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## A REVIEW OF FUNDAMENTAL EQUATIONS OF THE MIXTURE OF A GAS WITH SMALL SOLID PARTICLES

by

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# ABSTRACT

After a brief discussion of the various phases of the flow of a mixture of a gas and small solid particles and the solid particle behavior in the mixture, the fundamental equations of the mixture of the gas and the pseudo-fluid of solid particles are derived from both the continuum theory point of view and the simple kinetic theory point of view. Some new insights about various terms due to the effects of the mixture are obtained. For instance, the partial pressure of the pseudo-fluid of solid particles is not negligible if the volume fraction of the solid particles is not negligible. The internal energy of the pseudo-fluid of solid particles contains two parts; one part depends on the temperature of the solid particles and its specific heat and the other part depends on the random motion of the particles. The internal energy of the pseudo-fluid of the solid particles due to the random motion per unit volume is of the same order of magnitude as the partial pressure of the pseudo-fluid of solid particles. The relations between these two points of view are given and discussed by comparing the corresponding terms. Finally we discuss two simplified cases of the fundamental equations which consist of (i) the thermodynamics of the mixture of solid particles and a gas and (ii) the case of negligibly small volume fraction of solid particles.

# SYMBOLS

| | |
|---|---|
| $a$ | sound speed of the gas |
| $a_m$ | equilibrium speed of sound of the mixture |
| $b$ | a constant |
| $c_i$ | i th component of the instantaneous velocity of the particles |
| $c_s$ | specific heat of the solid particle at constant volume |
| $c_{vp}$ | effective specific heat at constant volume of the solid particles due to its random translational motion |
| $c_{sp}$ | total specific heat of the particles |
| $c_v$, $c_p$ | specific heat of the gas at constant volume and pressure resp. |
| $c_{vm}$, $c_{pm}$ | specific heat of the mixture at constant volume and pressure |
| $c_d$ | drag coefficient |
| $d$ | diameter of sphere particles |
| $D$ | drag force |
| $\vec{F}_b$ | body force |
| $\vec{F}_{bpg}$, $\vec{F}_{bgg}$ | gravitational force of the particles and gas resp. |
| $\vec{F}$ | interaction force between particles and gas |
| $F_p$ | distribution function of the solid particles |

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|---|---|
| $h$ | film coefficient |
| $H_m$ | enthalpy of the mixture per unit mass |
| $k_1, k_2, k_3$ | constants |
| $k_p$ | mass concentration of the solid particles |
| $L_f$ | mean free path of the fluid |
| $L_t$ | characteristic length of the temperature relaxation |
| $L_v$ | characteristic length of the velocity relaxation |
| $M$ | total mass |
| $\overline{m}$ | average mass of solid particles |
| $m$ | mass |
| $M_p^{(+)}, M_p^{(-)}$ | increase or decrease of the number of particles due to collisions resp. |
| $n_p$ | number density of the solid particles |
| $N_u$ | Nusselt number |
| $p$ | total pressure |
| $p_r$ | pressure of the rth species |
| $P_e$ | Peclet number |
| $P_r$ | Prandtl number |
| $\vec{q}$ | velocity vector |
| $q_{cpi}$ | i th component of energy flux of the pseudo fluid of solid particles due to the random motion |

| | |
|---|---|
| $q_{pi}$ | ith component of energy flux of the pseudo-fluid of solid particles |
| $q_r$ | velocity vector of the rth species |
| $Q_h$ | heat transfer rate from a single solid particle |
| $Q_{cr}$ | heat conducting flux of the r th species |
| $r_p$ | radius of spherical particles |
| R | gas constant |
| $R_e$ | Reynolds number |
| $R_m$ | effective gas constant of the mixture |
| $S_{ij}$ | stress tensor of the pseudo-fluid of solid particles |
| t | time |
| $t_t$ | characteristic time of the temperature relaxation |
| $t_v$ | characteristic time of the velocity relaxation |
| $T_r$ | temperature of the rth species |
| U | velocity |
| $u^i$ | ith component of the velocity |
| $U_{mr}$ | internal energy of the rth species |
| $U_{mp1}$ | internal energy of the particles due to the internal degree of freedom |
| $U_{mp2}$ | internal energy of the particles due to random motion |
| V | volume |

| | |
|---|---|
| $x_i$ | ith spatial coordinate |
| $Z$ | volume fraction of the solid particles |
| $\gamma$ | ratio of $c_p$ and $c_v$ |
| $\Gamma$ | ratio of $c_{pm}$ and $c_{vm}$ |
| $\delta$ | ratio of $c_{sp}$ and $c_v$ |
| $\varepsilon_r$ | energy source of the rth species |
| $\eta$ | ratio of $k_p$ and $(1-k_p)$ |
| $\theta$ | instantaneous temperature of the particles |
| $\kappa$ | coefficient of thermal conductivity |
| $\kappa_t$ | thermal friction coefficient |
| $\mu_u$ | coefficient of viscosity of the rth species |
| $\mu_{rl}$ | bulk coefficient of viscosity of the species |
| $\nu$ | kinematic viscosity of the fluid |
| $\bar{\rho}_r$ | partial density of the rth species |
| $\rho_g$ | species density of the gas |
| $\rho_{sp}$ | species density of the solid particles |
| $\sigma_p$ | source function of the particles |
| $\tau_p$ | volume of a single solid particle |
| $\tau^{ij}$ | viscous stress tensor |
| $\phi_j$ | jth component of the force on a solid particle |

| | |
|---|---|
| $\phi_r$ | potential energy of the rth species |
| $\underline{\overline{\phi}}$ | work done due to random body force |

Subscript: the following subscripts are generally used except specifically defined

| | |
|---|---|
| f | fluid |
| g | gas |
| i, j, k | ith, jth or kth component of a vector |
| M | mixture |
| p | particles |

## 1. Introduction

The study of the fluid flow containing solid particles has been the subject of scientific and engineering research for a long time. In the early days, the sediment transport in open channel flow was one of the most interesting research problems since J. Boussinesq's time of 19th century.[1] The sediment transports by water and by air are important in pollution problems which are currently interesting problems. There are many other engineering problems which concern with the fluid flow with solid particles such as fluidized beds, centrifugal separation of particular matter from fluids, electrostatic precipitation of dust, many chemical processings, solid particle motion in rocket exhaust and dust flow in geophysical and astrophysical problems.

Even though the dynamics of the fluid-particle system has been extensively investigated for a long time, because of the complicated situation, most of the investigations are empirical in nature and only in the last few years systemmatical analytical treatments have been developed[2-7]. Since the physical situation is very complicated, it is not possible to develop a general analytic approach which covers all the phases of the fluid-particle system. We have to divide the fluid-particle system into several phases and each phase should be treated in a manner which is most suitable for it. It is desirable to describe briefly the various phases of the fluid-particle system, particularly for the gas-solid particle mixture first and to define the phases which will be studied in this paper.

The various phases of the gas-solid particles can be described clearly by considering the burning of a well packed gun powder or by considering the fluidized beds of well packed particles. The flow conditions for these two cases may be divided into the following five phases:

### (I) Porous medium phase.

For simplicity, we consider a gas flowing through a well packed particle bed by burning the particles as the case of gun powder or by means of a pressure gradient as in the case of fluidized bed. If the rate of the gas flow is very small, the well packed solid particles will not be disturbed, the gas motion is the same as that through a porous medium. We may consider the solid particles as fixed in space. This phase is usually called the fixed bed stage. We shall not consider this phase in this paper.

### (II) Sedimentation Phase.

When the flow rate increases, some of the small particles may first move with the gas flow. As the flow rate increases, the amount of the solid particles moved with the gas flow increases. From now on, we may call it the sedimentation phase in whch the solid particles are transported by the fluid flow. At first the solid particles may not be considered as a pseudo-fluid and the individual properties of these solid particles in the gas flow play important role in the gas flow. We shall not discuss this phase in this paper.

### (III). Fluidization Phase.

As the rate of the fluid flow reaches a critical value, called the flow for fluidization, at which the character of the solid particles changes abruptly to a pseudo-fluid, waves can be set in the solid particle bed. The Pseudo-fluid has similar behavior as ordinary fluid such as to form a level surface. It is usually called the dense phase of the fluidized bed. Ordinarily the overall density of the mixture of the solid and the gas decreased only fractionally - say 10% to 50% are compared with the fixed bed. In other words, the volume occupied by the solid in the mixture would be 10% to 50% of the total volume of the mixture. This is one of the phases which we are going to consider.

### (IV). Slugging Phase.

Further increase of the fluid flow rate would cause the flow of the mixture irregular. For instance, bubbles of the gas rise through the packed solid particles and burst and more and more particles will be carried out by the gas. In principle, the behavior of the mixture is the same as that in the fluidization phase but the actual motion may be due to some instabilities of the flow of the mixture[5]. Hence this is also the case which we are going to consider.

### (V) Two Phase Flow of A Mixture of Solid Particles and Fluid.

For still further increases of the flow rate, the solid particles occupy less than 5% of the total volume and mix well with the gas in the flow field. This is known as the dilute phase of the two phase flow of a mixture of solid particles and gas. Or we simply call it the two phase flow of a mixture of solid particles and gas in a narrow sense. Many literature (reference 3,4 and 6) of gas-solid mixture discuss this dilute phase. This is also a phase which we are interested in.

In principle, the fundamental equations of the mixture of the gas and solid particles are the same as long as the solid particles may be considered as a pseudo-fluid. Hence it is possible to derive a set of fundamental equations of the mixture of the gas and solid particles which covers all the three phases; fluidization, slugging and dilute phases of the mixture of gas and solid particles. The fundamental equations of the mixture of gas and solid particles have been discussed by many authors, particularly by Marble (reference 4, 1963), Murray (reference 5, 1965), Soo (reference 6, 1967) and Vasiliev (reference 7, 1969). Those early attempts for derivations of the fundamental equations of the mixture of gas and solid particles were reviewed by Murray[5] and Soo[6] and we are not going to repeat them here. After studying this literature, particularly references 4-7, we find that there are many fundamental concepts which are not clearly discussed and some of which are misunderstood. We do feel that a systemmatic derivation of the fundamental equations of the mixture of gas and solid particles is necessary before we may study the flow problems of the mixture of gas and solid particles.

The paper by Marble[4] is the first attempt to apply the modern technique of fluid mechanics to the research of the two phase flow of gas and solid particles. Marble introduced many important concepts of the problem in his analysis such as the introduction of the temperature of solid particles and the diameter of solid particles in the distribution function of solid particles. These are very important in the development of fundamental equations of the mixture of gas and solid particles. However, Marble is limited to the case where the volume fraction of solid particles $Z$ is negligibly small in the mixture. Hence his results are applicable to the limiting case of the dilute phase only. It has been shown clearly by Rudinger[8] that the volume fraction of solid particles $Z$ may have significant effect on the flow field of the mixture of gas and solid particles. We are going to extend Marble's analysis for the case of finite volume fraction of particles $Z$ so that our fundamental equations may be applicable for all three phases; fluidization, slugging and dilute phases which we are interested in.

Murray[5] was the first one who derived systemmatically the fundamental equations of the mixture fluid (or gas) and solid particles with finite volume fraction $Z$ from the continuum theory.

Some interesting results have been obtained. However, since he did not use the modern technique of fluid mechanics, he made some fundamental errors such that he assumed that the partial pressure of solid particles is negligible and arbitrarily replaced the partial pressure of gas by the total pressure of the mixture. The partial pressure of the solid particles is directly proportional to the volume fraction Z which has been shown in reference 8 by Rudinger. We plan to derive the fundamental equations of the mixture of gas and solid particles by both the continuum theory and the kinetic theory so that errors made by Murray may be shown clearly and some insight about the properties of the pseudo-fluid of solid particles, such as the internal energy of the pseudo-fluid of solid particles which depends on both the temperature of the solid particles and the random motion of the particles, may be obtained.

In his book,[6] Soo emphasized the importance of the distribution of the size of solid particles in the fundamental equations of the mixture of the gas and solid particles. Hence Soo wrote formally and arbitrarily a set of fundamental equations of multifluid theory with solid particles of each size as a species in the mixture. This set of equations is useless because in general the distribution of size of the solid particles is more or less continuous and thus we cannot use infinite number of sizes. We feel that Marble's analysis by considering the diameter of the solid particles as a parameter in the distribution function of the solid particles is a much better treatment in considering the effects of distribution of size of the solid particles. Soo did not discuss how one could determine the species of the solid particles for any known distribution of size of solid particle in order that his equations may be useful. We shall use a statistical average size of the solid particles in our analysis. Hence we consider the solid particles as one species in our mixture with its diameter as a parameter in the distribution function of the solid particles.

Vasiliev in reference 7 discussed some theories of two phase flow of fluid and solid particles by Russian scientists. Since Vasiliev's main interest is hydraulics, he considered only the incompressible fluid case, i.e., the mixture of a liquid and solid particles. Hence his energy equation is the kinetic energy of the liquid and the solid particles which is not an independent relation and which may be derived from the equation of motion. He did not discuss the energy equation for the temperature distribution of the mixture, which is an independent relation from the equation of motion and which depends on the temperature of the solid particles and the random motion of the solid particles. However, Vasiliev discussed some interesting results of turbulent flow of the two phase flow which we are not going to discuss in this paper. After we derive the fundamental equations of the laminar flow of the mixture of the gas and solid particles, it is possible to derive the corresponding turbulent flow equations in the standard manner.

In the modern technique of fluid mechanics, we study the fundamental equations of fluid mechanics from the microscopic (kinetic theory) and the macroscopic (the continuum theory) points of view. Hence we shall discuss the fundamental equations of the mixture of the gas and the solid particles from the continuum theory §3 and the simple kinetic theory

§4. The complete kinetic theory of a mixture of gas and solid particles has not been developed. It is not the intention of the author to develop a complete kinetic theory of the mixture of gas and solid particles in this paper. What we would like to do is to derive the fundamental equations of the mixture of gas and solid in the continuum theory from the transfer equations of the Boltzmann equation of the mixture so that we may have some insight of various terms in the fundamental equations which cannot be obtained from the continuum theory alone such as the partial pressure of the solid particles and the internal energy of the solid particles.

In development of both the continuum theory of the pseudo-fluid of the solid particle and the kinetic theory of the random motion of the solid particle, we have to use the behaviours of the solid particles in the gas flow as a basis. In fact, the solid particle motion in a fluid is very complicated[2,3,6,9,10]. We shall briefly discuss the particle motion, the drag and heat transfer in §2.

## 2. Particle behaviour in the mixture of a gas and solid particles.

In the two phase flow of solid particles and a fluid, we usually assume that the solid particles are of small size. For instance, the mean particle size in the exhaust of a rocket engine is of the order of a micron in diameter. However, this size is still much larger than the molecular size. We cannot use the usual approach in kinetic theory of gases such that the motion of the particles depends on the force potential around a molecule. The behaviour of these solid particles in the fluid flow should be determined by the ordinary fluid dynamics of a fluid flow around a body. This problem is very complicated itself and we do not understand it in detail yet, because the details of the local flow field depend on the particle Reynolds number, particle Mach number, Knudsen number, shape and size of the particles and the interaction between particles. We have to make some approximations before we could develop any reasonable theory of the two phase flow of gas and solid particles. Even though the actual size of the particles is not uniform, in the two phase theory, we may use some statistical average size of the particles as a first approximation. In other words, in a first approximation, we consider that the size of the particles are the same. For a more accurate theory, we should include the particle size distribution function in the theory. The next complicated point is the shape of the particles. Bodies of different shape have different drag coefficient and heat transfer rate. In general, the shapes of these particles are not identical. As a first approximation, we may assume that the particles are all of the sperical shape. We may consider the drag coefficient of a sphere of diameter d as the average drag coefficient of all particles. We shall discuss some other effects of the shape of the particles later. The third difficulty in the treatment of the particle motion is the interaction between the particles. At the present time, only the theory of particle-fluid system without particle interaction has been developed. These theories, strictly speaking, hold true only for the dilute phase of the mixture of gas and solid particles. It is interesting to notice that the theory of dilute phase of a mixture of gas and solid particles may be used for the case where the particles are small liquid droplets as a first approximation. It would be better if some effects of the interaction between particles, even empirically, are taken into account for the case of dense phase of the two phase flow.

From our experience on ordinary fluid dynamics, the Navier-Stokes equations which may be derived from kinetic theory of gases are applicable to liquid. Hence it is usually assumed that the fundamental equation of the dilute phase of the dynamics of the mixture of gas and particles may be used for the dense phase too as long as the particles may be considered as a pseudo-fluid. When we derive the fundamental equations for the dynamics of the particle-fluid system from the continuum theory, we do not make any specific assumption about the number density of the particles. Hence the fundamental equations should be valid for both the dense and the dilute phase, except that the average drag force of the particles and the transport coefficients of the mixture should be different in these two phases.

From the above approximations, we need to know the drag coefficient and the heat transfer rate of a sphere of diameter d to develop the theory of the dynamics of particle-fluid system. When we examine the literature for the drag coefficient and the heat transfer rate of a sphere, we find that our knowledge is far from complete. Reference 9 gives an excellent review of our knowledge from this problem and it shows many gaps of our knowledge on this subject. In short, the drag coefficient of a sphere depends on the Reynolds number, Mach number and Knudsen number of the sphere as well as the flow condition, such as the turbulence in the free stream, the acceleration of the sphere and other factors. Because of the interactions between many of these factors, it is not possible to give a complete account of all the factors. In order to see the essential features of these factors, we are going to examine these effects separately.

The most important factor for the drag coefficient is the Reynolds number of the sphere $R_e = Ud/\nu$ which is defined as the ratio of the diameter d of the sphere times a typical velocity U divided by the coefficient of kinematic viscosity of the fluid $\nu$. In order to see the effects of Reynolds number on the drag coefficient, we consider a uniform steady flow of velocity U passing over a sphere of diameter d in an infinite domain. We further assume that the fluid is incompressible and isothermal so that the effect of Mach number will not be considered. We further assume that the Knudsen number K which is the ratio of the mean free path of the fluid $L_f$ to the diameter of the sphere is much smaller than unity, the rarefication effect of the fluid is then negligible. For this simple case, the drag coefficient of the sphere is a function of Reynolds number only as shown in Fig. 1. This curve is obtained experimentally. There is no complete theoretical result of the whole curve, even though we know fairly well the flow pattern at various Reynolds number ranges.

When the Reynolds number $R_e$ is very small, say below 0.1, the inertial terms are negligible and the drag coefficient of a sphere was first obtained by Stokes[9] who gave the following formula:

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2} = \frac{3\pi U d\mu}{\frac{1}{2}\rho U^2 \frac{1}{4} d^2 \pi} = \frac{24}{Ud/\nu} = \frac{24}{R_e} . \tag{1}$$

At very low Reynolds numbers, the flow field about the sphere has a fore and aft symmetry. As the Reynolds number increases, inertial force begins to play a role. Above $R_e = 0.1$, there will be a wake behind the sphere. For Reynolds numbers less than unity, we may use the Oseen's approximation[11] and the drag coefficient is given by the following formula:

$$C_D = \frac{24}{R_e} \left(1 + \frac{3R_e}{16}\right) \qquad (2)$$

As the Reynolds number $R_e$ is larger than unity, the drag curve of Fig. 1 is entirely empirical in nature. The flow pattern around the sphere becomes complicated. As the Reynolds number increases above 0.1, vortex rings may be formed at the rear of the sphere and as $R_e$ increases further, we have the well known Karman vortex street behind the sphere which will eventually be developed into a wake. For high Reynolds numbers, there will be a boundary layer flow on the sphere. At Reynolds number less than the critical Reynolds number $R_c$ which depends on the free stream turbulence and which is usually of the order of $10^5$ to $10^6$, there is a large wake due to the separation of the laminar boundary layer on the sphere. The drag coefficient is a constant over a large range of Reynolds number at a value of 0.5. At the critical Reynolds number, the drag coefficient of the sphere drops from 0.5 to 0.1 and then remains at a constant value of 0.1 for a large range of Re above $R_c$. Above the critical Reynolds number, the boundary layer on the sphere is turbulent. Since the ability to resist separation by the turbulent boundary layer is larger than that of the laminar boundary layer, the flow will separate from the sphere at a later point. As a result, the wake behind the sphere is smaller and the drag of the sphere is also smaller.

The drag curve given in Fig. 1 is for the steady flow in an infinite domain. In actual two phase flow, the velocity of the fluid is not uniform. Hence the particles are moving in an accelerating stream. Hence, strictly speaking, we should consider the acceleration effects on the drag coefficient of the sphere. The unsteady drag associated with the fluid and particle accelerations depends on the flow field. In other words, it depends on the change of magnitude and direction of the flow velocity around the particle. No general solution of this problem is available. Hinze and Tchen discussed the case for slow motion of a spherical particle in an accelerating flow,[12,13] which is difficult to use in the development of the theory of two phase flow. Fortunately, because of the small size particles and low relative acceleration between fluid and particles in many practical problems, we may neglect the acceleration effects and use the stokes formula as a first approximation to develop the theory of the dynamics of the fluid-particle system.

When the volume fraction of the solid particles Z is above 0.1, it is advisable to modify the expression of drag coefficient of the particles in the mixture of gas and solid particles. There is no theoretical analysis for such modification but some empirical formulas have been suggested by various authors (see chapter 5 of reference 6). When the particle Reynolds number is below 0.1, the following drag formula is recommended:[19]

$$\begin{aligned} &\text{When} \quad Z > 0.1 : \quad C_D = 200 \frac{Z}{(1-Z)^2 R_e} \\ &\text{and When} \quad Z < 0.1 : \quad C_D = 24/R_e \end{aligned} \tag{3}$$

We shall use Eq. (3) in some of numerical solution for our study in the lunar ash flow.[17] In these numerical calculations, we will see the effects of the drag coefficient due to volume fraction of the solid particles.

Besides the effects of Reynolds number on the particles, there are many other factors which would affect the drag coefficient of the sphere. These factors are as follows:

(i) Mach number of the particle. The Mach number gives the effects of compressibility on the drag of the sphere. The flow pattern would change as the Mach number increases, particularly when the flow is changed from low subsonic to a supersonic case. As a result, the drag of the sphere would change too. However, since the velocity which determines the Mach number of the sphere in the two phase flow case is the difference of velocities of the particle and that of the fluid. Thus in general, we do not expect a large difference of these velocities. The drag coefficient for low Mach number such as that given by Fig. 1 should be sufficiently accurate.

(ii) Free stream turbulence. The free stream turbulence has a large effect on the drag of the sphere if the Reynolds number is near the critical value $R_c$. If the Reynolds number of the particle is very small, the effect of the free stream turbulence should be small as far as the interaction of the particle and the fluid is concerned.

(iii) Rotation of the particles. If the sphere is rotating in the fluid flow, it would produce a lift which is known as the Magnus effect and then the interaction of the particle and the fluid should be modified accordingly. For particles of shape other than that of a sphere, the rotation of the body would introduce a moment and thus modify the interaction of the particle and the fluid accordingly.[10]

(iv) Surface roughness of the particle. The drag coefficient given by Fig. 1 is for a sphere of smooth surface. If the surface of the sphere is rough, the drag coefficient may be different from that given by Fig. 1, particularly when the size of the particle is not too small and the Reynolds number is near its critical value. Ordinarily, this effect is small in two phase flow.

(v) Rarefication effect. If the diameter of the sphere is of the order of the mean free path of the gas in which the sphere is moving, rarefication effect will not be negligible. At first, we may have slip flow condition for the sphere. If the diamter of the sphere is much smaller than the mean free path, we have free molecule flow for the sphere. If we operate in these conditions, the drag force should be modified accordingly.[14]

Even though there are many uncertainties about the drag coefficient of the sphere, the author believes that by proper choosing of the size of the sphere, the modified Stokes law (3) should be able to represent average drag force of the sphere in the development of the theory of the dynamics of solid-particle gas system which has been used by many authors[2-8] and which will be used in this report.

Another important property of the particle which is essential in the development of the theory of two phase flow of particle-and gas is the heat transfer rate of the particles. The status of our knowledge for the heat transfer of a sphere is just as uncertain or empirical as that for the drag coefficient, because the heat transfer rate depends on the local flow field which is essential for the determination of the drag. Similar to the case of drag coefficient, we have only theoretical results for low Reynolds number or the Stokes regime only. The Nusselt number for a sphere is two when the diameter of the sphere is used as a typical length. For the second approximation, we may use the following formula:

$$\text{Nusselt No.} = N_u = \frac{hd}{\kappa} = 2 + \tfrac{1}{2}P_e + \text{---} \tag{4}$$

where $h$ is the film coefficient of the sphere or the heat conducted per unit area per unit time into the sphere, $d$ is the diameter of the sphere, $\kappa$ is the coefficient of thermal conductivity of the fluid and $P_e = P_r \cdot R_e$ is the Peclet number, $P_r = c_p \mu/\kappa$ is the Prandtl number and $R_e = Ud/\nu$ is the Reynolds number of the sphere.

For high Reynolds number range, Eq. (4) may be modified by empirical formula such as

$$N_u = 2 + k_1 (R_e)^{k_2} \cdot (P_r)^{k_3} \tag{5}$$

where $k_1$, $k_2$ and $k_3$ are constants determined by fitting the empirical data in the range of Reynolds number $R_e$ and Prandtl number $P_r$ which are of interest. Those factors mentioned above which have influence on the drag coefficient would also have influence on the heat transfer rate too. It should be noticed that if we consider only the case of small particle Reynolds number, $N_u = 2$ should be of sufficient accuracy.

For a first approximation, we may use an analogy similar to Reynolds analogy of ordinary fluid dynamics for the heat transfer and interaction force due to drag of the solid particles as follows:

If we write the interaction force due to drag of particles as

$$\vec{F} = K_D(z)\ (\vec{q}_g - \vec{q}_p)$$

the heat transfer rate between the two species in the mixture is then

$$Q_T = \frac{2}{3} \frac{\kappa_p}{\mu_g} K_D(z) \ (T_p - T_g)$$

where $K_D(z)$ is a function of volume fraction $z$ (see Eq. (3)) and $\kappa_p$ is the coefficient of thermal conductivity of the solid particles and $\mu_g$ is the coefficient of viscosity of the gas [see Eq. (37)].

The drag force of the particle determines the slip in velocity between the average velocity of the solid particles and the fluid velocity while the heat transfer rate of the particle determines the difference of the temperature of the solid particles and that of the fluid. By dimensional analysis and the fundamental properties of the solid particles and those of the fluid, we may determine the characteristic times and lengths which show how long the time and / or the distance that the velocity and the temperature of the particles will reach those corresponding values of the fluid. We are going to determine these characteristic times and lengths as follows:[4]

(i) Relaxation in velocity. The properties of the solid particles are the average diameter $d$ of the sphere, the average mass $m$ of a solid particle and the specific heat $c_s$ of the particle. The physical properties of the fluid or the gas are the coefficient of viscosity $\mu$ of the fluid, the coefficient of thermal conductivity $\kappa$ of the fluid, the specific heat at constant pressure of the fluid $c_p$ and the flow velocity $U$ of the fluid. Now the drag force $D$ of the solid particle should be of the order of magnitude of its inertial force for the final equilbrium condition. By dimensional analysis, we have

$$D \sim d\mu U \sim \text{inertial force} \sim mU/t_v = mU^2/L_v \qquad (6)$$

where $t_v$ is the characteristic time of velocity relaxation while $L_v$ is the characteristic length of velocity relation. From Eq. (6), we have

$$t_v = \text{constant} \cdot \left( \frac{m}{d\mu} \right) = L_v/U \quad . \qquad (7)$$

Marble[4] suggested that the constant in Eq. (7) may take the value of $\frac{1}{3\pi}$ if we use the Stokes formula for the drag force. Equation (7) gives us a formula to estimate the relaxation time and distance to reach the equilibrium of velocity between the particles and the fluid.

(ii) Relaxation in temperature. By dimensional analysis, the equilibrium of the heat transfer by conduction and the increase of temperature of the solid gives:

$$\text{Total amount of heat conducted} \sim \text{area} \times \text{film coefficient} \times \text{time} \sim$$
$$\sim d^2 \cdot h \cdot t_T \sim d^2 \frac{\kappa}{d} t_T = \frac{\kappa d L_T}{U} \sim c_s\, m \qquad (8)$$

where $t_T$ is the characteristic time of temperature relaxation while $L_T$ is the characteristic length of temperature relaxation. From Eq. (8), we have

$$t_T = (\text{constant}) \cdot \frac{c_s m}{d\kappa} = L_T/U \quad . \qquad (9)$$

Marble[4] suggested that the constant in Eq. (9) may take the value of $1/2\ \pi$ if we use $N_u = 2$ for the heat transfer rate. Equation (9) gives us a formula to estimate the relaxation time and distance to reach the equilibrium of temperature between the particles and the fluid. Besides the heat conduction, the thermal radiation of the particles may not be negligible in some cases.

## 3. Fundamental equations of the mixture of a gas and solid particles from continuum theory point of view.

We consider only the case that the particles may be considered as a pseudo-fluid. The solid particles are spheres of identical mass $m_p$, radius $r_p$ and specific heat $c_s$. We may consider the mixture as the mixture of two fluid: one of the real fluid, gas or liquid and the other is the pseudo-fluid of the solid particles. For each species $r$ in the mixture, we would like to know its velocity vector $\vec{q}_r$, its temperature $T_r$, its pressure $p_r$ and its density $\bar{\rho}_r$. Since there are some confusions about the definition of density of the solid and the fluid, we first clarify this point, as follows:

We consider an element of the mixture of a fluid $f$ and solid particles $p$ with total mass $M = M_f + M_p$ and with total volume $V = V_f + V_p$ where the subscript $f$ refers to the value of the fluid, gas or liquid, and the subscript $p$ refers to that of solid particles. It is convenient to introduce the number density of the solid particles $n_p$ which is the number of solid particles per unit volume at a point in the flow field. The volume occupied by the solid particles $V_p$ is then

$$V_p = n_p \cdot V \cdot \bar{\tau}_p \tag{10}$$

where $\bar{\tau}_p = \frac{4}{3} \pi r_p^3$ is the volume of a solid particle in the mixture. The mass of the solid particles in the volume $V$ is

$$M_p = m_p n_p V \quad . \tag{11}$$

The species density of the solid particles is defined as

$$\rho_{sp} = \frac{M_p}{V_p} = \frac{m_p}{\bar{\tau}_p} \quad . \tag{12}$$

Hence the species density of the solid particles is a constant for a given problem.

In the analysis of the two-fluid theory, we should use the partial density of each species. The partial density of the pseudo-fluid of solid particles is defined as

$$\bar{\rho}_p = \frac{M_p}{V} = m_p n_p = Z \rho_{sp} = \rho_{sp} \bar{\tau}_p n_p \tag{13}$$

where $Z$ represents the fraction of volume of solid particles in the mixture which is one of the important variables in the treatment of two phase flow of gas and solid particles. From Eq. (13), we have

$$Z = \frac{V_p}{V} = n_p \bar{\tau}_p \quad . \tag{14}$$

In many analysis of two phase flow, we consider the case where $Z$ is very small in comparison with unity so that some simplification may be obtained (see §V). However, we shall first derive the fundamental equations for finite value of $Z$. The partial density of the pseudo-fluid of solid particles $\bar{\rho}_p$ is one of the fundamental variables in our analysis and it is proportional to $Z$ or $n_p$ .

Similarly, we have also the species density of the gas and the partial density of the gas too. The species density of the gas or fluid is defined as

$$\rho_g = \frac{M_g}{V_g} \tag{15}$$

and the partial density of the gas or fluid is defined as

$$\bar{\rho}_g = \frac{M_g}{V} = \frac{V_g}{V} \frac{M_g}{V_g} = (1 - Z) \rho_g \quad . \tag{16}$$

Only when the volume fraction of the particles $Z$ is negigibly small in comparison to unity, the partial density of the gas is approximately equal to the species density. The distinction between these two densities is very important in the analysis of two phase flow for finite values of $Z$.

Now we are going to discuss the fundamental equations for the mixture of a gas and a pseudo-fluid of solid particles based on the two fluid theory[15,16]. These equations are as follows:

(i) Equations of state.

For each species in the mixture of gas and pseudo-fluid of solid particles, we have one equation of state. For the gas, we may use the perfect gas law, we have

$$p_g = R\,\bar{\rho}_g\,T_g = R\,(1 - Z)\,\rho_g\,T_g = (1 - Z)\,p \tag{17}$$

where $p_g$ is the partial pressure of the gas in the mixture and $T_g$ is the partial temperature of the gas and $R$ is the gas constant. The total pressure of the mixture is $p$ which is obtained from the perfect gas law as follows:

$$p = R\,\rho_g\,T_g \tag{18}$$

Since the total pressure of the mixture is the sum of the partial pressure of the gas $p_g$ and the partial pressure of the pseudo-fluid of solid particles $p_p$, we have $p = p_g + p_p$. With the help of Eqs. (17) and (18), we find that the partial pressure of the solid particles must be

$$p_p = Z\,p \tag{19}$$

If $Z \ll 1$, we may assume that the partial pressure $p_p$ of the solid particles may be negligible as has been assumed in many literature. However, for finite value of $Z$, Eq. (19) should be used. Thus we show that Murray's assumption that $p_p$ is negligible for finite $Z$ is not correct.

The equation of state for the pseudo-fluid of solid particle is simply

$$\rho_{sp} = \text{constant} \tag{20}$$

Thus in the two fluid theory of the mixture of gas and solid particles we may consider one variable less than the theory of the mixture of two compressible fluids, as we shall show later.

(ii) Equations of continuity.

For each species in the mixture, we have one equation of continuity which gives the conservation of mass of that species.[15] For the pseudo-fluid of the solid particles, we have the equation of continuity:

$$\frac{\partial Z\rho_{sp}}{\partial t} + \frac{\partial}{\partial x^i}(Z\,\rho_{sp}\,u_p^i) = -\,\sigma_p \tag{21}$$

where $u_p^i$ is the ith component of the velocity of the pseudo-fluid of the solid particle $\vec{q}_p$ and $\sigma_p$ is the source function of the particles and the summation convention is used in Eq. (21). Ordinarily, we may take $\sigma_p = 0$ while in some problems such as lunar ash flow[17], we may take $\sigma_p = b\,\bar{\rho}_p = b\,Z\,\rho_{sp}$ where $b$ is a constant.

For the gas, we have the equation of continuity:

$$\frac{\partial}{\partial t}[(1 - Z)\,\rho_g] + \frac{\partial}{\partial x^i}[(1 - Z)\,\rho_g\,u_g^i] = \sigma_p \tag{22}$$

where $u_g^i$ is the ith component of the velocity of the gas $\vec{q}_g$ and $\rho_g$ is the species density of the gas, which is governed by Eq. (18).

Adding Eqs. (21) and (22), we have the equation of continuity of the mixture:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u^i}{\partial x^i} = 0 \tag{23}$$

where $\rho$ is the density of the mixture, i.e.,

$$\rho = Z\,\rho_{sp} + (1 - Z)\,\rho_g = \bar{\rho}_p + \bar{\rho}_g \tag{24}$$

and $u^i$ is the ith component of the flow velocity vector of the mixture $\vec{q}$ which is defined as

$$\vec{q} = \frac{1}{\rho}\,(\bar{\rho}_g\,\vec{q}_g + \bar{\rho}_p\,\vec{q}_p) \quad . \tag{25}$$

(iii) Equations of motion.

For each species, the conservation of momentum gives the corresponding equations of motion.

For the pseudo-fluid of solid particles, we have the equations of motion as follows:

$$Z\,\rho_{sp}\,\left(\frac{\partial}{\partial t} + u_p^i\,\frac{\partial}{\partial x^i}\right)\,\vec{q}_p = Z\,\rho_{sp}\,\frac{D_p\vec{q}_p}{Dt} = -\,\nabla p_p + \nabla\cdot\tau_p + \vec{F}_{bp} + \vec{F}_p - \sigma_p\vec{Z}_p \tag{26}$$

and for the gas, the equations of motion are:

$$(1-Z)\,\rho_g\left(\frac{\partial}{\partial t} + u_g^i\,\frac{\partial}{\partial x^i}\right)\,\vec{q}_g = (1-Z)\rho_g\,\frac{D_g\vec{q}_g}{Dt} = -\,\nabla p_g + \nabla\cdot\tau_g + \vec{F}_{bg} + \vec{F}_g + \sigma_p\vec{Z}_p \tag{27}$$

where the partial pressures of the pseudo-fluid of solid particles and that of the gas are respectively $p_p$ and $p_g$ while the corresponding viscous stresses tensors of the solid particles and gas are $\tau_p$ and $\tau_g$ . Even though the study of the viscous stresses of the pseudo-fluid of the solid particles is still meager, for a first approximation from the continuum theory point of view, we may take similar expression for both the pseudo-fluid of solid particles and that of the gas with the Navier-Stokes relations[4,5] as follows:

$$\tau_r^{ij} = \mu_r \left(\frac{\partial u_r^i}{\partial x^j} + \frac{\partial u_r^j}{\partial x^i}\right) + \mu_{r1} \left(\frac{\partial u_r^k}{\partial x^k}\right) \delta^{ij} \tag{28}$$

where $\delta^{ij} = 0$ , if $i \neq j$ , $\delta^{ij} = 1$ , if $i = j$ , subscript $r$ refers to the values of rth species, the summation convention does not apply to the indices of the species $r$ but only to the tensorial indices $i$ , $j$ or $k$ . The coefficient of viscosity of rth species is $\mu_r$ while the second or bulk coefficient of viscosity of rth species is $\mu_{r1}$ . The restrictions on the values of $\mu_r$ and $\mu_{r1}$ are $\mu_r \geqq 0$ and $2\mu_r + 3\mu_{r1} \geqq 0$ . Ordinarily, we may take $2\mu_r + 3\mu_{r1} = 0$ for a first approximation. If $Z$ is not negligibly small, we may not neglect the partial pressure of the pseudo-fluid of solid particles. The assumptions used by Murray in reference 5 that $p_g = p$ and $p_p = 0$ for arbitrary value of $Z$ are wrong. They are true only when $Z \to 0$ . In general, we should use the relations (17) and (19) for $p_g$ and $p_p$ in terms of $Z$ and the total pressure of the mixture.

The body forces of the species are $\vec{F}_{bp}$ and $\vec{F}_{bg}$ which consists of the gravitational forces and the electromagnetic forces and other body forces. For instance, for the gravitational forces, we have

$$\vec{F}_{bpg} = Z \rho_{sp} \vec{g} \quad , \quad \vec{F}_{bgg} = (1-Z) \rho_g \vec{g} \tag{29}$$

where $\vec{g}$ is the gravitational acceleration vector. If the body forces are the electromagnetic forces, we have to consider the electromagnetic field simultaneously with those fundamental equations discussed in this report. We are not going to consider the electromagnetic forces in this report.

The most difficult forces in the present analysis are the interaction forces between the fluid or gas and the pseudo-fluid of solid particles which are shown in Eqs. (26) and (27) as $\vec{F}_p$ and $\vec{F}_g$. As we have briefly discussed in section 2, the interaction forces consist of many terms such as the drag forces of the particles, the bouyancy force of the solid particles, the acceleration forces, etc. In many theoretical analyses, only the simple drag force due to viscous stress is considered. For first approximation, we may use the simple formula:

$$\vec{F}_p = n_p 6\pi r_p \mu_g (\vec{q}_g - \vec{q}_p) \tag{30}$$

where we assume that the solid particles are spheres of same radius $r_p$ and the interaction between particles are negligible and Stokes formula is used for the viscous stress. Of course, Eq. (30) is good only when the volume fraction of the solid particle is small, say $Z < 0.1$. For larger value of $Z$, some modification of Eq. (30) should be used[19] such as Eq. (3) or other formulas.

By Newton's third law of motion, we have always

$$\vec{F}_g = - \vec{F}_p \quad . \tag{31}$$

If we consider the gravitational force as a body force, we should also include the bouyancy force as a part of the interaction force. In this case, we should add the bouyancy force to the viscous drag force of Eq. (30) and the bouyancy force is simply

$$\vec{F}_{p(b)} = - Z \rho_g \vec{g} \tag{32}$$

It is interesting to notice that if we combine the bouyancy force with the corresponding body force due to gravitational acceleration, we have for the pseudo-fluid, the force due to gravitational acceleration

$$\vec{F}_{bpg} + \vec{F}_{p(b)} = Z (\rho_{sp} - \rho_g) \vec{g} \tag{33}$$

where for the gas (Eq. (27)), we have the force due to gravitational acceleration:

$$\vec{F}_{bgg} + \vec{F}_{g(b)} = (1-Z) \rho_g \vec{g} + Z \rho_g \vec{g} = \rho_g \vec{g} \tag{34}$$

where Eqs. (33) and (34) were used in references 5 and 6 without indication of the origins of these forces. Even though those other effects of interaction forces such as due to acceleration, rotation etc. have been discussed in literature, they have not been included in any successful treatment of theoretical analysis of the flow problems of particle-fluid system. We do not expect that they will be successfully included in any flow problems analysis in the near future and hence we shall not discuss them further.

The term $\sigma_p Z_p$ is the force associated with the momentum due to the source term $\sigma_p$.

If we add Eqs. (26) and (27) and use the definitions of the density and the flow velocity of the mixture (24) and (25), we may obtain an equation of motion for the mixture as a whole. If we use proper definition of the coefficient of viscosity or viscous stress tensor of the mixture including the diffusion terms, we may formally obtain the equation of motion of the mixture in exactly the same form as ordinary Navier-Stokes equation.[15] Because of the terms of diffusion in the definition of viscous stress and pressure, such an equation is not very useful in the analysis of flow problem when the diffusion terms is not negligibly small. Hence, we shall not write this equation of motion of the mixture here. But in some limiting cases, such an equation of motion of the mixture may be useful and we will discuss them in section V. Reference 15 may be referred to for the equation of motion of the mixture in a multi-fluid theory.

(iv) Equations of energy.

For each species, the conservation of energy gives the corresponding equation of energy.

For the pseudo-fluid of solid particles, we have the equation of energy as follows:

$$\frac{\partial[Z\,\rho_{sp}(U_{mp}+\tfrac{1}{2}q_p^2+\phi_p)]}{\partial t}+\frac{\partial}{\partial x^j}[Z\,\rho_{sp}\,u_p^j\,(U_{mp}+\tfrac{1}{2}q_p^2+\phi_p)-u_p^i\tau_p^{ij}+ \\ +\delta^{ij}\,u_p^i\,p_p-Q_{cp}^j]\;=\;\kappa_T(T_g-T_p)+\varepsilon_p \qquad (35)$$

and for the gas, the equation of energy is

$$\frac{\partial[(1-Z)\,\rho_g(U_{mg}+\tfrac{1}{2}q_g^2+\phi_g)]}{\partial t}+\frac{\partial}{\partial x^j}[(1-Z)\rho_g u_g^j(U_{mg}+\tfrac{1}{2}q_g^2+\phi_g)-u_g^i\tau_g^{ij}+ \\ +\delta^{ij}u_g^i p_g-Q_{cg}^j]\;=\;\kappa_T(T_p-T_g)+\varepsilon_g \qquad (36)$$

where $U_{mr}$ is the internal energy per unit mass of the rth species, $\tau_r^{ij}$ is the ijth component of the viscous stress tensor of the rth species; $Q_{cr}^j$ is the heat conducting flux of the rth species and $\phi_r$ is the potential energy of the rth species, $q_r$ is the magnitude of the velocity vector of the rth species; $\varepsilon_r$ is the energy source due to chemical reaction and/or electromagnetic forces and other heat addition terms of the rth species. We shall discuss the significance of some of these terms in the next section from the simple kinetic theory point of view. In general, they are of the same nature as ordinary fluid. $\kappa_T$ is the thermal friction coefficient between the solid particles and the gas. Our knowledge for the expression of $\kappa_T$ in terms of physical properties of the solid particles and various parameters is still very meager. It is more complicated than the expression of interaction forces. In our approximation with Stokes law for the interaction force between solid particles and the gas, we may take

$$\kappa_T = n_p \; 4\pi r_p \; \kappa_p \tag{37}$$

where $\kappa_p$ is the coefficient of thermal conductivity of the solid particles.

Similarly, we may add the two energy equations (35) and (36) to get an energy equation for the mixture of the gas and solid particles. This energy equation may be reduced to the same form as that of ordinary gasdynamics if we use complicated definition of viscous dissipation and heat conduction flux by including those complicated terms due to diffusion[15]. Since such an equation is not very useful in the treatment of problems, we shall not write it here. Reference 15 may be referred to for the energy equation of the mixture as a whole in the multi-fluid theory.

Since the density of the solid particles $\rho_{sp}$ is considered as a constant, we have eleven variables: $\vec{q}_p$ , $\vec{q}_g$ , $p$ , $Z$ , $\rho_g$ , $T_p$ and $T_g$

in our study of the flow problem of a mixture of a gas and a pseudo-fluid of solid particles which are governed by the eleven equations: (18), (21), (22), (26), (27), (35) and (36).

## 4. Fundamental equations of the mixture of a gas and solid particles from the simple kinetic theory point of view.

Similar to the case of ordinary gasdynamics, the kinetic theory of two phase flow of solid particles-gas system would give much more information than the continuum theory could and on the other hand because of many physical and mathematical difficulties, it is not possible at the present time to use the kinetic theory to treat any practical flow problem. However, the Boltzmann equation of the kinetic theory of gases serves two important aspects in the study of gasdynamics. In the first place, the fundamental equations for the macroscopic description may be derived from the Boltzmann equations as a first approximation. Thus we may have some guides about the validity of the fundamental equations of the continuum theory from the analysis of Boltzmann equation. In the second place, the Boltzmann equation may give us valuable information on the transport coefficients, such as the coefficients of viscosity, heat conductivity, etc. In the macroscopic analysis, these transport coefficients are simply introduced as known functions of state variables of the gas. We would expect that the Boltzmann equations of the two phase flow would serve in a similar manner in the dynamics of the particle-gas system.

Since the kinetic theory of particle-gas system is more complicated than that of gas alone, all the restrictions and difficulties in the development of kinetic theory of gases remain in that for the particle-gas system. Furthermore, we have to make additional approximations in the kinetic theory of particle-gas system. The additional complications in the kinetic theory of particle-gas system are (i) size distribution of the particles, (ii) the physical properties of the solid particles, and (iii) the collision process of the solid particles and the collision process of the solid particles with the gas. Hence the kinetic theory of the particle-gas system is still in an undeveloped stage and only a few preliminary attempts have been made. Thus, in this section, we can only give some sketch of these preliminary attempts [4,20] in order to get some insight of certain terms in the fundamental equation of the particle-gas system discussed in last section.

In the same spirit of kinetic theory of gas, the kinetic theory of particle-gas system has been treated on the basis of Boltzmann equation of single particle distribution function. We may define a molecular distribution function of the gas $F_g$ and a distribution function of the particles $F_p$ and each of these distribution functions is governed by a Boltzmann equation.

The distribution function of the gas molecules $F_g$ may be defined in in the same manner as that in ordinary kinetic theory of gases[16] but the Boltzmann equation for $F_g$ should contain the body force on the molecules by the solid particles and the collision terms should contain the collision between the gas molecules and the solid particles. A complete theoretical expression for these two additional terms is still not available at the present time.

The distribution function for the solid particles $F_p$ is different from that of the gas molecules because we have to take the different size, shape and physical properties of the solid particles into account. As suggested by Marble[4], if we consider only the case that all the particles are spheres, we may define the distribution function of the solid particles $F_p$ as follows:

The number of solid particles $dn_p$ with radius between $r_p$ and $r_p+dr_p$ , in a volume $x_i$ and $x_i+dx_i$ , having the particle instantaneous velocity $c_i$ in the range $c_i$ and $c_i+dc_i$ and the instantaneous temperature of the solid particles in the range $\theta$ and $\theta + d\theta$ is

$$F_p(x_i , c_j , \theta , r_p , t)\, d^3x_i\, d^3c_j\, d\theta\, dr_p = dn_p \cdot d^3x_i \tag{38}$$

where $x_i$ is the ith spatial coordinate of the point considered and $d^3x_i$ is the elementary volume at the point considered; $c_j$ is the jth component of the instantaneous velocity of a particle in the direction of $x_i$ and $d^3c_j$ is the volume in the velocity space; $\theta$ is the instantaneous temperature of the particle which may be varied from particle to particle and $r_p$ is the radius of a particle which may be different for different particles in the mixture.

The distribution function of the gas molecules $F_g$ may be defined in a similar manner as $F_p$ but $F_g$ is different from $F_p$ in two respects: (i) the radii of all molecules of a given gas are the same and hence we do not have the variation of the distribution function with the size of the particles $r_p$ and (ii) the temperature $\theta$ of the particle should be replaced by the internal energy of the molecule[16]. In the case of simple kinetic theory of monatomic gas, we may drop the variation of the distribution function with the temperature $\theta$ in the distribution function $F_g$. In general, our discussion on $F_p$ may be applied to $F_g$ with a little modification. Thus we shall consider only $F_p$ from now on.

The variation of the distribution function $F_p$ with the radius $r_p$, i.e., the size distribution of the solid particles, is different for different practical cases and we should assume that this variation is given in the development of the kinetic theory of the particle-gas system. The simplest approximation about the size distribution is that we replace the size distribution by an average size $\overline{r}_p$ and then the variation of $F_p$ with

$r_p$ may be omitted in our analysis. This is the case which we are going to consider. The second approximation for the size distribution is that we divide the particles into two groups and in each group, we use an average radius $\bar{r}_{pi}$ , $i = 1$ or $2$ . In this manner, we may consider the solid particles as a mixture of two kinds of spheres with radii $\bar{r}_{p1}$ and $\bar{r}_{p2}$ respectively for each group. We define a distribution function $F_{pi}$ . The distribution function so defined is independent of $\bar{r}_{pi}$ explicitly. This approach has been used in reference 20 in which some interesting results about the interaction of particles of different sizes were obtained. Of course, for more accurate approximation, we may divide the size of the particles into $N$ groups and for each group we use an average radius $\bar{r}_{pi}$ , $i = 1, 2, --- N$ . In such an analysis we have to use N-distribution functions for the particles, i.e., one each group $\bar{r}_{pi}$ . Since such an analysis may be formally extended from our analysis of one uniform size particle in a straightforward manner[16] just as in the case to extend the kinetic theory of a single gas to a mixture of gases.[16] We shall not consider this case of N-groups any more .

The distribution function $F_p$ is governed by the Boltzmann equation which shows that the total rate of change of $F_p$ with time must be equal to the change of number of the particles per unit volume in the range of variables considered, i.e.,

$$\frac{\partial F_p}{\partial t} + \frac{\partial}{\partial x_j} (c_j F_p) + \frac{\partial}{\partial c_j} \left(\frac{\phi_j F_p}{m}\right) + \frac{\partial}{\partial \theta} \left(\frac{Q_h F_p}{m c_s}\right) = M_p^{(+)} - M_p^{(-)} \qquad (39)$$

where $\phi_j$ is the jth component of the force on a solid particle of radius $r_p$ and a mass $m(r_p)$ which is a function of $r_p$ in general; $Q_h$ is the heat transfer rate from a particle of radius $r_p$ and $c_s$ is the specific heat of a particle, $M_p^{(+)}$ is the increase of the number of the particles per unit volume per unit time in the velocity range $c_j$ and $c_j + dc_j$ and temperature range $\theta$ and $\theta + d\theta$ and radius range $r_p$ and $r_p + dr_p$ due to the collision effects and $M_p^{(-)}$ is the corresponding decrease of the number of the particles in the range of the variables considered due to the collision effects. The exact expressions for $M_p^{(+)}$ and $M_p^{(-)}$ depend on the kinetic pictures of the motion and heat transfer of the solid particles. In general, we have to make various assumptions about the motion and the heat transfer of the particles before we may write down the expressions for $M_p^{(+)}$ and $M_p^{(-)}$.

Let us discuss a few fundamental difficulties in the development of a successful kinetic theory of particle-gas system as follows:

(i) The force $\phi_j$ on the particles are complicated functions of many gasdynamic parameters as we have discussed in sections 2. The force $\phi_j$ should be a function of the instantaneous velocity of the particles and the unsteady effect of the flow field may be important. At the present time, only the Stokes law of force for steady unlimited flow field has been used.[4,20] Hence we are still in the preliminary stage of simple kinetic theory of particle-gas system.

(ii) Similarly, the heat transfer rate $Q_h$ of the particles is very complicated as discussed in section 2 and only the heat transfer rate corresponding to the Stokes flow region has been used.

(iii) The collision terms $M_p^{(+)}$ and $M_p^{(-)}$ consist of the collisions between particles and the collisions between particles and gas molecules. The mechanism of these collisions are very complicated, particularly when the size of the particles varies. A part of the collision between the particles and the gas molecules is taken into account by the body force $\phi_j$. Since the body force $\phi_j$ is expressed in terms of the mean flow velocity of the gas around the particle, there are still remaining collision effects between particles and gas molecules due to peculiar velocity of the gas molecules. In reference 4, collision terms are completely neglected. In reference 20, the collision between particles and gas molecules are neglected except that part due to the mean flow which is expressed in the body force term $\phi_j$ according to Stokes law. In reference 20, only the long range collisions between two groups of particles were considered.

From the Boltzmann equation (39), we may formally derive the various transfer equations for the particles (pseudo-fluid) which are the fundamental equations of particle-gas system for macroscopic treatment. Before we derive these transfer equations, we have to list the formulas which show the macroscopic variables in terms of distribution function $F_p$.

(i) Density and number density of the pseudo-fluid of solid particles: The number density $n_p$ is

$$n_p = \int\int\int F_p(x_i\ , c_j\ , \theta\ , r_p\ , t)\ d^3c_j\ d\theta\ dr_p \tag{40}$$

If the particles are of the same radius, we may consider that the dependence of $F_p$ with $r_p$ is a delta function and the integration with respect to $r_p$ is unity. Hence the integration with respect to $r_p$ may be omitted. We have simply

$$n_p = \int F_p(x_i\ , c_j\ , \theta\ , r_p\ , t)\ d^3c_j\ d\theta \tag{40a}$$

Similarly, if the temperatures of all the particles are the same, we may consider that the dependence of $F_p$ with $\theta$ is a delta function and the integration with respect to $\theta$ is unity, and may be omitted. Hence for the case of constant radius particles with same temperature, Eq. (40) is reduced to the usual definition of ordinary kinetic theory of gas, i.e.,

$$n_p = \int F_p(x_i , c_j , \theta , r_p , t) \, d^3c_j \tag{40b}$$

The same concept of integration is applicable to all the integration in this section.

The mass density or simply density of the pseudo-fluid of solid particles is

$$\bar{\rho}_p = \int \int \int m_p(r_p) \, F(x_i , c_j , \theta , r_p , t) \, d^3c_j \, d\theta \, dr_p = \bar{m}_p(\bar{r}_p) \, n_p \tag{41}$$

where $\bar{\rho}_p$ is the partial density of the pseudo-fluid of solid particles and $\bar{m}_p$ is the mean mass of a particle in the fluid and $\bar{r}_p$ is the mean radius of a particle in the pseudo-fluid. Equation (41) may be used to determine the mean mass $\bar{m}_p$ and mean radius $\bar{r}_p$ of the particles if we replace the actual particles by the same number of particles of same size $\bar{r}_p$ .

(ii) Flow velocity of the pseudo-fluid of solid particles $u_{pi}$ is

$$u_{pi} = \frac{1}{\bar{m}_p n_p} \int \int \int m_p(r_p) \, c_i \, F_p \, d^3c_j \, d\theta \, dr_p \tag{42}$$

(iii) The temperature of the pseudo-fluid of solid particles $T_p$ is

$$T_p = \frac{1}{\bar{m}_p n_p} \int \int \int m_p(r_p) \, \theta \, F_p \, d^3c_j \, d\theta \, dr_p \quad . \tag{43}$$

It should be noticed that the temperature of the pseudo-fluid of particles is not the kinetic temperature of the particles. This is the temperature similar to those temperatures of internal energy other than the kinetic energy of random motion of the particles. For the gas, we should not use the expression of (43) for the temperature but the ordinary kinetic temperature of the gas.[16, 15]

(iv) The interaction force exerted upon the particles by the gas in a unit volume of gas-particle mixture is

$$F_{pi}(x_i \; , \; t) = \int\int\int \phi_i(x_i \; , \; c_j \; , \; \theta \; , \; r_p \; , \; t) \; F_p \; d^3c_j \; d\theta \; dr_p \quad . (44)$$

The difference of notations between the distribution function $F_p$ and the ith component of the total force acting on the particles by the gas $F_{pi}$ should be noticed. For a given expression of $\phi_i$ , we may calculate the interaction force. In the simplest case, we may have the Stokes formula (30).

(v) The work done on the gas by the entire particulate clouds in the elementary volume is

$$\Phi_p = \int\int\int (c_k - u_{pk}) \; \phi_k(x_i \; , \; c_j \; , \; \theta \; , \; r_p \; , \; T) \; F_p \; d^3c_j \; d\theta \; dr_p \quad (45)$$

This is the work done due to random body force in the random motion of the particles.

(vi) The total heat transfer rate to the gas by the particle cloud within the elementary volume is

$$Q_p = \int\int\int Q_h(x_i \; , \; c_j \; , \; \theta \; , \; r_p \; , \; t) \; F_p \; d^3c_j \; d\theta \; dr_p \quad (46)$$

There is no similar term of $Q_p$ in the ordinary kinetic theory of gas.

(vii) The stress tensor of the pseudo-fluid of solid particles is

$$S_{ij} = \int\int\int m_p(r_p)\,(c_i - u_{pi})\,(c_j - u_{pj})\,F_p\,d^3c_j\,d\theta\,dr_p \quad . \tag{47}$$

This stress tensor corresponds to the pressure tensor of ordinary kinetic theory of gases which consists of the partial pressure $p_p$ of the pseudo-fluid of solid particles and the effective viscous stress tensor of the solid particles due to their random motion. In addition to this stress tensor, we have the body force $\phi_i$ which is due to the difference of the velocities $c_j$ from the flow velocity of the gas $u_{gi}$ .

(viii) The energy flux of the pseudo-fluid of solid particles is

$$q_{pi} = \int\int\int m_p(r_p)\,c_s(\theta - T_p)\,(c_i - u_{pi})\,F_p\,d^3c_j\,d\theta\,dr_p \quad . \tag{48}$$

This energy flux is due to the random distribution of temperature and random motion of the solid particles in the mixture of gas and solid particles and it is the additional heat flux to the heat transfer $Q_p$ of Eq. (46) which is the heat transfer over the mean flow velocity and mean temperature of the gas. Furthermore, we may have additional heat conduction flux due to the third moment of the fluctuation velocity $c_i c_j c_k$ as in the case of ordinary kinetic theory of gases.

With the definitions of these macroscopic quantities of (40) to (48), we may obtain the transfer equations by taking moments of both sides of the Boltzmann equation (39) as follows:

(i) Equation of continuity. For the zeroth moment, we multiply Eq. (39) by $m_p(r_p)$ and integrate the resultant equations with respect to $d^3c_j$ over the whole velocity space, with respect to $d\theta$ over the whole temperature space and with respect to $r_p$ over the diameter space and we obtain the equation of the pseudo-fluid of solid particles in the same manner as that for gasdynamical equation of contuity:[15, 16]

$$\frac{\partial}{\partial t}( \bar{m}_p n_p ) + \frac{\partial}{\partial x_i}( \bar{m}_p n_p u_{pi} ) = - \sigma_p \tag{49}$$

where $\bar{m}_p n_p = Z \rho_{sp}$ by Eq. (13) and $- \sigma_p$ is the term due to the integration of the collision terms $M_p^{(+)} - M_p^{(-)}$. Hence Eq. (49) is identical to Eq. (21) except that we may calculate the source function $\sigma_p$ if we have the exact expression for the collision terms.

(ii) Equations of motion.

For the first moment, we multiply Eq. (39) by $m_p c_i$ and integrate the resulting equation with respect to $d^3 c_j \cdot d\theta \, dr_p$ over all these spaces as in the case of zeroth order moment and obtain the equation of motion of the pseudo-fluid of solid particles as follows;

$$\bar{m}_p n_p (\frac{\partial}{\partial t} + u_{pj} \frac{\partial}{\partial x_j}) u_{pi} = + F_{pi} - \frac{\partial S_{ij}}{\partial x_j} - \sigma_p Z_{pi} \tag{50}$$

where the stress tensor of the pseudo-fluid may be written as

$$S_{ij} = p_p \delta_{ij} - \tau_{pij} \quad . \tag{51}$$

Hence Eq. (50) is practically identical to Eq. (26) except that we do not consider the body force such as gravitational force in Eq. (50) nor in the Boltzmann equation (39). If we include the body force such as gravitational force in the expression $\phi_i$ of Eq. (39), we will have the corresponding force $F_{bpi}$ in Eq. (50) so that Eqs. (26) and (50) are identical. The term $\sigma_p Z_{pi}$ is given by the integration of the collision terms.

(iii) Equation of energy.

Because the temperature of the solid particles is not defined in terms of its random translational motion, but the internal degree of freedom (cf. Eq. (43)), the internal energy of the pseudo-fluid of solid particles may be considered of two parts as follows:

$$U_{mp} = U_{mp1} + U_{mp2} \tag{52}$$

where $U_{mp1} = c_s T_p$ = internal energy due to the internal degree of freedom of the particle whose average temperature of particles at $x_i$ and $t$ is $T_p$ , which is defined by Eq. (43).

and $$U_{mp2} = \frac{1}{\bar{m}_p n_p} \int\int\int \tfrac{1}{2} m_p (c_k - u_{pk})(c_k - u_{pk}) F_p \, d^3c_j \, d\theta \, dr_p$$

$= c_{vp} T_p$ where $c_{vp}$ may be considered as an effective specific heat at constant volume of the pseudo-fluid of solid particles due to random translational motion. For first approximation, we may assume that $c_{vp}$ is a constant.

It should be noticed that $U_{mp2}$ is of the same order of magnitude as $p_p/\bar{\rho}_p$ . Now we are going to derive an equation which governs $U_{mp1}$ and another equation which governs $U_{mp2}$ in order to get an equation for the variation of the total internal energy of the pseudo-fluid of solid particles $U_{mp}$ given in Eq. (35).

For $U_{mp1}$, we multiply Eq. (39) by $m_p c_s \theta$ and integrate the resultant equation with respect to $d^3c_j \cdot d\theta \cdot dr_p$ over the velocity, temperature and radius spaces and obtain the following equation for $U_{mp1}$ :

$$\frac{\partial}{\partial t}(\bar{m}_p n_p c_s T_p) + \frac{\partial}{\partial x_j}(\bar{m}_p n_p c_s u_{pj} T_p) = -\frac{\partial q_{pi}}{\partial x_i} + Q_p + \varepsilon_{p1} \tag{53}$$

where

$$\varepsilon_{p1} = \int\int\int m_p c_s \theta \, [M_p^{(+)} - M_p^{(-)}] \, d^3 c_j \, d\theta \, dr_p \tag{54}$$

and $q_{pi}$ and $Q_p$ are given respectively by Eqs. (48) and (46).

For $U_{mp2}$, we multiply Eq. (39) by $\frac{1}{2} m_p c_k^2$ and integrate the resultant equation with respect to $d^3 c_j \, d\theta \, dr_p$ over the whole velocity, temperature and radius spaces and obtain the following equation for $U_{mp2}$ :

$$\frac{\partial}{\partial t} ( \bar{m}_p n_p U_{mp2} + \frac{1}{2} \bar{m}_p n_p u_{pk}^2 ) + \frac{\partial}{\partial x_j} [u_{pj} \bar{m}_p n_p (U_{mp2} + \frac{1}{2} u_{pk}^2 )] = $$

$$= - \frac{\partial}{\partial x_j} (u_{pk} S_{jk}) - \frac{\partial q_{cpj}}{\partial x_j} + \Phi_p + u_i F_{pi} + \varepsilon_{p2} \tag{55}$$

where

$$q_{cpj} = \int\int\int m_p (c_k - u_{pk})^2 (c_j - u_{pj}) F_p \, d^3 cj \, d\theta \, dr_p \tag{56}$$

= heat conduction flux of the pseudo-fluid of solid particles due to random motion

$$\varepsilon_{p2} = \int\int\int \frac{1}{2} m_p c_k^2 \, [M_p^{(+)} - M_p^{(-)}] \, d^3 c_j \, d\theta \, dr_p \tag{57}$$

and $\Phi_p$ and $F_{pi}$ are given respectively by Eqs. (45) and (44).

If we add Eqs. (53) and (55), we obtain the energy equation for the pseudo-fluid of solid particles, which is identical to Eq. (35). If we neglect the potential energy which we do not include in our derivation of Boltzmann equation (39) and use the following relations:

$$Q_{cpj} = - (q_{cpj} + q_{pj}) \tag{58}$$

$$\kappa_T (T_g - T_p) + \varepsilon_p = Q_p + \Phi_p + u_{pi} F_{pi} + \varepsilon_{p1} + \varepsilon_{p2} \; . \tag{59}$$

Hence we have some confidence in Eq. (35) obtained from the continuum theory point of view and further we have more information about the terms such as the heat conduction flux of the pseudo-fluid of solid particles $Q_{cpj}$ which consists of two parts: one is due to the heat conduction from the solid partilces to the gas and the other is due to the random motion of the solid particles.

Formally, we may derive the equations for $S_{ij}$ , $q_{pi}$ etc., in a similar manner as in ordinary kinetic theory of gases as described in references 15 and 16. Since we do not know the exact form for the collision terms $[M_p^{(+)} - M_p^{(-)}]$ and we shall neglect the stress tensor of pseudo-fluid of solid particles $S_{ij}$ except the partial pressure $p_p$ which is proportional to $Z$ , we do not derive these equations here. It should be noticed that if we have additional body forces other than due to the interaction of solid particles and gas molecules, we may have other body forces in the equation of motion such as gravitational force in Eq. (26) and the potential energy terms in the energy equation (35).

## 5. Some simplifications of the fundamental equations of the mixture of a gas with small solid particles.

In this section, we are going to derive some simple relations for the mixture of a gas and pseudo-fluid of solid particles from the fundamental equations which we derive in the last two sections. First we consider a few basic thermodynamic relations of the mixture of pseudo-fluid of solid particles and a gas and then the case where the fraction volume $Z$ is very small.

### (a) Thermodynamics of the mixture of solid particles and gas.

We consider the mixture of small solid particles and a gas as a mixture of a pseudo-fluid of solid particles and a gas. We are going to find some thermodynamic relations of the mixture as a whole[8] in terms of properties of the two species and volume fraction $Z$ of the solid particles or the mass concentration of the solid particles $k_p$. The density of the mixture as a whole is:

$$\rho_M = \bar{\rho}_p + \bar{\rho}_g = m_p n_p + m_g n_g = Z\,\rho_{sp} + \rho_g(1 - Z) \tag{60}$$

where we assume $m_p = \bar{m}_p$.

We define the mass concentration of the pseudo-fluid of solid particles as follows:

$$k_p = \frac{\bar{\rho}_p}{\rho_M} = \frac{Z\rho_{sp}}{\rho_M} \tag{61}$$

The pressure of the mixture as a whole is

$$p_M = p = p_p + p_g \;. \tag{62}$$

From Eqs. (18), (60) and (61), we find a relation between the pressure of the mixture and the density of the mixture as follows:

$$p = \frac{1 - k_p}{1 - Z}\,\rho_M R T_g = \frac{\rho_M R_M T_g}{1 - Z} \tag{63}$$

$$R_M = (1 - k_p) R \tag{64}$$

where $R_M$ may be considered as an effective gas constant of the pseudo-fluid of solid particles. It is interesting to notice that if the volume fraction of the solid particles is negigibly small, the perfect gas law holds for the mixture when an effective gas constant $R_M$ is used. For many engineering problems, we do have very small value for $Z$ but $k_p$ is not negligible in comparison to unity. On the other hand, if $Z$ is not negligibly small in comparison to unity, the volume fraction of the solid particles does affect the equation of state of the mixture as a whole because $Z$ is a function of $\rho_M$ .

Since the temperature of the solid particles does not associate with the random kinetic energy of the particles and the temperature of the gas does relate with its random kinetic energy, it is not profitable to define a temperature of the mixture as a whole as in the usual treatment of a mixture of two gases.[15] Thus we retain the two temperatures: one for the gas $T_g$ and one for the pseudo-fluid of solid particles $T_p$ . In the equilibrium condition, these two temperature are, of course, equal, i.e., $T_g = T_p = T$ . In the general case, we shall write $T_g = T$ for simplicity and $T_p$ may be equal to or different from $T$ .

The internal energy of the mixture per unit mass $U_{mM}$ is related to the internal energies of the two species by the following relation:

$$\rho_M U_{mM} = Z \rho_{sp} c_{sp} T_p + (1 - Z) \rho_g c_V T \tag{65}$$

or

$$U_{mM} = k_p c_{sp} T_p + (1 - k_p) c_V T \tag{65a}$$

where $c_{sp} = c_s + c_{Vp}$ and we assume that $c_{sp}$ and $c_V$ are constants for simplicity. For equilibrium condition, we have the specific heat of the mixture at constnat volume $c_{VM}$ as follows:

$$c_{VM} = k_p c_{sp} + (1 - k_p) c_V \tag{66}$$

where $c_V$ is the specific heat of the gas at constant volume.

The enthalpy of the mixture per unit mass $H_M$ is

$$H_M = U_{mM} + \frac{p_M}{\rho_M} = k_p \left(c_{sp} T_p + \frac{p}{\rho_{sp}}\right) + (1 - k_p) c_p T \tag{67}$$

where $c_p$ is the specific heat of the gas at constant pressure. For equilibrium condition, the specific heat of the mixture at constant pressure is then

$$c_{pM} = k_p c_{sp} + (1 - k_p) c_p \quad . \tag{68}$$

The specific heats of the mixture are independent of the volume fraction $Z$ but depend on the mass concentration $k_p$ of the solid particles. The ratio of the specific heats of the mixture is

$$\Gamma = \frac{c_{pM}}{c_{VM}} = \frac{(1 - k_p) c_p + k_p c_{sp}}{(1 - k_p) c_V + k_p c_{sp}} = \gamma \frac{(1 + \frac{\eta\delta}{\gamma})}{1 + \eta\delta} \tag{69}$$

where $\gamma = c_p/c_V$ , $\eta = k_p(1 - k_p)$ and $\delta = c_{sp}/c_V$ . The ratio $\Gamma$ is always smaller than $\gamma$ if $k_p$ is different from zero and as $k_p = 1$ , $\Gamma = 1$ .

If we consider the mixture as a homogeneous medium, the first law of thermodynamics for the mixture may be written as follows:

$$dQ = dU_{mM} - \frac{1}{\rho_M^2} \, p \, d\rho_M \tag{70}$$

where $dQ$ is the heat addition to the mixture. Eq. (70) is the energy equation of the mixture as a whole.

For isentropic change of state of the gas-particle mixture, we have $dQ = 0$ and Eq. (70) gives

$$\frac{1}{\Gamma-1} \frac{dT}{T} = \frac{1}{1-Z} \frac{d\rho_M}{\rho_M} \tag{71}$$

Since $Z = k_p \rho_M / \rho_{sp}$ , for constnat $k_p$ , and $T_p = T$ , Eq. (71) may be integrated and gives

$$T\left(\frac{\rho_M}{1-Z}\right)^{-(\Gamma-1)} = \text{constant} \quad . \tag{72}$$

If $Z << 1$, the isentropic change of state of the mixture has a similar relation as that for a pure gas with the effective ratio of specific heats $\Gamma$. In general, the volume fraction $Z$ has some influence on the isentropic change of the mixture.

Similarly, from Eq. (63), for a given $k_p$ and $T_p = T$, we have

$$\frac{dp}{p} = \frac{dT}{T} + \frac{1}{1-Z}\,\frac{d\rho_M}{\rho_M} \quad . \tag{73}$$

From Eqs. (71) and (73), we have

$$\frac{dp}{p} = \frac{\Gamma}{1-Z}\,\frac{d\rho_M}{\rho_M} \tag{74}$$

or

$$p\left(\frac{\rho_M}{1-Z}\right)^{-\Gamma} = \text{constant} \quad . \tag{75}$$

Again, if $Z << 1$, Eq. (75) is identical in form for the corresponding relation of a pure gas but with the effective ratio of specific heats.

We may calculate the so-called equilibrium speed of sound of the mixture $a_M$ from Eq. (75) which is defined by the relation:

$$a_M^2 = \left(\frac{dp}{d\rho_M}\right) = \frac{\Gamma(1 - k_p)RT}{(1 - Z)^2} \tag{76}$$

The ratio of the equilibrium sound speed of the mixture $a_M$ to that of the gas alone $a = \gamma RT$ is

$$\frac{a_M}{a} = \frac{1}{1-Z}\left[\frac{\Gamma}{\gamma}(1-k_p)\right]^{\frac{1}{2}} = \frac{1}{1-Z}\left[\frac{(1+\frac{\eta\delta}{\gamma})(1-k_p)}{1+\eta\delta}\right]^{\frac{1}{2}} \quad . \tag{77}$$

(b) Fundamental equations for the mixture of gas and small solid particles when $Z << 1$ .

When the volume fraction $Z$ of the solid particles is negligibly small in the mixture, from Eq. (19), we may neglect the partial pressure of the solid particles $p_p$ . By the same order of magnitude estimate, we may also neglect the viscous stress of the pseudo-fluid of solid particles $\tau_{pij}$ and the internal energy $U_{mp2}$ . Furthermore, from Eq. (16), we have $\bar{\rho}_g = \rho_g$ . Even though $Z << 1$ , in order that the solid particles have some influence on the flow field of the mixture, the mass concentration $k_p$ of the pseudo-fluid of solid particles should not be negligibly small. As a result, from Eq. (13), we do not use $Z$ as a variable in the mixture but use the number density $n_p$ or the partial density $\bar{\rho}_p$ as a variable in the mixture, i.e.,

$$\bar{\rho}_p = \bar{m}_p n_p \quad . \tag{78}$$

Hence, we still have eleven variables: $p = p_g$ , $T_g = T$ , $T_p$ , $\rho_g$ , $\bar{\rho}_p$ (or $n_p$) , $u_{pi}$ and $u_{gi}$ where for simplicity we write $T_g = T$. The fundamental equations of the mixture of a gas and small solid particles may be simplified a little as follows:

(i) Equation of state of the gas: (see Eq. (18))

$$p = R\rho_g T \tag{79a}$$

(ii) Equation of continuity for the pseudo-fluid of solid particles. From Eq. (49), we have

$$\frac{\partial}{\partial t}(\bar{m}_p n_p) + \frac{\partial}{\partial x_i}(\bar{m}_p n_p u_{pi}) = -\sigma_p \quad . \tag{79b}$$

(iii) Equation of continuity for the gas. Eq. (22) with $Z << 1$ gives

$$\frac{\partial}{\partial t}(\rho_g) + \frac{\partial}{\partial x_i}(\rho_g u_{gi}) = \sigma_p \quad . \tag{79c}$$

(iv) Equations of motion for the pseudo-fluid of solid particles. By neglecting the partial pressure $p_p$ and viscous stress $p_{ij}$ of the pseudo-fluid of solid particles, Eq. (26)becomes:

$$\bar{m}_p n_p \left( \frac{\partial}{\partial t} + u_{pi} \frac{\partial}{\partial x_i} \right) \vec{q}_p = \vec{F}_p + \vec{F}_{bp} - \sigma_p \vec{Z}_p \tag{79d}$$

(v) Equations of motion for the gas. For $Z << 1$, Eq. (27) becomes

$$\rho_g \left( \frac{\partial}{\partial t} + u_{gi} \frac{\partial}{\partial x_i} \right) \vec{q}_g = -\nabla p + \nabla \cdot \tau_g + \vec{F}_g + \vec{F}_{bg} + \sigma_p \vec{Z}_p \tag{79e}$$

(vi) Equation of energy of the pseudo-fluid of solid particles. Since $U_{mp2}$ is negligible, we may use Eq. (53) for the internal energy of the pseudo-fluid of the solid particles. With the help of Eq. (49), Eq. (53) gives the equation for the internal energy of the pseudo-fluid of the solid particles as follows:

$$\bar{m}_p n_p \left(\frac{\partial}{\partial t} + u_{pi} \frac{\partial}{\partial x_i}\right) (c_s T_p) = - \frac{\partial q_{pi}}{\partial x_i} + Q_p + \varepsilon_{pl} + \sigma_p c_s T_p \qquad (79f)$$

(vii) Equation of energy of the gas. Eq. (36) with $Z << 1$ becomes

$$\begin{aligned} &\frac{\partial \rho_g (U_{mg} + \tfrac{1}{2} q_g^2 + \phi_g)}{\partial t} + \frac{\partial}{\partial x_j} \Big[ \rho_g u_{gi} (U_{mg} + \tfrac{1}{2} q_g^2 + \phi_g) - u_{gi} \tau_{gij} + \\ &\qquad + \delta^{ij} u_{gi} p - Q_{cgj} \Big] = k_T (T_p - T_g) + \varepsilon_g \quad . \qquad (79g) \end{aligned}$$

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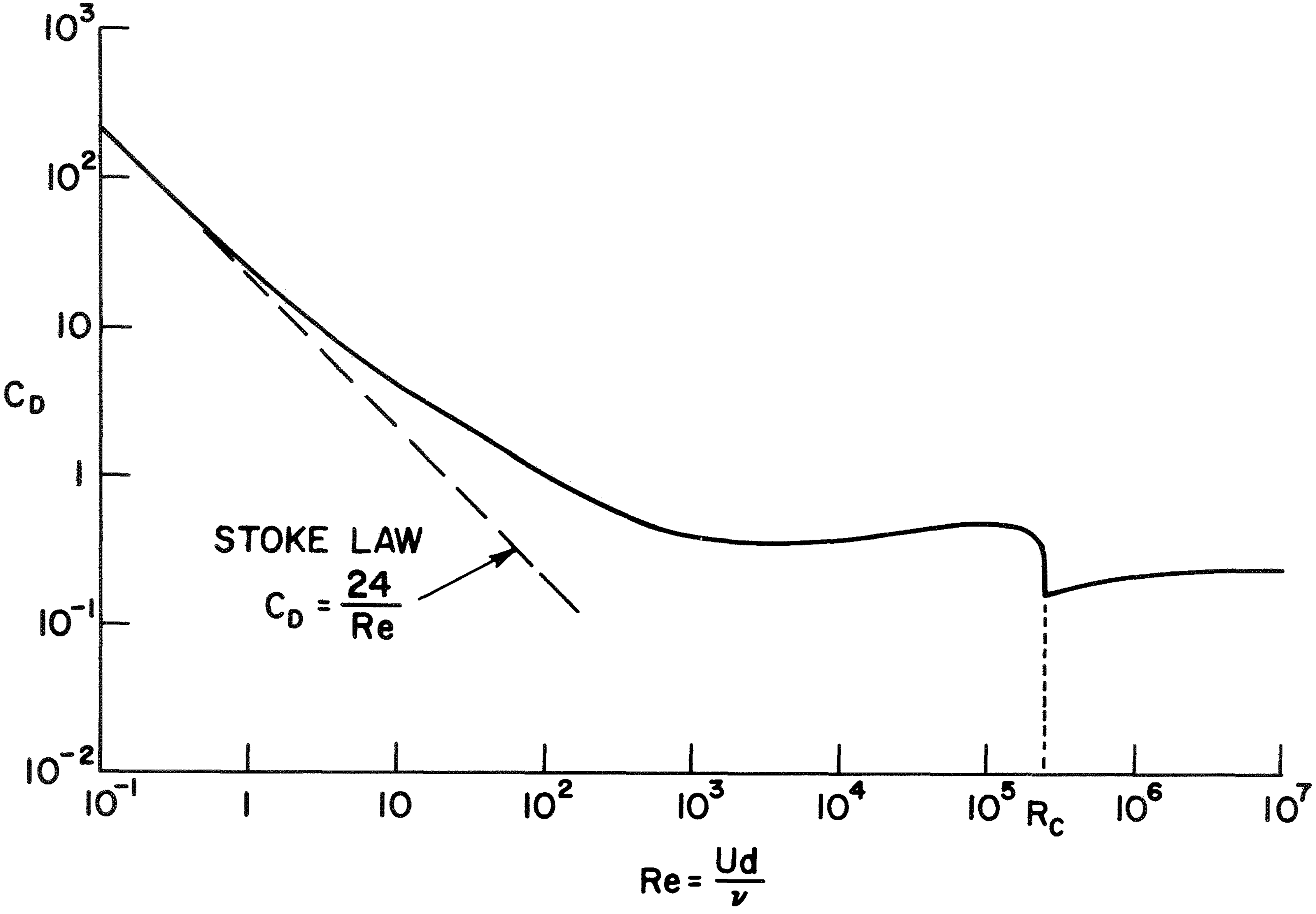

Fig. 1. STANDARD DRAG CURVE OF A SPHERE.