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MATHEMATICAL MODELLING BY SYMBOLIC MATHEMATICAL COMPUTATION – A COSMOLOGICAL APPLICATION

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16. Abstract

The advantages of symbolic mathematical computation are most evident in problems analogous to those described in this report; that is, the application of the method to reconstructing several existing cosmological models and their associated trajectory equations. The field equations that govern the trajectories of bodies in space have, in general, large numbers of terms, with each term a complicated mathematical expression. The evaluation of these terms and the derivation of the equations of the geodesics that describe the trajectories of bodies in space, require a substantial amount of algebraic manipulation and symbolic differentiation. For the models considered, the required operations were executed with speed and efficiency on an IBM 360/67 computer. For example, in the case of the nonhomogeneous Schwarzschild model, the computer times required to formulate the field and trajectory equations were 0.74 and 0.30 minutes, respectively. By mechanization of the procedure in the manner described, man hours are saved, the possibility of error is reduced, and the scope of the inquiry may be extended.

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NOMENCLATURE

A(I)	$\frac{d^2x^i}{ds^2}$
a	constant
$\bar{a}_i$	system of base vectors
$\bar{a}^i$	system of base vectors reciprocal to $\bar{a}_i$
ET(I,J,K)	energy momentum tensor in computer notation
G	computer notation for g
G(I,J)	covariant metric tensor in computer notation
g	determinant of covariant metric tensor matrix
g_{ij}	covariant metric tensor
g^{ij}	contravariant metric tensor
L·(x(1))	function of x^1 in computer notation
$L^{(1)}$·(x(1))	first derivative of L with respect to x^1
$L^{(1)}$·^{2}(x(1))	square of first derivative of L with respect to x^1
$L^{(2)}$·(x(1))	second derivative of L with respect to x^1
M·(x(1))	function of x^1 in computer notation
$M^{(1)}$·(x(1))	first derivative of M with respect to x^1
$M^{(1)}$·^{2}(x(1))	square of the first derivative of M with respect to x^1
$M^{(2)}$·(X(1))	second derivative of M with respect to x^1
m	mass
R	Ricci scalar
R(I,J)	Ricci tensor in computer notation
R_{ij}	covariant form of Ricci tensor
$R^i{}_{\cdot j}$	mixed form of Ricci tensor

s	line element
T	contracted form of the energy momentum tensor
T(I,J,K)	Christoffel symbol of the second kind in computer notation
T_{ij}	covariant form of energy momentum tensor
$T^{i}{}_{j}$	mixed form of energy momentum tensor
t	time
V(I)	$\frac{dx^{i}}{ds}$
x^{i}	system coordinates
$[ij,k]$	Christoffel symbol of first kind
$\left\{ {i \atop jk} \right\}$	Christoffel symbol of second kind
α	integer with values ranging from 1 to 4
β	integer with values ranging from 1 to 4
δ^{i}_{j}	Kronecker delta
κ	constant
ρ	density
ϕ	potential function
$\nabla\phi$	gradient of potential function
Λ	cosmological constant

Subscripts

I,J,K	indices of covariance in computer notation
i,j,k	indices of covariance

Superscripts

I,J,K	indices of contravariance and identification in computer notation
i,j,k	indices of contravariance and identification

MATHEMATICAL MODELLING BY SYMBOLIC MATHEMATICAL COMPUTATION – A COSMOLOGICAL APPLICATION

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SUMMARY

The advantages of symbolic mathematical computation are most evident in problems analogous to those described in this report; that is, the application of the method to reconstructing several existing cosmological models and their associated trajectory equations. The field equations that govern the trajectories of bodies in space have, in general, large numbers of terms, with each term a complicated mathematical expression. The evaluation of these terms and the derivation of the equations of the geodesics that describe the trajectories of bodies in space, require a substantial amount of algebraic manipulation and symbolic differentiation. For the models considered, the required operations were executed with speed and efficiency on an IBM 360/67 computer. For example, in the case of the nonhomogeneous Schwarzschild model, the computer times required to formulate the field and trajectory equations were 0.74 and 0.30 minutes, respectively. By mechanization of the procedure in the manner described, man hours are saved, the possibility of error is reduced, and the scope of the inquiry may be extended.

INTRODUCTION

Many problems in mathematical physics involve the formulation of complex models of physical systems or processes and the manipulation of large sets of nonlinear partial differential equations. The manual performance of these operations is time consuming, subject to human error, and in certain cases impossible.

In recent years mathematical manipulation and the formulation of mathematical models have been facilitated by the use of digital computers equipped with formula manipulation compilers (FORMAC) and by the development of computer programs designed to perform a variety of non-numeric operations (ref. 1). In references 2 and 3, it is shown how symbolic mathematical computation was used to obtain the Christoffel symbols of the first and second kinds for 12 orthogonal, curvilinear coordinate systems. In references 4 and 5 it is shown how an IBM 7094 digital computer, equipped with FORMAC, can be used to derive the equations of motion of a particle in any curvilinear coordinate system requested by the user. In reference 6 it is indicated how these methods can facilitate the derivation of the Navier-Stokes equations of fluid motion and the continuity equation.

The present report indicates how symbolic mathematical computation can be used to formulate a variety of cosmological models. Each model is determined by the metric of the Riemannian Space, and the only inputs required are the metric coefficients of the fundamental quadratic form. For illustrative purposes, only spherically symmetric static models are considered. It should be emphasized, however, that the method described is equally applicable to less restricted models, and nonstatic models can be formulated with equal facility. The determination of the geodesics that describe the trajectories of bodies in space requires that the appropriate potential functions be known. The relativistic analog of Poisson's equation, which in the Newtonian theory connects a single gravitational potential function with the density of matter, is a relation between the potential functions and the components of the energy momentum tensor. In general, this relationship gives rise to 10 nonlinear partial differential equations. The solution of these equations then yields the potential functions and must precede any attempt to obtain the corresponding trajectories.

ANALYSIS

Field and Trajectory Equations

Consider the equation of motion of a particle that is moving under the influence of gravitational forces. When the equation is written in the notation of the tensor calculus, it assumes the following form (ref. 5):

$$m\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right)\bar{a}_i = \nabla\phi \tag{1}$$

where $i,j,k = 1,2,3$

and

$$\nabla\phi = \frac{\partial\phi}{\partial x^j}\bar{a}^j = g^{ij}\frac{\partial\phi}{\partial x^j}\bar{a}_i \tag{2}$$

In these equations the summation convention is assumed. That is to say, if in any term an index occurs twice, the term is to be summed with respect to that index for all admissible values of the index.

In relativistic mechanics, equation (1) is replaced by the following trajectory equation (ref. 7):

$$\left(\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds}\frac{dx^k}{ds}\right) = 0 \tag{3}$$

$$i,j,k = 1,2,3,4$$

where the line element ds satisfies the fundamental quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{4}$$

The Newtonian theory of gravitation connects a single potential function ϕ with the density of matter. In this theory, the gravitational potential function is required to satisfy Poisson's equation (ref. 7).

$$\nabla^2\phi = -4\pi\rho \tag{5}$$

At all points of space devoid of matter $\rho = 0$ and Poisson's equation reduces to Laplace's equation

$$\nabla^2\phi = 0 \tag{6}$$

The relativistic analog of Poisson's equation is the following tensor equation (ref. 8).

$$R_{ij} - \frac{1}{2} Rg_{ij} + \Lambda g_{ij} = -\kappa T_{ij} \tag{7}$$

By raising indices, the field equations can be written in the alternative form

$$R^{i}_{\cdot j} - \frac{1}{2}\delta^i_j R + \delta^i_j \Lambda = -\kappa T^{i}{}_{\cdot j} \tag{8}$$

Contraction of equation (8) yields

$$R - 4\Lambda = \kappa T \tag{9}$$

In regions of space devoid of matter, all the components of the energy momentum tensor are zero, and equation (9) simplifies accordingly. In this case

$$R = 4\Lambda \tag{10}$$

When this result is substituted in equation (7), the field equations assume the form

$$R_{ij} = \Lambda g_{ij} \tag{11}$$

Nevertheless, in empty space, the trajectories of bodies moving within the solar system correspond with great precision to the simpler field equations (ref. 8).

$$R_{ij} = 0 \tag{12}$$

where

$$R_{ij} = \left(\frac{\partial^2}{\partial x^i \partial x^j} \log \sqrt{|g|} - \frac{\partial}{\partial x^\alpha} \left\{ {\alpha \atop ij} \right\} + \left\{ {\alpha \atop \beta j} \right\} \left\{ {\beta \atop i\alpha} \right\} - \left\{ {\beta \atop ij} \right\} \frac{\partial}{\partial x^\beta} \log \sqrt{|g|} \right) \tag{13}$$

$$\alpha, \beta, i, j = 1,2,3,4$$

$$g = \left| g_{ij} \right| \tag{14}$$

$$[ij,k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{15}$$

$$\left\{ {\alpha \atop ij} \right\} = g^{\alpha k} [ij,k] \tag{16}$$

Equation (12) is the relativistic analog of the Lapace equation. It represents 10 nonlinear partial differential equations for the 10 unknown functions g_{ij}. Once a set of potential functions g_{ij} satisfying equation (8) or (12) is found, the corresponding trajectory equations can be formulated.

COMPUTER APPLICATIONS

For the purpose of illustrating the modelling capability of symbolic mathematical computation, a spherically symmetric static field is assumed. This assumption implies that the metric tensors g_{ij} are spherically symmetric and independent of the time. Moreover, the metric tensors must be chosen in such a way that the line element will reduce to the special relativity form for flat space-time. These considerations led to the adoption of the following set of metric tensors for anisotropic space.

$$\left. \begin{aligned} g_{11} &= -e^{L(x^1)} \\ g_{22} &= -(x^1)^2 \\ g_{33} &= -(x^1 \sin x^2)^2 \\ g_{44} &= e^{M(x^1)} \end{aligned} \right\} \tag{17}$$

where the implicit functions $L(x^1)$ and $M(x^1)$ can be adjusted to account for the distortion of space in the presence of matter. The corresponding space-time interval is

$$ds^2 = \left[-e^{L(x^1)}(dx^1)^2 - (x^1\ dx^2)^2 - (x^1 \sin x^2\ dx^3)^2 + e^{M(x^1)}(dx^4)^2\right] \tag{18}$$

where

$$dx^4 = c\ dt$$

and

x^1 radial coordinate

x^2 polar angle coordinate

x^3 azimuthal angle coordinate

and for convenience the velocity of light is assumed equal to 1.

If the space is assumed to be isotropic, the metric tensors are modified as follows:

$$\left.\begin{aligned} g_{11} &= -e^{L(x^1)} \\ g_{22} &= -e^{L(x^1)}(x^1)^2 \\ g_{33} &= -e^{L(x^1)}(x^1 \sin x^2)^2 \\ g_{44} &= e^{M(x^1)} \end{aligned}\right\} \tag{19}$$

The space-time interval in the isotropic case is

$$ds^2 = \left\{-e^{L(x^1)}\left[(dx^1)^2 + (x^1\ dx^2)^2 + (x^1 \sin x^2\ dx^3)^2\right] + e^{M(x^1)}(dx^4)^2\right\} \tag{20}$$

In order to demonstrate the feasibility of using symbolic mathematical computation to obtain different models of the universe, a computer program[1] was written that required only the postulated metric tensors as inputs.

[1]The computer program and related documentation are available from Computer Software Management and Information Center (COSMIC), Barrow Hall, University of Georgia, Athens, Georgia, 30601.

Applications to Cosmological Models

Anisotropic model— The field equations and the corresponding trajectory equations for this condition can be obtained by using the tensors defined in equations (17). With these tensors as inputs to a 360/67 digital computer which was programmed to derive models of the universe, the following output was obtained.

The metric coefficients determine the gravitational model being studied. In order that each run be identified with the correct inputs, the postulated metric coefficients are printed out before the main results. In the case under consideration, these have the following values

$$G(1,1) = -E^{L\cdot(X(1))}$$

$$G(2,2) = -X(1)^2$$

$$G(3,3) = -SIN^2(X(2))X(1)^2$$

$$G(4,4) = E^{M\cdot(X(1))}$$

The program uses the metric tensor inputs to evaluate the Christoffel symbols of the first and second kinds. In order to reduce the amount of output, the Christoffel symbols of the first kind are not printed out. In terms of the system coordinates and the unknown functions L and M, the Christoffel symbols of the second kind are

$$T(1,1,1) = (1/2)L^{(1)}\cdot(X(1))$$

$$T(1,1,2) = 0$$

$$T(1,1,3) = 0$$

$$T(1,1,4) = 0$$

$$T(1,2,1) = 0$$

$$T(1,2,2) = 1/X(1)$$

$$T(1,2,3) = 0$$

$$T(1,2,4) = 0$$

$$T(1,3,1) = 0$$

$T(1,3,2) = 0$

$T(1,3,3) = 1/X(1)$

$T(1,3,4) = 0$

$T(1,4,1) = 0$

$T(1,4,2) = 0$

$T(1,4,3) = 0$

$T(1,4,4) = (1/2)M^{(1)} \cdot (X(1))$

$T(2,1,1) = 0$

$T(2,1,2) = 1/X(1)$

$T(2,1,3) = 0$

$T(2,1,4) = 0$

$T(2,2,1) = -E^{-L \cdot (X(1))} X(1)$

$T(2,2,2) = 0$

$T(2,2,3) = 0$

$T(2,2,4) = 0$

$T(2,3,1) = 0$

$T(2,3,2) = 0$

$T(2,3,3) = COS(X(2))/SIN(X(2))$

$T(2,3,4) = 0$

$T(2,4,1) = 0$

$T(2,4,2) = 0$

$T(2,4,3) = 0$

$T(2,4,4) = 0$

$T(3,1,1) = 0$

$T(3,1,2) = 0$

$T(3,1,3) = 1/X(1)$

$T(3,1,4) = 0$

$T(3,2,1) = 0$

$T(3,2,2) = 0$

$T(3,2,3) = COS(X(2))/SIN(X(2))$

$T(3,2,4) = 0$

$T(3,3,1) = -E^{-L\cdot(X(1))}SIN^{2}(X(2))X(1)$

$T(3,3,2) = -COS(X(2))SIN(X(2))$

$T(3,3,3) = 0$

$T(3,3,4) = 0$

$T(3,4,1) = 0$

$T(3,4,2) = 0$

$T(3,4,3) = 0$

$T(3,4,4) = 0$

$T(4,1,1) = 0$

$T(4,1,2) = 0$

$T(4,1,3) = 0$

$T(4,1,4) = (1/2)M^{(1)}\cdot(X(1))$

$T(4,2,1) = 0$

$T(4,2,2) = 0$

$T(4,2,3) = 0$

$T(4,2,4) = 0$

$T(4,3,1) = 0$

$T(4,3,2) = 0$

$T(4,3,3) = 0$

$T(4,3,4) = 0$

$T(4,4,1) = (1/2)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))$

$$T(4,4,2) = 0$$

$$T(4,4,3) = 0$$

$$T(4,4,4) = 0$$

Once the Christoffel symbols of the second kind are known the components of the Ricci tensor can be derived. The individual components are

$$R(1,1) = -L^{(1)}\cdot(X(1))/X(1)-(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))$$
$$+(1/4)M^{(1)}\cdot^{2}(X(1))+(1/2)M^{(2)}\cdot(X(1))$$

$$R(1,2) = 0$$

$$R(1,3) = 0$$

$$R(1,4) = 0$$

$$R(2,1) = 0$$

$$R(2,2) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))X(1)+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))X(1)$$
$$+E^{-L\cdot(X(1))}-1$$

$$R(2,3) = 0$$

$$R(2,4) = 0$$

$$R(3,1) = 0$$

$$R(3,2) = 0$$

$$R(3,3) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)$$
$$+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)+E^{-L\cdot(X(1))}SIN^{2}(X(2))$$
$$-SIN^{2}(X(2))$$

$$R(3,4) = 0$$

$$R(4,1) = 0$$

$$R(4,2) = 0$$

$$R(4,3) = 0$$

$$R(4,4) = -E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$-(1/4)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot{}^{2}(X(1))$$

$$+(1/4)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))$$

$$-(1/2)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(2)}\cdot(X(1))$$

G(I,J) and R(I,J) are both known at this stage of the program; therefore the Ricci scalar can be obtained. It is given by the following equation

$$R = -2E^{-L\cdot(X(1))}/X(1)^{2}+2E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)$$

$$-2E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)-(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot{}^{2}(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))-E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))+2/X(1)^{2}$$

The preceding information is next used to obtain the field equations. The individual equations are

$$ET(1,1) = E^{-L\cdot(X(1))}/X(1)^{2}+L^{(1)}\cdot(X(1))/(E^{L\cdot(X(1))})X(1)$$

$$-E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$-(1/4)M^{(1)}\cdot{}^{2}(X(1))/(E^{L\cdot(X(1))})$$

$$+(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))/(E^{L\cdot(X(1))})-(1/2)M^{(2)}\cdot(X(1))/(E^{L\cdot(X(1))})$$

$$+(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot{}^{2}(X(1))-(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))-1/X(1)^{2}$$

$$ET(1,2) = 0$$

$$ET(1,3) = 0$$

$$ET(1,4) = 0$$

$$ET(2,1) = 0$$

$$ET(2,2) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$+(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot^{2}(X(1))-(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))$$

$$ET(2,3) = 0$$

$$ET(2,4) = 0$$

$$ET(3,1) = 0$$

$$ET(3,2) = 0$$

$$ET(3,3) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$+(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot^{2}(X(1))-(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))$$

$$ET(3,4) = 0$$

$$ET(4,1) = 0$$

$$ET(4,2) = 0$$

$$ET(4,3) = 0$$

$$ET(4,4) = E^{-L\cdot(X(1))}/X(1)^{2}-E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)-1/X(1)^{2}$$

The choice of an energy momentum tensor completes the specification of this type of model. Solution of the resulting equations gives rise to the components of the potential function. In the case under consideration, the solution yields the unknown functions $L(x^1)$ and $M(x^1)$. In terms of the postulated metric tensor inputs, the computer derives the equation of the trajectories as follows:

$$A(1) = -(1/2)L^{(1)}\cdot(X(1))V(1)^{2}+E^{-L\cdot(X(1))}X(1)V(2)^{2}$$

$$+E^{-L\cdot(X(1))}SIN^{2}(X(2))X(1)V(3)^{2}$$

$$-(1/2)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))V(4)^{2}$$

$$A(2) = -2V(2)V(1)/X(1)+COS(X(2))SIN(X(2))V(3)^2$$

$$A(3) = -2V(3)V(1)/X(1)-2COS(X(2))V(3)V(2)/SIN(X(2))$$

$$A(4) = -M^{(1)}\cdot(X(1))V(4)V(1)$$

Isotropic model— If the universe is isotropic, the line element will assume the form of equation (20). When the corresponding metric tensors (eq. (19)) were used as inputs to the computer program, the following output was obtained.

The metric coefficients determine the gravitational model being studied. In order that each run be identified with the correct inputs, the postulated metric coefficients are printed out before the main results. In the case under consideration, these have the following values:

$$G(1,1) = -E^{L\cdot(X(1))}$$

$$G(2,2) = -E^{L\cdot(X(1))}X(1)^2$$

$$G(3,3) = -E^{L\cdot(X(1))}SIN^2(X(2))X(1)^2$$

$$G(4,4) = E^{M\cdot(X(1))}$$

The program uses the metric tensor inputs to evaluate the Christoffel symbols of the first and second kinds. In order to reduce the amount of output, the Christoffel symbols of the first kind are not printed out. In terms of the system coordinates and the unknown functions L and M, the Christoffel symbols of the second kind are:

$$T(1,1,1) = (1/2)L^{(1)}\cdot(X(1))$$

$$T(1,1,2) = 0$$

$$T(1,1,3) = 0$$

$$T(1,1,4) = 0$$

$$T(1,2,1) = 0$$

$$T(1,2,2) = 1/X(1)+(1/2)L^{(1)}\cdot(X(1))$$

$$T(1,2,3) = 0$$

$$T(1,2,4) = 0$$

$$T(1,3,1) = 0$$

$T(1,3,2) = 0$

$T(1,3,3) = 1/X(1)+(1/2)L^{(1)}\cdot(X(1))$

$T(1,3,4) = 0$

$T(1,4,1) = 0$

$T(1,4,2) = 0$

$T(1,4,3) = 0$

$T(1,4,4) = (1/2)M^{(1)}\cdot(X(1))$

$T(2,1,1) = 0$

$T(2,1,2) = 1/X(1)+(1/2)L^{(1)}\cdot(X(1))$

$T(2,1,3) = 0$

$T(2,1,4) = 0$

$T(2,2,1) = -X(1)-(1/2)L^{(1)}\cdot(X(1))X(1)^2$

$T(2,2,2) = 0$

$T(2,2,3) = 0$

$T(2,2,4) = 0$

$T(2,3,1) = 0$

$T(2,3,2) = 0$

$T(2,3,3) = COS(X(2))/SIN(X(2))$

$T(2,3,4) = 0$

$T(2,4,1) = 0$

$T(2,4,2) = 0$

$T)2,4,3) = 0$

$T(2,4,4) = 0$

$T(3,1,1) = 0$

$T(3,1,2) = 0$

$T(3,1,3) = 1/X(1)+(1/2)L^{(1)}\cdot(X(1))$

$T(3,1,4) = 0$

$T(3,2,1) = 0$

$T(3,2,2) = 0$

$T(3,2,3) = COS(X(2))/SIN(X(2))$

$T(3,2,4) = 0$

$T(3,3,1) = -SIN^2(X(2))X(1)-1/2L^{(1)}\cdot(X(1))SIN^2(X(2))X(1)^2$

$T(3,3,2) = -COS(X(2))SIN(X(2))$

$T(3,3,3) = 0$

$T(3,3,4) = 0$

$T(3,4,1) = 0$

$T(3,4,2) = 0$

$T(3,4,3) = 0$

$T(3,4,4) = 0$

$T(4,1,1) = 0$

$T(4,1,2) = 0$

$T(4,1,3) = 0$

$T(4,1,4) = (1/2)M^{(1)}\cdot(X(1))$

$T(4,2,1) = 0$

$T(4,2,2) = 0$

$T(4,2,3) = 0$

$T(4,2,4) = 0$

$T(4,3,1) = 0$

$T(4,3,2) = 0$

$T(4,3,3) = 0$

$T(4,3,4) = 0$

$T(4,4,1) = (1/2)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))$

$$T(4,4,2) = 0$$

$$T(4,4,3) = 0$$

$$T(4,4,4) = 0$$

Once the Christoffel symbols of the second kind are known, the components of the Ricci tensor can be derived. The individual components are:

$$R(1,1) = L^{(1)}\cdot(X(1))/X(1)-(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))+(1/4)M^{(1)}\cdot{}^{2}(X(1))$$
$$+L^{(2)}\cdot(X(1))+(1/2)M^{(2)}\cdot(X(1))$$

$$R(1,2) = 0$$

$$R(1,3) = 0$$

$$R(1,4) = 0$$

$$R(2,1) = 0$$

$$R(2,2) = (3/2)L^{(1)}\cdot(X(1))X(1)+(1/2)M^{(1)}\cdot(X(1))X(1)+(1/4)L^{(1)}\cdot{}^{2}(X(1))X(1)^{2}$$
$$+(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))X(1)^{2}+(1/2)L^{(2)}\cdot(X(1))X(1)^{2}$$

$$R(2,3) = 0$$

$$R(2,4) = 0$$

$$R(3,1) = 0$$

$$R(3,2) = 0$$

$$R(3,3) = (3/2)L^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)+(1/2)M^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)$$
$$+(1/4)L^{(1)}\cdot{}^{2}(X(1))SIN^{2}(X(2))X(1)^{2}$$
$$+(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)^{2}$$
$$+(1/2)L^{(2)}\cdot(X(1))SIN^{2}(X(2))X(1)^{2}$$

$$R(3,4) = 0$$

$$R(4,1) = 0$$

$$R(4,2) = 0$$

$$R(4,3) = 0$$

$$\begin{aligned}
R(4,4) = {} & -E^{-L\cdot(X(1))+M\cdot(X(1))} M^{(1)}\cdot(X(1))/X(1) \\
& -(1/4)E^{-L\cdot(X(1))+M\cdot(X(1))} M^{(1)}\cdot{}^{2}(X(1)) \\
& -(1/4)E^{-L\cdot(X(1))+M\cdot(X(1))} M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1)) \\
& -(1/2)E^{-L\cdot(X(1))+M\cdot(X(1))} M^{(2)}\cdot(X(1))
\end{aligned}$$

G(I,J) and R(I,J) are both known at this stage of the program; therefore the Ricci scalar can be obtained. It is given by the following equation

$$\begin{aligned}
R = {} & -4E^{-L\cdot(X(1))} L^{(1)}\cdot(X(1))/X(1)-2E^{-L\cdot(X(1))} M^{(1)}\cdot(X(1))/X(1) \\
& -(1/2)E^{-L\cdot(X(1))} L^{(1)}\cdot{}^{2}(X(1))-(1/2)E^{-L\cdot(X(1))} M^{(1)}\cdot{}^{2}(X(1)) \\
& -(1/2)E^{-L\cdot(X(1))} M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))-2E^{-L\cdot(X(1))} L^{(2)}\cdot(X(1)) \\
& -E^{-L\cdot(X(1))} M^{(2)}\cdot(X(1))
\end{aligned}$$

The preceding information is next used to obtain the field equations. The individual equations are

$$\begin{aligned}
ET(1,1) = {} & -L^{(1)}\cdot(X(1))/(E^{L\cdot(X(1))})X(1)+2E^{-L\cdot(X(1))} L^{(1)}\cdot(X(1))/X(1) \\
& +E^{-L\cdot(X(1))} M^{(1)}\cdot(X(1))/X(1)-(1/4)M^{(1)}\cdot{}^{2}(X(1))/(E^{L\cdot(X(1))}) \\
& +(1/4)M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1))/(E^{L\cdot(X(1))})-L^{(2)}\cdot(X(1))/(E^{L\cdot(X(1))}) \\
& -(1/2)M^{(2)}\cdot(X(1))/(E^{L\cdot(X(1))})+(1/4)E^{-L\cdot(X(1))} L^{(1)}\cdot{}^{2}(X(1)) \\
& +(1/4)E^{-L\cdot(X(1))} M^{(1)}\cdot{}^{2}(X(1))+(1/4)E^{-L\cdot(X(1))} M^{(1)}\cdot(X(1))L^{(1)}\cdot(X(1)) \\
& +E^{-L\cdot(X(1))} L^{(2)}\cdot(X(1))+(1/2)E^{-L\cdot(X(1))} M^{(2)}\cdot(X(1))
\end{aligned}$$

$$ET(1,2) = 0$$

$$ET(1,3) = 0$$

$$ET(1,4) = 0$$

$$ET(2,1) = 0$$

$$ET(2,2) = (1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$+(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot^{2}(X(1))+(1/2)E^{-L\cdot(X(1))}L^{(2)}\cdot(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))$$

$$ET(2,3) = 0$$

$$ET(2,4) = 0$$

$$ET(3,1) = 0$$

$$ET(3,2) = 0$$

$$ET(3,3) = (1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+(1/2)E^{-L\cdot(X(1))}M^{(1)}\cdot(X(1))/X(1)$$

$$+(1/4)E^{-L\cdot(X(1))}M^{(1)}\cdot^{2}(X(1))+(1/2)E^{-L\cdot(X(1))}L^{(2)}\cdot(X(1))$$

$$+(1/2)E^{-L\cdot(X(1))}M^{(2)}\cdot(X(1))$$

$$ET(3,4) = 0$$

$$ET(4,1) = 0$$

$$ET(4,2) = 0$$

$$ET(4,3) = 0$$

$$ET(4,4) = 2E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+ (1/4)E^{-L\cdot(X(1))}L^{(1)}\cdot^{2}(X(1))$$

$$+E^{-L\cdot(X(1))}L^{(2)}\cdot(X(1))$$

The trajectory equations for the isotropic case are:

$$A(1) = -(1/2)L^{(1)}\cdot(X(1))V(1)^{2}+X(1)V(2)^{2}+(1/2)L^{(1)}\cdot(X(1))X(1)^{2}V(2)^{2}$$

$$+SIN^{2}(X(2))X(1)V(3)^{2}+(1/2)L^{(1)}\cdot(X(1))SIN^{2}(X(2))X(1)^{2}V(3)^{2}$$

$$-(1/2)E^{-L\cdot(X(1))+M\cdot(X(1))}M^{(1)}\cdot(X(1))V(4)^{2}$$

$$A(2) = -2V(2)V(1)/X(1)-L^{(1)}\cdot(X(1))V(2)V(1)+COS(X(2))SIN(X(2))V(3)^2$$

$$A(3) = -2V(3)V(1)/X(1)-L^{(1)}\cdot(X(1))V(3)V(1)-2COS(X(2))V(3)V(2)/SIN(X(2))$$

$$A(4) = -M^{(1)}\cdot(X(1))V(4)V(1)$$

Static homogeneous models— In the case of a static homogeneous universe, it is evident that coordinates can be chosen so that the line element will exhibit spherical symmetry around any desired origin, since all parts of the universe are permanently alike. Hence, the line element may be taken in the spherically symmetric static form of equation (18). In obtaining this form of line element, local irregularities in the gravitational field, which would occur in the immediate neighborhood of individual stars or stellar systems, are neglected.

For the system described, it can be shown that the components of the energy momentum tensor are:

$$\left.\begin{aligned} &ET(1,1) = ET(2,2) = ET(3,3) = 8\pi p_o \\ &ET(4,4) = -8\pi\rho_o \\ &ET(I,J) = 0 \quad \text{for} \quad I \neq J \end{aligned}\right\} \tag{21}$$

where p_0 and ρ_0 are the pressure and density, respectively, as measured by an observer who is at least momentarily at rest with respect to the spatial axes. The solution of these equations gives rise to the components of the potential function. In the case of the field being considered, the solution yields the unknown functions $L(x^1)$ and $M(x^1)$.

In order to satisfy the conditions of static homogeneity, it can be shown that the implicit functions $L(x^1)$ and $M(x^1)$ are subject to the following constraints: If the model is homogeneous, the pressure as measured by a local observer will be the same everywhere. Again, owing to the assumed homogeneity of the model, the density will be the same everywhere. Moreover, the line element must reduce to the special relativity form, for flat space-time, owing to the known validity of the special theory in such regions. By imposing these conditions, it can be shown that there are only three possibilities for a static homogeneous model (ref. 8):

$$M = 0 \tag{22}$$

$$L + M = 0 \tag{23}$$

$$L = M = 0 \tag{24}$$

These conditions lead respectively to the Einstein, the de Sitter, and the special relativity line elements.

The Einstein model universe— Substitution from equation (22) in equation (18) yields the following metric for a homogeneous model which is not isotropic.

$$ds^2 = \left[-e^{L(x^1)}(dx^1)^2 - (x^1\ dx^2)^2 - (x^1 \sin x^2\ dx^3)^2 + (dx^4)^2\right] \quad (25)$$

If the model were assumed to be homogeneous and isotropic, it would be necessary to use equation (20) subject to the constraint equation (22).

Cosmological considerations led Einstein to consider a universe defined by the metric (25). When the metric coefficients were supplied as input to the computer program, the following output was obtained.

The metric coefficients determine the gravitational model being studied. In order that each run be identified with the correct inputs, the postulated metric coefficients are printed out before the main results. In the case under consideration, these have the following values:

$$G(1,1) = -E^{L\cdot(X(1))}$$

$$G(2,2) = -X(1)^2$$

$$G(3,3) = -SIN^2(X(2))X(1)^2$$

$$G(4,4) = 1$$

The program uses the metric tensor inputs to evaluate the Christoffel symbols of the first and second kinds. In order to reduce the amount of output, the Christoffel symbols of the first kind are not printed out. In terms of the system coordinates and the unknown functions L and M, the Christoffel symbols of the second kind are

$$T(1,1,1) = (1/2)L^{(1)}\cdot(X(1))$$

$$T(1,1,2) = 0$$

$$T(1,1,3) = 0$$

$$T(1,1,4) = 0$$

$$T(1,2,1) = 0$$

$$T(1,2,2) = 1/X(1)$$

$$T(1,2,3) = 0$$

$$T(1,2,4) = 0$$

$$T(1,3,1) = 0$$

$T(1,3,2) = 0$

$T(1,3,3) = 1/X(1)$

$T(1,3,4) = 0$

$T(1,4,1) = 0$

$T(1,4,2) = 0$

$T(1,4,3) = 0$

$T(1,4,4) = 0$

$T(2,1,1) = 0$

$T(2,1,2) = 1/X(1)$

$T(2,1,3) = 0$

$T(2,1,4) = 0$

$T(2,2,1) = -E^{-L\cdot(X(1))}X(1)$

$T(2,2,2) = 0$

$T(2,2,3) = 0$

$T(2,2,4) = 0$

$T(2,3,1) = 0$

$T(2,3,2) = 0$

$T(2,3,3) = COS(X(2))/SIN(X(2))$

$T(2,3,4) = 0$

$T(2,4,1) = 0$

$T(2,4,2) = 0$

$T(2,4,3) = 0$

$T(2,4,4) = 0$

$T(3,1,1) = 0$

$T(3,1,2) = 0$

$T(3,1,3) = 1/X(1)$

T(3,1,4) = 0

T(3,2,1) = 0

T(3,2,2) = 0

T(3,2,3) = COS(X(2))/SIN(X(2))

T(3,2,4) = 0

T(3,3,1) = $-E^{-L\cdot(X(1))}SIN^{2}(X(2))X(1)$

T(3,3,2) = -COS(X(2))SIN(X(2))

T(3,3,3) = 0

T(3,3,4) = 0

T(3,4,1) = 0

T(3,4,2) = 0

T(3,4,3) = 0

T(3,4,4) = 0

T(4,1,1) = 0

T(4,1,2) = 0

T(4,1,3) = 0

T(4,1,4) = 0

T(4,2,1) = 0

T(4,2,2) = 0

T(4,2,3) = 0

T(4,2,4) = 0

T(4,3,1) = 0

T(4,3,2) = 0

T(4,3,3) = 0

T(4,3,4) = 0

T(4,4,1) = 0

$$T(4,4,2) = 0$$

$$T(4,4,3) = 0$$

$$T(4,4,4) = 0$$

Once the Christoffel symbols of the second kind are known, the components of the Ricci tensor can be derived. The individual components are

$$R(1,1) = -L^{(1)}\cdot(X(1))/X(1)$$

$$R(1,2) = 0$$

$$R(1,3) = 0$$

$$R(1,4) = 0$$

$$R(2,1) = 0$$

$$R(2,2) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))X(1)+E^{-L\cdot(X(1))}-1$$

$$R(2,3) = 0$$

$$R(2,4) = 0$$

$$R(3,1) = 0$$

$$R(3,2) = 0$$

$$R(3,3) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))SIN^2(X(2))X(1)+E^{-L\cdot(X(1))}SIN^2(X(2)) -SIN^2(X(2))$$

$$R(3,4) = 0$$

$$R(4,1) = 0$$

$$R(4,2) = 0$$

$$R(4,3) = 0$$

$$R(4,4) = 0$$

G(I,J) and R(I,J) are both known at this stage of the program; therefore the Ricci scalar can be obtained. It is given by the following equation

$$R = -2E^{-L\cdot(X(1))}/X(1)^2+2E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)+2/X(1)^2$$

The preceding information is next used to obtain the field equations. The individual equations are

$$ET(1,1) = E^{-L\cdot(X(1))}/X(1)^2 + L^{(1)}\cdot(X(1))/(E^{L\cdot(X(1))})X(1)$$

$$-E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1) - 1/X(1)^2$$

$$ET(1,2) = 0$$

$$ET(1,3) = 0$$

$$ET(1,4) = 0$$

$$ET(2,1) = 0$$

$$ET(2,2) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)$$

$$ET(2,3) = 0$$

$$ET(2,4) = 0$$

$$ET(3,1) = 0$$

$$ET(3,2) = 0$$

$$ET(3,3) = -(1/2)E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1)$$

$$ET(3,4) = 0$$

$$ET(4,1) = 0$$

$$ET(4,2) = 0$$

$$ET(4,3) = 0$$

$$ET(4,4) = E^{-L\cdot(X(1))}/X(1)^2 - E^{-L\cdot(X(1))}L^{(1)}\cdot(X(1))/X(1) - 1/X(1)^2$$

The equations of the corresponding trajectories are:

$$A(1) = E^{-L\cdot(X(1))}SIN^2(X(2))X(1)V(3)^2 - (1/2)L^{(1)}\cdot(X(1))V(1)^2 + E^{-L\cdot(X(1))}X(1)V(2)^2$$

$$A(2) = -2V(2)V(1)/X(1) + COS(X(2))SIN(X(2))V(3)^2$$

$$A(3) = -2V(1)V(3)/X(1) - 2COS(X(2))V(2)V(3)/SIN(X(2))$$

$$A(4) = 0$$

The de Sitter model— As already indicated, the only other general relativistic model that is static and homogeneous is the de Sitter universe. In the next section the Schwarzschild model will be considered. It will be found to have the same form, although not the same content as the de Sitter model. Although the Schwarzschild universe is inhomogeneous, the implicit functions $L(x^1)$ and $M(x^1)$ that satisfy its field equations also satisfy equation (23). In view of these considerations the de Sitter model will not be formulated.

A nonhomogeneous case— The Schwarzschild model represents a specially important application of relativity theory, since it provides a treatment of the gravitational field surrounding the sun. This problem was first studied by Schwarzschild in 1916, and the results obtained were used to distinguish between the predictions of the Newtonian theory of gravitation and the more exact predictions of relativity theory. Since the space surrounding the sun is assumed to be devoid of matter, all the components of the energy momentum tensor are zero. In this case, the field equations have been shown to satisfy equation (12). That is:

$$R_{ij} = 0 \tag{26}$$

Therefore, the components of the Ricci tensor obtained for the anisotropic model and satisfying equation (26) yield the components of the potential function for the field surrounding a single attracting mass, which is spherically symmetric.

In terms of conventional mathematical symbolism, the Schwarzschild field equations assume the following form:

$$R_{11} = \left[\frac{1}{2}\frac{d^2M}{dx^1\,dx^1} - \frac{1}{4}\frac{dL}{dx^1}\frac{dM}{dx^1} + \frac{1}{4}\left(\frac{dM}{dx^1}\right)^2 - \frac{1}{x^1}\frac{dL}{dx^1}\right] = 0 \tag{27}$$

$$R_{22} = \left\{e^{-L}\left[1 + \frac{1}{2}x^1\left(\frac{dM}{dx^1} - \frac{dL}{dx^1}\right)\right] - 1\right\} = 0 \tag{28}$$

$$R_{33} = \sin^2 x^2 \left\{e^{-L}\left[1 + \frac{x^1}{2}\left(\frac{dM}{dx^1} - \frac{dL}{dx^1}\right)\right] - 1\right\} = 0 \tag{29}$$

$$R_{44} = e^{M-L}\left[\frac{1}{4}\frac{dL}{dx^1}\frac{dM}{dx^1} - \frac{1}{2}\frac{d^2M}{dx^1\,dx^1} - \frac{1}{x^1}\frac{dM}{dx^1} - \frac{1}{4}\left(\frac{dM}{dx^1}\right)^2\right] = 0 \tag{30}$$

$$R_{ij} = 0 \quad \text{for} \quad i \neq j$$

The corresponding trajectory equations are:

$$\frac{d^2x^1}{ds^2} = \left[-\frac{1}{2}\frac{dL}{dx^1}\left(\frac{dx^1}{ds}\right)^2 + e^{-L}x^1\left(\frac{dx^2}{ds}\right)^2 \right.$$

$$\left. + e^{-L}\sin^2 x^2 \cdot x^1\left(\frac{dx^3}{ds}\right)^2 - \frac{1}{2}e^{M-L}\frac{dM}{dx^1}\left(\frac{dx^4}{ds}\right)^2 \right] \tag{31}$$

$$\frac{d^2x^2}{ds^2} = \left[-\frac{2}{x^1}\frac{dx^1}{ds}\frac{dx^2}{ds} + \sin x^2 \cos x^2\left(\frac{dx^3}{ds}\right)^2 \right] \tag{32}$$

$$\frac{d^2x^3}{ds^2} = \left(-\frac{2}{x^1}\frac{dx^1}{ds}\frac{dx^3}{ds} - 2\cot x^2 \frac{dx^2}{ds}\frac{dx^3}{ds} \right) \tag{33}$$

$$\frac{d^2x^4}{ds^2} = \left(-\frac{dM}{dx^1}\frac{dx^1}{ds}\frac{dx^4}{ds} \right) \tag{34}$$

It is seen that

$$R_{33} = \sin^2 x^2 \cdot R_{22}$$

and there are therefore only three equations in L and M. In this connection, it should be noted that the 10 equations given by equations (8) or (12) are not all independent since, theoretically at least, they would then determine completely the metric tensor and would restrict the choice of reference system. Therefore, there can be no more than six independent conditions between the components of R_{ij} to permit a free choice of coordinate system in four-dimensional space (ref. 8).

The system of 10 nonlinear partial differential equations

$$R_{ij} = 0$$

for the 10 unknown functions g_{ij} is very complicated. The general solution of this system is not known. However, for the case considered in this paper, it is possible to obtain a closed-form solution. It can easily be deduced that

$$L = -M$$

and

$$e^M = 1 + \frac{a}{x^1} = e^{-L}$$

Hence,

$$\left.\begin{aligned} g_{11} &= -1/[1 + (a/x^1)] \\ g_{22} &= -(x^1)^2 \\ g_{33} &= -(x^1 \sin x^2)^2 \\ g_{44} &= 1 + (a/x^1) \end{aligned}\right\} \tag{35}$$

If $a = -2m$, the metric (35) is consistent with the existence of one gravitating mass (m) situated at the origin and surrounded by empty space.

If the metric tensor inputs (eqs. (17)) consisting of unknown functions of x^1 are now replaced by the known functions (35), and the program re-run, the trajectory equations are obtained in the following form:

$$\begin{aligned} &\left\{\frac{d^2x^1}{ds^2} + \frac{a}{2(x^1)^2[1 + (a/x^1)]}\left(\frac{dx^1}{ds}\right)^2 - x^1\left(1 + \frac{a}{x^1}\right)\left(\frac{dx^2}{ds}\right)^2\right. \\ &\qquad \left. - x^1 \sin^2 x^2\left(1 + \frac{a}{x^1}\right)\left(\frac{dx^3}{ds}\right)^2 - \frac{a[1 + (a/x^1)]}{2(x^1)^2}\left(\frac{dx^4}{ds}\right)^2\right\} = 0 \end{aligned} \tag{36}$$

$$\left[\frac{d^2x^2}{ds^2} + \frac{2}{x^1}\frac{dx^1}{ds}\frac{dx^2}{ds} - \sin x^2 \cos x^2\left(\frac{dx^3}{ds}\right)^2\right] = 0 \tag{37}$$

$$\left(\frac{d^2x^3}{ds^2} + \frac{2}{x^1}\frac{dx^1}{ds}\frac{dx^3}{ds} + 2 \cot x^2 \frac{dx^2}{ds}\frac{dx^3}{ds}\right) = 0 \tag{38}$$

$$\left\{\frac{d^2x^4}{ds^2} - \frac{a}{(x^1)^2[1 + (a/x^1)]}\frac{dx^1}{ds}\frac{dx^4}{ds}\right\} = 0 \tag{39}$$

CONCLUSIONS

Symbolic mathematical computation can facilitate the formulation of mathematical models. This has been demonstrated by using the method to reconstruct several existing cosmological models and their associated trajectory equations. It has been shown that such models can be derived with speed and efficiency on present generation computers, provided they are equipped with formula manipulation compilers. For example, in the case of the Einstein and de Sitter models, the computer times required to formulate the field and trajectory equations were 0.66 and 0.32 minutes, respectively. For the nonhomogeneous Schwarzschild model, the corresponding times were 0.74 and 0.30 minutes, respectively. In addition to saving man-hours and the errors to which humans are prone, the method facilitates the study of a greater variety of models.

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