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GYROSCOPIC CONTROL OF A RIGID BODY CONSTRAINED TO ROTATE ABOUT A FIXED AXIS

by

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1. Introduction

This investigation is concerned with the possibility of using gyroscopic effects to control either the spin rate or the orientation of a rigid body which is constrained to rotate about a fixed axis. The motivation for undertaking the study was to establish a theoretical basis for an experimental exploration of man's ability to employ gyroscopic effects while maneuvering in free space.

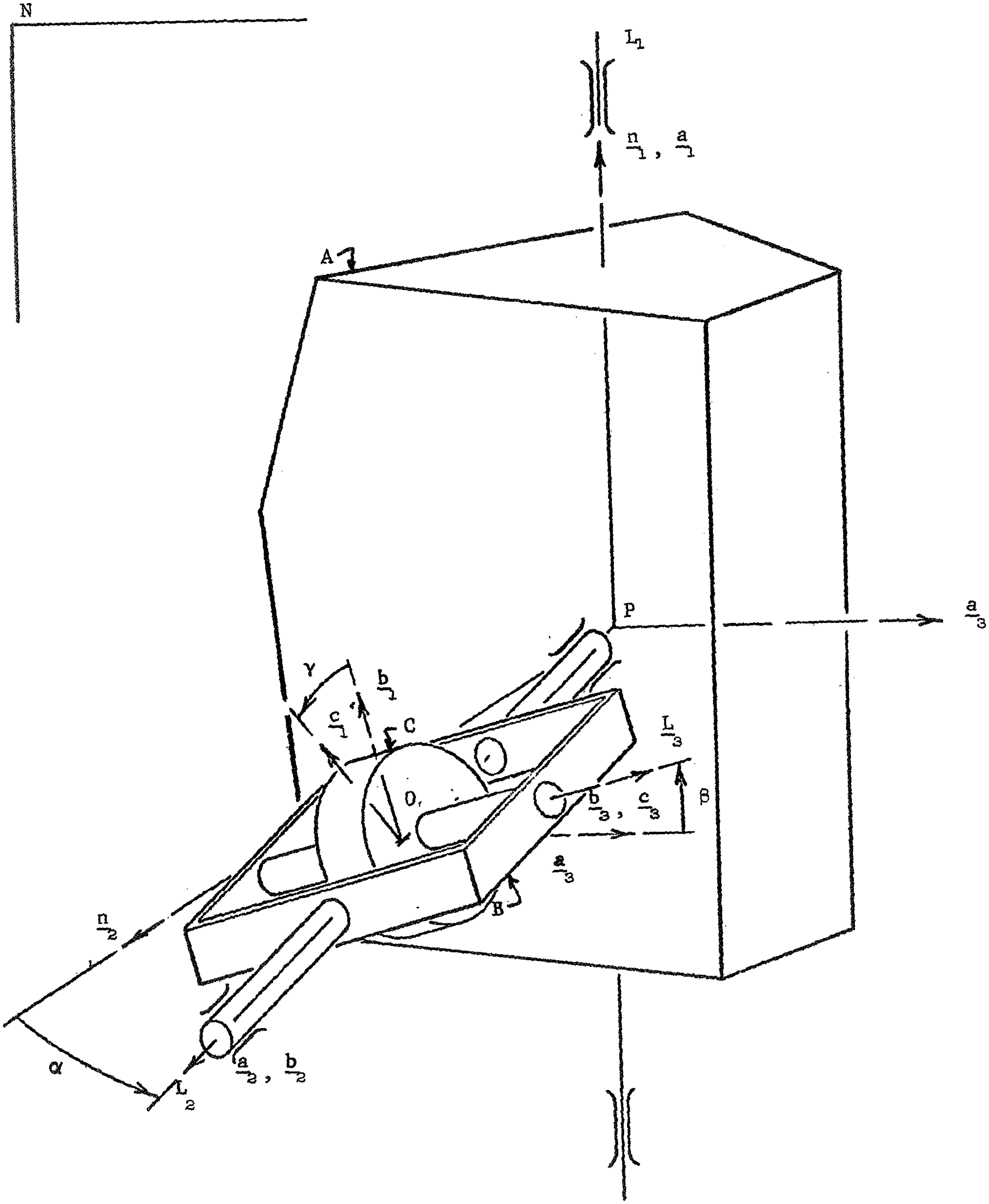
In Sec. 2 the system to be analyzed is described in detail, and equations of motion are formulated. Secs. 3 and 4 deal respectively with open loop and closed loop control. Finally, the use of these results as the basis for an experiment involving human subjects is discussed in Sec. 5.

2. Equations of Motion

The system to be analyzed consists of a rigid body A, a gimbal ring B, and a cylindrical rotor C, as shown in Fig. 1. Body A is free to rotate about a fixed vertical line L_1; B can be made to rotate about a line L_2 that is fixed in A, is perpendicular to L_1, and intersects L_1 at point P; C is free to rotate about a line L_3 that is fixed in B, is perpendicular to L_2, and intersects L_2 at point 0.

Three angles, α, β, γ, are used to describe the configuration of the system. These are defined as follows: Let $\underline{a}_i$, $\underline{b}_i$, $\underline{c}_i$ and $\underline{n}_i$ $(i = 1,2,3)$ denote right-handed sets of orthogonal unit vectors fixed in A, B, C, and a Newtonian reference frame N, respectively, orienting these vectors as indicated in Fig. 1. Then α is the angle between $\underline{n}_2$ and $\underline{a}_2$, β is the angle between $\underline{a}_3$ and $\underline{b}_3$, and γ is the angle between $\underline{b}_1$ and $\underline{c}_1$.

The system has the following inertia properties: Body A has a moment of inertia I about L_1. Point 0 is the common mass center of B and C. The unit vectors $\underline{b}_i$ and $\underline{c}_i$ are parallel to principal


N
L1
n1 , a1
A
P
a3
γ
b1
c1
C
L3
O
b3 , c3
β
a3
B
n2
α
a2 , b2
L2

axes of inertia of B and C for point O , and the associated moments of inertia are H_i and J_i (i = 1, 2, 3), respectively. Body C is presumed to be axially symmetric, with $\underline{c}_3$ parallel to the axis of symmetry. Hence the moment of inertia of C about any line passing through O and perpendicular to L_3 has the value J_1 . Finally, bodies B and C have masses M_B and M_C , respectively, and the distance between point O and line L_1 is r .

If K denotes the kinetic energy of the system, then

$$2K = \dot{\alpha}^2[I + r^2(M_B + M_C) + s^2\beta(H_3 + J_3) + c^2\beta(H_1 + J_1)]$$
$$+ \dot{\beta}^2[H_2 + J_1] + \dot{\gamma}^2[J_3] + \dot{\gamma}\dot{\alpha}[2J_3 s\beta] \tag{1}$$

where s and c are abbreviations for sine and cosine.

As the potential energy of the system remains constant, and as α and β can be regarded as cyclic coordinates of a two degree of freedom system, the associated Lagrange equations can be expressed as

$$\frac{\partial K}{\partial \dot{\alpha}} = \text{constant} \tag{2}$$

and

$$\frac{\partial K}{\partial \dot{\gamma}} = \text{constant} \tag{3}$$

or, after performing the indicated differentiations, †

$$[I + r^2(M_B + M_C) + s^2\beta(H_3 + J_3) + c^2\beta(H_1 + J_1)]\dot{\alpha} + [J_3 s\beta]\dot{\gamma}$$
$$\underset{(1,2)}{=} [I + r^2(M_B + M_C) + H_1 + J_1]\dot{\alpha}^* \tag{4}$$

and

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† Numbers beneath the equality symbols refer to corresponding equations.

$$\dot{\gamma} + s\beta\dot{\alpha} \underset{(1,3)}{=} \dot{\gamma}^* \tag{5}$$

where $\dot{\alpha}^*$ and $\dot{\gamma}^*$ denote the values of $\dot{\alpha}$ and $\dot{\gamma}$ for $\beta = 0$.

Using Eq. (5) to eliminate $\dot{\gamma}$ from Eq. (4), one obtains

$$[I + r^2(M_B + M_C) + J_1 + H_1 - (J_1 + H_1 - H_3)s^2\beta]\dot{\alpha} + J_3\dot{\gamma}^* s\beta$$
$$\underset{(4,5)}{=} [I + r^2(M_B + M_C) + J_1 + H_1]\dot{\alpha}^* \tag{6}$$

and after defining inertia ratios R and E as

$$R = \frac{J_3}{I + r^2(M_B + M_C) + J_1 + H_1} \tag{7}$$

$$E = \frac{J_1 + H_1 - H_3}{I + r^2(M_B + M_C) + J_1 + H_1} \tag{8}$$

one arrives at

$$\dot{\alpha} \underset{(6,7,8)}{=} \frac{\dot{\alpha}^* - R\dot{\gamma}^* s\beta}{1 - Es^2\beta} \tag{9}$$

For all physical systems of interest, E is small in comparison with unity. Consequently Eq. (9) may be replaced with

$$\dot{\alpha} \underset{(9)}{=} \dot{\alpha}^* - R\dot{\gamma}^* s\beta \tag{10}$$

For example, for a man standing on a light platform which permits him to rotate freely about a line through his mass center while he holds a bicycle wheel mounted in a light gimbal, R and E typically have values such as

$$R = 0.113, \qquad E = 0.056 \tag{11}$$

and the approximate equation of motion is

$$\dot{\alpha} \underset{(10,11)}{=} \dot{\alpha}^* - 0.113\dot{\gamma}^* s\beta \tag{12}$$

3. Open Loop Control

Given an arbitrary initial state of rotation of body A, one may wish to cause body **A** to acquire either a specified final spin rate, $\dot{\alpha}_f$, or a specified final orientation, α_f. Open loop control laws intended to accomplish these objectives will now be formulated.

The spin rate problem possesses a solution if and only if $\dot{\alpha}_f$ is an "attainable" value, i.e., if the equation

$$\beta_f \underset{(10)}{=} \sin^{-1}\left[\frac{\dot{\alpha}^* - \dot{\alpha}_f}{R\dot{\gamma}^*}\right] \tag{13}$$

yields real values of the angle β_f. When this is the case, varying β in any manner whatsoever from its initial value to β_f results in the desired spin rate, $\dot{\alpha}_f$.

The reorientation problem can be solved provided $\dot{\alpha} = 0$ is attainable, that is, if the equation

$$\beta_f \underset{(10)}{=} \sin^{-1}\left(\frac{\dot{\alpha}^*}{R\dot{\gamma}^*}\right) \tag{14}$$

yields real values of the angle β_f. One may then proceed in two steps:

First, vary β from its initial value to β_f as given in Eq. (14), thus attaining $\dot{\alpha} = 0$ and placing A in an orientation which may be characterized, without loss of generality, as $\alpha = 0$. Next, designating as α_f the value of α corresponding to the desired orientation, and letting τ be a positive constant having the dimensions of time, vary β in accordance with

$$\beta \underset{(10,14,15)}{=} \sin^{-1}\left[\sin(\beta_f) - \frac{\pi\alpha_f}{2\tau R\dot{\gamma}^*}\sin\left(\frac{\pi}{\tau}t\right)\right] \tag{15}$$

As may be verified by integrating Eq. (10) after using Eq. (15) to express $s\beta$ as an explicit function of t, α is then given by

$$\alpha = \frac{\alpha_f}{2}\left(1 - \cos\frac{\pi}{\tau}t\right) \tag{16}$$

which shows that α attains the value α_f at $t=\tau$.

4. Closed Loop Control Laws

It is possible to formulate closed loop control laws such that, for any initial state of motion, body A acquires either a constant spin rate, $\dot{\alpha}_f$, or a specified orientation, α_f.

The spin rate problem has a solution only if $\dot{\alpha}_f$ is attainable, i.e. if Eq. (13) has a real solution. Assuming that this condition is satisfied, consider the control law

$$\dot{\beta} = F(\dot{\alpha} - \dot{\alpha}_f) \tag{17}$$

where F is a constant. Substituting from Eq. (10) into Eq. (17) yields

$$\dot{\beta} \underset{(10,17)}{=} F(\dot{\alpha}^* - \dot{\alpha}_f) - FR\dot{\gamma}^* s\beta \tag{18}$$

which has the solution

$$\frac{\left|\sin\frac{\beta_f - \beta}{2}\right|}{\left|\cos\frac{\beta_f + \beta}{2}\right|} \underset{(18,13)}{=} De^{-c\beta_f R\dot{\gamma}^* Ft} \tag{19}$$

where D is a constant of integration. This solution shows that, when t approaches infinity, either β approaches β_f or $(\beta_f+\beta)$ approaches $\pm\pi$. In either case $s\beta$ approaches $s\beta_f$ and $\dot{\alpha}$ consequently approaches $\dot{\alpha}_f$ as desired.

As for the closed loop reorientation problem, assume that Eq.(14) yields real values of β_f, and consider the control law

$$\beta = \beta_f + N(\alpha - \alpha_f) \tag{20}$$

where N is a constant and α_f is the desired final orientation. Substituting for β in Eq.(10) then yields

$$\frac{\dot{\alpha}}{R\dot{\gamma}^*} \underset{(10,20)}{=} \sin\beta_f - \sin[\beta_f + N(\alpha - \alpha_f)] \tag{21}$$

which has the solution

$$\frac{\left|\sin\frac{N}{2}(\alpha - \alpha_f)\right|}{\left|\cos[\beta_f + \frac{N}{2}(\alpha - \alpha_f)]\right|} \underset{(20,21)}{=} De^{-c\beta_f RN\dot{\gamma}^* t} \tag{22}$$

where D is a constant of integration. As t approaches infinity, α approaches α_f if $c\beta_f N\dot{\gamma}^* > 0$ and $|N| \leq 2$. (Eq. (7) shows that R is intrinsically positive.)

The closed loop reorientation problem may thus be solved in two steps: After using Eq. (17) with $\dot{\alpha}_f = 0$, one designates β as β_f and uses Eq. (20).

It is also possible to devise a control law which permits reorientation of the system in one step. Let

$$\dot{\beta} = P\dot{\alpha} + Q(\alpha - \alpha_f) \tag{23}$$

where P and Q are constants. Substituting for $\dot{\alpha}$ in accordance with Eq. (10) yields

$$\dot{\beta} = R\dot{\gamma}^* P(s\beta_f - s\beta) + Q(\alpha - \alpha_f) \tag{24}$$

Eqs. (10) and (25) constitute a system of two first-order differential equations which possess the particular solution

$$\beta = \beta_f \quad , \quad \alpha = \alpha_f \tag{25}$$

Linearizing the equations about this particular solution yields

$$\dot{\hat{\beta}} \underset{(24)}{=} -PR\dot{\gamma}^* c\beta_f \hat{\beta} + Q\hat{\alpha} \tag{26}$$

$$\dot{\hat{\alpha}} \underset{(10)}{=} -R\dot{\gamma}^* \hat{\beta} c\beta_f \tag{27}$$

where

$$\hat{\beta} = \beta - \beta_f \tag{28}$$

and

$$\hat{\alpha} = \alpha - \alpha_f \tag{29}$$

The characteristic equation associated with Eqs. (26) and (27) is

$$\lambda^2 + \lambda(PR\dot{\gamma}^* c\beta_f) + QR\dot{\gamma}^* c\beta_f = 0 \tag{30}$$

Consequently, necessary and sufficient conditions for the asymptotic stability of the particular solution are

$$P\dot{\gamma}^* c\beta_f \underset{(30)}{>} 0 \tag{31}$$

and

$$Q\dot{\gamma}^* c\beta_f \underset{(30)}{>} 0 \tag{32}$$

Note that $\beta_f = \sin^{-1} \dfrac{\dot{\alpha}^*}{R\dot{\gamma}^*}$ is satisfied for two values of β; for one of these $c\beta_f > 0$, whereas, for the other, $c\beta_f < 0$. Thus the particular solution for one value of β_f is always stable if P and Q are of the same sign.

5. Proposed Experiment

The feasibility of using gyroscopic effects to control the spin rate or orientation of body A has been established in Secs. 3 and 4. The ability of a man to utilize gyroscopic effects can now be explored by letting a human subject play the role of a portion of the body A and attempt to produce suitable variations of the angle β. More specifically, the experiment requires a light platform capable of supporting a man in both a lying and standing position and carrying a gimbal containing a rotor attached as shown in Fig. 1. To minimize frictional effects high quality bearings must be used for the platform and rotor mounting, and the platform must have a leveling device to insure elimination of undesirable gravitational effects. As for instrumentation, the rate control problem requires that the spin rate of the platform be measured and displayed to the subject, and for the reorientation problem a simple device capable of measuring angular displacement must be provided.

The reason for requiring that the subject be able both to lie and to stand on the platform is that this makes it possible to explore differences in man's ability to perform yaw, roll, and pitch maneuvers. This information may be expected to be particularly useful in connection with the attempt to devise a six-degree-of-freedom maneuvering scheme involving gyroscopic devices.

APPENDIX

Derivation of the Equations of Motion for a System with the Rotor Driven at a Constant Rate

Consider a system as shown in Figure 1 (page 4). The rotor is driven at a constant rate, $\dot{\gamma}_o$, and the angle β is a prescribed function of time.

The kinetic energy of this single-degree-of-freedom system is given in Eq.(1), and the generalized active force is zero. Hence the equation of motion obtained from

$$\frac{d}{dt}\frac{K}{\partial\dot{\alpha}} - \frac{K}{\partial\alpha} = 0$$

is

$$[I + r^2(M_B + M_C) + s^2\beta(H_3 + J_3) + c^2\beta(H_1 + J_1)]\dot{\alpha} + \dot{\gamma}_o J_3 s\beta$$
$$= [I + r^2(M_B + M_C) + H_1 + J_1]\dot{\alpha}^*$$

and by introducing the inertia ratios

$$R = \frac{J_3}{I + r^2(M_B + M_C) + H_1 + J_1}$$

and

$$E' = \frac{J_1 + H_1 - J_3 - H_3}{I + r^2(M_B + M_C) + H_1 + J_1}$$

one arrives at

$$\dot{\alpha} = \frac{\dot{\alpha}^* - R\dot{\gamma}_o s\beta}{1 - E's^2\beta}$$

This equation is of the same form as Eq. (9). Hence, all previous analysis may be applied to the case of the driven rotor simply by replacing E and $\dot{\gamma}^*$ by E' and $\dot{\gamma}_o$, respectively.