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GENERALIZATIONS OF LAGRANGE'S EXPANSION COMBINED WITH LIGHTHILL'S TECHNIQUE FOR UNIFORMIZING SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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16. Abstract

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SUMMARY

Use of Lighthill's technique to uniformize approximate solutions of partial differential equations is simplified by incorporating it into a perturbation-expansion scheme based on a higher dimensional generalization of Lagrange's expansion. Lighthill's technique is made easier to apply by use of explicit formulas for the uniformized solution in terms of the previously determined nonuniform solution. Results of this study also indicate that Lighthill's technique can be more useful than realized previously. As one example, uniformly valid thin-airfoil solutions can be obtained, by the direct procedure, to any higher order of approximation.

INTRODUCTION

Since Lighthill introduced his technique for uniformizing approximate solutions of physical problems (ref. 1), the method has been widely used, especially in fluid and gas dynamics, for both subsonic and supersonic flow problems. (See refs. 2-4 for discussion and references.) The basic principles of the method have also been incorporated into extensions and other related approaches (refs. 3, 5, 6, 7).

It has been widely believed for a number of years that the use of Lighthill's technique should be restricted to ordinary and hyperbolic differential equations. No way has been found to apply Lighthill's technique in many problems governed by parabolic equations where the nonuniformity is due to an essential singularity, which can only be eliminated by "stretching" or magnifying the coordinate (cf. ref. 8). Difficulty in applying Lighthill's technique to solutions of elliptic equations has also caused the belief that its use was not generally valid there. Although Lighthill's technique had been used early to cope with the nonuniformity at the leading edge in thin-airfoil theory governed by an elliptic equation (ref. 9), it was subsequently presumed that Lighthill's treatment of thin airfoils was a special case, and that the method could not yield an improved uniform solution beyond the second order. Tsien (ref. 3) (among others) stated (incorrectly, as we shall see) that "a solution uniformly valid to *all orders* is not possible." This was cited as a "failure of the PLK method." Very recently, however, Hoogstraten (ref. 10) was able to use Lighthill's technique in conjunction with a special conformal-mapping technique to find uniformly valid thin-airfoil solutions for round-nosed and sharp-nosed airfoils. Essentially the same method has been used independently by Bollheimer and Weissinger (ref. 11). These significant results obtained by Hoogstraten and by Bollheimer and Weissinger apply only to thin-airfoil theory governed by the Laplace equation. However, it will be shown that uniform higher order thin-airfoil solutions can be obtained by a *direct* application of Lighthill's technique in this elliptic problem. In the case considered, the solution is obtained quite easily in a variety of ways, and is possible to all orders. The procedure is not limited to Laplace's equation and so should be applicable to many other elliptic problems. It is applied in essentially the same manner to hyperbolic problems. (Conditions for applicability of Lighthill's technique are not investigated here. The procedures given are intended to apply only when Lighthill's technique *is* applicable.[1])

It appears that the most simple and direct way of applying Lighthill's technique is to use a Lagrange expansion-perturbation scheme and to specify the terms according to Lighthill's principle that higher order solutions shall be no more singular than the first. The procedure has been given in reference 13 for one independent variable. For extension of this approach to problems where more than one independent variable must be strained simultaneously, a higher dimensional Lagrange expansion is needed. Several forms of the generalization have been given in references 14, 15, and 16. Those necessarily involved derivations provide the general term and convergence criteria. Two independent derivations of simple procedures for obtaining explicitly the terms to any desired order of the N-dimensional Lagrange expansion in simplest form are given in chapter V of reference 17. The needed perturbation expansions based on Lagrange's expansion also have been given by Sack (ref. 18) in rather complicated form, in terms of multiple summations of a general term. A simple derivation of the needed terms to any desired order is given in reference 17 in a form most conveniently used for present purposes.

Accordingly, the purposes of this paper are to outline a relatively simple and direct procedure for obtaining, to any order, the terms of a higher dimensional (vector) generalization of Lagrange's expansion; to give, in simplest form, a general perturbation-expansion scheme based on that generalization; to combine the expansion scheme with Lighthill's uniformization technique for a considerable simplification of its use for any number of independent variables; and to illustrate the simplified application of Lighthill's technique in uniformizing the approximate solutions of partial differential equations.

The special advantages of this approach to applying Lighthill's technique are: (a) By providing explicit formulas for the uniformized solution in terms of the original functions that were not uniformly valid, it eliminates much of the tedious procedure normally followed; and (b) it extends this simplification to higher dimensions, when more than one variable must be strained simultaneously. For one independent variable, results of a previous paper (ref. 13) are extended to give a more general, and hence more flexible, expression of

[1]Comstock (ref. 12), in studying a problem posed by C. C. Lin, showed certain limitations to Lighthill's technique in ordinary differential equations that presumably can apply as well to partial differential equations.

the explicit formulas for the uniformized solution. Because of this generalization, natural choices of constants in the uniformizing transformation are often evident, and can lead more readily to the appropriate uniformization. As a result of these factors it is believed that Lighthill's technique is made easier to apply, and more useful than previously, by the developments to follow.

Helpful comments by Profs. M. J. Lighthill and R. A. Sack are gratefully acknowledged.

VECTOR GENERALIZATION OF LAGRANGE'S EXPANSION

Lagrange's Expansion in One Variable

The standard form of Lagrange's expansion for one implicitly defined independent variable,

$$Z = Z(\zeta,\varepsilon) \equiv \zeta + \varepsilon\mu(Z) \tag{1a}$$

is

$$f(Z) = f(\zeta) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} \left[\mu^n(\zeta) \frac{df(\zeta)}{d\zeta}\right] \tag{1b}$$

Vector and Tensor Definitions and Notation

For the treatment in higher dimensions, consider the N-dimensional space with orthogonal unit base vectors $\mathbf{e}_k$ $(k = 1, 2, . . ., N)$:

$$\left.\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} &= 1 \quad \text{for} \quad i = j \\ &= 0 \quad \text{for} \quad i \neq j \end{aligned}\right\} \quad (i,j = 1, 2, . . ., N) \tag{2}$$

Let $\boldsymbol{\zeta}$, $\mathbf{Z}$, and the function $\boldsymbol{\mu}(\mathbf{Z})$ be N-dimensional vectors in this space such that

$$\mathbf{Z} = \mathbf{Z}(\boldsymbol{\zeta},\varepsilon) \equiv \boldsymbol{\zeta} + \varepsilon\boldsymbol{\mu}(\mathbf{Z}) \tag{3}$$

where

$$\boldsymbol{\zeta} = \sum_{k=1}^{N} \mathbf{e}_k \zeta_k \tag{4a}$$

$$\mathbf{Z} = \sum_{k=1}^{N} \mathbf{e}_k Z_k(\zeta_1, \zeta_2, . . ., \zeta_N, \varepsilon) \tag{4b}$$

$$\mu(\mathbf{Z}) = \sum_{k=1}^{N} \mathbf{e}_k \mu_k(Z_1, Z_2, \ldots, Z_N) \tag{4c}$$

For any arbitrary differentiable function $F(\boldsymbol{\zeta}, \varepsilon)$, define the vector operators ∇_ζ and ∇_Z:

$$\nabla_\zeta F(\boldsymbol{\zeta}, \varepsilon) \equiv \sum_{k=1}^{N} \mathbf{e}_k \frac{\partial}{\partial \zeta_k} F(\boldsymbol{\zeta}, \varepsilon) \tag{5a}$$

$$\nabla_Z F(\mathbf{Z}, \varepsilon) \equiv \sum_{k=1}^{N} \mathbf{e}_k \frac{\partial}{\partial Z_k} F(\mathbf{Z}, \varepsilon) \tag{5b}$$

where the partial derivatives are taken holding all other components of the argument fixed.

For arbitrary N-dimensional vectors

$$\mathbf{A} \equiv \sum_{k=1}^{N} \mathbf{e}_k A_k , \qquad \mathbf{B} \equiv \sum_{k=1}^{N} \mathbf{e}_k B_k \tag{6}$$

we use an extension of the notion of an N-dimensional dyadic (second-order tensor):

$$\mathbf{AB} \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{e}_i \mathbf{e}_j A_i B_j \tag{7}$$

to define the nth-order tensors

$$\mathbf{A}^{(n)} \equiv \underbrace{\mathbf{AAA} \ldots \mathbf{A}}_{\text{n times}} , \qquad \mathbf{B}^{(n)} \equiv \underbrace{\mathbf{BBB} \ldots \mathbf{B}}_{\text{n times}} \tag{8}$$

We might call $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ "polyadics," since the special cases for $n = 2$, 3, and 4 are known, respectively, as dyadics, triadics, and tetradics (cf. ref. 19). The following defined *scalar products* then follow quite naturally from equation (2):

$$\mathbf{A} \cdot \mathbf{B} \equiv \sum_{i=1}^{N} A_i B_i \tag{9a}$$

$$\mathbf{AA}:\mathbf{BB} \equiv \mathbf{A}\cdot(\mathbf{A}\cdot\mathbf{BB}) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_iA_jB_jB_i \tag{9b}$$

$$\mathbf{AAA}\vdots\mathbf{BBB} \equiv \mathbf{A}\cdot[\mathbf{A}\cdot(\mathbf{A}\cdot\mathbf{BBB})]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} A_iA_jA_kB_kB_jB_i \tag{9c}$$

and, in general, define the *nth scalar product*:

$$\mathbf{A}^{(n)} \binom{n}{\cdot} \mathbf{B}^{(n)} \equiv \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} A_{i_1}A_{i_2}\cdots A_{i_n}B_{i_n}B_{i_{n-1}}\cdots B_{i_1} \tag{9d}$$

Particular examples of nth-order tensors to be used are:

$$[\mu(\zeta)]^{(n)} \equiv \underbrace{\mu(\zeta)\mu(\zeta)\ \cdots\ \mu(\zeta)}_{n\ \text{times}}$$

$$= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} \mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n}\mu_{i_1}(\zeta)\cdots\mu_{i_n}(\zeta) \tag{10a}$$

and

$$\nabla_\zeta^{(n)} \equiv \nabla_\zeta\nabla_\zeta\cdots\nabla_\zeta = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} \mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n}\frac{\partial^n}{\partial\zeta_{i_1}\cdots\partial\zeta_{i_n}} \tag{10b}$$

N-Dimensional Lagrange Expansion

The well-known N-dimensional Taylor expansion for a function $f(\mathbf{Z})$ about the point ζ is

$$f(Z_1,Z_2,\ldots,Z_N) = f_1(\zeta_1,\zeta_2,\ldots,\zeta_N) + \sum_{i=1}^{N}(Z_i - \zeta_i)\frac{\partial f(\zeta_1,\zeta_2,\ldots,\zeta_N)}{\partial \zeta_i}$$

$$+ \frac{1}{2!}\sum_{i=1}^{N}\sum_{j=1}^{N}(Z_i - \zeta_i)(Z_j - \zeta_j)\frac{\partial^2 f(\zeta_1,\zeta_2,\ldots,\zeta_N)}{\partial\zeta_i\partial\zeta_j} + \ldots \tag{11}$$

In the above-defined notation, with $\mathbf{Z}$ defined by (3), the complete expansion (11) is *equivalent* to simply

$$f(\mathbf{Z}) = f(\boldsymbol{\zeta}) + \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}[\mu(\mathbf{Z})]^{(n)}(\overset{n}{\cdot})\nabla_\zeta^{(n)} f(\boldsymbol{\zeta}) \tag{12a}$$

Then, also,

$$\mu(\mathbf{Z}) = \mu(\boldsymbol{\zeta}) + \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}[\mu(\mathbf{Z})]^{(n)}(\overset{n}{\cdot})\nabla_\zeta^{(n)}\mu(\boldsymbol{\zeta}) \tag{12b}$$

These two equations determine the higher dimensional Lagrange expansion to any desired order in ε, as shown by the following example to order ε^2:

$$f(\mathbf{Z}) = f(\boldsymbol{\zeta}) + \varepsilon\mu(\mathbf{Z})\cdot\nabla_\zeta f(\boldsymbol{\zeta}) + \frac{1}{2}\varepsilon^2[\mu(\mathbf{Z})\mu(\mathbf{Z})]:\nabla_\zeta\nabla_\zeta f(\boldsymbol{\zeta}) + O(\varepsilon^3) \tag{13a}$$

where

$$\begin{aligned}\mu(\mathbf{Z}) &= \mu(\boldsymbol{\zeta}) + \varepsilon\mu(\mathbf{Z})\cdot\nabla_\zeta\mu(\boldsymbol{\zeta}) + O(\varepsilon^2)\\ &= \mu(\boldsymbol{\zeta}) + \varepsilon\mu(\boldsymbol{\zeta})\cdot\nabla_\zeta\mu(\boldsymbol{\zeta}) + O(\varepsilon^2)\end{aligned} \tag{13b}$$

Thus

$$f(\mathbf{Z}) = f(\boldsymbol{\zeta}) + \varepsilon\mu(\boldsymbol{\zeta})\cdot\nabla_\zeta f(\boldsymbol{\zeta}) + \varepsilon^2\{[\mu(\boldsymbol{\zeta})\cdot\nabla_\zeta\mu(\boldsymbol{\zeta})]\cdot\nabla_\zeta f(\boldsymbol{\zeta})$$

$$+ \frac{1}{2}[\mu(\boldsymbol{\zeta})\mu(\boldsymbol{\zeta})]:\nabla_\zeta\nabla_\zeta f(\boldsymbol{\zeta})\} + O(\varepsilon^3) \tag{13c}$$

To order ε^3, this standard form of Lagrange's expansion extended to N-dimensions is[2]

$$f(\mathbf{Z}) = f(\boldsymbol{\zeta}) + \varepsilon\{\boldsymbol{\mu} \cdot \nabla_{\zeta} f\} + \frac{\varepsilon^2}{2!}\{\boldsymbol{\mu} \cdot \nabla_{\zeta}(\boldsymbol{\mu} \cdot \nabla_{\zeta} f) + (\boldsymbol{\mu} \cdot \nabla_{\zeta}\boldsymbol{\mu}) \cdot \nabla_{\zeta} f\}$$

$$+ \frac{\varepsilon^3}{3!}\{\boldsymbol{\mu} \cdot \nabla_{\zeta}[\boldsymbol{\mu} \cdot \nabla_{\zeta}(\boldsymbol{\mu} \cdot \nabla_{\zeta} f) + (\boldsymbol{\mu} \cdot \nabla_{\zeta}\boldsymbol{\mu}) \cdot \nabla_{\zeta} f]$$

$$+ 2(\boldsymbol{\mu} \cdot \nabla_{\zeta}\boldsymbol{\mu}) \cdot \nabla_{\zeta}(\boldsymbol{\mu} \cdot \nabla_{\zeta} f) + [\boldsymbol{\mu} \cdot \nabla_{\zeta}(\boldsymbol{\mu} \cdot \nabla_{\zeta}\boldsymbol{\mu})$$

$$+ (\boldsymbol{\mu} \cdot \nabla_{\zeta}\boldsymbol{\mu}) \cdot \nabla_{\zeta}\boldsymbol{\mu}] \cdot \nabla_{\zeta} f\} + O(\varepsilon^4) \tag{14}$$

where the argument of each function on the right side of equation (14) is $\boldsymbol{\zeta}$, and where $\mathbf{Z}$ is defined implicitly by equation (3). To obtain the form (14) from the procedure illustrated in equations (13), several identities are needed. However, this is not essential, as the form (13) and higher orders of that form, with the definitions (9) and (10), are also readily usable. (An alternative, completely independent derivation of eq. (14) is given in ref. 17. The general terms are obtainable from the more complex derivations and results of ref. 16.)

A PERTURBATION-EXPANSION SCHEME

Any number of additional parameters may be included as arguments of f, $\boldsymbol{\mu}$, and $\mathbf{Z}$. Thus, for example, with only one additional parameter α, if

$$\mathbf{Z} = \mathbf{Z}(\boldsymbol{\zeta},\varepsilon,\alpha) \equiv \boldsymbol{\zeta} + \varepsilon\boldsymbol{\mu}(\mathbf{Z},\alpha) \tag{15}$$

then equations (12) may be replaced by

$$f(\mathbf{Z},\alpha) = f(\boldsymbol{\zeta},\alpha) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!}[\boldsymbol{\mu}(\mathbf{Z},\alpha)]^{(n)} (\overset{n}{\cdot}) \nabla_{\zeta}^{(n)} f(\boldsymbol{\zeta},\alpha) \tag{16a}$$

[2]Note that in equations such as (13c) and (14) an operator does not act beyond the closing bracket of a pair inside which it is located.

where

$$\mu(\mathbf{Z},\alpha) = \mu(\boldsymbol{\zeta},\alpha) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!}\,[\mu(\mathbf{Z},\alpha)]^n (\overset{n}{\cdot}) \nabla_\zeta^{(n)} \mu(\boldsymbol{\zeta},\alpha) \tag{16b}$$

The more explicit form to order ε^3 is the same as equation (14) except that the left side is $f(\mathbf{Z},\alpha)$ and the functions μ and f on the right side have arguments $(\boldsymbol{\zeta},\alpha)$.

Now suppose that $f(\mathbf{Z},\alpha)$ and $\mu(\mathbf{Z},\alpha)$ can be expanded in power series in α:

$$f(\mathbf{Z},\alpha) = \sum_{k=1}^{\infty} \alpha^{k-1} f_k(\mathbf{Z}); \qquad \mu(\mathbf{Z},\alpha) = \sum_{k=1}^{\infty} \alpha^{k-1} \mu_k(\mathbf{Z}) \tag{17}$$

If these expressions are substituted into equations (16), or into the generalization of (14), and if the results are specialized to the case where $\alpha = \varepsilon$, one obtains a useful perturbation-expansion scheme. To order ε^2, the results are:

$$f(\mathbf{Z},\varepsilon) = f_1(\mathbf{Z}) + \varepsilon f_2(\mathbf{Z}) + \varepsilon^2 f_3(\mathbf{Z}) + O(\varepsilon^3) \tag{18a}$$

and

$$\begin{aligned} f(\mathbf{Z},\varepsilon) = {} & f_1(\boldsymbol{\zeta}) + \varepsilon\{f_2(\boldsymbol{\zeta}) + \mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta f_1(\boldsymbol{\zeta})\} \\ & + \varepsilon^2 \Big\{ f_3(\boldsymbol{\zeta}) + \mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta f_2(\boldsymbol{\zeta}) + \mu_2(\boldsymbol{\zeta}) \cdot \nabla_\zeta f_1(\boldsymbol{\zeta}) \\ & + \left(\frac{1}{2}\right) \mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta [\mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta f_1(\boldsymbol{\zeta})] \\ & + \left(\frac{1}{2}\right) [\mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta \mu_1(\boldsymbol{\zeta})] \cdot \nabla_\zeta f_1(\boldsymbol{\zeta})] \Big\} + O(\varepsilon^3) \end{aligned} \tag{18b}$$

where

$$\mathbf{Z} = \boldsymbol{\zeta} + \varepsilon \mu_1(\boldsymbol{\zeta}) + \varepsilon^2\{\mu_2(\boldsymbol{\zeta}) + \mu_1(\boldsymbol{\zeta}) \cdot \nabla_\zeta \mu_1(\boldsymbol{\zeta})\} + O(\varepsilon^3) \tag{18c}$$

For the special case where $\boldsymbol{\zeta}$, μ, and $\mathbf{Z}$ have only one component each, ζ, μ, Z, the results to all orders, derived from (1), are:

$$f(Z,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^{n-1} f_n(Z) \tag{19a}$$

and

$$f(Z,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^{n-1} f_n(\zeta)$$

$$+ \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{\partial^{n-1}}{\partial \zeta^{n-1}} \left\{ \left[\sum_{k=1}^{\infty} \varepsilon^{k-1} \mu_k(\zeta) \right]^n \left[\sum_{k=1}^{\infty} \varepsilon^{k-1} f_k'(\zeta) \right] \right\} \tag{19b}$$

where

$$Z = \zeta + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{\partial^{n-1}}{\partial \zeta^{n-1}} \left\{ \left[\sum_{k=1}^{\infty} \varepsilon^{k-1} \mu_k(\zeta) \right]^n \right\} \tag{19c}$$

As noted in the Introduction, equations (18) and (19) should be obtainable from the more complicated general forms in reference 18. These results have been derived here more simply and stated in these forms for convenient use later.

COMBINATION OF LAGRANGE EXPANSION METHOD WITH LIGHTHILL'S TECHNIQUE

For a problem in which the terms of the approximate solution in the form of equation (18a) or (19a) have been found to be not uniformly valid, the basic principles of Lighthill's uniformization technique are:

(1) To reformulate the expanded solution, transformed in terms of initially undetermined "strained coordinates," and

(2) To specify the straining transformation in a manner that removes the nonuniformity, according to Lighthill's principle:

> Higher approximations shall be
> no more singular than the first.

For an arbitrary straining transformation (arbitrary $\mu_j(\zeta)$ in eq. (18c)), any transformed expanded solution is given by equation (18b) to order ε^2, where equation (18a) is the expanded solution in the "unstrained" coordinates. Thus, part (1) of Lighthill's technique is done. It remains to specify the terms so that Lighthill's principle is satisfied.

It is useful to consider first why the principle achieves uniformization. The expansion of the exact solution will have successively higher order nonuniformities in the terms that are of higher order in ε if the exact solution contains ε in such a way that its expansion shifts a singularity. For example, suppose the exact solution contains $g = (Z + \varepsilon)^{-1}$. This expression is singular at $Z = -\varepsilon$. When $\varepsilon \to 0$, the singularity moves to $Z = 0$. This is exhibited in the expansion about $\varepsilon = 0$:

$$g = g(Z + \varepsilon) \equiv (Z + \varepsilon)^{-1} = Z^{-1} - \varepsilon Z^{-2} + \varepsilon^2 Z^{-3} \mp \ldots \tag{20}$$

In a perturbation solution that would attempt to find g as a function of Z and ε, such singular terms will appear. However, if one made a transformation

$$\zeta = Z + \varepsilon \tag{21}$$

the complete term representing g is simply

$$g = g(\zeta) = \zeta^{-1} \tag{22}$$

where

$$Z = \zeta - \varepsilon \tag{23}$$

with which g is uniformly valid at $Z = 0$. The transformation simply *removed* ε from an expression (g) that is singular at a value of Z depending on ε. Thus, equation (20) contains ε, but equation (22) does not, so that "solution" for g and Z in terms of ζ, expanded in powers of ε, is uniformly valid at $Z = 0$. With this observation, for the general problem, one then simply *seeks a transformation* that precludes the occurrence of the higher order nonuniformities (and therefore that eliminates the shifting of the singularity when the solution is expanded) by *removing* ε from terms whose expansion would shift the singularity. This is accomplished by Lighthill's principle. It should be obvious then that Lighthill's technique should not be expected to remove the nonuniformity in a solution that is caused by an *essential singularity*, which is removable only by properly magnifying the variables and not by a simple straining. Thus, in an extension of Lighthill's technique (see ref. 3), Kuo combined a "stretching" with a straining of the coordinate in a boundary-layer problem. The nonuniformity caused by an (exponential) essential singularity occurs in problems where the highest order derivatives are multiplied by the small parameter, and so are lost in the perturbation solution (cf. refs. 2 and 8).

In the classes of problems for which Lighthill's technique can be used, Lighthill's principle can be applied as follows: To first order as $\varepsilon \to 0$, $f(Z,\varepsilon) = f_1(\zeta)$ in equation (18b). To satisfy the principle then, one can simply determine each $\mu_n(\zeta)$ so that the coefficient of each ε^n in equation (18b) is equal to some constant a_n times $f_1(\zeta)$. Thus, specify

$$f(Z,\varepsilon) = p(\varepsilon) f_1(\zeta) \tag{24a}$$

where

$$p(\varepsilon) \equiv 1 + \sum_{n=1}^{\infty} a_n \varepsilon^n \tag{24b}$$

The constants a_n are chosen for convenience in each problem. They might all be taken to be zero, but it is just as easy to leave them unspecified *a priori* for possibly greater convenience. Thus, from equations (18b) and (24), one may evaluate the $\mu_n(\zeta)$ to satisfy:

$$f_2(\zeta) + \mu_1(\zeta) \cdot \nabla_\zeta f_1(\zeta) = a_1 f_1(\zeta) \tag{25a}$$

$$f_3(\zeta) + \mu_1(\zeta) \cdot \nabla_\zeta f_2(\zeta) + \mu_2(\zeta) \cdot \nabla_\zeta f_1(\zeta) + \left(\frac{1}{2}\right) \mu_1(\zeta) \cdot \nabla_\zeta [\mu_1(\zeta) \cdot \nabla_\zeta f_1(\zeta)]$$

$$+ \left(\frac{1}{2}\right) [\mu_1(\zeta) \cdot \nabla_\zeta \mu_1(\zeta)] \cdot \nabla_\zeta f_1(\zeta) = a_2 f_1(\zeta) \tag{25b}$$

etc.

For the reduction of equations (25) and (18), the following notation is convenient. Let the component of $\mu(\zeta,\varepsilon)$ in the direction of ζ_i be $\mu_i(\zeta_1,\zeta_2, \ldots ,\zeta_N,\varepsilon)$, which has the expansion

$$\mu_i(\zeta_1,\ldots,\zeta_N,\varepsilon) = \mu_{i1}(\zeta_1,\ldots,\zeta_N) + \varepsilon\mu_{i2}(\zeta_1,\ldots,\zeta_N) + \ldots \tag{26}$$

Thus, in μ_{ij}, subscript i indicates the component direction and subscript j indicates the order of approximation. One result is

$$\mu_j(\zeta) \cdot \nabla_\zeta \equiv \sum_{i=1}^{N} \mu_{ij}(\zeta_1,\ldots,\zeta_N) \frac{\partial}{\partial \zeta_i} \tag{27}$$

for use in equations (25) and (18). In the special cases where only *one component* of the independent variable (say, Z_1) need be strained, denote:

$$\left.\begin{aligned} \mu_{ij} &\equiv 0 && \text{for } i \neq 1 \\ &\equiv \mu_j(\zeta_1,\ldots,\zeta_N) && \text{for } i = 1 \end{aligned}\right\} \tag{28a}$$

If $F'(\zeta)$ is then defined as

$$F'(\zeta) \equiv \frac{\partial F(\zeta)}{\partial \zeta_1} \tag{28b}$$

we find, from equations (19b) and (24), the special case of equation (25) when equation (28a) applies:

$$\mu_1 = \left(\frac{-1}{f_1'}\right) (f_2 - a_1 f_1) \tag{28c}$$

$$\mu_2 = \left(\frac{-1}{f_1'}\right) \left[f_3 - a_2 f_1 + \mu_1 f_2' + \left(\frac{1}{2}\right) (\mu_1^2 f_1')'\right] \tag{28d}$$

$$\mu_3 = \left(\frac{-1}{f_1'}\right) \left[f_4 - a_3 f_1 + \mu_1 f_3' + \mu_2 f_2' + \left(\frac{1}{2}\right) (\mu_1^2 f_2' + 2\mu_1\mu_2 f_1')' + \left(\frac{1}{6}\right) (\mu_1^3 f_1')''\right] \tag{28e}$$

$$\mu_4 = \left(\frac{-1}{f_1'}\right) \left[f_5 - a_4 f_1 + \mu_1 f_4' + \mu_2 f_3' + \mu_3 f_2' + \left(\frac{1}{2}\right) (\mu_1^2 f_3' + 2\mu_1\mu_2 f_2' + 2\mu_1\mu_3 f_1' + \mu_2^2 f_1')' + \left(\frac{1}{6}\right) (\mu_1^3 f_2' + 3\mu_1^2\mu_2 f_1')'' + \left(\frac{1}{24}\right) (\mu_1^4 f_1')'''\right] \tag{28f}$$

etc. In this special case, equation (18c) becomes (cf. eq. (19c)):

$$Z_1 = \zeta_1 + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{\partial^{n-1}}{\partial \zeta_1^{n-1}} \left\{\left[\sum_{k=1}^{\infty} \varepsilon^{k-1} \mu_k(\zeta_1, \ldots, \zeta_N)\right]^n\right\} \tag{28g}$$

The use of these equations for direct application of Lighthill's technique is illustrated below. With a nonuniform result in the form of equation (18a) if more than one independent variable is to be strained, equations (24), (25), (27), and (18c) are used; if only one coordinate is to be strained, equations (24) and (28) are convenient.

APPLICATIONS IN IDEAL FLOW OVER A THIN BODY (ELLIPTIC EQUATION)

The problem of flow over a thin airfoil of elliptical cross section is used for illustration here because it has been discussed extensively in the past in regard to the use of, and the previously presumed limitations of, Lighthill's technique. Also, the exact solution is available in a simple form for comparison of the results.

The elliptical-airfoil surface is represented by

$$y = y_b(x) = \pm\varepsilon(1 - x^2)^{1/2}, \qquad (-1 \le x \le 1) \tag{29}$$

where x and y are inertial Cartesian coordinates, made dimensionless with respect to the length of the semimajor axis of the elliptical body. The flow is assumed to be steady, inviscid, and incompressible. The flow velocity is assumed to be tangent to the body surface and, far from the body, to approach a uniform stream velocity with magnitude U and direction parallel to $y = 0$.

Consider the complex potential $F(Z)$ and complex velocity $W(Z)$, which are analytic functions of the complex variable $Z = x + iy$:

$$F(Z) = \phi(x,y) + i\psi(x,y) \tag{30a}$$

$$\left.\begin{aligned} F'(Z) \equiv W(Z) &= u(x,y) - iv(x,y) \\ &= \phi_x - i\phi_y = \psi_y + i\psi_x \end{aligned}\right\} \tag{30b}$$

where subscripts x and y indicate partial derivatives. The velocity potential ϕ and the stream function ψ then satisfy the elliptic equations representing conservation of mass and momentum:

$$\left.\begin{aligned} &\phi_{xx} + \phi_{yy} = 0 \\ &\phi \sim Ux + o(1) \qquad \text{as} \quad x^2 + y^2 \to \infty \\ &\frac{v}{u} \equiv \frac{\phi_y}{\phi_x} = \frac{dy_b(x)}{dx} \quad \text{on} \quad y = y_b(x) \end{aligned}\right\} \tag{31a}$$

and

$$\left.\begin{aligned} &\psi_{xx} + \psi_{yy} = 0 \\ &\psi \sim Uy + o(1) \qquad \text{as} \quad x^2 + y^2 \to \infty \\ &\frac{v}{u} \equiv -\frac{\psi_x}{\psi_y} = \frac{dy_b(x)}{dx}, \quad \text{or} \quad \psi = 0, \quad \text{on} \quad y = y_b(x) \end{aligned}\right\} \tag{31b}$$

Consider the solution in three forms: $F(Z)$, of which ϕ and ψ are the real and imaginary parts; $W(Z)$, of which u and $-v$ are the real and imaginary parts; and

$$Q_b(x) \equiv \left[(u^2 + v^2)^{1/2}\right]_{y=y_b(x)} \tag{32}$$

the velocity magnitude on the body surface. Define the dimensionless perturbation-solution functions $f(Z;\varepsilon)$, $w(Z;\varepsilon)$, and $q_b(x;\varepsilon)$ and the function $g(x;\varepsilon)$ by:

$$F(Z;\varepsilon) \equiv U[Z + \varepsilon f(Z;\varepsilon)] \tag{33a}$$

$$W(Z;\varepsilon) \equiv U[1 + \varepsilon w(Z;\varepsilon)] \tag{33b}$$

$$Q_b(x;\varepsilon) \equiv U[1 + \varepsilon q_b(x;\varepsilon)] \tag{33c}$$

$$\equiv U[1 + \varepsilon + \varepsilon^2 g(x;\varepsilon)] \tag{33d}$$

A number of textbooks give the procedure for finding the approximate thin-airfoil solutions of equations (31). With $y_b(x)$ given by equation (29), one finds

$$\left.\begin{aligned} f(Z;\varepsilon) &= f_1(Z) + \varepsilon f_2(Z) + \varepsilon^2 f_3(Z) + \dots \\ w(z;\varepsilon) \equiv f'(Z;\varepsilon) &= f_1'(Z) + \varepsilon f_2'(Z) + \varepsilon^2 f_3'(Z) + \dots \\ g(x;\varepsilon) &= g_1(x) + \varepsilon g_2(x) + \varepsilon^2 g_3(x) + \dots \end{aligned}\right\} \tag{34}$$

where

$$\left.\begin{aligned} &f_1(Z) = Z - (Z^2 - 1)^{1/2} \ ; && f_2(Z) = f_1(Z) \ ; \\ &f_3(Z) = f_1(Z) - \left(\frac{1}{2}\right)(Z^2 - 1)^{-1/2} \ ; && f_4(Z) = f_3(Z) \cdot \ ; \\ &f_5(Z) = f_3(Z) + \left(\frac{1}{8}\right)(Z^2 - 1)^{-3/2} \ ; && \text{etc.} \end{aligned}\right\} \tag{35}$$

(cf. p. 72 of ref. 4) and

$$\left.\begin{aligned} &g_1(x) = -\left(\frac{1}{2}\right)x^2(1 - x^2)^{-1} \ ; && g_2(x) = g_1(x) \ ; \\ &g_3(x) = \left(\frac{3}{8}\right)x^4(1 - x^2)^{-2} \ ; && g_4(x) = g_3(x) \ ; \\ &g_5(x) = -\left(\frac{5}{16}\right)x^6(1 - x^2)^{-3} \ ; && \text{etc.} \end{aligned}\right\} \tag{36}$$

(cf. p. 52 of ref. 4).

For later comparison, note that the exact solutions (which can be found, e.g., from ref. 20, p. 429) are:

$$f(Z;\varepsilon) = (1 - \varepsilon)^{-1}[Z - (Z^2 - 1 + \varepsilon^2)^{1/2}] \tag{37a}$$

$$w(Z;\varepsilon) = (1 - \varepsilon)^{-1}[1 - Z(Z^2 - 1 + \varepsilon^2)^{-1/2}] \tag{37b}$$

$$g(x;\varepsilon) = \frac{1 + \varepsilon}{\varepsilon^2}\left[\left(\frac{1 - x^2}{1 - x^2 + \varepsilon^2 x^2}\right)^{1/2} - 1\right] \tag{37c}$$

We see that the approximate solutions (34), with (35) and (36), are singular at $Z = \pm 1$, the leading and trailing edges, and the higher approximations are successively more singular at $Z = \pm 1$. We therefore consider use of Lighthill's technique by the direct procedure given above. To show the generality of the method, it is applied in the following paragraphs to this problem in a variety of ways.

First Solution

If Lighthill's technique in the form of equations (24) and (28), with the single independent variable Z, is applied to the complex potential F having the nonuniform solution given by (33a) with (34) and (35),

$$\left(\frac{1}{\varepsilon}\right)\left(\frac{F}{U} - Z\right) \equiv f(Z;\varepsilon) = p(\varepsilon)[\zeta - (\zeta^2 - 1)^{1/2}] \tag{38}$$

If we now arbitrarily take all $a_n = 0$ in equation (24b), so that $p(\varepsilon) = 1$, equations (28c) to (28g) give the solution directly as

$$\left.\begin{aligned} f(Z;\varepsilon) &= \zeta - (\zeta^2 - 1)^{1/2} \\ Z &= \zeta + \varepsilon(\zeta^2 - 1)^{1/2} \end{aligned}\right\} \tag{39a}$$

It is easily verified that this solution with only two terms in the expansion of Z is, in fact, equivalent to the *exact* solution (37a). Equations (39a) further reduce to

$$\left.\begin{aligned} \frac{F}{U} &= (1 + \varepsilon)\zeta \\ Z &= \zeta + \varepsilon(\zeta^2 - 1)^{1/2} \end{aligned}\right\} \tag{39b}$$

which has also been given by C. Jacob (see ref. 10). To find the expression for the corresponding velocity, one simply replaces $f(Z;\varepsilon)$ by $w = f'(Z;\varepsilon)$ and each $f_n(\zeta)$ by $f_n'(\zeta)$ in equations (19). The result for w in (33b) is

$$w = 1 - \zeta(\zeta^2 - 1)^{-1/2} + \varepsilon[1 - \zeta(\zeta^2 - 1)^{-1/2} - (\zeta^2 - 1)^{-1}]$$
$$+ \varepsilon^2[1 - \zeta(\zeta^2 - 1)^{-1/2} - (\zeta^2 - 1)^{-1}] + O(\varepsilon^3) \tag{40}$$

Note that there is no straining at $Z = \pm 1$, so the corresponding velocity would *not* be made uniformly valid by this straining transformation. Thus, even though this particular straining transformation (fixed by taking $p(\varepsilon) = 1$) is successful in uniformizing the complex potential, it is not useful for the velocity.

Second Solution

If, instead of arbitrarily taking all $a_n = 0$, one observes from equations (35) and (28c) through (28f) that it appears *most natural* to choose $a_n = 1$ for all n (since it substantially reduces eqs. (28) for this problem), so that $p(\varepsilon) = (1 - \varepsilon)^{-1}$, one obtains directly from equations (24) and (28):

$$\left.\begin{aligned} f(Z;\varepsilon) &= (1 - \varepsilon)^{-1}[\zeta - (\zeta^2 - 1)^{1/2}] \\ Z &= \zeta - \left(\frac{\varepsilon^2}{2}\right)[\zeta - (\zeta^2 - 1)^{1/2}]^{-1} \end{aligned}\right\} \tag{41}$$

It is easily verified that this solution, with only two nonvanishing terms in the expansion of Z, is also equivalent to the *exact solution* (37a). Furthermore, there is a finite straining at $Z = \pm 1$. The corresponding velocity field, which then remains uniformly valid including the points $Z = \pm 1$, is given by (33b), with

$$\begin{aligned} w(Z;\varepsilon) = 1 &- \zeta(\zeta^2 - 1)^{-1/2} + \varepsilon[1 - \zeta(\zeta^2 - 1)^{-1/2}] \\ &+ \varepsilon^2\left[1 - \zeta(\zeta^2 - 1)^{-1/2} - \left(\frac{1}{2}\right)(\zeta^2 - 1)^{-1}\right] \\ &+ \varepsilon^3\left[1 - \zeta(\zeta^2 - 1)^{-1/2} - \left(\frac{1}{2}\right)(\zeta^2 - 1)^{-1}\right] + O(\varepsilon^4) \end{aligned} \tag{42}$$

Third Solution

If the velocity field is of prime interest, one might instead apply the uniformization directly to the complex velocity, W. The nonuniform solution is given by (33b) with (34), in terms of the derivatives of the functions listed in (35). The uniformly valid solution can be represented by (24) and (28) with all $f_n(\zeta)$ replaced by $w_n(\zeta) \equiv f_n'(\zeta)$. Thus

$$\left(\frac{1}{\varepsilon}\right)\left(\frac{W}{U} - 1\right) \equiv w(Z;\varepsilon) = p(\varepsilon)f_1'(\zeta) \tag{43}$$

where $p(\varepsilon)$ is given by (24b) and where the corresponding expansion of Z is given by (28g). It appears most natural in this problem (in using (28c) through (28f) with all $f_n(\zeta)$ replaced by $w_n(\zeta)$) to set $a_n = 1$ for all n, as this substantially reduces the results. One then obtains directly

$$w(Z;\varepsilon) = (1 - \varepsilon)^{-1}[1 - \zeta(\zeta^2 - 1)^{-1/2}] \tag{44a}$$

$$Z = \zeta - \left(\frac{1}{2}\right)\varepsilon\zeta - \left(\frac{1}{8}\right)\varepsilon^2\zeta - \ldots \tag{44b}$$

It is easily shown, in fact, that the *exact* solution (37b) is equivalent to (44a) with $Z = \zeta(1 - \varepsilon^2)^{1/2}$, the expansion of which is (44b).

Fourth Solution

If one is interested in only the surface speed, Q_b (or the surface pressure coefficient, $C_p = 1 - (Q_b/U)^2$) as a function of x, one can use the above procedure directly to make Q_b uniformly valid. The nonuniform solution is given by (33d) with (34) and (36). The uniformized solution can then be found from equations (24) and (28) with Z and ζ replaced, respectively, by x and ξ, and with f, f_n, and their derivatives replaced by g, g_n, and their derivatives. Thus

$$\left(\frac{1}{\varepsilon^2}\right)\left[\frac{Q_b}{U} - (1 + \varepsilon)\right] \equiv g(x;\varepsilon) = p(\varepsilon)g_1(\xi) \tag{45}$$

where $p(\varepsilon)$ is given by (24b) and where the corresponding expansion for x as a function ξ and ε is found from the form of (28g). In using (28c) through (28f) one observes that it is convenient to let $a_1 = 1$ and $a_n = 0$ for $n \geq 2$. One then obtains directly

$$g(x;\varepsilon) = (1 + \varepsilon)\left(\frac{-1}{2}\right)\xi^2(1 - \xi^2)^{-1} \tag{46a}$$

$$x = \xi + \varepsilon^2\left(\frac{3\xi^2}{8}\right) + \varepsilon^4(1 - \xi^2)^{-1}\,\frac{23\xi^5 - 27\xi^7}{128} + O(\varepsilon^6) \tag{46b}$$

Not every possible choice of the straining transformation is useful in achieving the uniformization. The constants a_n must be chosen to give useful results.

Remarks on Elliptic Problems

Because of the variety of ways the technique has been used (i.e., variety of straining transformations) in a straightforward manner in the perturbation solution for potential flow over a thin elliptical airfoil, it appears that the technique should be applicable to a broad class of elliptic problems. Other simple examples of elliptic problems that have been worked out by the writer using the perturbation technique include potential flow over a thin parabolic cylinder, a thin round-nosed semi-infinite slab, a thin Rankine oval cylinder, and a slender axisymmetric paraboloid. Of course, it should not be implied that the procedure can apply directly to all elliptic perturbation problems. It cannot. The limitations of the method in elliptic problems are not definitely known, but are found to be apparently not so severe as believed previously.

ILLUSTRATIVE APPLICATION TO A HYPERBOLIC EQUATION

For illustration of the procedure when more than one independent variable must be strained to achieve a uniformly valid approximation, consider the problem:

$$f_{xx} - f_{yy} + \left(\frac{2}{xy + \varepsilon}\right)\left(yf_x - xf_y\right) = 0, \qquad (\varepsilon > 0) \tag{47a}$$

$$y = 1, \quad \text{all} \quad x: \quad f = 1 \tag{47b}$$

$$\frac{\partial f}{\partial y} = \frac{\varepsilon - x}{\varepsilon + x} \tag{47c}$$

This problem is chosen for illustration because it is especially simple, and it combines several features that may be encountered in more complicated problems. Also, the approximate solution can be compared with the exact solution. The partial differential equation (47a) is hyperbolic. Even though this particular problem could be simplified before solving it and the complete solution found, let us proceed to apply the perturbation technique directly, as one would need to do in more complicated problems.

First, the perturbation solution is sought in the form:

$$f(x,y,\varepsilon) = f_1(x,y) + \varepsilon f_2(x,y) + \varepsilon^2 f_3(x,y) + \ldots \tag{48}$$

In the limit as $\varepsilon \to 0$, one obtains the problem for f_1, which can be written in the form

$$\frac{\partial^2}{\partial x^2}(xyf_1) - \frac{\partial^2}{\partial y^2}(xyf_1) = 0 \tag{49a}$$

$$y = 1: \quad f_1 = 1, \qquad \frac{\partial f_1}{\partial y} = -1 \tag{49b}$$

The wave equation, (49a), with conditions (49b), has the solution

$$f_1(x,y) = y^{-1} \tag{50a}$$

Similarly, the problem for f_2 is found to be:

$$\frac{\partial^2}{\partial x^2}(xyf_2) - \frac{\partial^2}{\partial y^2}(xyf_2) = \frac{2}{y^3}$$

$$y = 1: \quad f_2 = 0, \qquad \frac{\partial f_2}{\partial y} = \frac{2}{x}$$

which has the solution

$$f_2(x,y) = x^{-1}(1 - y^{-2}) \tag{50b}$$

Further, one finds

$$f_3(x,y) = -x^{-2}y^{-1}(1 - y^{-2}) \tag{50c}$$

The perturbation solution (48) with (50) is singular on both lines $x = 0$ and $y = 0$, which are within the domain of interest. We therefore consider use of Lighthill's technique according to the simplified approach outlined on pages 10 and 11. In particular, if the solution near the origin is of interest, both coordinates must be strained simultaneously.

To apply this simplified procedure, let $N = 2$ in equations (4a) and (4b) and let

$$Z_1 \equiv x, \qquad Z_2 \equiv y \tag{51}$$

Then the expansion (48) corresponds to (18a). The coordinate transformation (18c) implies (18b). To achieve the desired uniformization, equations (18) are also equivalent to (24a) with (24b) and with the functions μ_j determined by (25). Thus

$$f(x,y,\varepsilon) = (1 + a_1\varepsilon + a_2\varepsilon^2 + \ . \ . \ .)\left(\frac{1}{\zeta_2}\right) \tag{52}$$

where

$$x \equiv Z_1 = \zeta_1 + \varepsilon\mu_{11} + \varepsilon^2 \left(\mu_{12} + \mu_{11} \frac{\partial\mu_{11}}{\partial\zeta_1} + \mu_{21} \frac{\partial\mu_{11}}{\partial\zeta_2}\right) + O(\varepsilon^3) \tag{53a}$$

$$y \equiv Z_2 = \zeta_2 + \varepsilon\mu_{21} + \varepsilon^2 \left(\mu_{22} + \mu_{11} \frac{\partial\mu_{21}}{\partial\zeta_1} + \mu_{21} \frac{\partial\mu_{21}}{\partial\zeta_2}\right) + O(\varepsilon^3) \tag{53b}$$

and where

$$f_2 + \mu_{11} \frac{\partial f_1}{\partial\zeta_1} + \mu_{21} \frac{\partial f_1}{\partial\zeta_2} = a_1 f_1 \tag{54a}$$

and

$$\begin{aligned} f_3 &+ \left(\mu_{11} \frac{\partial f_2}{\partial\zeta_1} + \mu_{21} \frac{\partial f_2}{\partial\zeta_2}\right) + \left(\mu_{12} \frac{\partial f_1}{\partial\zeta_1} + \mu_{22} \frac{\partial f_1}{\partial\zeta_2}\right) \\ &+ \frac{1}{2}\left(\mu_{11} \frac{\partial}{\partial\zeta_1} + \mu_{21} \frac{\partial}{\partial\zeta_2}\right)\left(\mu_{11} \frac{\partial f_1}{\partial\zeta_1} + \mu_{21} \frac{\partial f_1}{\partial\zeta_2}\right) \\ &+ \frac{1}{2}\left(\mu_{11} \frac{\partial\mu_{11}}{\partial\zeta_1} + \mu_{21} \frac{\partial\mu_{11}}{\partial\zeta_2}\right) \frac{\partial f_1}{\partial\zeta_1} + \frac{1}{2}\left(\mu_{11} \frac{\partial\mu_{21}}{\partial\zeta_1} + \mu_{21} \frac{\partial\mu_{21}}{\partial\zeta_2}\right) \frac{\partial f_1}{\partial\zeta_2} = a_2 f_1 \end{aligned} \tag{54b}$$

(the arguments of each f_j and μ_{ij} are ζ_1,ζ_2).

Equation (54a) gives an equation for μ_{21} in which there is no apparent advantage in having a_1 not zero; so for the simplest treatment, take

$$a_1 = 0 \tag{55}$$

The result for use in (53b) is

$$\mu_{21}(\zeta_1,\zeta_2) = \frac{\zeta_2^2 - 1}{\zeta_1} \tag{56}$$

If the determined functions (f_1, f_2, and μ_{21}) are then put into (54b), and if

$$a_2 = 0 \tag{57}$$

for simplest treatment, the result for equation (53b) to order ε^2 is

$$y = \zeta_2 + \varepsilon\left(\frac{\zeta_2^2 - 1}{\zeta_1}\right) + \varepsilon^2\left(\frac{\zeta_2 - \mu_{11}}{\zeta_1}\right)\left(\frac{\zeta_2^2 - 1}{\zeta_1}\right) + O(\varepsilon^3) \tag{58}$$

The function $\mu_{11}(\zeta_1,\zeta_2)$ is as yet undetermined, but it can be conveniently specified from the following observation. We see that the expansion (58) for $y \equiv Z_2$ is singular at $\zeta_1 = 0$ and that higher order terms in (58) would be more singular at $\zeta_1 = 0$ if $x \equiv Z_1$ were not strained appropriately. If we took $\mu_{11} = 0$, then Z_1 would be the same as ζ_1 to order ε, and the higher order singularity at $\zeta_1 = 0$ in equation (58) would not be acceptable. We therefore invoke Lighthill's principle in a second sense in this problem, and specify μ_{11} to make higher order terms in equation (58) no more singular than $(\zeta_2^2 - 1)/\zeta_1$ at $\zeta_1 = 0$. Thus, we may let

$$\frac{\zeta_2 - \mu_{11}}{\zeta_1} = \alpha = \text{constant in equation (58)}$$

or

$$\mu_{11}(\zeta_1,\zeta_2) = \zeta_2 - \alpha\zeta_1 \tag{59}$$

and obtain the uniformized second-order solution:

$$f(x,y,\varepsilon) = \frac{1}{\zeta_2} \tag{60a}$$

$$x = \zeta_1 + \varepsilon(\zeta_2 - \alpha\zeta_1) + O(\varepsilon^2) \tag{60b}$$

$$y = \zeta_2 + \varepsilon\left(\frac{\zeta_2^2 - 1}{\zeta_1}\right) + O(\varepsilon^2) \tag{60c}$$

For the solution to be real very near the origin α must be less than zero; so we take

$$\alpha = \alpha_1 = -1 \tag{61}$$

Equations (60b) and (60c) are easily combined to obtain ζ_2 as a function of x, y, and ε, so the uniform second-order solution for f is obtained directly as[3]

[3]The appropriate algebraic sign on the radical in equation (62) or in (65) must be chosen, but this detail is considered incidental to the illustration, as it is not of concern in equations (60) or (63). For both $x \geq 0$ and $y \geq 0$, the + sign is used.

$$f(x,y,\varepsilon) \approx \left\{\frac{x + \varepsilon y}{-2\varepsilon^2} \pm \frac{1}{2\varepsilon^2} \left[(x + \varepsilon y)^2 + 4\varepsilon^2(xy + \varepsilon + \varepsilon^2)\right]^{1/2}\right\}^{-1} \tag{62}$$

If the above procedure is continued to order ε^2 (leaving α as an arbitrary constant in equation (59)), the uniformized third-order solution is obtained:

$$f(x,y,\varepsilon) = \frac{1}{\zeta_2} \tag{63a}$$

$$x = \zeta_1 + \varepsilon(\zeta_2 - \alpha\zeta_1) + \varepsilon^2\alpha^2\zeta_1 + O(\varepsilon^3) \tag{63b}$$

$$y = \zeta_2 + \varepsilon\left(\frac{\zeta_2^2 - 1}{\zeta_1}\right) + \varepsilon^2\alpha\left(\frac{\zeta_2^2 - 1}{\zeta_1}\right) + O(\varepsilon^3) \tag{63c}$$

To this order, for the solution to be real very near the origin, α must be greater than zero; so we take

$$\alpha = \alpha_2 = 1 \tag{64}$$

Equations (63b) and (63c) are easily combined to obtain ζ_2, so that the uniform third-order solution is expressed explicitly as[4]

$$f \approx \left\{\frac{x + \varepsilon y}{-2\varepsilon^4} \pm \frac{1}{2\varepsilon^4} \left[(x + \varepsilon y)^2 + 4\varepsilon^4(xy + \varepsilon + \varepsilon^4)\right]^{1/2}\right\}^{-1} \tag{65}$$

For comparison, it is easily verified that the exact solution of the problem (47) is

$$f(x,y,\varepsilon) = \frac{x + \varepsilon y}{xy + \varepsilon} \tag{66}$$

and that the approximate solutions (60) or (62), and (63) or (65), approach (66) uniformly as $\varepsilon \to 0$ for all $|x|$ and $|y| < \infty$.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., August 12, 1970

[4]The appropriate algebraic sign on the radical in (62) or in (65) must be chosen, but this detail is considered incidental to the illustration, as it is not of concern in equations (60) or (63). For both $x \geq 0$ and $y \geq 0$, the + sign is used.

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