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MAPPING BY COMPUTER GRAPHICS: SATELLITE ANTENNA COVERAGE

N. C. Ostrander

A Report prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Rand
SANTA MONICA, CA. 90406

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PREFACE

The potential of directive satellite antennas has been investigated as part of Rand's studies of advanced communication satellite technology for the National Aeronautics and Space Administration. An important facet of these studies is the assessment of earth coverage attainable with a given satellite, antenna characteristics, and pointing direction. The coverage of geographic and political areas is often displayed and examined by superposing contours of marginal radiation intensity upon maps. The display of coverage patterns is a recurring problem generally treated by constructing overlays conforming to existing Mercator or cylindrically projected maps. On these maps, a unique overlay is required for almost all antenna pointing directions.

An alternative to the construction of a large number of overlays is to construct a map projection (polar perspective) upon which a single overlay (representing a beam of given angular cross section) is valid over the entire map. The projection is virtually a picture of the earth as seen from the satellite, and each contemplated satellite position requires the preparation of a corresponding map. A number of such maps were constructed and proved to be highly useful in preparing the illustrative example of a worldwide television system which was used in R-524-NASA, *Television From Satellites: New Possibilities For Worldwide Use*.

Given an adequate data bank, it is obviously possible to construct many kinds of maps, and indeed other maps have been generated to supply the background upon which intersectional contours have been plotted. The use of a computer in generating special-purpose maps or in providing background upon which to display geometrically definable contours seems extremely attractive. With illustrative examples, this report describes some of the more common map types generated from a data tape of 10,000 points.

The report should be of interest to those dealing with study and display of geometric problems related to geography.

SUMMARY

The problem of displaying geometrically definable contours upon geographic maps arises in many contexts. If the parameters defining the contours are to be varied to achieve some desired relation between the contours and the geography, and if lack of symmetry precludes the use of general-purpose overlays, then computer graphics become very attractive. The computer with graphic output capability can be used to construct overlays for use with existing maps, or if provided with map data, can draw contours upon a map. In the second case, the choice of scale and map projection is at the user's discretion. Sometimes it is possible to construct a special map projection possessing the features essential to the use of general-purpose overlays.

The contours to be mapped are usually definable as the intersection of some geometric surface with the earth's surface. It is assumed here that the computation provides the coordinates of intersectional points to be mapped and the process of mapping these points is identical to that of mapping earth geographic points. Thus, in the preparation of overlays, the mapping transformation must be the same as that used in preparing the map upon which the overlay is to be used.

Overlays are useful when the problem can be treated with a small number of overlays; for some problems it may be economical to exercise the required care in handling the scaling and registration problems and to accept the restrictions imposed by existing map scales and projections. Such cases are probably not numerous, and the alternative of generating the map by computer graphics will generally be favored. Although the focus of this report is upon the generation of maps from a data bank of points, the section dealing with map transformations may be useful to those concerned with overlay construction.

A computer-generated map is produced by connecting sequences of points with line segments. A string of points may represent an island, lake, political boundary, or a portion of continental coastline. A particular mapping or map projection is created by individually transforming the coordinates of the stored points to a cartesian coordinate equivalent to the desired mapping and connecting the points sequentially

with straight line segments. The principal questions which arise in map generation are the number of points required and the form in which coordinates are stored.

A stored map of about 10,000 points appears to be adequate to produce maps of scale 1:40,000,000. At this scale, a world map with rectilinear grids is about 40 in. wide. At greater enlargement the geometric substructure of straight line segments becomes evident but may not preclude the use of such maps for some problems. At scales smaller than 1:80,000,000, some points tend to coalesce and give a darkening effect detrimental to the appearance of the map. The effect is especially pronounced on those regions of some maps where the scale becomes very small. Rejection of some closely spaced points can be employed to improve the appearance of small scale maps or map regions.

A brief discussion of geographic and geocentric coordinates is presented to introduce the subject of coordinate forms (angular versus unit vector representation) and to indicate some limits in the validity of the spherical earth approximation with respect to mapping.

Several map types are illustrated: rectilinear, azimuthal equidistant, and polar perspective. The corresponding coordinate transformations are described. The transformation equations depend upon the form of the stored coordinates; with rectilinear maps the most efficient coordinate form is angular (latitude and longitude), whereas unit vectors (direction cosines) are favored for azimuthal type maps.

The efficiency of coordinates is partially in simplifying and speeding the mapping transformations but is more clearly evident when mapping limited regions. With limited regions, efficiency implies rapid identification of those points which fall within the domain of interest or rapid elimination of points exterior to the domain of interest. With rectilinear maps, the domain of interest is defined by latitude and longitude extremes and point coordinates defined by latitude and longitude can be rapidly sorted upon the bounding extremes. With azimuthal maps, the domain of interest is defined by the cosine of a polar angle. For each point, the cosine of the polar angle (relative to an arbitrary origin) can be rapidly calculated as the scalar product of the unit vector to the origin and the unit vector to the point. By comparing that cosine with the limiting cosine, the point can be rapidly accepted or rejected.

The efficiency of sorting points can be improved by limiting the interval (latitude-longitude or polar angle) spanned by a data string. If the maximum interval spanned by a string is ℓ, then only those strings with the first point lying within $\pm\ell$ of a boundary require point-by-point testing; other strings are either totally within the boundaries or totally exterior to the boundaries.

Intersectional contours can often be calculated approximately by using a spherical model of the earth's surface. However, if ray directions are specified at a point in space, the precise values of intersectional coordinates will be sensitive to the assumed shape of the earth. For rays originating at a space point several earth radii from the center of the earth, coordinate errors of hundreds of miles arise in substituting a sperical earth model for an oblate earth model. The large errors occur with rays which are nearly tangent to the earth. When rays pass over the horizon, the intersectional contour defined by intersectional points ceases to exist, but a second contour is introduced which defines the earth point locations at which the space point is just on the horizon. This horizon plane and the local vertical are fundamental to the geographic coordinate system and for a space point at synchronous distance, an assumed identity of geocentric coordinates on a sphere and geographic coordinates upon the oblate earth introduces error of no more than 8 n mi.

ACKNOWLEDGMENTS

The author gratefully acknowledges the helpful assistance and sage advice of Rand colleagues Worthie Doyle, Carroll Lindholm, and John Rieber in constructing efficient programs for the generation of maps.

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I. INTRODUCTION

Those features of the earth's surface which can be represented by contours (e.g., coastlines, political boundaries, and lines of constant elevation) can be represented approximately by points connected by line segments. By storing sequences of points (say latitude and longitude coordinates) upon tape or disk, a great variety of mappings or projections can be computer-generated to provide work aids or illustrations for particular geographic problems. The stored map is a data bank which provides geographic coordinates of earth surface points. This report illustrates the use of a small data bank (10,500 points) stored on magnetic tape for constructing several kinds of maps.

Computer-generated maps can be useful in providing a background upon which various curves are plotted. The curves might define satellite or missile ground tracks, the region within which a geostationary satellite is at least n degrees about the horizon, or the swath of earth visible to a high-flying aircraft.

If the parameters describing the curve are fixed and known in advance, then direct plotting of the curve upon existing maps is likely to be more efficient than using a computer-generated map. However, if the parameters describing the curve are to be studied in order to achieve some desired relation between the curves and the existing geography, then trial-and-error plotting might be abandoned in favor of superposing movable overlays on the map. Overlays are useful in many problems, but fail if there is insufficient inherent symmetry to permit the construction of general-purpose (movable) overlays. If general-purpose overlays cannot be constructed, the computer-generated maps become very attractive.

There are several ways of using a computer to treat mapping problems. In a direct and obvious way, the computer generates overlays conforming to existing map scales and need not necessarily generate the geographic background data. This scheme can be useful if scaling and registration are treated carefully (i.e., if the computed overlays are distortion-free relative to the map on which they are used). Several advantages appear if the background geography is computer-generated; the scaling, registration, and relative distortion problems vanish, and the computer-generated picture is not limited to the scale choices or projections dictated by available maps.

For some problems, special-purpose maps may present the desired information in directly scalable form. An example arises in plotting great circle routes from some point of origin. If, for instance, it is desired to plot great circle routes from Seattle, Washington, to a number of earth points, then these routes can obviously be reduced to sequences of points and plotted on any map of the world. Alternatively, an azimuthal projection of the world with Seattle at the pole directly displays great circle routes to every other earth point.

A less obvious example of the use of special-purpose maps occurs in studies of the area covered by narrow antenna beams directed from satellites toward specific earth points. The antenna beams are idealized to cones of circular or elliptical cross section, with the cone dimensions reflecting some critical or marginal intensity of radiation. The intersection of these cones with the surface of the earth is a space curve which can be plotted on a variety of maps; however, most such maps introduce distortions peculiar to the map itself. However, by creating a map or picture of the earth as seen from the satellite, the intersections retain their geometric simplicity (circles or ellipses) and the characteristic dimensions are approximately valid for all pointing directions. Maps or pictures of this kind are termed polar perspective projections.

In the previous example and in many others, geometric contours on the earth's surface result from delineating those earth points lying within some solid angle defined at a point in space. A related, but somewhat different, problem delineates the region of the earth's surface from which a space point can be viewed under some limitations of elevation and possibly azimuth. In either case, the problem is reducible to the calculation of intersecting rays with the surface of the earth. The rays represent lines of sight, and successive intersectional points may be connected to construct the desired contour.

For rays defined by a solid angle at a space point, intersections with the earth's surface do not necessarily exist for all rays within the solid angle. The intersectional coordinates are sensitive to the assumed shape of the earth, and large errors can arise in substituting a spherical model for an oblate earth model. For rays which originate at the earth's surface with directions specified in a frame defined by the local vertical, the error introduced by choosing a spherical earth model is small, approaching zero as the distance to the space point becomes large.

The illustrative map-pictures that appear in the text were produced with an IBM-360/65 computer coupled to a Stromberg Datagraphix, Inc., S-C 4060 graphic output device. A data tape of 8300 geographic points was supplied through the courtesy of the UAIDE library operated by Stromberg Datagraphix, Inc. These data were supplemented with 2200 political boundary points read from maps (AMS stock numbers 1125x1, 1125x2, and 1125x3).

II. POINT DENSITY AND NUMBER OF POINTS

The map illustrations shown in this report are based upon a data bank of about 10,500 points defined by geographic coordinates. Point coordinates are stored in a string of alternate latitude and longitude values. The end of a string is indicated by (0,0), which signals the computer that a contour (island, lake, political boundary, etc.) is completed. The data set contains 374 such strings. The points within a string have nearest-neighbor spacings not less than 5 n mi. Maximum spacings of 150 n mi occur infrequently and only along geometrically defined political boundaries.

Figure 1 shows a map based upon the 9475 points (exclusive of zero signals) which lie between -75° and $+75^{\circ}$ latitude. The line width (0.0065 in.) defines a resolution unit; two points spaced closer than a resolution unit are not distinguishable from each other. Although the eye at normal viewing distances is capable of resolving small points at somewhat smaller spacing (0.004 in.), the resolution unit defined by line width is more appropriate to the present discussion. The scale of Fig. 1 is approximately 1:124,000,000 along the equator and at this scale the resolution unit of 0.0065 in. transforms to approximately 11 n mi. Evidently the minimum point spacing of 5 n mi is not resolvable at this scale.

Figure 2 shows the same map constructed by rejecting points which lie within six resolution units (0.039 in.) of both nearest neighbors. The rejection rule reduced the number of points used from 9475 to 4372. Small islands are retained by overriding the rejection rule for the first three points of a string. Comparison of Figs. 1 and 2 indicates that there is some loss of detail -- the available point density is perhaps a factor of two greater than necessary to draw maps at this scale.

Points are treated sequentially along a string and rejection rules can be constructed based upon nearest neighbors, next nearest neighbors, and so forth. However, adjacent strings and "goosenecks" within strings can create small spacings of a nonsequential nature. This problem is aggravated in some perspective mappings which view regions at low

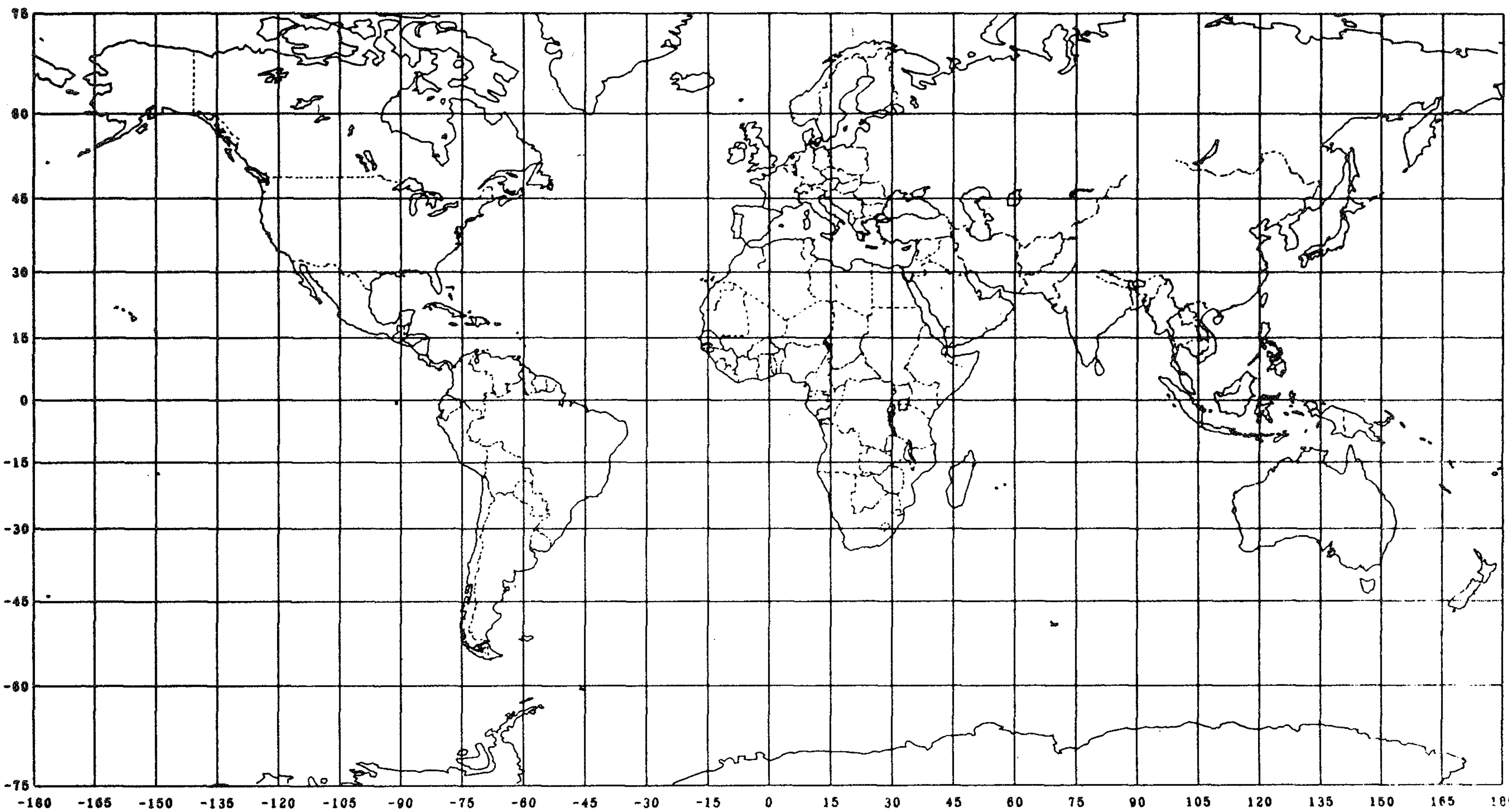

Fig. 1 - World map based upon 9475 points

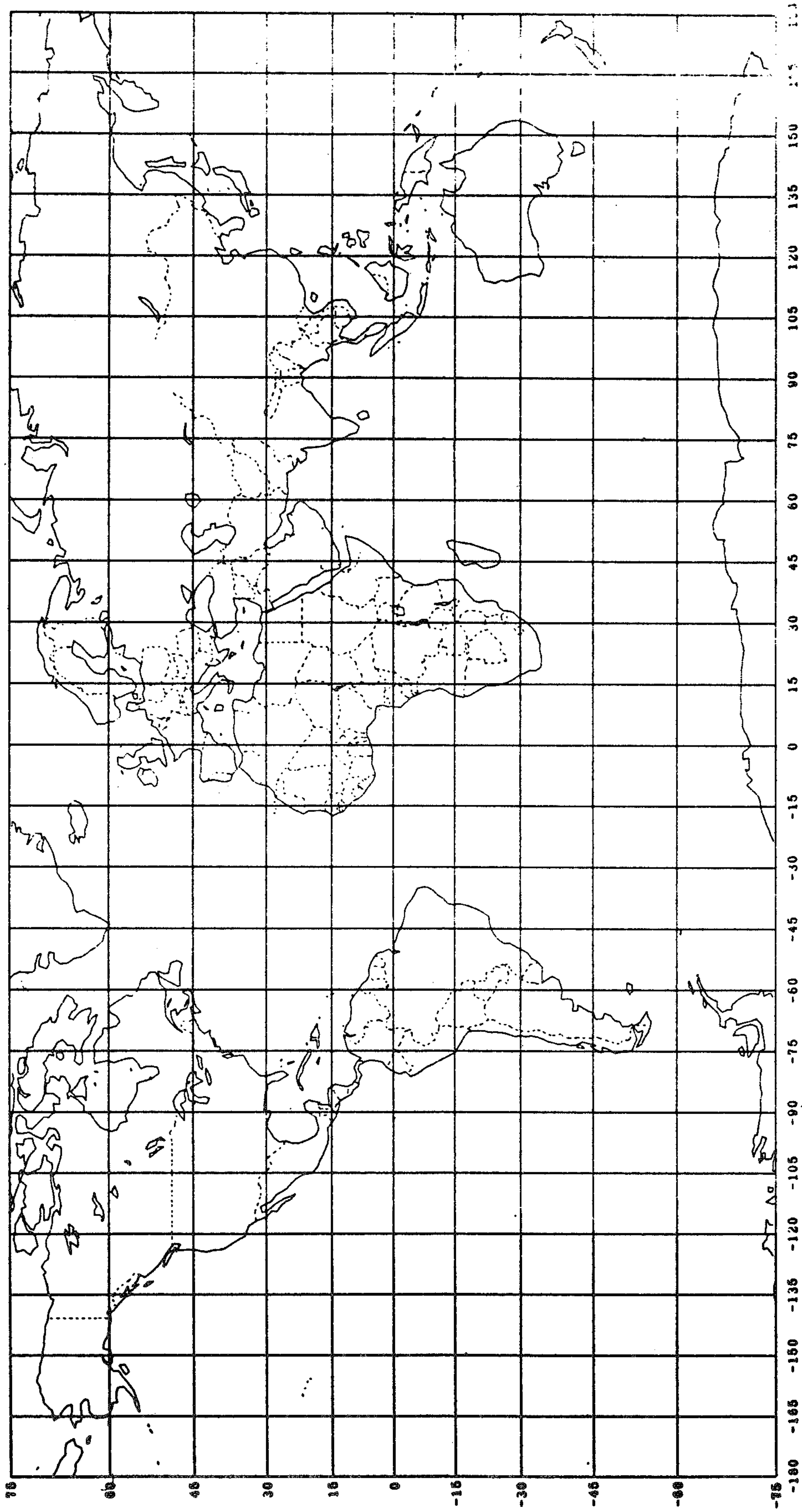

Fig. 2 - World map based upon 4372 points

incidence angles and consequently decrease the apparent distances between points. An example of this phenomenon is shown in the following section in connection with the polar perspective projection.

The foregoing concerns problems arising from an over-density of points. At the other end of the point density spectrum is the question of the minimum useful number of points. Figures 3 and 4 represent successive enlargements of an area centered on 15°E and 45°N. In Fig. 3 the scale is approximately 1:40,000,000 (at 45° latitude) and the resolution element defined by a line width is about 4 mi. The point density is evidently sufficient to draw a good map at this scale. In Fig. 4 (scale $\sim$ 1:20,000,000) the resolution element is decreased to 2 mi and some geographic features appear schematized. At this scale a higher point density could provide a better appearance but the general area is recognizable and the map detail may be adequate for some purposes.

The examples and discussion above indicate that maps of good appearance require point spacing which sometimes approaches the resolution limits of the plotting equipment and the human eye. Over-densities by a factor of two are probably acceptable. If the minimum spacing falls to one-third of the resolution element, then rejection of closely spaced points can improve map appearance. Loss of detail is apparent where numerous point spacings exceed about two resolution elements, but useful maps of somewhat schematic appearance may be produced.

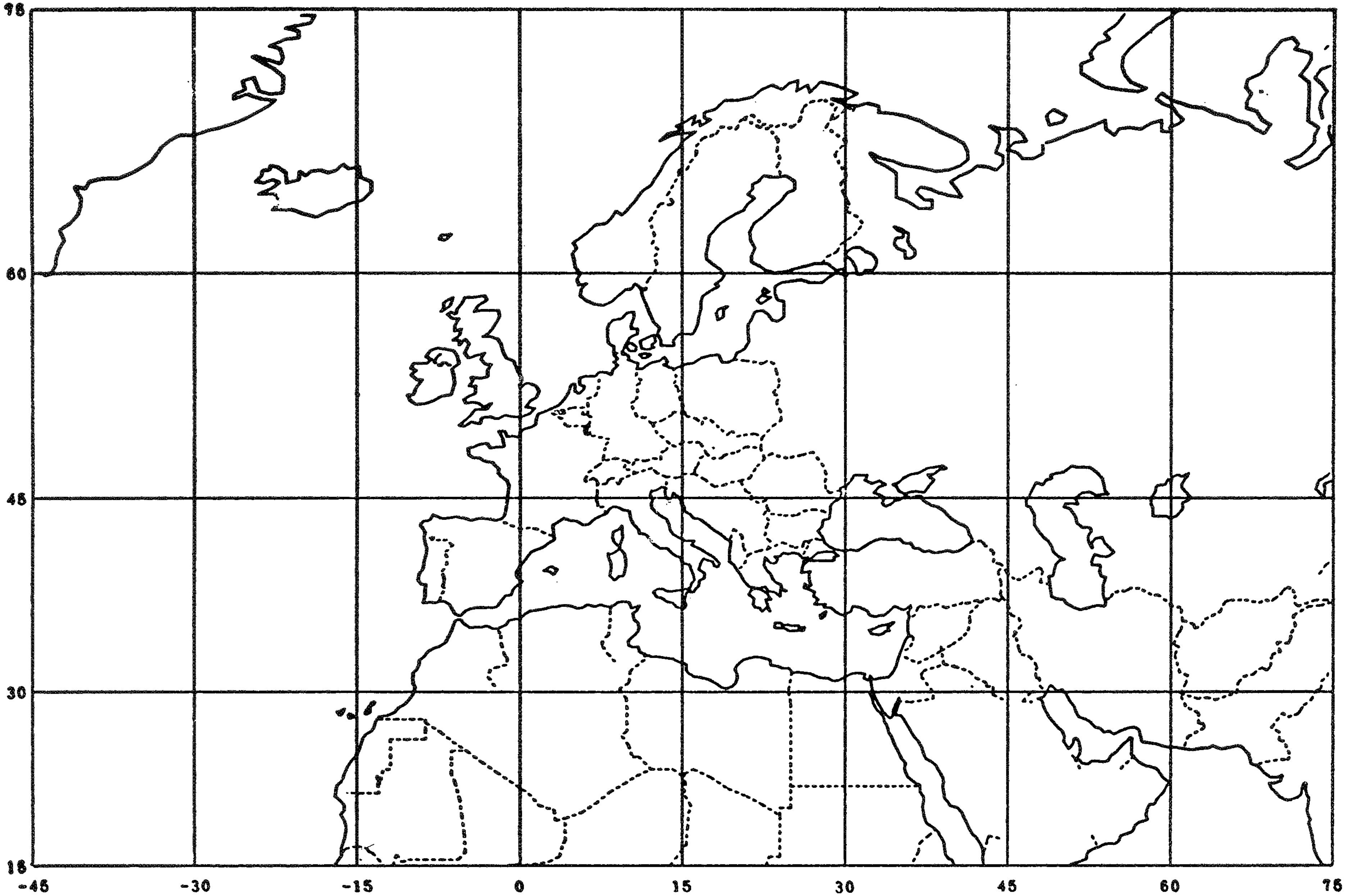

Fig. 3—Europe and vicinity at 1:40,000,000 scale

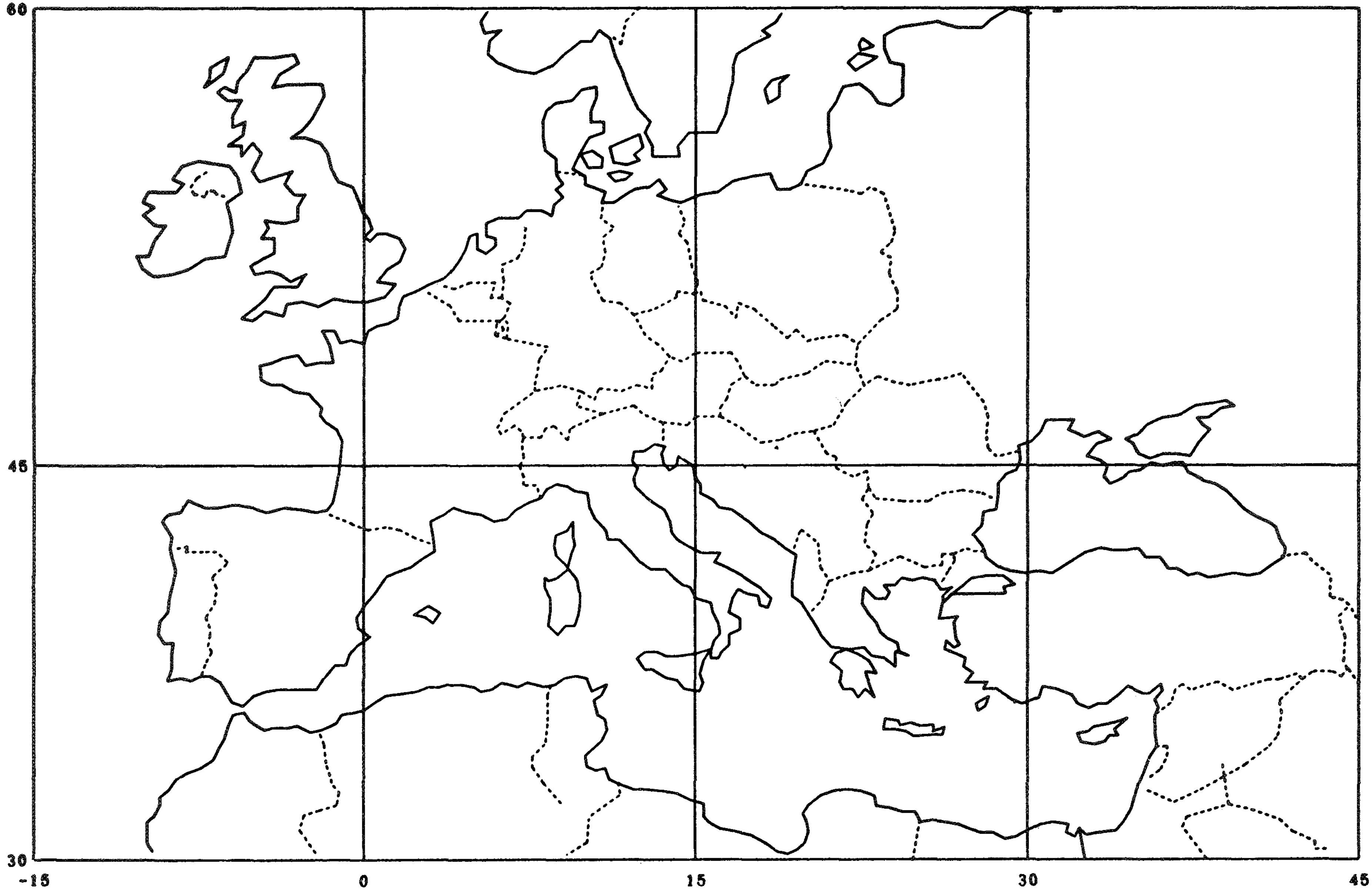

Fig. 4—Europe at 1:20,000,000 scale

III. COORDINATES AND TRANSFORMATIONS

GEOGRAPHIC AND GEOCENTRIC COORDINATES

An observer at earth point P can determine his latitude by measuring the elevations of stars relative to his local horizon plane. By plotting his position on a map constructed with grid lines of geographic latitude, he identifies his position relative to other points similarly defined by observation. The geographic latitude ϕ defined in this way is distinct from geocentric latitude ϕ' except on the equator and at the poles (see Fig. 5). The direct connection between observation and geographic latitude is useful to navigators and surveyors and geographic coordinates are used on all common maps. The geographic coordinates are sometimes implicit as, for example, in a computer-stored map providing the direction cosines of the unit vector normal to the reference ellipsoid ($\vec{n}$ of Fig. 5).

In the construction of maps, the distinction between geocentric and geographic coordinates can often be ignored. In particular, an assumed identity of geocentric and geographic latitude yields maximum error distances of about 11.6 n mi (at 45° latitude), an error that is only marginally discernible when the map scale is smaller than about 1:120,000,000.

For all problems in which mapping transformations are applied to both the points and the grids, the resultant map will show points correctly spaced relative to the grids. Thus if a mapping transformation is derived by projective geometry using a spherical earth model, the transformation may be reduced to an algebraic equation and applied to geographic coordinates which had no meaning in the derivation of the transformation. The points are still correctly placed relative to the transformed coordinate lines, and the geographic latitude of points can be accurately located with respect to the transformed latitude scale.

Earth oblateness and the related distinction between geocentric and geographic coordinates is primarily significant when calculating contours or points defined geometrically by the intersection of surfaces or rays with the earth's surface. The generation of such intersectional contours is discussed in Section IV. The assumed shape of

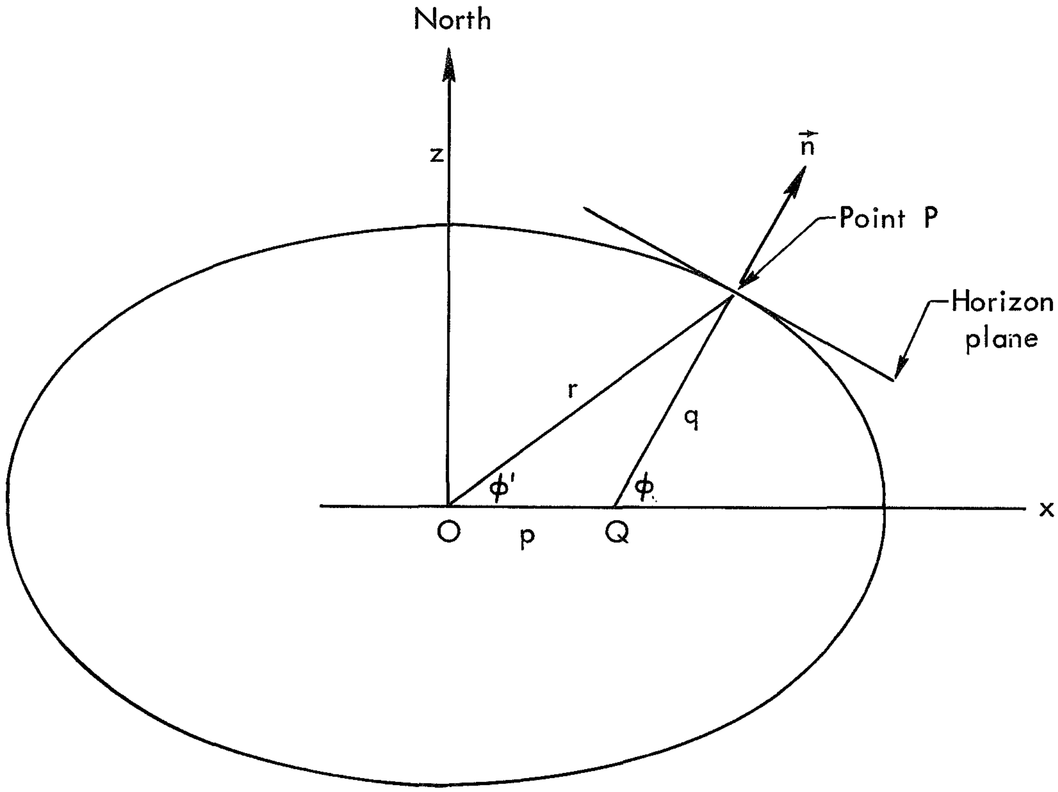

Fig. 5—Geographic latitude ϕ and geocentric latitude ϕ'

the earth (spherical or oblate) can be a significant factor in defining the intersectional coordinates. The purpose of the present discussion is to point out that geographic coordinates are the natural coordinates of maps and that intersectional contours should be similarly defined to be compatible with conventional mappings.

COORDINATE FORMS

The precise form of a mapping transformation is determined partially by the form of the input coordinates. The two most important general forms of input coordinates are (1) angular coordinates (latitude and longitude) and (2) unit vectors (the three-direction cosines of the point to be mapped). In describing mapping transformations it is convenient to assume the most natural form of input coordinates for the desired mapping. In the notation to be used here, the direction cosines are related to longitude λ and latitude ϕ by:

$$\begin{aligned} u_x &= \cos\phi \cos\lambda \\ u_y &= \cos\phi \sin\lambda \\ u_z &= \sin\phi \end{aligned} \tag{1}$$

The inverse relations are:

$$\begin{aligned} \lambda &= \tan^{-1} \frac{u_y}{u_x} \\ \phi &= \tan^{-1} \frac{u_z}{\sqrt{1 - u_z^2}} \end{aligned} \tag{2}$$

These substitutions can be made in all of the following mapping transformations to describe the transformation in terms of the alternative input coordinates. It will usually be found, however, that the natural choices provide simpler and more rapid computation, and it may be economical to provide data banks of both types.

For some applications it may be desirable to store specialized coordinates. If, for example, it is desired to produce a large number of maps all having the same general transformation law but differing in scale and domain of interest, then it may be possible to introduce coordinates proportional to the map distances and avoid the repetitious computation of the entire mapping transformation. The possible economy of the precomputation of linearly scalable coordinates is self-evident but dependent upon expected output volume. The following description is limited to general situations which can be specialized at the discretion of the user.

MAPS WITH RECTILINEAR GRIDS

Maps with rectilinear coordinates can be generated by direct plotting of latitude versus longitude, by conformal Mercator mapping, by cylindrical projection, and by somewhat arbitrary rules of vertical scale nonlinearity. The direct plotting of latitude versus longitude is convenient for interpolation but is less commonly used than maps intended to preserve (not necessarily achieving) local scaling. The conformal Mercator map does preserve local scale and direction by expanding the latitude interval to compensate for the parallel presentation of meridians. The Mercator transformation yielding abcissa x and ordinate y from longitude λ and latitude ϕ is

Mercator map

$$\begin{aligned} x &= K\lambda \\ y &= \frac{K}{2} \ln \frac{1 + \sin \phi}{1 - \sin \phi} \end{aligned} \tag{3}$$

with ϕ small, $y \to K\phi$, and it is convenient to regard K as a factor converting radians to map distances (inches or centimeters).

Mercator maps and maps derived by cylindrical projections are very common and present the land masses in a familiar but distorted appearance.

Unless the conformal property of the Mercator map is specifically required, the cylindrical projection can provide a familiar appearance and avoid the awkwardness of the Mercator mapping at extreme latitudes. A class of cylindrical projections can be constructed by projecting points upon a tangent cylinder as indicated in Fig. 6. The projection point P lies in the plane defined by the tangent circle and also lies in the meridian plane of the point being projected. The tangent circle is the equator in familiar maps, but oblique projections could be used to portray, for instance, the swath of earth visible on a great circle airline route. For the cylinder tangent at the equator, the mapping transformation is

Cylindrical Projection

$$x = K\lambda$$
$$y = \frac{K(1 + k) \sin \phi}{k + \cos \phi} \tag{4}$$

for ϕ small, $y \to K\phi$ at all k.

The properties of the cylindrical projection map depend upon the value of the parameter k. If $k \to \infty$, latitude lines are parallel projected onto the cylinder and the map is called an equal-area projection. Areas are preserved but at the sacrifice of large distortions in height/width ratios. The case $k = 0$ gives more distortion upon approaching either pole than does the Mercator map and is a seldom-used projection. The choice $k \simeq 0.8$ gives a map of good proportions when mapping the world to latitudes approaching (or reaching) the poles.

Maps of the Army Map Service, which are designated as Series 1107, are cylindrical projections available at several scales. These maps are useful in conjunction with overlays. The axes of the small scale sheet are well described by Eq. (4) with $k = 0.77$. The same map at a scale of 1:40,000,000 shows discrepancies of one or two millimeters in the vertical axis and the fit can be improved by using

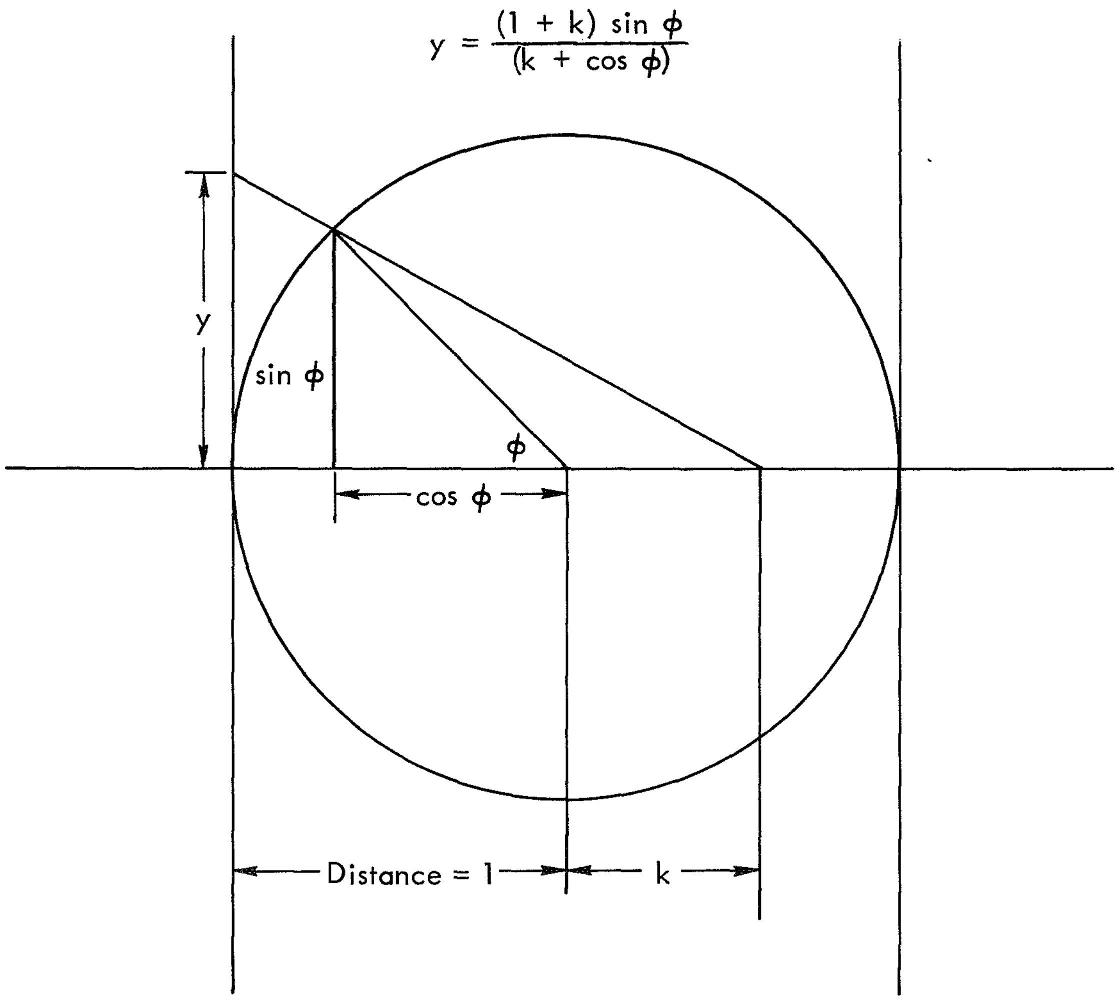

Fig. 6—Geometry of the cylindrical projection (spherical earth)

$$y = \frac{K(1 + k)(1 - 2f) \sin \phi}{k(1 - f \sin^2 \phi) + \cos \phi} \tag{5}$$

The added refinement which arises because the map is projected with an oblate rather than a spherical model reduces the scale errors to a few tenths of a millimeter. At this error level some scale inconsistencies can be observed by comparing vertical scales on the left and right sides of the map or in the positive-negative latitude regions. These discrepancies, possibly due to dimensional changes of the paper, impose a limit upon the accuracy of overlays. The problem is aggravated by the fact that overlays are usually constructed on a backing different from the map paper stock and long-term changes produce differential errors between map and overlay.

The problem of paper dimensional stability and precise representation of a somewhat arbitrary projection are unimportant if map points, grids, and overlay contours are generated by the same transformation rule upon a computer. If maps with rectilinear grids are to be used only for background with curves or contours superposed according to the *same* latitude transformation law, ease of computation and familiar appearance may be the dominant factors determining the latitude transformation. The qualitative characteristic of familiar appearance can be achieved with a simple quadratic representation of the vertical scale expansion.

Quadratic Scale Factor

$$\begin{aligned} x &= K\lambda \\ y &= K\phi(1 + a\phi^2) \end{aligned} \tag{6}$$

If a is chosen as ~ 0.15, the latitude dimension of the map over the interval 75°S to 75°N will be nearly the same as that of a cylindrical projection with scale factor K and parameter $k = 0.8$. Figures 7 and 8 illustrate a side-by-side comparison of the western hemisphere mapped according to the cylindrical projection ($k = 0.8$) and the quadratic

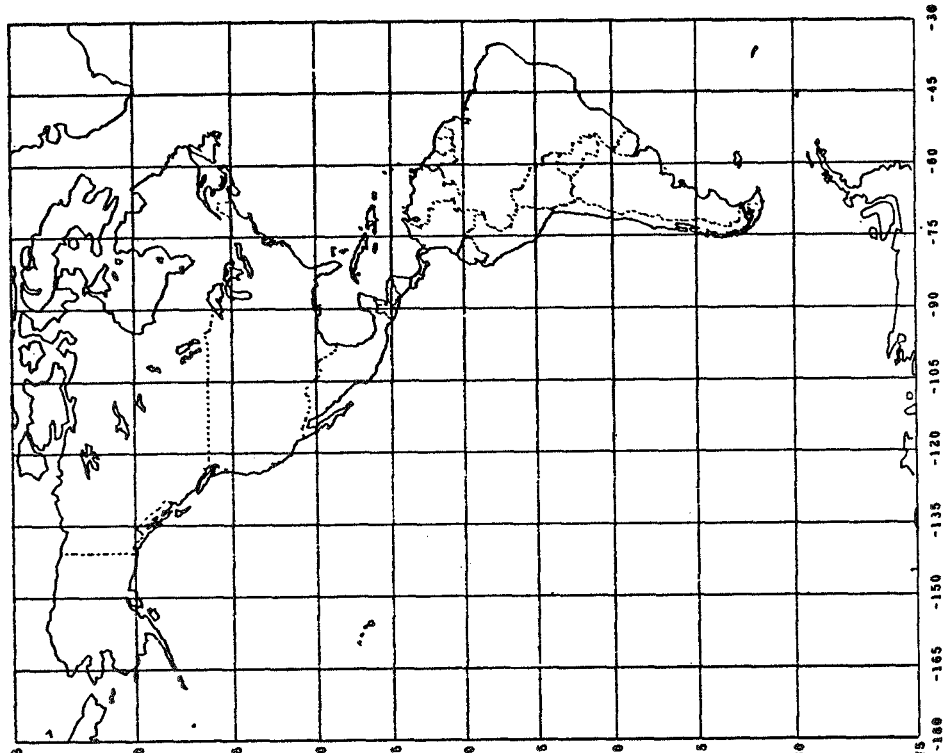

Fig. 7—Western hemisphere, cylindrical projection

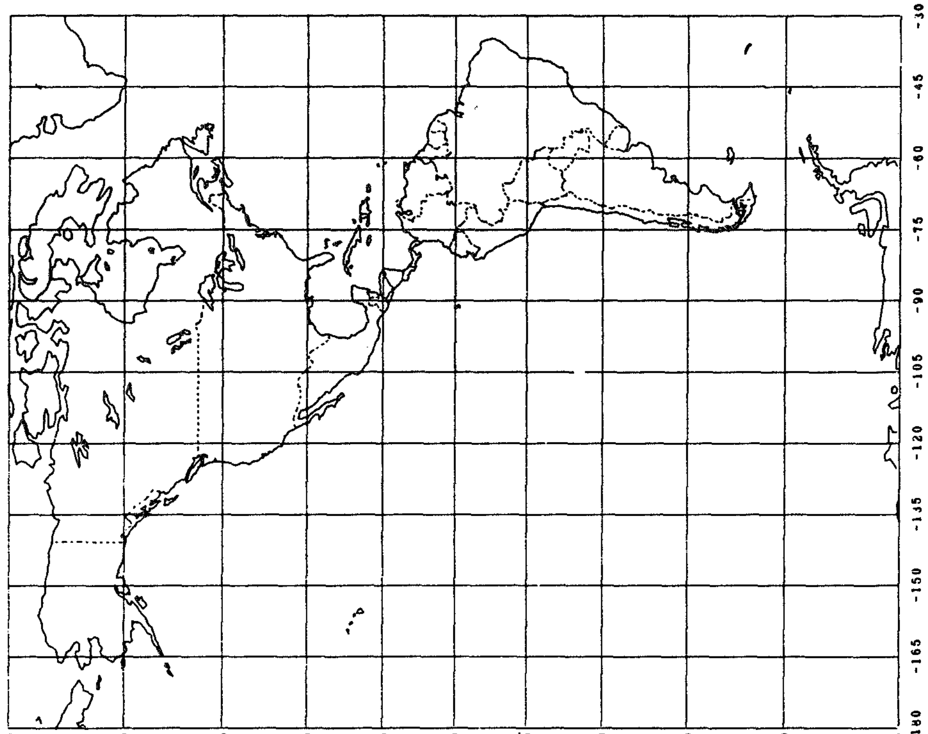

Fig. 8—Western hemisphere, quadratically expanded latitude scale

expansion ($a = 0.15$). All of the other rectilinear map illustrations of this report are based upon the quadratic expansion with $a = 0.15$.

Azimuthal Maps

With azimuthal maps, all great circles through the origin are straight (radial) lines. Directions from the point of origin are correctly represented, but distances along the radial great circle routes are not uniformly scaled in maps derived by geometric projection. An azimuthal map with uniform radial scale can be readily derived, however, and the example given here is based on this choice.

The great circle arc distance R from the origin (ϕ_o = latitude, λ_o = longitude) to a point ϕ, λ is a fundamental quantity. For the equidistant (or equal interval) map, the arc R may be uniformly scaled. The same arc could be scaled in other ways, e.g., by rules derived from projective geometry, to present other radial dependencies.

For stored points defined by direction cosines, the direction cosines of the point of origin are fixed quantities given by:

$$
\begin{aligned}
u_{xo} &= \cos\phi_o \cos\lambda_o \\
u_{yo} &= \cos\phi_o \sin\lambda_o \\
u_{zo} &= \sin\phi_o
\end{aligned}
\tag{7}
$$

The arc R to a point defined by u_x, u_y, and u_z is

$$\cos R = u_{xo}u_x + u_{yo}u_y, u_{zo}u_z \tag{8a}$$

The arc R must satisfy $0 < r \leq \pi$ and no sign difficulties arise in taking the arc cosine of the above expression. The plane defined by the point (u_{xo}, u_{yo}, u_{zo}), (u_x, u_y, u_z), $(0, 0, 0)$ contains the arc R. Azimuth can be defined as the inclination of this plane to the meridian plane containing the point of origin, and the inclination is calculated as the angle between the normals of these planes.

The normal to the meridian plane has direction cosines proportional to:

$$
\begin{aligned}
u'_{xo} &= -u_{yo} \\
u'_{yo} &= u_{xo} \\
u'_{zo} &= 0 \\
N_o &= \sqrt{u_{xo}^2 + u_{yo}^2}
\end{aligned}
\tag{8b}
$$

The normal to the plane containing R has direction cosines proportional to:

$$
\begin{aligned}
u'_x &= u_{yo}u_z - u_{zo}u_y \\
u'_y &= u_{zo}u_x - u_{xo}u_z \\
u'_z &= u_{xo}u_y - u_{yo}u_x \\
N &= \sqrt{u_x'^2 + u_y'^2 + u_z'^2}
\end{aligned}
\tag{8c}
$$

The azimuth is given by

$$
\begin{aligned}
\cos A &= \frac{u'_{xo}u'_x + u'_{yo}u'_y}{N_o N} \\
\sin A &= \sqrt{1 - \cos^2 A}
\end{aligned}
\tag{8d}
$$

where sin A is positive if $0 < \lambda - \lambda_o \leq \pi$, and otherwise is negative.

The cartesian coordinate equivalent of the azimuthal map is easily generated by

$$
\begin{aligned}
x &= KR \sin A \\
y &= KR \cos A
\end{aligned}
\tag{8e}
$$

where R is found by Eq. (8a) and K is a scale factor.

The computer operations indicated by the foregoing steps are quite simple and fast relative to the number of trigonometric functions which would have to be computed with latitude and longitude input coordinates. Figure 9 illustrates an azimuthal, equidistant map of the world with Seattle, Washington, at the origin.

Polar Perspective Maps

Polar perspective maps are special azimuthal maps which portray the earth as seen from some point in space. The apparent angular separation between pairs of points is approximately preserved in this mapping. A projection is first made onto the interior surface of a sphere centered at the viewer's position in space (see Fig. 10). The preservation of angular directions is not exact because of the impossibility of transforming the spherical projection surface into a plane picture. However, if the observer is more than a few earth radii from the center of the earth, the pilot representation yields only small distortion.

The arc R is the same as used in the azimuthal projection, but must here be limited to $R \leq \sin^{-1}(1/s)$ where s is the observer's distance (earth radii) from the earth's center. The angle T is related to the arc R by

$$\tan T = \frac{\sin R}{s - \cos R} \tag{9}$$

Azimuths are calculated as in the azimuthal projection and radial distances are mapped proportional to the angle T. Figure 11 shows a polar perspective projection based upon an observer located over the middle of the North American continent at distance s = 6.5 earth radii from the center of the earth.

In this mapping the radial scale approaches zero near the picture perimeter and points become spaced very closely together. Figure 11 was produced by rejecting all points spaced within three resolution units of the preceding neighbor. No points were plotted on the boundary circle or upon the circle one resolution unit interior to the boundary.

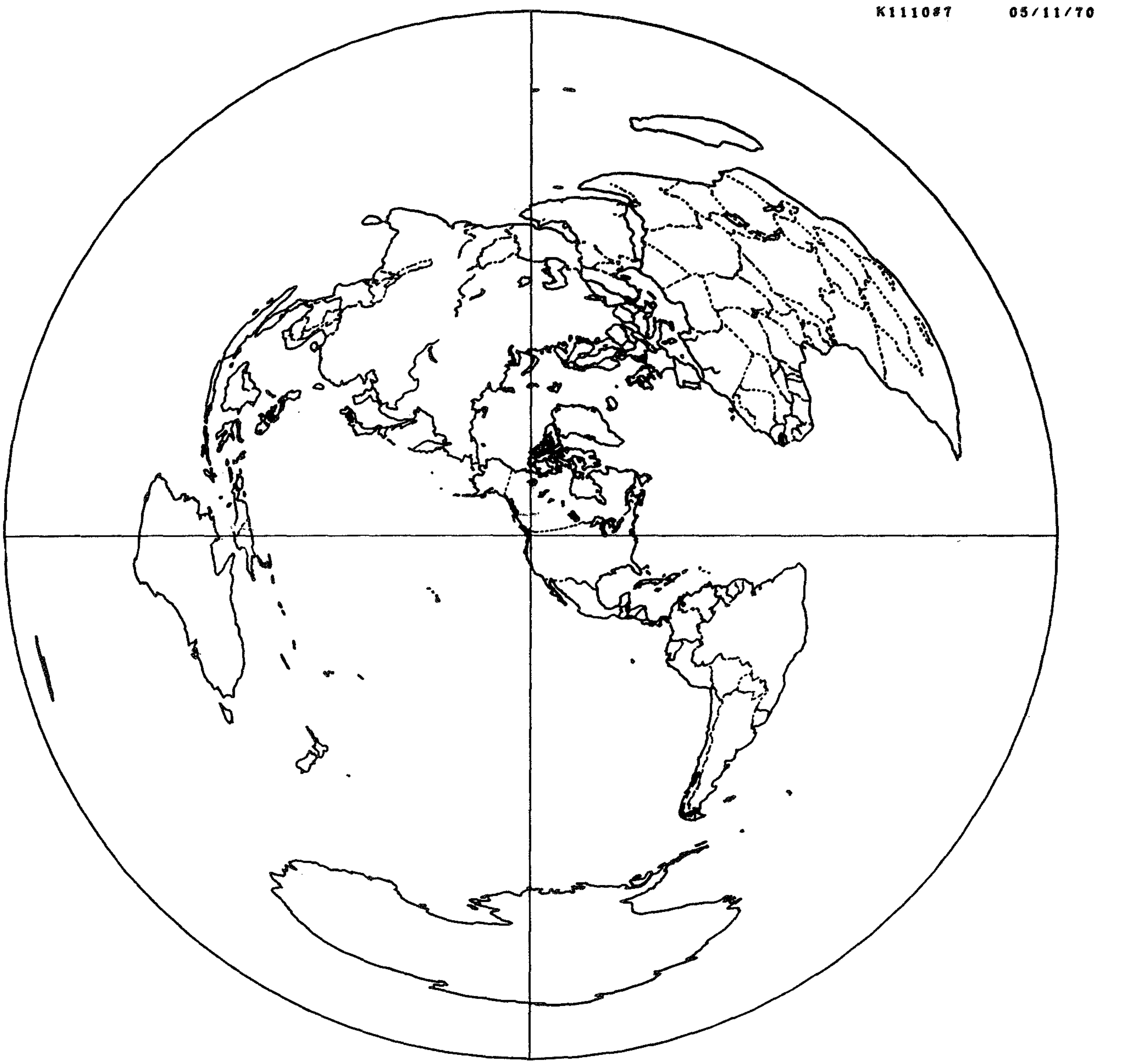

Fig. 9 - Azimuthal, equidistant projection centered at Seattle, Washington

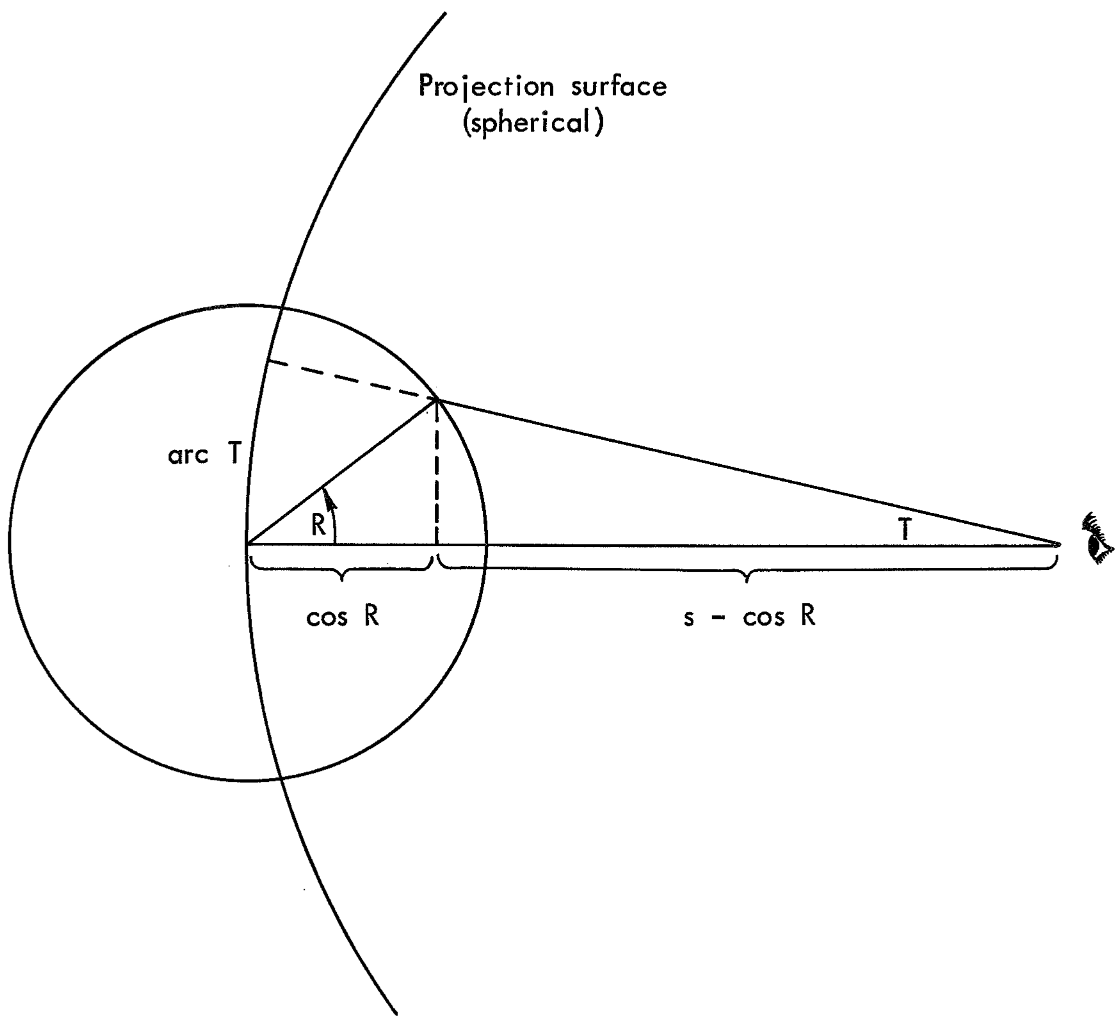

Fig. 10—The polar perspective (geometry)

Fig. 11—Polar perspective mapping over North America

MAPS OF LIMITED REGIONS: COORDINATE STORAGE CONSIDERATIONS

When mapping a limited region with a data bank of world points, the data set may be considered as consisting of two subsets: (1) the subset of points interior to the region of interest, and (2) the complementary subset of exterior points. For large data banks it is desirable to provide either rapid access to interior points or rapid rejection of exterior points. The limited region maps (Figs. 3, 4, and 11) were produced with a point-by-point test applied to every point of the data set.

Earlier reference was made to the use of natural coordinates--latitude and longitude for rectilinear maps, and direction cosines for azimuthal maps. Although the choice of the natural coordinates will be efficient in generating world maps, the choice of appropriate coordinates is more important when mapping limited regions. For rectilinear maps, the desired boundaries will be described by latitude and longitude limits and exterior points can be rapidly rejected in a series of (at most) four comparisons if the points are described by latitude and longitude. If the coordinates were described by direction cosines, the transformation of Eqs. (2) would have to be applied prior to the comparison tests. Computation of the transformation consumes more time than does the cylindrical mapping transformation.

For azimuthal maps, the domain of interest is a circular region of polar angle R centered at an arbitrary origin (the pole). Points are rejected if cos R is less than a specified value, cos R_{max}. When coordinates are given by direction cosines, cos R is formed by three multiplications and two additions. The subsequent comparison is very rapid. If the points are specified by latitude and longitude, then cos R is given by the form:

$$\cos R = a \sin \phi + b \cos \phi \cos (\lambda - \lambda_o) \tag{10}$$

where a and b are the fixed values sin ϕ_o, cos ϕ_o of the origin. The computation of cos R requires the computation of a sine and two cosines, a process about 12 times longer on the IBM 360/65 than the three multiplications and two additions required by the unit vector description.

Evidently point-by-point rejection could be replaced by block rejection. The construction of regional blocks of points involves a compromise between the desirability of minimizing the overlap between adjacent regional blocks and the undesirability of many stops and starts associated with boundary crossings. It will probably be easier and perhaps as efficient to limit strings to some maximum span (latitude-longitude intervals for rectilinear maps, polar angle interval for direction cosine data). If the first point of a string falls within the span distance of a boundary, then every point of the string must be tested, but strings remote from the boundary by more than the maximum span can be identified as either totally outside or totally inside the boundary.

IV. GEOMETRIC CONTOURS ON THE EARTH'S SURFACE

The geometric contours described here are construed as the intersection of a cone with the surface of the earth. The elements (generators) of the geometric surface are straight lines or rays, and the contours can be developed by connecting the points at which the rays intersect the surface of the earth. The coordinates of the intersectional points can be drawn on a map in the same fashion as are the mapping points already discussed. It is convenient to first describe the intersections using a spherical model of the earth's surface. These approximate geocentric coordinates are rather easily found and provide the necessary data for the computations required to obtain geographic coordinates upon the oblate earth.

INTERSECTIONS WITH THE SPHERICAL EARTH

The basic geometry which describes either the polar perspective mapping or the problem of intersections is illustrated in Fig. 12. Point P is at λ relative to the longitude of S and is at latitude ϕ. In this relative longitude system, the longitude of S is zero, the latitude is δ, and the geocentric distance of S is s. The plane SPO defined by S, P, and the center of the earth intersects the earth in a great circle arc, $R = \widehat{QP}$. Plane SPO forms dihedral angle A with the meridian plane of S and Z with the meridian plane of P. The spherical triangle QPN defined by the meridian planes and by arc R is shown in Fig. 13. The dihedral angles A and Z are azimuths, conventionally measured positive eastward from the northern direction of the meridian. In the spherical earth model, it is convenient to take the earth radius q as unity and measure the distance s in units of earth radius.

The elevation H is the angle between ray SP and the tangent to arc R at point P. Angles H, T, and R are all measured in the plane SPO. Figure 14 illustrates these angles in this plane. The angles are not independent, since, in fact, $H + T + R = \pi/2$. Figure 14 is the same as that used in the description of the polar perspective mapping, Fig. 10, except that it is slightly generalized to allow for a nonzero latitude for point S.

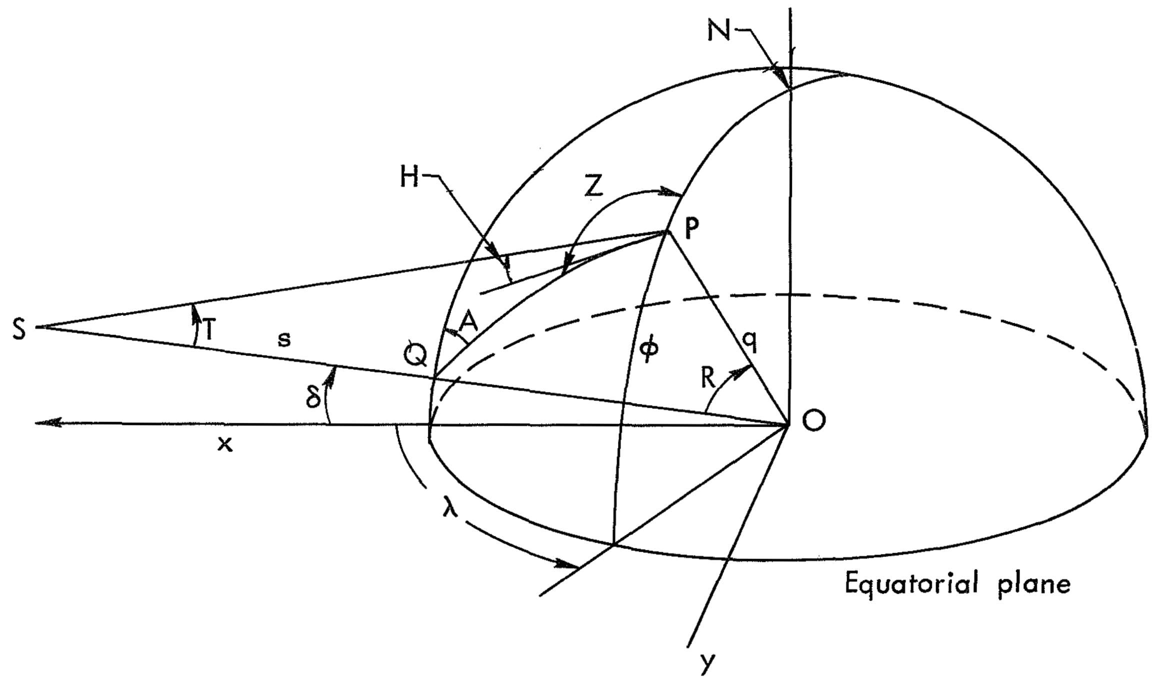

Fig. 12—Intersecting ray SP in three dimensions

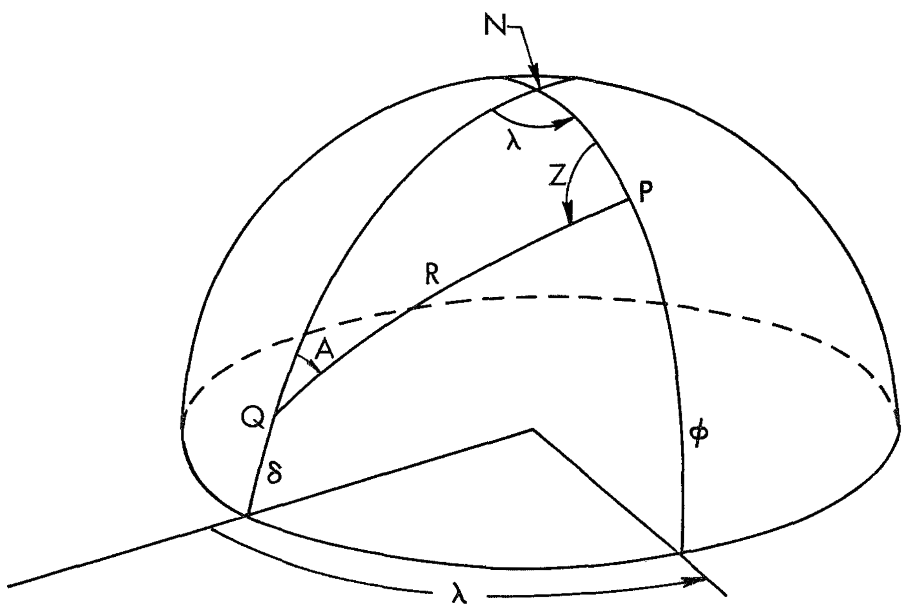

Fig. 13—The spherical triangle of the intersection problem

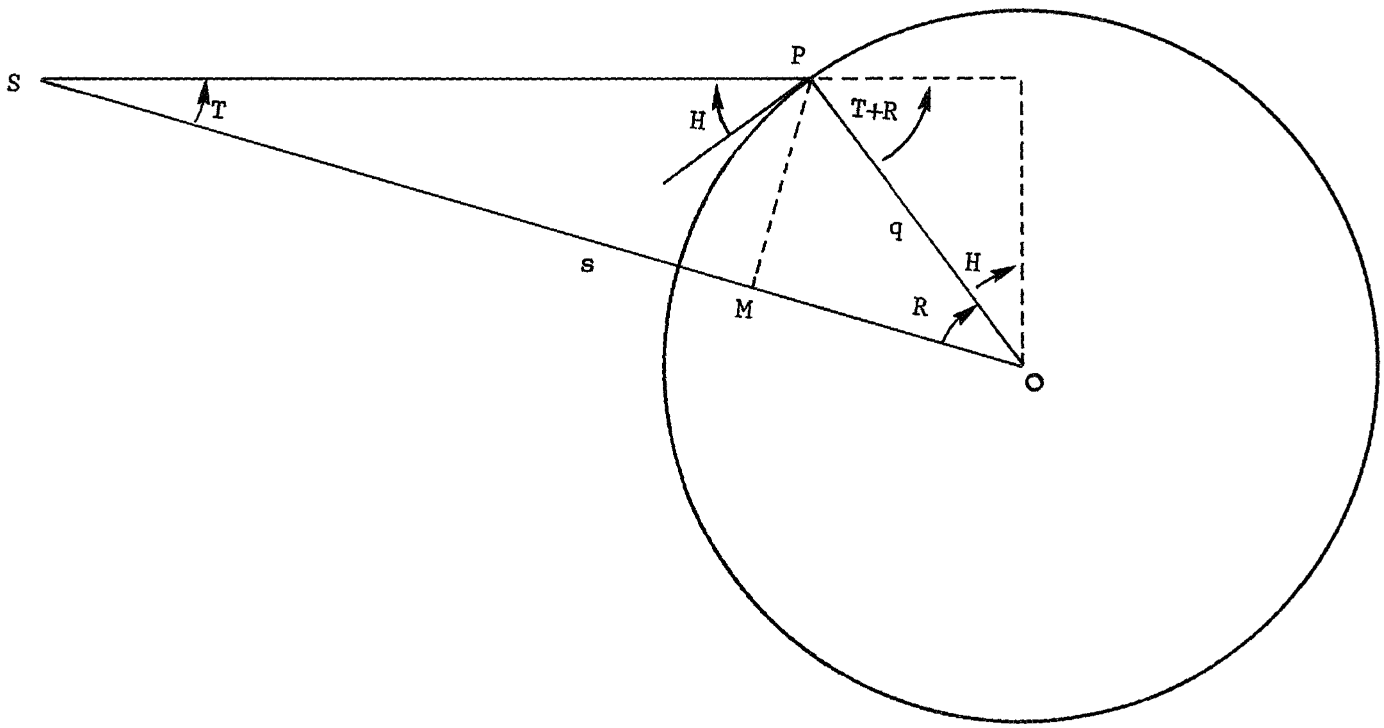

Fig. 14—The plane containing arc R

It is convenient to categorize mapping (polar perspective) and two kinds of intersection problems according to the characteristic input coordinates and the desired output coordinates. In mapping, the coordinates of the point to be mapped are known whereas intersectional problems seek the coordinates to be mapped when these coordinates are defined indirectly by constraints.

In polar perspective mapping, the point $P(\lambda, \phi)$ to be mapped is described by latitude ϕ and relative longitude λ. From the spherical triangle NQP (Fig. 13):

$$\cos R = \sin \delta \sin \phi + \cos \delta \cos \phi \cos \lambda \tag{11}$$

$$\tan A = \frac{\cos \phi \sin \lambda}{\sin \phi \cos \delta - \cos \phi \cos \lambda \sin \delta} \tag{12}$$

From the plane triangle (using construction line PM; see Fig. 14),

$$\tan T = \frac{\sin R}{s - \cos R} \tag{13}$$

In the case of intersectional problems, the coordinates of the intersection (λ, ϕ) are desired. Two kinds of intersection problems may be distinguished, and in both kinds the coordinates of S are known. For intersections of the first kind, the direction cosines (or T, A) of the ray from S to P are prescribed. Then, from the plane triangle

$$s \sin T = \sin (T + R) \tag{14}$$

or

$$R = -T + \sin^{-1} (s \sin T) \tag{15}$$

Using the spherical triangle NPQ (see Fig. 13),

$$\sin \phi = \sin \delta \cos R + \cos \delta \sin R \cos A \tag{16}$$

$$\sin\lambda = \frac{\sin R \sin A}{\cos\phi}$$
$$\cos\lambda = \frac{\cos R - \sin\delta \sin\phi}{\cos\delta \cos\phi} \tag{17}$$

Intersections of the second kind arise when finding the coordinates λ, ϕ of points from which S can be observed at an elevation angle H and azimuth Z. From the plane triangle

$$\cos H = s \cos (R + H) \tag{18}$$

or

$$R = -H + \cos^{-1}\left(\frac{\cos H}{s}\right) \tag{19}$$

In the spherical triangle

$$\sin\lambda = \frac{\sin R \sin Z}{\cos\delta} \tag{20}$$

Using the law of cosines in the spherical triangle

$$\sin\delta = \sin\phi \cos R + \cos\phi \sin R \cos Z \tag{21}$$

By rearranging and squaring, it is possible to form a quadratic equation for either $\sin\phi$ or $\cos\phi$. A somewhat easier solution can be found by forming an auxiliary triangle as shown in Fig. 15.

The arc R' is given by

$$\cos R' = \cos\delta \cos\lambda = \cos\phi \cos R - \sin\phi \sin R \cos Z \tag{22}$$

This equation can be solved jointly with the previous equation containing $\sin\phi$, $\cos\phi$. Eliminating $\cos\phi$ between the two equations

$$\sin\phi = \frac{\sin\delta \cos R - \cos\delta \cos\lambda \sin R \cos Z}{1 - \sin^2 R \sin^2 Z} \tag{23}$$

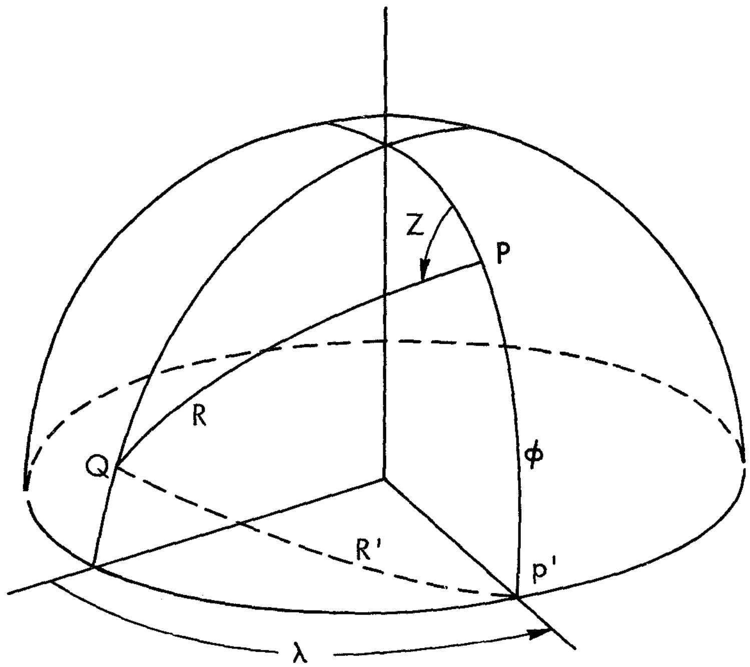

Fig. 15—The auxilliary spherical triangle

The preceding equations are based upon treating the earth as a sphere.

THE OBLATE EARTH

Figure 5 shows a sketch of the cross section of an ellipsoid of revolution representing the oblate earth (greatly exaggerated). The cross section is taken through the meridian plane of point P. The semimajor axis of the ellipse is a = 1 where a is the equatorial radius of the earth. The semiminor axis (polar radius) is b. The earth flattening factor is defined by

$$f = \frac{a - b}{a} \simeq \frac{1}{297} \tag{24}$$

In Fig. 5, the geocentric latitude and radius of point P are ϕ' and r, respectively.* The geographic latitude of P is ϕ and is the inclination of the local vertical (at P) to the equatorial plane. The ellipse is described by:

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{25}$$

The local vertical is normal to the ellipsoid of revolution so that angle ϕ is given by

$$\tan \phi = - \frac{dx}{dz} = \frac{a^2}{b^2} \frac{z}{x} = \frac{z/x}{(1 - f)^2} \tag{26}$$

---

*The prime is appended to ϕ to designate geocentric latitudes only when the distinction is required. In a spherical earth model, the distinction cannot be made ($\phi' = \phi$). In the description of mapping transformations of Section III, the symbol ϕ (unprimed) always represents geographic latitude.

Noting that $z/x = \tan \phi'$

$$\tan \phi = (1 - f)^{-2} \tan \phi' \simeq (1 + 2f) \tan \phi' \tag{27}$$

In this and the following, the quantity f^2 will be ignored. Because f is small, let $\phi' = \phi - \delta\phi$, and

$$\tan \phi = (1 + 2f) \tan (\phi - \delta\phi) = (1 + 2f) \left(\frac{\tan \phi - \tan \delta\phi}{1 + \tan \phi \tan \delta\phi} \right) \tag{28}$$

This can be rearranged and simplified to

$$\delta\phi = f \sin 2\phi \tag{29}$$

The quantity f is about 11.6 arc-min and the maximum difference between geocentric and geographic latitude occurs at 45° latitude.

The quantities p and q in Fig. 5 are useful in describing intersections of rays with the oblate earth. For the point P(x, y)

$$\begin{aligned} x &= p + q \cos \phi \\ z &= q \sin \phi \end{aligned} \tag{30}$$

Substituting these in Eqs. (25) and (26) and dropping terms of degree 2 and higher in f, the definitions of p and q are obtained:

$$\begin{aligned} p &= 2f \cos \phi \\ q &= 1 - 2f + f \sin^2 \phi \end{aligned} \tag{31}$$

Using the values of p, q, the cartesian coordinates on the ellipse are given in terms of geographic latitude by

$$\begin{aligned} x &= (1 + f \sin^2 \phi) \cos \phi \\ z &= (1 - 2f + f \sin^2 \phi) \sin \phi \end{aligned} \tag{32}$$

The geocentric radius r of P(x, z) is

$$r = 1 - f \sin^2 \phi \tag{33}$$

INTERSECTIONS WITH THE OBLATE EARTH

Intersections of arbitrary surfaces with the surface of the earth are often adequately approximated with a spherical representation of the earth's surface. Corrections can be derived for converting spherically calculated intersectional coordinates to geographical coordinates. The following paragraphs describe the correction procedure for space point S, which lies in the equatorial plane, and the numerical values which follow are further specialized to points which are at the geosynchronous radius of 6.6166 earth radii. The magnitudes of such corrections provide a useful indication of the adequacy of the spherical earth model for a synchronous, equatorial satellite.

As in the previous description of intersections on the sphere, two cases are distinguished: (1) the ray is described by an origin and direction at a point S exterior to the earth's surface, and (2) the ray direction is specified on the earth's surface with a termination specified at point S. The two cases are complementary, and both often arise within a particular problem. An example which illustrates the two cases in their complementary form is found in treating the ground coverage of a narrow beam directed from a communication satellite. The beam is conical, and the generators of the cone are rays with directions specified relative to the satellite body axes. The satellite position and orientation are specified relative to the earth, and consequently the direction of every ray is specified in the overall geometry. Ray intersections, when they exist, can be calculated for spherical and oblate earth models. Some rays may not intersect the spherical earth, and some which intersect the spherical earth will pass the oblate earth. For such nonintersecting rays, the rays will have passed over the horizon of the earth as seen from the satellite.

If a portion of a beam passes over the horizon, then the coverage contour displayed upon a rectilinear map is closed by the horizon line. More generally, adequate coverage may require that the satellite be seen at some minimum elevation H above the horizon. We are then interested in the geometry of case 2, which seeks the coordinates of an observer such that the satellite can be viewed at a certain elevation. By treating azimuth, Z, as a parameter, the contour of constant elevation can be computed.

In either of the two geometries, the earth may be treated as spherical and corresponding coordinates calculated for the intersectional contours. Corrections developed to convert the spherically calculated intersections to geographical coordinates can usually be associated with a spherically calculated pair of coordinates. The corrections will not necessarily be the same in the two geometric cases, as evidenced by the fact that in the first case some spherically calculated intersections have no corresponding intersections on the oblate earth and hence no real corrections.

Case 1: Ray Origin and Direction Specified at an Exterior Point

The ray direction at point S is specified by tilt T and azimuth A (Figs. 13 and 16). From these angles and knowledge of the position S, tentative coordinates λ, ϕ are calculated according to Eqs. (15)-(17) of the spherical earth model. These coordinates index the point to be treated and corrections are to be applied to these coordinates. For the oblate earth, the radius of the intersection is not unity but is given approximately by

$$r \doteq 1 - f \sin^2 \phi \tag{34}$$

The intersection of the ray with a sphere of radius r satisfies

$$s \sin T = r \sin (R + T) \tag{35}$$

In correcting from the unit sphere to a sphere of radius r, the angle T remains constant but

$$\delta R = - \tan (R + T)\ \delta r = \tan (R + T)\ f \sin^2 \phi \tag{36}$$

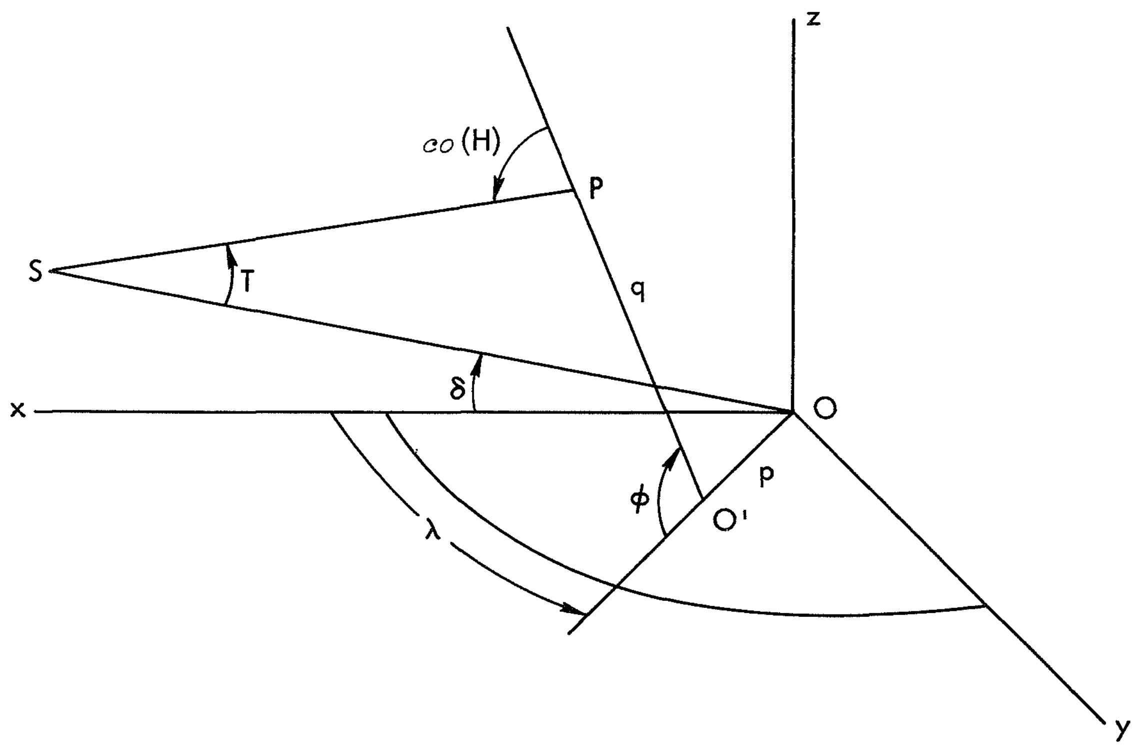

Fig. 16—Intersecting ray SP on the oblate earth

In the following, consideration is limited to points S which lie on the equator, and the longitude of the intersection is measured relative to the longitude of point S. For the spherical triangle thus restricted, using the previous notation (Fig. 13),

$$\begin{aligned} \tan \lambda &= \sin A \tan R \\ \sin \phi &= \cos A \sin R \end{aligned} \tag{37}$$

Differentiating with azimuth A constant

$$\begin{aligned} \delta\lambda &= \frac{\sin A}{\cos^2 \phi} \; \delta R \\ \delta\phi &= \cos A \cos \lambda \; \delta R \end{aligned} \tag{38}$$

Eliminating A in favor of the spherical values of λ, ϕ, subtituting the previous expression for δR, and adding the correction to convert to geographic latitude

$$\begin{aligned} \delta\lambda &= \frac{f \sin^2 \phi \sin \lambda \tan (R + T)}{\sin R \cos \phi} \\ \delta\phi &= f \left[\frac{\sin 2\phi + \cos \lambda \sin^3 \phi \tan (R + T)}{\sin R} \right] \end{aligned} \tag{39}$$

It is desired to represent these corrections entirely as functions of the spherically calculated values of λ, ϕ, and for this purpose

$$\begin{aligned} R &= \cos^{-1} (\cos \lambda \cos \phi) \\ T &= \tan^{-1} \left[\frac{\sin R}{s - \cos R} \right] \end{aligned} \tag{40}$$

Table 1 shows the corrections $\delta\lambda$ and $\delta\phi$ as functions of λ, ϕ. The table applies to a synchronous, equatorial satellite with s = 6.6166.

Table 1

LATITUDE AND LONGITUDE CORRECTIONS (ARC-MIN) FOR INTERSECTIONS DEFINED BY TILT AND AZIMUTH

Latitude	Longitude						
	0	15	30	45	60	75	80
0	0.00[a] 0.00[b]	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00
5	0.00 2.02	0.03 2.02	0.06 2.02	0.11 2.02	0.22 2.02	0.80 2.03	3.98 2.07
10	0.00 4.03	0.11 4.03	0.25 4.03	0.46 4.04	0.90 4.05	3.30 4.11	17.56 4.49
15	0.00 6.03	0.27 6.04	0.59 6.04	1.07 6.05	2.09 6.09	7.84 6.31	47.63 7.89
20	0.00 8.03	0.49 8.03	1.09 8.05	1.98 8.08	3.92 8.17	15.12 8.74	117.85 14.12
25	0.00 10.02	0.82 10.03	1.80 10.06	3.29 10.13	6.54 10.31	26.41 11.58	359.79 33.17
30	0.00 12.05	1.26 12.06	2.79 12.12	5.12 12.24	10.27 12.59	44.21 15.15	
35	0.00 14.15	1.88 14.17	4.16 14.27	7.68 14.48	15.58 15.10	73.76 20.16	
40	0.00 16.40	2.74 16.44	6.09 16.60	11.30 16.97	23.32 18.03	127.95 28.28	
45	0.00 18.94	3.98 19.01	8.87 19.26	16.59 19.87	35.02 21.68	248.00 44.80	
50	0.00 21.98	5.82 22.10	13.03 22.51	24.63 23.53	53.75 26.68	670.17 99.82	
55	0.00 25.94	8.70 26.13	19.59 26.82	37.63 28.56	86.45 34.33		
60	0.00 31.57	13.54 31.91	30.80 33.12	60.65 36.29	152.09 48.05		
65	0.00 40.61	22.65 41.24	52.35 43.60	107.70 50.12	323.77 80.47		
70	0.00 57.76	43.15 59.20	103.00 64.78	232.95 82.31	1302.15 249.06		
75	0.00 102.66	109.24 107.71	285.76 129.53	925.57 237.18			
80	0.00 495.02	1008.18 647.40					

[a]Longitude correction, $\delta\lambda$.

[b]Latitude correction, $\delta\phi$.

Case 2: Ray Origin at the Earth's Surface with Specified Direction and Termination

With prescribed values of elevation H and azimuth Z, the coordinates of the point of origin are tentatively calculated using the spherical model (Eqs. (19), (20), (23)). H and Z are rigorously defined at the (unknown) point of origin in a coordinate system with x the outward normal to the earth's surface, z tangent to the local meridian line in the northern direction, and y orthogonal (east) to form a right-handed (x, y, z) coordinate system. If this coordinate system is rotated clockwise about y through the angle f sin 2ϕ, then the original direction cosines of the ray will be converted to direction cosines in a geocentrically oriented coordinate system. In this system, the direction cosines are:

$$\begin{aligned} n'_z &= \cos (H + \delta H) \cos (Z + \delta Z) = \cos H \cos Z + f \sin 2\phi \sin H \\ n'_y &= \cos (H + \delta H) \sin (Z + \delta Z) = \cos H \sin Z \\ n'_x &= \sin (H + \delta H) = -f \sin 2\phi \cos H \cos Z + \sin H \end{aligned} \tag{41}$$

The corrections δH and δZ are very small, and these equations can be solved to obtain

$$\begin{aligned} \delta H &= -f \sin 2\phi \cos Z \\ \delta Z &= -f \sin 2\phi \tan H \sin Z \end{aligned} \tag{42}$$

By definition, the arc R from which the tentative values of λ, ϕ were computed satisfied the equation

$$s \cos (R + H) = \cos H \tag{43}$$

This equation is valid on the unit sphere. On the sphere of radius r, δR will satisfy

$$s \cos (R + \delta R + H + \delta H) = (1 - f \sin^2 \phi) \cos (H + \delta H) \tag{44}$$

The two preceding equations can be combined to obtain

$$\delta R = -\left(1 - \frac{\sin H}{s \sin (R + H)}\right) \delta H + \frac{\cos H}{s \sin (R + H)} f \sin^2 \phi \tag{45}$$

For convenience, let

$$\begin{aligned} \alpha &= \frac{\cos H}{s \sin (R + H)} \\ \beta &= \frac{\sin H}{s \sin (R + H)} \end{aligned} \tag{46}$$

The quantity s sin (R + H) is on the order of the distance from point S to the intersection, and if s is appreciably larger than the radius of the earth, both α and β do not exceed about $1/s$. With the previous definition of δH in terms of f,

$$\delta R = f \left[(1 - \beta) \sin 2\phi \cos Z + \alpha \sin^2 \phi\right] \tag{47}$$

Assuming that point S is an equatorial point (at latitude zero), the variables λ, ϕ, Z, and R are related by

$$\begin{aligned} \sin \lambda &= \sin Z \sin R \\ \tan \phi &= -\cos Z \tan R \end{aligned} \tag{48}$$

Taking derivatives with respect to R and Z

$$\begin{aligned} \delta\lambda &= \frac{\cos Z \sin R}{\cos \lambda} \delta Z + \frac{\sin Z \cos R}{\cos \lambda} \delta R \\ \delta\phi &= \tan \lambda \cos \phi \, \delta Z - \frac{\cos Z}{\cos^2 \lambda} \delta R \end{aligned} \tag{49}$$

Inserting δZ from Eq. (42), δR from Eq. (47) and appending f sin (2ϕ) to $\delta\phi$ to convert to geographic latitude, the final equations for the corrections in this case are

$$\delta\lambda = f\left[\frac{(-\beta \cos R + 1/s) \sin 2\phi \cos Z}{\cos \lambda} + \alpha \cos \phi \sin^2 \phi\right] \sin Z$$

$$\delta\phi = \frac{f}{\cos^2 \lambda}\left[\left(\beta \cos^2 Z + \frac{\sin^2 Z \cos R}{s}\right) \sin 2\phi - \alpha \cos Z \sin^2 \phi\right] \tag{50}$$

To evaluate these in terms of λ, ϕ

$$R = \cos^{-1} (\cos \lambda \cos \phi)$$

$$Z = \tan^{-1} \frac{\sin \lambda}{-\sin \phi \cos \lambda} \tag{51}$$

Values of the corrections are listed in Table 2 as functions of λ and ϕ. The various corrective terms are all proportional to 1/s so that for a truly distant point S, the corrections vanish, as they should since H and Z observations of a star yield the correct geographic coordinates at point P.

Table 2

LATITUDE AND LONGITUDE CORRECTIONS (ARC-MIN) FOR INTERSECTIONS DEFINED BY ELEVATION AND AZIMUTH

	Longitude						
Latitude	0	15	30	45	60	75	80
0	0.00[a] 0.00[b]	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00
5	0.00 0.30	0.00 0.31	0.01 0.35	0.01 0.43	0.01 0.61	0.01 1.17	0.01 1.75
10	0.00 0.60	0.02 0.62	0.03 0.69	0.04 0.84	0.05 1.20	0.05 2.31	0.05 3.44
15	0.00 0.87	0.03 0.90	0.06 1.01	0.09 1.23	0.11 1.75	0.11 3.38	0.11 5.04
20	0.00 1.12	0.06 1.15	0.11 1.29	0.15 1.58	0.18 2.24	0.19 4.34	0.19 6.48
25	0.00 1.33	0.08 1.37	0.16 1.53	0.22 1.88	0.26 2.67	0.28 5.18	0.29 7.74
30	0.00 1.50	0.11 1.55	0.21 1.74	0.30 2.13	0.35 3.03	0.38 5.88	
35	0.00 1.63	0.14 1.69	0.26 1.89	0.37 2.33	0.44 3.31	0.47 6.44	
40	0.00 1.72	0.16 1.79	0.31 2.00	0.43 2.46	0.51 3.51	0.55 6.84	
45	0.00 1.78	0.18 1.85	0.34 2.07	0.47 2.55	0.57 3.64	0.61 7.10	
50	0.00 1.81	0.19 1.88	0.36 2.10	0.50 2.60	0.60 3.71	0.65 7.24	
55	0.00 1.81	0.19 1.88	0.36 2.11	0.51 2.60	0.61 3.72		
60	0.00 1.80	0.18 1.87	0.35 2.09	0.49 2.59	0.59 3.70		
65	0.00 1.78	0.17 1.85	0.32 2.07	0.45 2.56	0.54 3.65		
70	0.00 1.76	0.14 1.82	0.28 2.04	0.39 2.52	0.47 3.60		
75	0.00 1.74	0.11 1.80	0.22 2.02	0.31 2.49			
80	0.00 1.73	0.08 1.79					

[a]Longitude correction, $\delta\lambda$.

[b]Latitude correction, $\delta\phi$.