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CONTENTS

A SURVEY ON SOME MATHEMATICAL MODELS OF THE VERY LOW FREQUENCY WAVE PROPAGATION

I. INTRODUCTION

1. Historical Background

The development of the theory of the very low frequency wave propagation may be divided roughly into two periods. The first period was ushered in with the introduction of long distance communications by means of the wireless at the turn of the century. The underlying principle in almost all the earlier analyses was the diffraction concept advanced by Rayleigh in 1904 [1].

In such a diffraction model, the existence of the ionosphere was ignored, and the earth was considered as a homogeneous conducting sphere. Among the earlier investigators were Poincaré [2], MacDonald [3], Nicholson [4], Sommerfeld [5], Love [6], March [7], Rybezynski [8], and others. Most of the theoretical investigations made use of the theory of the zonal harmonics, either by a direct summation or by forming an integral from the series, to explain the facts concerning the propagation of radio waves around the earth's curvature. However, the series then formulated was difficult to sum because of its slow convergence for the radio problem.

Poincaré [2] and Nicholson [4] attempted to replace the series by an integral and then obtained an approximate value for the integral by means of the calculus of residues. Their scheme, though substantially sound, was exceedingly elaborate, and their approximations became invalid in the vicinity of the antipodes of the transmitter.

The method used by MacDonald [3] was to approximate the original series by a modified series which in turn was replaced by an integral. This procedure was questioned from a purely mathematical point of view, though worked in most practical circumstances.

Love [6] undertook the laborious task of evaluating the series for the radio problem by numerically adding the terms of the series taken in groups. His memoir contained a complete bibliography of the problem up to 1915.

The results of the aforementioned authors did not explain the propagation phenomenon round the earth's curvature; nevertheless, a correct conclusion was reached that the diffracted field decayed exponentially from the transmitter.

Sommerfeld [5] investigated the problem using a planar model and concluded that the field decayed approximately as $1/\sqrt{d}$ or at most $1/d$ where d is the distance of propagation from the transmitter, which checked fairly well with the observed facts. Upon his suggestion, March [7] reformulated the problem using a spherical model. He cast his solution in an integral in order to obtain an answer more consistent with observed facts. Indeed, he found that the field decayed approximately as $1/\sqrt{\theta \sin \theta} \sim 1/d$ where θ is the angular distance between the transmitter and the receiver. March concluded that the waves were propagated without exponential attenuation contrary to the results obtained by other investigators. His method was severely questioned. However, von Rybezynski [8] extended March's method and pointed out that the exponential factor was neglected by March.

In reference to these investigations, Smith [9] remarked that "many years of work of the most famous mathematicians were inadequate to derive this task, which at the first glance seems very simple, from that state with respect to which Nicholson said that out of all mathematical problems it is the only one that has caused such a great divergence of opinions." It was against this background that Watson [10], in 1918, succeeded in his first of two classical papers on VLF wave propagation in making the final conclusion that we could not explain the fact of the long distance propagation of long waves around the terrestrial sphere using only the diffraction model.

The second period was initiated in 1919 by the work of Watson [10]. In the second of his two papers on the radio problem, Watson postulated a concentric plasma of high conductivity to account for the ionosphere which is mainly responsible for the VLF propagation over great distances. Watson succeeded in reformulating the integral used by others by means of now the well known Watson transformation. Instead of working with the slowly convergent series he was able to replace it by a rapidly converging series. He introduced the earth ionosphere waveguide concept. The radio waves excited by a vertical Hertzian dipole propagated in the atmosphere between the earth and the ionosphere in a way similar to the microwave in modern wave guides.

Among the earlier investigators in the second period in the development of the theory of the propagation of LF and VLF waves, aside from Watson, are Rydbeck [11], Van der Pol and Bremmer [12], Eckersley [13], Bremmer [14], Kendrick [15], Weyrick [16], Brekhovskikh [17], and Al'pert [18], Fock [19], and others. The results of their analyses indicated needs for further refinement of the model such as the introduction of terrestrial sphere stratification, anisotropy, continuous ionospheric stratification, polarization coupling and anisotropy at the ionosphere and the effects of ions, both heavy and light.

2. Recent Investigations

Among the most recent and active investigators are Budden [20], Wait [21], Johler [22], and Kranushkin [23].

Wait is a proponent of the mode theory (or the residue theory). To account for the behaviors of the fields in the presence of an ionosphere, he considered successively the ionosphere as sharply bounded, gradual, and then stratified; isotropic and the anisotropic; uniform and non-uniform. The stumbling block of the mode theory was the transcendental model equation in Hankel functions of complex order for determining the modes of the propagating waves. In its solutions, various approximations were employed. In view of this difficulty, Johler attempted the solution of the VLF propagation problem using only the original zonal harmonic series. With the improved summation method, the series could readily be summed on a large computer in spite of a large number of terms needed. Budden, on the other hand, advocated a direct and numerical solution of the full-wave equation taking into account of the coupling caused by the anisotropic and nonuniform nature of the ionosphere.

Kranushkin resolved the problem using the theory of non-self-adjoint differential operator eigenfunction expansion. He called his method the normal wave solution. He considered the ionosphere as stratified media with the boundaries being weakly angle dependent.

We will consider in the next section the four mathematical models currently being used; namely,

i. the full-wave solution,

ii. the mode theory,

iii. the zonal harmonic series,

iv. the normal wave method.

3. Theoretical Background

Most of the theoretical considerations of problems involving the propagation of radio waves are based on the assumption that the electromagnetic field satisfies Maxwell's equation in the form

$$\nabla \times \vec{E} = -i \omega \mu_0 \vec{H}$$

$$\nabla \times \vec{H} = i \omega \epsilon \vec{E}$$

In the above equations, a time harmonic factor $\exp(i\omega t)$ is understood; $\vec{E}$ and $\vec{H}$ are the complex representations of the electric and magnetic field vectors; μ_0 and ϵ are the permeability and permittivity of the material medium. For most non-ferromagnetic media, the permeability equals to that of the free space. For the ionosphere, the permittivity is a tensor. If ϵ is put equal to $\epsilon_0 \kappa$, where ϵ_0 is the permittivity of the free space, then κ, the dielectric tensor can be written as

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{X}{U(U^2 - Y^2)}$$

$$\times \begin{pmatrix} U^2 - \ell^2 Y^2 & i\,n\,Y\,U - \ell\,m\,Y^2 & -i\,m\,Y\,U - \ell\,n\,Y^2 \\ -i\,n\,Y\,U - \ell\,m\,Y^2 & U^2 - m^2 Y^2 & i\,\ell\,Y\,U - m\,n\,Y^2 \\ i\,m\,Y\,U - \ell\,n\,Y^2 & -i\,\ell\,Y\,U - m\,n\,Y^2 & U^2 - n^2 Y^2 \end{pmatrix}$$

The standard URSI notation is used here. X is the square of the ratio of plasma frequency to wave frequency; $U = 1 - iZ$, where Z is the ratio of the electron collision frequency to the wave frequency; $Y(\ell, m, n)$ are the components of the vector $\vec{Y}$ referred to some rectangular cartesian coordinates. Y is the magnitude of the ratio of the gyromagnetic frequency of electrons to the wave frequency in the direction of the earth's magnetic field.

The dielectric tensor is Hermitian. It implies that the ionosphere is an anisotropic medium. The changes in the electron density and collision frequency and their variations with height contributing to the inhomogeneity to the property of the ionosphere of the VLF wave propagation.

In all the problems considered in the sequel, the primary source is a Hertzian vertical dipole.

II. THE DIFFRACTION MODEL

We will consider first the diffraction problem together with Van der Pol and Bremmer.

An electric dipole is situated at Q, a distance b away from the center of the sphere of radius a. The observation point is at P, a distance r away from the center of the sphere. The points Q and P are separated by a distance R.

The electromagnetic fields, $\vec{E}$ and $\vec{H}$ satisfy Maxwell's equations. A time factor $\exp(-i\omega t)$ is used. The analytic problem consists of finding an electric Hertzian vector potential function $\pi\vec{r}$, where $\vec{r}$ is a position vector, such that it satisfies the following differential equations and boundary conditions:

(1) $(\nabla^2 + k_1^2)\,\pi = 0 \quad (r > a),$

$(\nabla^2 + k_2^2)\,\pi = 0 \quad (r < a).$

(2) $\frac{\partial}{\partial r}(r\pi)$ and $k^2\pi$ are continuous at $r = a$,

(3) A singularity at Q of the form $\frac{e^{ik_1R}}{ik_1R}$ which is further called the primary field.

The electric and magnetic fields are given by

$$E_r = \left(k^2 + \frac{\partial^2}{\partial r^2}\right)(r\,\pi)$$

$$E_\theta = \frac{1}{r}\,\frac{\partial^2}{\partial r\,\partial\theta}(r\,\pi)$$

$$H_\phi = -\frac{ic}{\omega}\,k^2\,\frac{\partial\pi}{\partial\theta}$$

and

$$k_{1,2}^2 = \frac{\epsilon_{1,2}\,\omega^2 + i\,\sigma_{1,2}\,\omega}{c^2}$$

1. The Zonal Harmonic Series

The solution of the total Hertzian potential function in the form of the zonal harmonic series for $r > a$ is,

$$\pi_{tot} = \frac{e^{ik_1 R}}{i k_1 R}$$

$$+ \sum_{n=0}^{\infty} (2n+1) R_n \frac{\psi_n(k_1 a)}{\zeta_n^{(1)}(k_1 a)} \zeta_n^{(1)}(k_1 b)\, \zeta_n^{(1)}(k_1 r)\, P_n(\cos\theta) \quad (0.1a)$$

and for $r < a$ is

$$\pi_{tot} = \frac{k_1^2}{k_2^2} \sum_{n=0}^{\infty} (2n+1)(R_n + 1) \frac{\psi_n(k_1 a)}{\psi_n(k_2 a)} \zeta_n^{(1)}(k_1 b)\, \psi_n(k_2 r)\, P_n(\cos\theta) \quad (0.1b)$$

where

$$\zeta_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+1/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{ix}\right)$$

$$\zeta_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+1/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{i e^{-ix}}{x}\right)$$

$$\psi_n(x) = \frac{1}{2} \left\{\zeta_n^{(1)}(x) + \zeta_n^{(2)}(x)\right\}$$

$$= \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

(H = Hankel function, J = Bessel function) and where

$$R_n = \frac{-\left[\frac{1}{x}\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_1 a} + \left[\frac{1}{x}\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_2 a}}{\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(1)}(x)\}\right]_{x=k_1 a} - \left[\frac{1}{x}\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_2 a}} \quad (0.2)$$

It is well known that

$$\frac{e^{ik_1R}}{ik_1R} = \sum_{n=0}^{\infty} (2n+1)\, \zeta_n^{(1)} (k\ \ r)\, \psi_n (k_1 b)\, P_n (\cos\theta)\ (r > b) \tag{0.3a}$$

$$\frac{e^{ik_1R}}{ik_1R} = \sum_{n=0}^{\infty} (2n+1)\, \zeta_n^{(1)} (k, b)\, \psi_n (k_1 r)\, P_n (\cos\theta)\ (r < b) \tag{0.3b}$$

By substituting 0.3b into 0.1a, 1.1a can be expressed explicitly as a zonal harmonic series. Furthermore, on the surface of the earth were $r = a$, it can be reduced to the following

$$\pi_{tot} = \frac{i}{(k_1a)^3} \sum_{n=0}^{\infty} (2n+1) \frac{\zeta_n^{(1)} (k_1 b)}{\zeta_n^{(1)} (k_1 a)} \frac{1}{N_n} P_b (\cos\theta) \tag{0.4}$$

where

$$N_n (x, y) = \left[\frac{1}{x} \frac{d}{dx} \log \{x\, \zeta_n^{(1)} (x)\}\right]_{x = k_1 a}$$

$$- \left[\frac{1}{y} \frac{d}{dy} \log \{y \psi_n (y)\}\right]_{y = k_2 a} \tag{0.5}$$

In the reduction to eq. (0.4), use is made of the Wronskian:

$$\psi_n (x)\, \zeta_n^{(1)\prime} (x) - \zeta_n^{(1)} (x)\, \psi_n' (x) = \frac{i}{x^2} \tag{0.6}$$

2. The Watson Transformation and the Residue Series

Following Watson, we transform (0.4) into an integral over n leading to

$$\pi_{tot} = \frac{1}{(k_1 a)^3} \int_{c_1} \frac{n\, dn}{\cos (n\pi)} \frac{\zeta_{n-1/2}^{(1)} (k_1 b)}{\zeta_{n-1/2}^{(1)} (k_1 a)} \frac{1}{N_{n-1/2}} P_{n-1/2} [\cos (\pi - \theta)] \tag{0.7a}$$

where C_1 is the contour which enclosed the positive real axis in a counter-clockwise direction. By a judicial manipulation of C_1, eq. (0.7a) is equivalent to the following:

$$\pi_{tot} = \frac{1}{(k_1 a)^3} \int_L \frac{n\,dn}{\cos(n\pi)} \frac{\zeta^{(1)}_{n-1/2}(k_1 b)}{\zeta^{(1)}_{n-1/2}(k_1 a)} \frac{1}{N_{n-1/2}} P_{n-1/2}[\cos(\pi-\theta)]$$

$$+ \int_{-i\infty}^{i\infty} f(n)\,dn \tag{0.7b}$$

where f(n) is the integrand of (0.7a). The line integral in (0.7b) is usually considered negligible. L is the contour encloses all the poles in the integrand of (0.7a) other than the ones due to $\cos(n\pi)$.

The first term in (0.7b) enables us to obtain a physical interpretation. Let us expand $1/\cos(n\pi)$ as

$$\frac{1}{\cos(n\pi)} = 2\, e^{i\pi n} \sum_{m=0}^{\infty} e^{i\pi m(2n+1)} \tag{0.8}$$

We can write (0.7b) over the integration path L as

$$\pi_{tot} \cong \sum_{m=0}^{\infty} \pi_m \tag{0.9}$$

where

$$\pi_m = \frac{2\, e^{i\pi m}}{(k_1 a)^3} \int_L n\,dn \quad e^{i\pi n(2m+1)} \frac{\zeta^{(1)}_{n-1/2}(k_1 b)}{\zeta^{(1)}_{n-1/2}(k_1 a)} \frac{1}{N_{n-1/2}}$$

$$\times P_{n-1/2}[\cos(\pi-\theta)] \tag{0.10}$$

The integral in (0.10) can now be represented as the sum of the residues of all the enclosed poles in L, and it is written as

$$\pi_m = \frac{4\pi i\, e^{i\pi m}}{(k_1 a)^3} \sum_{s=0}^{\infty} n_s\, e^{i\pi n_s(2m+1)} \frac{\zeta^{(1)}_{n_s-1/2}(k_1 b)}{\zeta^{(1)}_{n_s-1/2}(k_1 a)}$$

$$\times \frac{1}{\left(\frac{\partial}{\partial n} N_{n-1/2}\right)_{n=n_s}} P_{n_s-1/2}[\cos(\pi-\theta)] \tag{0.11}$$

Upon using a well-known asymptotic formula for the spherical harmonics of complex order with positive imaginary part, (if θ is not too near 0 or π),

$$P_{n-1/2}\left[\cos(\pi-\theta)\right] \sim \frac{1}{\sqrt{2\pi n \sin\theta}}\, e^{-in(\pi-\theta)+i\pi/4} \tag{0.12}$$

$$\pi_m \sim \frac{2\sqrt{2\pi}}{(k_1 a)^3}\, \frac{e^{i\pi(m+3/4)}}{\sqrt{\sin\theta}}$$

$$\times \sum_{s=0}^{\infty} \frac{\zeta^{(1)}_{n_s-1/2}(k_1 b)}{\zeta^{(1)}_{n_s-1/2}(k_1 a)}\, \frac{n_s^{1/2}}{\left(\frac{\partial N_{n-1/2}}{\partial n}\right)_{n=n_s}}\, e^{in_s(\theta+2\pi m)} \tag{0.13}$$

The last factor in (0.13) becomes

$$e^{in_s(\theta+2\pi m)} = e^{(i\alpha_s-\beta_s)(\theta+2\pi m)} \tag{0.14}$$

where

$$n_s = \alpha_s + i\,\beta_s \qquad (\beta_s > 0)$$

Hence (0.13) represents a wave travelling round the sphere m times before reaching the receiver with a phase velocity and attenuation determined by α_s and β_s. Because of the fact that a wave travelling m times round the earth becomes greatly attenuated, only the m = 0 term contribute significantly to the fields. Thus,

$$\pi_{tot} \sim \pi_0$$

$$\sim \frac{2\sqrt{2\pi}\, e^{i3\pi/4}}{(k_1 a)^3 \sqrt{\sin\theta}}$$

$$\times \sum_{s=0}^{\infty} \frac{\zeta^{(1)}_{n_s-1/2}(k_1 b)}{\zeta^{(1)}_{n_s-1/2}(k_1 a)}\, \frac{n_s^{1/2}}{\left(\frac{\partial N_{n-1/2}}{\partial n}\right)_{n=n_s}}\, e^{in_s\theta} \tag{0.15}$$

3. The Eigenfunction Method

Van der Pol and Bremmer also consider the possibility of casting the radio wave propagation as an eigenvalue problem. In this case eq's (0.1a) and (0.1b) are determined by the following considerations:

(1) $(\nabla^2 + k_1^2 \lambda_j^2)\, \phi_j = 0 \quad (r > a)$

$(\nabla^2 + k_2^2 \lambda_j^2)\, \phi_j = 0 \quad (r < a)$

(2) $k^2 \phi_j$ and $\frac{\partial}{\partial r}(r\, \phi_j)$ are continuous at $r = a$.

(3) $k_1^2 \int_{r=a}^{\infty} \phi_j^2 \, d\tau + k_2^2 \int_{r=0}^{a} \phi_j^2 \, d\tau = 1$

where ϕ_j's are the eigenfunctions corresponding to the eigenvalues λ_j's. The primary field pertaining to a point source leads to a three dimensional Dirac function $4\pi i\, \delta(R)/k_1$ at Q.

The eigenfunctions are of the form

$$\phi_{n,j} = \begin{cases} A_{n,j}\, \zeta_n^{(1)}(\lambda_{n,j} k_1 r)\; P_n(\cos\theta) & (r > a) \\ B_{n,j}\, \psi_n(\lambda_{n,j} k_1 r)\; P_n(\cos\theta) & (r < a) \end{cases} \tag{0.16}$$

where n is a positive integer. The total field satisfying conditions (1) through (3) becomes

$$\pi_{tot} = \frac{4\pi i}{k_1} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{A_{n,j}^2}{(1-\lambda_{n,j}^2)}\, \zeta_n^{(1)}(\lambda_{n,j} k_1 b)\, \zeta_n^{(1)}(\lambda_{n,j} k_1 r)\, P_n(\cos\theta) \qquad (r > a) \tag{0.17a}$$

$$\pi_{tot} = \frac{4\pi i}{k_1} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{A_{n,j} B_{n,j}}{(1-\lambda_{n,j}^2)}\, \zeta_n^{(1)}(\lambda_{n,j} k_1 b)\, \psi_n(\lambda_{n,j} k_2 r)\, P_n(\cos\theta) \qquad (r < a) \tag{0.17b}$$

where

$$A^2_{n,j} = \frac{(2n+1)}{2\pi k_1^2 a^3 \{\zeta_n^{(1)}(x)\}^2 \left[\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)\log\{x\,\zeta_n^{(1)}(x)\} - \frac{k_1^2}{k_2^2}\left(\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right)\log\{y\,\psi_n(y)\}\right]} \tag{0.18a}$$

and

$$B^2_{n,j} = \frac{(2n+1)}{2\pi k_2^2 a^3 \{\psi_n(y)\}^2 \left[\frac{k_2^2}{k_1^2}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)\log\{x\,\zeta_n^{(1)}(x)\} - \left(\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right)\log\{y\,\psi_n(y)\}\right]} \tag{0.18b}$$

The difference between the earlier method and the method of the eigenfunctions is clear. In the latter the summation is done over the integers whereas in the former it is done over the complex roots of the modal equation.

To sum up, we have seen in this section that there are now clearly three different methods used in the solution of the VLF propagation problems. They are (a) the zonal harmonic series method; (b) the residue series method; and (c) the eigenfunction method. In the next two decades, most of the analyses are done following these three methods.

III. THE EARTH-IONOSPHERE MODEL

To account for the effects on the VLF wave propagation due to the presence of the ionosphere, additional boundary conditions were introduced. The ionosphere was first considered as a sharply bounded homogeneous medium, then as stratified media. Magneto-ionic theory was also introduced to take care of both the isotropic case where the geomagnetic field was ignored and the anisotropic case where the geomagnetic field is included. The additional conditions introduced led to a great deal of complexity; nevertheless, the mathematical models employed had their genesis in the diffraction model.

1. The Mode Theory

Among the most active proponent of the mode theory is J. R. Wait. We shall look at the problem with him.

A. Isotropic, uniform, sharply bounded ionosphere. The ionosphere is treated as a concentric homogeneous sharply bounded ionosphere. The source is a verticle dipole. The fields are expressed in terms of a Hertzian vector which has only a radial component U. In the region between the surface of the earth ($r = a$) and the lower edge of the ionosphere ($r = c$, or $a + h$),

$$E_r = \left(k^2 + \frac{\partial^2}{\partial r^2}\right)(rU),$$

$$E_\theta = \frac{1}{r}\,\frac{\partial^2}{\partial r\,\partial\theta}\,(rU),$$

$$H_\phi = -\,i\,\epsilon\,\omega\,\frac{\partial U}{\partial\theta},$$

$$E_\phi = H_r = H_\theta = 0. \tag{1.1}$$

The time factor $\exp(i\,\omega t)$ is understood.

ϵ and μ are the electric constants and $k = (\epsilon\mu)^{1/2}\omega$. Subscripts g and i are added to the field quantities where reference is made to the region $r < a$ and $r > c$, respectively. To account for the singularity introduced by the source at $r = b$, the Hertzian potential satisfies the inhomogeneous wave equation,

$$E_\theta^{(q)} = Z_i^{(q)} H_\phi^{(q)} \qquad \text{at } r = c \tag{1.9}$$

where

$$Z_g^{(q)} = \frac{1}{i\,\epsilon\,\omega} \frac{\frac{\partial}{\partial r}\left[r\, j_q\,(k_g\, r)\right]}{r\, j_q\,(k_g\, r)} \tag{1.10}$$

and

$$Z_i^{(q)} = -\frac{1}{i\,\epsilon\,\omega} \frac{\frac{\partial}{\partial r}\left[r\, h_q^{(2)}\,(k_i\, r)\right]}{r\, h_q^{(2)}\,(k_i\, r)} \tag{1.11}$$

These can be rewritten upon replacing kr by x as

$$\frac{1}{x}\frac{\partial}{\partial x}(x U) = i\,(Z_g^{(q)}/\eta)\, U \quad \text{for } x = k\, a \tag{1.12}$$

$$\frac{1}{x}\frac{\partial}{\partial x}(x U) = -\, i\,(Z_i^{(q)}/\eta)\, U \quad \text{for } x = k\, c \tag{1.13}$$

The solution for $a \leq r \leq c$ is

$$U = \frac{i\,k\,C}{4\,\pi}\sum_{q=0}^{\infty}(2\,q+1)\, h_q^{(2)}\,(k\,b)\, h_q^{(1)}\,(k\,r)\, P_q\,(\cos\theta)\frac{F_q}{D_q} \tag{1.14}$$

where

$$F_q = \left[1 + R_g^{(q)}\frac{h_q^{(1)}\,(k\,a)\, h_q^{(2)}\,(k\,r)}{h_q^{(2)}\,(k\,a)\, h_q^{(1)}\,(k\,r)}\right]\left[1 + R_i^{(q)}\frac{h_q^{(2)}\,(k\,c)\, h_q^{(1)}\,(k\,b)}{h_q^{(1)}\,(k\,c)\, h_q^{(2)}\,(k\,b)}\right] \tag{1.15}$$

$$D_q = 1 - R_g^{(q)}\, R_i^{(q)}\frac{h_q^{(1)}\,(k\,a)\, h_q^{(2)}\,(k\,c)}{h_q^{(2)}\,(k\,a)\, h_q^{(1)}\,(k\,c)} \tag{1.16}$$

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$$R_g^{(q)} = \frac{\frac{d}{dx}\,[\ln x\, h_q^{(1)}(x)]_{x=ka} - i\, Z_g^{(q)}/\eta}{-\frac{d}{dx}\,[\ln x\, h_q^{(2)}(x)]_{x=ka} - i\, Z_g^{(q)}/\eta} \tag{1.17}$$

$$R_i^{(q)} = -\frac{\frac{d}{dx}\,[\ln x\, h_q^{(2)}(x)]_{x=kc} + i\, Z_i^{(q)}/\eta}{\frac{d}{dx}\,[\ln x\, h_q^{(1)}(x)]_{x=kc} + i\, Z_i^{(q)}/\eta} \tag{1.18}$$

By means of the Watson transformation, the solution becomes

$$U = -\, i\, k\, C \sum_{\nu} \frac{\left(\nu + \frac{1}{2}\right)}{\sin \nu \pi}\, h_\nu^{(2)}(kb)\, h_\nu^{(1)}(kr)\, \frac{F_\nu}{D'_\nu} P_\nu\, [\cos(\pi - \theta)] \tag{1.19}$$

where $D'_\nu = \partial D_\nu / \partial \nu$. All the values in the summand are to be evaluated at the poles of $f(\nu)$ which are the roots of the modal equation

$$D_\nu = 0 \tag{1.20}$$

If use is made of the Airy integral as expounded by V. A. Fock in place of the Hankel function, the radial component of the electric field can be written in the form

$$E_r \simeq \frac{e^{-ika\theta}}{a\,(\theta \sin\theta)^{1/2}}\, V \tag{1.21}$$

apart from a constant factor. And

$$V = e^{i\pi/4} \left(\frac{x}{\pi}\right)^{1/2} \oint \frac{e^{-ixt}\,[w_1(t-y) + A(t)\, w_2(t-y)]}{[w'_1(t) - q w'_1(t)][1 - A(t)\, B(t)]}\, dt \tag{1.22}$$

where

$$x = (ka/2)^{1/2}\,\theta, \quad y = (2/ka)^{1/3}\, k\,(r - a) \tag{1.23}$$

Now the contour is to enclose the complex poles t_n's which are solutions of

$$1 - A(t)\, B(t) = 0 \tag{1.24}$$

where

$$A(t) = -\left[\frac{w_1'(t - y_0) + q_i\, w_1(t - y_0)}{w_2'(t - y_0) + q_i\, w_2(t - y_0)}\right], \tag{1.25}$$

$$B(t) = -\left[\frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)}\right], \tag{1.26}$$

$$y_0 = (2/k\,a)^{1/3}\, k\,h, \quad q_i = -\,i\,(k\,a/2)^{1/3}\, z/\eta_0, \quad \eta_0 = 120\,\pi$$

and

$$q = -\,i\,(k\,a/2)^{1/3}\left(\frac{i\,\epsilon_0\,\omega}{\sigma_g + i\,\epsilon_g\,\omega}\right)^{1/2}\left(1 - \frac{i\,\epsilon_0\,\omega}{\sigma_g + i\,\epsilon_g\,\omega}\right)^{1/2}$$

The residue or mode series is

$$v = -\,2\,(\pi\,x)^{1/2}\, e^{-i\pi/4} \sum_{n=0}^{\infty} \frac{e^{-ixt_n}\,[w_1(t_n - y) + A(t_n)\, w_2(t_n - y)]}{[w_1'(t_n) - q\, w_1(t_n)]\left[\frac{\partial}{\partial t} A(t)\, B(t)\right]_{t=t_n}} \tag{1.27}$$

It may also be written as

$$v = \frac{4\,(\pi\,x)^{1/2}}{y_0}\, e^{-i\pi/4} \sum_{n=0}^{\infty} e^{-izt_n}\, G_n(\hat{y})\, G_n(y)\, \Lambda_n \tag{1.28}$$

where $\hat{y} = (2/ka)^{1/3}\, kZ_h$. The functions G_n are height functions and they are normalized to unity for y or y_0 equal to zero. Explicitly

$$G_n(y) = \frac{f(t_{n,\,y})}{f(t_{n,\,0})} \tag{1.29}$$

$$f(t_n, y) = w_1(t_n - y) + A(t_n)\, w_2(t_n - y) \tag{1.30}$$

The function Λ_n is a modal excitation factor. It is normalized so that, in the limit of zero curvature and perfectly reflecting boundaries, it becomes unity for all modes. Explicitly,

$$\Lambda_n = \frac{y_0}{2}\left[(t_n - q^2) - \frac{(t_n - y_0 - q_i^2)\,[w_2'(t_n) - q\, w_2(t_n)]^2}{[w_2'(t_n - y_0) + q_i\, w_2(t_n - y_0)]^2}\right]^{-1} \tag{1.31}$$

b. <u>Anisotropic Ionosphere.</u> For the case of an anisotropic ionosphere where the earth magnetic field is not ignored, the electromagnetic field components in the homogeneous space ($a \le r \le c$) are expressed in terms of purely TM and TE waves. They are derived from two scalar functions U and V as follows

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right) (r\,U)$$

$$E_\theta = \frac{1}{r}\,\frac{\partial^2}{\partial\theta\,\partial r}\,(r\,U) - \frac{i\,\mu\,\omega}{r\sin\theta}\,\frac{\partial}{\partial\phi}\,(r\,V)$$

$$E_\phi = \frac{1}{r\sin\theta}\,\frac{\partial^2}{\partial\phi\,\partial r}\,(r\,U) + \frac{i\,\mu\,\omega}{r}\,\frac{\partial}{\partial\theta}\,(r\,V) \tag{1.32}$$

$$H_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right) (r\,V)$$

$$H_\theta = \frac{1}{r}\,\frac{\partial^2}{\partial\theta\,\partial r}\,(r\,V) + \frac{i\,\epsilon\,\omega}{r\sin\theta}\,\frac{\partial}{\partial\phi}\,(r\,U)$$

$$H_\phi = \frac{1}{r\sin\theta}\,\frac{\partial^2}{\partial r\,\partial\phi}\,(r\,V) - \frac{i\,\epsilon\,\omega}{r}\,\frac{\partial}{\partial\theta}\,(r\,U)$$

The functions U and V satisfy

$$(\nabla^2 + k^2)\begin{Bmatrix} U \\ V \end{Bmatrix} = 0 \tag{1.33}$$

in source free region. The source again is a vertical dipole. The primary field of such a source is of a purely TM character. It is derived from a single scalar function U_e satisfying the following inhomogeneous wave equation,

$$(\nabla^2 + k^2)\,U_e = C\,\frac{\delta\,(r-b)\;\delta\,(\theta)}{2\,\pi\,r^2\sin\theta} \tag{1.34}$$

The boundary conditions, in terms of surface impedance, are

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(xU) = i\,\Delta_g\, xU \\[2em] &\frac{\partial}{\partial x}(x\overline{V}) = (i/\Delta_g)\, x\overline{V} \end{aligned}\right|_{x=ka} \tag{1.35}$$

where

$$\Delta_g = Z_g/\eta, \quad U = U_e + U_s, \quad \overline{V} = \eta V;$$

and

$$\left.\begin{aligned} \frac{\partial}{\partial x}(xU) &= \Delta_{11}\frac{\partial}{\partial x}(x\overline{V}) - i\,\Delta_{12}(xU) \\[2em] -\,i\,(x\overline{V}) &= \Delta_{21}\frac{\partial}{\partial x}(x\overline{V}) - i\,\Delta_{22}(xU) \end{aligned}\right|_{x=kc} \tag{1.36}$$

where

and

$$\Delta_{11} = Z_{11}/\eta, \quad \Delta_{12} = Z_{12}/\eta, \quad \Delta_{21} = Z_{21}/\eta$$

$$\Delta_{22} = Z_{22}/\eta,$$

The radial field in the concentric waveguide region written in the matrix form is

$$\begin{bmatrix} E_r \\ \eta H_r \end{bmatrix} \simeq \frac{e^{-ika\theta}}{a(\theta \sin\theta)^{1/2}} \begin{bmatrix} F_e \\ F_h \end{bmatrix} \frac{i\,\mu_0\,\omega\, I\, ds}{4\pi} \tag{1.37}$$

where

$$\begin{bmatrix} F_e \\ F_h \end{bmatrix} = -2(\pi x)^{1/2} e^{-i\pi/4} \sum_{n=0}^{\infty} \frac{e^{-ixt_n}\,[w_1(t_n - y) + A(t_n)\, w_2(t_n - y)]}{[w_1'(t_n) - q\, w_1(t_n)]\left[\frac{\partial}{\partial t} A(t)\,B(t)\right]_{t_n}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.38}$$

The roots t_n's are determined from the matrix equation

$$[A(t)]\ [B(t)] = I \tag{1.39}$$

where

$$A(t) = \begin{bmatrix} {}_{\parallel}R_{\parallel} & {}_{\perp}R_{\parallel} \\ {}_{\parallel}R_{\perp} & {}_{\perp}R_{\perp} \end{bmatrix} \exp\left[-i\,(4/3)\,(y_0 - t)^{3/2} - i\,\pi/2\right] \tag{1.40}$$

and

$$B(t) = \begin{bmatrix} -\left[\dfrac{w_2'(t) - q\,w_2(t)}{w_1'(t) - q\,w_1(t)}\right] & 0 \\ 0 & -\left[\dfrac{w_2'(t) - \hat{q}\,w_2(t)}{w_1'(t) - \hat{q}\,w_1(t)}\right] \end{bmatrix} \tag{1.41}$$

$$q = -i\,(k\,a/2)^{1/3}\,\Delta_g,\ \text{with}\ \Delta_g = \left(\frac{i\,\epsilon_0\,\omega}{\sigma_g + i\,\epsilon_g\,\omega}\right)^{1/2}\left(1 - \frac{i\,\epsilon_0\,\omega}{\sigma_g + i\,\epsilon_g\,\omega}\right)^{1/2}$$

and

$$\hat{q} = -i\,(k\,a/2)^{1/3}\,\Delta_g^h,\ \text{with}\ \Delta_g^h = \left(\frac{\sigma_g + i\,\epsilon_g\,\omega}{i\,\epsilon_0\,\omega} - 1\right)^{1/2}$$

${}_{\parallel}R_{\parallel}$, ${}_{\perp}R_{\parallel}$, ${}_{\parallel}R_{\perp}$, and ${}_{\perp}R_{\perp}$ are the reflection coefficients, where $\parallel$ and $\perp$ refer to the direction of the electric vector, relative to the plane of incidence, of the incident (first suffix) and reflected (second suffix) wave.

2. The Zonal Harmonic Series

The stumbling block of the mode theory is the rather formidable transcendental modal equation to solve. Recently Wait and Spies made detailed computation of the characteristics of the least attenuated modes. However, a single mode is not adequate to describe the complete field satisfying Maxwell's equations at either intermediate and great distances for daytime models of the lower ionosphere. In the case of a nighttime model even more modes are needed at all distances.

With the speed of the modern digital computers, the argument once held against the zonal harmonic solution of the VLF propagation problem is no longer convincing. Granted that the zonal harmonic series converges rather slow. Johler showed that with improved summation technique, the results of the computations indicate that full rigor can be achieved with comparative ease at frequencies less than about 50 kc/s, leaving only the assumed model for the transmitter and the propagation medium and avoiding the complications of the Watson transformation.

For a simple model of the terrestrial sphere of radius a surrounded by concentric ionosphere from $r = c$ to $r = \infty$, the field components are expressed in terms of the Hertzian potential functions, π^e and π^m as follows, with $\exp(iwt)$ understood,

$$E_r = \frac{-1}{rb\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\pi^e}{\partial\theta}\right)$$

$$E_\theta = \frac{1}{rb}\frac{\partial^2}{\partial r\,\partial\theta}(r\,\pi^e)$$

$$H_\phi = \frac{1}{b}\frac{k^2}{\mu_0 i\omega}\frac{\partial\pi^e}{\partial\theta}$$

$$E_\phi = \frac{\mu_0 i\omega}{b}\frac{\partial\pi^m}{\partial\theta}$$

$$H_r = \frac{-1}{rb\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\pi^m}{\partial\theta}\right)$$

$$H_\theta = \frac{1}{rb}\frac{\partial^2}{\partial r\,\partial\theta}(r\,\pi^m) \qquad (2.1)$$

where π^e and π^m satisfy the wave equations

$$(\nabla^2 + k^2)\begin{Bmatrix}\pi^e\\ \pi^m\end{Bmatrix} = 0 \qquad (2.2)$$

The media of propagation are characterized by their electric constants in the wave number, k. Thus, for air, with index of refraction, η, or dielectric constant $\epsilon_1 = \eta_1^2$

$$k^2 = k_1^2 = \frac{\omega^2}{c^2} \epsilon_1$$

or

$$k_1 = \frac{\omega}{c} \eta_1 \tag{2.3}$$

For the ground

$$k = k_2 = \frac{\omega}{c} \sqrt{\epsilon_2 - i \frac{\sigma \mu_0 c^2}{\omega}} \tag{2.4}$$

where σ is the ground conductivity and ϵ_2 is the relative dielectric constant.

The wave number of the ionosphere is rather intricate.

$$k = k_2 = \frac{\omega}{c} \eta_{n,e}^{m,r} \tag{2.5}$$

There are four distinct propagation components with the complex index of refraction, η_n^m, where n equals to 0 or e, corresponding to ordinary wave (0), and extraordinary wave (e); and m equals to i or r corresponding to upgoing wave (i), and downgoing wave (r). They are related to the roots of a Booker type quartic in the parameters, ζ, which is the result of the simultaneous solution of the Langevin equations of charged particles and Maxwell's equation. The index of refraction, η,

$$\eta^2 = \zeta^2 + \sin^2 \phi_i \tag{2.6}$$

where ϕ_i is the angle of incidence on the lower ionosphere plasma.

With a vertical electric dipole transmitter, $\vec{\Omega}\pi^e_o$ located in medium of wave number k_1, the boundary conditions are

$$\left.\begin{aligned}
k_1^2(\pi_0^e + \pi_1^e) &= k_2^2(\pi_2^e) \\
\frac{\partial}{\partial r}(r\pi_0^e + r\pi_1^e) &= \frac{\partial}{\partial r}(r\pi_2^e) \\
\pi_1^m &= \pi_2^m \\
\frac{\partial}{\partial r}(r\pi_1^m) &= \frac{\partial}{\partial r}(r\pi_2^m)
\end{aligned}\right|_{r=a} \tag{2.7}$$

$$\left.\begin{aligned}
k_1^2(\pi_0^e + \pi_1^e) &= \mu_0\, i\,\omega\, Q_{em}^{-1}\,\frac{1}{r}\,\frac{\partial}{\partial r}(r\pi_3^m) \\
\frac{1}{r}\,\frac{\partial}{\partial r}(r\pi_0^e + r\pi_1^e) &= \mu_0\, i\,\omega\, Q_{me}^{-1}\,\pi_3^m \\
\pi_1^m &= Q_{me}\,\frac{1}{r}\,\frac{\partial}{\partial r}(r\pi_3^e) \\
\mu_0\, i\,\omega\,\frac{\partial}{\partial r}(r\pi_1^m) &= Q_{em}\, k_3^2\,\pi_3^e
\end{aligned}\right|_{r=c} \tag{2.8}$$

where in the quasi-longitudinal approximation,

$$Q_{me} = \frac{E_\phi}{E_\theta} = \pm\,\frac{i\,k_3/k_1}{\sqrt{k_3^2/k_1^2 - \rho}}$$

$$P_{me} = \frac{E_r}{E_\theta} = \frac{-\sin\phi_i}{\zeta} \tag{2.9}$$

$\rho = \sin^2 \phi_2$ is the complex direction sine squared of propagation, and the minus sign (-) is taken with the ordinary wave, $k_{3,0}$ and the plus sign (+) is taken with the extraordinary wave $k_{3,e}$. The solutions are

$$\pi_0^e = \frac{1}{k_1^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, \zeta_n(k_1 b)\, \psi_n(k_1 r)\, P_n(\cos\theta), \ (r < b)$$

$$= \frac{1}{k_1^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, \zeta_n(k_1 r)\, \psi_n(k_1 b)\, P_n(\cos\theta), \ (r > b)$$

$$\pi_1^e = \frac{1}{k_1^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, [b_n^e\, \zeta_n(k_1 r) + c_n^e\, \psi_n(k_1 r)]\, P_n(\cos\theta)$$

$$\pi_1^m = \frac{1}{k_1^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, [b_n^m\, \zeta_n(k_1 r) + c_n^m\, \zeta_n(k_1 r)]\, P_n(\cos\theta)$$

$$\pi_2^e = \frac{1}{k_2^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, a_n^e\, \psi_n(k_2 r)\, P_n(\cos\theta)$$

$$\pi_2^m = \frac{1}{k_2^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, a_n^m\, \psi_n(k_2 r)\, P_n(\cos\theta)$$

$$\pi_3^e = \frac{1}{k_3^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, d_n^e\, \zeta_n(k_3 r)\, P_n(\cos\theta)$$

$$\pi_3^m = \frac{1}{k_3^2\, r b} \sum_{n=0}^{\infty} (2n+1)\, d_n^m\, \zeta_n(k_3 r)\, P_n(\cos\theta) \tag{2.10}$$

The constants a_n^e, b_n^e and the like are determined by the boundary conditions.

For a stratified ionosphere model, the matching for boundary conditions could be a rather formidal task. (For details see Reference 22).

3. The Full-wave Theory

It is seen that when coupling is introduced into the wave propagation problem, no matter what mathematical model one uses whether it is based on the residue series or the zonal harmonic series, one often resorts to the digital computer for numerical computations. Budden is one among many of the English school who rely more and more on the direct solution of the full wave equation using computers. This approach is formally of complete generality and does not depend on the assumption of a special ionospheric model.

In deriving the differential equations, Budden uses the cartesian coordinates with the z-axis directed vertically upwards. A plane wave is incident at angle θ_i to the verticle on the ionosphere from below. Let $s = \sin\theta_i$, $c = \cos\theta_i$. Let the ionosphere be stratified such that in each of which the medium is considered homogeneous. For the waves in each stratum $\partial/\partial x(\ ) \equiv -iks(\ )$; $\partial/\partial y(\ ) \equiv 0$. Then Maxwell's equations give,

$$-\frac{dE_y}{dz} = -ikH_x$$

$$\frac{dE_x}{dz} + iksE_z = -ikH_y$$

$$-iksE_y = -ikH_z \tag{3.1a}$$

$$-\frac{dH_y}{dz} = \frac{ik}{\epsilon_0}D_x$$

$$\frac{dH_x}{dz} + iksH_z = \frac{ik}{\epsilon_0}D_y$$

$$-iksH_y = \frac{ik}{\epsilon_0}D_z \tag{3.1b}$$

From the magnetoionic theory, after the z-components of the fields are eliminated, we can write the following matrix equation,

$$\frac{de}{dz} = - i k T e \tag{3.2}$$

where

$$e = \begin{pmatrix} E_x \\ -E_y \\ H_x \\ H_y \end{pmatrix} \tag{3.3}$$

and T is a 4×4 matrix whose elements are related to the parameters describing the ionosphere, such as the electron density, the electron collision frequency, the earth's magnetic field and others.

The matrix T has four characteristic roots or eigenvalues q_i $(i = 1, 2, 3, 4)$ which satisfy the characteristic equation

$$\det (T - qI) = 0 \tag{3.4}$$

where I is the 4×4 unit matrix.

In order to solve the matrix equation, it is convenient to introduce a new column matrix f, thus

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{3.5}$$

such that

$$e = S f \tag{3.6}$$

where S is some 4×4 matrix whose elements are functions of z to be determined.

Assuming that S is non-singular so that its inverse S^{-1} exists. Substitution of (3.6) in (3.2) and premultiplication of S^{-1} gives

$$f' + i\,k\,S^{-1}\,TS\,f = -\,S^{-1}\,S'\,f \tag{3.7}$$

where the dash indicates the derivative with respect to z.

The S is so chosen as to make f_i the only element of f appearing on the left hand side of the i'th equation in (3.7). This means that $S^{-1}TS$ must be a diagonal matrix, this can be done if the characteristic roots q_i of T are all distinct. It can then be shown that

$$S^{-1}\,TS = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \equiv \Delta \tag{3.8}$$

and eq. (3.7) becomes

$$f' + i\,k\,\Delta\,f = -\,S^{-1}\,S'\,f \tag{3.9}$$

Equations (3.9) are the four, first order, coupled differential equations. For a homogeneous medium $S' = 0$ and (3.9) becomes

$$f' + i\,k\,\Delta\,f = 0 \tag{3.10}$$

or

$$f_i' + i\,k\,q_i\,f_i = 0 \quad (i = 1, 2, 3, 4) \tag{3.11}$$

The independent characteristic waves in a homogeneous medium are determined by

$$f_i = e^{-ikq_i z} \quad (i = 1, 2, 3, 4) \tag{3.12}$$

Now, premultiply (3.9) by any 4×4 matrix, say F, (3.9) then can be written as

$$(F\,f)' + (i\,k\,F\,\Delta - F')\,f = -\,F\,S^{-1}\,S'\,f \tag{3.13}$$

Choose F so that

$$F' = i\,k\,F\,\Delta, \tag{3.14}$$

and

$$(F\ f)' = -F\,S^{-1}\,S'\,f \tag{3.15}$$

and integrate (3.15) in the form

$$f = F^{-1} - F^{-1}\int^{z} F\,S^{-1}\,S'\,f\,dz \tag{3.16}$$

From (3.14) F may be taken as the diagonal matrix whose diagonal elements are $\exp(ik\int^{z} q_i\,dz)$; hence, if f_0 is the fundamental solution of (3.10) namely

$$f_{0i} = \exp\left[-ik\int^{z} q_i\,dz\right] \tag{3.17}$$

(3.16) becomes

$$f = f_0 - f_0\int^{z} f_0^{-1}\,S^{-1}\,S'\,f_0\,dz \tag{3.18}$$

The evaluation of $S^{-1}S'$ and its properties can be found in Budden's book. The thing of interest to us here is that we have a set of first order differential equations (3.15), which can be integrated numerically choosing various ionospheric models. A work of this kind has been undertaken by Budden and his co-workers in the Cavandish Laboratory. Their method of attack is to assume some plausible law for the variation of electron density and collision frequency with height in the ionosphere, and to calculate the reflecting properties of this model ionosphere for long and very long waves. This is repeated for a series of models, and those models are selected which have properties most closely resembling the experimental results.

4. The Normal Wave Solution

In all the previous methods of solution of the VLF wave propagation problem, numerical checks were made by assuming a prior knowledge of the parameters used. The selections of the parameters though based on sound and scientific judgment, were, to say the least, arbitrary. Furthermore, these methods could account for only homogeneous tracks (night or day) up to now. The diurnal changes in both the phase and the amplitude of the VLF waves remained mostly unexplained. In view of the inadequacy in these approaches, Krasnushkin proposed a mixed method. He suggested to use the data known in the short range propagation to evaluate some of the unknown parameters. This is basically an inverse method. With the unknown parameters determined, a direct method is used to evaluate the fields at great distances. In doing so, some of the difficulties encountered in evaluating the roots of a transcendental equation as in the mode theory could be circumvented.

The mathematical models he used are based on the theory of non-self adjoint differential operator eigenfunction expansions. He called these functions normal waves.

a. Stratified ionosphere. In a spherically stratified media, let the dielectric tensors be as follows,

$$\|\epsilon_k\| = \begin{Vmatrix} \epsilon^k_{rr} & 0 & 0 \\ 0 & \epsilon^k_{\theta\theta} & \epsilon^k_{\theta\phi} \\ 0 & -\epsilon^k_{\theta\phi} & \epsilon^k_{\phi\phi} \end{Vmatrix} \tag{4.1}$$

$$r_{k-1} \le r \le r_k \quad k = 0, 1, 2, \ldots, N$$

where, for the zero layer (k = 0), or ($r_{-1} = 0, r_0$)

$$\epsilon^0_{rr} = \epsilon^0_{\theta\theta} = \epsilon^1 + i \frac{4\pi\sigma_0}{\omega}, \; \epsilon^0_{\theta\phi} = 0 \tag{4.2}$$

for the first layer (k = 1); ($r_0 = a, r_1$)

$$\epsilon^1_{rr} = \epsilon^1_{\theta\theta} = 1 \quad \epsilon^1_{\theta\phi} = 0 \tag{4.3}$$

and for the other n - 1 layers (k = 2, 3, . . . , N)

$$(r_1, r_2); (r_2, r_3); \; . \; . \; . \; . \; ; (r_{N-1}, r_N = \infty)$$

the tensor $\|\epsilon_k\|$ is as follows,

$$\epsilon_{rr} = 1 + \frac{i\,\omega_0^2}{\omega(\nu_{eff} - i\,\omega)},$$

$$\epsilon_{\theta\theta} = 1 + \frac{i\,\omega_0^2(\nu_{eff} - i\,\omega)}{\omega[(\nu_{eff} - i\,\omega)^2 + \omega_H^2]} \tag{4.4}$$

$$\epsilon_{\theta\phi} = -\frac{i\,\omega_0^2\,\omega_H}{\omega[(\nu_{eff} - i\,\omega)^2 + \omega_H^2]}$$

where

$$\omega_0^2 = \frac{4\pi N e^2}{m}$$

$$\omega_H = -\frac{e H_0}{m c} \tag{4.5}$$

The components of the field in each layer are

$$\frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, H_\phi^k) = -\frac{i\omega}{c} \epsilon_{rr}^k E_r^k + \frac{4\pi}{c} I_r^k,$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r H_\phi^k) = \frac{i\omega}{c} [\epsilon_{\theta\theta}^k E_\theta^k + \epsilon_{\theta\phi}^k E_\phi^k],$$

$$\frac{1}{r} \left[\frac{\partial}{\partial r} (r H_\theta^k) - \frac{\partial H_r^k}{\partial\theta}\right] = -\frac{i\omega}{c} [-\epsilon_{\theta\phi}^k E_\theta^k + \epsilon_{\phi\phi}^k E_\phi^k],$$

$$\frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, E_\phi^k) = \frac{i\omega}{c} H_r^k,$$

$$-\frac{1}{r} \frac{\partial}{\partial r} (r E_\phi^k) = \frac{i\omega}{c} H_\theta^k,$$

$$\frac{1}{r} \left[\frac{\partial}{\partial r} (r E_\theta^k) - \frac{\partial E_r^k}{\partial\theta}\right] = \frac{i\omega}{c} H_\phi^k. \tag{4.6}$$

They are subject to the following conditions:

(1) Tangential components E_θ, E_ϕ, H_θ, H_ϕ are continuous on the spherical surfaces r_k = const., k = 1, 2, . . . , N.

(2) In the center of the earth ($r = r_{-1} = 0$) the fields are bounded and

(3) At infinity they tend to zero.

Now, introduce the Hertzian potential function

$$\begin{vmatrix} B\,(r,\theta) \\ A\,(r,\theta) \end{vmatrix} \tag{4.7}$$

where

$$H_\phi = \frac{1}{r}\frac{\partial B}{\partial\theta}, \quad E_\phi = \frac{1}{r}\frac{\partial A}{\partial\theta},$$

$$E_r = \frac{i}{k_0\,\epsilon_{rr}\,r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial B}{\partial\theta}\right),$$

$$E_\theta = -\frac{i}{k_0\,\epsilon_{\theta\theta}\,r}\,\frac{\partial^2 B}{\partial r\,\partial\theta} - \frac{\epsilon_{\theta\phi}}{r\,\epsilon_{\theta\theta}}\,\frac{\partial A}{\partial\theta},$$

$$H_r = -\frac{i}{k_0\,r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial A}{\partial\theta}\right),$$

$$H_\theta = \frac{i}{k_0\,r}\,\frac{\partial^2 A}{\partial r\,\partial\theta}, \tag{4.8}$$

where $k_0 = \omega/c$ is the wave number of the free space.

Eq. (4.6) can be expressed in terms of A and B

$$\ell_r^{(k)}\begin{vmatrix} B_k \\ A_k \end{vmatrix} + \ell_\theta^{(k)}\begin{vmatrix} B_k \\ A_k \end{vmatrix} = \frac{4\pi}{c}\,r^2\begin{vmatrix} I_r \\ 0 \end{vmatrix} \tag{4.9}$$

where

$$
\ell_r^{(k)} = \begin{vmatrix} \epsilon_{rr}^k \, r^2 \dfrac{\partial}{\partial r} \left[\dfrac{1}{\epsilon_{\theta\theta}^k}\right] + k_0^2 \, \epsilon_{rr}^k \, r^2 & - i \, k_0 \, \epsilon_{rr}^k \, r^2 \dfrac{\partial}{\partial r} \left[\dfrac{\epsilon_{\theta\phi}^k}{\epsilon_{\theta\theta}^k}\right] \\ \dfrac{i \, k_0 \, \epsilon_{\theta\phi}^k}{\epsilon_{\theta\theta}^k} \, r^2 \dfrac{\partial}{\partial r} & r^2 \dfrac{\partial^2}{\partial r^2} + k_0^2 \, r^2 \, \epsilon_{\phi\phi}^k + \dfrac{(\epsilon_{\theta\phi}^k)^2}{\epsilon_{\theta\theta}^k} \end{vmatrix} \tag{4.10}
$$

and

$$
\ell_\theta^{(k)} = \begin{vmatrix} \dfrac{1}{\sin\theta} \dfrac{\partial}{\partial\theta} \left(\sin\theta \dfrac{\partial}{\partial\theta}\right) & 0 \\ 0 & \dfrac{1}{\sin\theta} \dfrac{\partial}{\partial\theta} \left(\sin\theta \dfrac{\partial}{\partial\theta}\right) \end{vmatrix} \tag{4.11}
$$

The continuity conditions for E_θ, H_θ, E_ϕ, H_ϕ at $r = r_k$ become

$$
\Delta^{(k)} \begin{vmatrix} B_k \\ A_k \end{vmatrix}_{r = r_k} = \Delta^{(k+1)} \begin{vmatrix} B_{k+1} \\ A_{k+1} \end{vmatrix}_{r = r_k}
$$

$$
\begin{vmatrix} B_k \\ A_k \end{vmatrix}_{r = r_k} = \begin{vmatrix} B_{k+1} \\ A_{k+1} \end{vmatrix}_{r = r_k} \tag{4.12}
$$

where

$$
\Delta^{(k)} = \begin{vmatrix} \dfrac{1}{\epsilon_{\theta\theta}^k} \dfrac{\partial}{\partial r} & - \dfrac{i \, k_0 \, \epsilon_{\theta\phi}^k}{\epsilon_{\theta\theta}^k} \\ 0 & \dfrac{\partial}{\partial r} \end{vmatrix} \tag{4.13}
$$

Now decompose (4.7) into the product

$$\begin{vmatrix} B \\ \\ A \end{vmatrix} = \begin{vmatrix} Y(r) \\ \\ Z(r) \end{vmatrix} \psi(\theta) \tag{4.14}$$

Then $\ell_r^{(k)}$ will act only on $\left|\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right|$ and $\ell_\theta^{(k)}$ only on θ. After replacing $\partial/\partial r$ by d/dr one obtains the differential operator L_r defined by $\ell_r^{(k)}$. Similarly, one obtains L_θ defined by $\ell_\theta^{(k)}$. Then one has a single operational equation:

$$L_r \begin{vmatrix} B \\ A \end{vmatrix} + L_\theta \begin{vmatrix} B \\ A \end{vmatrix} = \frac{4\pi}{c} r^2 \begin{vmatrix} I_r \\ 0 \end{vmatrix} \tag{4.15}$$

To find the solution of (4.15), we will find first the solution of the homogenous equation

$$L_r \begin{vmatrix} B \\ A \end{vmatrix} + L_\theta \begin{vmatrix} B \\ A \end{vmatrix} = 0 \tag{4.16}$$

From (4.14), we have

$$\frac{L_r \begin{vmatrix} Y(r) \\ Z(r) \end{vmatrix}}{\begin{vmatrix} Y(r) \\ Z(r) \end{vmatrix}} = - \frac{L_\theta \psi(\theta)}{\psi(\theta)}$$

or

$$L_r \begin{vmatrix} Y(r) \\ Z(r) \end{vmatrix} = x \begin{vmatrix} Y(r) \\ Z(r) \end{vmatrix} \tag{4.17}$$

$$L_\theta \psi(\theta) + x \psi(\theta) = 0 \tag{4.18}$$

The solution of (4.17) exists for eigenvalues x_j's (j = 1, 2, . . .) corresponding to the eigenfunctions

$$\begin{vmatrix} Y_j \\ \\ Z_0 \end{vmatrix}.$$

Equation (4.18) is the usual Legendre equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_j}{d\theta}\right)+\nu_j(\nu_j+1)\psi_j=0 \tag{4.19}$$

where

$$\nu_j(\nu_j+1)=x_j \tag{4.20}$$

For the case of a verticle dipole source placed at $r = b$ we may write the solution of (4.15) as follows,

$$\begin{vmatrix} B \\ A \end{vmatrix} = -\frac{\pi P}{c b^2}\sum_{j=1}^{\infty}\begin{vmatrix} Y_j\ (r) \\ Z_j\ (r) \end{vmatrix}\frac{Y_j\ (b)}{N_j\ \sin\nu_j\pi}P\nu_j\ [\cos(\pi-\theta)] \tag{4.21}$$

where P is the dipole moment of the source and

$$N_j=\int_0^{\infty}\frac{Y_j^2}{\epsilon_{rr}\ r^2}\,dr+\int_0^{\infty}\frac{Z_j^2}{r^2}\,dr \tag{4.22}$$

Using the asymptotical approximation of the Legendre function, and expanding $1/\sin\nu_j\pi$, we write (4.21) as follows,

$$\begin{vmatrix} B \\ A \end{vmatrix} \simeq \frac{2P}{c b^2}\sqrt{\frac{\pi}{2\sin\theta}}\sum_{j=1}^{\infty}\begin{vmatrix} Y_j\ (r) \\ Z_j\ (r) \end{vmatrix}\frac{Y_j\ (b)\ \nu_j^{1/2}}{N_j}$$

$$\cdot\left\{\sum_{n=0}^{\infty}\exp\left[i\ (n+1)\ 2\pi\nu_j - i\left(\nu_j^*\theta+\frac{\pi}{4}\right)\right]\right.$$

$$\left.+\exp\left[i\ 2\pi n\nu_j + i\left(\nu_j^*\theta+\frac{\pi}{4}\right)\right]\right\} \tag{4.23}$$

where $\nu_j^* = \nu_j + 1/2$. The separate terms in the braces of (4.23) denote the signals reaching the receiver after passing round the earth n times. Because of the attenuation only the waves with $n = 0$ arriving at the receiver along the shortest path should be considered. Thus,

$$\left|\begin{matrix} B \\ A \end{matrix}\right| \simeq \frac{2P}{cb^2} \sqrt{\frac{\pi}{2 \sin \theta}} \sum_{j=1}^{\infty} Y_j \text{ (b)} \left|\begin{matrix} Y_j \text{ (r)} \\ Z_j \text{ (r)} \end{matrix}\right| \frac{e^{i(\nu_j^* \theta + \pi/4)}}{N_j \sqrt{\nu_j}} \tag{4.24}$$

The field components are

$$H_\phi = A(\theta)\, i \sum_{j=1}^{\infty} Y_j(r)\, B_j(\theta),$$

$$E_r = -\frac{A(\theta)\, i}{k_0 r} \sum_{j=1}^{\infty} \nu_j\, Y_j(r)\, B_j(\theta),$$

$$E_\theta = \frac{A(\theta)}{k_0} \sum_{j=1}^{\infty} \frac{d\, Y_j(r)}{d\, r} B_j(\theta),$$

$$E_\phi = A(\theta)\, i \sum_{j=1}^{\infty} Z_j(r)\, B_j(\theta),$$

$$H_r = \frac{A(\theta)\, i}{k_0 r} \sum_{j=1}^{\infty} \nu_j\, Z_j(r)\, B_j(\theta),$$

$$H_\theta = -\frac{A(\theta)}{k_0} \sum_{j=1}^{\infty} \frac{d\, Z_j(r)}{d\, r} B_j(\theta), \tag{4.25}$$

where

$$A(\theta) = \frac{2P}{cb^2 r} \sqrt{\frac{\pi}{2 \sin \theta}}$$

$$B_j(\theta) = \frac{Y_j(b)}{N_j} \nu_j^{1/2} \exp\left[i\left(\nu_j^* \theta + \frac{\pi}{4}\right)\right].$$

For calculating the field in the atmospheric layer, a further simplification of formula (4.2) is needed. For the details involved, see reference [23].

b. Stratified medium with relief. To account for the diurnal variations of the field, Krasnushkin proposed another model to take into consideration of the slow variations of the dielectric tensor (4.1) along the path of radio waves. He assumed that

$$\| \epsilon \| = f(r, \mu\theta, \mu\phi) \tag{4.26}$$

where $90° - \theta$ and ϕ are latitude and longitude and μ is some small parameter. The operational equation now takes the following form

$$L_r \begin{vmatrix} B \\ A \end{vmatrix} + L_{\theta\phi} \begin{vmatrix} B \\ A \end{vmatrix} = \frac{4\pi}{c} r^2 \begin{vmatrix} I_r \\ 0 \end{vmatrix}. \tag{4.27}$$

This equation is not separable due to dependence of L_r on θ and ϕ. However, for small μ one may expand L_r in powers of μ. The first approximation yields the following eigenvalue equation:

$$L_r \begin{vmatrix} Y \\ Z \end{vmatrix} = x(\mu\theta, \mu\phi) \begin{vmatrix} Y \\ Z \end{vmatrix} \tag{4.28}$$

From whence one obtains the set of eigenvalues $x_j(\mu\theta, \mu\phi)$, $j = 1, 2, \ldots \infty$ corresponding to the eigenfunctions

$$\begin{vmatrix} Y_j(r; \mu\theta\, \mu\phi) \\ Z_j(r; \mu\theta\, \mu\phi) \end{vmatrix} \tag{4.29}$$

Thus we assume that

$$\begin{vmatrix} B \\ A \end{vmatrix} = \sum_{j=0}^{\infty} \Phi_j(\theta, \phi) \begin{vmatrix} Y_j(r; \mu\theta, \mu\phi) \\ Z_j(r; \mu\theta, \mu\phi) \end{vmatrix}. \tag{4.30}$$

For $\Phi_j\ (\theta,\phi)$ one obtains the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi_j}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\Phi_j}{\partial\phi^2}+x_j\,\Phi_j$$

$$=\frac{4\pi}{c\,N_j}\int_0^\infty\frac{I_r\,Y_j}{\epsilon_{rr}}\,d\,r. \tag{4.31}$$

If one seeks the solution in the form

$$\Phi_j = U_j\ (\theta,\ \phi)\ \exp\ [i\ S_j\ (\theta,\ \phi)] \tag{4.32}$$

where U_j is a slowly varying function of θ and ϕ. After substituting (4.32) into (4.31), one finally obtains for $\left|\begin{smallmatrix}B\\A\end{smallmatrix}\right|$ the following expressions:

$$\left|\begin{matrix}B\\A\end{matrix}\right| = \frac{2P}{c\,b^2}\sqrt{\frac{\pi}{2\sin\theta}}\sum_{j=0}^{\infty}Y_j\ (b)\left|\begin{matrix}Y_j\ (r)\\Z_j\ (r)\end{matrix}\right|\frac{\nu_j^{-1/2}}{\sqrt{N_j\ (O)}\ \sqrt{N_j\ (P)}}$$

$$\cdot\ \exp\ [iS_j(\theta,\ \phi)] \tag{4.33}$$

where normalization factors N_j (O) and N_j (P) are evaluated at the sender and the receiver points. The path of the wave is defined by the eikonal function S_j evaluated from the equation

$$\left(\frac{\partial S_j}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S_j}{\partial\phi}\right)^2 = x_j\ (\mu\theta,\ \mu\phi) \tag{4.34}$$

and

$$S_j = \int_O^P\sqrt{x_j}\ d\,s \tag{4.35}$$

If the geodetic line is followed, then

$$S_j = \int_{\theta_O}^{\theta_P}\sqrt{x_j}\ d\theta \tag{4.36}$$

After substituting (4.36) into (4.33), one finally obtains the radial component of the electric field

$$E_r = \sqrt{\frac{W}{\sin\theta}} \exp\left[i\,\frac{\pi}{4}\right]\left\{\sum_{j=0} n_j(0)\, n_j(p)\, \exp\left[i \int_{\theta_0}^{\theta_p} \sqrt{x_j}\, d\theta\right]\right.$$

$$\left. + \sum_{K=0} \eta_K(0)\, \eta_K(p)\, \exp\left[i \int_{\theta_0}^{\theta_p} \sqrt{x_R}\, d\theta\right]\right\} \qquad (4.37)$$

IV. SUMMARY

So far we have seen four mathematical models in the study of the VLF wave propagation; namely (1) waveguide mode theory, (2) zonal harmonic series, (3) eigenfunction method, and (4) full wave theory. Of the four, the last stresses more on the computational aspects of the problem whereas the first three give analytic expressions as solutions of the problem. Models one and three are more elegant in form, each is capable of explaining the propagation problem in simple and tractable terms using mostly one or two modes. Model (1) is based on the concept of the classical residue series, whereas model (2) is formulated on the modern theory of the non-self adjoint operators. The stumbling block in the mode theory is the rather formidable transcendental equation whose solution is necessary to account for the various modes needed in evaluating the electromagnetic fields. There is a corresponding transcendental equation in the eigenvalue method, however, Krasnushkin showed that the difficulty can be circumvented by means of what he called the mixed method; whereby he used the data of the short range fields and some of the known parameters to determine the unknown parameters. Then he used the known and the determined parameters to evaluate the long range propagation phenomenon.

In contrast to the mode theory model and the eigenfunction model, the zonal series model relies only on the original harmonic series for its solution. The speed of the electronic computers offsets the objection of the direct summation of the slowly convergent harmonic series. A complete solution is thus made possible leaving only the assumed model for the transmitter and the propagation medium. The advantage of a descriptive discussion that a simple formula affords is lost. Perhaps when all the details of the attributes of the ionosphere are included, graphical solutions based on numerical computation are the only answers. In which case, the full wave theory model as is handled by Budden may be the best approach. Unfortunately, Budden's is a planar model. To satisfy the long range propagation, the earth's curvature must be taken into consideration.

The abundant experimental materials cumulated over the past fifty years in the field of VLF propagation have established many facts that the current theories are still incapable of explaining, such as

(1) the diurnal and the seasonal variations of the long distance VLF field;

(2) the dependence of both the field intensity and the phase on a heterogenous path of various illuminations;

(3) the effect of the solar flare on both the field strength and the phase.

It appears that some work could be done on the formulation of a new model which does not depend only harmonically on time in order to resolve some of the aforementioned observed facts.

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