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Equivalent Spring-Mass System for Normal Modes

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16. Abstract

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This memorandum describes a technique that can be used to generate equivalent spring-mass models for the normal modes of a structural subsystem when the generalized mass matrix and resonant frequencies are available. Where modal truncation is employed, the residual mass matrix is used to preserve the correctness of the rigid-body mass properties.

Applications of the modeling technique and the residual mass matrix are discussed.

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PREFACE

The work described in this report was performed by the Engineering Mechanics Division of the Jet Propulsion Laboratory.

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CONTENTS

ABSTRACT

Since the lower resonant frequencies are of interest in most structural problems, a significant reduction of independent variables is possible by the use of the normal modes of structural subsystems as independent variables.

This memorandum describes a technique that can be used to generate equivalent spring-mass models for the normal modes of a structural subsystem when the generalized mass matrix and resonant frequencies are available. Where modal truncation is employed, the residual mass matrix is used to preserve the correctness of the rigid-body mass properties.

Applications of the modeling technique and the residual mass matrix are discussed.

INTRODUCTION

Since the lower resonant frequencies are of interest in many structural problems, a significant reduction of independent variables is made possible by the use of the normal modes of structural subsystems as independent variables. The representation of the normal modes of the subsystems as uncoupled single degree-of-freedom spring-mass systems simplifies the task of combining two structural subsystems which are attached at a common point and it puts the information in such a form that most structural analysis computer programs can be used to evaluate the normal modes of the total structure.

This report uses well-known concepts to develop a technique of obtaining an equivalent spring-mass system for each normal mode when the dynamic characteristics of the structural subsystem are available as a generalized mass matrix and associated resonant frequencies. The dynamic characteristics of a continuous subsystem can thus be combined with a discrete system because the continuous subsystem can be represented by a set of mutually independent single-degree-of-freedom systems.

A description of the procedure is to renormalize each normal mode such that its reactions are represented by those of a corresponding single degree-of-freedom equivalent system. Incremental inertia properties must be added in the model to represent the rigid-body contribution of the truncated normal modes. To help illustrate the ideas, a simple model is qualitatively described.

The model is a representation of structure A, a cantilevered beam, attached to a structure B as shown in Figure 1.

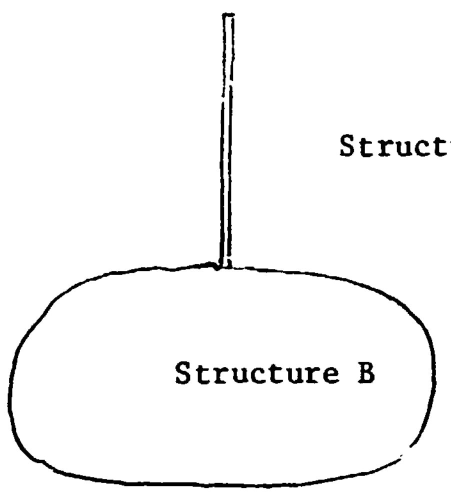

Fig. 1 INITIAL STRUCTURE

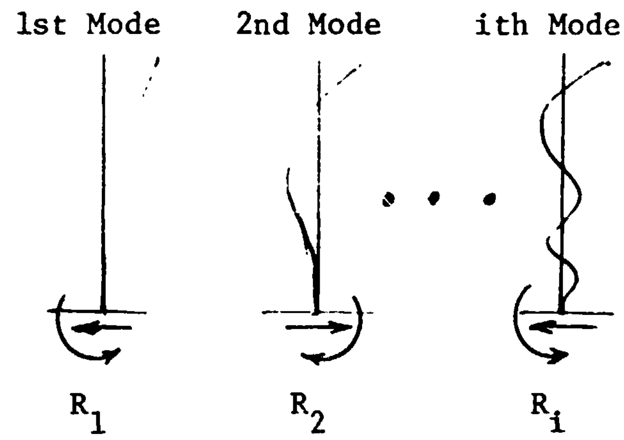

Fig. 2 CANTILEVER MODES OF STRUCTURE A

The equivalent single degree-of-freedom system for each normal mode of Structure A (Fig. 2) is normalized and represented such that each reaction, R_i, is properly simulated when combined with Structure B as shown in Fig. 3.

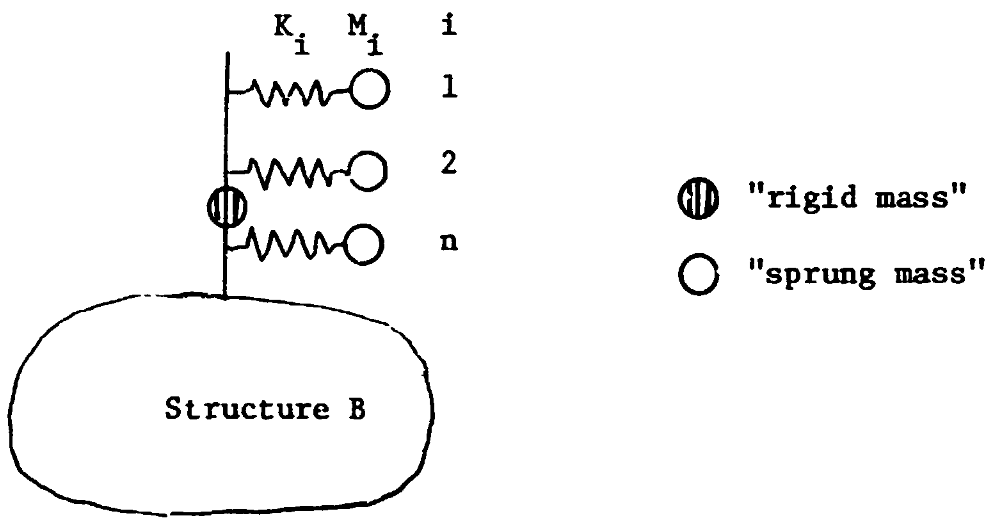

Fig. 3 MODELED STRUCTURE

The "rigid mass" in Fig. 3 represents the contribution of the truncated modes to the rigid-body mass properties of Structure A.

The procedure to define equivalent single degree-of-freedom systems is developed for the general case wherein six base reactions are represented by three orthogonal force components and three orthogonal moment components. Appendices 1 and 2 discuss systems with fewer base reactions to aid in the explanation of the ideas.

ANALYSIS

The equilibrium Equations

The equilibrium equations for any discrete or continuous linear, undamped, structural subsystem in terms of its rigid-body modes and its arbitrarily-normalized characteristic modes are, in matrix form:

$$\begin{bmatrix} M_{RR} & \overline{M}_{RN} \\ \overline{M}_{NR} & \overline{M}_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{P}_R \\ \ddot{P}_N \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \overline{K}_{NN} \end{bmatrix} \begin{Bmatrix} P_R \\ P_N \end{Bmatrix} = \begin{Bmatrix} F \end{Bmatrix} \tag{1}$$

Where

$\left[M_{RR}\right]$ = mass matrix, rigid-body modes

$\left[\overline{M}_{RN}\right] = \left[\overline{M}_{NR}\right]^T$ = inertial coupling matrix, rigid-body and normal modes

$\left[\overline{M}_{NN}\right]$ = diagonal matrix of generalized masses in normal modes

$\left[\overline{K}_{NN}\right] = \left[\omega_N^2 \overline{M}_{NN}\right]$ = the diagonal stiffness matrix

$\{P_R\}$ = vector of generalized displacements in rigid-body modes

$\{P_N\}$ = vector of generalized displacements in normal modes

$(\ddot{\ })$ = second time derivative of ( )

$\{F\}$ = generalized force vector

Eq. (1) may apply to a structural subsystem with inertia relief in some rigid-body degrees of freedom, in which case the related $\overline{M}_{RN}$ and $\overline{M}_{NR}$ terms are zero. Here the normal modes are to be regarded as applying to fully cantilevered structures.

Let x_i $(i = 1,2,3)$ denote a set of orthogonal reference axes with origin at the base point*. The vectors, u_i $(i = 1,2,3)$, represent translational

---

*The base point is the point where the model of the structural subsystem under discussion is attached to another structural subsystem, effectively at a point.

displacements (Fig. 4) and the vectors, $u_i \; (i = 4, 5, 6)$ represent rotational displacements. Unit vectors are $\bar{i}$, $\bar{j}$, and $\bar{k}$.

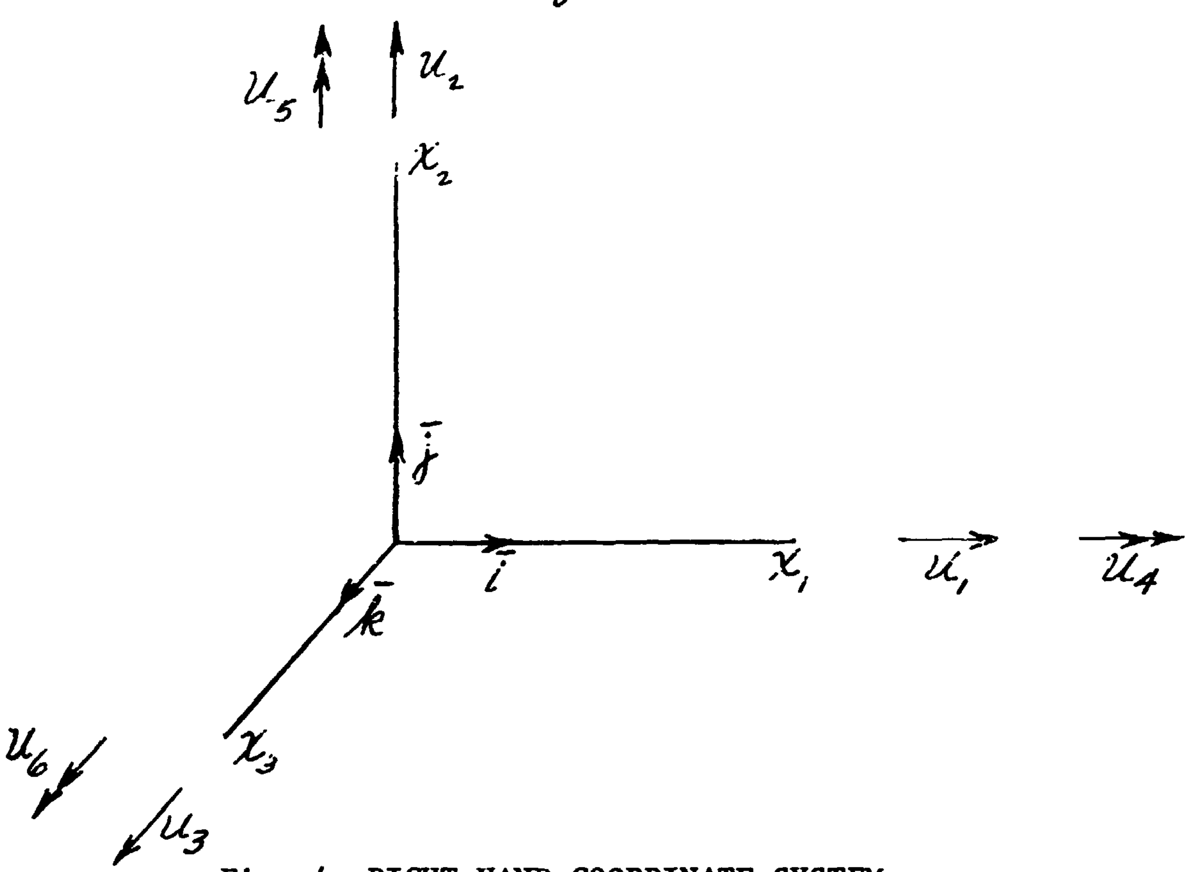

Fig. 4 RIGHT-HAND COORDINATE SYSTEM

The displacement vector of a point on the structural subsystem is

$$\{u\} = \left[\phi_R \,\vdots\, \phi_N\right] \begin{Bmatrix} P_R \\ \hdashline P_N \end{Bmatrix} \tag{2}$$

where

$[\phi_R]$ = rigid-body transformation matrix

$[\phi_N]$ = $R \times S$ matrix of normal-mode displacements at the point, where $R \leq 6$, and S = number of normal modes.

The original equilibrium equations can be written in a form also applying to a system of single-degree-of-freedom spring-mass elements, each of which represents a normal mode. The following analysis leads to such a system.

The Mass Matrix

For the N^{th} normal mode, a renormalization factor is defined:

$$q_N = \left(\bar{M}_{1N}^2 + \bar{M}_{2N}^2 + \bar{M}_{3N}^2\right)^{1/2} \Big/ \bar{M}_{NN} \tag{3}$$

(If the numerator is zero, use $\bar{M}_{4N}, \bar{M}_{5N}, \bar{M}_{6N}$)

The renormalized elements in the complete matrix are*

$$\left.\begin{aligned} M_{NN} &= q_N^2 \bar{M}_{NN} \\ M_{iN} &= q_N \bar{M}_{iN} \qquad (i = 1, \ldots 6) \end{aligned}\right\} \tag{4}$$

It follows that

$$\left(M_{1N}^2 + M_{2N}^2 + M_{3N}^2\right)^{1/2} = M_{NN} \tag{5}$$

If the modes are initially so normalized, then q_N is unity, and subsequent application of the criteria [Eqs. (3) and (4)] produces no change.

The lumped-mass system equivalent to the continuous or discrete structural subsystem is described in terms of selected normal modes, with the complete model represented by the sum of such modal models.

A mass with magnitude M_{NN} is restrained to move parallel to the vector $M_{1N}\bar{i} + M_{2N}\bar{j} + M_{3N}\bar{k}$. The line of action of M_{NN} is placed such that moment arms, $r_{(i+3)N}$, about the x_i axes are

$$\begin{aligned} r_{4N} &= M_{4N} \Big/ \left(M_{2N}^2 + M_{3N}^2\right)^{1/2} \\ r_{5N} &= M_{5N} \Big/ \left(M_{3N}^2 + M_{1N}^2\right)^{1/2} \\ r_{6N} &= M_{6N} \Big/ \left(M_{1N}^2 + M_{2N}^2\right)^{1/2} \end{aligned} \tag{6}$$

*Mass terms with a bar represent the original mass terms, and without a bar the renormalized mass terms.

If $M_{iN} = 0$ (i = 1, 2, 3), the location of the rotatory inertia can be chosen arbitrarily; however, this inertia is constrained to rotate about an axis parallel to the vector $M_{4N}\bar{i} + M_{5N}\bar{j} + M_{6N}\bar{k}$.

For the present, a distinction is made between a normal mode of the original distributed (D) structure and its equivalent lumped (L) system by use of letter superscripts. The motion of the modal mass, M^D_{NN} corresponding to a unit rigid-body translation is

$$\phi^L_{Ni} = M^L_{Ni} / M^D_{NN} \qquad (i = 1, 2, 3)$$

For a unit rigid-body rotation about, say, the x_3 axis

$$\phi^L_{N6} = r_6 \left(M^{D^2}_{1N} + M^{D^2}_{2N} \right)^{1/2} \Big/ M^D_{NN} = M^D_{N6} / M^D_{NN}$$

Thus, in general,

$$\phi^L_{Ni} = M^D_{Ni} / M^D_{NN} \qquad (i = 1, \ldots 6) \tag{7}$$

By definition,

$$M^L_{ij} = \phi^L_{Ni} M^D_{NN} \phi^L_{Nj} = M^D_{iN} M^D_{Nj} / M^D_{NN} \qquad (i,j = 1, \ldots 6) \tag{8}$$

Moreover, since $\phi^L_{NN} = 1$,

$$M^L_{iN} = \phi^L_{Ni} M^D_{NN} \phi^L_{NN} = M^D_{iN} \tag{9}$$

Subsequently, the superscripts, L and D, will be omitted.

From Eqs. (8) and (9) the complete mass matrix for the N^{th} normal-mode equivalent may be written as

$$[M]^{(N)} = \frac{1}{M_{NN}} \left[\begin{array}{cccccc|c} M_{1N}M_{N1} & M_{1N}M_{N2} & \cdots & & & M_{1N}M_{N6} & M_{1N}M_{NN} \\ & M_{2N}M_{N2} & \cdots & & & M_{2N}M_{N6} & M_{2N}M_{NN} \\ & & \cdots & & & M_{3N}M_{N6} & M_{3N}M_{NN} \\ & & & \cdots & & M_{4N}M_{N6} & M_{4N}M_{NN} \\ & (SYM.) & & & \cdot & M_{5N}M_{N6} & M_{5N}M_{NN} \\ & & & & & M_{6N}M_{N6} & M_{6N}M_{NN} \\ \hline & & & & & & M_{NN}^2 \end{array}\right] \quad (10a)$$

or

$$[M]^{(N)} = \left[\begin{array}{c|c} M_{RR}^{(N)} & M_{RN} \\ \hline M_{NR} & M_{NN} \end{array}\right] \quad (10b)$$

The Stiffness Matrix

After renormalization, the stiffness matrix for the N^{th} normal mode is

$$[K_{NN}]^{(N)} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K_{NN} \end{array}\right] \tag{11}$$

where

$$K_{NN} = \omega_N^2 M_{NN}$$

The original structural subsystem represented by S normal modes can be replaced by an equivalent discrete model comprised of S independent single-degree-of-freedom spring mass elements. For the N^{th} mode, the mass is constrained to move along a prescribed line of action. Elastic restraint is provided by a spring of stiffness $\omega_N^2 M_{NN}$.

The solution can be simplified by eliminating the calculations necessary to determine the vector description of the line of action. Moreover, it can be made adaptable to structural-analysis computer programs that allow restraints in only one coordinate system.

Accordingly, the displacement system of Eq. (2), which provides for absolute base-point motion, is used.

Let $\mathcal{U}_B$ = base-motion vector

$\mathcal{U}_N$ = modal displacement, in the absolute reference system, of the point mass representing the N^{th} normal mode

For the single point mass, M_{NN},

$$\phi_N = 1$$

and, for base motion

$$[\phi_{RR}] = [I]$$

In parallel with Eq. (7), the row matrix

$\lfloor \phi_{NR} \rfloor = \dfrac{\lfloor M_{NR} \rfloor}{M_{NN}}$, where M_{NN} is scalar.

Then

$$\begin{Bmatrix} u_B \\ \hdashline u_N \end{Bmatrix} = \left[\begin{array}{c:c} I & 0 \\ \hdashline \phi_{NR} & 1 \end{array}\right] \begin{Bmatrix} P_R \\ \hdashline P_N \end{Bmatrix} \tag{12}$$

Premultiplying both sides of Eq. (12) by $\begin{bmatrix} I & 0 \\ -\phi_{NR} & 1 \end{bmatrix}$ gives

$$\begin{Bmatrix} P_R \\ \hdashline P_N \end{Bmatrix} = \left[\begin{array}{c:c} I & 0 \\ \hdashline -\phi_{NR} & 1 \end{array}\right] \begin{Bmatrix} u_B \\ \hdashline u_N \end{Bmatrix} \tag{13a}$$

or,

$$\{P\} = [T]\{u\} \tag{13b}$$

The contribution of the N^{th} normal mode to Eq. (1) is now subjected to a coordinate transformation. For the N^{th} normal mode, the new stiffness matrix* is

---

*This same result can be obtained by deriving the force required to give the base a unit displacement, successively, in each degree of freedom while holding the modal mass, M_{NN}, fixed in the absolute reference system. It can also be obtained by requiring null forces for rigid-body motion; i.e.,

$$\left[\begin{array}{c:c} K_{BB}^{(N)} & K_{BN} \\ \hdashline K_{NB} & K_{NN} \end{array}\right] \left[\begin{array}{c} I \\ \hdashline \phi_{NR} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hdashline 0 \end{array}\right]$$

from which

$$\lfloor K_{NB} \rfloor = -K_{NN} \lfloor \phi_{NR} \rfloor$$

$$\{K_{BN}\} = -\{\phi_{NR}\}^T K_{NN}$$

$$\left[K_{BB}^{(N)}\right] = -\{K_{BN}\}\lfloor \phi_{NR} \rfloor = \{\phi_{RN}\} K_{NN} \lfloor \phi_{NR} \rfloor$$

$$
\left[\mathcal{K}\right]^{(N)} = \left[T\right]^{T}\left[K^{(N)}\right]\left[T\right]
$$

$$
= \left[\begin{array}{c|c} I & -\phi_{RN} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K_{NN} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline -\phi_{NR} & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} \phi_{RN} K_{NN} \phi_{NR} & -\phi_{RN} K_{NN} \\ \hline -K_{NN} \phi_{NR} & K_{NN} \end{array}\right]
$$

$$
= \omega_N^{2} \left[\begin{array}{c|c} \dfrac{M_{RN} M_{NR}}{M_{NN}} & -M_{RN} \\ \hline -M_{NR} & M_{NN} \end{array}\right] \tag{14a}
$$

Eq. (14a) serves to define the following terms

$$
\left[\mathcal{K}\right]^{(N)} = \left[\begin{array}{c|c} K_{BB}^{(N)} & -K_{BN} \\ \hline -K_{NB} & K_{NN} \end{array}\right] \tag{14b}
$$

The new mass matrix for the N^{th} normal mode is

$$
[\mathcal{M}]^{(N)} = [T]^T [M^{(N)}] [T]
$$

$$
= \left[\begin{array}{c|c} M_{RR}^{(N)} - \dfrac{M_{RN} M_{NR}}{M_{NN}} & 0 \\ \hline 0 & M_{NN} \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{NN} \end{array}\right] \tag{15}
$$

The Complete Mathematical Model

The equivalent discrete model representing S normal modes of the original structural subsystem has the stiffness matrix,

$$
[\mathcal{K}] = \left[\begin{array}{c|ccccc}
\sum\limits_{N=1}^{N=S} K_{BB}^{(N)} & K_{B1} & K_{B2} & \cdot & \cdot & \cdot \quad K_{BS} \\
\hline
K_{B1}^{T} & K_{11} & & & & \\
K_{B2}^{T} & & K_{22} & & & \\
\cdot & & & \cdot & & \\
\cdot & & & & \cdot & \\
\cdot & & & & & \cdot \\
K_{BS}^{T} & & & & & K_{SS}
\end{array}\right] \tag{16}
$$

and the mass matrix

$$
[\mathcal{M}] = \left[\begin{array}{c|ccccc}
M_{RES.} & & & 0 & & \\
\hline
 & M_{11} & & & & \\
 & & M_{22} & & & \\
0 & & & \cdot & & \\
 & & & & \cdot & \\
 & & & & \cdot & \\
 & & & & & M_{SS}
\end{array}\right] \tag{17}
$$

where the "residual mass matrix,"*

$$
\left[M_{RES}\right] = \left[M_{RR}\right] - \sum_{N=1}^{N=S} \frac{M_{RN} M_{NR}}{M_{NN}} \tag{18}
$$

Eq. (1) becomes transformed to

$$
[\mathcal{M}]\{\ddot{u}\} + [\mathcal{K}]\{u\} = \{Q\} \tag{19}
$$

where

$$
\{Q\} = [T]^{T}\{F\}
$$

* See footnote, page 18.

The Residual Mass Matrix

The mass matrix of Eq. (1) is commonly obtained from an initial finite-element mathematical model. Its submatrices may be identified as

$$[M_{RR}] = [\phi_R]^T [m] [\phi_R] \tag{20}$$

$$[\bar{M}_{RN}] = [\phi_R]^T [m] [\bar{\phi}_N] \tag{21}$$

$$[\bar{M}_{NN}] = [\bar{\phi}_N]^T [m] [\bar{\phi}_N] \tag{22}$$

where $[m]$ is the initial mass matrix of r^{th} order, and, in general, is non-diagonal

$[\phi_R]$ is the $(r \times R)$ modal matrix for rigid body modes

$[\bar{\phi}_N]$ is the modal matrix for the normal modes.

If all r modes are chosen for use in Eq. (1), $[\bar{\phi}_N]$ is a square non-singular matrix, and hence can be inverted.

It can be deduced from the form of $M_{RR}^{(N)}$ in Eq. (10) that if all of the modes are retained, then

$$[\bar{M}_{RN}] [\bar{M}_{NN}]^{-1} [\bar{M}_{NR}] = [M_{RR}] \tag{23}$$

Indeed, if Eqs. (21) and (22) are substituted in Eq. (23),

$$[M_{RR}] = [\phi_R]^T [m] [\bar{\phi}_N] [\bar{\phi}_N]^{-1} [m]^{-1} [\bar{\phi}_N^T]^{-1} [\bar{\phi}_N^T] [m] [\phi_R]$$

$$= [\phi_R]^T [m] [\phi_R] \quad \text{(as defined in Eq. 20)}$$

If, as a matter of engineering judgment, S modes are chosen ($S < r$), there is a "residual mass matrix,"

$$[M_{RES}] = [M_{RR}] - [\bar{M}_{RN}] [\bar{M}_{NN}]^{-1} [\bar{M}_{NR}] \tag{24}$$

Eq. (24) is completely general; its validity does not depend on the renormalization decribed by Eqs. (3) and (4).

The residual mass matrix must be added at the base to provide simulation of the total rigid-body mass properties. This addition may be done directly if the computer program to be used accommodates non-diagonal mass matrices. However, two alternate methods are described.

The first and philosophically most obvious is to choose a sufficient number of normal modes to make the residual mass terms negligible. This is not necessarily a practical or desirable course.

An alternative method is to add as many additional spring-mass elements as there are rigid-body degrees of freedom; and to assign to each a frequency well above the modal frequencies of interest or of validity for the total structural system. The $\left[M_{RN}\right]$ matrix for the added systems is contained in the relation

$$\left[M_{RES}\right] = \left[M_{RN}\right]\left[M_{NN}\right]^{-1}\left[M_{NR}\right] \tag{25}$$

where, for the j^{th} normal mode and the i^{th} rigid body mode [from Eq.(5)]

$$M_{jj} = \sqrt{\sum_{k=1}^{k=3} M_{kj}^{2}} \tag{26}$$

(use $k = 4 \ldots 6$ if $\sum = 0$)

Eq. (25) may be written as

$$\left[M_{kj}\right]\left[M_{jj}\right]^{-1/2}\left[M_{jj}\right]^{-1/2}\left[M_{ji}\right] = \left[M_{RES}\right] \tag{27}$$

where the indices, k , i , pertain to rigid-body modes and where the row and column indices have been included in matrix labels to maintain the correspondence to the physical problem.

The matrix equation (27) can be written as

$$\left[M_{RES}\right] = \left[D\right]^{T}\left[D\right] \tag{28}$$

Let [D] be the Choleski decomposition of $[M_{RES}]$.

$$\left[D\right] \equiv \left[M_{\ddot{\gamma}\ddot{\gamma}}\right]^{-1/2}\left[M_{\ddot{\gamma}\ddot{i}}\right]$$

Then

$$\left[M_{\ddot{\gamma}\ddot{i}}\right] = \left[M_{\ddot{\gamma}\ddot{\gamma}}\right]^{1/2}\left[D_{\ddot{\gamma}\ddot{i}}\right] \tag{29}$$

The matrix element

$$M_{\ddot{\gamma}\ddot{i}} = M_{\ddot{i}\ddot{\gamma}} \tag{30}$$

Thus, use of Eqs. (29) and (30) in Eq. (26) gives

$$M_{\ddot{\gamma}\ddot{\gamma}} = \sqrt{\sum_{k=1}^{k=3} M_{\ddot{\gamma}\ddot{\gamma}} D_{\ddot{\gamma}k}^{2}}$$

or

$$M_{\ddot{\gamma}\ddot{\gamma}}^{1/2} = \sqrt{\sum_{k=1}^{k=3} D_{\ddot{\gamma}k}^{2}} \tag{31}$$

Substitution of Eq. (31) into elements of Eq. (29) gives, with use of Eq. (30),

$$M_{\ddot{i}\ddot{\gamma}} = D_{\ddot{\gamma}\ddot{i}}\sqrt{\sum_{k=1}^{k=3} D_{\ddot{\gamma}k}^{2}} \tag{32}$$

(use $k = 4 \ldots 6$ in Eqs. (31) and (32) if $\sum = 0$)

Thus the added "high-frequency" modes, of number equal to the rank of $\left[M_{RES}\right]$, completely account for the residual mass elements resulting from truncation. The generalized masses for these normal modes are obtained from Eq. (31), and the "rigid-elastic" coupling matrix is obtained from Eq. (32).

APPLICATIONS

The first documented application of the renormalization concept described herein appears in Ref. (1). The only available mathematical models of the cantilevered Ranger and Surveyor spacecraft were obtained from modal surveys by application of Eq. (14), wherein the ϕ_N were the measured mode shapes associated with physical items of known mass properties. In the case of Ranger, only the first cantilever torsion mode was of interest. In the case of Surveyor, nine modes having significant coupling with "rigid-body roll" were chosen. Since the launch vehicles in both cases were represented by discrete torsion-spring and moment-of-inertia elements, it was expedient to represent the normal modes of the spacecraft by equivalent spring-mass systems.

The Atlas booster engines and the Centaur main engines were remodeled in a manner to assure only antisymmetric motions. Each engine mode was coupled with rigid-body roll of the axis containing the two gimbal blocks; product-of-inertia terms provided the coupling. On renormalization, each antisymmetric mode of the engine pair was representable as a single spring-mass system, and a residual mass (moment-of-inertia) was attached to the node in the plane of the gimbal blocks to preserve the total rigid-body properties.

Perhaps the most useful facet of the concepts presented herein is the physical "feel" provided to the analyst by associating the M_{RN} vector with base reactions and by using the residual mass matrix as a guide in truncation.

As an example, a mathematical model of a spacecraft and its structural adapter to the launch vehicle was developed for use with the SAMIS Computer Program (Refs. 2 through 5). The mass matrix of this model was of 139th order. The first 42 normal modes were computed, along with the M_{RN} and M_{RR} matrices. Additionally, the residual mass matrices were computed for the number of consecutive modes from mode 1 through mode 42. Table 1 lists the main-diagonal elements, M^R_{ii} of the residual mass matrices through the 14th mode, and lists, also, the ΔM^R_{ii} associated with each added mode. In the first mode, the generalized mass associated with the x_2 (translational) coordinate (i.e., M_{22}) accounts for 82% of the rigid-body mass; and the generalized mass associated with the x_4 (rotational) coordinate (i.e., M_{44}) accounts for 72% of the rigid-body moment-of-inertia about the x_1 axis (see Fig. 4).

The first ten modes contain little effective mass in the longitudinal (x_3) direction; the 11th and 12th modes account for 85% of the rigid-body mass effective in the x_3 direction.

The ΔM_{ii} of Table 1 permit qualitative description of the character of a mode without any detailed knowledge of mode shape. Those modes for which all ΔM_{ii} are small are termed "local modes," in which some component or appendage is the principal contributor.

This type of information can be of use to the analyst who is undertaking a dynamic loads analysis of the entire space vehicle. If the objective is to obtain a preliminary assessment of loads on the launch vehicle, economies can be effected by deleting "local modes," with placement of the appropriate residual masses at the base of the spacecraft.

During a modal survey of the Mariner Mars '71 Development Test Model, an on-site computer terminal was used to make an orthogonality check of measured modes after completion of the survey of the second and each subsequent mode. The residual mass matrix was also computed. Inspection of this matrix served as a guide in shaker placement for excitation of a new mode. It also served as a practical indicator of the number of modes to be surveyed.

CONCLUSION

The concepts presented herein are intended to aid the physical understanding of the dynamic influence of a complex substructure at its point of attachment to another substructure. There may be instances in which it is convenient, for one reason or another, to adopt the equivalent spring-mass approach through the described renormalization process.

The concept of the residual mass matrix, which is not dependent on any particular modal normalization, is, nonetheless, a byproduct of the modeling concept. It has proved to be very useful in the exercise of engineering judgments relating to modal truncation in analysis and to the requisite completeness of a modal vibration survey.

*Footnote:

After the completion of the draft of this memorandum, it came to the authors' attention that Schwendler and MacNeal, in Ref. 11, define and use "residual flexibility matrices." Moreover, in writings not in the open literature, MacNeal has used the term "residual mass matrix" with exactly the same definition as given herein.

	RESIDUAL MASS MATRIX ELEMENTS, % OF CORRESPONDING RIGID-BODY ELEMENT													
No. of Modes	M^R_{11}	ΔM^R_{11}	M^R_{22}	ΔM^R_{22}	M^R_{33}	ΔM^R_{33}	M^R_{44}	ΔM^R_{44}	M^R_{55}	ΔM^R_{55}	M^R_{66}	ΔM^R_{66}	Mode Description	Mode No.
0	100.0		100.0		100.0		100.0		100.0		100.0		"Rigid-Body"	
		5.2		81.8				71.5		3.7				
1	94.8		18.2		100.0		28.5		96.3		95.4		"1st Bending"	1
		79.6		6.0				7.9		57.1				
2	15.2		12.2		100.0		20.6		39.2		95.3		"2nd Bending"	2
				8.5				3.7				38.1		
3	14.9		3.7		100.0		16.9		39.0		57.2		"1st Torsion"	3
		9.0								35.8				
4	5.9		3.7		99.7		16.9		3.2		56.9		"3rd Bending"	4
												1.9		
5	5.9		3.7		99.7		16.9		3.2		55.0		"Local"	5
6	5.9		3.5		99.7		16.7		3.2		54.9		"Local"	6
						2.8						1.6		
7	5.8		3.5		96.9		16.4		3.1		50.3		"Local"	7
												38.1		
8	5.8		3.5		96.7		13.6		3.1		12.2		"2nd Torsion"	8
				.7				11.5				4.7		
9	5.5		2.8		96.4		2.1		3.0		7.5		"4th Bending"	9
		3.1								1.3		1.8		
10	2.4		2.7		96.4				1.7		5.7			10
11	2.3		2.6		96.4						4.9		"Local"	11
						55.5								
12	2.3		2.5		40.9						4.9		"1st Longitudinal"	12
						29.3								
13	2.3		2.1		11.6						4.9		"2nd Longitudinal"	[illegible]
14	2.3		2.1		10.6						4.9		"Local"	14

Table 1. Residual Mass Matrix Elements Applying to Consecutive Numbers of Normal Modes of a Spacecraft Model Having an Initial Dynamic Matrix of 139th Order.

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APPENDIX 1

Illustrative Case of a Single Base-Reaction Component

Consider the longitudinal modes of a "fixed-free" uniform bar of length ℓ and weight per unit length, γ. From Ref. (6), it can be shown (using Timoshenko's notation) that the circular frequency of the ith normal mode is

$$p_i = \frac{\pi}{2}\left(\frac{a}{\ell}\right) i \qquad (i = 1, 3, 5, \ldots) \tag{1.1}$$

where

$$a \equiv \left(\frac{Eg}{\gamma}\right)^{1/2}$$

The modal amplitude at a distance x from the root is

$$X = D_i \sin\left(\frac{i\pi x}{2\ell}\right) \qquad (i = 1, 3, 5, \ldots) \tag{1.2}$$

where D_i is, in general, an arbitrary normalization factor.

Here it will be shown that, for each mode, a value of D_i exists to give base reaction equivalence between the continuous system and a simple spring-mass model.

The generalized mass of the ith mode is

$$\overline{M}_{ii} = \int_0^{\ell} X_i^2 \mu \, dx \tag{1.3}$$

where $\mu \equiv \gamma/g$. Use of Eq. (1.2) in Eq. (1.3) leads to

$$\overline{M}_{ii} = \frac{\mu \ell}{2} D_i^2 \tag{1.4}$$

The axial load in the ith mode at station x is

$$\overline{P}_i = p_i^2 \int_x^{\ell} X_i \mu \, dx \tag{1.5}$$

At the base,

$$\bar{P}_i = \frac{p_i^2}{X_r} \int_0^{\ell} X_r X_i \mu \, dx \tag{1.6}$$

where X_r is an arbitrary rigid-body displacement.

The integral in Eq. (1.6) is recognized as the "rigid-elastic" coupling term in the partitioned matrix

$$[M] = \left[\begin{array}{c|c} M_{rr} & \bar{M}_{ri} \\ \hline \bar{M}_{ir} & \bar{M}_{ii} \end{array}\right]$$

wherein $$M_{rr} = \int_0^{\ell} X_r^2 \mu \, dx$$

With $X_r = 1.0$,

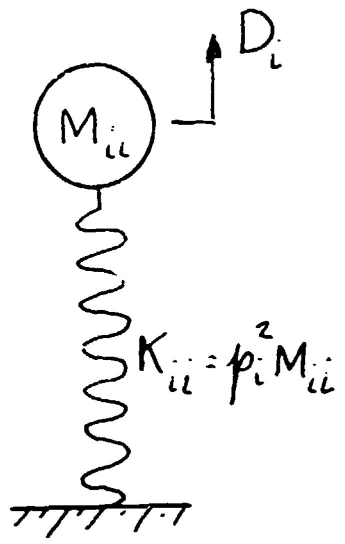
$$M_{rr} = \mu \ell = m$$

$$\bar{P}_i = p_i^2 \bar{M}_{ri} \tag{1.7}$$

Use of Eq. (1.2) in the integral of Eq. (1.6) gives

$$\bar{M}_{ri} = \frac{2\mu\ell}{i\pi} D_i \tag{1.8}$$

To replace the ith mode of the continuous system with a simple system as sketched, it is necessary only to specify a modal renormalization that equates the inertia force to the force acting on the base: i.e.



$$p_i^2 \bar{M}_{ii} = p_i^2 \bar{M}_{ri} \tag{1.9}$$

Thus, equating (1.4) and (1.8) gives

$$D_i = \frac{4}{i\pi} \tag{1.10}$$

$$\left.\begin{aligned} M_{ri} &= D_i \bar{M}_{ri} = \frac{8\mu l}{i^2\pi^2} \\ M_{ii} &= D_i \bar{M}_{ii} = \frac{8\mu l}{i^2\pi^2} \end{aligned}\right\} \tag{1.11}$$

$$(i = 1, 3, 5, \ldots)$$

and the mass matrix becomes

$$[M] = \left[\begin{array}{c|ccccc} M_{rr} & M_{r1} & M_{r2} & \cdots & & M_{rn} \\ \hline M_{1r} & M_{11} & & & & \\ M_{2r} & & M_{22} & & & \\ \vdots & & & \ddots & & \\ M_{nr} & & & & & M_{nn} \end{array}\right]$$

It is convenient to rewrite Eq. (1.11) as

$$M_{ri} = M_{ii} = \frac{8m}{\pi^2(2i-1)^2} \qquad (i = 1, 2, 3, \ldots) \tag{1.12}$$

For the infinite set of elastic modes

$$\sum_{i=1}^{i=\infty} M_{ii} = \frac{8m}{\pi^2} \sum_{i=1}^{i=\infty} \frac{1}{(2i-1)^2} \tag{1.13}$$

From Ref. (7)

$$\int_0^1 \frac{\log(1+x)}{x} dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \quad \ldots = \frac{\pi^2}{12} \tag{1.14}$$

From Ref. (8), the sum of the infinite series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdot \qquad = \frac{\pi^2}{6} \tag{1.15}$$

Adding these series gives

$$2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{3\pi^2}{12}$$

or

$$\sum_{i=1}^{i=\infty} \frac{1}{(2i-1)^2} = \frac{\pi^2}{8} \tag{1.16}$$

Thus

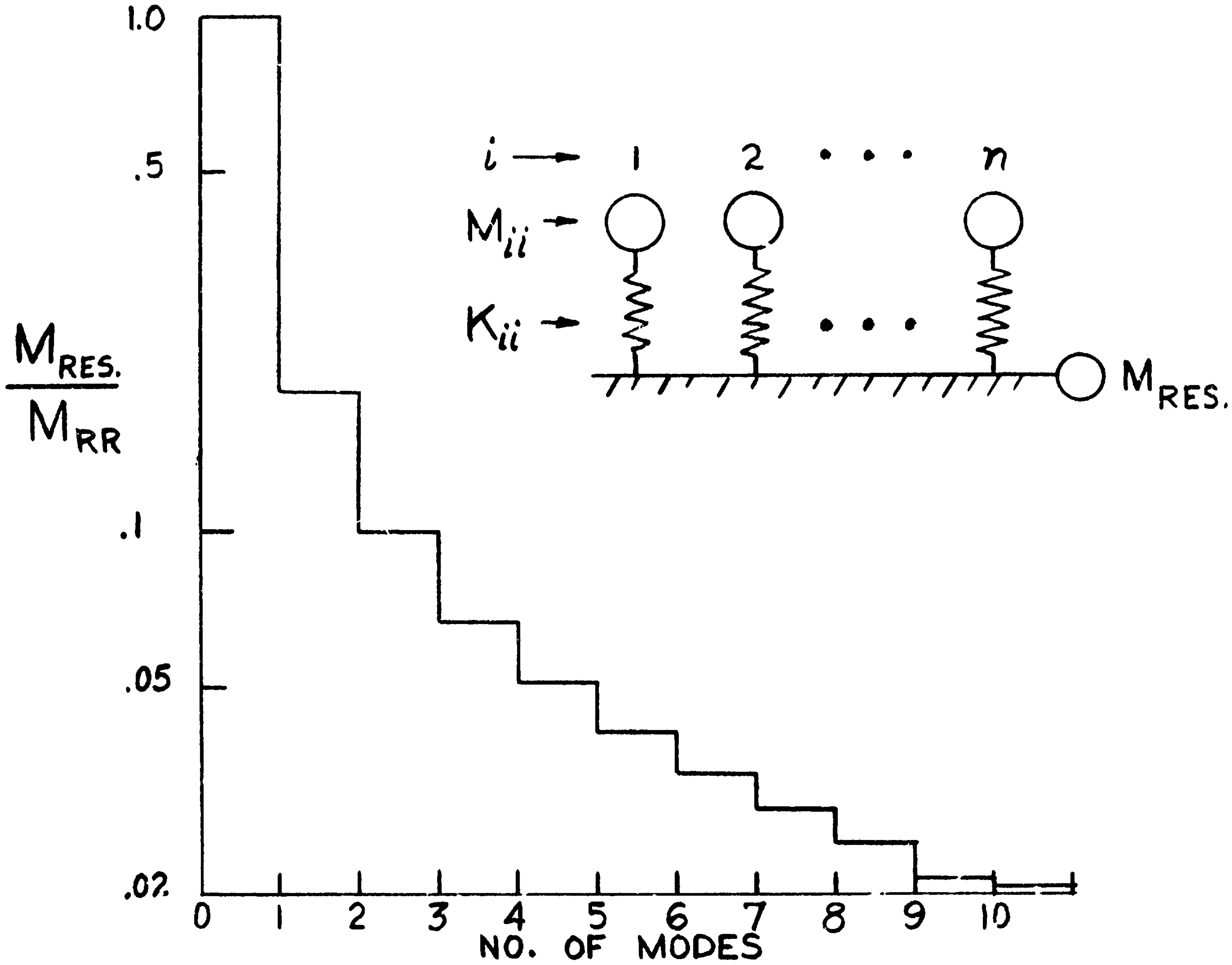
$$\sum_{i=1}^{i=\infty} M_{ii} = m \quad (= M_{rr}) \tag{1.17}$$

In most practical engineering situations, use of a relatively few of the lower modes suffices. Thus, in order to keep the rigid-body mass properties correct, a residual mass is added to the base.

$$M_{RES} = M_{rr} - \sum_{i=1}^{i=n} M_{ii} \tag{1.18}$$

The equivalent mathematical model of the "fixed-free" uniform bar is sketched in Figure 1.1, which also shows the decrease in M_{RES}/M_{rr} with increasing number of modes selected.

Fig. 1.1 Axial modes ~ uniform bar



APPENDIX 2

Illustrative Case of Two Base-Reaction Components

Consider the bending modes of a cantilevered uniform beam, for which there will be transverse shear and bending moment to be reacted by the base.

References 9 and 10 deal with characteristic functions of uniform beams with various boundary conditions. The modal normalization has been chosen such that, for the nth mode,

$$\int_0^{\ell} \phi_n^2(x)\, dx = \ell \tag{2.1}$$

where ℓ is the beam length. Let μ be the mass per unit length. Then the generalized mass in the nth mode is

$$\overline{M}_{nn} = \mu \int_0^{\ell} \phi_n^2(x)\, dx = \mu\ell = m \tag{2.2}$$

The shear at the beam root is

$$\begin{aligned} \overline{V}_n &= \omega_n^2 \int_0^{\ell} \phi_n(x)\mu\, dx \\ &= \frac{\omega_n^2}{\Delta} \int_0^{\ell} \phi_T \phi_n \mu\, dx \end{aligned} \tag{2.3}$$

where $\phi_T \equiv \Delta$ is an arbitrary rigid-body translation.

In this context, the integral of Eq. (2.3) is the "translation-elastic" coupling term, $\bar{M}_{Tn}$ in . ` .ass matrix

$$
[\bar{M}] = \begin{bmatrix}
M_{TT} & M_{TR} & \bar{M}_{T1} & \bar{M}_{T2} & \cdot & \cdot & \bar{M}_{Ts} \\
M_{RT} & M_{RR} & \bar{M}_{R1} & \bar{M}_{R2} & \cdot & \cdot & \bar{M}_{Rs} \\
\bar{M}_{1T} & \bar{M}_{1R} & \bar{M}_{11} & & & & \\
\bar{M}_{2T} & \bar{M}_{2R} & & \bar{M}_{22} & & & \\
\vdots & \vdots & & & \ddots & & \\
\bar{M}_{sT} & \bar{M}_{sR} & & & & & \bar{M}_{ss}
\end{bmatrix} \tag{2.4}
$$

The bending moment at the root is

$$
\begin{aligned}
\mathcal{M}_r &= \omega_n^2 \int_0^\ell x\phi_n \mu\, dx \\
&= \frac{\omega_n^2}{\theta_R} \int_0^\ell \phi_R \phi_n \mu\, dx
\end{aligned} \tag{2.5}
$$

where $\phi_R = \theta_R x$ the transverse displacement of an element of the beam at station x for an arbitrary rotation, θ_R , about the root. Thus, the integral of Eq. (2.5) is the "rotation-elastic" coupling term

$$
\bar{M}_{Rn} = \int_0^\ell \phi_R \phi_n \mu\, dx \tag{2.6}
$$

Since the normalization of the rigid-body modes is optional, let θ_R and Δ be unity. Then

$$\overline{M}_{Tn} = \int_0^{\ell} \phi_n \mu \, dx \tag{2.7}$$

$$\overline{M}_{Rn} = \int_0^{\ell} x \phi_n \mu \, dx \tag{2.8}$$

Now

$$\left.\begin{aligned} M_{TT} &= \int_0^{\ell} \phi_T^2 \mu \, dx = m \\ M_{TR} &= \int_0^{\ell} \phi_T \phi_R \mu \, dx = \frac{m\ell}{2} \\ M_{RR} &= \int_0^{\ell} \phi_R^2 \mu \, dx = \frac{m\ell^2}{3} \end{aligned}\right\} \tag{2.9}$$

From Integral 1 of Ref. 10

$$\mu \int_0^{\ell} \phi_n(x) \, dx = 2m\left(\frac{\alpha_n}{\beta_n \ell}\right) \tag{2.10}$$

From Integral 25 of Ref. 10

$$\mu \int_0^{\ell} x \phi_n(x) \, dx = 2m\ell\left(\frac{1}{\beta_n \ell}\right)^2 \tag{2.11}$$

Ref. 9 tabulates values of α_n and $\beta_n \ell$.

Now, to represent the nth normal mode by a simple spring-mass system, first choose a renormalization factor

$$g_n = \frac{\overline{M}_{Tn}}{\overline{M}_{nn}}$$

$$= 2\frac{\alpha_n}{\beta_n \ell}$$

Then the new matrix elements are

$$\left.\begin{aligned} M_{Tn} &= g_n \overline{M}_{Tn} = 4m\left(\frac{\alpha_n}{\beta_n \ell}\right)^2 \\ M_{nn} &= g_n^2 \overline{M}_{nn} = 4m\left(\frac{\alpha_n}{\beta_n \ell}\right)^2 \end{aligned}\right\} \qquad (2.12)$$

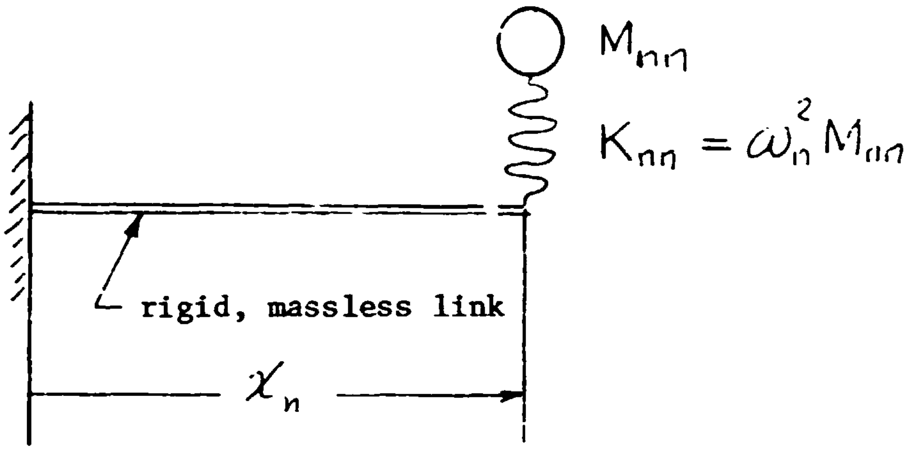
(The shear at the base is now "matched.")

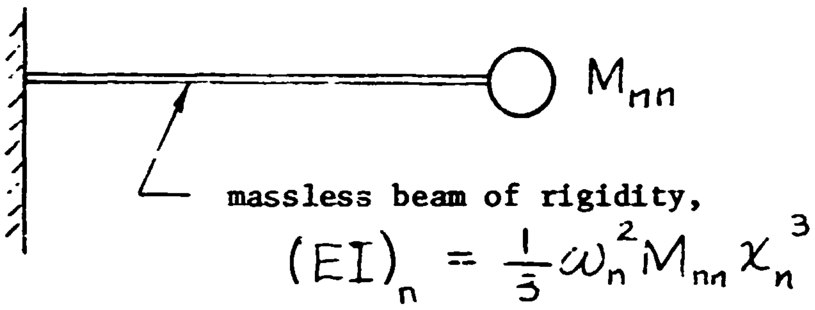
$$M_{Rn} = g_n \overline{M}_{Rn} = 4m\ell\alpha_n\left(\frac{1}{\beta_n \ell}\right)^3 \qquad (2.13)$$

The base moment may be matched by placing the lumped mass at station

$$x_n = \frac{M_{Rn}}{M_{nn}} = \frac{1}{\alpha_n(\beta_n \ell)}\ell \qquad (2.14)$$

In the model, this placement may be done in at least two ways:





It can be shown that

$$\lim_{N\to\infty} \sum_{1}^{N} M_{nn} = M_{TT} \tag{2.15}$$

$$\lim_{N\to\infty} \sum_{1}^{N} x_n M_{nn} = M_{TR} \tag{2.16}$$

$$\lim_{N\to\infty} \sum_{1}^{N} x_n^2 M_{nn} = M_{RR} \tag{2.17}$$

For a finite number of modes, N , a residual mass must be attached to the base.

$$M_{RES.} = M_{TT} - \sum_{1}^{N} M_{nn} \tag{2.18}$$

By placing this mass at

$$x_{RES.} = \frac{\sum_{1}^{N} M_{Rn}}{\sum_{1}^{N} M_{Tn}} \tag{2.19}$$

the static-moment equivalence is preserved.

In general, the rigid-body moment-of-inertia equivalence may require the addition of a centroidal moment-of-inertia.

$$(I_o)_{RES} = M_{RR} - \sum_{1}^{N} M_{nn} x_n^2 - M_{RES.} x_{RES.}^2$$

or

$$\frac{(I_o)_{RES}}{M_{RR}} = 1 - \frac{3\left[\sum_{1}^{N} M_{nn}\left(\frac{x_n}{l}\right)^2 + M_{RES}\left(\frac{x_{RES}}{l}\right)^2\right]}{m} \tag{2.20}$$

The $\left[\overline{M}\right]$ and $\left[M\right]$ matrices for the first five normal modes of a uniform cantilever beam are presented below:

$$
\left[\bar{M}\right] = m \left[\begin{array}{cc:ccccc}
1 & \frac{\ell}{2} & .78299 & .43394 & .25443 & .18190 & .14147 \\
 & \frac{\ell^2}{3} & .56883\ell & .090767\ell & .032416\ell & .016542\ell & .010007\ell \\
\hdashline
 & & 1 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 \\
 & \text{(SYM)} & & & & 1 & 0 \\
 & & & & & & 1
\end{array}\right] \qquad (2.2
$$

$$
\left[M\right] = m \left[\begin{array}{cc:ccccc}
1 & \frac{\ell}{2} & .61307 & .18830 & .064735 & .033088 & .020014 \\
 & \frac{\ell^2}{3} & .44539\ell & .039387\ell & .0082476\ell & .0030089\ell & .0014157\ell \\
\hdashline
 & & .61307 & 0 & 0 & 0 & 0 \\
 & & & .18830 & 0 & 0 & 0 \\
 & & & & .064735 & 0 & 0 \\
 & & \text{(SYM)} & & & .033088 & 0 \\
 & & & & & & .020014
\end{array}\right] \qquad (2.2
$$

Table 2.1 gives a description of residual mass requirements as a function of the number, N, of uniform-beam elastic modes chosen. It illustrates the approach of the residual mass to zero as N increases.

A model representation is given in Fig. 2.1.

N	$\frac{M_{RES}}{M_{TT}}$	$\frac{X_{RES}}{\ell}$	$\frac{(I_o)_{RES}}{M_{RR}}$
1	.3869	.1411	.00616
2	.1986	.0767	.00103
3	.1339	.0521	.00031
4	.1008	.0394	.00010
5	.0808	.0316	.00004

TABLE 2.1 RESIDUAL MASS DESCRIPTIONS AS A FUNCTION OF THE NUMBER OF ELASTIC MODES CHOSEN.

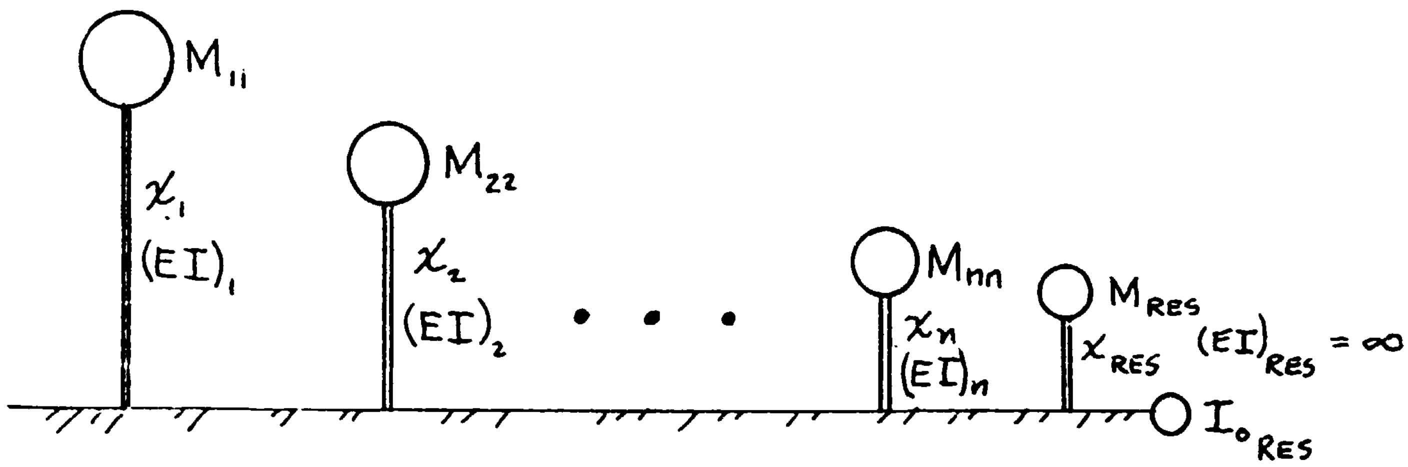


FIG. 2.1 MATHEMATICAL MODEL OF THE FIRST N NORMAL MODES OF A UNIFORM CANTILEVER BEAM.

The stiffness matrix associated with this system is

$$[K] = \omega_1^2 \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Omega_N^2 M_{NN} \end{array}\right]$$

where $\Omega_N^2 \equiv (\omega_N/\omega_1)^2$

From numerical values of Ω_N^2 listed in Ref. 9 , the stiffness matrix for the first five modes is

$$[K] = K' \left[\begin{array}{c|ccccc} 0 & & & 0 & & \\ \hline & .6131 & & & & \\ & & 7.395 & & & \\ & & & 19.33 & & \\ 0 & & & & 39.12 & \\ & & & & & 64.66 \end{array}\right]$$

where $K' = \omega_1^2 m$

Application of the coordinate transformation of Eqs. (14) and (15) leads to

$$[\mathcal{K}] = K' \left[\begin{array}{cc|ccccc} 131.7 & 12.66\ell & -.6131 & -7.395 & -19.93 & -39.12 & -64.67 \\ & 1.618\ell^2 & -.4454\ell & -1.547\ell & -2.540\ell & -3.556\ell & -4.574\ell \\ \hline & & .6131 & & & & \\ & & & 7.395 & & & \\ & & & & 19.94 & & \\ & & (\mathrm{SYM}) & & & 39.12 & \\ & & & & & & 64.67 \end{array}\right]$$

and

$$[\mathcal{M}] = m \left[\begin{array}{cc|ccccc} .0808 & .00255\ell & & & & & \\ .00255\ell & .00010\ell^2 & & & 0 & & \\ \hline & & .6131 & & & & \\ & & & .1883 & & & \\ 0 & & & & .06474 & & \\ & & & & & .03309 & \\ & & & & & & .02001 \end{array}\right]$$

As a further illustration, assume that only the first three normal modes of the uniform beam are to be chosen, and that the truncated modes are to be represented by two additional spring mass systems of arbitrarily high frequency.

From Eq. (28), the Choleski decomposition matrix is written as the upper triangular matrix

$$[D] = \begin{bmatrix} D_{4T} & D_{4R} \\ 0 & D_{5R} \end{bmatrix}$$

(T: translation)

(R: rotation)

The residual mass matrix, determined from values listed in either Eq. (2.21) or (2.22) by use of Eq. (24), is

$$[M_{RES}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} .13389 & .00697 \\ .00697 & .00047 \end{bmatrix}$$

Then, by Eq. (28)

$$\begin{bmatrix} D_{4T} & 0 \\ D_{4R} & D_{5R} \end{bmatrix} \begin{bmatrix} D_{4T} & D_{4R} \\ 0 & D_{5R} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

from which

$$D_{4T}^2 = a_{11} \quad ; \quad D_{4T} = \sqrt{a_{11}}$$

$$D_{4R} D_{4T} = a_{21} \quad ; \quad D_{4R} = \frac{a_{21}}{\sqrt{a_{11}}}$$

$$D_{4R}^2 + D_{5R}^2 = a_{22} \quad ; \quad D_{5R} = \sqrt{a_{22} - \frac{a_{21}^2}{a_{11}}}$$

From Eq. (32), and the above,

$$M_{T4} = D_{4T}\sqrt{D_{4T}^2} = a_{11}$$

$$M_{R4} = D_{4R}\sqrt{D_{4T}^2} = a_{21}$$

$$M_{T5} = D_{T5}\sqrt{D_{5R}^2} = 0$$

$$M_{R5} = D_{5R}\sqrt{D_{5R}^2} = a_{22} - \frac{a_{21}^2}{a_{11}}$$

From Eq. (31),

$$M_{44} = D_{R4}^2 = a_{11}$$

$$M_{55} = D_{R5}^2 = a_{22} - \frac{a_{21}^2}{a_{11}}$$

In matrix form, for the two added degrees of freedom,

$$\left[\begin{array}{c} M_{\ddot{\gamma}\ddot{i}} \\ \hline M_{\ddot{\gamma}\ddot{\gamma}} \end{array}\right] = m \left[\begin{array}{cc} .13389 & 0 \\ .00697\ell & .00010\ell^2 \\ \hline .13389 & 0 \\ 0 & .00010\ell^2 \end{array}\right]$$

Using these $M_{\ddot{\gamma}\ddot{i}}$, $M_{\ddot{\gamma}\ddot{\gamma}}$, and $M^T_{\ddot{\gamma}\ddot{i}}$ matrices to replace the last two rows and columns of Eq. (2.22), and making the coordinate transformation indicated by Eqs. (15) and (14) leads to

$$[\mathcal{M}] = m \left[\begin{array}{c|ccccc} 0 & & & 0 & & \\ \hline & .6131 & & & & \\ & & .1883 & & & \\ 0 & & & .0674 & & \\ & & & & .13389 & \\ & & & & & .00010\ell^2 \end{array}\right]$$

and, with the original frequency ratios of modes 4 and 5 arbitrarily retained for the two residual-mass modes,

$$
[K] = K' \left[\begin{array}{cc|ccccc}
186.3 & 12.77\ell & -.6131 & -7.395 & -19.93 & -158.31 & 0 \\
 & 1.729\ell^2 & -.4454\ell & -1.547\ell & -2.540\ell & -8.241\ell & -.3463\ell^2 \\
\hline
 & & .6131 & & & & \\
 & & & 7.395 & & & \\
 & & & & 19.93 & & \\
 & \text{(SYM)} & & & & 158.31 & \\
 & & & & & & .3463\ell^2
\end{array} \right]
$$

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