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FOR AXIALLY SYMMETRIC CIRCULAR CYLINDERS

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Robert F. Melworm

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Bernard W. Shaffer

Department of Mechanical Engineering

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Robert F. Melworm
Research Associate
New York University
and
Engineering Systems Analyst
American Bureau of Shipping

Irwin Berman
Head of Solid Mechanics Department
Foster Wheeler Corporation

Bernard W. Shaffer
Professor of Mechanical Engineering
New York University

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Abstract

An elastic shell theory is formulated for axisymmetric circular cylinders by consistent application of the assumptions of classical structural theory. In formulating the theory, the ratio of the cylinder's wall thickness to its mean radius is not neglected in comparison to unity and cognizance is taken of the actual surfaces on which loads are applied. The resulting set of equations are reduced, by consistently neglecting certain powers of the thickness to mean radius ratio as compared to unity, to other governing sets of equations. Three such reductions are considered, the final one being an extended form of the commonly used thin-shell theory.

Introduction

Different theoretical formulations of the boundary value problems of structural elements are available in the literature. These may be either mathematically rigorous or approximate depending upon whether they are expressed in terms of the mathematical theory of elasticity or on simplifying engineering assumptions. The approximate formulations are usually for a special class of structural elements such as shells, plates, and beams each of which is characterized by the fact that it contains certain relatively small dimensional parameters.

The present paper will be concerned with analysis of hollow circular cylinders made of homogeneous, isotropic time independent elastic materials and subject to axisymmetric loadings. Even though many formulations for this structural element exist, because each is based on a different set of simplifying assumptions, it had become necessary to understand each set of assumptions and evaluate the consistency of each theory before considering its application in a rocket motor assembly investigation in which the shell plays an important role. The foregoing eventually led to the formulation of a consistent theory based on the use of classical structural theory assumptions. Other formulations, including an extended form of thin-shell theory, are arrived at as reductions of this theory.

It may be broadly considered that two methods have been used in an attempt to arrive at solutions for shells[1]*. These methods are, of necessity, approximate in terms of elasticity solutions but they lead to results that may be considered adequate.

The first is a mathematical method in which the stresses and displacements are expressed in terms of series containing essentially small dimensional parameters. The applicable equations, and thus its basis, are traceable from the mathematical theory of elasticity. Different solutions can be obtained, depending upon the terms of each series that are retained. When more than certain leading series terms are retained, the adjectives 'higher order' are generally associated with the theory.

The second is a physical method in which simplifying assumptions are added to the equations of elasticity. The resulting formulation has been called structural theory. In the case of straight beams and flat plates, the assumptions of structural theory are known to lead to quite accurate predictions of elastic behavior. These assumptions, herein designated as classical structural assumptions, may be stated as follows: for the stress (a) the normal stress in a direction of small dimension is small compared to the normal stresses in the direction of the other dimensions and may therefore be neglected in the stress-strain relations; for the displacements plane sections initially normal to the middle surface in the undeformed state remain (b) plane (c) normal and (d) unextended in the deformed state. The

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* Superscript numbers in brackets [ ] designate references at the end of the paper.

previous assumptions are also made when curvature is present. However, additional assumptions have been made as well. The foregoing displacement assumptions are associated with the names of Euler-Bernoulli for straight beams, with Winkler for curved beams, with Kirchhoff[2] for plates, and with Love-Kirchoff for shells.

A relationship between the mathematical and physical methods may be established if the appropriate number of terms of the first method is chosen to correspond to the form of the second method. Conceivably this approach may even lead from shell theory to membrane theory (in which bending is neglected). Perhaps one may even be able to account for physical effects neglected in structural theories, such as transverse shear and extensional deformations, by the use of additional terms in the mathematical theory. Whereas the concept being advanced seems to fit together neatly, problems can arise because of the convergence of all series representations have not been established. In fact, it is disturbing to note that some series have been shown to diverge[3,4]. The addition of terms to such a series will not necessarily lead to better results unless it can be shown to do so by physical justification.

The preceding ideas will be specifically linked herein with developments in the theories of axisymmetric cylinders. First, a literature survey is made of existing theories which may be considered to be a simplification of the theory of elasticity as it applies to hollow, torsionless, axisymmetric cylinder problems. The survey initially considers theories based on structural and additional assumptions and then considers the abandonment of one or more of the structural assumptions that lead to higher order theories.

Then a derivation of a generalized shell theory for the axisymmetric cylinder problem is presented based upon the classical assumptions of structural theory. It is different than the usual formulation because of its consideration of the following two items: First, the ratio of cylinder wall thickness to mean radius is not neglected as compared to unity. Second, in considering boundary conditions and body forces for the formulating of shell theory, cognizance is taken of the actual surfaces upon which each load acts. Elements of both these items have been considered previously. To the best of the authors' knowledge, however, the use of both items in a consistent fashion has not been done. They will be considered in the present paper.

Equations for the solution of torsionless axially symmetric problems will be derived from the theory of elasticity by introducing the assumptions of classical structure theory. Equilibrium equations in terms of middle surface displacements may then be obtained by the usual introduction of stress resultants and middle surface displacements. The resulting set of equations are said to describe behavior within what is called a generalized shell theory. A procedure is summarized for determining the unknown middle surface displacements of generalized shell theory.

The equations of generalized shell theory are consistently reduced by neglecting certain powers of the wall thickness to mean radius ratio when compared to unity. In the course of such reductions shell theories are obtained which may be more accurate for thick cylinders than theories for thin shells. Continued reduction of the equations represent a unique way of obtaining the equations associated with a more complete version of commonly used thin-shell theory.

Literature Survey of Existing Theories

The formulation of theories for the approximate solution for axisymmetric cylinder problems may be listed in the order of increasing complexity, as: thin-shell theory, Flügge theory, shear deformation theories, Reissner-Naghdi theory and then other more general higher order shell theories. The first two theories fall within the category of classical structure theory. Shear deformation and Reissner-Naghdi theories and the more general higher order theories can be viewed as attempts to extend the range of structural theory to account for effects neglected in classical structural theory.

The general developments of structural theory can be traced in a textbook type of reference such as Timoshenko[5]. Detailed bibliographies and categorization of general developments in shell theory are given in recent works of Novozhilov[1], Naghdi[6] and Nash[7]. Bibliographies of cylindrical shell development are given in Novozhilov[1] and Flügge[8]. Recent reviews of the theoretical and practical solution of the general cylindrical shell problem are given in Holand[9] and Simmonds[10]. The axisymmetric situation is a reduction of the general cylindrical shell problem and is frequently encountered in practice. Nevertheless, reviews of theoretical developments do not seem to be restricted to this important case.

Thin-shell theory is based upon the classical structural theory assumptions delineated in the Introduction, as (a), (b), (c) and (d). In addition it is assumed in thin-shell theory that (e) the ratio of the thickness, t, to the mean radius, R, is negligible with respect to unity. Among the initial contributors to the developments of thin-shell theory are Aron[11] and Love[12,13]. Subsequent developments are traced in Novozhilov[1] and Naghdi[6].

The formulation of a theory which describes the behavior of a thin cylindrical shell under general loading conditions has been considered by a number of authors[14-17]. The general nonsymmetric equations of both Donnell[14] and Timoshenko[15] reduce to forms often used to solve axisymmetric cylinder problems. Their names are, therefore, usually associated with the axisymmetric theory. Axisymmetric theory, in the absence of axial loads, appears in Timoshenko and Woinowsky-Krieger[15]. It has also been described in Hetenyi[18] by way of an analogy with the behavior of a beam on an elastic foundation.

Flügge's theory for cylinder problems is also based upon classical structural assumptions. In addition, the theory assumes that certain but not all t/R and $(t/R)^2$ terms are negligible in comparison to unity. The specific assumptions and their inconsistencies will be noted later in the course of the development of the generalized shell theory, based only on classical assumptions.

An attempt to abandon the thin-shell assumption (e) was made by Lur'e[19,20] and Byrne[12] who advanced a shell theory written in terms of a coordinate system imbedded in the surface of the shell. The more general version of their theory as it applies to asymmetric loadings was considered by Flügge[8] and by Biezeno and Grammel[22]. A summary of the latter work was given by Biezeno and Koch[23]. An approximation to this general cylinder theory was considered by Morley[24]. The reduction of this formulation to the axisymmetric case appears in the work of Klosner and Kempner[25].

In the following theories one or more of the classical structural assumptions are abandoned. These theories, therefore, can no longer be considered structural theories.

When the assumption that sections normal to the middle surface remain normal to the deformed middle surface, designated as assumption (c), is abandoned shear deformation effects may be considered. The inclusion of shear deformation has been done for beams by Timoshenko[26-28] for plates, by Reissner[29-31] and likewise may be considered for shells. Hildebrand, Reissner and Thomas[32] as well as Green and Zerna[33] modified shell theory to include this effect in their formulation of a general shell theory. Cooper[34] included shear deformation in a formulation for cylindrical shells. In the axisymmetric case, shear deformation is given as an addition to thin-shell theory by Klosner[25,35] who extended Flügge's shell theory. The modified concepts were designated as Timoshenko-type and Flügge-type shear deformation theories, respectively.

In a further attempt to extend shell theory, not only is the assumption concerned with the normality of a cross section neglected, but in addition the assumptions that the normal radial stress is negligibly small and that normals to the middle surface are unextendable, designated as assumptions (a) and (d) respectively, are also abandoned. Reissner[36] and Naghdi[37] included such considerations in the development of their general shell theory. A generalized form of this theory for orthotropic cylinders is given by Crouzet-Pascal and Pifko[38,39]. The Reissner-Naghdi theory, adapted to the axisymmetric cylindrical shell problems with t/R and $(t/R)^2$ terms retained in comparison to unity appears in the work of Klosner and Levine[35].

General investigations which include higher order terms have been made by a number of authors. Cauchy[40] and Poisson[41] considered the series expansion, in terms of the thickness parameter, for plates. Basset[42] initially considered the series expansion for shells. Other series expansions have been considered for plates[43-47] and shells[4,48-58]. The most recent work is that of Hu[58] who categorizes and discusses the forms of series. The general cylinder problem has been considered by Reiss[59,60] as well as by Bazarenko and Vorovitch[61]; the axisymmetric problem has been considered by Johnson and Reissner[4], Reiss[62] and Prokopov[63].

Other approximate theories have been formulated in an attempt to make more adequate predictions as to the behavior of thick-walled cylinders.

MacGregor and Coffin[64] as well as Bijlaard and Dohrmann[65] have approached such theories from the viewpoint of an extension to the theory of a beam on an elastic foundation. The former authors used a semi-empirical approach while the latter authors' development applied to axisymmetrically loaded shells of revolution. Lee[66] considered series solutions for stresses based upon the Lamé solution in which no axial variation of load exists.

Elasticity Formulation

The small displacement, time and temperature independent elasticity equations for an axisymmetric homogeneous isotropic circular cylinder with static external loading and body forces will be derived for the circular cylinder shown in Figure 1, whose coordinates in the radial, circumferential and axial directions are z, θ, and x, respectively. The coordinate z is considered to be positive when measured radially outward from the middle surface. The strain-displacement equations may be written [12]

$$\varepsilon_{zz} = \frac{\partial w_z}{\partial z} \quad ; \quad \varepsilon_{yy} = \frac{w_z}{R+z} \quad ; \quad \varepsilon_{xx} = \frac{\partial u_z}{\partial x} \quad ; \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w_z}{\partial x} + \frac{\partial u_z}{\partial z}\right) \qquad \text{(1a-1d)}$$

where ε_{zz}, ε_{yy}, ε_{xx}, ε_{zx} are the radial, circumferential, axial and shear strain components respectively; w_z and u_z are the radial and axial displacements respectively. The stress-strain relations are[78]

$$\varepsilon_{zz} = \frac{\sigma_{zz}-\mu(\sigma_{xx}+\sigma_{yy})}{E} + \alpha T \quad ; \quad \varepsilon_{yy} = \frac{\sigma_{yy}-\mu(\sigma_{xx}+\sigma_{zz})}{E} + \alpha T \qquad \text{(2a,2b)}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}-\mu(\sigma_{yy}+\sigma_{zz})}{E} + \alpha T \quad ; \quad \varepsilon_{zx} = \frac{1+\mu}{E} \tau_{zx} \qquad \text{(2c,2d)}$$

where σ_{zz}, σ_{yy}, σ_{xx}, τ_{zx} are the four stress components which do not vanish identically, namely the radial, circumferential, axial and shear stress components respectively and E and μ are, respectively, the modulus of elasticity and Poisson's ratio of the material. The quantity α is the coefficient of thermal expansion and T the temperature distribution which is a function of both z and x. The stress components satisfy the equilibrium relations[12]

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} - \frac{\sigma_{yy}-\sigma_{zz}}{R + z} + Z = 0 \quad ; \quad \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\tau_{zx}}{R+z} + X = 0 \qquad (3a,3b)$$

where X and Z are prescribed body forces per unit volume (these may include inertia forces). The use of Equations (1) through (3) with appropriate stress and/or displacement boundary conditions constitute formulation of the problem for the ten unknown stress, strain and displacement functions of x and z. Different procedures, each dependent upon the nature of the boundary conditions, may be followed to arrive at solutions for the unknowns. For displacement boundary conditions it is desirable to substitute the strain-displacement relations of Equations (1) into the stress-strain relations of Equations(2) to get the stress-displacement relations. These may then be substituted into the equilibrium requirement of Equation (3) to obtain equilibrium equations in terms of the displacements. This approach, generally attributed to Navier, is sometimes referred to as the second boundary value problem[67]. A similar approach will be used in the structural theory procedure that follows to arrive at the shell theory problem formulation.

Stress Assumption and Definitions of Force Resultants

The stress assumption of structural theory as applied to the cylinder problem under consideration is that $\sigma_{zz} << \sigma_{xx}$ and σ_{yy} and may therefore be neglected in the stress-strain relations of Equations (2a)-(2c). Rather than deal directly with the remaining unknown stresses σ_{xx}, σ_{yy}, τ_{zx}, stress resultants are defined. The resultants are forces or moments defined per unit length of middle surface. The normal force resultants in the x and θ directions, called N_x and N_y, are prescribed by the definitions

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx}(1 + \frac{z}{R})\ dz\ ;\ N_y = \int_{-t/2}^{t/2} \sigma_{yy}\ dz \tag{4a,4b}$$

The shear force resultant, Q_x, is defined

$$Q_x = -\int_{-t/2}^{t/2} \tau_{zx}(1 + \frac{z}{R})\ dz \tag{5a}$$

The bending moment resultants M_x and M_y are defined

$$M_x = -\int_{-t/2}^{t/2} \sigma_{xx}(1 + \frac{z}{R})\ zdz\ \ ;\ \ M_y = -\int_{-t/2}^{t/2} \sigma_{yy}\ zdz \tag{6a,6b}$$

The positive directions of the foregoing force and moment resultants per unit length are indicated in Figure 2.

When a circular cylinder is viewed from the structural theory or shell theory point of view, a problem of reduced dimension results. In addition to the foregoing stress resultants, other forces on the cylinder should be considered in terms of resultants that act on the middle surface. These resultants are the effect of other forces that act at different radii through the wall thickness. Even though the middle surface is used as a reference for force and moment resultants, it is possible to verify that the formulation and solution of a problem is independent of the particular surface chosen; so that any other surface could have been chosen as the reference surface.

In view of the assumption that σ_{zz} is negligible, it is expected that Equation (3a), which expresses equilibrium in the z direction, is not applicable. Body forces that act at different radii through the wall thickness and loads which act at the inner and/or outer surfaces of the cylinder must be considered in terms of their effect on the middle surface when the equilibrium of a shell element is considered.

Boundary conditions which specify the displacements on the inner and/or outer surfaces of a shell can also be treated in the foregoing formulation--but only in an artificial way. Fictitious loads are assumed to exist on the middle surface of the cylinder. Their magnitude and distribution are determined so that the associated displacements satisfy prescribed boundary conditions. The fictitious forces as well as actual body forces are incorporated into the equilibrium equations of the middle surface element.

The surface forces and body forces should be defined as resultants per unit length of the middle surface in the same sense that the other stress resultants were defined. If the surface stresses are σ_{zz} and τ_{zx}, the radial, axial and moment load resultants per unit area of a middle surface element are denoted by p_r, p_x and m respectively and expressed

$$p_r = \left[\sigma_{zz}(1 + \frac{z}{R})\right]_{-t/2}^{t/2} ; \quad p_x = \left[\tau_{zx}(1+ \frac{z}{R})\right]_{-t/2}^{t/2} ; \quad m = \left[\tau_{zx}(1+ \frac{z}{R})z\right]_{-t/2}^{t/2} \qquad (7a\text{-}7c)$$

The upper limit is introduced for z and the stress components when stresses act on the outer surface of the cylinder, whereas the lower limit for z and the stress components introduced when stresses act on the internal surface of the cylinder. Similarly if the radial, axial and moment body force resultants are denoted by B_r, B_x and M these may be given by:

$$B_r = \int_{-t/2}^{t/2} Z(1+ \frac{z}{R})\, dz ; \quad B_x = \int_{-t/2}^{t/2} X(1+ \frac{z}{R})\, dz ; \quad M = \int_{-t/2}^{t/2} X(1+ \frac{z}{R})\, zdz \qquad (8a\text{-}8c)$$

The positive directions associated with the resultants of Equations (7) and (8) are indicated in Figure 2. These definitions constitute a reduction of the equations presented in Hildebrand, Reissner and Thomas[32]. A more recent general presentation in tensor notation in which the terminology used herein is used can be found in Section 5.1 of Naghdi[6].

Equilibrium Considerations

Variation in the axial direction of the stress, load and body force resultants over a differential element of the middle surface is shown in Figure 2. Equilibrium of forces in the z and x directions requires that

$$\frac{N_y}{R} + Q_{x,x} - (p_r + B_r) = 0 \quad ; \quad N_{x,x} + (p_x + B_x) = 0 \tag{9a,9b}$$

where the notation $,_x$ denotes differentiation with respect to the x coordinate. Equilibrium of moments about an axis in the θ direction requires

$$M_{x,x} - Q_x - (m + M) = 0 \tag{9c}$$

The foregoing shows that the shear force resultant, Q_x, is dependent upon the other terms in accordance with the relation

$$Q_x = M_{x,x} - (m + M) \tag{10}$$

Substitution of Equation (10) into (9b) indicates that

$$M_{x,xx} + \frac{N_y}{R} - (m + M),_x - (p_r + B_r) = 0 \tag{11}$$

Equations (9a) and (11) constitute the equilibrium equations for a cylinder. Derivation of these equations has been referred to as the direct method[4]. They may also be derived by direct integration of the elasticity equilibrium relations of Equation (3). This method was considered by Herrmann and Mirsky[68] and Hu[58]. Still other methods exist for the determination of these equations, but they will not be described in this paper.

Displacement Assumptions, Middle Surface Displacements and Constitutive Equations

The displacement assumptions of classical structural theory have been enumerated in the Introduction and prefixed with the letters (b), (c) and (d). From the inextensibility assumption (d) it is concluded that the radial displacement w_z at a distance z from the middle surface can be expressed in terms of the middle surface radial displacement w, as

$$w_z = w \tag{12}$$

From assumptions (b) and (c), that the deformed state remains plane and normal to the middle surface, the axial displacement u_z of an element a distance z from the middle surface can be expressed in terms of the radial displacement w and the axial displacement u of the middle surface, namely

$$u_z = u - zw,_x \tag{13}$$

Substitution of Equations (12) and (13) into the strain displacement relations of Equations (1) leads to the conclusion that

$$\varepsilon_{zz} = 0 \quad ; \quad \varepsilon_{yy} = \frac{w}{R+z} \quad ; \quad \varepsilon_{xx} = u,_x - zw,_{xx} \quad ; \quad \varepsilon_{zx} = 0 \tag{14a-14d}$$

From the stress-strain relation of Equations (2) and the stress assumption that σ_{zz} may be neglected in comparison to the other normal stresses, the axial and circumferential stresses can be written

$$\sigma_{xx} = \frac{E}{1-\mu^2}[\varepsilon_{xx}+\mu\varepsilon_{yy}-(1+\mu)\alpha T] \quad ; \quad \sigma_{yy} = \frac{E}{1-\mu^2}[\varepsilon_{yy}+\mu\varepsilon_{xx}-(1+\mu)\alpha T] \tag{15a,15b}$$

respectively. The stress-strain equations which involve components in the z direction, namely Equations (2a) and (2d), are obviously violated. This is an expected consequence of the additional structural theory assumptions that have been made for σ_{zz} and w_z.

Substitution of the relation between strain and displacement of the middle surface, namely Equations (14b) and (14c), into the stress-strain relation of Equations (15) gives the relation between stress and the displacement of the middle surface, namely

$$\sigma_{xx} = \frac{E}{1-\mu^2}[u,_x - zw,_{xx} + \frac{\mu w}{R+z} - (1+\mu)\alpha T] \tag{16a}$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}[\frac{w}{R+z} + \mu u,_x - \mu zw,_{xx} - (1+\mu)\alpha T] \tag{16b}$$

Substitution of these equations into the stress resultant-stress component definitions of Equations (4) and (6) shows after integration, the relations between the stress resultant and the displacement of the middle surface, namely

$$N_x = \frac{D}{kR^2}[u,_x + \mu\frac{w}{R} - kRw,_{xx}] - \frac{[N_T + M_T/R]}{1-\mu} \tag{17a}$$

$$N_y = \frac{D}{kR^2}[\mu u,_x + (1+k+c)\frac{w}{R}] - \frac{N_T}{1-\mu} \tag{17b}$$

$$M_x = D[w,_{xx} - \frac{u,_x}{R}] + \frac{[M_T + S_T/R]}{1-\mu} \tag{17c}$$

$$M_y = D[\mu w,_{xx} + (1+\frac{c}{k})\frac{w}{R^2}] + \frac{M_T}{1-\mu} \tag{17d}$$

where, D is called the flexural rigidity of the shell and is defined

$$D = \frac{Et^3}{12(1-\mu^2)} \tag{18a}$$

The term k is a small nondimensional parameter defined

$$k = \frac{t^2}{12R^2} \tag{18b}$$

and c is an even smaller nondimensional parameter defined

$$c = \frac{R}{t}\ \ln\left(\frac{R+\frac{t}{2}}{R-\frac{t}{2}}\right) - (1+k) \tag{19}$$

The quantities N_T, M_T, and S_T are defined by

$$N_T = E\alpha\int_{-t/2}^{t/2} T dz\ ;\quad M_T = E\alpha\int_{-t/2}^{t/2} Tz dz\ ;\quad S_T = E\alpha\int_{-t/2}^{t/2} Tz^2 dz \tag{20a-20c}$$

The shear force resultant may also be expressed in terms of the middle surface displacement by the substitution of Equation (17c) into Equation (10b). It is then seen that

$$Q_x = D[w,_{xxx} - \frac{u,_{xx}}{R}] + \frac{[M_T + S_T/R]}{1-\mu} - (m + M) \tag{21}$$

The parameter c is of particular interest. A Taylor series expansion of the natural logarithm term in Equation (19) shows[69] that, for $\frac{t}{2R} < 1$,

$$\ln \left(\frac{R+\frac{t}{2}}{R-\frac{t}{2}}\right) = 2\left[\frac{t}{2R} + \frac{1}{3}\left(\frac{t}{2R}\right)^3 + \frac{1}{5}\left(\frac{t}{2R}\right)^5 + \frac{1}{7}\left(\frac{t}{2R}\right)^7 + \ldots\right] \tag{22a}$$

When the non-dimensional parameter k, defined in Equation (18b) is introduced, the series can be rewritten

$$\ln \left(\frac{R+\frac{t}{2}}{R-\frac{t}{2}}\right) = \frac{t}{R}\left(1 + k + \frac{9}{5}k^2 + \frac{27}{7}k^3 + \ldots\right) \tag{22b}$$

Substitution of Equation (22b) into Equation (20) indicates that

$$c = \frac{(3k)^2}{5} + \frac{(3k)^3}{27} + \ldots \tag{23}$$

For a solid cylinder the quantity k approaches 1/3 and c approaches the divergent series 1/5 + 1/7 . . .

In the general (nonsymmetric) cylinder problem studied by Flügge[8], integrations to find the resultants N_y and M_y, as well as other resultants that do not appear in the axisymmetric case, are performed by considering an expansion such as that given in Equations (22). In the analysis, terms past the first two are considered negligible--i.e. terms such as k^2 are neglected in comparison to k and unity. Yet, when the equilibrium equations for the asymmetric case were written in terms of displacement, terms such as k^2 were retained in the same coefficient with terms such as k and unity. This fact may be somewhat obscured in the original work of Flügge where each equilibrium equation contains more than one displacement component. This situation may be viewed more easily in the work of Kempner[70] who was able to rewrite the equilibrium equations in such a way that the displacements are uncoupled. It is worth noting the inconsistency in the retention of higher orders of k even though the point may be academic and in reality would not affect numerical solutions to practical shell problems.

In the present derivation the expression for c given by Equation (19) will be initially considered. Any subsequent simplification containing the ratio t/R or powers of t/R as compared to unity will be approached consistently in all expressions.

Procedure for Obtaining Solutions

Substitution of Equations (17) into Equations (9a) and (11) leads to two coupled ordinary differential equations

$$R^2u,_{xx}+\mu Rw,_x-kR^3w,_{xxx}+[p_x+B_x-(\frac{N_{T,x}+M_{T,x}/R}{1-\mu})]\frac{R^4k}{D}=0 \qquad (24a)$$

$$w,_{xxxx}+[\frac{1+k+c}{kR^4}]w-\frac{u,_{xxx}}{R}+\frac{u,_x}{kR^3}+\frac{[\frac{M_{T,xx}+(S_{T,xx}-N_T)/R}{1-\mu}]-[(m+M),_x+(p_r+B_r)]}{D}=0 \qquad (24b)$$

which express equilibrium in terms of displacements. Rather than attempt to solve them directly, an alternate procedure will be suggested.

The solution for the axial force resultant N_x from Equation (9b), a linear ordinary differential equation of the first order, requires only one boundary condition and is, therefore, easily obtained. When Equation (17a) is rearranged, the derivative of the middle surface axial displacement u can be expressed in terms of the known value of N_x, and functions of the radial displacement w, namely

$$u,_x=\frac{kR^2[N_x+\frac{N_T+M_T/R}{1-\mu}]}{D}-\mu\frac{w}{R}+kRw,_{xx} \qquad (25)$$

Substitution of Equation (25) into Equation (24b) leads to

$$w,_{xxxx}+\frac{2\mu}{(1-k)R^2}w,_{xx}+[\frac{1-\mu^2+k+c}{kR^4(1-k)}]w=\frac{1}{D(1-k)}[p_r+B_r\pm(m+M),_x-\mu\frac{N_x}{R}+kRN_{x,xx}$$

$$+\frac{N_T}{R}+\frac{kR}{1-\mu}N_{T,xx}-\frac{\mu}{1-\mu}\frac{M_T}{R^2}-M_{T,xx}(\frac{1-k}{1-\mu})-\frac{S_{T,xx}}{(1-\mu)R}] \qquad (26)$$

a single fourth order ordinary differential equation for w with the non-homogeneous term containing the applied loads, the body forces and N_x. Equation (26) is the basic equation that must be solved in order to obtain the complete solution of the axisymmetric cylinder problem. The differential equation is similar to that of classical thin shell theory. Its solution is straightforward and requires four boundary conditions.

Once the middle surface radial displacement has been determined, the middle surface axial displacement can be evaluated from Equation (25). Only one additional boundary condition is required. The heretofore unknown force and moment resultants, expressed in terms of the middle surface displacements by Equations (17b), (17c), (17d) and (21), can then also be determined.

In the present paper it is of interest to obtain expressions for the stresses and/or strains and displacements so that they may be compared with those obtained from the classical theory of elasticity. To this end the circumferential and axial stresses and strains, as well as the radial and axial displacements of each element of the cross-section, have been expressed in terms of the middle surface displacements by Equations (16), (14b), (14c), (12) and (13) respectively. The radial and shear stresses may be evaluated from Equations (3) with the accompaniment of a single boundary condition. Ordinarily, these stresses are specified on both the inner and outer surfaces of the cylinder. Hence there appears to be an overspecification of these stresses in the present formulation. An additional assumption would be required to solve for these stresses explicitly. Since such an assumption could be beyond that normally made in structural theory, and it is the authors' intent to confine the current paper to the assumptions of structural theory, no attempt will be made to specify how to evaluate either the radial or the shear stress.

The strains ϵ_{zz} and ϵ_{zx} have been shown to vanish by Equations (14a) and (14d). To check the consistency of the assumptions that lead to this result, one could check the stress-strain relations of Equations (2a) and (2d). A check of some of the other assumptions can be made through the stress-stain relations of Equations (2b) and (2c). In either case it is apparent that the check cannot be made unless all stress components have been evaluated.

For easy reference, the equations derived in the present section for the solution of circular cylinder problems have been repeated in the first column of Table 1. This theory, based on the assumptions of classical structural theory is designated - Generalized Shell Theory.

Reduction of Equations for Different t/R Ratios

Since the wall thickness to mean radius ratio may be particularly small in comparison to unity certain additional simplifying assumptions or reduction of terms may be made in the derivation of approximate relations. A decision has to be made as the limit of the ratio R/t for which an approximation may be made to still provide acceptable computed solutions. It is observed in the derived equations that a form of the ratio R/t appears with unity in the terms t/2R, k and c. These terms are numerically equal to 0.1 and hence would introduce 10% variation from unity should t/2R,k and c be neglected in comparison to unity when R/t is greater than 5, .913, and .667 respectively. However, the mere maintenance of a fixed variation in the computation of each term does not necessarily reflect itself in an equivalent accuracy in the computed solution of the complete problem. In seeking an acceptable limit for thin shell theory, Novozhilov[1] suggested that $R/t \geq 5$ be used, even though he was considering an "accuracy" of 5%, while Kraus[71] suggested an R/t of 10. Neither suggestion is traceable from theory as developed up to this point.

In the present paper three particular reductions are to be considered. When a quantity such as $1 + (t/R)^n$ appears with $n > 2$, the first reduction states that the $(\frac{t}{R})^n$ term with $n > 2$ is negligible with respect to unity. Equations associated with this additional assumption are said to belong to a "very thick theory." The quantity c defined by Equation (23) is then negligible with respect to unity. As a consequence, simplifications occur in the expressions for the circumferential force and moment resultants and for the radial displacement, so that they may be written

$$N_y = \frac{D}{kR^2}[\mu u,_x + (1+k)\frac{w}{R}] - \frac{N_T}{1-\mu} \tag{27}$$

$$M_y = D[\mu w,_{xx} + (1 + \frac{9k}{5})\frac{w}{R^2}] + \frac{M_T}{1-\mu} \tag{28}$$

$$w,_{xxxx} + \frac{2\mu}{(1-k)R^2} w,_{xx} + [\frac{1-\mu^2+k}{kR^4(1-k)}] w = \frac{1}{D(1-k)}[p_r + B_r + (m+M),_x - \mu\frac{N_x}{R} + kRN_{x,xx}$$
$$+ \frac{N_T}{R} + \frac{kR}{1-\mu}N_{T,xx} - \frac{\mu}{1-\mu}\frac{M_T}{R^2} - M_{T,xx}(\frac{1-k}{1-\mu}) + \frac{S_{T,xx}}{(1-\mu)R}] \tag{29}$$

instead of Equations (17b), (17d) and (26) respectively.

The second reduction occurs when a quantity such as $1 + (\frac{t}{R})^n$ appears with $n \geq 2$, for then the term $(\frac{t}{R})^n$ with $n \geq 2$ is taken to be negligible with respect to unity. The associated equations are said to belong to thick wall theory. The quantities c and k defined by Equations (19) and (23) are then negligible with respect to unity with resulting simplifications in the expressions for the circumferential force and moment resultants and for the radial displacement, namely

$$N_y = \frac{D}{kR^2}[\mu u,_x + \frac{w}{R}] - \frac{N_T}{1-\mu} \tag{30}$$

$$M_y = D[\mu w,_{xx} + \frac{w}{R^2}] + \frac{M_T}{1-\mu} \tag{31}$$

$$w,_{xxxx}+\frac{2\mu}{R^2}w,_{xx}+\frac{1-\mu^2}{kR^4}w=\frac{1}{D}[p_r+B_r+(m+M),_x-\mu\frac{N_x}{R}+kRN_{x,xx}$$
$$+\frac{N_T}{R}+\frac{kR}{1-\mu}N_{T,xx}-\frac{\mu}{1-\mu}\frac{M_T}{R^2}-\frac{M_{T,xx}}{1-\mu}-\frac{S_{T,xx}}{(1-\mu)R}] \qquad (32)$$

It is of importance to note that Equations (7) and (8) for the load and body force resultants remain unchanged in the reductions to very thick theory and thick theory. That which has been called very thick theory is basically the Flügge Theory as used by other authors[25,35,72] except for a difference in Equation (7a). The term t/R was neglected in comparison to unity in previous work but is not neglected in the current presentation for reasons of consistency. It is also noteworthy that the stress distributions computed from both theories are not linear through the wall thickness.

The third and final reduction to be considered is that associated with the neglect of $\frac{t}{R}$ with respect to unity, resulting in what is called thin-shell theory. This theory was reviewed in the section on Literature Survey. The simplified equations of thin shell theory may be derived from the equations thus far presented in two different ways. The first is to drop the terms $2\frac{t}{R}$ and $\frac{z}{R}$ terms compared to unity in Equations (4a), (5), (6a), (7), (8) and (14b) before proceeding to the derivation of the stress-middle surface displacement equations. This procedure can be seen in Timoshenko and Woinowsky-Krieger[15] in the absence of axial forces and will not be repeated. A second approach for obtaining the equations of thin shell theory is to reduce all expressions of thick wall theory. The direct reduction of the differential equation for the radial displacement is not evident; a reduction of its solution, however, will be more fruitful. To this end, let Equations (29) and (32) be rewritten as

$$w,_{xxxx}+\Gamma w,_{xx}+\Theta w=\Psi \qquad (33)$$

where Ψ is a function of x. For thick wall theory the terms Γ, Θ and Ψ of Equation (33) are

$$\Gamma_t = \frac{2\mu}{R^2} \quad ; \quad \Theta_t = \frac{1 - \mu^2}{kR^4} \tag{34a,34b}$$

$$\Psi_t = \frac{p_r + B_r + (m+M)_{,x} - \mu\frac{N_x}{R} + kRN_{x,xx} + \frac{N_T}{R} + \frac{kR}{1-\mu}N_{T,xx} - \frac{\mu}{1-\mu}\frac{M_T}{R^2} - \frac{M_{T,xx}}{1-\mu} - \frac{S_{T,xx}}{(1-\mu)R}}{D} \tag{34c}$$

where the subscript t has been appended to identify the theory with which it is associated. For very thick walled theory

$$\Gamma_v = \frac{2\mu}{(1-k)R^2} \quad ; \quad \Theta_v = \frac{1-\mu^2+k}{kR^4(1-k)} \tag{35a,35b}$$

$$\Psi_v = \frac{p_r + B_r + (m+M)_{,x} - \mu\frac{N_x}{R} + kRN_{x,xx} + \frac{N_T}{R} + \frac{kR}{1-\mu}N_{T,xx} - \frac{\mu}{1-\mu}\frac{M_T}{R^2} - M_{T,xx}\left(\frac{1-k}{1-\mu}\right) - \frac{S_{T,xx}}{(1-\mu)R}}{D(1-k)} \tag{35c}$$

where the subscript v has been appended to identify the theory.

With the foregoing nomenclature, one may write the homogeneous solution w_h of the fourth order differential equation[73]

$$w_h = C_1 e^{\beta_1 x}\cos\beta_2 x + C_2 e^{\beta_1 x}\sin\beta_2 x + C_3 e^{-\beta_1 x}\cos\beta_2 x + C_4 e^{-\beta_1 x}\sin\beta_2 x \tag{36}$$

with β_1 and β_2 defined

$$\beta_1 = \sqrt{\frac{\Theta^{1/2}}{2} - \frac{\Gamma}{4}} \quad ; \quad \beta_2 = \sqrt{\frac{\Theta^{1/2}}{2} + \frac{\Gamma}{4}} \tag{37a,37b}$$

where again subscripts t and v have to be added to identify the thick walled theory and the very thick walled theory respectively; C_1, C_2, C_3 and C_4 are constants. When Equations (34a) and (34b) are substituted into Equation (36), and recognition is taken of the definition of k as given by Equation (18b), the coefficients of Equation (37) may be written

$$(\beta_1)_t = \left[\frac{3\sqrt{1-\mu^2} - \mu\left(\frac{t}{2R}\right)}{tR}\right]^{1/2} \quad ; \quad (\beta_2)_t = \left[\frac{3\sqrt{1-\mu^2} + \mu\left(\frac{t}{2R}\right)}{tR}\right]^{1/2} \tag{38a,38b}$$

where again the subscript t was appended to identify thick walled theory. The second term in the numerator of each of the preceding equations may be dropped in thin shell theory where t/R is taken to be negligible with respect to unity. Both expressions are then identical and reduce to the quantity β commonly used in axisymmmetric thin walled shell theory, where

$$\beta = \left[\frac{3(1 - \mu^2)}{t^2R^2}\right]^{1/4} \tag{39}$$

When $(\beta_1)_t$ and $(\beta_2)_t$ are equal to β, the homogeneous solution of Equation (33) as given by Equation (36), is identical to the solution of the differential equation in which Γ and hence the term $\Gamma w,_{xx}$ vanishes. Hence vanishing of the term $\Gamma w,_{xx}$ is apparently equivalent to the neglect of t/R with respect to terms of the order of unity.

A reduction in the right hand side of Equation (33) is also possible. To this end consider the sum of $(m + M),_x$ and $kRN_{x,xx}$. From Equations (7b), (7c), (8b), (8c) and (9b)

$$(m+M),_x + kRN_{x,xx} = [\gamma\tau_{zx,x}]_{-t/2}^{t/2} + \int_{-t/2}^{t/2} \gamma X,_x dx \tag{40}$$

where the quantity γ is defined

$$\gamma = [(1 + \frac{z}{R})z - kR(1 + \frac{z}{R})] \tag{41a}$$

By introducing Equation (18b), which defines the quantity k, the newly defined parameter γ may be written

$$\gamma = z[1 + \frac{z}{R} - (\frac{t}{2z})\frac{t}{6R} - k\ ] \tag{41b}$$

Within thin shell theory in which $\frac{t}{R}$ is negligible with respect to unity and $|z| \leq \frac{t}{2}$, the parameter γ reduces to the radial coordinate z. Furthermore, within thin shell theory, Equations (7c) and (8c) for the bending moment per unit length and the body moment per unit length reduce to

$$m = [z\tau_{zx}]_{-t/2}^{t/2} \quad ; \quad M = \int_{-t/2}^{t/2} zXdz \tag{42a,42b}$$

Hence when γ equal to z is inserted in the right hand side of Equation (40) it may be written $m,_x + M,_x$, so that obviously $kRN_{x,xx}$ is a negligible term. Further reduction in the right hand side of Equation (33) is possible in the temperature dependent terms. For this consider the following two sums of terms taken from Equation (34c)

$$\frac{N_T}{R} - \frac{\mu}{1-\mu}\frac{M_T}{R^2} = \frac{E\alpha}{R}\left[\int_{-t/2}^{t/2} \left(1 - \frac{\mu}{1-\mu}\frac{z}{R}\right)T dz\right] \tag{43a}$$

$$\frac{kR}{1-\mu} N_{T,xx} - \frac{M_{T,xx}}{1-\mu} - \frac{S_{T,xx}}{(1-\mu)R} = \frac{E\alpha R}{1-\mu}\left[\int_{-t/2}^{t/2} \left(k - \frac{z}{R} - \frac{z^2}{R^2}\right) T,_{xx}\, dz\right] \tag{43b}$$

The expressions on the right in Equations (43) are arrived at based on the definitions given in Equations (20). It is possible to show that the second term and first and third term, in the parentheses on the right of the first and second equation respectively, are negligible for thin shell theory.

Thus the final form of Equation (33) for the radial displacement may be given for thin shell theory as

$$w,_{xxxx} + 4\beta^4 w = \frac{p_r + B_r + (m+M),_x - \mu\frac{N_x}{R} + \frac{N_T}{R} - \frac{M_{T,xx}}{1-\mu}}{D} \tag{44}$$

where β is defined by Equation (39). In the absence of body forces, axial load and temperature effects, this equation is analogous to the expression for the displacement of a beam on an elastic foundation[18]. A similar equation is commonly used in the analysis of thin cylindrical shells in the absence of body forces and temperature (i.e., $B_r = 0$, $M = 0$ and $T = 0$) but then the term $m,_x$ does not appear. The term does appear when Donnell's[16] general cylinder equations are reduced, but to the best of the authors' knowledge it has not been previously noted or used for axisymmetric problems.

To show any additional reduction in the expressions, it is necessary to consider the complete solution for the radial displacement. This may be arrived at by application of the method of variation of parameters[73] to find the particular solution to Equation (33) in view of the availability of Equation (36). Application of this method is straightforward, but lengthy and tedius[74]. Only the results are given, namely, that the radial displacement w may be expressed in the form

$$w = F_1 y_1 + F_2 y_2 + F_3 y_3 + F_4 y_4 \tag{45}$$

where y_1, y_2, y_3, and y_4 are products of the exponential and trigonometric functions. Specifically, for thin-shell theory in which β_1 and β_2 reduce to β defined by Equation (39),

$$y_1 = e^{\beta x}\cos\beta x \quad ; \quad y_2 = e^{\beta x}\sin\beta x \quad ; \quad y_3 = e^{-\beta x}\cos\beta x \; ; \; y_4 = e^{-\beta x}\sin\beta x \tag{46a-46d}$$

whereas F_1, F_2, F_3 and F_4 are functions of x which may be expressed in terms of integrals of Ψ,

$$F_1 = -\frac{1}{8\beta^3}\int(y_3 + y_4)\Psi dx \quad ; \quad F_2 = \frac{1}{8\beta^3}\int(y_3 - y_4)\Psi dx \tag{47a,47b}$$

$$F_3 = \frac{1}{8\beta^3}\int(y_1 - y_2)\Psi dx \quad ; \quad F_4 = \frac{1}{8\beta^3}\int(y_1 + y_2)\Psi dx \tag{47c,47d}$$

The radial displacement may then be expressed

$$w = w_1 + w_2 \tag{48}$$

where

$$w_1 = \frac{1}{8\beta^3}\left[-y_1\int y_4\Psi dx - y_3\int y_2\Psi dx + y_2\int y_3\Psi dx + y_4\int y_1\Psi dx\right] \tag{49a}$$

$$w_2 = \frac{1}{8\beta^3}\left[-y_1\int y_3\Psi dx - y_2\int y_4\Psi dx + y_3\int y_1\Psi dx + y_4\int y_2\Psi dx\right] \tag{49b}$$

It is then possible to verify the fact that the second derivative of w with respect to x

$$w,_{xx} = 2\beta^2 \, [w_1 - w_2] \tag{50}$$

These results will be used for further reduction of the thick wall to thin-shell expressions.

Consideration is next given to a particular sum of w and its second derivative, namely $-(\mu/R)w + kRw,_{xx}$. In view of Equations (48) and (50) it is seen that

$$-\frac{\mu}{R} w + kRw,_{xx} = [-\frac{\mu}{R} + 2\beta^2 kR]w_1 + [-\frac{\mu}{R} - 2\beta^2 kR]w_2 \tag{51}$$

When the definition of β as given by Equation (39) is introduced to the right hand side of Equation (51), the bracketed terms may be written

$$-\frac{\mu}{R} \pm 2\beta^2 kR = -\frac{\mu}{R}[1 \mp \frac{\sqrt{1-\mu^2}}{2\sqrt{3}\,\mu}\frac{t}{R}] \tag{52}$$

Since the second term on the right side is negligible within thin-shell theory, as long as μ is not especially small, it becomes $-\mu/R$. Hence in view of the reduced form of Equation (52) together with Equations (47) and (50) it is seen that $kRw,_{xx}$ is negligible in comparison to $-(\mu/R)w$ and, therefore, Equation (25) for the derivative of the middle surface axial displacement becomes

$$u,_x = \frac{kR^2[N_x + \frac{N_T}{1-\mu}]}{D} - \mu\frac{w}{R} \tag{53}$$

The quantity M_T does not appear in this expression since it is easily shown from the definitions of N_T and M_T given by Equations (20a) and (20b) that M_T/R is negligible with respect to N_T for this shell theory.

Similarly Equations (49) and (50) enable us to write

$$\frac{1}{R^2} w + \mu w,_{xx} = [\frac{1}{R^2} + 2\beta^2\mu]w_1 + [\frac{1}{R^2} - 2\beta^2\mu]w_2 \tag{54}$$

Use of Equation (39) for the bracketed terms on the right shows the coefficients of w_1 and w_2 expressible in the form

$$\frac{1}{R^2} \pm 2\beta^2\mu = \frac{1}{tR}\left[\frac{t}{R} \pm 2\mu\sqrt{3(1-\mu^2)}\right] \tag{55}$$

The first term on the right side is negligible for thin-shell theory again, as long as μ is not especially small. Therefore, Equation (31) may be reduced to read

$$M_y = \mu D w,_{xx} + \frac{M_T}{1-\mu} \tag{56}$$

The reduction of Equations (24), which express equilibrium in terms of the middle surface displacements, will now be considered. From the discussion following Equation (52) it is seen that the term $kR^3w,_{xxx}$ may be dropped in Equation (24a). In Equation (24b), the terms k and c in the coefficient of w may also be dropped. When the term $u,_{xxx}/R$ is evaluated from Equation (53) and substituted into the reduced form of Equation (24b) it may be seen that this term must be negligible to lead to Equation (44). Therefore, the reduced form of Equations (24) for thin-shell theory may be given

$$R^2u,_{xx} + \mu Rw,_x + (1-\mu^2)\frac{R^2}{Et}\left[p_x + B_x - \frac{N_{T,x}}{1-\mu}\right] = 0 \tag{57a}$$

$$\mu Ru,_x + w + \frac{R^2t^2}{12}w,_{xxxx} - (1-\mu^2)\frac{R^2}{Et}\left[(m+M),_x + p_r + B_r - \frac{M_{T,xx} - N_T/R}{1-\mu}\right] = 0 \tag{57b}$$

The quantities $M_{T,x}$ and $S_{T,xx}$ do not appear in the right hand side of Equation (57a) and (57b) respectively since these are easily shown to be negligible with respect to $N_{T,x}$ and $M_{T,xx}$ respectively for thin-shell theory, based on the definitions of Equations (20). Previous use of the foregoing form of the thin-shell equations[75,76] in the presence of axial loads, have not included the term [a] $m,_x$. The particular form of Equations (57)

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[a] The previous equations were expressed in the absence of the body forces B_r, B_x and M.

has been investigated[77] in the examination of the effect of a band load on an cylindrical shell filled with an elastic core. In that investigation the effect of the term $m,_x$ was not significant.

It was shown in arriving at Equations (57) that the term $u,_{xxx}/R$ can be dropped from thick shell theory as expressed by Equations (24) because it is negligible with respect to the surviving terms. In particular, $u,_{xxx}/R$ may be considered negligible with respect to $w,_{xxxx}$. It then follows that both $u,_{xxx}/R$ and $u,_x/R$ are negligible with respect to $w,_{xxx}$ and $w,_{xx}$ respectively. As a consequence, the expressions for axial bending moment and shear resultants M_x and Q_x, previously given by Equations (17c) and (21) reduce to

$$M_x = Dw,_{xx} + \frac{M_T}{1-\mu} \quad ; \quad Q_x = Dw,_{xxx} - (m+M) + \frac{M_T}{1-\mu} \tag{58a,58b}$$

In addition, reductions in Equations (14b), (16a), (16b) and (36) for ε_{yy}, σ_{xx}, σ_{yy} and τ_{xy} are easily observed and may be written as

$$\varepsilon_{yy} = \frac{w}{R} \tag{59}$$

$$\sigma_{xx} = \frac{E}{1-\mu^2}\left[u,_x - zw,_{xx} + \mu\frac{w}{R} - (1+\mu)\alpha T\right] \tag{60a}$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}\left[\frac{w}{R} + \mu u,_x - \mu zw,_{xx} - (1+\mu)\alpha T\right] \tag{60b}$$

$$\tau_{zx,z} + \tau_{zx} = -(\sigma_{xx,x} + X) \tag{61}$$

Equations (58), (59), (60) and (61) may be used with the basic axially symmetric thin-walled cylinder relations as expressed either by Equation (57) or by Equation (44).

A summary of the equations of both thick walled and thin-walled cylinder theory are given in Table 1. The procedure used by previous authors to obtain solutions for stresses, strains, displacements and stress resultants for the Flügge and thin-shell theories has, for the most part, been the same as the procedure given herein.

Conclusions

Structural theories for torsionless axisymmetric hollow cylinders have been discussed, Emphasis has been placed on their reduction to commonly used thin-shell theory. In the course of such a derivation and the reduction, which is accomplished in a unique way, differences that exist in the theories derived and those available have been appropriately noted. Three such differences existed. The first involved a question of consistency in the Flügge equations, for general cylindrical shells, in which k^2 terms neglected at one point in comparison to unity appear in the final Flügge equations. Since k^2 is a very small quantity this point is mainly academic. The second difference arose in the treatment of the load and body force resultants. In previous work in which Flügge theory was used for an axisymmetric solution t/R was neglected in comparison to unity in the radial load resultant. This overlooks the actual surface at which the load is applied. The third difference involved the presence of the moment load resultant in thin-shell theory. The effect of these latter two items can be considered in terms of differences in numerical results by the presence, or lack therof, in the equations used for solution of several problems. The evaluation of these results and the results of higher order theories as compared to structural theory and its reductions should be made, where possible, in light of available elasticity and/or experimental results.

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TABLE 1

SUMMARY OF EQUATIONS FOR SOLUTION OF CIRCULAR CYLINDRICAL SHELL PROBLEMS

GENERALIZED, VERY THICK, THICK AND THIN SHELL THEORY

GENERALIZED THEORY	VERY THICK THEORY	THICK THEORY	THIN THEORY
(9b) $N_{x,x} = -(p_x + B_x)$	→	→	→
(26) $w_{,xxxx} + \frac{2\mu}{(1-k)R^2} w_{,xx} + \left[\frac{1-\mu^2+k+c}{kR^4(1-k)}\right] w = \frac{1}{D(1-k)}\left[p_r + B_r + (m+M)_{,x} - \mu\frac{N_x}{R} + kRN_{x,xx}\right]$	(29) $w_{,xxxx} + \frac{2\mu}{(1-k)R^2} w_{,xx} + \left[\frac{1-\mu^2+k}{kR^4(1-k)}\right] w = \frac{1}{D(1-k)}\left[p_r + B_r + (m+M)_{,x} - \mu\frac{N_x}{R} + kRN_{x,xx}\right]$	(32) $w_{,xxxx} + \frac{2\mu}{R^2} w_{,xx} + \frac{1-\mu^2}{kR^4} w = \frac{1}{D}\left[p_r + B_r + (m+M)_{,x} - \frac{\mu N_x}{R} + kRN_{x,xx}\right]$	(44) $w_{,xxxx} + \frac{1-\mu^2}{kR^4} w = \frac{1}{D}\left[p_r + B_r + (m+M)_{,x} - \mu\frac{N_x}{R}\right]$
(25) $u_{,x} = \frac{kR^2N_x}{D} - \mu\frac{w}{R} + kRw_{,xx}$	→	→	(53) $u_{,x} = \frac{kR^2N_x}{D} - \mu\frac{w}{R}$
(12) $w_z = w$	→	→	→
(13) $u_z = u - zw_{,x}$	→	→	→
(14c) $\epsilon_{xx} = u_{,x} - zw_{,xx}$	→	→	→
(14b) $\epsilon_{yy} = \frac{w}{R+z}$	→	→	(59) $\epsilon_{yy} = \frac{w}{R}$
(14a) $\epsilon_{zz} = 0$	→	→	→
(14d) $\epsilon_{zx} = 0$	→	→	→
(16a) $\sigma_{xx} = \frac{E}{1-\mu^2}\left[u_{,x} - zw_{,xx} + \frac{\mu w}{R+z}\right]$	→	→	(60a) $\sigma_{xx} = \frac{E}{1-\mu^2}\left[u_{,x} - zw_{,xx} + \mu\frac{w}{R}\right]$
(16b) $\sigma_{yy} = \frac{E}{1-\mu^2}\left[\frac{w}{R+z} + \mu u_{,x} - \mu zw_{,xx}\right]$	→	→	(60b) $\sigma_{yy} = \frac{E}{1-\mu^2}\left[\frac{w}{R} + \mu u_{,x} - \mu zw_{,xx}\right]$
— $\sigma_{zz} = 0$	→	→	→
(3b) $\tau_{zx,z} + \frac{\tau_{zx}}{R+z} = -(\sigma_{xx,x} + X)$	→	→	(61) $\tau_{zx,z} + \frac{\tau_{zx}}{R} = -(\sigma_{xx,x} + X)$
(17c) $M_x = D\left[w_{,xx} - \frac{u_{,x}}{R}\right]$	→	→	(58a) $M_x = Dw_{,xx}$
(17a) $M_y = D\left[\mu w_{,xx} + (1+\frac{c}{k})\frac{w}{R^2}\right]$	(28) $M_y = D\left[\mu w_{,xx} + (1+\frac{9}{5}k)\frac{w}{R^2}\right]$	(31) $M_y = D\left[\mu w_{,xx} + \frac{w}{R^2}\right]$	(56) $M_y = \mu Dw_{,xx}$
(17b) $N_y = \frac{D}{kR^2}\left[\mu u_{,x} + (1+k+c)\frac{w}{R}\right]$	(27) $N_y = \frac{D}{kR^2}\left[\mu u_{,x} + (1+k)\frac{w}{R}\right]$	(30) $N_y = \frac{D}{kR^2}\left[\mu u_{,x} + \frac{w}{R}\right]$	→
(21) $Q_x = D\left[w_{,xxx} - \frac{u_{,xx}}{R}\right] - (m+M)$	→	→	(58b) $Q_x = Dw_{,xxx} - (m+M)$

The Equation numbers are given in the lower left hand corner of each box.
These equations do not include thermal effects.

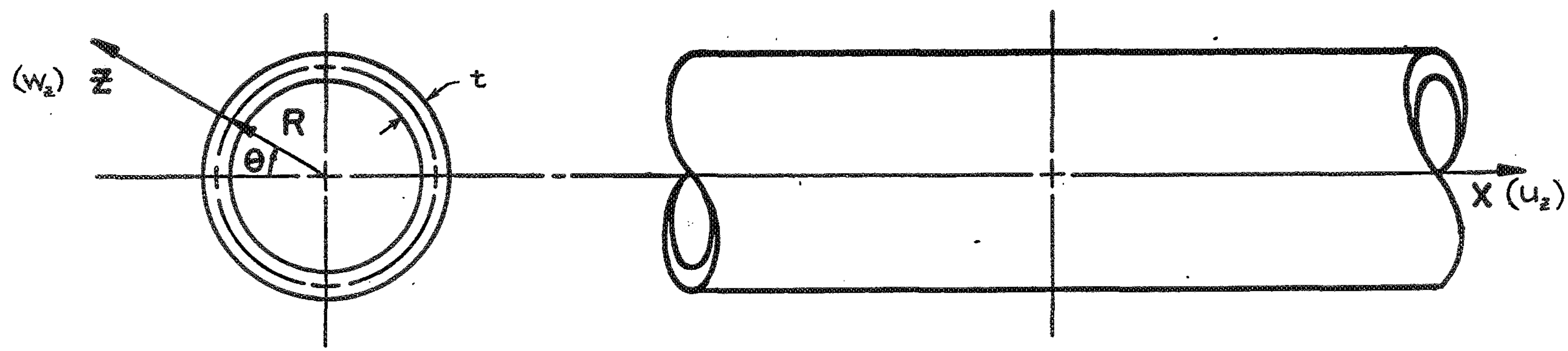

Figure 1. CYLINDRICAL COORDINATE SYSTEM FOR HOLLOW CYLINDER

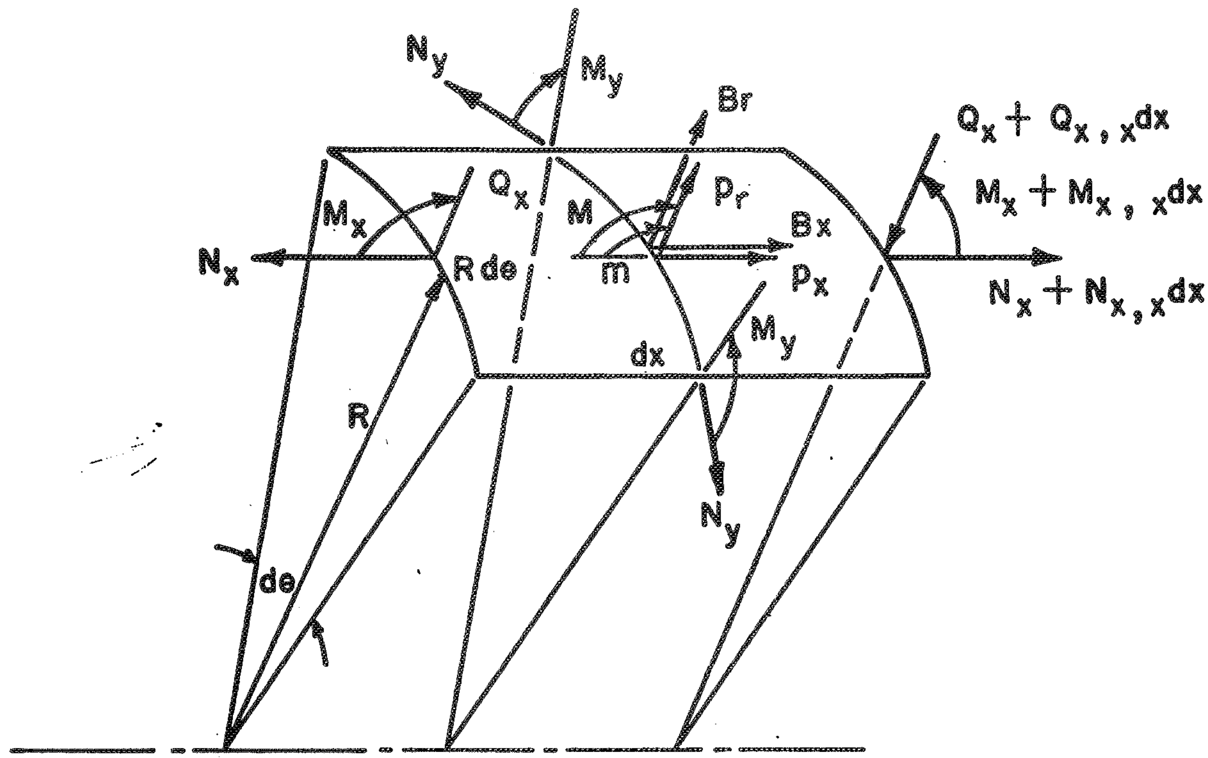

Figure 2. STRESS, LOAD AND BODY FORCE RESULTANTS ACTING ON MIDDLE SURFACE ELEMENT AND THEIR VARIATION