delta(\lambda)$ may not change sign.
 If $IIPP > 1$; XLAM is an eigenvalue as located by the
 preceding method, and ready to calculate the coef-
 ficients of eigenfunctions.

Input

Card 1; IIPP, NMAX, LMAX (Format 315)

Card 2; XLAM (Format F15.0)

Output and Definition

$\left. \begin{matrix} NMAX \\ LMAX \end{matrix} \right\}$: input data previously described

IIMAX : number of linear homogeneous equations, for the odd
 solutions; $IIMAX=LMAX+NMAX$; for the even solution:
 $IIMAX=LMAX+NMAX+2$

ALPHA : 2α

Output and Definition

PPHI : $\Phi = 2(\pi - \alpha)$
XNUI : $\nu_0 = \pi/\Phi$
XMU : $\mu_0 = \cos(\beta)$ where β is the latitude of upper surface
XLAM : previously described
DET : the value of determinant for a given λ ; $\Delta c(\lambda) = |a_{ij}|$
BMIN(J), : coefficients in the series representation of eigen-
DMIN(L) functions,

for even function First $M_c + 1$ output are A_m 's
 Next $J_c + 1$ output are C_j 's

for odd function First M_s are B_m 's
 Next J_s are D_j 's

These A_m 's, C_j 's, B_m 's, D_j 's are defined by eq. (25).

Second Program: Determination of the coefficient K_λ of the eigen-
function expansion from the initial data.

Input Definition

II : control constant
II=1; to calculate K_λ for odd function
II equal to any integer other than one; to calculate
 K_λ for even function

NMAX }
LMAX } : previously described
XLAM }

BMIN(J) }
DMIN(L) } : these are the coefficients calculated from program
one (A) for odd function or program one (B) for even
function

Input

II=1, NMAX, LMAX (3 I5)

Set { XLAM (F15.0)
1 { BMIN(J) (5F15.0)
 DMIN(L) (5F15.0)

Set { }
.
.
.
END FILE

Input

II=2, NMAX, LMAX

{ }

{ }

·
·
·

{ }

END FILE

Note that the bracket indicates a set of data composed of a given λ and its corresponding coefficients BMIN(J) and DMIN(L). Note also that these sets are then listed into a group of odd functions and a group of even functions separated by an end file card.

This use of the end file card facilitates the orderly calculations of all odd functions first and then all even functions.

Output and Definition

XLAM
NMAX, LMAX, IIMAX, ALPHA,
PPHI, XNUI, XMU
BMIN(J)
DMIN(L)

These are
input data which
are listed
for identification
and verification

K(Lamda)

K_λ the coefficient of the
eigenfunction expansion from
the initial data

Remark: the subroutine FSF and subroutine CON2D prepare the boundary data on the sonic sphere for the problem described in 6. For a different problem only these two subroutines should be changed.

Third Program: determination of the pressure distribution on the surfaces.

Input same as the second program, except the first card of each set of input data contains both λ and its corresponding K_λ which was calculated from the second program.

Output and Definition

XLAM : λ

AAO : defined in eq. (39)

K(Lamda) : K_λ

$\left. \begin{matrix} \text{PHI} \\ \text{XMU} \end{matrix} \right\} : \begin{matrix} \varphi & \text{where } \theta \text{ and } \varphi \text{ are spherical coordinates} \\ \mu = \cos \theta & \text{for points on the surfaces} \end{matrix}$

$P_1 : \text{ pressure on surface OAB on figure (4) where } \theta = \beta, \mu = \cos \beta; \varphi_N = \pi - \alpha + \frac{2\alpha}{4} (N-1), N = 1, 2, \dots, 5$

$P_2 : \text{ pressure on surface OAD where } \varphi = \frac{3\pi}{4}; \theta_N = \frac{\pi}{2} + \frac{\pi}{8} (N-1) \text{ where } N = 1, 2, 3, 4, 5$

$P_3 : \text{ pressure on surface OBD where } \varphi = \frac{-3\pi}{4}; \theta_N = \frac{\pi}{2} + \frac{\pi}{8} (N-1), N = 1, 2, \dots, 5$

Note: P_1, P_2, P_3 are defined in eq. (34)

Output Format

Odd function

NMAX, LMAX, IIMAX,
ALPHA, PPHI, XNUI, XMU

Set 1 $\left\{ \begin{matrix} \text{XLAM, AAO, K(Lamda)} \\ \text{BMIN(J)} \\ \text{DMIN(J)} \end{matrix} \right\}$

Set 2 $\left\{ \quad \right\}$

·
·
·

Even function

NMAX, LMAX, IIMAX,
ALPHA, PPHI, XNUI, XMU

Sect. 1 $\left\{ \quad \right\}$

Sect. 2 $\left\{ \quad \right\}$

·
·
·

At the last page of the output, a table of PHI, P_1 , XMU, P_2 , P_3 is tabulated in five columns with Format (2E16.5, 15X, 3E16.5).

Remark: In this program, there are 10 $Z(\zeta)$'s where ζ 's are 0.111, 0.211, to 0.9989. In case of a different increment of ζ , the subroutine T102 must then use a different value of XTEST.

C FIRST PROGRAM (A)

C ODD FUNCTION TO CALCULATE DETERMINANT AND COEFFICIENTS

```

    DIMENSION AEL(50,50),B(50,50)
    DIMENSION ADET(50,50),EEEE(50,50)
    DIMENSION SUMA(101),SUMB(101),BMIN(30),DMIN(30),SBMIN(30),
1SDMIN(30)
    LLMAX=50
    READ(5,5) IIPP,NMAX,LMAX
5  FORMAT(3I5)
    IIMAX=LMAX+NMAX
    ITTEM=IIMAX
    PI=3.1415926
    ALPHA=PI/2.
    PPHI=2.*(PI-ALPHA/2.)
    XNU1=PI/PPHI
    INTERV=30
    LEP=LMAX
    JEP=NMAX
    UO=0.
    XMU=0.
    XMUFIX=XMU
    WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
502  FORMAT(      *  NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,      *
1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
400  READ(5,31) XLAM
31  FORMAT(F14.0)
    IF(ENDFILE 5) 5559,5556
5556  CONTINUE
C  CALCULATION OF ELEMENTS OF DETERMINANT
    DO 303 J=1,IIMAX
    DO 300 K=1,IIMAX
    IF(J-NMAX ) 10,10,200
10  IF(K-NMAX ) 11,11,30
11  CONTINUE
    EFL=1.
14  IF(J-K) 16,15,16
15  DELTA=1.
    GO TO 17
16  DELTA=0.
17  SS=K
    CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
    AEL(J,K)=DP*EFL*DELTA
    GO TO 300
30  SS=(2*(K-NMAX)-1)*XNU1
    CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
    LH=K-NMAX
    LI=J
    CALL XLALL(LH,LI,ALPHA,XNU1,XII)
    AEL(J,K)=DP*XII
    GO TO 300
200  IF(K-NMAX ) 212,212,230
212  LH=J-NMAX
    LI=K
    CALL XLALL(LH,LI,ALPHA,XNU1,XII)
    SS=K
    CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
    AEL(J,K)=PF*XII
    GO TO 300

```

```

230 IF(J-K) 232,232,233
232 DELTA=1.
    GO TO 234
233 DELTA=0.
234 CONTINUE
    EEL=1.
237 SS=(2*(K-NMAX)-1)*XNU1
    CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
    AEL(J,K)=-((1.-ALPHA/(2.*PI))*EEL*PP*DELTA
300 CONTINUE
303 CONTINUE
    DO 307 I=1,ITTEM
    DO 306 J=1,ITTEM
    B(I,J)=0.
    EEEE(I,J)=AEL(I,J)
306 CONTINUE
307 CONTINUE
    CALL LEQ(AEL,B,IIMAX,0,50,50,DET)
    WRITE(6,32) XLAM ,DET
32  FORMAT(*      XLAM=*,E15.8,5X,*DET=*,E15.8)
    IF (IIPP) 400,400,1002
1002 CONTINUE
C  CALCULATION OF COFACTORS
    DO 903 I=1,ITTEM
    DO 902 J=1,ITTEM
    ADET(I,J)=EEEE(I,J)
902 CONTINUE
903 CONTINUE
    ICORD=1
    DO 511 J=1,JEP
    JCORD=J
    CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
    BMIN(J)=-((-1.)*JCORD*DETCOE)
511 CONTINUE
    DO 512 L=1,LEP
    JCORD=L+JEP
    CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
    DMIN(L)=-((-1.)*JCORD*DETCOE)
512 CONTINUE
    DELU=(1.-U0)/INTERV
    ISTEP=INTERV+1
    DO 611 I=1,ISTEP
    XMM=(I-1)*DELU
    SUM10=0.
    DO 601 J=1,JEP
    SS=J
    CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
    SUM10=SUM10+(BMIN(J)*PP)**2
601 CONTINUE
    SUMA(I)=SUM10
611 CONTINUE
    DELV=(1.+U0)/INTERV
    DO 711 I=1,ISTEP
    XMM=(I-1)*DELV
    SUM11=0.
    DO 603 L=1,LEP
    SS=(2*L-1)*XNU1
    CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)

```

```

SUM11=SUM11+(DMIN(L)*PP)**2
603 CONTINUE
SUMB(I)=SUM11
604 FORMAT(15X,I5,3E20.8)
711 CONTINUE
TOSUM1=SUMA(1)
TOSUM2=SUMB(1)
DO 622 I=2,INTERV
COEF=3.+(-1.)*I
TOSUM1=TOSUM1+COEF*SUMA(I)
TOSUM2=TOSUM2+COEF*SUMB(I)
622 CONTINUE
TOSUM1=(TOSUM1+SUMA(ISTEP))*DELU/3.
TOSUM2=(TOSUM2+SUMB(ISTEP))*DELV/3.
ODDN=TOSUM1*PI/2.+PPHI*TOSUM2/4.
641 FORMAT(9X,3E20.8)
DO 633 I=1,JEP
SBMIN(I)=BMIN(I)/SQRT(ODDN)
BMIN(I)=SBMIN(I)
633 CONTINUE
C CALCULATION OF NORMALIZED COFACTORS --COEFFICIENT OF EIGEN FUNCTION
WRITE(6,6)
6 FORMAT(// * COEFFICIENTS OF ODD FUNCTION*)
304 FORMAT(5E20.8/(5X,5E20.8))
WRITE(6,304) ( BMIN(I),I=1,JEP)
DO 634 L=1,LEP
SDMIN(L)=DMIN(L)/SQRT(ODDN)
DMIN(L)=SDMIN(L)
634 CONTINUE
WRITE(6,304) ( DMIN(L),L=1,LEP)
GO TO 400
5559 STOP
END

```

```

SUBROUTINE XLLLL(LH,LI,ALPHA,XNU1,XII)
  XH=LH
  XI=LI
  EPSIL=1.E-09
  PI=3.1415926
  PPHI=PI/XNU1
  VC=(2.*XH-1.)*XNU1
  TEST=ABS(VC-XI)-(1.E-08)
  IF(TEST)20,20,10
10 VAL=XI/(2.*XNU1)
  VAA=VAL+0.5
  J2V0=VAA+EPSIL
  IF(ABS(J2V0-VAA)-2.*EPSIL)14,11,11
11 JZ=LH+1
  XII=2.*(-1.)**JZ*COS(VAL*PI)*XI/(PI*(VC**2-XI**2))
  GO TO 30
14 XII=0
  GO TO 30
20 IF(LI)21,21,23
21 EEL=2.
  GO TO 24
23 EEL=1.
24 XII=PPHI/(2.*PI)*EEL
30 RETURN
  END

```

```

SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
XNU=SS
DD=1.
TEM1=0.
TEM2=0.
ZX=(1.-XMU)/2.
IF(ABS(ZX)-0.000001) 35,35,31
31 CONTINUE
DO 40 L=1,LLMAX
ZL=L
TEM1=TEM1+DD*ZX**(L-1)
TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
DD=DD*(ZL-XLAM-1.)*(ZL+XLAM)/(ZL**2+ZL*XNU)
40 CONTINUE
ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
PP=ARTFL*TEM1
DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
GO TO 41
35 PP=0.
DP=0.
41 CONTINUE
RETURN
END

```

CLEQ	SUBROUTINE LEQ(A,B,NEQS,NSOLNS,IA,IB,DET)	LEQF0020
C	LINEAR EQUATIONS SOLUTIONS FORTRAN II VERSION	LEQF0030
C	SOLVE A SYSTEM OF LINEAR EQUATIONS OF THE FORM AX=B BY A MODIFIED	LEQF0040
C	GAUSS ELIMINATION SCHEME	LEQF0050
C		LEQF0060
C	NEQS = NUMBER OF EQUATIONS AND UNKNOWNNS	LEQF0070
C	NSOLNS = NUMBER OF VECTOR SOLUTIONS DESIRED	LEQF0080
C	IA = NUMBER OF ROWS OF A AS DEFINED BY DIMENSION STATEMENT ENTRY	LEQF0090
C	IB = NUMBER OF ROWS OF B AS DEFINED BY DIMENSION STATEMENT ENTRY	LEQF0100
C	ADET = DETERMINANT OF A, AFTER EXIT FROM LEQ	LEQF0110
C		LEQF0120
	DIMENSION A(IA,IA),B(IB,IB)	LEQF0130
	NSIZ = NEQS	LEQF0140
	NBSIZ = NSOLNS	LEQF0150
C	NORMALIZE EACH ROW BY ITS LARGEST ELEMENT. FORM PARTIAL DETERNT	LEQF0160
	DET=1.0	LEQF0170
	DO 1 I=1,NSIZ	LEQF0180
	BIG=A(I,1)	LEQF0190
51	IF(NSIZ-1)50,50,51	LEQF0200
	DO 2 J=2,NSIZ	LEQF0210
	IF(ABSF(BIG)-ABSF(A(I,J))) 3,2,2	LEQF0220
3	BIG=A(I,J)	LEQF0230
2	CONTINUE	LEQF0240
	BG=1.0/BIG	LEQF0250
	DO 4 J=1,NSIZ	LEQF0260
4	A(I,J)=A(I,J)*BG	LEQF0270
	DO 41 J=1,NBSIZ	LEQF0280
41	B(I,J)=B(I,J)*BG	LEQF0290
	DET=DET*BIG	LEQF0300
1	CONTINUE	LEQF0310
C	START SYSTEM REDUCTION	LEQF0320
	NUMSYS=NSIZ-1	LEQF0330
	DO 14 I=1,NUMSYS	LEQF0340
C	SCAN FIRST COLUMN OF CURRENT SYSTEM FOR LARGEST ELEMENT	LEQF0350
C	CALL THE ROW CONTAINING THIS ELEMENT, ROW NBGRW	LEQF0360
	NN=I+1	LEQF0370
	BIG=A(I,I)	LEQF0380
	NBGRW=I	LEQF0390
	DO 5 J=NN,NSIZ	LEQF0400
	IF(ABSF(BIG)-ABSF(A(J,I))) 6,5,5	LEQF0410
6	BIG=A(J,I)	LEQF0420
	NBGRW=J	LEQF0430
5	CONTINUE	LEQF0440
	BG=1.0/BIG	LEQF0450
C	SWAP ROW I WITH ROW NBGRW UNLESS I=NBGRW	LEQF0460
	IF(NBGRW-I) 7,10,7	LEQF0470
C	SWAP A-MATRIX ROWS	LEQF0480
7	DO 8 J=I,NSIZ	LEQF0490
	TEMP=A(NBGRW,J)	LEQF0500
	A(NBGRW,J)=A(I,J)	LEQF0510
8	A(I,J)=TEMP	LEQF0520
	DET = -DET	LEQF0530
C	SWAP B-MATRIX ROWS	LEQF0540
	DO 9 J=1,NBSIZ	LEQF0550
	TEMP=B(NBGRW,J)	LEQF0560
	B(NBGRW,J)=B(I,J)	LEQF0570
9	B(I,J)=TEMP	LEQF0580
C	ELIMINATE UNKNOWNNS FROM FIRST COLUMN OF CURRENT SYSTEM	

10	DO 13 K=NN,NSIZ	LEQF0590
C	COMPUTE PIVOTAL MULTIPLIER	LEQF0600
	PMULT=-A(K,I)*BG	LEQF0610
C	APPLY PMULT TO ALL COLUMNS OF THE CURRENT A-MATRIX ROW	LEQF0620
	DO 11 J=NN,NSIZ	LEQF0630
11	A(K,J)=PMULT*A(I,J)+A(K,J)	LEQF0640
C	APPLY PMULT TO ALL COLUMNS OF MATRIX B	LEQF0650
	DO 12 L=1,NBSIZ	LEQF0660
12	B(K,L)=PMULT*B(I,L)+B(K,L)	LEQF0670
13	CONTINUE	LEQF0680
14	CONTINUE	LEQF0690
C	DO BACK SUBSTITUTION	LEQF0700
C	WITH B-MATRIX COLUMN = NCOLB	LEQF0710
50	DO 15 NCOLB=1,NBSIZ	LEQF0720
C	DO FOR ROW = NROW	LEQF0730
	DO 19 I=1,NSIZ	LEQF0740
	NROW=NSIZ+1-I	LEQF0750
	TEMP=0.0	LEQF0760
C	NUMBER OF PREVIOUSLY COMPUTED UNKNOWNNS = NX	LEQF0770
	NXS=NSIZ-NROW	LEQF0780
C	ARE WE DOING THE BOTTOM ROW	LEQF0790
	IF(NXS) 16,17,16	LEQF0800
C	NO	LEQF0810
16	DO 18 K=1,NXS	LEQF0820
	KK=NSIZ+1-K	LEQF0830
18	TEMP=TEMP+B(KK,NCOLB)*A(NROW,KK)	LEQF0840
17	B(NROW,NCOLB)=(B(NROW,NCOLB)-TEMP)/A(NROW,NROW)	LEQF0850
C	HAVE WE FINISHED ALL ROWS FOR B-MATRIX COLUMN = NCOLB	LEQF0860
19	CONTINUE	LEQF0870
C	YES	LEQF0880
C	HAVE WE JUST FINISHED WITH B-MATRIX COLUMN NCOLB=NSIZ	LEQF0890
15	CONTINUE	LEQF0900
C	YES	LEQF0910
C	NOW FINISH COMPUTING THE DETERMINANT	LEQF0920
	DO 20 I=1,NSIZ	LEQF0930
20	DET=DET*A(I,I)	LEQF0940
C	WE ARE ALL DONE NOW	LEQF0950
C	WHEW...	LEQF0960
	RETURN	LEQF0970
	END	LEQF0980

```

      SUBROUTINE COFACT(ADET,IIMAX,ICORD,JCORD,DETCOE)
      DIMENSION ADET(50,50),BDET(50,50),CDET(50,50)
      M=1
      MM=1
10   N=1
      MN=1
      IF(M-ICORD) 20,12,20
12   MM=M+1
20   IF(N-JCORD) 23,22,23
22   MN=N+1
23   BDET(M,N)=ADET(MM,MN)
      N=N+1
      MN=MN+1
      IF(N-IIMAX+1) 20,20,31
31   M=M+1
      MM=MM+1
      IF(M-IIMAX+1) 10,10,35
35   ITERM=IIMAX-1
      DO 37 I=1,ITERM
      DO 36 J=1,ITERM
      CDET(I,J)=0.
36   CONTINUE
37   CONTINUE
      CALL LEQ(BDET,CDET,ITERM,0,50,50,DETCOE)
      RETURN
      END

```

```

C FIRST PROGRAM(B)
C EVEN FUNCTION TO CALCULATE DETERMINANT AND COEFICIENTS
  DIMENSION AEL(50,50),B(50,50)
  DIMENSION ADET (50,50),EEEE(50,50)
  DIMENSION SUMA(101),SUMB(101),BMIN(30),DMIN(30),SBMIN(30),
1SDMIN(30)
  LLMAX=50
  READ(5,5) IIPP,NMAX,LMAX
5 FORMAT(3I5)
  IIMAX=LMAX+NMAX+2
  ITTEM=IIMAX
  PI=3.1415926
  ALPHA=PI/2.
  PPHI=2.*(PI-ALPHA/2.)
  XNU1=PI/PPHI
  INTERV=30
  LEP=LMAX+1
  JEP=NMAX+1
  UO=0.
  XMU=0.
  XMUFIK=XMU
  WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
502 FORMAT( * NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X, *
1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
400 READ(5,31) XLAM
31 FORMAT(F14.0)
  IF(ENDFILE 5) 5559,5556
5556 CONTINUE
C CALCULATION OF ELEMENTS OF DETERMINANT
  DO 303 J=1,ITTEM
  DO 300 K=1,ITTEM
  IF(J-NMAX-1) 10,10,200
10 IF(K-NMAX-1) 11,11,30
11 IF(J-1) 12,12,13
12 EEL=2.
  GO TO 14
13 EEL=1.
14 IF(J-K) 16,15,16
15 DELTA=1.
  GO TO 17
16 DELTA=0.
17 SS=K-1
  CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
  AEL(J,K)=DP*EEL*DELTA
  GO TO 300
30 SS=2*(K-NMAX-2) *XNU1
  CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
  LH=K-NMAX-2
  LI=J-1
  CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
  AEL(J,K)=DP*XII
  GO TO 300
200 IF(K-NMAX-1) 212,212,230
212 LH=J-NMAX-2
  LI=K-1
  CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
  SS=K-1
  CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)

```

```

      AEL(J,K)=PP*XII
      GO TO 300
230 IF(J-K) 233,232,233
232 DELTA=1.
      GO TO 234
233 DELTA=0.
234 IF(J-NMAX-2) 236,235,236
235 EEL=2.
      GO TO 237
236 EEL=1.
237 SS=2*(K-NMAX-2)*XNU1
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      AEL(J,K)=-((1.-ALPHA/(2.*PI))*EEL*PP*DELTA
300 CONTINUE
303 CONTINUE
      DO 307 I=1,ITTEM
      DO 306 J=1,ITTEM
      B(I,J)=0.
      EEEE(I,J)=AEL(I,J)
306 CONTINUE
307 CONTINUE
      CALL LEQ(AEL,B,ITTEM,0,50,50,DET)
      WRITE(6,32) XLAM ,DET
32  FORMAT(*      XLAM=*,E15.8,5X,*DET=*,E15.8)
      IF(IIPP) 400,400,1002
1002 CONTINUE
C   CALCULATION OF COFACTORS
      DO 903 I=1,ITTEM
      DO 902 J=1,ITTEM
      ADET(I,J)=EEEE(I,J)
902 CONTINUE
903 CONTINUE
      ICORD=1
      DO 511 J=1,JEP
      JCORD=J
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      BMIN(J)=-((-1.)*JCORD*DETCOE)
526 CONTINUE
511 CONTINUE
      DO 512 L=1,LFP
      JCORD=L+JEP
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      DMIN(L)=-((-1.)*JCORD*DETCOE)
536 CONTINUE
512 CONTINUE
      DELU=(1.-U0)/INTERV
      ISTEP=INTERV+1
      DO 611 I=1,ISTEP
      XMM=(I-1)*DELU
      SS=0.
      CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
      SUM10=2.*(BMIN(1)*PP)**2
      DO 601 J=2,JEP
      SS=J-1
      CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
      SUM10=SUM10+(BMIN(J)*PP)**2
601 CONTINUE
      SUMA(I)=SUM10

```

```

611 CONTINUE
DELV=(1.+U0)/INTERV
DO 711 I=1,ISTFP
XMM=(I-1)*DELV
SS=0.
CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
SUM11=2.*(DMIN(1)*PP)**2
DO 603 L=2,LEP
SS=2.*(L-1)*XNUI
CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
SUM11=SUM11+(DMIN(L)*PP)**2
603 CONTINUE
SUMB(I)=SUM11
711 CONTINUE
TOSUM1=SUMA(1)
TOSUM2=SUMB(1)
DO 622 I=2,INTERV
COFF=3.+(-1.)**I
TOSUM1=TOSUM1+COFF*SUMA(I)
TOSUM2=TOSUM2+COFF*SUMB(I)
622 CONTINUE
TOSUM1=(TOSUM1+SUMA(ISTEP))*DELU/3.
TOSUM2=(TOSUM2+SUMB(ISTEP))*DFLV/3.
ODDN=TOSUM1*PI/2.+PPHI*TOSUM2/4.
C  CALCULATION OF NORMALIZED COFACTORS --COEFFICIENT OF EIGEN FUNCTION
DO 633 I=1,JEP
SBMIN(I)=BMIN(I)/SQRT(ODDN)
BMIN(I)=SBMIN(I)
633 CONTINUE
WRITE(6,6)
6  FORMAT( /* COEFFICIENTS OF EVEN FUNCTION*)
304 FORMAT(5E20.8/(5X,5E20.8))
WRITE(6,304) (BMIN(I),I=1,JEP)
DO 634 L=1,LEP
SDMIN(L)=DMIN(L)/SQRT(ODDN)
DMIN(L)=SDMIN(L)
634 CONTINUE
WRITE(6,304) (DMIN(L),L=1,LEP)
WRITE(6,401)
401 FORMAT(////)
GO TO 400
5559 STOP
END

```

```

      SUBROUTINE XLLLL(LH,LI,ALPHA,XNU1,XII)
C
C
C   THIS SUBROUTINE IS FOR 'BETA-'
C
      XH=LH
      XI=LI
      EPSIL=1.E-09
      PI=3.1415926
      PPHI=PI/XNU1
      VC=2.*XH*XNU1
      TEST=ABS(VC-XI)-(1.E-08)
      IF(TEST)20,20,10
10  VAL=XI/(2.*XNU1)
      J2V0=VAL+EPSIL
      IF(ABS(J2V0-VAL)-2.*EPSIL)14,11,11
11  JZ=LH+1
      XII=2.*(-1.)**JZ*SIN(VAL*PI)*XI/(PI*(VC**2-XI**2))
      GO TO 30
14  XII=0
      GO TO 30
20  IF(LI)21,21,23
21  EEL=2.
      GO TO 24
23  EEL=1.
24  XII=PPHI/(2.*PI)*EEL
30  RETURN
      END

```

```

SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
XNU=SS
DD=1.
TEM1=0.
TEM2=0.
ZX=(1.-XMU)/2.
IF(ABS(ZX)-0.000001) 35,35,31
31 CONTINUE
DO 40 L=1,LLMAX
ZL=L
TEM1=TEM1+DD*ZX**(L-1)
TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
DD=DD*(ZL-XLAM-1.)*(ZL+XLAM)/(ZL**2+ZL*XNU)
40 CONTINUE
ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
PP=ARTFL*TEM1
DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
GO TO 41
35 PP=0.
DP=0.
41 CONTINUE
RETURN
END

```

C	LEQ	SUBROUTINE LEQ(A,B,NEQS,NSOLNS,IA,IB,DET)	LEQF0020
C		LINEAR EQUATIONS SOLUTIONS FORTRAN II VERSION	LEQF0030
C		SOLVE A SYSTEM OF LINEAR EQUATIONS OF THE FORM AX=B BY A MODIFIED	LEQF0040
C		GAUSS ELIMINATION SCHEME	LEQF0050
C		NEQS = NUMBER OF EQUATIONS AND UNKNOWNNS	LEQF0060
C		NSOLNS = NUMBER OF VECTOR SOLUTIONS DESIRED	LEQF0070
C		IA = NUMBER OF ROWS OF A AS DEFINED BY DIMENSION STATEMENT ENTRY	LEQF0080
C		IB = NUMBER OF ROWS OF B AS DEFINED BY DIMENSION STATEMENT ENTRY	LEQF0090
C		ADET = DETERMINANT OF A, AFTER EXIT FROM LEQ	LEQF0100
C			LEQF0110
		DIMENSION A(IA,IA),B(IB,IB)	LEQF0120
		NSIZ = NEQS	LEQF0130
		NBSIZ = NSOLNS	LEQF0140
C		NORMALIZE EACH ROW BY ITS LARGEST ELEMENT. FORM PARTIAL DETERNT	LEQF0150
		DET=1.0	LEQF0160
		DO 1 I=1,NSIZ	LEQF0170
		BIG=A(I,1)	LEQF0180
		IF(NSIZ-1)50,50,51	LEQF0190
51		DO 2 J=2,NSIZ	LEQF0200
		IF(ABSF(BIG)-ABSF(A(I,J))) 3,2,2	LEQF0210
3		BIG=A(I,J)	LEQF0220
2		CONTINUE	LEQF0230
		BG=1.0/BIG	LEQF0240
		DO 4 J=1,NSIZ	LEQF0250
4		A(I,J)=A(I,J)*BG	LEQF0260
		DO 41 J=1,NBSIZ	LEQF0270
41		B(I,J)=B(I,J)*BG	LEQF0280
		DET=DET*BIG	LEQF0290
1		CONTINUE	LEQF0300
C		START SYSTEM REDUCTION	LEQF0310
		NUMSYS=NSIZ-1	LEQF0320
		DO 14 I=1,NUMSYS	LEQF0330
C		SCAN FIRST COLUMN OF CURRENT SYSTEM FOR LARGEST ELEMENT	LEQF0340
C		CALL THE ROW CONTAINING THIS ELEMENT, ROW NBGRW	LEQF0350
		NN=I+1	LEQF0360
		BIG=A(I,I)	LEQF0370
		NBGRW=I	LEQF0380
		DO 5 J=NN,NSIZ	LEQF0390
		IF(ABSF(BIG)-ABSF(A(J,I))) 6,5,5	LEQF0400
6		BIG=A(J,I)	LEQF0410
		NBGRW=J	LEQF0420
5		CONTINUE	LEQF0430
		BG=1.0/BIG	LEQF0440
C		SWAP ROW I WITH ROW NBGRW UNLESS I=NBGRW	LEQF0450
		IF(NBGRW-I) 7,10,7	LEQF0460
C		SWAP A-MATRIX ROWS	LEQF0470
7		DO 8 J=I,NSIZ	LEQF0480
		TEMP=A(NBGRW,J)	LEQF0490
		A(NBGRW,J)=A(I,J)	LEQF0500
8		A(I,J)=TEMP	LEQF0510
		DET = -DET	LEQF0520
C		SWAP B-MATRIX ROWS	LEQF0530
		DO 9 J=1,NBSIZ	LEQF0540
		TEMP=B(NBGRW,J)	LEQF0550
		B(NBGRW,J)=B(I,J)	LEQF0560
9		B(I,J)=TEMP	LEQF0570
C		ELIMINATE UNKNOWNNS FROM FIRST COLUMN OF CURRENT SYSTEM	LEQF0580

10	DO 13 K=NN,NSIZ	LEQF0590
C	COMPUTE PIVOTAL MULTIPLIER	LEQF0600
C	APPLY PMULT TO ALL COLUMNS OF THE CURRENT A-MATRIX ROW	LEQF0620
	PMULT=-A(K,I)*BG	LEQF0610
	DO 11 J=NN,NSIZ	LEQF0630
C	APPLY PMULT TO ALL COLUMNS OF MATRIX B	LEQF0650
11	A(K,J)=PMULT*A(I,J)+A(K,J)	LEQF0640
	DO 12 L=1,NBSIZ	LEQF0660
12	B(K,L)=PMULT*B(I,L)+B(K,L)	LEQF0670
13	CONTINUE	LEQF0680
14	CONTINUE	LEQF0690
C	DO BACK SUBSTITUTION	LEQF0700
C	WITH B-MATRIX COLUMN = NCOLB	LEQF0710
50	DO 15 NCOLB=1,NBSIZ	LEQF0720
C	DO FOR ROW = NROW	LEQF0730
	DO 19 I=1,NSIZ	LEQF0740
	NROW=NSIZ+1-I	LEQF0750
	TEMP=0.0	LEQF0760
C	NUMBER OF PREVIOUSLY COMPUTED UNKNOWNNS = NXS	LEQF0770
	NXS=NSIZ-NROW	LEQF0780
C	ARE WE DOING THE BOTTOM ROW	LEQF0790
	IF(NXS) 16,17,16	LEQF0800
C	NO	LEQF0810
16	DO 18 K=1,NXS	LEQF0820
	KK=NSIZ+1-K	LEQF0830
18	TEMP=TEMP+B(KK,NCOLB)*A(NROW,KK)	LEQF0840
17	B(NROW,NCOLB)=(B(NROW,NCOLB)-TEMP)/A(NROW,NROW)	LEQF0850
C	HAVE WE FINISHED ALL ROWS FOR B-MATRIX COLUMN = NCOLB	LEQF0860
19	CONTINUE	LEQF0870
C	YES	LEQF0880
C	HAVE WE JUST FINISHED WITH B-MATRIX COLUMN NCOLB=NSIZ	LEQF0890
15	CONTINUE	LEQF0900
C	YES	LEQF0910
C	NOW FINISH COMPUTING THE DETERMINANT	LEQF0920
	DO 20 I=1,NSIZ	LEQF0930
C	WE ARE ALL DONE NOW	LEQF0950
C	WHEW...	LEQF0960
20	DET=DET*A(I,I)	LEQF0940
	RETURN	LEQF0970
	END	LEQF0980

```

SUBROUTINE COFACT(ADET,IIMAX,ICORD,JCORD,DETCOE)
DIMENSION ADET(50,50),BDET(50,50),CDET(50,50)
M=1
MM=1
10 N=1
MN=1
IF(M-ICORD) 20,12,20
12 MM=M+1
20 IF(N-JCORD) 23,22,23
22 MN=N+1
23 BDET(M,N)=ADET(MM,MN)
N=N+1
MN=MN+1
IF(N-IIMAX+1) 20,20,31
31 M=M+1
MM=MM+1
IF(M-IIMAX+1) 10,10,35
35 ITERM=IIMAX-1
DO 37 I=1,ITERM
DO 36 J=1,ITERM
CDET(I,J)=0.
36 CONTINUE
37 CONTINUE
CALL LEQ(BDET,CDET,ITERM,0,50,50,DETCOE)
RETURN
END

```

```

C SECOND PROGRAM
C DETERMINATION OF THE COEFFICIENTS OF THE EIGENFUNCTION EXPANSION FROM
C THE INITIAL DATA
  DIMENSION F(70),FUP(70,70),FBT(70,70),SFL(70,70)
  DIMENSION SUMJ(70),SUML(70),BMIN(70),DMIN(70)
  LLMAX=50
  IDIM=70
  MAXU1=60
  MAXTOP=60
  MAXU2=60
  MAXBOT=60
1000 READ(5,1001) II,NMAX,LMAX
1001 FORMAT(3I5)
  IF(ENDFILE 5) 9999,1002
1002 CONTINUE
  IF(II .EQ. 1) 101,103
C
C
C ODD FUNCTION
  101 IIMAX=LMAX+NMAX
  WRITE(6,1003)
1003 FORMAT(1H1* ODD FUNCTION*)
  NCALL=NMAX
  LCALL=LMAX
  GO TO 104
C
C EVEN FUNCTION
  103 IIMAX=LMAX+NMAX+2
  WRITE(6,1004)
1004 FORMAT(1H1,* EVEN FUNCTION*)
  NCALL=NMAX+1
  LCALL=LMAX+1
  104 CONTINUE
  PI=3.1415926
  ALPHA=PI/2.
  XMU=0.
  PPHI=2.*(PI-ALPHA/2.)
  XNU1=PI/PPHI
  HALPHI=PPHI/2.
  EPS =0.000001
  WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
502 FORMAT( * NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X, *
  1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
400 READ(5,100)XLAM
  IF(ENDFILE 5) 1000,1111
1111 CONTINUE
  100 FORMAT(5F15.0)
  READ(5,100) (BMIN(I),I=1,NCALL)
  READ(5,100) (DMIN(L),L=1,LCALL)
  UG=0.
  MAXPUS=MAXTOP+1
  INDEX=1
  WRITE(6,32) XLAM
  32 FORMAT( * XLAM=*,E15.8)
  WRITE(6,503)
503 FORMAT(//* COEFFICIENTS OF EIGENFUNCTION*)
  WRITE(6,304) (BMIN(I),I=1,NCALL)
  WRITE(6,304) (DMIN(L),L=1,LCALL)

```

```

304 FORMAT(5E20.8/(5X,5E20.8))
   DELU1=(1.-U0-2.*EPS )/MAXU1
   MAXUST=MAXU1+1
   UK=U0+EPS
   DO 5 I=1,MAXUST
   TOTARG=0.
   CALL FFF(UK,INDEX,MAXTOP,F,II,PPHI,IDIM)
   DO 4 J=1,MAXPUS
   FUP(I,J)=F(J)
4 CONTINUE
   IROW=I
   DO 13 J=1,NCALL
   IF(II .EQ. 1) 111,113
C
C   ODD FUNCTION
111 CONTINUE
   XVAL=J
   CALL SININT(FUP,XVAL,IROW,MAXTOP,PI,0,IDIM ,TRIINT)
   GO TO 114
C
C   EVEN FUNCTION
113 CONTINUE
   XVAL=J-1
   CALL COSINT(FUP,XVAL,IROW,MAXTOP,PI,0,IDIM ,TRIINT)
114 CONTINUE
   SS=XVAL
   CALL PPDD(SS,XLAM,UK,LLMAX,PP,DP)
   TOTARG=TOTARG+BMIN(J)*PP*TRIINT
13 CONTINUE
   SUMJ(I)=TOTARG
   UK=UK+DELU1
5 CONTINUE
   TOTEM1=SUMJ(1)
   DO 19 I=2,MAXU1
   COEF=3.+(-1.)**I
   TOTEM1=TOTEM1+COEF*SUMJ(I)
19 CONTINUE
   TOTEM1=(TOTEM1+SUMJ(MAXUST))*DELU1/3.
   MAXPUS=MAXPOT+1
   INDEX=2
   DELU2=(1.+U0-2.*EPS )/MAXU2
   MAXUSM=MAXU2+1
   UK=-1.+EPS
   DO 9 I=1,MAXUSM
   TOTARG=0.
   CALL FFF(UK,INDEX,MAXBOT,F,II,PPHI,IDIM)
   DO 8 J=1,MAXPUS
   FBT(I,J)=F(J)
8 CONTINUE
   IROW=I
   DO 23 L=1,LCALL
   IF(II .EQ. 1) 121,123
C
C   ODD FUNCTION
121 CONTINUE
   XVAL=(2*L-1)*XNU1
   CALL SININT(FBT,XVAL,IROW,MAXBOT,HALPHI,0,IDIM ,TRIINT)
   GO TO 124

```

```

C
C  EVEN FUNCTION
123  CONTINUE
      XVAL=2*(L-1)*XNU1
      CALL COSINT(FBT,XVAL,IROW,MAXBOT,HALPHI,0,IDIM ,TRIINT)
124  CONTINUE
      SS=XVAL
      UF=-UK
      CALL FPDD(SS,XLAM,UF,LLMAX,PP,DP)
      TOTARG=TOTARG+DMIN(L)*PP*TRIINT
23   CONTINUE
      SUML(I)=TOTARG
      UK=UK+DELU2
9    CONTINUE
      TOTEM2=SUML(1)
      DO 29 I=2,MAXU2
      COEF=3.+(-1.)**I
      TOTEM2=TOTEM2+COEF*SUML(I)
29   CONTINUE
      TOTEM2=(TOTEM2+SUML(MAXUSM))*DELU2/3.
      EEEM=TOTEM1+TOTEM2
      WRITE(6,11)EEEM
11   FORMAT( /20X,*K(LAMDA)=*,E15.8)
      WRITE(6,401)
401  FORMAT(////)
      GO TO 400
9999 STOP
      END

```

```

      SUBROUTINE SININT(FP,XVAL,IROW,MAXST,UPLIM,BOTLIM,IDIM,TRIINT)
C   FILON'S METHOD FOR THE NUMERICAL EVALUATION OF TRIGONOMETRICAL
C   INTEGRALS--(INTEGRAND=FP(P)*SIN(X*P))
      DIMENSION FP(IDIM,IDIM)
      HH=(UPLIM-BOTLIM)/MAXST
      S2S=0.5*FP(IROW,1)*SIN(XVAL*BOTLIM)
      DO 14 J=3,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2S=S2S+FP(IROW,J)*SIN(XVAL*P)
14  CONTINUE
      J=MAXST+1
      S2S=S2S+0.5*FP(IROW,MAXST+1)*SIN(XVAL*UPLIM)
      S2SM=0.
      DO 16 J=2,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2SM=S2SM+FP(IROW,J)*SIN(XVAL*P)
16  CONTINUE
      THE=XVAL*HH
      IF (THE-0.2) 25,21,21
21  ALPHA=(THE**2+THE*SIN(THE)*COS(THE)-2.*SIN(THE)**2)/THE**3
      BETA=2.*(THE*(1.+COS(THE)**2)-2.*SIN(THE)*COS(THE))/THE**3
      GARM=4.*(SIN(THE)-THE*COS(THE))/THE**3
      GO TO 31
25  ALPHA=2.*THE**3/45.-2.*THE**5/315.+2.*THE**7/4725.
      BETA=2./3.+2.*THE**2/15.-4.*THE**4/105.+2.*THE**6/567.
      GARM=4./3.-2.*THE**2/15.+THE**4/210.-THE**6/11340.
31  FA=FP(IROW,1)
      FB=FP(IROW,MAXST+1)
      TRIINT=HH*(-ALPHA*(FB*COS(XVAL*UPLIM)-FA*COS(XVAL*BOTLIM))+BETA*
1S2S+GARM*S2SM)
      RETURN
      END

```

```

      SUBROUTINE COSINT(FP,XVAL,IROW,MAXST,UPLIM,BOTLIM,IDIM,TRIINT)
C   FILON'S METHOD FOR THE NUMERICAL EVALUATION OF TRIGONOMETRICAL
C   INTEGRALS--(INTEGRAND=FP(P)*COS(X*P))
      DIMENSION FP(IDIM,IDIM)
      HH=(UPLIM-BOTLIM)/MAXST
      S2S=0.5*FP(IROW,1)*COS(XVAL*BOTLIM)
      DO 14 J=3,MAXST,2
        P=BOTLIM+(J-1)*HH
        S2S=S2S+FP(IROW,J)*COS(XVAL*P)
14    CONTINUE
      J=MAXST+1
      S2S=S2S+0.5*FP(IROW,MAXST+1)*COS(XVAL*UPLIM)
      S2SM=0.
      DO 16 J=2,MAXST,2
        P=BOTLIM+(J-1)*HH
        S2SM=S2SM+FP(IROW,J)*COS(XVAL*P)
16    CONTINUE
      THE=XVAL*HH
      IF (THE-0.2) 25,21,21
21    ALPHA=(THE**2+THE*SIN(THE)*COS(THE)-2.*SIN(THE)**2)/THE**3
      BETA=2.*(THE*(1.+COS(THE)**2)-2.*SIN(THE)*COS(THE))/THE**3
      GARM=4.*(SIN(THE)-THE*COS(THE))/THE**3
      GO TO 31
25    ALPHA=2.*THE**3/45.-2.*THE**5/315.+2.*THE**7/4725.
      BETA=2./3.+2.*THE**2/15.-4.*THE**4/105.+2.*THE**6/567.
      GARM=4./3.-2.*THE**2/15.+THE**4/210.-THE**6/11340.
31    FA=FP(IROW,1)
      FB=FP(IROW,MAXST+1)
      TRIINT=HH*( ALPHA*(FB*SIN(XVAL*UPLIM)-FA*SIN(XVAL*BOTLIM))+BETA*
1    S2S+GARM*S2SM)
      RETURN
      END

```

```

      SUBROUTINE FFF(UU,INDEX,MAX,F,II,PPHI,IDIM)
      DIMENSION F(IDIM)
      PI=3.1415926
      IF(INDEX .EQ. 1) 4,6
4     DELPSI=PI/FLOAT(MAX)
      GO TO 7
6     DELPSI=PPHI/FLOAT(2*MAX)
7     CONTINUE
      PSI=0.
      MAXPUS=MAX+1
      DO 19 N=1,MAXPUS
      BPSI=PSI-PPHI/2.
      CALL FSF(BPSI,UU,FS ,PPHI)
      FA=FS
      BPSI=2.*PI-PSI-PPHI/2.
      CALL FSF(BPSI,UU,FS ,PPHI)
      FB=FS
      IF(II .EQ. 1) 14,16
14    F(N)=(FA-FB)/2.
      GO TO 17
16    F(N)=(FA+FB)/2.
17    CONTINUE
      PSI=PSI+DELPsi
19    CONTINUE
      RETURN
      END

```

```

SUBROUTINE FSF(BPSI,UU,FS ,PPHI)
PI=3.1415926
PI23=2.*PI/3.
PI43=4.*PI/3.
CONS23=2./3.
HALPI=PI/2.
HALPHI=PPHI/2.
TEST1=2.*PI-HALPHI
TEST2=2.*PI-PPHI
IF(UU) 11, 1,1
1 F0=1.
GO TO 31
11 IF(BPSI .LE. TEST1 .AND. BPSI .GE. HALPI .OR. BPSI .LT. -HALPI
1 .AND. BPSI .GE. (-HALPHI-0.0000001)) 1,14
12 F0=0.
GO TO 31
14 IF(BPSI .LE. 0 .AND. BPSI .GT. -HALPI) 12,15
15 IF(BPSI .LT. HALPI .AND. BPSI .GT. 0) 16,18
16 WRITE(6,17) UU,BPSI
17 FORMAT(* PRORAM STOP IN CASE 1 U=*,E12.5,5X,*BPSI=*,E12.5)
STOP
18 IF(ABS(BPSI+HALPI) -0.00001) 19,19,20
19 F0=0.5
GO TO 31
20 WRITE(6,21) UU,BPSI
21 FORMAT(* PRORAM STOP IN CASE 2 U=*,E12.5,5X,*BPSI=*,E12.5)
STOP
31 ABSPSI=ABS(BPSI)
IF(ABSPSI .GT. HALPI) 32,33
32 F1=0.
GO TO 51
33 RHO=SQRT(UU**2+(1.-UU**2)*SIN(BPSI)**2)
TAU0=ASIN(ABS(UU)/RHO)
IF(UU .GE. 0) 34,37
34 IF(BPSI .GE. 0) 35,36
35 TAU=TAU0
GO TO 42
36 TAU=PI-TAU0
GO TO 42
37 IF(BPSI .LE. 0) 38,39
38 TAU=PI+TAU0
GO TO 42
39 WRITE(6,40) UU,BPSI
40 FORMAT(* PRORAM STOP IN CASE 3 U=*,E12.5,5X,*BPSI=*,E12.5)
STOP
42 CALL CON2D(RHO,TAU,PI23,PI43,CONS23,FIF)
IF(ABSPSI-HALPI) 45,43,43
43 F1=FIF/2.
GO TO 51
45 F1=FIF
51 IF(UU ) 56,54,53
53 F2=0.
GO TO 71
54 IF(BPSI .LT. HALPI .AND. BPSI .GT. 0) 55,56
55 F2=0.
GO TO 71
56 RHO=SQRT(1.-UU**2)
IF(BPSI .LE. TEST1 .AND. BPSI .GE. HALPI) 57,58

```

```

57 TAU=BPSI-HALPI
   GO TO 62
58 IF(BPSI .LE. 0 .AND. BPSI .GE. -HALPHI) 59,60
59 TAU=BPSI+3.*PI/2.
   GO TO 62
60 WRITE(6,61) UU,BPSI
61 FORMAT(* PRORAM STOP IN CASE 4  U=*,E12.5,5X,*BPSI=*,E12.5)
   STOP
62 CALL CON2D(RHO,TAU,PI23,PI43,CONS23,FIF)
   IF(UU ) 66,64,55
64 F2=0.5*FIF
   GO TO 71
66 F2=FIF
71 FS =F0+F1+F2
   RETURN
   END

```

```

SUBROUTINE CON2D(RHO,TAU,W1,W2,SS,PC)
PI=3.1415926
EPSIL=0.000001
CON1=1.-RHO**2
IF(CON1+EPSIL) 10,10,14
10 WRITE(6,11)
11 FORMAT(* 1-RHO**2+EPSIL .LE. ZERO, CHECK THE PROGRAM *)
STOP
14 IF(CON1 )15,15,17
15 GG=1.
GO TO 21
17 GG=(RHO/(1.+SQRT(CON1)))*SS
21 BTAU=SS*TAU
DD=(1.-GG**2)*SIN(0.5*(W2-W1))
CC=(1.+GG**2)*COS(0.5*(W2-W1)) -2.*GG*COS(BTAU-(W2+W1)*0.5)
SQCD=SQRT(CC**2+DD**2)
IF(SQCD-EPSIL) 23,25,25
23 PC=0.5
GO TO 30
25 IF(CC) 28,26,26
26 PC=ASIN(DD/SQCD)/PI
GO TO 30
28 PC=ASIN(DD/SQCD)/PI*(-1.)+1.
30 CONTINUE
RETURN
END

```

```

SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
XNU=SS
DD=1.
TEM1=0.
TEM2=0.
ZX=(1.-XMU)/2.
IF(ABS(ZX)-0.000001) 35,35,31
31 CONTINUE
DO 40 L=1,LLMAX
ZL=L
TEM1=TEM1+DD*ZX**(L-1)
TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
DD=DD*(ZL-XLAM-1.)*(ZL+XLAM)/(ZL**2+ZL*XNU)
40 CONTINUE
ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
PP=ARTFL*TEM1
DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
GO TO 41
35 PP=0.
DP=0.
41 CONTINUE
RETURN
END

```

```

C   THIRD PROGRAM
C   DETERMINATION OF THE PRESURE DISTRIBUTION ON SURFACES
      DIMENSION EDIM(20),P1(20,20),P2(20,20),P3(20,20),G1(20,20),
      1G2(20,20),G3(20,20) ,BMIN(20),DMIN(20)
      LLMAX=50
      PI=3.1415926
      ALPHA=PI/2.
      PPHI=2.*(PI-ALPHA/2.)
      XNU1=PI/PPHI
      HALPHI=PPHI/2.
      DO 106 M=1,10
      DO 105 N=1,5
      P1(M,N)=0.
      P2(M,N)=0.
      P3(M,N)=0.
105  CONTINUE
106  CONTINUE
1000 READ(5,1001) II,NMAX,LMAX
1001 FORMAT(3I5)
      IF(ENDFILE 5) 9999,1002
1002 CONTINUE
      IF(II .EQ. 1) 101,103
C
C   ODD FUNCTION
101  IIMAX=LMAX+NMAX
      WRITE(6,2)
      2  FORMAT(1H1,//*   ODD FUNCTION*)
      NCALL=NMAX
      LCALL=LMAX
      GO TO 104
C
C   EVEN FUNCTION
103  IIMAX=LMAX+NMAX+2
      WRITE(6,3)
      3  FORMAT(1H1,//*   EVEN FUNCTION *)
      NCALL=NMAX+1
      LCALL=LMAX+1
104  CONTINUE
      XMU=0.
      WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
502  FORMAT(      *   NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,      *
      1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
110  READ(5,100) XLAM,EEEM
100  FORMAT(5F15.0)
      IF(ENDFILE 5) 1000,1111
1111 CONTINUE
      READ(5,100) (BMIN(I),I=1,NCALL)
      READ(5,100) (DMIN(L),L=1,LCALL)
      XMU=0.
      CALL T102(XLAM,1.,EDIM,BBBBBBB)
      AAAAAA=BBBBBB
      WRITE(6,32) XLAM,AAAAAA,EEEM
      32  FORMAT(      *   XLAM=*,F14.9,5X,*AA0=*,F14.9,5X,*K(LAMDA)=*,F14.9)
      WRITE(6,503)
503  FORMAT(// *   COEFFICIENTS OF EIGEN-FUNCTIONS*)
      WRITE(6,304) (BMIN(I),I=1,NCALL)
      WRITE(6,304) (DMIN(L),L=1,LCALL)
304  FORMAT(5E20.8/(5X,5E20.8))

```

```

        WRITE(6,401)
401  FORMAT(////)
        DELPHI=(PI-HALPHI)/2.
        IF(II .EQ. 1) 200,203
C
C      ODD FUNCTION
200  DO 202 I=1,5
        PHI=PI+(I-3)*DELPHI
        PPUS=0.
        DO 201 J=1,NCALL
            SS=J
            CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
            SINFU=SIN(J*PHI)
            PPUS=PPUS+BMIN(J)*PP*SINFU
201  CONTINUE
        G1(I)=PPUS
202  CONTINUE
        GO TO 209
C
C      EVEN FUNCTION
203  CONTINUE
204  DO 207 I=1,5
        PHI=PI+(I-3)*DELPHI
        PPUS=0.
        DO 206 J=1,NCALL
            SS=J-1
            CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
            COSFU=COS(SS*PHI)
            PPUS=PPUS+BMIN(J)*PP*COSFU
206  CONTINUE
        G1(I)=PPUS
207  CONTINUE
209  CALL T102(XLAM,AAAAAA,EDIM,BBBBBB)
        DO 228 M=1,10
            DO 227 N=1,5
                P1(M,N)=P1(M,N)+EDIM(M)*G1(N)*EEEM
227  CONTINUE
228  CONTINUE
        MMP=1
        PHI=HALPHI
        DELTH=PI/8. -0.0005
300  IF(II .EQ. 1) 301,400
C
C      ODD FUNCTION
301  DO 309 I=1,5
        THE=PI/2.+(I-1)*DELTH
        XMU=COS(THE)
        PMIN=0.
        DO 302 L=1,LCALL
            SS=(2*L-1)*XNU1
            CALL PPDD(SS,XLAM,-XMU,LLMAX,PP,DP)
            SINFU=SIN(SS*PHI)
            PMIN=PMIN+DMIN(L)*PP*SINFU
302  CONTINUE
303  G2(I)=PMIN
305  G3(I)=PMIN*(-1.)
309  CONTINUE
        GO TO 500

```

```

C
C  EVEN FUNCTION
400 DO 409 I=1,5
    THE=PI/2.+(I-1)*DELTH
    XMU=COS(THE)
    PMIN=0.
    DO 402 L=1,LCALL
    SS=2*(L-1)*XNU1
    CALL PPDD(SS,XLAM,-XMU,LLMAX,PP,DP)
    COSFU=COS(SS*PHI)
    PMIN=PMIN+DMIN(L)*PP*COSFU
402 CONTINUE
403 G2(I)=PMIN
404 G3(I)=PMIN
409 CONTINUE
500 CONTINUE
501 DO 505 M=1,10
    DO 504 N=1,5
    P2(M,N)=P2(M,N)+EDIM(M)*G2(N)*EEEM
    P3(M,N)=P3(M,N)+EDIM(M)*G3(N)*EEEM
504 CONTINUE
505 CONTINUE
    GO TO 110
9999 CONTINUE
    WRITE(6,9998)
9998 FORMAT(/ /8X,*PHI*,14X,*P1*,28X,*XMU*,14X,*P2*,14X,*P3*)
601 DO 605 N=1,5
    PHI=PI+(N-3)*DELPHI
    THE=PI/2.+(N-1)*DELTH
    XMU=COS(THE)
    DO 604 M=1,10
    WRITE(6,607) PHI,P1(M,N),XMU, P2(M,N),P3(M,N)
607 FORMAT(2E16.5,15X,3E16.5)
604 CONTINUE
605 CONTINUE
    STOP
    END

```

```

SUBROUTINE T102(XLAM,AAAAAA,EDIM,BBBBBB)
DIMENSION EDIM(20)
COMMON XXXXL
XXXXL=XLAM
XMAX=0.9989
X0=0.001
XOXL=F3(XLAM,X0)
XOXLP=F3(XLAM+1.,X0)
XOXLM=F3(XLAM-1.,X0)
XOXLM2=F3(XLAM+2.,X0)
10 F0=(XOXL+XLAM*(XLAM+1.)*XOXLM2/(4.*XLAM+6.))*AAAAAA
G0=(XLAM*XOXLM+(XLAM+1.)*(XLAM+2.)*XLAM*XOXLP/(4.*XLAM+6.))*AAAAAA
DELX=0.001
X=X0
F=F0
G=G0
I=1
XTEST=0.101
20 XK1=F1(X,F,G)*DELX
XL1=F2(X,F,G)*DELX
XK2=F1(X+DELX/2., F+XK1/2., G+XL1/2.)*DELX
XL2=F2(X+DELX/2., F+XK1/2., G+XL1/2.)*DELX
XK3=F1(X+DELX/2., F+XK2/2., G+XL2/2.)*DELX
XL3=F2(X+DELX/2., F+XK2/2., G+XL2/2.)*DELX
XK4=F1(X+DELX, F+XK3, G+XL3)*DELX
XL4=F2(X+DELX, F+XK3, G+XL3)*DELX
DELF=1./6.*(XK1+2.*XK2+2.*XK3+XK4)
DELG=1./6.*(XL1+2.*XL2+2.*XL3+XL4)
X=X+DELX
F=F+DELF
G=G+DELG
IF(X .GE. XTEST) 201,203
201 EDIM(I)=F
XTFST=XTFST+0.1
I=I+1
203 CONTINUE
IF(X-0.01) 20,20,31
31 IF(X-0.99) 32,32,33
32 DELX=0.01
GO TO 20
33 IF(X-0.9989) 34,40,40
34 DELX=0.001
GO TO 20
40 ZETA=1.-X
C1=-XLAM*(XLAM+1.)/4.
C2=C1**2
C3=((XLAM*(XLAM+1.)-2.)*C1**2+2.*C1)/6.
GEND=F/(1.+C1*ZETA+C2*ZETA**2+C3*ZETA**3*ALOG(ZETA))
41 BBBBBB=1./GEND
EDIM(I)=GEND
RETURN
END

```

```
FUNCTION F1(A,B,C)
COMMON XLAM
F1=C
RETURN
END
```

```
FUNCTION F2(A,B,C)
COMMON XLAM
F2=(-2.*A*(A**2-1.)*C-XLAM*(XLAM+1.)*B)/(A**2*(A**2-1.))
RETURN
END
```

```
FUNCTION F3(D,E)
F3=EXP(D*ALOG(E))
RETURN
END
```

```

SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
XNU=SS
DD=1.
TEM1=0.
TEM2=0.
ZX=(1.-XMU)/2.
IF(ABS(ZX)-0.000001) 35,35,31
31 CONTINUE
DO 40 L=1,LLMAX
ZL=L
TEM1=TEM1+DD*ZX**(L-1)
TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
DD=DD*(ZL-XLAM-1.)*(ZL+XLAM)/(ZL**2+ZL*XNU)
40 CONTINUE
ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
PP=ARTFL*TEM1
DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
GO TO 41
35 PP=0.
DP=0.
41 CONTINUE
RETURN
END

```

TABLE I

Comparison of numerical results with the exact eigenvalues for a two dimensional corner.

Exact Solution	Numerical Solution			
	M $\neq$ J		M = J	
	even	odd	even	odd
	M _C =8 J _C =4	M _S =9 J _S =5	M _C =6 J _C =6	M _S =7 J _S =7
0.0	0.0		0.0	
2/3	.667394		.669715	
1.0		1.0		1.0
1+1/3	1.333342		1.333518	
1+2/3		1.668060		1.672477
2.0*	2.0*		2.0*	
2+1/3		2.333356		2.333793
2+2/3*	{2.666667 2.669734		{2.666669 2.679591	
3.0*		3.0*		3.0*

* Double root

TABLE II

Some exact eigensolutions for the three dimensional corner with $\beta = \pi/2$, $\alpha = \pi/4$.

Eigenvalue	Eigensolutions	
λ	Even solutions	Odd solutions
0	P ₀ (μ)	2
2	P ₂ (μ)	P ₂ ² (μ)sin 2 φ
4	P ₄ (μ), P ₄ ⁴ (μ)cos 4 φ	P ₄ ² (μ)sin 2 φ

TABLE III

Tabulations of the eigenvalues and the coefficients in the series representations of the eigenfunctions ($2\alpha = \pi/2$, $\beta_j = \pi/2$).

	even	
	Coef. A_m	Coef. C_j
$\lambda = 0.0$	.426487	.426487
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	
	0.0	0.0
	0.0	0.0
	0.0	

	odd	
	Coef. B_m	Coef. D_j
$\lambda = .839948$	-1.135600	-1.444600
	.073230	-.097647
	.004226	.031486
	-.021135	-.016324
	.017872	.010371
	-.007779	-.007452
	-.001831	.005917
	.007199	-.005446
	.007525	
	.004085	

$\lambda = .840341$	.496980	-.662630
	-.85376	-.394590
	.19722	.070879
	-.086956	-.032794
	.024596	.020205
	.009953	-.014725
	-.022866	.012976
	.020424	
	-.009732	

$\lambda =$ 1.807053	2.676300	-2.692700
	-.141180	.188240
	.224060	.027981
	-.131070	-.019274
	.059485	.013975
	-.010481	-.011068
	-.016294	.009643
	.024045	-.010008
	-.018772	
	.007506	

	even	
	Coef. Am	Coef. Cj
$\lambda=1.206357$	.379050	-.505400
	1.656200	1.188100
	.143140	-.038965
	-.010898	.011436
	-.008577	-.005016
	.008833	.002579
	-.004721	-.001293
	.000934	
	.000970	

	odd	
	Coef. Bm	Coef. Dj
$\lambda=2.0$	0.0	0.0
	3.303556	3.303556
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0

$\lambda=1.805246$	-.172240	.229650
	-1.263800	2.980100
	2.193400	.529750
	-.359600	-.135050
	.101280	.065806
	.009582	-.041720
	-.050099	.032932
	.048334	
	-.024699	

$\lambda =$ 2.814159	-4.072100	-2.750300
	3.765900	5.021200
	1.818200	.698380
	-.358120	-.103210
	.135310	.043530
	-.032647	-.025606
	-.016271	.018233
	.031678	-.015960
	-.026615	
	.011970	

	even	
	Coef. Am	Coef. Cj
$\lambda = 2.0$	.953654	.953654
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	
	0.0	

	odd	
	Coef. Bm	Coef. Dj
$\lambda = 2.868927$	2.584400	3.142500
	-4.047200	5.396300
	3.291700	1.576600
	-.657200	-.249830
	.283730	.108690
	-.081520	-.064265
	-.027367	.045417
	.067622	-038976
	-.060812	
	.029232	

$\lambda = 2.446688$	-.208840	.278460
	2.725500	-3.976900
	4.206800	1.747500
	.358770	-.103810
	.077859	.032182
	-.065595	-.014013
	.030583	.006305
	-.005977	
	-.004729	

	even	
	Coef. Am	Coef. Cj
$\lambda=2.813928$	.530570	-.707430
	-2.247100	-.855920
	1.003000	5.289000
	4.135700	.342700
	-.257020	-.116920
	-.039995	.064339
	.097445	-.047471
	-.079136	
	.035604	

$\lambda=2.958670$	-.507640	.676780
	2.214800	4.076500
	-3.983000	5.162800
	3.733900	.323730
	-.242800	-.065636
	.101760	.024029
	-.035758	-.010174
	.003411	
	.007631	

TABLE IV

Comparison of the eigenvalues for a three dimensional corner with those for a circular cone of the same solid angle $\Theta = \cos^{-1}(3/4)$.

Three Dimensional Corner		Circular Cone	
even	odd	$\Theta = \cos^{-1}(3/4)$	n
.840341	.839948	.863382	1
1.206357		1.210120	0
1.805246	1.807053	1.893798	2
2.0	2.0	1.961940	1
2.446688		2.477387	0
2.813928	2.814159	2.865298	2
2.958670	2.868927	2.936459	3
3.205219	3.205133	3.190234	1