NASA TECHNICAL NOTE

NASA TN D-6209

c. 1

NASA TN D-6209

VARIANCE REDUCTION BY IMPORTANCE SAMPLING AND THE METHOD OF SPLITTING IN MONTE CARLO CALCULATIONS

by Burt M. Rosenbaum

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1971

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TN D-6209		

4. Title and Subtitle	5. Report Date
VARIANCE REDUCTION BY IMPORTANCE SAMPLING AND THE METHOD OF SPLITTING IN MONTE CARLO CALCULATIONS	March 1971
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Burt M. Rosenbaum	E-5793
	10. Work Unit No.
	129-01
9. Performing Organization Name and Address	11. Contract or Grant No.
Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135	
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Technical Note
National Aeronautics and Space Administration Washington, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The two techniques of variance reduction that are considered are (1) importance sampling and (2) splitting and Russian roulette. Based on the value of the variance, optimum biasing sampling procedures are investigated and it is determined when adjoint biasing yields the minimum variance. It is shown that the method of Russian roulette may lead to an increase, rather than a decrease, in variance. A short example illustrates the methods used.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Variance reduction; mathematical model; Monte Carlo; computer method; importance sampling; Russian roulette; splitting	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	77	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

CONTENTS

VARIANCE REDUCTION BY IMPORTANCE SAMPLING AND THE METHOD OF SPLITTING IN MONTE CARLO CALCULATIONS

by Burt M. Rosenbaum

Lewis Research Center

SUMMARY

The two techniques of variance reduction that are considered are (1) importance sampling and (2) splitting and Russian roulette. Based on the value of the variance, optimum biasing sampling procedures are investigated and it is determined when adjoint biasing yields the minimum variance. It is shown that the method of Russian roulette may lead to an increase, rather than a decrease, in variance. A short example illustrates the methods used.

INTRODUCTION

Before the advent of the computer, when a problem involved a large number of members or participants (e.g., when dealing with, say, a collection of molecules), analyses usually could not be carried out for the general case. Because of the complexities, analytical solutions could only be found in limiting situations or where simplifying assumptions could be made. In the regions where such assumptions could not be made, realistic theoretical analyses could not be accomplished and so-called educated guesses were resorted to.

The computer enables an investigator to compare the theoretical behavior of his conceptual model with experimental data in the complicated intermediate region where standard analyses break down. When the Monte Carlo method is used, the possibilities of occurrence are encoded into the computer program and the behavior of a large number of sample particles is simulated by computer decisions as the sample particles are followed through the system. On the basis of the data thus generated average behavior patterns may be calculated. According to the information needed, the computer may be instructed to spew out local densities, total kinetic energy densities, heat and mass transfer rates, pressures, probabilities of penetration through a barrier, fission rates, chemical reaction rates, and so forth.

In an attempt to decrease the computation time necessary for answers as well as to enable one to handle problems which originally would overload the computer capacity, techniques were evolved which established more efficient simulation processes than direct simulation; that is, better accuracy could be obtained for a given sample size. The systemization of such error-reducing procedures was due in large measure, to the work of H. Kahn (refs. 1 to 4).

This report concerns itself with two of these techniques: (1) importance sampling and (2) splitting and Russian roulette. The theory has been extensively treated in the literature and applications of the two techniques, used separately or in combination, abound in computer programs (refs. 5 to 13). In this report, both techniques are formulated in a unique manner and equations for the variance resulting from their use are derived. These equations are generalized to apply to any number of sets of random variables and optimum procedures are developed. In addition, a brief example is posed and analyzed to illustrate the method. It is hoped that the formulation of the problem as presented herein serves to clarify the basic concepts involved.

IMPORTANCE SAMPLING

The following problem is considered. Suppose we are given a function g dependent on three or more sets of random variables $\vec{x} = x_1, x_2, \ldots, x_n$ where x_i represents all variables in the i^{th} set of random variables and we wish to determine the mean or expectation value of g:

$$E[g] = \int \ldots \int g(\vec{x})f(\vec{x})d\vec{x} \tag{1}$$

In equation (1), the multivariate probability density function is denoted by $f(\vec{x}) = f(x_1, x_2, \ldots, x_n)$. When the relation

$$f(\vec{x}) = f(x_1, x_2)f(x_3, \ldots, x_n/x_1, x_2) \tag{2}$$

is used where the symbol $f(x_3, \ldots, x_n/x_1, x_2)$ is the conditional probability density function of the random variables $x_3, \ldots, x_n$ given that x_1, x_2 have taken on fixed values, equation (1) can be written

$$\begin{aligned} E[g] &= \iint f(x_1, x_2)dx_1\, dx_2 \int \ldots \int g(\vec{x})f(x_3, \ldots, x_n/x_1, x_2)dx_3 \ldots dx_n \\ &= \iint E[g/x_1, x_2]f(x_1, x_2)dx_1\, dx_2 \end{aligned} \tag{3}$$

It can be seen, therefore, that $E[g/x_1, x_2]$, the expectation value of g given that x_1 and x_2 have been fixed, satisfies

$$E[g/x_1, x_2] = \int \cdots \int g(\vec{x}) f(x_3, \ldots, x_n/x_1, x_2) dx_3 \ldots dx_n \tag{4}$$

The variance of g for fixed x_1 and x_2 is given by

$$\sigma_g^2/x_1, x_2 = \int \cdots \int \left(g(\vec{x}) - E[g/x_1, x_2]\right)^2 f(x_3, \ldots, x_n/x_1, x_2) dx_3 \ldots dx_n \tag{5}$$

If we pick from a population distributed in accordance with the probability density function $f(x_1, x_2)$, then a weight function $w(x_1, x_2)$ can be incorporated into the density function which may act to decrease the value of the variance $\sigma_g^2/x_1, x_2$ while keeping the value of $E[g]$ invariant. The new probability density function $f(x_1, x_2)/w(x_1, x_2)$ satisfies the relation

$$\iint \frac{f(x_1, x_2)}{w(x_1, x_2)} dx_1\, dx_2 = 1 \tag{6}$$

and equation (1) becomes

$$E[g] = \int \cdots \int w(x_1, x_2) g(\vec{x}) \frac{f(x_1, x_2, \ldots, x_n)}{w(x_1, x_2)} dx_1\, dx_2 \ldots dx_n \tag{7}$$

The method just described whereby a weight function is employed is called "importance sampling" and, by equation (7), we see that the expectation value of $g(\vec{x})$ when sampling from a population with probability density function $f(\vec{x})$ is equal to the expectation value of $w(x_1, x_2)g(\vec{x})$ when sampling from a population described by the density function $f(\vec{x})/w(x_1, x_2)$.

The variance of $w(x_1, x_2)g(\vec{x})$ associated with the probability density function $f(\vec{x})/w(x_1, x_2)$ is given by

$$\sigma^2_{w(x_1, x_2)g(\vec{x})}\Bigg|_{\frac{f(\vec{x})}{w(x_1, x_2)}} = \int \cdots \int \left(w(x_1, x_2)g(\vec{x}) - E[g]\right)^2 \frac{f(\vec{x})}{w(x_1, x_2)}\, dx_1\, dx_2 \ldots dx_n$$

$$= \iint w(x_1, x_2)E\left[g^2/x_1, x_2\right]f(x_1, x_2)dx_1\, dx_2 - E^2[g]$$

$$= E\left[w(x_1, x_2)E\left[g^2/x_1, x_2\right]\right] - E^2[g] \tag{8}$$

where

$$E\left[g^2/x_1, x_2\right] = \int \cdots \int g^2(\vec{x})f(x_3, x_4, \ldots, x_n/x_1, x_2)dx_3 \ldots dx_n$$

$$= \sigma^2_g/x_1, x_2 + E^2[g/x_1, x_2] \tag{9}$$

The particular weight function $\hat{w}(x_1, x_2)$ that minimizes the variance of $w(x_1, x_2)g(\vec{x})$ subject to the constraint given by equation (6) satisfies the equation

$$\hat{w}(x_1, x_2) = \frac{E\left[\sqrt{E\left[g^2/x_1, x_2\right]}\right]}{\sqrt{E\left[g^2/x_1, x_2\right]}} \tag{10}$$

and the minimum value of the variance is given by

$$\hat{\sigma}^2_{w(x_1, x_2)g(\vec{x})}\Bigg|_{\frac{f(\vec{x})}{w(x_1, x_2)}} = E^2\left[\sqrt{E\left[g^2/x_1, x_2\right]}\right] - E^2[g] \tag{11}$$

This formulation, as mentioned before, is the method of importance sampling in the two variable sets x_1 and x_2.

It may be stated here that if N measurements of $w(x_1, x_2)g(\vec{x})$ were made, then the variance of the average is merely equal to the value given by equation (8) divided by N.

The concept of stages is now introduced and it will be employed throughout this report. We assume that in the first stage we sample from the x_1-distribution and in the second stage from the x_2-distribution. We write

$$f(x_1, x_2) = f(x_1)f(x_2/x_1) \tag{12}$$

$$w(x_1, x_2) = u(x_1)u(x_1, x_2) \tag{13}$$

where $u(x_1)$ and $u(x_1, x_2)$ satisfy

$$\int \frac{f(x_1)}{u(x_1)} dx_1 = 1 \tag{14a}$$

$$\int \frac{f(x_2/x_1)}{u(x_1, x_2)} dx_2 = 1 \tag{14b}$$

It should be noted that a knowledge of the function $w(x_1, x_2)$ uniquely determines the functions $u(x_1)$ and $u(x_1, x_2)$. Substitution of equation (13) into equation (14b) results in the relation

$$u(x_1) = \left(\int \frac{f(x_2/x_1)}{w(x_1, x_2)} dx_2 \right)^{-1} \tag{15}$$

so that the function $u(x_1)$ and, hence, the function $u(x_1, x_2)$ may be directly solved for once $w(x_1, x_2)$ is known.

Picking from a population with density function $f(x_1, x_2)/w(x_1, x_2)$ is equivalent to first choosing x_1 from a population with density function $f(x_1)/u(x_1)$ and then choosing x_2 from a population with density function $f(x_2/x_1)/u(x_1, x_2)$ where the value of x_1 for the second population has been set at the value chosen from the first population. Equation (8) can now be written in the more revealing form, namely,

$$\sigma^2_{u(x_1)u(x_1,x_2)g(\vec{x})} \frac{f(\vec{x})}{u(x_1)u(x_1,x_2)} = \iint u^2(x_1)u^2(x_1,x_2)\left(\sigma^2_g/x_1,x_2\right)\frac{f(x_1,x_2)}{u(x_1)u(x_1,x_2)}\,dx_1\,dx_2$$

$$+ \iint\left(u(x_1)u(x_1,x_2)E[g/x_1,x_2] - u(x_1)E[g/x_1]\right)^2 \frac{f(x_1,x_2)}{u(x_1)u(x_1,x_2)}\,dx_1\,dx_2$$

$$+ \iint\left(u(x_1)E[g/x_1] - E[g]\right)^2 \frac{f(x_1)}{u(x_1)}\,dx_1 \tag{8a}$$

In equation (8a), the variance has been broken down into components where each of the integrals on the right side can be interpreted in the following way: (1) the first integral is due to the variation in g when x_1 and x_2 are both fixed and the other variables are allowed to vary; (2) the second is due to the variation in $E[g/x_1,x_2]$ when x_1 alone has been fixed and x_2 is allowed to vary; and (3) the third integral is due to the variation in $E[g/x_1]$ when x_1 is allowed to vary. Note that it would be an easy task to generalize equation (8a) to any number of sets of pertinent variables.

FORMULATION OF THE METHOD OF IMPORTANCE SAMPLING FOR DISCRETE VARIABLES

For the sake of mathematical simplicity, assume that each of the sets of random variables $x_1, \ldots, x_n$ are discrete and let us set up the foregoing problem by using a notation in keeping with this assumption. The results obtained can readily be modified to apply to the case when the variables are continuous, and the simpler discrete model will be used to establish the relations that arise when considering the method of splitting.

First the notation is defined. Let the possible value sets of x_1 be put into one-to-one correspondence with the index $i=1, 2, 3, \ldots$; let the possible value sets of x_2 be put into one-to-one correspondence with the index $j=1, 2, 3, \ldots$, etc. Whether the total number of possible value sets for any variable is finite or infinite does not change the problem. Set p_i equal to the probability that x_1 takes on its i^{th} value set and set p_{ij} equal to the conditional probability that x_2 takes on its j^{th} value set given that x_1 has its i^{th} value set. We have

$$E[g] = \sum_{i, j, k, \ldots} p_i p_{ij} p_{ijk} \cdot \cdot \cdot g_{ijk \ldots} = \sum_i p_i E_i[g] \tag{1'}$$

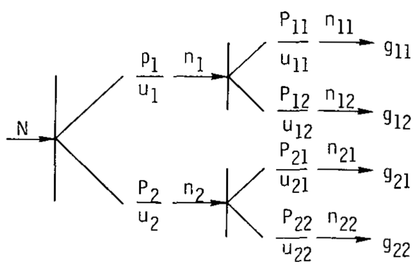
$$E_i[g] = \sum_{j, k, l, \ldots} p_{ij} p_{ijk} p_{ijkl} \cdot \cdot \cdot g_{ijkl \ldots} = \sum_j p_{ij} E_{ij}[g] \tag{3'}$$

$$E_{ij}[g] = \sum_{k, l, \ldots} p_{ijk} p_{ijkl} \cdot \cdot \cdot g_{ijkl \ldots} \tag{4'}$$

$$\sigma^2_{g_{ij}} = \sum_{k, l, \ldots} p_{ijk} p_{ijkl \ldots} \left(g_{ijkl \ldots} - E_{ij}[g]\right)^2 \tag{5'}$$

(Primed and unprimed equations of the same number are analogous.)

The scheme by which the average of g is obtained is illustrated in sketch (a) where



(a)

the number of possible x_1 and x_2 (for each x_1 value set) value sets are shown as two in the sketch. A sample size N is first pulled from the x_1-population where the probability of getting the i^{th} value set of x_1 is p_i/u_i. The number in the sample possessing the i^{th} value set of x_1 is designated as n_i and the expectation value of n_i is $N(p_i/u_i)$. Then this sample is further subdivided by picking from that x_2-population corresponding to the value set of x_1 where the probability of getting the j^{th} value set of x_2 given that the i^{th} value set of x_1 has been chosen is p_{ij}/u_{ij}. (The u_i and u_{ij} are weight functions that play the same role as $u(x_1)$ and $u(x_1, x_2)$ did in the case where the random variables were considered as continuous.) The number of members in the sample possessing the i^{th} value set of x_1 and the j^{th} value set of x_2 is denoted as n_{ij}. We note that

$$\sum_i \frac{p_i}{u_i} = 1 \tag{14a'}$$

$$\sum_j \frac{p_{ij}}{u_{ij}} = 1 \tag{14b'}$$

Let $(g_{ij})_\alpha$ be the α^{th} measurement of g where x_1 has been fixed at its i^{th} value and x_2 at its j^{th} value regardless of the values of the other sets of variables. There are n_{ij} such measurements. Consider the random variable

$$Z = \frac{\sum\limits_{i,j} \sum\limits_{\alpha=1}^{n_{ij}} u_i u_{ij} (g_{ij})_\alpha}{N} \tag{16}$$

Equation (16) can be written

$$Z = \sum_{i,j} Z_{ij}$$

where

$$\left.\begin{aligned} Z_{ij} &= \frac{u_i u_{ij}}{N} \sum_{\alpha=1}^{n_{ij}} (g_{ij})_\alpha \\ &= \frac{u_i u_{ij}}{N} n_{ij} \overline{(g_{ij})} \end{aligned}\right\} \tag{17}$$

Taking the expectation value of Z_{ij} gives

$$E[Z_{ij}] = \frac{u_i u_{ij}}{N} E\left[n_{ij} \overline{(g_{ij})}\right] = \frac{u_i u_{ij}}{N} E\left[n_{ij} E\left[\overline{(g_{ij})}/n_{ij}\right]\right]$$

where $E\left[\overline{(g_{ij})}/n_{ij}\right]$ is the expectation of the average measurement of g_{ij} when the number of measurements is n_{ij}. Because this expectation value is independent of the number of measurements, the equation becomes

$$E[Z_{ij}] = \frac{u_i u_{ij}}{N} E_{ij}[g] N \frac{p_i}{u_i} \frac{p_{ij}}{u_{ij}} = p_i p_{ij} E_{ij}[g] \tag{18}$$

so that

$$E[Z] = \sum_{i,j} E[Z_{ij}] = \sum_{i,j} p_i p_{ij} E_{ij}[g] = E[g] \tag{19}$$

Hence Z is an unbiased estimator of $E[g]$.

We wish to find the variance σ_Z^2 of Z. The following relation holds:

$$\sigma_Z^2 = E[Z^2] - E^2[Z] = \sum_{i,j,i',j'} E[Z_{ij} Z_{i'j'}] - E^2[g]$$

$$= \sum_{i,j} E\left[Z_{ij}^2\right] + \sum_{\substack{i,j,j' \\ (j \neq j')}} E[Z_{ij} Z_{ij'}] + \sum_{\substack{i,j,i',j' \\ (i \neq i')}} E[Z_{ij} Z_{i'j'}] - E^2[g] \tag{20}$$

Considering the terms of the first summation on the right side of equation (20) gives

$$E\left[Z_{ij}^2\right] = \frac{u_i^2 u_{ij}^2}{N^2} E\left[n_{ij}^2 \overline{(g_{ij})}^2\right] = \frac{u_i^2 u_{ij}^2}{N^2} E\left[n_{ij}^2 E\left[\overline{(g_{ij})}^2/n_{ij}\right]\right] = \frac{u_i^2 u_{ij}^2}{N^2} E\left[n_{ij}^2\left(\frac{\sigma_{g_{ij}}^2}{n_{ij}} + E_{ij}^2[g]\right)\right]$$

$$= \frac{u_i^2 u_{ij}^2}{N^2} \sigma_{g_{ij}}^2 E[n_{ij}] + \frac{u_i^2 u_{ij}^2}{N^2} E_{ij}^2[g] E\left[n_{ij}^2\right]$$

The relation

$$E[n_{ij}] = \frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}$$

has already been employed in obtaining equation (18). To find an expression of $E\left[n_{ij}^2\right]$, we employ the fact that, for a specified value of n_i, the quantity n_{ij} is distributed as a binomial variable with probability p_{ij}/u_{ij} of "success" where the number of trials is equal to n_i. Hence

$$E\left[n_{ij}^2\right] = E\left[E\left[n_{ij}^2/n_i\right]\right]$$

$$= \frac{p_{ij}^2}{u_{ij}^2} E\left[n_i^2\right] + \frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)E[n_i]$$

$$= \frac{p_{ij}^2}{u_{ij}^2}\left[N^2\frac{p_i^2}{u_i^2} + N\frac{p_i}{u_i}\left(1 - \frac{p_i}{u_j}\right)\right] + N\frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)$$

$$= N^2\frac{p_i^2}{u_i^2}\frac{p_{ij}^2}{u_{ij}^2} + N\frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}\right)$$

Thus, the terms of the first summation are given by

$$E\left[Z_{ij}^2\right] = p_i^2 p_{ij}^2 E_{ij}^2[g] + \frac{1}{N} p_i p_{ij} u_i u_{ij} \sigma_{g_{ij}}^2 + \frac{1}{N}\frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_i}{u_i}\frac{p_{ij}}{u_{ij}}\right)u_i^2 u_{ij}^2 E_{ij}^2[g] \tag{21}$$

Considering the terms of the second summation gives, for $j \neq j'$,

$$E[Z_{ij}Z_{ij'}] = \frac{u_i^2 u_{ij} u_{ij'}}{N^2} E_{ij}[g]E_{ij'}[g]E[n_{ij}n_{ij'}]$$

where we have used the fact that $(\overline{g_{ij}})$ and $(\overline{g_{ij'}})$ involve measurements made on two different groups of the sample and, hence, are independent variables. We have

$$E[n_{ij}n_{ij'}] = E\left[n_{ij}E[n_{ij'}/n_{ij}]\right]$$

Now with n_{ij} fixed, $n_{ij'}$ is a binomial variable, where the number of trials is $(n_i - n_{ij})$ and the probability per trial of taking on the j'^{th} value set of x_2 is $(p_{ij'}/u_{ij'})/[1 - (p_{ij}/u_{ij})]$. The denominator $1 - (p_{ij}/u_{ij})$ is needed as a normalization factor modifying the probability $p_{ij'}/u_{ij'}$ because, with n_{ij} fixed, none of the $(n_i - n_{ij})$ trials can give rise to a member having the j^{th} value set of x_2. Thus

$$E[n_{ij'}/n_{ij}] = \frac{(n_i - n_{ij})\dfrac{p_{ij'}}{u_{ij'}}}{1 - \dfrac{p_{ij}}{u_{ij}}}$$

and

$$E[n_{ij}n_{ij'}] = \frac{\dfrac{p_{ij'}}{u_{ij'}}}{1 - \dfrac{p_{ij}}{u_{ij}}} E\left[n_i n_{ij} - n_{ij}^2\right]$$

Continuing,

$$E[n_{ij}n_{ij'}] = \frac{\dfrac{p_{ij'}}{u_{ij'}}}{1 - \dfrac{p_{ij}}{u_{ij}}} E\left[\frac{p_{ij}}{u_{ij}} n_i^2 - \left(\frac{p_{ij}^2}{u_{ij}^2} n_i^2 + \frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)n_i\right)\right]$$

$$= \frac{p_{ij'}}{u_{ij'}} \frac{p_{ij}}{u_{ij}} E\left[n_i^2 - n_i\right]$$

$$= \frac{p_{ij'}}{u_{ij'}} \frac{p_{ij}}{u_{ij}} \frac{p_i^2}{u_i^2} N^2\left(1 - \frac{1}{N}\right)$$

Hence, for $j \neq j'$,

$$E[Z_{ij}Z_{ij'}] = p_i^2 p_{ij} p_{ij'} E_{ij}[g] E_{ij'}[g] \left(1 - \frac{1}{N}\right) \tag{22}$$

Considering the terms of the third summation of equation (20), we have, for $i \neq i'$,

$$E[Z_{ij}Z_{i'j'}] = \frac{u_i u_{i'} u_{ij} u_{i'j'}}{N^2} E_{ij}[g] E_{i'j'}[g] E[n_{ij} n_{i'j'}]$$

where

$$E[n_{ij}n_{i'j'}] = E\left[n_{ij} E[n_{i'j'}/n_{ij}]\right]$$

$$= E\left[n_{ij} \frac{(N - n_{ij}) \dfrac{p_{i'}}{u_{i'}} \dfrac{p_{i'j'}}{u_{i'j'}}}{1 - \dfrac{p_i}{u_i} \dfrac{p_{ij}}{u_{ij}}}\right]$$

$$= \frac{\dfrac{p_{i'}}{u_{i'}} \dfrac{p_{i'j'}}{u_{i'j'}}}{1 - \dfrac{p_i}{u_i} \dfrac{p_{ij}}{u_{ij}}} E\left[Nn_{ij} - n_{ij}^2\right] = \frac{p_{i'}}{u_{i'}} \frac{p_{i'j'}}{u_{i'j'}} \frac{p_i}{u_i} \frac{p_{ij}}{u_{ij}} N^2 \left(1 - \frac{1}{N}\right)$$

Hence, for $i \neq i'$,

$$E[Z_{ij}Z_{i'j'}] = p_i p_{i'} p_{ij} p_{i'j'} E_{ij}[g] E_{i'j'}[g] \left(1 - \frac{1}{N}\right) \tag{23}$$

Substituting equations (21) to (23) into equation (20) and using the relations

$$E_{ij}[g^2] = \sigma_{g_{ij}}^2 + E_{ij}^2[g] \tag{9'}$$

and

$$w_{ij} \equiv u_i u_{ij} \tag{13'}$$

yield

$$N\sigma_Z^2 = \sum_{i,j} p_i p_{ij} u_i u_{ij} \sigma_{g_{ij}}^2 + \sum_{i,j} \frac{p_i}{u_i} \frac{p_{ij}}{u_{ij}} \left(1 - \frac{p_i}{u_i} \frac{p_{ij}}{u_{ij}}\right) u_i^2 u_{ij}^2 E_{ij}^2[g]$$

$$- \sum_{\substack{i,j,j \\ (j \neq j')}} p_i^2 p_{ij} \rho_{ij'} E_{ij}[g] E_{ij'}[g] - \sum_{\substack{i,j,i',j' \\ (i \neq i')}} p_i p_{i'} p_{ij} p_{i'j'} E_{ij}[g] E_{i'j'}[g]$$

$$= \sum_{i,j} p_i p_{ij} u_i u_{ij} \sigma_{g_{ij}}^2 + \sum_{i,j} p_i p_{ij} u_i u_{ij} E_{ij}^2[g] - \left(\sum_{i,j} p_i p_{ij} E_{ij}[g]\right)^2$$

$$= \sum_{i,j} w_{ij} E_{ij}[g^2] p_i \rho_{ij} - E^2[g] \tag{8'}$$

Equation (8'), which is analogous to equation (8), can be written in a form analogous to equation (8a) as

$$N\sigma_Z^2 = \sum_{i,j} w_{ij}^2 \sigma_{g_{ij}}^2 \frac{p_i p_{ij}}{w_{ij}} + \sum_{i,j} \left(w_{ij} E_{ij}[g] - u_i E_i[g]\right)^2 \frac{p_i p_{ij}}{w_{ij}}$$

$$+ \sum_i \left(u_i E_i[g] - E[g]\right)^2 \frac{p_i}{u_i} \tag{8a'}$$

The analogous equations to equations (10), (11), and (15) are

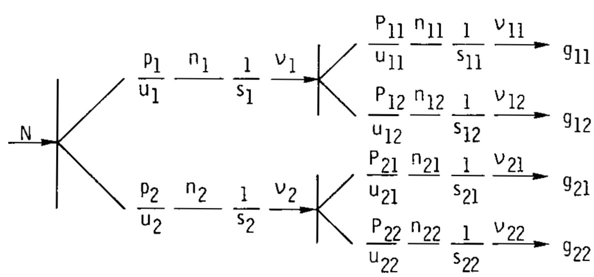
$$\hat{w}_{ij} = \frac{E\left[\left(E_{ij}[g^2]\right)^{1/2}\right]}{\sqrt{E_{ij}[g^2]}} \tag{10'}$$

$$N\hat{\sigma}_Z^2 = E^2\left[\sqrt{E_{ij}[g^2]}\right] - E^2[g] \tag{11'}$$

$$\frac{1}{\hat{u}_i} = \sum_j \frac{p_{ij}}{\hat{w}_{ij}} = \frac{E_i\left[\left(E_{ij}[g^2]\right)^{1/2}\right]}{E\left[\left(E_{ij}[g^2]\right)^{1/2}\right]} \tag{15'}$$

MODIFICATION TO INCLUDE THE METHOD OF SPLITTING

We now modify the scheme by which the average of g is obtained in accordance with sketch (b).



(b)

Again the possible x_1 and x_2 (for each x_1 value set) value sets are shown in sketch (b) as two in number.

The first step in the sampling procedure remains the same; a sample of size N is first picked from the x_1-population where the probability of getting the i^{th} value set of x_1 is ρ_i/u_i. The number in the sample possessing the i^{th} value set of x_1 is designated as n_i and, for a sample of size N, the expectation value of n_i equals $(p_i/u_i)N$. This number n_i is then multiplied by the splitting factor $1/s_i$ to yield the number $\nu_i = (1/s_i)n_i$ that now possesses the i^{th} value set of x_1. (It may be noted at this point that no longer does the sample necessarily consist of N members because $\sum_i \nu_i \neq N$ in general.) The sample is further subdivided on the basis of the variable x_2 and n_{ij} represents the number of members in the sample with the i^{th} value set of x_1 and the j^{th} value set of x_2 where the probability that a member possessing the i^{th} value set of x_1 also has the j^{th} value set of x_2 is p_{ij}/u_{ij}. Again, the splitting factor $1/s_{ij}$ is introduced and the number of members possessing the i^{th} value set of x_1 and the j^{th} value set of x_2 is changed to $\nu_{ij} = n_{ij}/s_{ij}$. Finally, ν_{ij} measurements of g_{ij} are made where the α^{th} measurement is denoted as $(g_{ij})_\alpha$.

It is observed that two steps are included in each stage: (1) importance sampling where the weight factor $1/u_i$ or $1/u_{ij}$ alters the selection probabilities and (2) splitting where the splitting factor $1/s_i$ or $1/s_{ij}$ alters the numbers selected. Such a stage will be designated as a "composite" stage. In general, whenever a splitting step is present, the total number of members in the sample changes.

The random variable that is of interest now is

$$\mathfrak{z} = \frac{\sum_{i,j} \sum_{\alpha=1}^{\nu_{ij}} u_i s_i u_{ij} s_{ij} (g_{ij})_\alpha}{N} \tag{24}$$

Proceeding in the same way as before, we write

$$\mathfrak{z} = \sum_{i,j} \mathfrak{z}_{ij}; \quad \mathfrak{z}_{ij} = \frac{u_i s_i u_{ij} s_{ij}}{N} \sum_{\alpha=1}^{\nu_{ij}} (g_{ij})_\alpha = \frac{u_i s_i u_{ij} s_{ij} \nu_{ij} \overline{(g_{ij})}}{N} \tag{25}$$

$$E[\mathfrak{z}_{ij}] = \frac{u_i s_i u_{ij} s_{ij}}{N} E\left[\sum_{\alpha=1}^{\nu_{ij}} (g_{ij})_\alpha\right] = \frac{u_i s_i u_{ij} s_{ij}}{N} E_{ij}[g] E[\nu_{ij}] = p_i p_{ij} E_{ij}[g] \tag{26}$$

$$E[\mathfrak{z}] = \sum_{i,j} E[\mathfrak{z}_{ij}] = \sum_{i,j} p_i p_{ij} E_{ij}[g] = E[g] \tag{27}$$

Hence, $\mathfrak{z}$ is an unbiased estimator of $E[g]$. Continuing, we have

$$\sigma^2_{\mathcal{J}} = E[\mathcal{J}^2] - E^2[\mathcal{J}]$$

$$= \sum_{i,j} E\left[\mathcal{J}^2_{ij}\right] + \sum_{\substack{i,j,j' \\ (j \neq j')}} E[\mathcal{J}_{ij}\mathcal{J}_{ij'}] + \sum_{\substack{i,j,i',j' \\ (i \neq i')}} E[\mathcal{J}_{ij}\mathcal{J}_{i'j'}] - E^2[g] \tag{28}$$

As before, each of the summation terms of the right side of equation (28) will be considered in turn. For the first summation terms we get

$$E\left[\mathcal{J}^2_{ij}\right] = \frac{u_i^2 s_i^2 u_{ij}^2 s_{ij}^2}{N^2} E\left[\nu^2_{ij}\overline{(g_{ij})^2}\right] = \frac{u_i^2 s_i^2 u_{ij}^2 s_{ij}^2}{N^2} \sigma^2_{g_{ij}} E[\nu_{ij}] + \frac{u_i^2 s_i^2 u_{ij}^2 s_{ij}^2}{N^2} E^2_{ij}[g] E\left[\nu^2_{ij}\right]$$

where

$$E[\nu_{ij}] = N \frac{p_i}{u_i s_i} \frac{p_{ij}}{u_{ij} s_{ij}}$$

and

$$E\left[\nu^2_{ij}\right] = \frac{1}{s^2_{ij}} E\left[n^2_{ij}\right]$$

$$= \frac{1}{s^2_{ij}} E\left[\frac{p^2_{ij}}{u^2_{ij}} \nu_i^2 + \frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)\nu_i\right]$$

$$= \frac{p^2_{ij}}{u^2_{ij}s^2_{ij}} \frac{1}{s^2_i}\left[N^2 \frac{p_i^2}{u_i^2} + N \frac{p_i}{u_i}\left(1 - \frac{p_i}{u_i}\right)\right] + \frac{p_{ij}}{u_{ij}s^2_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right) N \frac{p_i}{u_i s_i}$$

This yields

$$E\left[\mathcal{Z}_{ij}^{2}\right] = p_i^2 p_{ij}^2 E_{ij}^2[g] + \frac{1}{N} p_i p_{ij} u_i s_i u_{ij} s_{ij} \sigma^2_{g_{ij}}$$

$$+ \frac{E_{ij}^2[g]}{N} p_i p_{ij}\left[p_{ij} u_i (1 - s_i) - p_i p_{ij} + u_i s_i u_{ij}\right] \tag{29}$$

For the second summation $j \neq j'$ and

$$E[\mathcal{Z}_{ij}\mathcal{Z}_{ij'}] = \frac{u_i^2 s_i^2 u_{ij} u_{ij'} s_{ij} s_{ij'}}{N^2} E_{ij}[g] E_{ij'}[g] E[\nu_{ij}\nu_{ij'}]$$

where

$$E[\nu_{ij}\nu_{ij'}] = \frac{1}{s_{ij}s_{ij'}} E[n_{ij}n_{ij'}] = \frac{1}{s_{ij}s_{ij'}} E\left[n_{ij}\frac{(\nu_i - n_{ij})\frac{p_{ij'}}{u_{ij'}}}{1 - \frac{p_{ij}}{u_{ij}}}\right]$$

$$= \frac{\frac{p_{ij'}}{u_{ij'}}}{s_{ij}s_{ij'}\left(1 - \frac{p_{ij}}{u_{ij}}\right)} E\left[\nu_i n_{ij} - n_{ij}^2\right]$$

$$= \frac{p_{ij'}}{u_{ij'}s_{ij}s_{ij'}\left(1 - \frac{p_{ij}}{u_{ij}}\right)} E\left\{\nu_i^2 \frac{p_{ij}}{u_{ij}} - \left[\nu_i^2 \frac{p_{ij}^2}{u_{ij}^2} + \nu_i \frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)\right]\right\}$$

$$= \frac{p_{ij'}p_{ij}}{u_{ij}u_{ij'}s_{ij}s_{ij'}} E\left[\nu_i^2 - \nu_i\right]$$

and we use the method of Lagrangian multipliers. Equations (34a) and (34c) merely require that the altered probability distributions be true probability distributions whereas equations (34b) and (34d) require that, on the average, the number of members in the sample after the splitting is changed by the factor m_1 for the x_1-variable stage and m_2 for the x_2-variable stage. (From this point of view, the factors m_1 and m_2 can be regarded as "magnification" factors.)

Multiplying each of the equations of constraints by so-called Lagrangian multipliers and adding these terms to the expression for $N\sigma_{\mathcal{g}}^2$ defines the quantity

$$L = N\sigma_{\mathcal{g}}^2 + \lambda_I^2\left(\sum_i \frac{p_i}{u_i} - 1\right) + \lambda_{II}^2\left(\sum_i \frac{p_i}{w_i} - m_1\right) + \sum_i \lambda_i^2 p_i\left(\sum_j \frac{p_{ij}}{u_{ij}} - 1\right)$$

$$+ \lambda_{III}^2\left(\sum_{i,j} \frac{p_i p_{ij}}{w_{ij}} - m_1 m_2\right)$$

where λ_I^2, λ_{II}^2, $\lambda_i^2 (i=1, 2, \ldots)$, λ_{III}^2 denote the Lagrangian multipliers. This expression is minimized with respect to each of the four variables u_i, w_i, w_{ij}, and w_{ij} by setting the appropriate partial derivatives to zero:

$$\left.\begin{aligned}
\left.\frac{\partial L}{\partial u_i}\right|_{\hat{u}_i, \hat{w}_i, \hat{u}_{ij}, \hat{w}_{ij}} &= p_i E_i^2[g] - \lambda_I^2 \frac{p_i}{\hat{u}_i^2} = 0 \\
\left.\frac{\partial L}{\partial w_i}\right|_{\hat{u}_i, \hat{w}_i, \hat{u}_{ij}, \hat{w}_{ij}} &= p_i \sum_j p_{ij} \hat{u}_{ij} E_{ij}^2[g] - p_i E_i^2[g] - \lambda_{II}^2 \frac{p_i}{\hat{w}_i^2} = 0 \\
\left.\frac{\partial L}{\partial u_{ij}}\right|_{\hat{u}_i, \hat{w}_i, \hat{u}_{ij}, \hat{w}_{ij}} &= p_i p_{ij} \hat{w}_i E_{ij}^2[g] - \lambda_i^2 \frac{p_i p_{ij}}{\hat{u}_{ij}^2} = 0 \\
\left.\frac{\partial L}{\partial w_{ij}}\right|_{\hat{u}_i, \hat{w}_i, \hat{u}_{ij}, \hat{w}_{ij}} &= p_i p_{ij} \sigma_{g_{ij}}^2 - \lambda_{III}^2 \frac{p_i p_{ij}}{\hat{w}_{ij}^2} = 0
\end{aligned}\right\} \quad (35)$$

Solving equations (34) and (35) for the optimum values of $\hat{u}_i$, $\hat{w}_i$, $\hat{u}_{ij}$, and $\hat{w}_{ij}$ gives

$$\frac{1}{\hat{u}_i} = \frac{E_i[g]}{E[g]} \tag{36a}$$

$$\frac{1}{\hat{u}_{ij}} = \frac{E_{ij}[g]}{E_i[g]} \tag{36b}$$

$$\frac{1}{\hat{w}_{ij}} = \frac{m_1 m_2 \sigma_{g_{ij}}}{E\left[\sigma_{g_{ij}}\right]} \tag{36c}$$

If u_i, u_{ij}, and w_{ij} take on the values given by equations (36), then it turns out that $N\sigma_{\mathcal{Z}}^2$ is independent of the form of w_i so that the simplest expression for $1/s_i$ in accordance with the constraint given by equation (34b) is

$$\frac{1}{\hat{s}_i} = m_1$$

which yields

$$\frac{1}{\hat{w}_i} = \frac{m_1 E_i[g]}{E[g]} \tag{36d}$$

It can be seen from equations (36a) and (36b) that the optimum values $\hat{u}_i$ and $\hat{u}_{ij}$ of the weight factors correspond to "adjoint biasing" (refs. 1, 10, and 11) wherein the biasing as given by the reciprocal of the u's is proportional to the expected contribution of the member to the answer $E[g]$.

The minimum value of $N\sigma_{\mathcal{Z}}^2$ is found by substituting equations (36) into equation (32a)

$$\widehat{N\sigma_{\mathcal{Z}}^2} = \frac{E^2\left[\sigma_{g_{ij}}\right]}{m_1 m_2} \tag{37}$$

The previous equations can easily be converted to the case wherein x_1 and x_2 are continuous variables (or sets of continuous variables). The following relations hold

$$\mathcal{Z} = \frac{\sum\limits_{\alpha} u(x_1)s(x_1)u(x_1, x_2)s(x_1, x_2)\left(g(x_1, x_2)\right)_\alpha}{N} \tag{24'}$$

where the summation is taken over all measurements.

$$E[\mathcal{Z}] = \iint E[g/x_1, x_2]f(x_1, x_2)dx_1\, dx_2 = E[g] \tag{27'}$$

$$\begin{aligned} N\sigma^2_{\mathcal{Z}} &= \iint w(x_1, x_2)\left(\sigma^2_g/x_1, x_2\right)f(x_1, x_2)dx_1\, dx_2 \\ &+ \iint w(x_1)u(x_1, x_2)E^2[g/x_1, x_2]f(x_1, x_2)dx_1\, dx_2 \\ &- \int w(x_1)E^2[g/x_1]f(x_1)dx_1 \\ &+ \int u(x_1)E^2[g/x_1]f(x_1)dx_1 - E^2[g] \end{aligned} \tag{32a'}$$

$$w(x_1) \equiv u(x_1)s(x_1) \tag{33a'}$$

$$w(x_1, x_2) \equiv u(x_1)s(x_1)u(x_1, x_2)s(x_1, x_2) = w(x_1)u(x_1, x_2)s(x_1, x_2) \tag{33b'}$$

$$\int \frac{f(x_1)}{u(x_1)}\, dx_1 = 1 \tag{34a'}$$

$$\int \frac{f(x_1)}{w(x_1)}\, dx_1 = m_1 \tag{34b'}$$

$$\int \frac{f(x_2/x_1)}{u(x_1, x_2)}\, dx_2 = 1 \tag{34c'}$$

$$\iint \frac{f(x_1, x_2)}{w(x_1, x_2)}\, dx_1\, dx_2 = m_1 m_2 \tag{34d'}$$

$$\frac{1}{\hat{u}(x_1)} = \frac{E[g/x_1]}{E[g]} \tag{36a'}$$

$$\frac{1}{\hat{u}(x_1, x_2)} = \frac{E[g/x_1, x_2]}{E[g/x_1]} \tag{36b'}$$

$$\frac{1}{\hat{w}(x_1, x_2)} = \frac{m_1 m_2(\sigma_g/x_1, x_2)}{E[\sigma_g/x_1, x_2]} \tag{36c'}$$

$$\frac{1}{\hat{w}(x_1)} = \frac{m_1 E[g/x_1]}{E[g]} \tag{36d'}$$

$$N\widehat{\sigma^2_{\mathfrak{Z}}} = \frac{E^2[\sigma_g/x_1, x_2]}{m_1 m_2} \tag{37'}$$

CARRYING OUT OF THE SPLITTING PROCESS

The question now is how best to carry out the process of splitting. According to sketch (b), in the splitting part of the $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ stage, the number $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ of members is multiplied by the factor $\begin{Bmatrix} 1/s_i \\ 1/s_{ij} \end{Bmatrix}$ to yield the number $\begin{Bmatrix} \nu_i \\ \nu_{ij} \end{Bmatrix}$. This is a simplified concept of what can be done because $\begin{Bmatrix} 1/s_i \\ 1/s_{ij} \end{Bmatrix}$ is not, in general, an integer. Since, in a literal sense, we cannot work with fractional members, a method must be devised to give the necessary flexibility.

In some cases, the number $\begin{Bmatrix} \nu_i \\ \nu_{ij} \end{Bmatrix}$ of members corresponding to a given value of $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ is left to chance where the probability is so chosen that the required relations

$$E[\nu_i/n_i] = \frac{1}{s_i} n_i \tag{38a}$$

$$E[\nu_{ij}/n_{ij}] = \frac{1}{s_{ij}} n_{ij} \tag{38b}$$

hold. For example, suppose we write

$$\frac{1}{s_i} = I_i + r_i \tag{39a}$$

$$\frac{1}{s_{ij}} = I_{ij} + r_{ij} \tag{39b}$$

where $\begin{Bmatrix} I_i \\ I_{ij} \end{Bmatrix}$ is a non-negative integer and $0 \leq \begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix} < 1$. Then, if each of the $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ members gives rise to $\begin{Bmatrix} I_i \\ I_{ij} \end{Bmatrix}$ members $\begin{Bmatrix} 1-r_i \\ 1-r_{ij} \end{Bmatrix}$ of the time and $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix}$ members $\begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix}$ of the time, on the average, each of the $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ members generates $\begin{Bmatrix} I_i+r_i \\ I_{ij}+r_{ij} \end{Bmatrix}$ members, thereby satisfying equations (39). This process is an example of the "Russian roulette" method wherein particles are created or annihilated by chance. If this process is used as described to obtain the sampling as a function of the variables x_1 and x_2, then the expression for the variance of $\mathcal{J}$ as given by equation (32) no longer holds. The reason is that additional uncertainty has been introduced by the fact that the variance of $\begin{Bmatrix} \nu_i \\ \nu_{ij} \end{Bmatrix}$

corresponding to a given value of $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ is no longer zero. For this technique, the number of times $\mathscr{T}$ that the $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ members give rise to $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix}$ members is a binomial variable where $\begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix}$ is the probability of "success" and $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ is the number of trials. Because

$$\begin{Bmatrix} \nu_i \\ \nu_{ij} \end{Bmatrix} = \begin{Bmatrix} I_i n_i \\ I_{ij} n_{ij} \end{Bmatrix} + \mathscr{T}$$

it is easy to show that

$$E\left[\nu_i^2/n_i\right] = \frac{n_i^2}{s_i^2} + n_i r_i (1 - r_i) \tag{40a}$$

and

$$E\left[\nu_{ij}^2/n_{ij}\right] = \frac{n_{ij}^2}{s_{ij}^2} + n_{ij} r_{ij} (1 - r_{ij}) \tag{40b}$$

With the aid of these relations, the expression for the variance may be obtained in exactly the same way as carried through in the preceding section. The random variable $\mathscr{Z}$ is given by equation (24) and each term of equation (28) must be reevaluated for the situation under consideration. A point that might cause some difficulty is the evaluation of $E[\nu_{ij}\nu_{ij'}]$ for $j \neq j'$. This is accomplished as follows:

$$E[\nu_{ij}\nu_{ij'}] = \iiint\!\!\iiint \nu_{ij}\nu_{ij'} f(n_i, \nu_i, n_{ij}, n_{ij'}, \nu_{ij}, \nu_{ij'}) dn_i \, d\nu_i \, dn_{ij} \, dn_{ij'} \, d\nu_{ij} \, d\nu_{ij'}$$

where $f(n_i, \nu_i, n_{ij}, n_{ij'}, \nu_{ij}, \nu_{ij'})$ is the appropriate multivariant probability density function. This equation can be written as

$$E[\nu_{ij}\nu_{ij'}] = \iiiint f(n_i, \nu_i, n_{ij}, n_{ij'})dn_i\, d\nu_i\, dn_{ij}\, dn_{ij'}$$

$$\times \iint \nu_{ij}\nu_{ij'}f(\nu_{ij}, \nu_{ij'}/n_i, \nu_i, n_{ij}, n_{ij'})d\nu_{ij}\, d\nu_{ij'}$$

But because the random variables ν_{ij} and $\nu_{ij'}$ only depend on n_{ij} and $n_{ij'}$, respectively, we get

$$E[\nu_{ij}\nu_{ij'}] = \iiiint f(n_i, \nu_i, n_{ij}, n_{ij'})dn_i\, d\nu_i\, dn_{ij}\, dn_{ij'}$$

$$\times\left(\int \nu_{ij}f(\nu_{ij}/n_{ij})d\nu_{ij}\right)\left(\int \nu_{ij'}f(\nu_{ij'}/n_{ij'})d\nu_{ij'}\right)$$

$$= E\left[\left(E[\nu_{ij}/n_{ij}]\right)\cdot\left(E[\nu_{ij'}/n_{ij'}]\right)\right]$$

$$= \frac{1}{s_{ij}s_{ij'}}\, E[n_{ij}n_{ij'}] = \frac{1}{s_{ij}s_{ij'}}\, \frac{p_{ij}}{u_{ij}}\, \frac{p_{ij'}}{u_{ij'}}\, E\left[\nu_i^2 - \nu_i\right]$$

For the splitting technique involving Russian roulette as described, the increase $\Delta\left(E\left[\mathcal{J}_{ij}^2\right]\right)$ in $E\left[\mathcal{J}_{ij}^2\right]$ over that value given by equation (29) is found to be

$$\Delta\left(E\left[\mathcal{J}_{ij}^2\right]\right) = \frac{1}{N}\, p_i p_{ij}^2 w_i s_i r_i(1 - r_i)E_{ij}^2[g]$$

$$+ \frac{1}{N}\, p_i p_{ij} w_{ij} s_{ij} r_{ij}(1 - r_{ij})E_{ij}^2[g]$$

The increase $\Delta\left(E\,[\mathcal{J}_{ij}\mathcal{J}_{ij'}]\right)$ in $E\,[\mathcal{J}_{ij}\mathcal{J}_{ij'}]$ over that value given by equation (30) is found to be

$$\Delta\left(E[\mathcal{Z}_{ij}\mathcal{Z}_{ij'}]\right) = \frac{1}{N}\, p_i p_{ij} p_{ij'} w_i s_i r_i (1 - r_i) E_{ij}[g] E_{ij'}[g]$$

Equation (31) is found to remain unchanged. The increases result in an increase $\Delta\left(N\sigma^2_{\mathcal{Z}}\right)$ in $N\sigma^2_{\mathcal{Z}}$ over that value given by equation (32):

$$\Delta\left(N\sigma^2_{\mathcal{Z}}\right) = \sum_i p_i w_i s_i r_i (1 - r_i) E_i^2[g] + \sum_{i,j} p_i p_{ij} w_{ij} s_{ij} r_{ij} (1 - r_{ij}) E_{ij}^2[g] \tag{41}$$

To get an approximation of the magnitude of this increase, we make the tentative assumption that both r_i and r_{ij} are uniformly distributed over the interval (0, 1) so that

$$E[r(1 - r)] \approx \int_0^1 r(1 - r)dr = \frac{1}{6} \tag{42}$$

and equation (41) becomes

$$\Delta\left(N\sigma^2_{\mathcal{Z}}\right) \approx \frac{1}{6}\sum_i p_i w_i s_i E_i^2[g] + \frac{1}{6}\sum_{i,j} p_i p_{ij} w_{ij} s_{ij} E_{ij}^2[g] \tag{43}$$

In general, the increase in the variance of $\mathcal{Z}$ as given by this equation is not insignificant and, in some instances, could practically nullify the reduction obtained by splitting. Hence, the method of Russian roulette should be used with caution.

A technique whereby equations (38) are satisfied without the introduction of additional uncertainty in the final result is that in which, for each of the original $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ members, $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix}$ members are generated, the last member having a weight $\begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix}$ that of the first $\begin{Bmatrix} I_i \\ I_{ij} \end{Bmatrix}$ members. Hence, in this method, provided $\begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix} \neq 0$, we actually follow $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix}$ members but weight the last one differently. Here our relevant random variable has changed from that given by equation (24) to

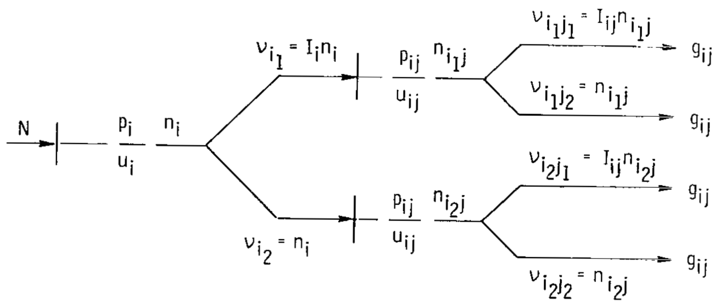
$$Y = \sum_{i,j} Y_{ij} \tag{44}$$

where

$$Y_{ij} = \frac{w_{ij}}{N} \left\{ \sum_{\alpha_{11}=1}^{\nu_{i_1 j_1}} (g_{ij})_{\alpha_{11}} + r_i \sum_{\alpha_{21}=1}^{\nu_{i_2 j_1}} (g_{ij})_{\alpha_{21}} + r_{ij} \sum_{\alpha_{12}=1}^{\nu_{i_1 j_2}} (g_{ij})_{\alpha_{12}} \right.$$

$$\left. + r_i r_{ij} \sum_{\alpha_{22}=1}^{\nu_{i_2 j_2}} (g_{ij})_{\alpha_{22}} \right\} \tag{45}$$

and we have subdivided the paths as shown in sketch (c). Hence



(c)

$$E\left[\nu_{i_1 j_1}\right] = N\left(\frac{p_i}{u_i}\right) I_i \left(\frac{p_{ij}}{u_{ij}}\right) I_{ij} \tag{46a}$$

$$E\left[\nu_{i_2 j_1}\right] = N\left(\frac{p_i}{u_i}\right)\left(\frac{p_{ij}}{u_{ij}}\right) I_{ij} \tag{46b}$$

$$E\left[\nu_{i_1 j_2}\right] = N\left(\frac{p_i}{u_i}\right) I_i \left(\frac{p_{ij}}{u_{ij}}\right) \tag{46c}$$

$$E\left[\nu_{i_2 j_2}\right] = N\left(\frac{p_i}{u_i}\right)\left(\frac{p_{ij}}{u_{ij}}\right) \tag{46d}$$

Note that

$$E[Y_{ij}] = \frac{w_{ij}}{N} E_{ij}[g] N \frac{p_i p_{ij}}{u_i u_{ij}} [I_i I_{ij} + r_i I_{ij} + r_{ij} I_i + r_i r_{ij}] = w_{ij} E_{ij}[g] \frac{p_i p_{ij}}{u_i u_{ij} s_i s_{ij}} = p_i p_{ij} E_{ij}[g]$$

and thus

$$E[Y] = E[g]$$

as desired.

This choice of random variable introduces no new uncertainties into the calculations and the fact that we are dealing with more members than previously (because $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix} \geq \begin{Bmatrix} I_i+r_i \\ I_{ij}+r_{ij} \end{Bmatrix} = \begin{Bmatrix} 1/s_i \\ 1/s_{ij} \end{Bmatrix}$) results in a reduction in the variance. In appendix A, it is shown that

$$N\sigma_Y^2 = N\sigma_{\mathcal{J}}^2 - \sum_{i,j} p_i p_{ij} w_{ij} \sigma_{g_{ij}}^2 \left\{1 - \left[1 - s_i r_i (1 - r_i)\right]\left[1 - s_{ij} r_{ij} (1 - r_{ij})\right]\right\}$$

$$- \sum_{i,j} p_i w_i s_i r_i (1 - r_i) \frac{p_{ij}}{u_{ij}} \left(u_{ij} E_{ij}[g] - E_i[g]\right)^2 \tag{47}$$

where $N\sigma_{\mathcal{J}}^2$ is given by equations (32). The disadvantage connected with this technique is that only the fractions r_i and r_{ij} of the measurements made on the last members of each stage are being used. In this sense, the method is not as efficient as it might be.

A more simple way of proceeding is to insist that $1/s_i$ and $1/s_{ij}$ are both integral for every i and j. In other words, even though the optimum values of these quantities are nonintegral, we always take, as the value to use for $1/s_i$ or $1/s_{ij}$, the smallest integer larger than or equal to the optimum value. In this way, we avoid both of the disadvantages associated with the two methods previously described - namely, (1) no new uncertainties arise and (2) each of the $(g_{ij})_\alpha$ measurements corresponding to particular values of i and j carries the same weight. However, if this method is used, then, as also occurs in the previous method, the number of particles in a splitting step never decreases.

Kahn (ref. 3) suggests two separate treatments in a splitting step and which of the two treatments an individual member receives depends on its values of the measured x-variables. Sample members in a splitting step are classed as belonging to either a type I or II region. In type I regions, the optimum value of $1/s_i$ or $1/s_{ij}$ is less than or equal to, say, 0.5 so it is desired to decrease the number sampled in these regions; in type II regions, the optimum value of $1/s_i$ or $1/s_{ij}$ is larger than 0.5 so the number sampled in these regions should not decrease. Kahn uses Russian roulette on type I region members whereby a member having an optimum value of $\left\{\begin{matrix} 1/s_i = r_i \\ 1/s_{ij} = r_{ij} \end{matrix}\right\}$ less than 0.5 is given the chance of $\left\{\begin{matrix} r_i \\ r_{ij} \end{matrix}\right\}$ of going on and the chance of $\left\{\begin{matrix} 1-r_i \\ 1-r_{ij} \end{matrix}\right\}$ of being killed, thus satisfying equations (38). For type II region members, "integral" splitting is used where the $\left\{\begin{matrix} s_i \\ s_{ij} \end{matrix}\right\}$ taken on is the reciprocal of the integer closest to the optimum value of $\left\{\begin{matrix} 1/s_i \\ 1/s_{ij} \end{matrix}\right\}$. After going through two stages of splitting in each of which a member can be classified into one of the two regions I and II, there are four classes of members, that is, I-I, I-II, II-I, and II-II. Equations (32) and (41) apply in this case to yield the proper expression for the variance.

EXTENSION OF ANALYSIS TO ANY NUMBER OF STAGES

Equation (32b) gives the expression for variance where two stages have been employed, each stage being a composite stage consisting of an importance sampling step followed by a splitting step. This equation may be easily generalized to apply to any number of stages. For example, if there were three composite stages, then the appropriate relation becomes

$$N\sigma^2 = \sum_{i,j,k} \rho_{ijk} w_{ijk} \sigma^2_{g_{ijk}} + \sum_{i,j,k} \frac{\rho_{ijk}}{w_{ij} u_{ijk}} \left(w_{ij} u_{ijk} E_{ijk}[g] - w_{ij} E_{ij}[g] \right)^2$$

$$+ \sum_{i,j} \frac{\rho_{ij}}{w_i u_{ij}} \left(w_i u_{ij} E_{ij}[g] - w_i E_i[g] \right)^2 + \sum_i \frac{\rho_i}{u_i} \left(u_i E_i[g] - E[g] \right)^2 \tag{48a}$$

where

$$\left.\begin{aligned} w_{ijk} &= w_{ij}u_{ijk}s_{ijk} \\ w_{ij} &= w_i u_{ij} s_{ij} \\ w_i &= u_i s_i \end{aligned}\right\} \quad (48b)$$

and where the symbol $\wp$ has been introduced to denote the unconditional probabilities - namely,

$$\left.\begin{aligned} &\wp_i = p_i = \text{probability that } x_1 \text{ takes on its } i^{th} \text{ value set} \\ &\wp_{ij} = p_i p_{ij} = \text{probability that } x_1 \text{ takes on its } i^{th} \text{ value set} \\ &\quad \text{and } x_2 \text{ its } j^{th} \text{ value set, simultaneously} \\ &\wp_{ijk} = \wp_{ij} p_{ijk} = p_i p_{ij} p_{ijk} = \text{probability that } x_1 \text{ takes on its} \\ &\quad i^{th} \text{ value set, } x_2 \text{ its } j^{th} \text{ value set, and } x_3 \text{ its } k^{th} \\ &\quad \text{value set, simultaneously} \end{aligned}\right\} \quad (48c)$$

In general, for any given number κ of successive composite stages, the equation for variance can be written

$$\begin{aligned} N\sigma^2 = &\sum_{i_1, i_2, \ldots, i_\kappa} \wp_{i_1 i_2 \ldots i_\kappa} w_{i_1 i_2 \ldots i_\kappa} \sigma^2_{g_{i_1 i_2 \ldots i_\kappa}} \\ &+ \sum_{\rho=1}^{\kappa} \sum_{i_1, i_2, \ldots, i_\rho} \frac{\wp_{i_1 i_2 \ldots i_\rho}}{w_{i_1 i_2 \ldots i_{\rho-1}} u_{i_1 i_2 \ldots i_\rho}} \\ &\times \Big(w_{i_1 i_2 \ldots i_{\rho-1}} u_{i_1 i_2 \ldots i_\rho} E_{i_1 i_2 \ldots i_\rho}[g] \\ &- w_{i_1 i_2 \ldots i_{\rho-1}} E_{i_1 i_2 \ldots i_{\rho-1}}[g] \Big)^2 \end{aligned} \quad (49a)$$

or

$$N\sigma^2 = \sum_{i_1, i_2, \ldots, i_\kappa} \rho_{i_1 i_2 \ldots i_\kappa} w_{i_1 i_2 \ldots i_\kappa} \sigma^2_{g_{i_1 i_2 \ldots i_\kappa}}$$

$$+\sum_{\rho=1}^{\kappa} \sum_{i_1, i_2, \ldots, i_\rho} \Big(\rho_{i_1 i_2 \ldots i_\rho} w_{i_1 i_2 \ldots i_{\rho-1}} u_{i_1 i_2 \ldots i_\rho} E^2_{i_1 i_2 \ldots i_\rho}[g]$$

$$- \rho_{i_1 i_2 \ldots i_{\rho-1}} w_{i_1 i_2 \ldots i_{\rho-1}} E^2_{i_1 i_2 \ldots i_{\rho-1}}[g] \Big) \tag{49b}$$

where ρ_{i_1} is the probability that x_1 takes on its i_1^{th} value set, $\rho_{i_1 i_2}$ is the probability that both x_1 takes on its i_1^{th} value set, and x_2 its i_2^{th} value set, etc.

Again, as noted just after equations (33), these expressions for the variance also hold for the cases where the stages are pure importance sampling stages (s=1) or pure splitting stages (u=1). (We shall designate a composite stage by the symbol "Ψ", a pure importance stage by "I", and a pure splitting stage by "S".) Also, as demonstrated previously, these expressions may be readily modified to apply to continuous variables. Finally, these expressions must be adjusted in accordance with the relations of the preceding section if applicable.

OPTIMIZATION OF THE WEIGHT FACTORS FOR NON-NEGATIVE g

If, for the moment, we ignore how the actual splitting process at any given stage is to be effected, expressions for optimum weight factors may be worked out. For example, in the case of two pure importance stages (I-I), the weight factors u_i and u_{ij} satisfy equations (14a') and (14b'), respectively, and the optimum choice of these weight factors is given by equations (10') and (15') where equation (13') applies. For the case of two composite stages, Ψ-Ψ where g is non-negative, the weight factors satisfy the conditions of equations (34) and their optimum choice is governed by equations (36).

For the general situation, it should first be noted that the number of stages in the sampling procedure does not necessarily change the minimum variance attainable. As an example, let us again restrict ourselves to non-negative g and consider the three stage sampling I-S-Ψ, that is, where the first stage is a pure importance stage, the second pure splitting, and the last a composite stage. In this case, s_i=1 and u_{ij}=1 so that

$$\left.\begin{aligned} w_i &= u_i \\ w_{ij} &= u_i s_{ij} \\ w_{ijk} &= u_i s_{ij} u_{ijk} s_{ijk} \end{aligned}\right\} \tag{50}$$

and equation (48a) can be written

$$N\sigma^2 = \sum_{i,j,k} p_{ijk} w_{ijk} \sigma^2_{g_{ijk}} + \sum_{i,j,k} p_{ijk} w_{ij} u_{ijk} E^2_{ijk}[g]$$

$$- \sum_{i,j} p_{ij} w_{ij} E^2_{ij}[g] + \sum_{i,j} p_{ij} u_i E^2_{ij}[g] - E^2[g] \tag{51}$$

The constraints to be satisfied by the weight factors are

$$\sum_i \frac{p_i}{u_i} = 1 \tag{52a}$$

$$\sum_{i,j} \frac{p_{ij}}{w_{ij}} = m_2 \tag{52b}$$

$$\sum_k \frac{p_{ijk}}{u_{ijk}} = 1 \tag{52c}$$

$$\sum_{i,j,k} \frac{p_{ijk}}{w_{ijk}} = m_2 m_3 \tag{52d}$$

Using the method of Langrangian multipliers to determine the minimum of $N\sigma^2$ subject to the constraints of equations (52), we find that the reciprocals of the optimum weight factors satisfy

$$\frac{1}{\hat{u}_i} = \frac{\epsilon_i}{E[\epsilon_i]} \tag{53a}$$

$$\frac{1}{\hat{u}_{ijk}} = \frac{E_{ijk}[g]}{E_{ij}[g]} \tag{53b}$$

$$\frac{1}{\hat{w}_{ijk}} = \frac{m_2 m_3 \sigma_{g_{ijk}}}{E\left[\sigma_{g_{ijk}}\right]} \tag{53c}$$

where ϵ_i^2 is defined by the equation

$$\epsilon_i^2 \equiv E_i\left[E_{ij}^2[g]\right] = \sum_j p_{ij} E_{ij}^2[g] \tag{54}$$

It turns out that the minimum value of $N\sigma^2$ is independent of the form of w_{ij} and is equal to

$$\widehat{N\sigma^2} = \frac{E^2\left[\sigma_{g_{ijk}}\right]}{m_2 m_3} + E^2[\epsilon_i] - E^2[g] \tag{55}$$

where the sum of the last two terms on the right is non-negative as shown by

$$\begin{aligned} E^2[\epsilon_i] - E^2[g] &= \left(E[\epsilon_i] + E[g]\right) \cdot \left(E[\epsilon_i] - E[g]\right) \\ &= \left(E[\epsilon_i] + E[g]\right) \cdot \left(E\left[\epsilon_i - E_i[g]\right]\right) \end{aligned}$$

coupled with the relation

$$\epsilon_i^2 - E_i^2[g] = E_i\left[\left(E_{ij}[g] - E_i[g]\right)^2\right] \geq 0$$

Another way of demonstrating that $\epsilon_i \geq E_i[g]$ is

$$\epsilon_i^2 - E_i^2[g] = \sum_j p_{ij} E_{ij}^2[g] - \left(\sum_j p_{ij} E_{ij}[g] \right)^2$$

$$= \sum_{j, j'} p_{ij} p_{ij'} \left(E_{ij}^2[g] - E_{ij}[g] E_{ij'}[g] \right)$$

$$= \frac{1}{2} \sum_{j, j'} p_{ij} p_{ij'} \left(E_{ij}[g] - E_{ij'}[g] \right)^2 \geq 0$$

However, if the splitting stage in I-S-Ψ is eliminated by changing to an I-$(\Psi)_{23}$ sampling where $(\Psi)_{23}$ denotes a composite stage for the two sets of variables x_2 and x_3 simultaneously, then, equation (32) applies where $j \rightarrow (j, k)$ so that

$$N\sigma^2 = \sum_{i, (j, k)} \rho_{i(jk)} w_{i(jk)} \sigma^2_{g_{i(jk)}} + \sum_{i, (j, k)} \rho_{i(jk)} u_i u_{i(jk)} E^2_{i(jk)}[g] - E^2[g] \qquad (56)$$

The constraints to be satisfied by the weight factors are

$$\sum_i \frac{p_i}{u_i} = 1 \qquad (57a)$$

$$\sum_{(j, k)} \frac{p_{i(jk)}}{u_{i(jk)}} = 1 \qquad (57b)$$

$$\sum_{i, (j, k)} \frac{\rho_{i(jk)}}{w_{i(jk)}} = m_{23} \qquad (57c)$$

The reciprocals of the optimum weight factors for non-negative g are given by

$$\frac{1}{\hat{u}_i} = \frac{E_i[g]}{E[g]} \tag{58a}$$

$$\frac{1}{\hat{u}_{i(jk)}} = \frac{E_{i(jk)}[g]}{E_i[g]} \tag{58b}$$

$$\frac{1}{\hat{w}_{i(jk)}} = \frac{m_{23}\sigma_{g_{i(jk)}}}{E\left[\sigma_{g_{i(jk)}}\right]} \tag{58c}$$

and the minimum variance for non-negative g is

$$N\widehat{\sigma^2} = \frac{E^2\left[\sigma_{g_{i(jk)}}\right]}{m_{23}} \tag{59}$$

Hence, if m_{23} of equation (59) is equal to $m_2 m_3$ of equation (55), then the minimum variance of an I-$(\Psi)_{23}$ sampling is less than or equal to the minimum variance of an I-S-Ψ sampling. (Of course, in order to attain the minimum variance in both cases, the answer $E[g]$ among other things must be known before sampling begins.) Note that the optimum biasing of the I stage in the I-$(\Psi)_{23}$ sampling is adjoint biasing whereas, in the I-S-Ψ sampling, the optimum biasing of the I stage does not correspond to adjoint biasing.

Another point may be discussed in connection with our problem. If we consider a $(\Psi)_{123}$ sampling, then the minimum variance expression is again given by equation (59) with m_{23} going over to m_{123} and the optimum biasing of the importance step of the composite $(\Psi)_{123}$ stage corresponds to adjoint biasing, namely,

$$\frac{1}{\hat{u}_{(ijk)}} = \frac{E_{(ijk)}[g]}{E[g]}$$

Again the minimum variance attainable is not affected adversely.

It must be mentioned here that a three-variable $(\Psi)_{123}$ sampling yields an optimum minimum variance that is, in general, less than a single-variable Ψ sampling (contain ing the same number of elements in the sample) wherein only the first variable x_1 is measured. For a Ψ sampling and non-negative g, the minimum variance is

$$N\hat{\sigma}^2 = \frac{E^2\left[\sigma_{g_i}\right]}{m_1} \tag{60}$$

whereas equation (59) with m_{123} replacing m_{23} holds for the $(\Psi)_{123}$ sampling. Inasmuch as

$$E\left[\sigma_{g_i}\right] \geq E\left[\sigma_{g_{ijk}}\right] \tag{61}$$

as demonstrated in appendix B, our contention is readily established.

For non-negative g, it turns out that with any given number κ of sampling stages, the minimum variance with the least number of members in the sample is obtained if the last stage is composite and all stages except the last are adjoint-biased pure importance sampling stages. The importance step of the composite stage should be adjoint biased and the splitting step should be biased in accordance with

$$\frac{1}{\hat{w}_{i_1 i_2 \ldots i_\kappa}} = \frac{m_\kappa \sigma_{g_{i_1 i_2 \ldots i_\kappa}}}{E\left[\sigma_{g_{i_1 i_2 \ldots i_\kappa}}\right]} \tag{62}$$

The minimum variance for this situation is

$$\hat{\sigma}^2 = \frac{E^2\left[\sigma_{g_{i_1 i_2 \ldots i_\kappa}}\right]}{m_\kappa N} \tag{63}$$

It is instructive to investigate the general problem of the optimum biasing of a stage for other cases than that just presented. It is found that, in order to set up the optimum biasing of a stage, one must know the nature of the stages following the stage in question. The optimum biasing results for non-negative g are depicted in the accompanying table I. The following notation is used in the table:

$$\tau^2 \equiv E[g^2]; \; \tau_i^2 \equiv E_i[g^2]; \; \tau_{ij}^2 \equiv E_{ij}[g^2]; \; . \; . \; . \tag{64}$$

$$\left.\begin{aligned} \epsilon^2 &\equiv E\left[E_i^2[g]\right] = \sum_i p_i E_i^2[g] \\ \epsilon_i^2 &\equiv E_i\left[E_{ij}^2[g]\right]; \; \epsilon_{ij}^2 \equiv E_{ij}\left[E_{ijk}^2[g]\right]; \; . \; . \; . \end{aligned}\right\} \tag{65}$$

$$\delta^2 \equiv \epsilon^2 - E^2[g]; \; \delta_i^2 \equiv \epsilon_i^2 - E_i^2[g]; \; \delta_{ij}^2 \equiv \epsilon_{ij}^2 - E_{ij}^2[g]; \; . \; . \; . \tag{66}$$

$$\left.\begin{aligned} \gamma^2(\square) &\equiv E^2[\square] - E^2[g] \\ \gamma_i^2(\square) &\equiv E_i^2[\square] - E_i^2[g] \\ \gamma_{ij}^2(\square) &\equiv E_{ij}^2[\square] - E_{ij}^2[g] \end{aligned}\right\} \tag{67}$$

where the symbol $\square$ stands for any quantity. For example, by equations (67),

$$\gamma_{ij}^2(\epsilon_{ijkl}) = E_{ij}^2[\epsilon_{ijkl}] - E_{ij}^2[g]$$

$$\gamma_i^2(\tau_{ij}) = E_i^2[\tau_{ij}] - E_i^2[g]$$

The symbol m_i is the magnification factor for the i^{th} stage. Of course, $m_i \equiv 1$ if the i^{th} stage is a pure importance sampling stage.

Equation (B1a) shows that

$$\tau_i^2 = E_i\left[\tau_{ij}^2\right] \tag{68}$$

and equation (B1b) shows that

$$\tau_{ij}^2 = E_{ij}\left[\tau_{ijk}^2\right] \tag{69}$$

so that

$$\tau_i^2 = E_i\left[\tau_{ijk}^2\right] = E_i\left[\tau_{ijkl}^2 \ldots\right] \tag{70}$$

As can be seen from table I, the optimum importance sampling step of a composite stage is always adjoint biased. Also, in general, the optimum choice of a weight factor for a particular stage depends on the nature of the stages following the stage under consideration - in particular, on whether the stages terminate before the first pure splitting or composite stage occurs. (Note that the results are the same for the pure splitting stage and the splitting step of the composite stage.) The blank spaces in the tabulation for the S or S_Ψ steps indicate that the minimum variance is independent of the biasing employed with these stages and, hence, for simplicity, uniform biasing is to be employed.

As an illustration of the use of table I, let us consider a four-variable sampling of non-negative g which is to proceed as Ψ-I-S-I. The following conditions apply in general to the weight factors:

$$\left.\begin{aligned}
&\sum_i \frac{p_i}{u_i} = 1, \sum_i \frac{\rho_i}{w_i} = m_1 \\
&\sum_j \frac{p_{ij}}{u_{ij}} = 1 \\
&\sum_{i,j,k} \frac{\rho_{ijk}}{w_{ijk}} = m_1 m_3 \\
&\sum_l \frac{p_{ijkl}}{u_{ijkl}} = 1
\end{aligned}\right\} \tag{71}$$

According to table I, the optimum weight factors are given by

1st stage: Ψ followed by I-S . . .

Importance step I_Ψ:

$$\frac{1}{\hat{u}_i} \approx \frac{E_i[g]}{E[g]} \tag{72a}$$

Splitting step S_Ψ:

$$\frac{1}{\hat{w}_i} = \frac{m_1 \gamma_i(\epsilon_{ij})}{E\left[\gamma_i(\epsilon_{ij})\right]} \tag{72b}$$

2^{nd} stage: I followed by S . . .

$$\frac{1}{\hat{u}_{ij}} = \frac{\epsilon_{ij}}{E_i[\epsilon_{ij}]} \tag{72c}$$

3^{rd} stage: S followed by I

$$\frac{1}{\hat{w}_{ijk}} = \frac{m_1 m_3 \gamma_{ijk}(\tau_{ijkl})}{E\left[\gamma_{ijk}(\tau_{ijkl})\right]} \tag{72d}$$

4^{th} stage: I (last stage)

$$\frac{1}{\hat{u}_{ijkl}} = \frac{\tau_{ijkl}}{E_{ijk}[\tau_{ijkl}]} \tag{72e}$$

(It may be noted that eq. (72e) arises from an "extrapolation" of table I.) Substituting these values into the expression for $N\sigma^2$ yields the result

$$\widehat{\sigma^2} = \frac{E^2\left[\gamma_i(\epsilon_{ij})\right]}{m_1 N} + \frac{E^2\left[\gamma_{ijk}(\tau_{ijkl})\right]}{m_1 m_3 N} \tag{73}$$

where each of the terms on the right side is non-negative.

OPTIMIZATION OF THE SIZE OF THE SAMPLE AT EACH STAGE

The magnification factors m_1 and m_3 may themselves be optimized. Suppose, on the average, the total cost T of conducting the Monte Carlo analysis for the example Ψ-I-S-I just considered is given by an expression of the form

$$T = c_{\Psi 1}N + c_{I2}(m_1N) + c_{S3}(m_1N) + c_{I4}(m_1m_3N) + c_g(m_1m_3N)$$

where

$c_{\Psi 1}$ = average cost per sample member processed through composite stage 1

c_{I2} = average cost per sample member processed through importance stage 2

c_{S3} = average cost per sample member processed through splitting stage 3

c_{I4} = average cost per sample member processed through importance stage 4

c_g = average cost per sample member of measuring g

The above equation can be written as

$$T = a_1^2N + a_2^2m_1N + a_3^2m_1m_3N \tag{74}$$

where

$$a_1^2 = c_{\Psi 1}$$

$$a_2^2 = c_{I2} + c_{S3}$$

$$a_3^2 = c_{I4} + c_g$$

In practice, $c_{\Psi 1}$ and c_{S3} are not constants independent of m_1 and m_3, respectively, but in this analysis we shall assume that such dependence can be neglected without introducing appreciable error. The problem as now set up is the minimization of $\hat{\sigma}^2$ given by equation (73) with respect to the variables N, m_1N, and m_1m_3N subject to the constraint that T of equation (74) is fixed. Carrying out this minimization process results in the theoretically optimum values

$$\left.\begin{aligned}
\hat{N} &= 0 \\
\widehat{(m_1 N)} &= \frac{TE\left[\gamma_i(\epsilon_{ij})\right]/a_2}{a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]} \\
\widehat{(m_1 m_3 N)} &= \frac{TE\left[\gamma_{ijk}(\tau_{ijkl})\right]/a_3}{a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]}
\end{aligned}\right\} \tag{75}$$

and the *minimum* variance for given T is

$$\widehat{(\sigma^2)} = \frac{1}{T}\left(a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]\right)^2 \tag{76}$$

where the corrections to be made dependent on how the splitting processes are carried out are not included in equation (76). Of course, since N must be a positive integer, the theoretically optimum result $\hat{N} = 0$ is not allowed and, in actuality, $\hat{N}$ should equal the smallest positive integer, namely, 1. The reason that $\hat{N}$ turns out to be zero is that the true optimum values of the weight factors are being employed. Appendix C shows that any deviations of the weight factors from their optimum values result in an increase in $\hat{N}$.

It was remarked previously that the corrections dependent on how the splitting process is effected are not considered in equation (76). As an example of how such corrections may be included, we turn again to the three-stage sampling I-S-Ψ considered previously. The minimum value of the variance is given by equation (55) as

$$\widehat{\sigma^2} = \frac{\gamma^2(\epsilon_i)}{N} + \frac{E^2\left[\sigma_{g_{ijk}}\right]}{m_2 m_3 N} \tag{55a}$$

where the notation of equations (65) and (67) had been incorporated in equation (55a). Here we also wish to obtain optimum values of N, m_2N, and m_2m_3N subject to the constraint that the quantity

$$T = b_1^2 N + b_2^2 m_2 N + b_3^2 m_2 m_3 N \tag{77}$$

It is realized that equation (51), from which we obtain equation (55a), does not include the additional variance terms (such as those given in eq. (41)) for the S and Ψ stages. Consequently, equation (55a) cannot represent a true minimum. Let us first consider the S stage. It is assumed that the added terms, that is, the second summation of equation (41), constitute only a small perturbation. If this is so, then equations (53a), (53b), and (53c) and the result that $\widehat{\sigma^2}$ is independent of the form of w_{ij} and, hence, on the form of s_{ij} can be considered as approximately correct. For simplicity, we can take s_{ij} to be a constant independent of i and j where, by equations (52a) and (52b), this constant must be $1/m_2$. If Kahn's procedure is followed, then m_2 is either a positive integer (corresponding to a type I region) or m_2 is a fraction less than 1 (corresponding to a type II region). In the first instance, m_2 is a positive integer and all r_{ij} are zero so that the added terms of equation (41) vanish and equation (55a) is exact. Now, since $\widehat{\sigma^2}$ is independent of (m_2N), if the process of minimization of $\widehat{\sigma^2}$ were mechanically carried out, the optimum value of (m_2N) would be zero. But, because m_2 is restricted to the positive integers, the lowest value for m_2 that can be chosen is unity and the splitting stage S is reduced to a unit stage U where $s_{ij} = 1$. In the second instance, m_2 is a fraction less than 1 and equation (51) must be modified to include the appropriate terms of equation (41). Minimization of the variance with such terms included show that the optimum value of m_2 for this case is unity. Thus, it has been proved that, to obtain the minimum variance, the splitting stage becomes a unit stage whereby the three-stage sampling process goes over to I-U-Ψ where equation (55a) is now

$$\widehat{\sigma^2} = \frac{\gamma^2(\epsilon_i)}{N} + \frac{E^2\left[\sigma_{g_{ijk}}\right]}{m_3N} \tag{78}$$

Equation (77) becomes

$$T = \left(b_1^2 + b_2^2\right)N + b_3^2 m_3 N \tag{79}$$

(It should be remarked that this result holds in general; that is, when the variance of the sampling process does not depend on the form of the splitting factor for a particular S stage or for the splitting step of a composite stage (blank spaces in table I), in order to obtain minimum variance, the splitting factor goes to unity.) Minimization of $\widehat{\sigma^2}$ of equation (78) with respect to N and m_3N subject to T of equation (79) being fixed yields

$$\left.\begin{aligned} \hat{N} &= \frac{T\gamma(\epsilon_i) \Big/ \sqrt{b_1^2 + b_2^2}}{\sqrt{b_1^2 + b_2^2\,\gamma(\epsilon_i) + b_3 E\left[\sigma_{g_{ijk}}\right]}} \\ \widehat{(m_3 N)} &= \frac{TE\left[\sigma_{g_{ijk}}\right] \Big/ b_3}{\sqrt{b_1^2 + b_2^2\,\gamma(\epsilon_i) + b_3 E\left[\sigma_{g_{ijk}}\right]}} \end{aligned}\right\} \qquad (80)$$

and

$$\hat{\hat{\sigma}}^2 = \frac{1}{T}\left(\sqrt{b_1^2 + b_2^2\,\gamma(\epsilon_i)} + b_3 E\left[\sigma_{g_{ijk}}\right]\right)^2 \qquad (81)$$

It must be remarked that the additional variance due to the splitting step of the Ψ stage which is equal to

$$\sum_{i,j,k} \rho_{ijk} w_{ijk} s_{ijk} r_{ijk} (1 - r_{ijk}) E_{ijk}^2[g]$$

has not been incorporated into equation (81) and the optimum values of r_{ijk} may have to be readjusted to keep the additional variance small.

REMOVAL OF NON-NEGATIVE RESTRICTION

Heretofore we have limited g to non-negative values. The hypothesis that g be non-negative meant that all expectation values of g are necessarily non-negative and permitted us to express the weight factors, which must themselves be non-negative quantities, in terms of these expectation values. Table I holds only for the case where g is non-negative.

If g can take on negative as well as positive values, then equations such as (36a) and (36b) cannot, in general, be written. To demonstrate the changes in the relations, let us treat the same problem (a Ψ-Ψ sampling) which led to equations (36a) and (36b) but now no longer regard g as being restricted to non-negative values. Equations (32), (34), and (35) still apply and the optimum values of $\hat{u}_i$, $\hat{w}_i$, $\hat{u}_{ij}$, and $\hat{w}_{ij}$ are given by

$$\frac{1}{\hat{u}_i} = \frac{\alpha_i}{E[\alpha_i]} \tag{82a}$$

$$\frac{1}{\hat{w}_i} = \frac{m_1\gamma_i(\alpha_{ij})}{E\left[\gamma_i(\alpha_{ij})\right]} \tag{82b}$$

$$\frac{1}{\hat{u}_{ij}} = \frac{\alpha_{ij}}{E_i[\alpha_{ij}]} \tag{82c}$$

$$\frac{1}{\hat{w}_{ij}} = \frac{m_1 m_2 \sigma_{g_{ij}}}{E\left[\sigma_{g_{ij}}\right]} \tag{82d}$$

where α_i denotes the absolute value of $E_i[g]$ and α_{ij} the absolute value of $E_{ij}[g]$; that is,

$$\alpha_i \equiv \left|E_i[g]\right|; \ \alpha_{ij} \equiv \left|E_{ij}[g]\right|; \ . \ . \ . \tag{83}$$

Note that

$$\alpha_i \equiv \left|E_i[g]\right| = \left|\sum_j p_{ij}E_{ij}[g]\right| \leq \sum_j p_{ij}\left|E_{ij}[g]\right| = E_i[\alpha_{ij}] \tag{84}$$

so that

$$\alpha_i \leq E_i[\alpha_{ij}] \leq E_i[\alpha_{ijk}] \leq . \ . \ . \tag{85a}$$

Similarly,

$$\alpha_{ij} \leq E_{ij}[\alpha_{ijk}] \leq E_{ij}[\alpha_{ijkl}] \leq . \ . \ . \tag{85b}$$

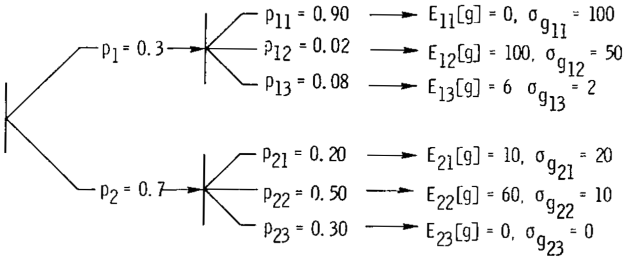
The minimum value of $N\sigma_{\mathcal{J}}^2$ is found by substituting equations (82) into equation (32a):

$$N\hat{\sigma}^2_{\mathcal{Z}} = \frac{E^2\left[\sigma_{g_{ij}}\right]}{m_1 m_2} + \frac{E^2\left[\gamma_i(\alpha_{ij})\right]}{m_1} + \gamma^2(\alpha_i) \tag{86}$$

Table II is a listing of the optimum values of the reciprocal weight factors when g can take on negative as well as positive values.

A SIMPLE EXAMPLE

We take a simple example to illustrate the concepts. We suppose the situation as shown in sketch (d). There are two stages of sampling, the parameters being given in



(d)

the sketch. The formula for the variance for the two-stage sampling is given as equation (32a) to which must be added equation (41) to take into account the added variance introduced by the splitting processes.

The variance resulting from straightforward sampling (U-U) for this example is

$$N\sigma^2_{UU} = E\left[\tau^2_{ij}\right] - E^2[g] = 3605 \tag{87}$$

We shall determine the minimum variance $\hat{\sigma}^2$ for each of the following four cases: (1) U-Ψ, (2) I-I, (3) Ψ-I, and (4) Ψ-Ψ. The magnification factor of any splitting stage or step will be set close to unity.

Case (1) U-Ψ

Equation (32a) becomes

$$N\sigma^2_{U\Psi} = \sum_{i,j} p_{ij} w_{ij} \sigma^2_{g_{ij}} + \sum_{i,j} p_{ij} u_{ij} E^2_{ij}[g] - E^2[g] \tag{88}$$

From table I, the optimum choices of the weight factors are

$$\frac{1}{\hat{u}_{ij}} = \frac{E_{ij}[g]}{E_i[g]}; \quad \frac{1}{\hat{w}_{ij}} = \frac{m_2 \sigma_{g_{ij}}}{E\left[\sigma_{g_{ij}}\right]} \tag{89}$$

Substituting these values into equation (88) results in

$$N\hat{\sigma}^2_{U\Psi} = \frac{E^2\left[\sigma_{g_{ij}}\right]}{m_2} + \delta^2 = \frac{1132}{m_2} + 183 \tag{90}$$

where a correction based on the splitting step is still to be added.

For $m_2 = 1$, the reduction in variance over that of the straightforward sampling situation can be written as

$$N\left(\hat{\sigma}^2_{UU} - \hat{\sigma}^2_{U\Psi}\Big|_{m_2=1}\right) = E\left[\tau^2_{ij}\right] - E^2\left[\sigma_{g_{ij}}\right] - \epsilon^2$$

$$= E\left[\left(\sigma_{g_{ij}} - E\left[\sigma_{g_{ij}}\right]\right)^2\right] + E\left[\left(E_{ij}[g] - E_i[g]\right)^2\right]$$

$$= 1673 + 616 = 2289$$

Calculating the optimum weight factors for $m_2 = 1$ from equations (89) and the relation

$$\frac{1}{s_{ij}} = \frac{\frac{1}{w_{ij}}}{\frac{1}{u_{ij}}}$$

gives

i	j	$1/\hat{u}_{ij}$	$1/\hat{w}_{ij}$	$1/\hat{s}_{ij}$
1	1	0^+	2.97	∞^-
	2	40.32	1.485	0.0368
	3	2.42	0.0594	0.0245
2	1	0.3126	0.594	1.90
	2	1.875	0.294	0.157
	3	0	0	------

Following Kahn's procedure as described previously, we take $1/s_{11}$ and $1/s_{21}$ to be integers. In particular, $1/s_{21}$ is taken as 2. This changes the numbers for i=2 and j=1 from those given previously to

$$\frac{1}{s_{21}} = 2, \quad \frac{1}{w_{21}} = \frac{1}{s_{21}} \frac{1}{\hat{u}_{21}} = 0.6252$$

and increases m_2 to slightly more than unity. Substituting the altered values for weight factors into equation (88) yields the new values of $N\sigma^2_{U\Psi}$ as 1310. To this figure must be added the increase in variance due to the Russian roulette process as given by equation (41):

$$\Delta(N\sigma^2) = \sum_{i,j} \rho_{ij} w_{ij} s_{ij} r_{ij} (1 - r_{ij}) E^2_{ij}[g]$$

$$= \rho_{12} w_{12} (1 - r_{12}) E^2_{12}[g] + \rho_{13} w_{13} (1 - r_{13}) E^2_{13}[g] + \rho_{22} w_{22} (1 - r_{22}) E^2_{22}[g]$$

$$= 38.9 + 14.2 + 3610 = 3663$$

Hence, the total variance using almost optimum weight factors is

$$N\sigma_{U\Psi}^2\Big|_{m_2=1.004} = 1310 + 3663 = 4973$$

which is a figure that is larger than that obtained with straightforward sampling. This large value is due to the Russian roulette process practiced on the i=2, j=2 branch which results in the addition of 3610 to $N\sigma^2$. The reason the term $p_{22}p_{22}w_{22}(1 - r_{22})E_{22}^2[g]$ is much larger than the other terms in $\Delta(N\sigma^2)$ is that the product $p_i p_{ij} E_{ij}^2[g]$ is much larger for the i=2, j=2 branch than for the other two branches in which Russian roulette is used.

In order to eliminate this large value of $\Delta(N\sigma^2)$, $1/s_{22}$ is taken as equal to 1 giving rise to the new values

$$\frac{1}{s_{22}} = 1, \quad \frac{1}{w_{22}} = \frac{1}{s_{22}} \quad \frac{1}{\hat{u}_{22}} = \frac{1}{\hat{u}_{22}} = 1.875$$

and thereby raising the value of m_2 to 1.551. Now the total variance becomes

$$N\sigma_{U\Psi}^2\Big|_{m_2=1.551} = 1263$$

where the contribution to this figure by the Russian roulette process in the splitting step is only about 50 in magnitude.

Case (2) I-I

Equation (32a) becomes

$$N\sigma_{II}^2 = \sum_{i,j} \rho_{ij} u_i u_{ij} \tau_{ij}^2 - E^2[g] \tag{91}$$

By table I, for this case,

$$\frac{1}{\hat{u}_i} = \frac{E_i[\tau_{ij}]}{E[\tau_{ij}]}; \quad \frac{1}{\hat{u}_{ij}} = \frac{\tau_{ij}}{E_i[\tau_{ij}]} \tag{92}$$

and

$$N\widehat{\sigma_{II}^2} = E^2[\tau_{ij}] - E^2[g] = 2194 \tag{93}$$

Here the reduction in variance is

$$N\left(\sigma_{UU}^2 - \widehat{\sigma_{II}^2}\right) = E\left[\tau_{ij}^2\right] - E^2[\tau_{ij}] = E\left[\left(\tau_{ij} - E[\tau_{ij}]\right)^2\right] = 1411$$

and the optimum weight factors as given by equations (92) are

i	j	$1/\hat{u}_i$	$1/\hat{u}_{ij}$
1	1	1.775	1.078
	2		1.206
	3		0.0682
2	1	0.668	0.640
	2		1.744
	3		0

Case (3) Ψ-I

By equation (32a)

$$N\sigma_{\Psi I}^2 = \sum_{i,j} \rho_{ij} w_i u_{ij} \tau_{ij}^2 - \sum_i \rho_i w_i E_i^2[g] + \sum_i \rho_i u_i E_i^2[g] - E^2[g] \tag{94}$$

and by table I

$$\frac{1}{\hat{u}_i} = \frac{E_i[g]}{E[g]}, \quad \frac{1}{\hat{w}_i} = \frac{m_1 \gamma_i(\tau_{ij})}{E\left[\gamma_i(\tau_{ij})\right]}, \quad \frac{1}{\hat{u}_{ij}} = \frac{\tau_{ij}}{E_i[\tau_{ij}]} \tag{95}$$

Substituting equations (95) into equation (94) results in

$$N\widehat{\sigma^2_{\Psi I}} = \frac{E^2\left[\gamma_i(\tau_{ij})\right]}{m_1} = \frac{1410}{m_1} \tag{96}$$

where the correction based on the splitting step of the composite stage is not yet incorporated.

For $m_1 = 1$,

$$N\left(\sigma^2_{UU} - \widehat{\sigma^2_{\Psi I}}\bigg|_{m_1=1}\right) = E\left[\tau^2_{ij}\right] - E^2[g] - E^2\left[\gamma_i(\tau_{ij})\right]$$

$$= E\left[\left(\tau_{ij} - E_i[\tau_{ij}]\right)^2\right] + E\left[\left(\gamma_i(\tau_{ij}) - E\left[\gamma_i(\tau_{ij})\right]\right)^2\right] + E\left[\left(E_i[g] - E[g]\right)^2\right]$$

$$= 708 + 1304 + 183 = 2195$$

The increase in variance due to the splitting step in the Ψ stage must be evaluated. We get, for $m_1 = 1$,

i	$1/\hat{u}_i$	$1/\hat{w}_i$	$1/\hat{s}_i$
1	0. 1071	2. 47	23. 04
2	1. 383	0. 37	0. 2675

Again, if Kahn's procedure is followed, then $1/s_1$ should be taken as 23 whereas $1/s_2$ is left at the value 0. 2675. Applying equation (41), we find

$$\Delta(N\sigma^2) = \sum_i p_i w_i s_i r_i (1 - r_i) E_i^2[g] = p_2 w_2 (1 - r_2) E_2^2[g] = 1419$$

This value, if left unchanged, would more than double the original value $N\widehat{\sigma^2_{\Psi I}}$. Hence, we take $(1/s_2) = 1$, which eliminates Russian roulette entirely. Our altered values are

i	j	$1/\hat{u}_i$	$1/s_i$	$1/w_i$	$1/\hat{u}_{ij}$
1	1	0.1071	23	2.463	1.078
	2				1.206
	3				0.0682
2	1	1.383	1	1.383	0.640
	2				1.744
	3				0

The value of m_1 has been raised to 1.707, and the total variance becomes

$$N\sigma_{\Psi I}^2\Big|_{m_1=1.707} = \sum_i p_i w_i \gamma_i^2(\tau_{ij})$$

$$= \frac{0.3\times 8595}{2.463} + \frac{0.7\times 193.1}{1.383} = 1047 + 98 = 1145$$

Case (4) Ψ-Ψ

The equations are

$$N\sigma_{\Psi\Psi}^2 = \sum_{i,j} p_i p_{ij} w_{ij} \sigma_{g_{ij}}^2 + \sum_{i,j} p_i p_{ij} w_i u_{ij} E_{ij}^2[g]$$

$$- \sum_i p_i w_i E_i^2[g] + \sum_i p_i u_i E_i^2[g] - E^2[g] \tag{97}$$

where, by table I,

$$\frac{1}{\hat{u}_i} = \frac{E_i[g]}{E[g]}, \quad \frac{1}{\hat{u}_{ij}} = \frac{E_{ij}[g]}{E_i[g]}, \quad \frac{1}{\hat{w}_{ij}} = \frac{m_1 m_2 \sigma_{g_{ij}}}{E\left[\sigma_{g_{ij}}\right]} \tag{98}$$

and the variance of the sampling process does not depend on the form of the splitting factor s_i. We get

$$N\widehat{\sigma}^2_{\Psi\Psi} = \frac{E^2\left[\sigma_{g_{ij}}\right]}{m_1 m_2} \tag{99}$$

As mentioned previously, for optimum results, the splitting step of the first composite stage is eliminated so that

$$\frac{1}{s_i} = 1, \quad i=1, 2 \tag{100}$$

and m_1 is 1. Thus, Ψ-Ψ goes over to I-Ψ and equation (99) now becomes

$$N\widehat{\sigma}^2_{\Psi\Psi} = N\widehat{\sigma}^2_{I\Psi} = \frac{E^2\left[\sigma_{g_{ij}}\right]}{m_2} = \frac{1133}{m_2} \tag{101}$$

For $m_2 = 1$,

$$N\left(\sigma^2_{UU} - \widehat{\sigma}^2_{I\Psi}\Big|_{m_2=1}\right) = E\left[\tau^2_{ij}\right] - E^2[g] - E^2\left[\sigma_{g_{ij}}\right]$$

$$= E\left[\left(\sigma_{g_{ij}} - E\left[\sigma_{g_{ij}}\right]\right)^2\right] + E\left[\left(E_{ij}[g] - E[g]\right)^2\right]$$

$$= 1673 + 799 = 2472$$

and

i	j	$\frac{1}{\hat{u}_i \hat{u}_{ij}}$	$\frac{1}{\hat{w}_{ij}}$	$\frac{1}{\hat{s}_{ij}}$
1	1	0^+	2.97	∞^-
	2	4.32	1.485	0.3435
	3	0.2593	0.0594	0.229
2	1	0.432	0.594	1.372
	2	2.593	0.297	0.1145
	3	0	0	0

Again, by Kahn's procedure, we take $1/s_{21}$ equal to unity. In addition, to avoid an unduly large variance due to the splitting method, $1/s_{22}$ is also set equal to unity. Hence, $1/w_{21} = 0.432$ and $1/w_{22} = 2.593$ and these values correspond to a value for m_2 of 1.781.

The formula for total variance now becomes

$$N\sigma_{I\Psi}^2\Big|_{m_2=1.781} = \sum_{i,j} \rho_{ij} w_{ij} \sigma_{g_{ij}}^2 + \sum_{i,j} \rho_{ij} w_{ij} s_{ij} r_{ij} (1 - r_{ij}) E_{ij}^2 \,[g]$$

$$= 1063 + 38 = 1101$$

SUMMARY AND CONCLUDING REMARKS

General formulas have been developed for the variance reduction obtained under different sampling schemes for the situation where the two techniques of importance sampling and splitting and Russian roulette are employed. Optimum biasing procedures have been determined for any number of independent random variables and the results are shown in tables I and II.

It is found that, when the random variable under consideration is non-negative, the minimum variance with the least number of members in the sample is obtained if the last random variable is sampled in a combination importance-splitting sampling manner and all preceding variables are sampled employing adjoint biasing. A short example has illustrated the possible increase in variance that may occur if indiscriminate Russian roulette practices are used.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 22, 1970,
129-01.

APPENDIX A

A SPLITTING TECHNIQUE

We are to consider the splitting method in which, for each of the original $\begin{Bmatrix} n_i \\ n_{ij} \end{Bmatrix}$ members, $\begin{Bmatrix} I_i+1 \\ I_{ij}+1 \end{Bmatrix}$ members are generated, the last member having a weight equal to $\begin{Bmatrix} r_i \\ r_{ij} \end{Bmatrix}$ that of the first $\begin{Bmatrix} I_i \\ I_{ij} \end{Bmatrix}$ members. Thus, an effective splitting factor equal to

$$\left\{\begin{matrix} \dfrac{1}{s_i} = I_i+r_i \\ \dfrac{1}{s_{ij}} = I_{ij}+r_{ij} \end{matrix}\right\} \text{is obtained.}$$

The random variable to be considered is that given in equations (44) and (45) of the text and repeated here:

$$Y = \sum_{ij} Y_{ij} \tag{44}$$

where

$$\begin{aligned} Y_{ij} = \frac{w_{ij}}{N} \Bigg\{ & \sum_{\alpha_{11}=1}^{\nu_{i_1 j_1}} \left(g_{ij}\right)_{\alpha_{11}} + r_i \sum_{\alpha_{21}=1}^{\nu_{i_2 j_1}} \left(g_{ij}\right)_{\alpha_{21}} \\ & + r_{ij} \sum_{\alpha_{12}=1}^{\nu_{i_1 j_2}} \left(g_{ij}\right)_{\alpha_{12}} + r_i r_{ij} \sum_{\alpha_{22}=1}^{\nu_{i_2 j_2}} \left(g_{ij}\right)_{\alpha_{22}} \Bigg\} \end{aligned} \tag{45}$$

The mechanics of carrying out the sampling procedure are as follows. A sample of size N is first picked from the x_1-population in accordance with the probability distribution function p_i/u_i and n_i designates the number of members possessing the i^{th} value set of x_1. These n_i members generate $\nu_{i_1} = I_i n_i$ members, each having an associated weight of w_i, and $\nu_{i_2} = n_i$ members, each having a weight of $w_i r_i$. The sample members are further subdivided in accordance with their x_2-values where the probability distribution function is p_{ij}/u_{ij}. Thus, of the members possessing the i^{th} value set of x_1 and the j^{th} value set of x_2, there are n_{i_1j} members with a weight of w_iu_i where $\sum_j n_{i_1j} = \nu_{i_1}$ and n_{i_2j} members with a weight of $w_ir_iu_i$ where $\sum_j n_{i_2j} = \nu_{i_2}$. Finally, each of these members in turn is split into $(I_{i_j}+1)$ members where the last member has a weight r_{ij} that of the others. Hence, for each i and j, there are four groups: (1) $\nu_{i_1j_1} = I_{ij}n_{i_1j}$ members with weight w_{ij}, (2) $\nu_{i_2j_1} = I_{ij}n_{i_2j}$ members with weight $w_{ij}r_i$, (3) $\nu_{i_1j_2} = n_{i_1j}$ members with weight $w_{ij}r_{ij}$, and (4) $\nu_{i_2j_2} = n_{i_2j}$ members with weight $w_{ij}r_ir_{ij}$.

We wish to determine the variance of Y. Proceeding as in equation (28), we have

$$\sigma_Y^2 = \sum_{i,j} E\left[Y_{ij}^2\right] + \sum_{\substack{i,j,j' \\ (j\neq j')}} E\left[Y_{ij}Y_{ij'}\right] + \sum_{\substack{i,j,i',j' \\ (i\neq i')}} E\left[Y_{ij}Y_{i'j'}\right] - E^2[g] \tag{A1}$$

The terms of the first summation on the right side of this summation may be written as

$$E\left[Y_{ij}^2\right] = \frac{w_{ij}^2}{N^2} E\left[\left(\nu_{i_1j_1}\overline{\left(g_{ij}\right)}_{11} + r_i\nu_{i_2j_1}\overline{\left(g_{ij}\right)}_{21} + r_{ij}\nu_{i_1j_2}\overline{\left(g_{ij}\right)}_{12} + r_ir_{ij}\nu_{i_2j_2}\overline{\left(g_{ij}\right)}_{22}\right)^2\right] \tag{A2}$$

There are ten terms to be evaluated, four squared terms and six cross products, and each can be handled by the methods employed previously. For example,

$$E\left[\nu_{i_1j_1}^2\overline{\left(g_{ij}\right)}_{11}^2\right] = E\left[\nu_{i_1j_1}^2\left(\frac{\sigma_{g_{ij}}^2}{\nu_{i_1j_1}} + E_{ij}^2[g]\right)\right]$$

$$= \sigma_{g_{ij}}^2 E\left[\nu_{i_1j_1}\right] + E_{ij}^2[g]\,E\left[\nu_{i_1j_1}^2\right]$$

$$= \sigma_{g_{ij}}^2 N\frac{p_i}{u_i}I_i\frac{p_{ij}}{u_{ij}}I_{ij} + E_{ij}^2[g]I_{ij}^2\,E\left[n_{i_1j}^2\right]$$

and

$$E\left[2r_{ij}\nu_{i_1j_1}\overline{\left(g_{ij}\right)_{11}}\nu_{i_1j_2}\overline{\left(g_{ij}\right)_{12}}\right] = 2r_{ij}E_{ij}^2[g]E\left[\nu_{i_1j_1}\nu_{i_1j_2}\right]$$

$$= 2r_{ij}E_{ij}^2[g]I_{ij}E\left[n_{i_1j}^2\right]$$

where

$$E\left[n_{i_1j}^2\right] = \left[N^2\frac{p_i^2}{u_i^2} + N\frac{p_i}{u_i}\left(1 - \frac{p_i}{u_i}\right)\right]I_i^2\frac{p_{ij}^2}{u_{ij}^2} + N\frac{p_i}{u_i}I_i\frac{p_{ij}}{u_{ij}}\left(1 - \frac{p_{ij}}{u_{ij}}\right)$$

Evaluating each of the terms entering into equation (A2), we find that

$$E\left[Y_{ij}^2\right] = E\left[\mathcal{J}_{ij}^2\right] - \frac{1}{N}p_ip_{ij}w_{ij}\sigma_{g_{ij}}^2\left\{1 - \left[1 - s_ir_i(1 - r_i)\right]\left[1 - s_{ij}r_{ij}(1 - r_{ij})\right]\right\}$$

$$- \frac{1}{N}p_ip_{ij}w_is_iu_{ij}\left(1 - \frac{p_{ij}}{u_{ij}}\right)r_i(1 - r_i)E_{ij}^2[g] \qquad \text{(A3)}$$

where the expression for $E\left[\mathcal{J}_{ij}^2\right]$ is given in equation (29). Continuing with the computations, we find that, for $j \neq j'$,

$$E\left[Y_{ij}Y_{ij'}\right] = \frac{w_{ij}w_{ij'}}{N^2}E\left[\left(\nu_{i_1j_1}\overline{\left(g_{ij}\right)_{11}} + r_i\nu_{i_2j_1}\overline{\left(g_{ij}\right)_{21}} + r_{ij}\nu_{i_1j_2}\overline{\left(g_{ij}\right)_{12}} + r_ir_{ij}\nu_{i_2j_2}\overline{\left(g_{ij}\right)_{22}}\right)\right.$$

$$\left.\cdot\left(\nu_{i_1j'_1}\overline{\left(g_{ij'}\right)_{11}} + r_i\nu_{i_2j'_1}\overline{\left(g_{ij'}\right)_{21}} + r_{ij'}\nu_{i_1j'_2}\overline{\left(g_{ij'}\right)_{12}} + r_ir_{ij'}\nu_{i_2j'_2}\overline{\left(g_{ij'}\right)_{22}}\right)\right]$$

$$= E\left[\mathcal{J}_{ij}\mathcal{J}_{ij'}\right] + \frac{1}{N}p_ip_{ij}p_{ij'}w_is_ir_i(1 - r_i)E_{ij}[g]E_{ij'}[g] \qquad \text{(A4)}$$

and, for $i \neq i'$,

$$E\left[Y_{ij}Y_{i'j'}\right] = E\left[\mathcal{Y}_{ij}\mathcal{Y}_{i'j'}\right] \tag{A5}$$

where the expressions for $E\left[\mathcal{Y}_{ij}\mathcal{Y}_{ij'}\right]$ and $E\left[\mathcal{Y}_{ij}\mathcal{Y}_{i'j'}\right]$ are given in equations (30) and (31), respectively.

Substituting equations (A3), (A4), and (A5) into equation (A1) results in

$$N\sigma_Y^2 = N\sigma_{\mathcal{Y}}^2 - \sum_{i,j} p_i p_{ij} w_{ij} \sigma_{g_{ij}}^2 \left\{1 - \left[1 - s_i r_i(1 - r_i)\right]\left[1 - s_{ij} r_{ij}(1 - r_{ij})\right]\right\}$$

$$- \sum_{i,j} p_i w_i s_i r_i (1 - r_i) \frac{p_{ij}}{u_{ij}} \left(u_{ij} E_{ij}[g] - E_i[g]\right)^2 \tag{47}$$

Similarly, it can be shown that, for the splitting method considered in this appendix, if there were only one sampling stage, then the change in variance would be given by

$$\Delta(N\sigma^2) = - \sum_i p_i w_i \sigma_{g_i}^2 s_i r_i (1 - r_i) \tag{A6}$$

and, if there were three stages of sampling, then

$$\Delta(N\sigma^2) = - \sum_{i,j,k} p_i p_{ij} p_{ijk} w_{ijk} \sigma_{g_{ijk}}^2 \left\{1 - \left[1 - s_i r_i(1 - r_i)\right]\left[1 - s_{ij} r_{ij}(1 - r_{ij})\right]\left[1 - s_{ijk} r_{ijk}(1 - r_{ijk})\right]\right\}$$

$$- \sum_{i,j,k} p_i p_{ij} w_{ij} s_{ij} r_{ij} (1 - r_{ij}) \frac{p_{ijk}}{u_{ijk}} \left(u_{ijk} E_{ijk}[g] - E_{ij}[g]\right)^2$$

$$- \sum_{i,j} p_i w_i s_i r_i (1 - r_i) \frac{p_{ij}}{u_{ij}} \left(u_{ij} E_{ij}[g] - E_i[g]\right)^2 \tag{A7}$$

The formula for any arbitrary number of stages should now be readily apparent.

APPENDIX B

PROOF OF INEQUALITY

We wish to show that

$$E\left[\sigma_{g_i}\right] \geq E\left[\sigma_{g_{ijk}}\right] \tag{61}$$

We can write

$$\sigma_{g_i}^2 = \sum_{j,k,l,\ldots} \left(g_{ijkl} - E_i[g]\right)^2 p_{ij}p_{ijk}\ldots$$

$$= \sum_{j,k,l,\ldots} \left[\left(g_{ijkl\ldots} - E_{ij}[g]\right) + \left(E_{ij}[g] - E_i[g]\right)\right]^2 p_{ij}p_{ijk}\ldots$$

$$= \sum_j p_{ij} \sum_{k,l,\ldots} \left(g_{ijkl\ldots} - E_{ij}[g]\right)^2 p_{ijk}p_{ijkl}\ldots$$

$$+ \sum_j p_{ij}\left(E_{ij}[g] - E_i[g]\right)^2$$

$$= \sum_j p_{ij}\left\{\sigma_{g_{ij}}^2 + \left(E_{ij}[g] - E_i[g]\right)^2\right\}$$

$$= E_i\left[\sigma_{g_{ij}}^2 + \left(E_{ij}[g] - E_i[g]\right)^2\right] \tag{B1a}$$

and similarly

$$\sigma_{g_{ij}}^2 = E_{ij}\left[\sigma_{g_{ijk}}^2 + \left(E_{ijk}[g] - E_{ij}[g]\right)^2\right] \tag{B1b}$$

Hence,

$$E\left[\sigma_{g_i}\right] = E\left[\sqrt{E_i\left[\sigma^2_{g_{ij}} + \left(E_{ij}[g] - E_i[g]\right)^2\right]}\right]$$

$$= E\left[\sqrt{E_i\left[\sigma^2_{g_{ijk}} + \left(E_{ijk}[g] - E_{ij}[g]\right)^2 + \left(E_{ij}[g] - E_i[g]\right)^2\right]}\right]$$

$$\geq E\left[\left(E_i\left[\sigma^2_{g_{ijk}}\right]\right)^{1/2}\right] \tag{B2}$$

In addition, we have the relation

$$E_i\left[\sigma^2_{g_{ijk}}\right] - E_i^2\left[\sigma_{g_{ijk}}\right] = E_i\left[\left(\sigma_{g_{ijk}} - E_i\left[\sigma_{g_{ijk}}\right]\right)^2\right] \geq 0$$

which can alternatively be shown by proceeding as follows:

$$E_i\left[\sigma^2_{g_{ijk}}\right] - E_i^2\left[\sigma_{g_{ijk}}\right] = \sum_{j,k} p_{ij}p_{ijk}\sigma^2_{g_{ijk}} - \left(\sum_{j,k} p_{ij}p_{ijk}\sigma_{g_{ijk}}\right)^2$$

$$= \sum_{j,k} p_{ij}p_{ijk}\sigma^2_{g_{ijk}} \sum_{j',k'} p_{ij'}p_{ij'k'}$$

$$- \sum_{j,k,j',k'} p_{ij}p_{ij'}p_{ijk}p_{ij'k'}\,\sigma_{g_{ijk}}\sigma_{g_{ij'k'}}$$

$$= \sum_{j,k,j',k'} p_{ij}p_{ij'}p_{ijk}p_{ij'k'}\left[\sigma^2_{g_{ijk}} - \sigma_{g_{ijk}}\,\sigma_{g_{ij'k'}}\right]$$

$$= \frac{1}{2}\sum_{j,k,j',k'} p_{ij}p_{ij'}p_{ijk}p_{ij'k'}\left[\sigma_{g_{ijk}} - \sigma_{g_{ij'k'}}\right]^2 \geq 0$$

Thus,

$$\left(E_i\left[\sigma^2_{g_{ijk}}\right]\right)^{1/2} \geq E_i\left[\sigma_{g_{ijk}}\right] \tag{B3}$$

Substituting inequality (B3) into inequality (B2) yields

$$E\left[\sigma_{g_i}\right] \geq E\left[E_i\left[\sigma_{g_{ijk}}\right]\right] = E\left[\sigma_{g_{ijk}}\right] \tag{B4}$$

and inequality (61) has been proved. It may be remarked that inequality (61) merely demonstrates the obvious fact that measurement of an additional two correlated variable sets in general reduces the expected standard deviation.

APPENDIX C

VARIANCE RELATION FOR NONOPTIMUM SAMPLING

In this appendix, we will determine, for a Ψ-I-S-I sampling, the optimum values of N, m_1N, and m_1m_3N that minimize the variance subject to the constraint given by equation (74). It will be assumed that the weight factors u_i, w_i, u_{ij}, w_{ijk}, and u_{ijkl} are not at the optimum values as given by equations (72).

Using equation (49b) where $\kappa = 4$ and the relations

$$\left.\begin{aligned} w_{ij} &= u_{ij}w_i \\ u_{ijk} &= 1 \\ w_{ijkl} &= u_{ijkl}w_{ijk} \end{aligned}\right\} \tag{C1}$$

that hold for a Ψ-I-S-I sampling, we obtain

$$N\sigma^2 = \sum_{i,j,k,l} \rho_{ijkl}w_{ijk}u_{ijkl}\tau^2_{ijkl} - \sum_{i,j,k} \rho_{ijk}w_{ijk}E^2_{ijk}[g] + \sum_{i,j,k} \rho_{ijk}w_iu_{ij}E^2_{ijk}[g]$$

$$- \sum_{i} \rho_i w_i E_i^2[g] + \sum_{i} \rho_i u_i E_i^2[g] - E^2[g] \tag{C2}$$

We can write each of the weight factors as a sum of its optimum value and its deviation from optimum

$$\left.\begin{aligned} u_i &= \hat{u}_i + \Delta u_i \\ w_i &= \hat{w}_i + \Delta w_i \\ u_{ij} &= \hat{u}_{ij} + \Delta u_{ij} \\ w_{ijk} &= \hat{w}_{ijk} + \Delta w_{ijk} \\ u_{ijkl} &= \hat{u}_{ijkl} + \Delta u_{ijkl} \end{aligned}\right\} \tag{C3}$$

where the optimum values are given in equations (72). Substituting equations (C3) into equation (C2) gives

$$N\sigma^2 = \sum_{i,j,k,l} \rho_{ijkl} \left[\left(\hat{w}_{ijk} + \Delta w_{ijk} \right) \Delta u_{ijkl} + \hat{u}_{ijkl}\, \Delta w_{ijk} \right] \tau^2_{ijkl}$$

$$- \sum_{i,j,k} \rho_{ijk}(\Delta w_{ijk}) E^2_{ijk}[g] + \sum_{i,j,k} \rho_{ijk} \left[\left(\hat{w}_i + \Delta w_i \right) \Delta u_{ij} + \hat{u}_{ij}\, \Delta w_i \right] E^2_{ijk}[g]$$

$$- \sum_i \rho_i(\Delta w_i) E_i^2[g] + \sum_i \rho_i(\Delta u_i) E_i^2[g] + \frac{E^2\left[\gamma_i(\epsilon_{ij})\right]}{m_1} + \frac{E^2\left[\gamma_{ijk}(\tau_{ijkl})\right]}{m_1 m_3} \tag{C4}$$

The sum of the last two terms on the right side of equation (C4) is $\widehat{N\sigma^2}$ (see eq. (73)) and, hence, in the event that all deviations from optimum go to zero, $N\sigma^2$ reduces to $\widehat{N\sigma^2}$ as it should.

We would like to show that $\widehat{N\sigma^2}$ is the smallest value that can be obtained for $N\sigma^2$ by establishing that the sum of the terms on the right side other than the last two terms is larger than or equal to zero. In order to accomplish this, let us define A^2, B^2, and C^2 by

$$A^2 = \sum_{i,j,k,l} \rho_{ijkl} \left(w_{ijk}\, \Delta u_{ijkl} + \hat{u}_{ijkl}\, \Delta w_{ijk} \right) \tau^2_{ijkl} - \sum_{i,j,k} \rho_{ijk}(\Delta w_{ijk}) E^2_{ijk}[g] \tag{C5}$$

$$B^2 = \sum_{i,j,k} \rho_{ijk} \left(w_i\, \Delta u_{ij} + \hat{u}_{ij}\, \Delta w_i \right) E^2_{ijk}[g] - \sum_i \rho_i(\Delta w_i) E_i^2[g] \tag{C6}$$

$$C^2 = \sum_i \rho_i(\Delta u_i) E_i^2[g] \tag{C7}$$

so that

$$N\sigma^2 = A^2 + B^2 + C^2 + \frac{E^2\left[\gamma_i(\epsilon_{ij})\right]}{m_1} + \frac{E^2\left[\gamma_{ijk}(\tau_{ijkl})\right]}{m_1 m_3} \tag{C8}$$

With the use of the constraining conditions on the weight factors as given by equations (71), it can be shown that each of the quantities A^2, B^2, and C^2 is larger than or equal to zero.

Let us treat the quantity C^2 in detail to illustrate the procedure. By equations (71), u_i satisfies the equation

$$\sum_i \frac{p_i}{u_i} = 1 \tag{C9}$$

and, consequently,

$$\sum_i \frac{p_i}{\hat{u}_i} = 1 \tag{C10}$$

Equation (C9) can be written

$$\sum_i p_i \left[\frac{1}{\hat{u}_i} + \Delta\left(\frac{1}{u_i}\right)\right] = 1 \tag{C11}$$

where

$$\Delta\left(\frac{1}{u_i}\right) = \frac{1}{u_i} - \frac{1}{\hat{u}_i} = -\frac{\Delta u_i}{\hat{u}_i u_i} \tag{C12}$$

Introducing equations (C10) and (C12) into equation (C11) gives

$$\sum_i \frac{p_i}{\hat{u}_i u_i} \Delta u_i = 0 \tag{C13}$$

which is the constraining condition that (Δu_i) must satisfy. Now equation (C7) can be expressed as

$$C^2 = E^2[g] \sum_i \frac{p_i \, \Delta u_i}{\hat{u}_i^2} \tag{C14}$$

where equation (72a) has been employed. We are now in the position where we can modify equation (C14) by using equation (C13) to yield

$$
C^2 = E^2[g]\left(\sum_i \frac{p_i\ \Delta u_i}{\hat{u}_i^2} - \sum_i \frac{p_i\ \Delta u_i}{\hat{u}_i u_i}\right)
$$

$$
= E^2[g]\sum_i \frac{p_i}{u_i \hat{u}_i^2}\ (\Delta u_i)^2 = \sum_i \frac{p_i}{u_i}\ E_i^2[g]\ (\Delta u_i)^2 \tag{C15}
$$

Equation (C15) is the desired relation that shows $C^2 \geq 0$.

Treating the quantities A^2 and B^2 in like manner, we have

$$
A^2 = \sum_{i,j,k,l} \wp_{ijkl} w_{ijk} (\Delta u_{ijkl}) \tau_{ijkl}^2 + \sum_{i,j,k} \wp_{ijk} (\Delta w_{ijk}) \left[\left(\sum_l p_{ijkl} \hat{u}_{ijkl} \tau_{ijkl}^2\right) - E_{ijk}^2[g]\right]
$$

$$
= \sum_{i,j,k,l} \wp_{ijkl} w_{ijk} (\Delta u_{ijkl}) \frac{E_{ijk}^2[\tau_{ijkl}]}{\hat{u}_{ijkl}^2} + \sum_{i,j,k} \wp_{ijk} (\Delta w_{ijk}) \gamma_{ijk}^2(\tau_{ijkl})
$$

$$
= \sum_{i,j,k} \wp_{ijk} w_{ijk} E_{ijk}^2[\tau_{ijkl}] \left[\sum_l \frac{p_{ijkl}}{\hat{u}_{ijkl}^2} (\Delta u_{ijkl}) - \sum_l \frac{p_{ijkl}}{\hat{u}_{ijkl} u_{ijkl}} (\Delta u_{ijkl})\right]
$$

$$
+ \frac{E^2\left[\gamma_{ijk}(\tau_{ijkl})\right]}{m_1^2 m_3^2} \left[\sum_{i,j,k} \wp_{ijk} \frac{(\Delta w_{ijk})}{\hat{w}_{ijk}^2} - \sum_{i,j,k} \frac{\wp_{ijk}}{\hat{w}_{ijk} w_{ijk}} (\Delta w_{ijk})\right]
$$

$$
= \sum_{i,j,k,l} \frac{\wp_{ijkl}}{u_{ijkl}} w_{ijk} \tau_{ijkl}^2 (\Delta u_{ijkl})^2 + \sum_{i,j,k} \frac{\wp_{ijk}}{w_{ijk}} \gamma_{ijk}^2(\tau_{ijkl}) (\Delta w_{ijk})^2 \tag{C16}
$$

and

$$
\begin{aligned}
B^2 &= \sum_{i,j,k} \rho_{ijk} w_i (\Delta u_{ij}) E^2_{ijk}[g] + \sum_i \rho_i (\Delta w_i) \left[\left(\sum_{j,k} p_{ij} p_{ijk} \hat{u}_{ij} E^2_{ijk}[g] \right) - E^2_i[g] \right] \\
&= \sum_{i,j} \rho_{ij} w_i (\Delta u_{ij}) \epsilon^2_{ij} + \sum_i \rho_i (\Delta w_i) \left[\left(\sum_j p_{ij} \hat{u}_{ij} \epsilon^2_{ij} \right) - E^2_i[g] \right] \\
&= \sum_{i,j} \rho_{ij} w_i (\Delta u_{ij}) \epsilon^2_{ij} + \sum_i \rho_i (\Delta w_i) \gamma^2_i (\epsilon_{ij}) \\
&= \sum_{i,j} \rho_{ij} w_i (\Delta u_{ij}) \frac{E^2_i[\epsilon_{ij}]}{\hat{u}^2_{ij}} + \sum_i \rho_i (\Delta w_i) \frac{E^2\left[\gamma_i(\epsilon_{ij})\right]}{m_1^2 \hat{w}_i^2} \\
&= \sum_i \rho_i w_i E^2_i[\epsilon_{ij}] \left(\sum_j \frac{p_{ij}}{\hat{u}^2_{ij}} \Delta u_{ij} - \sum_j \frac{p_{ij}}{\hat{u}_{ij} u_{ij}} \Delta u_{ij} \right) \\
&\quad + \frac{E^2\left[\gamma_i(\epsilon_{ij})\right]}{m_1^2} \left(\sum_i \frac{\rho_i}{\hat{w}_i^2} \Delta w_i - \sum_i \frac{\rho_i}{\hat{w}_i w_i} \Delta w_i \right) \\
&= \sum_{i,j} \frac{\rho_{ij}}{u_{ij}} w_i \epsilon^2_{ij} (\Delta u_{ij})^2 + \sum_i \frac{\rho_i}{w_i} \gamma^2_i (\epsilon_{ij}) (\Delta w_i)^2 \qquad \text{(C17)}
\end{aligned}
$$

We see that minimization of the variance as given by equation (C8) with respect to the variables N, $m_1 N$, and $m_1 m_3 N$ subject to the constraint expressed by equation (74) no longer yields zero as the optimum value of N but instead gives

$$\hat{N} = \frac{T\left(A^2 + B^2 + C^2\right)^{1/2} \Big/ a_1}{a_1\left(A^2 + B^2 + C^2\right)^{1/2} + a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]} \tag{C18a}$$

The optimum values of m_1N and m_1m_3N become

$$\left(\hat{m_1 N}\right) = \frac{TE\left[\gamma_i(\epsilon_{ij})\right] \Big/ a_2}{a_1\left(A^2 + B^2 + C^2\right)^{1/2} + a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]} \tag{C18b}$$

$$\left(m_1 \hat{m_3} N\right) = \frac{TE\left[\gamma_{ijk}(\tau_{ijkl})\right]}{a_1\left(A^2 + B^2 + C^2\right)^{1/2} + a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]} \tag{C18c}$$

and the minimum variance is

$$\left(\sigma^2\right)_{min} = \frac{1}{T}\left[a_1\left(A^2 + B^2 + C^2\right)^{1/2} + a_2 E\left[\gamma_i(\epsilon_{ij})\right] + a_3 E\left[\gamma_{ijk}(\tau_{ijkl})\right]\right]^2 \tag{C19}$$

It is to be noted that $\left(\sigma^2\right)_{min} \geq \left(\widehat{\widehat{\sigma^2}}\right)$ because $\left(\sigma^2\right)_{min}$ is not minimized with respect to the weight factors. To be redundant, if the weight factors take on their optimum values, then $A^2 + B^2 + C^2 = 0$ and $\left(\sigma^2\right)_{min} = \left(\widehat{\widehat{\sigma^2}}\right)$.

APPENDIX D

SYMBOLS

A^2	quantity defined by eq. (C5)
a_1^2, a_2^2, a_3^2	coefficients appearing in eq. (74)
B^2	quantity defined by eq. (C6)
b_1^2, b_2^2, b_3^2	coefficients appearing in eq. (77)
C^2	quantity defined by eq. (C7)
$c_{\Psi 1}, c_{I2}, c_{S3}, c_{I4}, c_g$	coefficients defined immediately before eq. (74)
$E[\square]$	expectation value of $\square$
$E[\square/\bigcirc]$	expectation value of $\square$ given that $\bigcirc$ is fixed at a certain value
$E_i[\square]$	expectation value of $\square$ given that x_1 takes on its i^{th} value set
$E_{ij}[\square]$	expectation value of $\square$ given that x_1 and x_2 take on their i^{th} and j^{th} value set, respectively
$f(\square)$	probability density function of $\square$
$f(\square/\bigcirc)$	probability density function of $\square$ given that $\bigcirc$ is fixed at a certain value
g	function dependent on a number of sets of random variables
$I_i, I_{ij}, \ldots$	integral portion of the reciprocal of $s_i, s_{ij}, \ldots$, respectively
I	pure importance sampling stage
I_Ψ	importance sampling step in a composite sampling stage
L	quantity employed when using the Langrangian-multiplier technique of minimization subject to constraints
m_i	magnification factor for i^{th} stage
N	original number of members in sample
n	number of sets of variables upon which g is dependent

n_i	number of members in sample possessing i^{th} value set of x_1 immediately after importance sampling on x_1-variable
n_{ij}	number of members in sample possessing i^{th} value set of x_1 and j^{th} value set of x_2, immediately after importance sampling on x_2-variable
$p_i, \mathscr{p}_i$	probability that x_1 takes on its i^{th} value set
p_{ij}	conditional probability that x_2 takes on its j^{th} value set given that x_1 has taken on its i^{th} value set
p_{ijk}	conditional probability that x_3 takes on its k^{th} value set given that x_1 and x_2 have taken on their i^{th} and j^{th} value sets, respectively
$\mathscr{p}_{ij}$	probability that x_1 and x_2 take on their i^{th} and j^{th} value sets, respectively
$\mathscr{p}_{ijk}$	probability that x_1, x_2, and x_3 take on their i^{th}, j^{th}, and k^{th} value sets, respectively
$r_i, r_{ij}, \ldots$	fractional portion of reciprocal of $s_i, s_{ij}, \ldots$, respectively
S	pure splitting sampling stage
S_ψ	splitting step of a composite sampling stage
$s(x_1, x_2, \ldots)$	splitting factor dependent on variable sets $x_1, x_2, \ldots$
s_i	splitting factor for x_1-stage of sampling
s_{ij}	splitting factor for x_2-stage of sampling
T	constraining quantity defined by eq. (74)
$\mathscr{T}$	binomial variable occurring in analysis of Russian roulette
U	unit sampling stage
$u(x_1, x_2, \ldots)$	importance sampling weight factor dependent on variable sets $x_1, x_2, \ldots$
u_i	importance sampling weight factor for x_1-stage of sampling
u_{ij}	importance sampling weight factor for x_2-stage of sampling
$w(x_1, x_2, \ldots)$	overall weight factor dependent on variable sets $x_1, x_2, \ldots$

w_i	overall weight factor for x_1-stage of sampling
w_{ij}	overall weight factor for x_1- and x_2-stages of sampling
$\vec{x}$	collection of sets of random variables upon which g is dependent
x_i	i^{th} set of random variables
Y	random variable defined by eq. (44)
Y_{ij}	random variable defined by eq. (45)
Z	random variable defined by eq. (16)
Z_{ij}	random variable defined by eq. (17)
$\mathcal{Z}$	random variable defined by eq. (24)
$\mathcal{Z}_{ij}$	random variable defined by eq. (25)
α_i	absolute value of $E_i[g]$
α_{ij}	absolute value of $E_{ij}[g]$
$\gamma^2(\square), \gamma_i^2(\square), \gamma_{ij}^2(\square), \ldots$	quantities defined by eqs. (67)
$\Delta(\square)$	change in $\square$
$\delta^2, \delta_i^2, \delta_{ij}^2, \ldots$	quantities defined by eqs. (66)
$\epsilon^2, \epsilon_i^2, \epsilon_{ij}^2, \ldots$	quantities defined by eqs. (54) and (65)
κ	number of stages
λ^2	Lagrangian multiplier
ν	number of members in sample immediately after splitting
σ^2	variance
$\sigma_{\square}^2$	variance of $\square$
$\sigma_g^2 / x_1, x_2$	variance of g when x_1 and x_2 are fixed at certain values
$\left(\sigma^2\right)_{min}$	minimum variance appearing in appendix C
$\tau^2, \tau_i^2, \tau_{ij}^2, \ldots$	quantities defined by eqs. (64)
Ψ	composite sampling stage
Ψ_{23}	composite sampling stage for variable sets x_1 and x_2 simultaneously

$\square, \bigcirc$	any quantity
Subscripts:	
i	x_1 fixed at its i^{th} value set
j	x_2 fixed at its j^{th} value set
k	x_3 fixed at its k^{th} value set
.	
.	
.	
α, ρ	dummy indices
Superscripts:	
' (prime)	differentiates from unprimed quantity
^	optimum value

REFERENCES

1. Kahn, H.: Modification of the Monte Carlo Method. P-132, Rand Corp., Nov. 14, 1949.
2. Kahn, H.; and Marshall, A. W.: Methods of Reducing Sample Size in Monte Carlo Computauions. J. Oper. Res. Soc. Amer., vol. 1, 1953, pp. 263-78.
3. Kahn, H.: Applications of Monte Carlo. RM-1237 - ARC, Rand Corp., April 27, 1956.
4. Kahn, H.: Use of Different Monte Carlo Sampling Techniques. Symp. on Monte Carlo Methods, Herbert A. Meyer, ed., John Wiley and Sons, Inc., 1956, pp. 146-90.
5. Goertzel, Gerald; and Kalos, Melvin H.: Monte Carlo Methods in Transport Problems. Vol. II of Progress in Nucl. Energy, ser. I, Physics and Mathematics, R. A. Charpie, J. Horowitz, D. J. Hughes, and D. J. Littler, Eds. Pergamon Press, 1958, pp. 315-69.
6. Kalos, M. H.: Importance Sampling in Monte Carlo Calculation of Thick Shield Penetration. Nucl. Sci. and Eng., vol. 2, Suppl. #1, 1959, pp. 34-5.
7. Clark, Francis H.: The Exponential Transform as an Importance-Sampling Device. A Review. ORNL-RSIC - 14, Oak Ridge National Lab., Jan. 1966.
8. Kalos, M. H.: Methods in Monte Carlo Solution of the Radiation Transport Problem. UNC-5014 (vol. A), United Nuclear Corp., Development Div., May 31, 1962.
9. Kalos, M. H.: Importance Sampling in Monte Carlo Shielding Calculations. I. Neutron Penetration Through Thick Hydrogen Slabs. Nucl. Sci. and Eng., vol. 16, 1963, pp. 227-234.
10. Coveyou, R. R.; Cain, V. R.; and Yost, K. J.: Adjoint and Importance in Monte Carlo Application. Nucl. Sci. and Eng., vol. 27, 1967, pp. 219-234.
11. Schmidt, F. A. R.; Straker, E. A.; and Cain, V. R.: Applications of Adjoint Flux Calculations to Monte Carlo Biasing. ORNL-TM-2454, Oak Ridge National Lab., Dec. 26, 1968.
12. Levitt, Leo B.: The Use of Self-Optimized Exponential Biasing in Obtaining Monte Carlo Estimates of Transmission Probabilities. Nuclear Sci. and Eng., vol. 31, 1968, pp. 500-504.
13. Spanier, Jerome: An Analytic Approach to Variance Reduction. SIAM, J. Appl. Math., vol. 18, no. 1, Jan. 1970, pp. 172-190.

TABLE I. - OPTIMUM RECIPROCAL WEIGHT FACTORS (g NON-NEGATIVE)

Followed by	Splitting stages (S or S_ψ)			Importance stages					
				I			I_ψ		
	1 $1/\hat{w}_i$	2 $1/\hat{w}_{ij}$	3 $1/\hat{w}_{ijk}$	1 $1/\hat{u}_i$	2 $1/\hat{u}_{ij}$	3 $1/\hat{u}_{ijk}$	1 $1/\hat{u}_i$	2 $1/\hat{u}_{ij}$	3 $1/\hat{u}_{ijk}$
(Stage is last)	$\frac{m_1\sigma_{g_i}}{E[\sigma_{g_i}]}$	$\frac{m_1m_2\sigma_{g_{ij}}}{E[\sigma_{g_{ij}}]}$	$\frac{m_1m_2m_3\sigma_{g_{ijk}}}{E[\sigma_{g_{ijk}}]}$	$\frac{\tau_i}{E[\tau_i]}$	$\frac{\tau_{ij}}{E_i[\tau_{ij}]}$	$\frac{\tau_{ijk}}{E_{ij}[\tau_{ijk}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I	$\frac{m_1\gamma_i(\tau_{ij})}{E[\gamma_i(\tau_{ij})]}$	$\frac{m_1m_2\gamma_{ij}(\tau_{ijk})}{E[\gamma_{ij}(\tau_{ijk})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\tau_{ijkl})}{E[\gamma_{ijk}(\tau_{ijkl})]}$	$\frac{E_i[\tau_{ij}]}{E[\tau_{ij}]}$	$\frac{E_{ij}[\tau_{ijk}]}{E_i[\tau_{ijk}]}$	$\frac{E_{ijk}[\tau_{ijkl}]}{E_{ij}[\tau_{ijkl}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I I	$\frac{m_1\gamma_i(\tau_{ijk})}{E[\gamma_i(\tau_{ijk})]}$	$\frac{m_1m_2\gamma_{ij}(\tau_{ijkl})}{E[\gamma_{ij}(\tau_{ijkl})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\tau_{ijklm})}{E[\gamma_{ijk}(\tau_{ijklm})]}$	$\frac{E_i[\tau_{ijk}]}{E[\tau_{ijk}]}$	$\frac{E_{ij}[\tau_{ijkl}]}{E_i[\tau_{ijkl}]}$	$\frac{E_{ijk}[\tau_{ijklm}]}{E_{ij}[\tau_{ijklm}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
S. . .	$\frac{m_1\delta_i}{E[\delta_i]}$	$\frac{m_1m_2\delta_{ij}}{E[\delta_{ij}]}$	$\frac{m_1m_2m_3\delta_{ijk}}{E[\delta_{ijk}]}$	$\frac{\epsilon_i}{E[\epsilon_i]}$	$\frac{\epsilon_{ij}}{E_i[\epsilon_{ij}]}$	$\frac{\epsilon_{ijk}}{E_i[\epsilon_{ijk}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I S. . .	$\frac{m_1\gamma_i(\epsilon_{ij})}{E[\gamma_i(\epsilon_{ij})]}$	$\frac{m_1m_2\gamma_{ij}(\epsilon_{ijk})}{E[\gamma_{ij}(\epsilon_{ijk})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\epsilon_{ijkl})}{E[\gamma_{ijk}(\epsilon_{ijkl})]}$	$\frac{E_i[\epsilon_{ij}]}{E[\epsilon_{ij}]}$	$\frac{E_{ij}[\epsilon_{ijk}]}{E_i[\epsilon_{ijk}]}$	$\frac{E_{ijk}[\epsilon_{ijkl}]}{E_{ij}[\epsilon_{ijkl}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I I S. . .	$\frac{m_1\gamma_i(\epsilon_{ijk})}{E[\gamma_i(\epsilon_{ijk})]}$	$\frac{m_1m_2\gamma_{ij}(\epsilon_{ijkl})}{E[\gamma_{ij}(\epsilon_{ijkl})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\epsilon_{ijklm})}{E[\gamma_{ijk}(\epsilon_{ijklm})]}$	$\frac{E_i[\epsilon_{ijk}]}{E[\epsilon_{ijk}]}$	$\frac{E_{ij}[\epsilon_{ijkl}]}{E_i[\epsilon_{ijkl}]}$	$\frac{E_{ijk}[\epsilon_{ijklm}]}{E_{ij}[\epsilon_{ijklm}]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
Ψ. . .	--------	--------	--------	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I Ψ. . .	--------	--------	--------	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$
I I Ψ. . .	--------	--------	--------	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$	$\frac{E_i[g]}{E[g]}$	$\frac{E_{ij}[g]}{E_i[g]}$	$\frac{E_{ijk}[g]}{E_{ij}[g]}$

TABLE II. - OPTIMUM RECIPRICAL WEIGHT FACTORS (NO RESTRICTIONS ON g)

Followed by	Splitting stages (S or S_Ψ)			Importance stages					
				I			I_Ψ		
	1 $1/\hat{w}_i$	2 $1/\hat{w}_{ij}$	3 $1/\hat{w}_{ijk}$	1 $1/\hat{u}_i$	2 $1/\hat{u}_{ij}$	3 $1/\hat{u}_{ijk}$	1 $1/\hat{u}_i$	2 $1/\hat{u}_{ij}$	3 $1/\hat{u}_{ijk}$
(Stage is last)	$\frac{m_1\sigma_{g_i}}{E[\sigma_{g_i}]}$	$\frac{m_1m_2\sigma_{g_{ij}}}{E[\sigma_{g_{ij}}]}$	$\frac{m_1m_2m_3\sigma_{g_{ijk}}}{E[\sigma_{g_{ijk}}]}$	$\frac{\tau_i}{E[\tau_i]}$	$\frac{\tau_{ij}}{E_i[\tau_{ij}]}$	$\frac{\tau_{ijk}}{E_{ij}[\tau_{ijk}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I	$\frac{m_1\gamma_i(\tau_{ij})}{E[\gamma_i(\tau_{ij})]}$	$\frac{m_1m_2\gamma_{ij}(\tau_{ijk})}{E[\gamma_{ij}(\tau_{ijk})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\tau_{ijkl})}{E[\gamma_{ijk}(\tau_{ijkl})]}$	$\frac{E_i[\tau_{ij}]}{E[\tau_{ij}]}$	$\frac{E_{ij}[\tau_{ijk}]}{E_i[\tau_{ijk}]}$	$\frac{E_{ijk}[\tau_{ijkl}]}{E_{ij}[\tau_{ijkl}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I I	$\frac{m_1\gamma_i(\tau_{ijk})}{E[\gamma_i(\tau_{ijk})]}$	$\frac{m\ m_2\gamma_{ij}(\tau_{ijkl})}{E[\gamma_{ij}(\tau_{ijkl})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\tau_{ijklm})}{E[\gamma_{ijk}(\tau_{ijklm})]}$	$\frac{E_i[\tau_{ijk}]}{E[\tau_{ijk}]}$	$\frac{E_{ij}[\tau_{ijkl}]}{E_i[\tau_{ijkl}]}$	$\frac{E_{ijk}[\tau_{ijklm}]}{E_{ij}[\tau_{ijklm}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
S. . .	$\frac{m_1\delta_i}{E[\delta_i]}$	$\frac{m_1m_2\delta_{ij}}{E[\delta_{ij}]}$	$\frac{m_1m_2m_3\delta_{ijk}}{E[\delta_{ijk}]}$	$\frac{\epsilon_i}{E[\epsilon_i]}$	$\frac{\epsilon_{ij}}{E_i[\epsilon_{ij}]}$	$\frac{\epsilon_{ijk}}{E_{ij}[\epsilon_{ijk}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I S. . .	$\frac{m_1\gamma_i(\epsilon_{ij})}{E[\gamma_i(\epsilon_{ij})]}$	$\frac{m_1m_2\gamma_{ij}(\epsilon_{ijk})}{E[\gamma_{ij}(\epsilon_{ijk})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\epsilon_{ijkl})}{E[\gamma_{ijk}(\epsilon_{ijkl})]}$	$\frac{E_i[\epsilon_{ij}]}{E[\epsilon_{ij}]}$	$\frac{E_{ij}[\epsilon_{ijk}]}{E_i[\epsilon_{ijk}]}$	$\frac{E_{ijk}[\epsilon_{ijkl}]}{E_{ij}[\epsilon_{ijkl}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I I S. . .	$\frac{m_1\gamma_i(\epsilon_{ijk})}{E[\gamma_i(\epsilon_{ijk})]}$	$\frac{m_1m_2\gamma_{ij}(\epsilon_{ijkl})}{E[\gamma_{ij}(\epsilon_{ijkl})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\epsilon_{ijklm})}{E[\gamma_{ijk}(\epsilon_{ijklm})]}$	$\frac{E_i[\epsilon_{ijk}]}{E[\epsilon_{ijk}]}$	$\frac{E_{ij}[\epsilon_{ijkl}]}{E_i[\epsilon_{ijkl}]}$	$\frac{E_{ijk}[\epsilon_{ijklm}]}{E_{ij}[\epsilon_{ijklm}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
Ψ. . .	$\frac{m_1\gamma_i(\alpha_{ij})}{E[\gamma_i(\alpha_{ij})]}$	$\frac{m_1m_2\gamma_{ij}(\alpha_{ijk})}{E[\gamma_{ij}(\alpha_{ijk})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\alpha_{ijkl})}{E[\gamma_{ijk}(\alpha_{ijkl})]}$	$\frac{E_i[\alpha_{ij}]}{E[\alpha_{ij}]}$	$\frac{E_{ij}[\alpha_{ijk}]}{E_i[\alpha_{ijk}]}$	$\frac{E_{ijk}[\alpha_{ijkl}]}{E_{ij}[\alpha_{ijkl}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I Ψ. . .	$\frac{m_1\gamma_i(\alpha_{ijk})}{E[\gamma_i(\alpha_{ijk})]}$	$\frac{m_1m_2\gamma_{ij}(\alpha_{ijkl})}{E[\gamma_{ij}(\alpha_{ijkl})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\alpha_{ijklm})}{E[\gamma_{ijk}(\alpha_{ijklm})]}$	$\frac{E_i[\alpha_{ijk}]}{E[\alpha_{ijk}]}$	$\frac{E_{ij}[\alpha_{ijkl}]}{E_i[\alpha_{ijkl}]}$	$\frac{E_{ijk}[\alpha_{ijklm}]}{E_{ij}[\alpha_{ijklm}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$
I I Ψ. . .	$\frac{m_1\gamma_i(\alpha_{ijkl})}{E[\gamma_i(\alpha_{ijkl})]}$	$\frac{m_1m_2\gamma_{ij}(\alpha_{ijklm})}{E[\gamma_{ij}(\alpha_{ijklm})]}$	$\frac{m_1m_2m_3\gamma_{ijk}(\alpha_{ijklmn})}{E[\gamma_{ijk}(\alpha_{ijklmn})]}$	$\frac{E_i[\alpha_{ijkl}]}{E[\alpha_{ijkl}]}$	$\frac{E_{ij}[\alpha_{ijklm}]}{E_i[\alpha_{ijklm}]}$	$\frac{E_{ijk}[\alpha_{ijklmn}]}{E_{ij}[\alpha_{ijklmn}]}$	$\frac{\alpha_i}{E[\alpha_i]}$	$\frac{\alpha_{ij}}{E_i[\alpha_{ij}]}$	$\frac{\alpha_{ijk}}{E_{ij}[\alpha_{ijk}]}$

NASA-Langley, 1971 —— 19 E-5793

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAI
NATIONAL AERONAUTICS
SPACE ADMINISTRATIO

03U 001 44 51 3DS 71043 00903
AIR FORCE WEAPONS LABORATORY /WLOL/
KIRTLAND AFB, NEW MEXICO 87117

ATT E. LOU BOWMAN, CHIEF,TECH. LIBRARY

POSTMASTER: If Undeliverable (Section 1 Postal Manual) Do Not Re

"*The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof.*"

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546