NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1516

Multi-Rigid-Body Attitude Dynamics Simulation

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Preface

The work encompassed by this report was performed under the cognizance of the Guidance and Control Division of the Jet Propulsion Laboratory and was accomplished through the support of the Guidance and Controls Branch of the Office of Advanced Research and Technology, NASA. Computer programs developed as a result of this support have already been successfully applied to attitude dynamics studies of the *Mariner* VI and VII spacecraft to Mars and to design analysis for a *Mariner* Mars 1971 orbiter and a *Mariner* Venus–Mercury 1973 spacecraft.

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Abstract

The results of attempts to put into practice the apparent advantages of the "barycenter formulation" of rigid-body rotational dynamics is described. The end product is a FORTRAN subroutine capable of computing the angular accelerations of each body in a system composed of several point-connected rigid bodies.

A 3-body system is used to illustrate the concept of the connection barycenter. Extension of the barycenter formulation of the dynamical equations to the general case of n bodies is then derived. Some discussion is devoted to the computational problem of handling interbody torques of constraint. An efficient procedure for accommodating the presence of symmetric rotors in the system is also developed.

Two space vehicle attitude dynamics and control simulations of some interest are used to illustrate the application of the computer subroutine MLTBDY: one example is a spacecraft, under three-axis control, subject to the perturbations of a mechanically scanning platform, while the other is a rigid space vehicle hinged to four large solar-cell panels and under the influence of a trajectory-correcting rocket engine.

Multi-Rigid-Body Attitude Dynamics Simulation

I. Introduction

Continuing developments in the field of spacecraft attitude control have led, in many cases, to the desire on the part of the designer and the analyst for more detailed and comprehensive mathematical models of dynamic performance. For example, the recent emergence of dual- and multi-spin spacecraft configurations seems to have been the catalyst for many efforts to devise dynamic models and computer simulations capable of describing complex vehicle attitude motion. Although system stability and performance can be estimated under certain idealizations of energy-loss mechanisms, simplified geometry, and linearization of vehicle dynamics, ultimately it becomes necessary to deal with a reasonably detailed model, including the complete nonlinear vehicle dynamic model and closed-loop control system. Of course, the digital (or analog-hybrid) computer then becomes the indispensable analytical tool for evaluating the complicated interactions of the vehicle and the control system, studying the effects of various parameters, and generally verifying system performance as predicted by the more simplified analyses.

The mathematical complexity resulting from the modeling process is simply a consequence of the fact that the system consists of several interconnected rigid, semirigid, and/or quite flexible bodies, and further that the motions cannot necessarily be assumed to be arbitrarily small. Thus, the detailed dynamical model, in general, becomes a large system of nonlinear ordinary differential equations with time-varying coefficients. In this report, a particular method (Ref. 1) of formulating the dynamical equations of motion of a system of interconnected rigid bodies is adopted and used for the computer simulation of space-vehicle attitude and pointing control systems. This method, which might be referred to as the "barycenter formulation," is illustrated in Section II with respect to a system of three rigid bodies. Subsequently, the generalization to a system of n rigid bodies is developed, the simplicity of which is due to the recognition of a characteristic point in each body known as the "connection barycenter."

Sections follow that deal with certain manipulations of the vehicle equations of motion in an effort to improve computational efficiency. It is desirable, for example, to eliminate from the equations certain unknown torques resulting from rigid constraints imposed at the connecting joints. Further, since such constraints decrease the number of system degrees of freedom, it is more efficient computationally to deal directly with the "free" variables and to obtain constrained variable values from algebraic relations among the free variables.

In addition, it is demonstrated that the presence of rigid symmetric rotors (e.g., momentum wheels) can be easily accommodated in the barycenter formulation.

The report concludes with a description of digital computer subroutines (written in FORTRAN IV) developed for the purpose of evaluating the vehicle's angular accelerations based on the barycenter formulation. Through the use of the subroutines as part of a larger digital simulation involving numerical integration of the kinematical equations as well as the control equations, two representative problems in spacecraft attitude control are presented. One of these is a 3-axis-stabilized craft (using celestial sensors and jets) that carries an actively driven platform and is required to maintain certain inertial attitude accuracies and/or instrument-pointing accuracies. The other configuration simulated is that of a spacecraft and multi-panel solar array as it responds to the thrust of a trajectory-correction engine.

II. A 3-Body System

The barycenter formulation as derived by Hooker and Margulies (see Ref. 1) and also described by Roberson and Wittenberg (Ref. 2) offers a highly systematic approach for mathematically describing the rotations of a vehicle that may be represented as a collection of interconnected rigid bodies. Restrictions placed on the rigid-body system are the following: (1) closed connection loops are prohibited and (2) the connecting joints allow only relative rotation between pairs of joined bodies.

To clarify the derivation of the desired equations (i.e., to avoid the necessity of the rather cumbersome notation used in the general case of n bodies), the development that follows is presented for the case of three bodies. A derivation for a specific number of bodies will not only serve to illuminate the concept of a barycenter but will retain sufficient complexity to enable a straightforward generalization to the n-body case.

A system of three interconnected rigid bodies is shown in Fig. 1, where point 0 represents the system center of mass and the vectors $\boldsymbol{\rho}_{1,2,3}$ join point 0 to the mass center of each of the bodies. Vectors $\mathbf{c}_{12}$, $\mathbf{c}_{13}$, $\mathbf{c}_{21}$, $\mathbf{c}_{23}$, $\mathbf{c}_{31}$, $\mathbf{c}_{32}$, or more generally $\mathbf{c}_{ij}$ $(i \neq j)$, connect the mass center of body i to that joint on body i that leads to body j. Finally, the vector $\mathbf{R}$ locates the system center of mass with respect

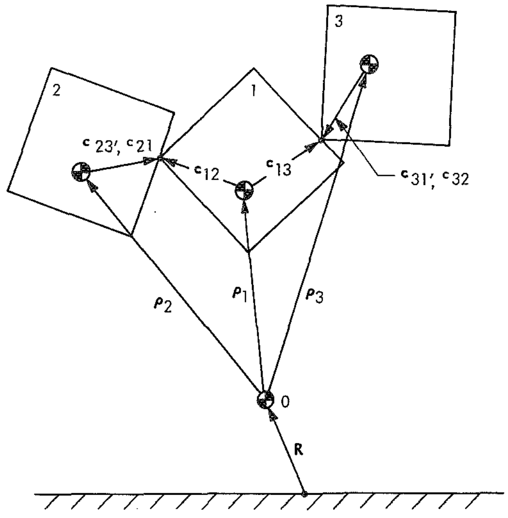

Fig. 1. A 3-body system

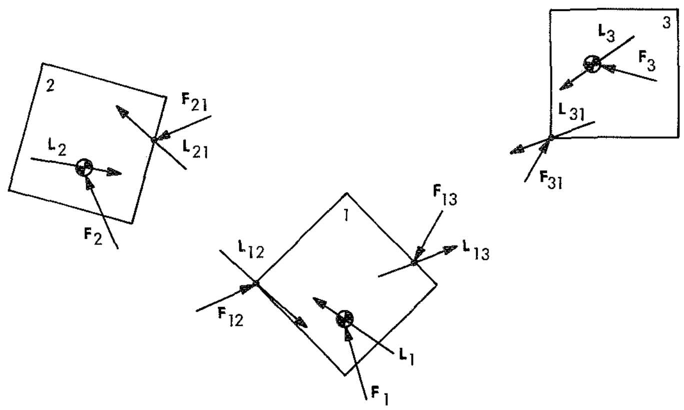

Fig. 2. Free-body diagram of 3-body system

to a fixed (inertial) reference. A free-body diagram of the system, given in Fig. 2, describes the force and torque relationship, with $\mathbf{F}_i$ and $\mathbf{L}_i$ representing vector sums of forces and torques applied directly to body i. The vectors $\mathbf{F}_{ij}$ and $\mathbf{L}_{ij}$ represent reaction forces and torques appearing at joints shared by bodies i and j.

Newton's and Euler's vector equations of motion for each of the bodies can therefore be written as

$$\mathbf{F}_1 + \mathbf{F}_{12} + \mathbf{F}_{13} = m_1 (\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_1) \tag{1}$$

$$\mathbf{F}_2 + \mathbf{F}_{21} = m_2 (\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_2) \tag{2}$$

$$\mathbf{F}_3 + \mathbf{F}_{31} = m_3 (\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_3) \tag{3}$$

$$\mathbf{L}_1 + \mathbf{L}_{12} + \mathbf{L}_{13} + \mathbf{c}_{12} \times \mathbf{F}_{12} + \mathbf{c}_{13} \times \mathbf{F}_{13} = \frac{d}{dt} (\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) \tag{4}$$

$$\mathbf{L}_2 + \mathbf{L}_{21} + \mathbf{c}_{21} \times \mathbf{F}_{21} = \frac{d}{dt}(\mathbf{I}_2 \cdot \boldsymbol{\omega}_2) \tag{5}$$

$$\mathbf{L}_3 + \mathbf{L}_{31} + \mathbf{c}_{31} \times \mathbf{F}_{31} = \frac{d}{dt}(\mathbf{I}_3 \cdot \boldsymbol{\omega}_3) \tag{6}$$

where overdots denote time derivatives in the inertial frame. The terms $\mathbf{I}_{1,2,3}$ are the inertia dyadics of each body about its center of mass, and $\boldsymbol{\omega}_{1,2,3}$ are the angular-velocity vectors of each body.

Equations (1–3) may now be used to eliminate the unknown reaction forces $\mathbf{F}_{12}$, $\mathbf{F}_{13}$, $\mathbf{F}_{21}$, and $\mathbf{F}_{31}$ from the rotational equations, Eqs. (4–6). Required, however, are the additional relations

$$\mathbf{F}_{12} = -\mathbf{F}_{21}, \qquad \mathbf{F}_{13} = -\mathbf{F}_{31}$$

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = m\ddot{\mathbf{R}}, \qquad \text{where } m = m_1 + m_2 + m_3$$

$$\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 = -\mathbf{c}_{12} + \mathbf{c}_{21}$$

$$\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3 = -\mathbf{c}_{13} + \mathbf{c}_{31}$$

$$m_1\boldsymbol{\rho}_1 + m_2\boldsymbol{\rho}_2 + m_3\boldsymbol{\rho}_3 = 0$$

As a result, $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, and $\boldsymbol{\rho}_3$ can be given as

$$\boldsymbol{\rho}_1 = \frac{\begin{vmatrix} (\mathbf{c}_{21} - \mathbf{c}_{12}) & -1 & 0 \\ (\mathbf{c}_{31} - \mathbf{c}_{13}) & 0 & -1 \\ 0 & m_2 & m_3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ m_1 & m_2 & m_3 \end{vmatrix}} = \frac{m_2}{m}(\mathbf{c}_{21} - \mathbf{c}_{12}) + \frac{m_3}{m}(\mathbf{c}_{31} - \mathbf{c}_{13})$$

$$\boldsymbol{\rho}_2 = \frac{m_1}{m}(\mathbf{c}_{12} - \mathbf{c}_{21}) + \frac{m_3}{m}(\mathbf{c}_{31} + \mathbf{c}_{12} - \mathbf{c}_{13} - \mathbf{c}_{21})$$

$$\boldsymbol{\rho}_3 = \frac{m_1}{m}(\mathbf{c}_{13} - \mathbf{c}_{31}) + \frac{m_2}{m}(\mathbf{c}_{21} + \mathbf{c}_{13} - \mathbf{c}_{12} - \mathbf{c}_{31})$$

By substitution into Eqs. (2) and (3),

$$\mathbf{F}_{21} = -\mathbf{F}_2 + \frac{m_2}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) + \frac{m_1 m_2}{m}(\ddot{\mathbf{c}}_{12} - \ddot{\mathbf{c}}_{21}) + \frac{m_2 m_3}{m}(\ddot{\mathbf{c}}_{31} + \ddot{\mathbf{c}}_{12} - \ddot{\mathbf{c}}_{13} - \ddot{\mathbf{c}}_{21}) \tag{7}$$

$$\mathbf{F}_{31} = -\mathbf{F}_3 + \frac{m_3}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) + \frac{m_1 m_3}{m}(\ddot{\mathbf{c}}_{13} - \ddot{\mathbf{c}}_{31}) + \frac{m_2 m_3}{m}(\ddot{\mathbf{c}}_{21} + \ddot{\mathbf{c}}_{13} - \ddot{\mathbf{c}}_{12} - \ddot{\mathbf{c}}_{31}) \tag{8}$$

The time derivatives of the $\mathbf{c}_{ij}$ vectors, fixed in body i, can be expressed in terms of the angular velocity of body i as follows:

$$\ddot{\mathbf{c}}_{12} = \dot{\boldsymbol{\omega}}_1 \times \mathbf{c}_{12} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{12}) \tag{9a}$$

$$\ddot{\mathbf{c}}_{13} = \dot{\boldsymbol{\omega}}_1 \times \mathbf{c}_{13} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{13}) \tag{9b}$$

$$\ddot{\mathbf{c}}_{21} = \dot{\boldsymbol{\omega}}_2 \times \mathbf{c}_{21} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21}) \tag{9c}$$

$$\ddot{\mathbf{c}}_{31} = \dot{\boldsymbol{\omega}}_3 \times \mathbf{c}_{31} + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31}) \tag{9d}$$

After the differentiations indicated in Eqs. (4–6) are performed as follows,

$$\frac{d}{dt}(\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) = \frac{{}^B d}{dt}(\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) + \boldsymbol{\omega}_1 \times (\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) = \mathbf{I}_1 \cdot \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times (\mathbf{I}_1 \cdot \boldsymbol{\omega}_1)$$

(where ${}^B d/dt$ denotes differentiation in body fixed frame), Eqs. (7–9) may be substituted, with the result:

$$\begin{aligned}
\mathbf{I}_1 \cdot \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times (\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) = \mathbf{L}_1 + \mathbf{L}_{12} + \mathbf{L}_{13} + \mathbf{c}_{12} \times \Big(& \mathbf{F}_2 - \frac{m_2}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) + \frac{m_1 m_2}{m}[\mathbf{c}_{12} \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{12}) \\
& - \mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21})] + \frac{m_2 m_3}{m}\{\mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31}) \\
& + (\mathbf{c}_{12} - \mathbf{c}_{13}) \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \times [\boldsymbol{\omega}_1 \times (\mathbf{c}_{12} - \mathbf{c}_{13})] - \mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21})\}\Big) \\
& + \mathbf{c}_{13} \times \Big(\mathbf{F}_3 - \frac{m_3}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) + \frac{m_1 m_3}{m}[\mathbf{c}_{13} \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{13}) \\
& - \mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31})] + \frac{m_2 m_3}{m}\{\mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 - \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21}) \\
& + (\mathbf{c}_{13} - \mathbf{c}_{12}) \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \times [\boldsymbol{\omega}_1 \times (\mathbf{c}_{13} - \mathbf{c}_{12})] - \mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31})\}\Big)
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathbf{I}_2 \cdot \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times (\mathbf{I}_2 \cdot \boldsymbol{\omega}_2) = \mathbf{L}_2 + \mathbf{L}_{21} & \\
+ \mathbf{c}_{21} \times \Big(& - \mathbf{F}_2 + \frac{m_2}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) + \frac{m_1 m_2}{m}[\mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 - \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21}) \\
& - \mathbf{c}_{12} \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{12})] + \frac{m_2 m_3}{m}\{\mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 - \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21}) \\
& + (\mathbf{c}_{13} - \mathbf{c}_{12}) \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \times [\boldsymbol{\omega}_1 \times (\mathbf{c}_{13} - \mathbf{c}_{12})] - \mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31})\}\Big)
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathbf{I}_3 \cdot \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times (\mathbf{I}_3 \cdot \boldsymbol{\omega}_3) = \mathbf{L}_3 + \mathbf{L}_{31} & \\
+ \mathbf{c}_{31} \times \Big(& - \mathbf{F}_3 + \frac{m_3}{m}(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) \\
& + \frac{m_1 m_3}{m}[\mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31}) - \mathbf{c}_{13} \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{c}_{13})] \\
& + \frac{m_2 m_3}{m}\{\mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{c}_{31}) + (\mathbf{c}_{12} - \mathbf{c}_{13}) \times \dot{\boldsymbol{\omega}}_1 - \boldsymbol{\omega}_1 \\
& \times [\boldsymbol{\omega}_1 \times (\mathbf{c}_{12} - \mathbf{c}_{13})] - \mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{c}_{21})\}\Big)
\end{aligned} \tag{12}$$

Since*

$$\mathbf{c} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{c})] = - \boldsymbol{\omega} \times [\mathbf{c} \times (\mathbf{c} \times \boldsymbol{\omega})]$$

---

*See Appendix A.

similar terms may be collected in Eq. (10) as follows:

$$
\mathbf{I}_1 \cdot \dot{\boldsymbol{\omega}}_1 - \mathbf{c}_{12} \times (\mathbf{d}_{12} \times \dot{\boldsymbol{\omega}}_1) - \mathbf{c}_{13} \times (\mathbf{d}_{13} \times \dot{\boldsymbol{\omega}}_1) + \boldsymbol{\omega}_1 \times (\mathbf{I}_1 \cdot \boldsymbol{\omega}_1) - \boldsymbol{\omega}_1 \times [\mathbf{c}_{12} \times (\mathbf{d}_{12} \times \boldsymbol{\omega}_1) + \mathbf{c}_{13} \times (\mathbf{d}_{13} \times \boldsymbol{\omega}_1)] =
$$

$$
\mathbf{L}_1 + \mathbf{L}_{12} + \mathbf{L}_{13} + \mathbf{c}_{12} \times \left[\mathbf{F}_2 - \frac{m_2}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) \right] + \mathbf{c}_{13} \times \left[\mathbf{F}_3 - \frac{m_3}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) \right] - \mathbf{d}_{12} \times (\mathbf{c}_{21} \times \dot{\boldsymbol{\omega}}_2)
$$

$$
- \mathbf{d}_{13} \times (\mathbf{c}_{31} \times \dot{\boldsymbol{\omega}}_3) - \boldsymbol{\omega}_2 \times [\mathbf{d}_{12} \times (\mathbf{c}_{21} \times \boldsymbol{\omega}_2)] - \boldsymbol{\omega}_3 \times [\mathbf{d}_{13} \times (\mathbf{c}_{31} \times \boldsymbol{\omega}_3)] \qquad (13)
$$

where

$$
\mathbf{d}_{12} = \frac{m_1 m_2}{m} \mathbf{c}_{12} + \frac{m_2 m_3}{m} \mathbf{c}_{12} - \frac{m_2 m_3}{m} \mathbf{c}_{13} \qquad \mathbf{d}_{21} = \frac{m_1 m_2}{m} \mathbf{c}_{21} + \frac{m_2 m_3}{m} \mathbf{c}_{21}
$$

$$
\mathbf{d}_{13} = \frac{m_1 m_3}{m} \mathbf{c}_{13} + \frac{m_2 m_3}{m} \mathbf{c}_{13} - \frac{m_2 m_3}{m} \mathbf{c}_{12} \qquad \mathbf{d}_{31} = \frac{m_1 m_3}{m} \mathbf{c}_{31} + \frac{m_2 m_3}{m} \mathbf{c}_{31}
$$

Similarly, in Eq. (11),

$$
\mathbf{I}_2 \cdot \dot{\boldsymbol{\omega}}_2 - \mathbf{c}_{21} \times (\mathbf{d}_{21} \times \dot{\boldsymbol{\omega}}_2) + \boldsymbol{\omega}_2 \times (\mathbf{I}_2 \cdot \boldsymbol{\omega}_2) - \boldsymbol{\omega}_2 \times [\mathbf{c}_{21} \times (\mathbf{d}_{21} \times \boldsymbol{\omega}_2)] =
$$

$$
\mathbf{L}_2 + \mathbf{L}_{21} + \mathbf{c}_{21} \times \left[-\mathbf{F}_2 + \frac{m_2}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) - \mathbf{d}_{12} \times \dot{\boldsymbol{\omega}}_1 - \mathbf{d}_{32} \times \dot{\boldsymbol{\omega}}_3 \right]
$$

$$
- \boldsymbol{\omega}_1 \times [\mathbf{c}_{21} \times (\mathbf{d}_{12} \times \boldsymbol{\omega}_1)] - \boldsymbol{\omega}_3 \times [\mathbf{c}_{21} \times (\mathbf{d}_{32} \times \boldsymbol{\omega}_3)] \qquad (14)
$$

where

$$
\mathbf{d}_{32} = \frac{m_2 m_3}{m} \mathbf{c}_{31} = \frac{m_2 m_3}{m} \mathbf{c}_{32}
$$

Finally, Eq. (12) becomes

$$
\mathbf{I}_3 \cdot \dot{\boldsymbol{\omega}}_3 - \mathbf{c}_{31} \times (\mathbf{d}_{31} \times \dot{\boldsymbol{\omega}}_3) + \boldsymbol{\omega}_3 \times (\mathbf{I}_3 \cdot \boldsymbol{\omega}_3) - \boldsymbol{\omega}_3 \times [\mathbf{c}_{31} \times (\mathbf{d}_{31} \times \boldsymbol{\omega}_3)] =
$$

$$
\mathbf{L}_3 + \mathbf{L}_{31} + \mathbf{c}_{31} \times \left[-\mathbf{F}_3 + \frac{m_3}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) - \mathbf{d}_{13} \times \dot{\boldsymbol{\omega}}_1 - \mathbf{d}_{23} \times \dot{\boldsymbol{\omega}}_2 \right]
$$

$$
- \boldsymbol{\omega}_1 \times [\mathbf{c}_{31} \times (\mathbf{d}_{13} \times \boldsymbol{\omega}_1)] - \boldsymbol{\omega}_2 \times [\mathbf{c}_{31} \times (\mathbf{d}_{23} \times \boldsymbol{\omega}_2)] \qquad (15)
$$

where

$$
\mathbf{d}_{23} = \frac{m_2 m_3}{m} \mathbf{c}_{21} = \frac{m_2 m_3}{m} \mathbf{c}_{23}
$$

Referring now to Fig. 3, it is possible to view each of the system's rigid bodies as having a mass concentrated at its joint(s) equal to the sum of the system mass connected to that joint. One can then define a new "center of mass" for each body in terms of these "fictitious" masses and its own mass. Thus,

For body 1:

$$
m_1 \mathbf{b}_{11} + m_2 \mathbf{b}_{12} + m_3 \mathbf{b}_{13} = 0, \qquad \mathbf{c}_{12} = \mathbf{b}_{12} - \mathbf{b}_{11}, \qquad \mathbf{c}_{13} = \mathbf{b}_{13} - \mathbf{b}_{11},
$$

For body 2:

$$
m_2 \mathbf{b}_{22} + m_1 \mathbf{b}_{21} + m_3 \mathbf{b}_{23} = 0, \qquad \mathbf{c}_{21} = \mathbf{c}_{23} = \mathbf{b}_{21} - \mathbf{b}_{22} = \mathbf{b}_{23} - \mathbf{b}_{22}
$$

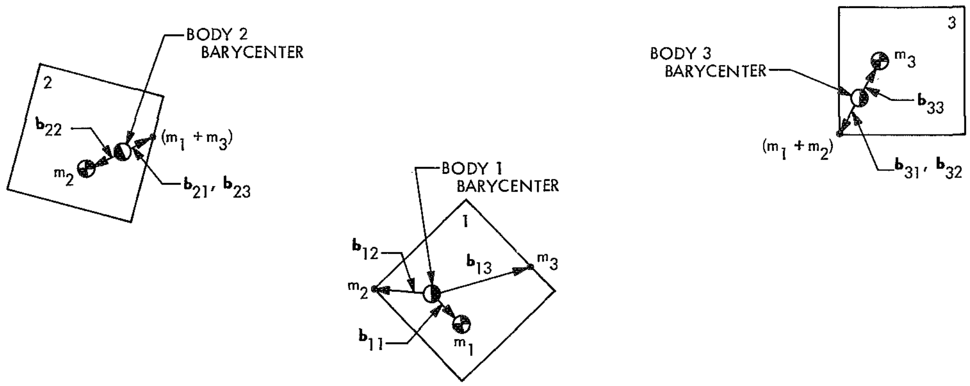

Fig. 3. Body barycenters for 3-body system

For body 3:

$$m_3\mathbf{b}_{33} + m_1\mathbf{b}_{31} + m_2\mathbf{b}_{32} = 0, \qquad \mathbf{c}_{31} = \mathbf{c}_{32} = \mathbf{b}_{31} - \mathbf{b}_{33} = \mathbf{b}_{32} - \mathbf{b}_{33}$$

The vectors $\mathbf{b}_{ij}$ then locate those joints on body i which lead to bodies j (when $i \neq j$) as well as the center of mass of body i (when $i = j$) with respect to the new center of mass, called the *barycenter* (or sometimes the *connection barycenter*) of body i. In general, one can verify from the relations above that

$$\mathbf{b}_{ii} = -\frac{1}{m}\sum_{j\neq i} m_j\mathbf{c}_{ij} \qquad \text{and} \qquad \mathbf{b}_{ij} = \mathbf{b}_{ii} + \mathbf{c}_{ij}$$

Furthermore, it is possible to define a new set of moments of inertia for the rigid body, now augmented by mass-loaded joints, about axes parallel to those originating at the normal center of mass but with origin now at the connection barycenter. The moments of inertia of "augmented body" 1 would then be given by the parallel axis theorem as

$$\phi_{1xx} = I_{1xx} + m_1(b_{11z}^2 + b_{11y}^2) + m_2(b_{12z}^2 + b_{12y}^2) + m_3(b_{13z}^2 + b_{13y}^2)$$

$$\phi_{1yy} = I_{1yy} + m_1(b_{11x}^2 + b_{11z}^2) + m_2(b_{12x}^2 + b_{12z}^2) + m_3(b_{13x}^2 + b_{13z}^2)$$

$$\phi_{1zz} = I_{1zz} + m_1(b_{11x}^2 + b_{11y}^2) + m_2(b_{12x}^2 + b_{12y}^2) + m_3(b_{13x}^2 + b_{13y}^2)$$

$$\phi_{1xy} = I_{1xy} + m_1b_{11x}b_{11y} + m_2b_{12x}b_{12y} + m_3b_{13x}b_{13y}$$

$$\phi_{1xz} = I_{1xz} + m_1b_{11x}b_{11z} + m_2b_{12x}b_{12z} + m_3b_{13x}b_{13z}$$

$$\phi_{1yz} = I_{1yz} + m_1b_{11y}b_{11z} + m_2b_{12y}b_{12z} + m_3b_{13y}b_{13z}$$

where the momental dyadics

$$\mathbb{I}_1 = \begin{bmatrix} \mathbf{i}_1\mathbf{i}_1 I_{1xx} & -\mathbf{i}_1\mathbf{j}_1 I_{1xy} & -\mathbf{i}_1\mathbf{k}_1 I_{1xz} \\ -\mathbf{j}_1\mathbf{i}_1 I_{1xy} & \mathbf{j}_1\mathbf{j}_1 I_{1yy} & -\mathbf{j}_1\mathbf{k}_1 I_{1yz} \\ -\mathbf{k}_1\mathbf{i}_1 I_{1xz} & -\mathbf{k}_1\mathbf{j}_1 I_{1yz} & \mathbf{k}_1\mathbf{k}_1 I_{1zz} \end{bmatrix} \qquad \text{and} \qquad \Phi_1 = \begin{bmatrix} \mathbf{i}_1\mathbf{i}_1 \phi_{1xx} & -\mathbf{i}_1\mathbf{j}_1 \phi_{1xy} & -\mathbf{i}_1\mathbf{k}_1 \phi_{1xz} \\ -\mathbf{j}_1\mathbf{i}_1 \phi_{1xy} & \mathbf{j}_1\mathbf{j}_1 \phi_{1yy} & -\mathbf{j}_1\mathbf{k}_1 \phi_{1yz} \\ -\mathbf{k}_1\mathbf{i}_1 \phi_{1xz} & -\mathbf{k}_1\mathbf{j}_1 \phi_{1yz} & \mathbf{k}_1\mathbf{k}_1 \phi_{1zz} \end{bmatrix}$$

are used to bring the moments of inertia into a vector equation context.

Now the terms in Eq. (13) involving $\dot{\boldsymbol{\omega}}_1$ and $\boldsymbol{\omega}_1$ are examined, and substitution for vectors $\mathbf{c}_{ij}$ is made in terms of the $\mathbf{b}_{ij}$ vectors:

$$\mathbf{d}_{12} = \frac{m_2}{m}\left[m_1\mathbf{b}_{12} - m_1\mathbf{b}_{11} + m_3\mathbf{b}_{12} - m_3\mathbf{b}_{13}\right] = \frac{m_2}{m}\left[m\mathbf{b}_{12} - m_1\mathbf{b}_{11} - m_2\mathbf{b}_{12} - m_3\mathbf{b}_{13}\right] = m_2\mathbf{b}_{12}$$

$$\mathbf{d}_{13} = \frac{m_3}{m}\left[m_1\mathbf{b}_{13} - m_1\mathbf{b}_{11} + m_2\mathbf{b}_{13} - m_2\mathbf{b}_{12}\right] = \frac{m_3}{m}\left[m\mathbf{b}_{13} - m_1\mathbf{b}_{11} - m_2\mathbf{b}_{12} - m_3\mathbf{b}_{13}\right] = m_3\mathbf{b}_{13}$$

Therefore,

$$\begin{aligned}
-\mathbf{c}_{12}\times(\mathbf{d}_{12}\times\dot{\boldsymbol{\omega}}_1) - \mathbf{c}_{13}\times(\mathbf{d}_{13}\times\dot{\boldsymbol{\omega}}_1) &= (\mathbf{b}_{11}-\mathbf{b}_{12})\times(m_2\mathbf{b}_{12}\times\dot{\boldsymbol{\omega}}_1) + (\mathbf{b}_{11}-\mathbf{b}_{13})\times(m_3\mathbf{b}_{13}\times\dot{\boldsymbol{\omega}}_1)\\
&= -m_2\mathbf{b}_{12}\times(\mathbf{b}_{12}\times\dot{\boldsymbol{\omega}}_1) - m_3\mathbf{b}_{13}\times(\mathbf{b}_{13}\times\dot{\boldsymbol{\omega}}_1) + m_2\mathbf{b}_{11}\times(\mathbf{b}_{12}\times\dot{\boldsymbol{\omega}}_1)\\
&\quad + m_3\mathbf{b}_{11}\times\left[\left(-\frac{m_1}{m_3}\mathbf{b}_{11} - \frac{m_2}{m_3}\mathbf{b}_{12}\right)\times\dot{\boldsymbol{\omega}}_1\right]\\
&= -m_1\mathbf{b}_{11}\times(\mathbf{b}_{11}\times\dot{\boldsymbol{\omega}}_1) - m_2\mathbf{b}_{12}\times(\mathbf{b}_{12}\times\dot{\boldsymbol{\omega}}_1) - m_3\mathbf{b}_{13}\times(\mathbf{b}_{13}\times\dot{\boldsymbol{\omega}}_1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&-\boldsymbol{\omega}_1\times[\mathbf{c}_{12}\times(\mathbf{d}_{12}\times\boldsymbol{\omega}_1) + \mathbf{c}_{13}\times(\mathbf{d}_{13}\times\boldsymbol{\omega}_1)] =\\
&\qquad -\boldsymbol{\omega}_1\times[m_1\mathbf{b}_{11}\times(\mathbf{b}_{11}\times\boldsymbol{\omega}_1) + m_2\mathbf{b}_{12}\times(\mathbf{b}_{12}\times\boldsymbol{\omega}_1) + m_3\mathbf{b}_{13}\times(\mathbf{b}_{13}\times\boldsymbol{\omega}_1)]
\end{aligned}$$

Terms of the type $\mathbf{b}\times(\mathbf{b}\times\boldsymbol{\omega})$ generate the following set of relations:

$$\mathbf{b}\times\boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_x & b_y & b_z\\ \omega_x & \omega_y & \omega_z \end{vmatrix} = (\omega_z b_y - \omega_y b_z)\,\mathbf{i} + (\omega_x b_z - \omega_z b_x)\,\mathbf{j} + (\omega_y b_x - \omega_x b_y)\,\mathbf{k}$$

$$\begin{aligned}
\mathbf{b}\times(\mathbf{b}\times\boldsymbol{\omega}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_x & b_y & b_z\\ (\omega_z b_y - \omega_y b_z) & (\omega_x b_z - \omega_z b_x) & (\omega_y b_x - \omega_x b_y) \end{vmatrix} &= [\omega_x(-b_y^2 - b_z^2) + \omega_y(b_x b_y) + \omega_z(b_x b_z)]\,\mathbf{i}\\
&\quad + [\omega_y(b_x b_y) + \omega_y(-b_x^2 - b_z^2) + \omega_z(b_y b_z)]\,\mathbf{j}\\
&\quad + [\omega_z(b_x b_z) + \omega_y(b_y b_z) + \omega_z(-b_x^2 - b_y^2)]\,\mathbf{k}
\end{aligned}$$

The result, of course, is that the terms $-m_1\mathbf{b}_{11}\times(\mathbf{b}_{11}\times\dot{\boldsymbol{\omega}}_1)$ or $-m_1\mathbf{b}_{11}\times(\mathbf{b}_{11}\times\boldsymbol{\omega}_1)$, etc., may be represented by the dot product of a momental dyadic with the angular velocity (or acceleration) vector:

$$-m_1\mathbf{b}_{11}\times(\mathbf{b}_{11}\times\dot{\boldsymbol{\omega}}_1) = \begin{bmatrix} \mathbf{i}_1\mathbf{i}_1 m_1(b_{11y}^2 + b_{11z}^2) & -\mathbf{i}_1\mathbf{j}_1 m_1 b_{11x}b_{11y} & -\mathbf{i}_1\mathbf{k}_1 m_1 b_{11x}b_{11z}\\ -\mathbf{j}_1\mathbf{i}_1 m_1 b_{11x}b_{11y} & \mathbf{j}_1\mathbf{j}_1 m_1(b_{11x}^2 + b_{11z}^2) & -\mathbf{j}_1\mathbf{k}_1 m_1 b_{11y}b_{11z}\\ -\mathbf{k}_1\mathbf{i}_1 m_1 b_{11x}b_{11z} & -\mathbf{k}_1\mathbf{j}_1 m_1 b_{11y}b_{11z} & \mathbf{k}_1\mathbf{k}_1 m_1(b_{11x}^2 + b_{11y}^2) \end{bmatrix}\cdot\dot{\boldsymbol{\omega}}_1 = \Phi_{11}\cdot\dot{\boldsymbol{\omega}}_1$$

Also,

$$-m_2\mathbf{b}_{12}\times(\mathbf{b}_{12}\times\dot{\boldsymbol{\omega}}_1)=\Phi_{12}\cdot\dot{\boldsymbol{\omega}}_1$$

$$-m_3\mathbf{b}_{13}\times(\mathbf{b}_{13}\times\dot{\boldsymbol{\omega}}_1)=\Phi_{13}\cdot\dot{\boldsymbol{\omega}}_1$$

so that Eq. (13) for body 1 (since $\Phi_1=\mathbb{I}_1+\Phi_{11}+\Phi_{12}+\Phi_{13}$), becomes

$$\begin{aligned}\Phi_1\cdot\dot{\boldsymbol{\omega}}_1+\boldsymbol{\omega}_1\times(\Phi_1\cdot\boldsymbol{\omega}_1)&=\mathbf{L}_1+\mathbf{L}_{12}+\mathbf{L}_{13}+\mathbf{c}_{12}\times\left[\mathbf{F}_2-\frac{m_2}{m}(\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3)\right]+\mathbf{c}_{13}\times\left[\mathbf{F}_3-\frac{m_3}{m}(\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3)\right]\\&-\mathbf{d}_{12}\times(\mathbf{c}_{21}\times\dot{\boldsymbol{\omega}}_2)-\mathbf{d}_{13}\times(\mathbf{c}_{31}\times\dot{\boldsymbol{\omega}}_3)-\boldsymbol{\omega}_2\times[\mathbf{d}_{12}\times(\mathbf{c}_{21}\times\boldsymbol{\omega}_2)]-\boldsymbol{\omega}_3\times[\mathbf{d}_{13}\times(\mathbf{c}_{31}\times\boldsymbol{\omega}_3)]\end{aligned}\tag{16}$$

Completing the substitution in Eq. (16) for $\mathbf{c}_{ij}$-type terms results in

$$-\mathbf{d}_{12}\times(\mathbf{c}_{21}\times\dot{\boldsymbol{\omega}}_2)=(-m_2\mathbf{b}_{12})\times\left[\left(\mathbf{b}_{21}+\frac{m_1}{m_2}\mathbf{b}_{21}+\frac{m_3}{m_2}\mathbf{b}_{21}\right)\times\dot{\boldsymbol{\omega}}_2\right]=-m\mathbf{b}_{12}\times(\mathbf{b}_{21}\times\dot{\boldsymbol{\omega}}_2)$$

$$-\mathbf{d}_{13}\times(\mathbf{c}_{31}\times\dot{\boldsymbol{\omega}}_3)=(-m_3\mathbf{b}_{13})\times\left[\left(\mathbf{b}_{31}+\frac{m_1}{m_3}\mathbf{b}_{31}+\frac{m_2}{m_3}\mathbf{b}_{31}\right)\times\dot{\boldsymbol{\omega}}_3\right]=-m\mathbf{b}_{13}\times(\mathbf{b}_{31}\times\dot{\boldsymbol{\omega}}_3)$$

and

$$\begin{aligned}&\mathbf{c}_{12}\times\left[\mathbf{F}_2-\frac{m_2}{m}(\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3)\right]\\&+\mathbf{c}_{13}\times\left[\mathbf{F}_3-\frac{m_3}{m}(\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3)\right]=(\mathbf{b}_{12}-\mathbf{b}_{11})\times\left(\frac{m_1}{m}\mathbf{F}_2+\frac{m_3}{m}\mathbf{F}_2-\frac{m_2}{m}\mathbf{F}_1-\frac{m_2}{m}\mathbf{F}_3\right)\\&\qquad+(\mathbf{b}_{13}-\mathbf{b}_{11})\times\left(\frac{m_1}{m}\mathbf{F}_3+\frac{m_2}{m}\mathbf{F}_3-\frac{m_3}{m}\mathbf{F}_1-\frac{m_3}{m}\mathbf{F}_2\right)\\&\qquad=\frac{1}{m}(-m_2\mathbf{b}_{12}-m_3\mathbf{b}_{13}+m_2\mathbf{b}_{11}+m_3\mathbf{b}_{11})\times\mathbf{F}_1\\&\qquad+\frac{1}{m}(m_1\mathbf{b}_{12}+m_3\mathbf{b}_{12}-m_1\mathbf{b}_{11}-m_3\mathbf{b}_{13})\times\mathbf{F}_2\\&\qquad+\frac{1}{m}(-m_2\mathbf{b}_{12}+m_1\mathbf{b}_{13}+m_2\mathbf{b}_{13}-m_1\mathbf{b}_{11})\times\mathbf{F}_3\\&\qquad=\mathbf{b}_{11}\times\mathbf{F}_1+\mathbf{b}_{12}\times\mathbf{F}_2+\mathbf{b}_{13}\times\mathbf{F}_3\end{aligned}$$

Finally, Eq. (16) becomes

$$\begin{aligned}\Phi_1\cdot\dot{\boldsymbol{\omega}}_1+\boldsymbol{\omega}_1\times\Phi_1\cdot\boldsymbol{\omega}_1=\mathbf{L}_1+\mathbf{L}_{12}+\mathbf{L}_{13}+\mathbf{b}_{11}\times\mathbf{F}_1+\mathbf{b}_{12}\times\mathbf{F}_2+\mathbf{b}_{13}\times\mathbf{F}_3-m\{\mathbf{b}_{12}\times(\mathbf{b}_{21}\times\dot{\boldsymbol{\omega}}_2)\\+\mathbf{b}_{13}\times(\mathbf{b}_{31}\times\dot{\boldsymbol{\omega}}_3)-\boldsymbol{\omega}_2\times[\mathbf{b}_{12}\times(\mathbf{b}_{21}\times\boldsymbol{\omega}_2)]-\boldsymbol{\omega}_3\times[\mathbf{b}_{13}\times(\mathbf{b}_{31}\times\boldsymbol{\omega}_3)]\}\end{aligned}$$

or

$$\Phi_1\cdot\dot{\boldsymbol{\omega}}_1+\boldsymbol{\omega}_1\times\Phi_1\cdot\boldsymbol{\omega}_1=\mathbf{L}_1+\mathbf{L}_{12}+\mathbf{L}_{13}+\mathbf{b}_{11}\times\mathbf{F}_1+\mathbf{b}_{12}\times\mathbf{F}_2+\mathbf{b}_{13}\times\mathbf{F}_3+m(\mathbf{b}_{12}\times\ddot{\mathbf{b}}_{21}+\mathbf{b}_{13}\times\ddot{\mathbf{b}}_{31})\tag{17}$$

where

$$\ddot{\mathbf{b}}_{21}=\dot{\boldsymbol{\omega}}_2\times\mathbf{b}_{21}+\boldsymbol{\omega}_2\times(\boldsymbol{\omega}_2\times\mathbf{b}_{21})\qquad\text{and}\qquad\ddot{\mathbf{b}}_{31}=\dot{\boldsymbol{\omega}}_3\times\mathbf{b}_{31}+\boldsymbol{\omega}_3\times(\boldsymbol{\omega}_3\times\mathbf{b}_{31})$$

In the same manner, Eqs. (14) and (15) can be simplified by substitution of the barycenter vectors $\mathbf{b}_{ij}$ and the use of augmented inertia dyadics to obtain

$$\Phi_2 \cdot \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \Phi_2 \cdot \boldsymbol{\omega}_2 = \mathbf{L}_2 + \mathbf{L}_{21} + \mathbf{b}_{21} \times \mathbf{F}_1 + \mathbf{b}_{22} \times \mathbf{F}_2 + \mathbf{b}_{23} \times \mathbf{F}_3 + m(\mathbf{b}_{21} \times \ddot{\mathbf{b}}_{12} + \mathbf{b}_{23} \times \ddot{\mathbf{b}}_{32}) \tag{18}$$

$$\Phi_3 \cdot \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times \Phi_3 \cdot \boldsymbol{\omega}_3 = \mathbf{L}_3 + \mathbf{L}_{31} + \mathbf{b}_{31} \times \mathbf{F}_1 + \mathbf{b}_{32} \times \mathbf{F}_2 + \mathbf{b}_{33} \times \mathbf{F}_3 + m(\mathbf{b}_{31} \times \ddot{\mathbf{b}}_{13} + \mathbf{b}_{32} \times \ddot{\mathbf{b}}_{23})$$

$$\mathbf{L}_{12} = -\mathbf{L}_{21}, \qquad \mathbf{L}_{13} = -\mathbf{L}_{31} \tag{19}$$

where Φ_2 and Φ_3 are obtained by substituting subscripts in the expression already given for Φ_1.

Euler's equations for the given 3-body system, as embodied finally in Eqs. (17–19), when joined with the appropriate differential equations relating angular velocities and positions (kinematical equations), may be integrated by a computer for the dynamic solution. In general, connecting joints in the system will not necessarily allow three degrees of rotational freedom. As a result, unknown torques due to such "rigid constraints" will appear within the $\mathbf{L}_{ij}$ terms.

III. The Generalized *n*-Body Equations

The set of vector equations developed for the system of Fig. 1 and formulated in terms of body barycenters is, with one minor modification, the set applying to all 3-body systems. If all the possible $\mathbf{L}_{ij}$ terms ($i = 1, 2, 3$; $j = 1, 2, 3$; $i \neq j$) are included in each vector equation, the set becomes perfectly general. Specialization of the set to a particular system would then require some of the $\mathbf{L}_{ij}$ to be dropped since no closed-connection loops are permitted.

It is, in fact, rather obvious from the subscript pattern how the equations would be extended to larger systems. The proof of the generalization to n bodies is pursued in much the same manner as the development for three bodies. Beginning with the general equation of rotation for body λ,

$$\frac{d}{dt}\mathbf{I}_\lambda \cdot \boldsymbol{\omega}_\lambda = \mathbf{L}_\lambda + \sum_{\beta \neq \lambda} \mathbf{L}_{\lambda\beta} + \sum_{\beta \neq \lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta} \tag{20}$$

where $\lambda = 1, 2, \cdots, n$ and where β denotes those bodies directly sharing a joint with body λ. Generalizing from Eqs. (1–3) results in the force equation

$$\mathbf{F}_\lambda + \sum_{\beta \neq \lambda} \mathbf{F}_{\lambda\beta} = m_\lambda (\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_\lambda) \tag{21}$$

and

$$\sum_\alpha \mathbf{F}_\alpha = m\ddot{\mathbf{R}}$$

where

$$m = \sum_\alpha m_\alpha, \qquad \alpha = 1, 2, 3, \cdots, n$$

A general expression that may be substituted in Eq. (20) is required for $\mathbf{F}_{\lambda\beta}$. Equation (21) provides the needed relation, but it may be helpful to examine a particular system of connected bodies, e.g., the one shown in Fig. 4.

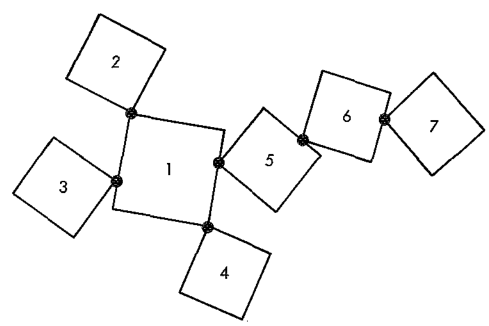

Fig. 4. An arbitrary 7-body system

The set of force equations applying to this system and corresponding to Eq. (21) becomes:

Body 1: $\mathbf{F}_1 + \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{15} = m_1(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_1)$

Body 2: $\mathbf{F}_2 + \mathbf{F}_{21} = m_2(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_2)$

Body 3: $\mathbf{F}_3 + \mathbf{F}_{31} = m_3(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_3)$

Body 4: $\mathbf{F}_4 + \mathbf{F}_{41} = m_4(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_4)$

Body 5: $\mathbf{F}_5 + \mathbf{F}_{51} + \mathbf{F}_{56} = m_5(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_5)$

Body 6: $\mathbf{F}_6 + \mathbf{F}_{65} + \mathbf{F}_{67} = m_6(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_6)$

Body 7: $\mathbf{F}_7 + \mathbf{F}_{76} = m_7(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}_7)$

To solve for $\mathbf{F}_{65}$, for example, one could add equations for body 6 and body 7 and, realizing that $\mathbf{F}_{\lambda\beta} = -\mathbf{F}_{\beta\lambda}$, obtain

$$\mathbf{F}_{65} = -\mathbf{F}_6 - \mathbf{F}_7 + (m_6 + m_7)\ddot{\mathbf{R}} + m_6\ddot{\boldsymbol{\rho}}_6 + m_7\ddot{\boldsymbol{\rho}}_7$$

However, since $\mathbf{F}_{65} = -\mathbf{F}_{56}$, equations for bodies 1–5 could be added to give

$$\begin{aligned}\mathbf{F}_{65} = -\mathbf{F}_{56} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 + \mathbf{F}_5 \\ &\quad - (m_1 + m_2 + m_3 + m_4 + m_5)\ddot{\mathbf{R}} \\ &\quad - m_1\ddot{\boldsymbol{\rho}}_1 - m_2\ddot{\boldsymbol{\rho}}_2 - m_3\ddot{\boldsymbol{\rho}}_3 - m_4\ddot{\boldsymbol{\rho}}_4 - m_5\ddot{\boldsymbol{\rho}}_5\end{aligned}$$

That the two expressions for $\mathbf{F}_{65}$ are equivalent can be easily verified, since

$$\sum_\alpha m_\alpha\ddot{\boldsymbol{\rho}}_\alpha = 0$$

The latter expression gives the reaction force exerted on body 6 through the joint leading to bodies 1–5, in terms of applied and inertial forces due to these same bodies 1–5. To generalize, then, for $\mathbf{F}_{\lambda\beta}$, one can write

$$\mathbf{F}_{\lambda\beta} = \sum_\eta [\mathbf{F}_\eta - m_\eta\ddot{\mathbf{R}} - m_\eta\ddot{\boldsymbol{\rho}}_\eta] \tag{22}$$

where η includes those bodies which are directly or indirectly connected, through an intermediate chain of bodies, to body λ through the joint it shares with body β.

Therefore the term

$$\sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta}$$

in Eq. (20) may be expanded as

$$\sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta} = \sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \left[\sum_\eta (\mathbf{F}_\eta - m_\eta\ddot{\mathbf{R}} - m_\eta\ddot{\boldsymbol{\rho}}_\eta)\right] \tag{23}$$

Again, note that all the vectors $\mathbf{c}_{\lambda\alpha}$, $\alpha = 1, 2, 3, \cdots, n$, $\lambda = 1, 2, 3, \cdots, n$ $(\lambda \neq \alpha)$, exist since $\mathbf{c}_{\lambda\alpha}$ is simply the vector from body λ's center of mass to the joint directly or indirectly connecting λ to body α. Thus, in the system of Fig. 4, $\mathbf{c}_{65} = \mathbf{c}_{61} = \mathbf{c}_{62} = \mathbf{c}_{63} = \mathbf{c}_{64}$. As a consequence of this built-in redundancy and since, in Eq. (23), η ultimately covers all bodies except λ, Eq. (23) can be rewritten as

$$\begin{aligned}\sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta} &= \sum_{\alpha\neq\lambda} \mathbf{c}_{\lambda\alpha} \times (\mathbf{F}_\alpha - m_\alpha\ddot{\mathbf{R}} - m_\alpha\ddot{\boldsymbol{\rho}}_\alpha), \qquad \alpha = 1, 2, \cdots, n \\ &= \sum_{\alpha\neq\lambda} \mathbf{c}_{\lambda\alpha} \times \left[\mathbf{F}_\alpha - \frac{m_\alpha}{m}\left(\sum_\nu \mathbf{F}_\nu\right) - m_\alpha\ddot{\boldsymbol{\rho}}_\alpha\right], \qquad \nu = 1, 2, \cdots, n \\ &= \sum_{\alpha\neq\lambda} \mathbf{c}_{\lambda\alpha} \times \left[\mathbf{F}_\alpha - \frac{m_\alpha}{m}\mathbf{F}_\lambda - \frac{m_\alpha}{m}\left(\sum_{\alpha\neq\lambda} \mathbf{F}_\alpha\right) - m_\alpha\ddot{\boldsymbol{\rho}}_\alpha\right]\end{aligned} \tag{24}$$

Since the $\mathbf{c}$ vectors are related to the barycentric vectors $\mathbf{b}$ by

$$\mathbf{b}_{\lambda\lambda} = -\frac{1}{m}\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \qquad \text{and} \qquad \mathbf{b}_{\lambda\alpha} = \mathbf{b}_{\lambda\lambda} + \mathbf{c}_{\lambda\alpha} \tag{25}$$

Eq. (24) may be expressed by

$$\sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta} = -\sum_{\alpha\neq\lambda} \frac{m_\alpha}{m}\mathbf{c}_{\lambda\alpha} \times \mathbf{F}_\lambda + \sum_{\alpha\neq\lambda}\left(\mathbf{c}_{\lambda\alpha} - \sum_{\alpha\neq\lambda}\frac{m_\alpha}{m}\mathbf{c}_{\lambda\alpha}\right) \times \mathbf{F}_\alpha - \sum_{\alpha\neq\lambda} \mathbf{c}_{\lambda\alpha} \times m_\alpha\ddot{\boldsymbol{\rho}}_\alpha$$

$$\sum_{\beta\neq\lambda} \mathbf{c}_{\lambda\beta} \times \mathbf{F}_{\lambda\beta} = \mathbf{b}_{\lambda\lambda} \times \mathbf{F}_\lambda + \sum_{\alpha\neq\lambda} \mathbf{b}_{\lambda\alpha} \times \mathbf{F}_\alpha - \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha \tag{26}$$

The first two terms on the right-hand side of Eq. (26) provide the needed general expressions for the effects of the applied forces. In fact, they fit exactly with what one might have inferred from the 3-body example. In the third term, it is desirable to express $\boldsymbol{\rho}_\alpha$ in terms of $\mathbf{c}$ vectors and, one hopes, barycenter vectors $\mathbf{b}$.

In general, for any two bodies λ and μ in the set of bodies numbered from 1 to n,

$$\boldsymbol{\rho}_\lambda - \boldsymbol{\rho}_\mu = \mathbf{c}_{\mu\lambda} - \mathbf{c}_{\lambda\mu} + \sum_i (\mathbf{c}_{i\lambda} - \mathbf{c}_{i\mu}) \tag{27}$$

where i now refers to those bodies in the chain connecting body μ to body λ.

Then if, as is done in Ref. 1, both sides of Eq. (27) are multiplied by m_μ/m and the summation over all $\mu \neq \lambda$ is made, one has

$$\boldsymbol{\rho}_\lambda \sum_{\mu\neq\lambda} \frac{m_\mu}{m} - \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \boldsymbol{\rho}_\mu = \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \left[\mathbf{c}_{\mu\lambda} - \mathbf{c}_{\lambda\mu} + \sum_i (\mathbf{c}_{i\lambda} - \mathbf{c}_{i\mu}) \right]$$

$$\boldsymbol{\rho}_\lambda \left(1 - \frac{m_\lambda}{m}\right) - \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \boldsymbol{\rho}_\mu = \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \left[\mathbf{c}_{\mu\lambda} - \mathbf{c}_{\lambda\mu} + \sum_i (\mathbf{c}_{i\lambda} - \mathbf{c}_{i\mu}) \right]$$

since, by definition of the $\boldsymbol{\rho}$ vectors,

$$\sum_{k=1}^{n} \frac{m_k}{m} \boldsymbol{\rho}_k = 0$$

$$\boldsymbol{\rho}_\lambda = \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \left[\mathbf{c}_{\mu\lambda} - \mathbf{c}_{\lambda\mu} + \sum_i (\mathbf{c}_{i\lambda} - \mathbf{c}_{i\mu}) \right]$$

or

$$\boldsymbol{\rho}_\lambda = \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \left[\mathbf{c}_{\mu\lambda} + \sum_i (\mathbf{c}_{i\lambda} - \mathbf{c}_{i\mu}) \right] \tag{28}$$

Further, it proves useful to extend the summation over bodies i to a summation over bodies α, where α includes all bodies in the system except μ and λ. This does not change Eq. (28) since, for those bodies not included in the chain connecting μ to λ, $\mathbf{c}_{\alpha\lambda} = \mathbf{c}_{\alpha\mu}$ and nothing is contributed to the summation:

$$\sum_\alpha (\mathbf{c}_{\alpha\lambda} - \mathbf{c}_{\alpha\mu})$$

As a result,

$$\boldsymbol{\rho}_\lambda = \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \mathbf{c}_{\mu\lambda} + \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \left[\sum_{\substack{\alpha\neq\mu \\ \alpha\neq\lambda}} (\mathbf{c}_{\alpha\lambda} - \mathbf{c}_{\alpha\mu}) \right] \tag{29}$$

Since the subscripts α and μ both range over all the system bodies except λ, subject to the given restrictions, they can be interchanged in the final term of Eq. (29) and, if the order of the summations is interchanged as well, one has

$$\boldsymbol{\rho}_\lambda = \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \mathbf{c}_{\mu\lambda} + \sum_{\mu\neq\lambda} \left[\sum_{\substack{\alpha\neq\lambda \\ \alpha\neq\mu}} \frac{m_\alpha}{m} (\mathbf{c}_{\mu\lambda} - \mathbf{c}_{\mu\alpha}) \right]$$

But

$$\sum_{\substack{\alpha\neq\lambda \\ \alpha\neq\mu}} \frac{m_\alpha}{m} \mathbf{c}_{\mu\lambda} = \frac{m - m_\lambda - m_\mu}{m} \mathbf{c}_{\mu\lambda}$$

$$= \mathbf{c}_{\mu\lambda} - \frac{m_\mu}{m} \mathbf{c}_{\mu\lambda} - \frac{m_\lambda}{m} \mathbf{c}_{\mu\lambda}$$

Thus

$$\boldsymbol{\rho}_\lambda = \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \left(\mathbf{c}_{\mu\lambda} - \frac{m_\lambda}{m} \mathbf{c}_{\mu\lambda} - \sum_{\substack{\alpha\neq\lambda \\ \alpha\neq\mu}} \frac{m_\alpha}{m} \mathbf{c}_{\mu\alpha} \right)$$

$$= \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \left(\mathbf{c}_{\mu\lambda} - \sum_{\alpha\neq\mu} \frac{m_\alpha}{m} \mathbf{c}_{\mu\alpha} \right)$$

$$\boldsymbol{\rho}_\lambda = \mathbf{b}_{\lambda\lambda} + \sum_{\mu\neq\lambda} \mathbf{b}_{\mu\lambda} \tag{30}$$

The last term in Eq. (26) can then be expanded as

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha = -\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \left(\ddot{\mathbf{b}}_{\alpha\alpha} + \sum_{\mu\neq\alpha} \ddot{\mathbf{b}}_{\mu\alpha} \right) \tag{31}$$

From the relations of Eq. (25) one can derive the following:

$$m_\alpha \mathbf{b}_{\alpha\alpha} + \sum_{\mu\neq\lambda} m_\mu \mathbf{b}_{\alpha\mu} = -\frac{m_\alpha}{m} \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\alpha\mu} - \sum_{\mu\neq\alpha} \frac{m_\mu}{m} \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\alpha\mu} + \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\alpha\mu}$$

$$= \left(-\frac{m_\alpha}{m} - \sum_{\mu\neq\alpha} \frac{m_\mu}{m} + 1 \right) \cdot \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\alpha\mu}$$

Therefore,

$$m_\alpha \mathbf{b}_{\alpha\alpha} + \sum_{\mu\neq\alpha} m_\mu \mathbf{b}_{\alpha\mu} = 0 \tag{32}$$

Equation (31) can therefore be written as

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha = -\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \left(-\frac{1}{m_\alpha} \sum_{\mu\neq\alpha} m_\mu \ddot{\mathbf{b}}_{\alpha\mu} + \ddot{\mathbf{b}}_{\lambda\alpha} + \sum_{\substack{\mu\neq\alpha \\ \mu\neq\lambda}} \ddot{\mathbf{b}}_{\mu\alpha} \right)$$

$$= \sum_{\alpha\neq\lambda} \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\alpha\mu} - \sum_{\alpha\neq\lambda} \sum_{\substack{\mu\neq\alpha \\ \mu\neq\lambda}} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\mu\alpha} - \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

Subscripts α and μ may be interchanged in the second term of the preceding equation to give the following:

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha = \sum_{\alpha\neq\lambda} \sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\alpha\mu} - \sum_{\mu\neq\lambda} \sum_{\substack{\alpha\neq\mu \\ \alpha\neq\lambda}} m_\mu \mathbf{c}_{\lambda\mu} \times \ddot{\mathbf{b}}_{\alpha\mu} - \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

$$= \sum_{\alpha\neq\lambda} \left(\sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\alpha\mu} - \sum_{\substack{\mu\neq\lambda \\ \mu\neq\alpha}} m_\mu \mathbf{c}_{\lambda\mu} \times \ddot{\mathbf{b}}_{\alpha\mu} \right) - \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha} \tag{33}$$

For the expression within the parentheses above, α and λ are of course fixed, and μ ranges over the system bodies under the restrictions shown. However, for those bodies μ that are directly or indirectly connected to a joint on body λ which also leads, directly or indirectly, from body λ to body α, $\mathbf{c}_{\lambda\alpha} = \mathbf{c}_{\lambda\mu}$. For example, in Fig. 5, if $\alpha = 6$ and $\lambda = 2$, $\mathbf{c}_{25} = \mathbf{c}_{26} = \mathbf{c}_{27}$, so that for $\mu = 5, 7$ no net term is contributed in Eq. (33) from the summations over μ.

Furthermore, $\mathbf{b}_{\alpha\lambda} = \mathbf{b}_{\alpha\mu}$ for those remaining bodies μ which *do* contribute nonzero terms to Eq. (33). The result can then be rewritten as

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha = \sum_{\alpha\neq\lambda} \left[\left(\sum_{\mu\neq\alpha} m_\mu \mathbf{c}_{\lambda\alpha} - \sum_{\substack{\mu\neq\lambda \\ \mu\neq\alpha}} m_\mu \mathbf{c}_{\lambda\mu} \right) \times \ddot{\mathbf{b}}_{\alpha\lambda} \right]$$

$$- \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

$$= \sum_{\alpha\neq\lambda} \left[(m - m_\alpha)\, \mathbf{c}_{\lambda\alpha} - \sum_{\substack{\mu\neq\lambda \\ \mu\neq\alpha}} m_\mu \mathbf{c}_{\lambda\mu} \right] \times \ddot{\mathbf{b}}_{\alpha\lambda}$$

$$- \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

$$= \sum_{\alpha\neq\lambda} \left(m\mathbf{c}_{\lambda\alpha} - \sum_{\mu\neq\lambda} m_\mu \mathbf{c}_{\lambda\mu} \right) \times \ddot{\mathbf{b}}_{\alpha\lambda}$$

$$- \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

$$= \sum_{\alpha\neq\lambda} m \left(\mathbf{c}_{\lambda\alpha} - \sum_{\mu\neq\lambda} \frac{m_\mu}{m} \mathbf{c}_{\lambda\mu} \right) \times \ddot{\mathbf{b}}_{\alpha\lambda}$$

$$- \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

and, finally,

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\boldsymbol{\rho}}_\alpha = \sum_{\alpha\neq\lambda} m\mathbf{b}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\alpha\lambda} - \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha} \tag{34}$$

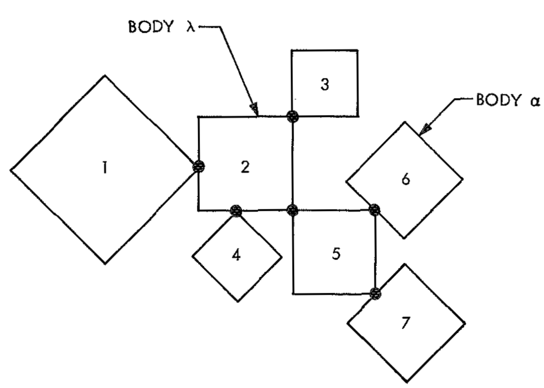

Fig. 5. Another 7-body system configuration

Returning to Eq. (20), one can now write it as

$$\frac{d}{dt}\mathbf{I}_\lambda \cdot \boldsymbol{\omega}_\lambda + \sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha} = \mathbf{L}_\lambda + \sum_{\beta\neq\lambda} \mathbf{L}_{\lambda\beta} + \mathbf{b}_{\lambda\lambda} \times \mathbf{F}_\lambda + \sum_{\alpha\neq\lambda} \mathbf{b}_{\lambda\alpha} \times (\mathbf{F}_\alpha + m\ddot{\mathbf{b}}_{\alpha\lambda}) \tag{35}$$

where β is restricted to those bodies directly sharing a joint with λ.

Since $\mathbf{b}_{\lambda\alpha}$ is a vector fixed in body λ, it is natural to express $\ddot{\mathbf{b}}_{\lambda\alpha}$ in terms of $\boldsymbol{\omega}_\lambda$ and $\dot{\boldsymbol{\omega}}_\lambda$:

$$\ddot{\mathbf{b}}_{\lambda\alpha} = \dot{\boldsymbol{\omega}}_\lambda \times \mathbf{b}_{\lambda\alpha} + \boldsymbol{\omega}_\lambda \times (\boldsymbol{\omega}_\lambda \times \mathbf{b}_{\lambda\alpha})$$

Likewise,

$$\ddot{\mathbf{b}}_{\alpha\lambda} = \dot{\boldsymbol{\omega}}_\alpha \times \mathbf{b}_{\alpha\lambda} + \boldsymbol{\omega}_\alpha \times (\boldsymbol{\omega}_\alpha \times \mathbf{b}_{\alpha\lambda})$$

Also, the second term on the left of Eq. (35) can be expanded as

$$\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha} = \sum_{\alpha\neq\lambda} m_\alpha (\mathbf{b}_{\lambda\alpha} - \mathbf{b}_{\lambda\lambda}) \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

$$= \sum_{\alpha\neq\lambda} m_\alpha \mathbf{b}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha} - \mathbf{b}_{\lambda\lambda} \times \sum_{\alpha\neq\lambda} m_\alpha \ddot{\mathbf{b}}_{\lambda\alpha}$$

Using Eq. (32), one obtains

$$\sum_{\alpha\neq\lambda} m_\alpha \ddot{\mathbf{b}}_{\lambda\alpha} = -m_\lambda \ddot{\mathbf{b}}_{\lambda\lambda}$$

Therefore,

$$\sum_{\alpha\neq\lambda} m_\alpha \mathbf{c}_{\lambda\alpha} \times \mathbf{b}_{\lambda\alpha} = m_\lambda \mathbf{b}_{\lambda\lambda} \times \ddot{\mathbf{b}}_{\lambda\lambda} + \sum_{\alpha\neq\lambda} m_\alpha \mathbf{b}_{\lambda\alpha} \times \ddot{\mathbf{b}}_{\lambda\alpha}$$

Again, one is faced with the terms of the type

$$\mathbf{b} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] = -\boldsymbol{\omega} \times [\mathbf{b} \times (\mathbf{b} \times \boldsymbol{\omega})]$$

and

$$\mathbf{b} \times (\dot{\boldsymbol{\omega}} \times \mathbf{b}) = -\mathbf{b} \times (\mathbf{b} \times \dot{\boldsymbol{\omega}})$$

and, as shown earlier, each of these can be written as a dot product of a dyadic with an angular velocity (or acceleration) vector. Thus,

$$-\boldsymbol{\omega}_\lambda \times [m_\lambda \mathbf{b}_{\lambda\lambda} \times (\mathbf{b}_{\lambda\lambda} \times \boldsymbol{\omega}_\lambda)] = \boldsymbol{\omega}_\lambda \times \Phi_{\lambda\lambda} \cdot \boldsymbol{\omega}_\lambda$$

$$-m_\lambda \mathbf{b}_{\lambda\lambda} \times (\mathbf{b}_{\lambda\lambda} \times \dot{\boldsymbol{\omega}}_\lambda) = \Phi_{\lambda\lambda} \cdot \dot{\boldsymbol{\omega}}_\lambda$$

$$-\sum_{\alpha\neq\lambda} \boldsymbol{\omega}_\lambda \times [m_\alpha \mathbf{b}_{\lambda\alpha} \times (\mathbf{b}_{\lambda\alpha} \times \boldsymbol{\omega}_\lambda)] = \boldsymbol{\omega}_\lambda \times \sum_{\alpha\neq\lambda} \Phi_{\lambda\alpha} \cdot \boldsymbol{\omega}_\lambda$$

$$-\sum_{\alpha\neq\lambda} m_\alpha \mathbf{b}_{\lambda\alpha} \times (\mathbf{b}_{\lambda\alpha} \times \dot{\boldsymbol{\omega}}_\lambda) = \sum_{\alpha\neq\lambda} \Phi_{\lambda\alpha} \cdot \dot{\boldsymbol{\omega}}_\lambda$$

Consequently, Eq. (35) can be reduced to

$$\Phi_\lambda \cdot \dot{\boldsymbol{\omega}}_\lambda + \boldsymbol{\omega}_\lambda \times \Phi_\lambda \cdot \boldsymbol{\omega}_\lambda = \mathbf{L}_\lambda + \sum_{\beta\neq\lambda} \mathbf{L}_{\lambda\beta} + \mathbf{b}_{\lambda\lambda} \times \mathbf{F}_\lambda + \sum_{\alpha\neq\lambda} \mathbf{b}_{\lambda\alpha} \times (\mathbf{F}_\alpha + m\ddot{\mathbf{b}}_{\alpha\lambda}) \tag{36}$$

where

$$\Phi_\lambda = \mathbf{I}_\lambda + \sum_\alpha \Phi_{\lambda\alpha}, \qquad \alpha = 1, 2, 3, \cdots, n$$

and

$$\Phi_{\lambda\alpha} = m_\alpha \begin{bmatrix} \mathbf{i}_\lambda \mathbf{i}_\lambda (b^2_{\lambda\alpha y} + b^2_{\lambda\alpha z}) & -\mathbf{i}_\lambda \mathbf{j}_\lambda b_{\lambda\alpha x} b_{\lambda\alpha y} & -\mathbf{i}_\lambda \mathbf{k}_\lambda b_{\lambda\alpha x} b_{\lambda\alpha z} \\ -\mathbf{j}_\lambda \mathbf{i}_\lambda b_{\lambda\alpha x} b_{\lambda\alpha y} & \mathbf{j}_\lambda \mathbf{j}_\lambda (b^2_{\lambda\alpha x} + b^2_{\lambda\alpha z}) & -\mathbf{j}_\lambda \mathbf{k}_\lambda b_{\lambda\alpha y} b_{\lambda\alpha z} \\ -\mathbf{k}_\lambda \mathbf{i}_\lambda b_{\lambda\alpha x} b_{\lambda\alpha z} & -\mathbf{k}_\lambda \mathbf{j}_\lambda b_{\lambda\alpha y} b_{\lambda\alpha z} & \mathbf{k}_\lambda \mathbf{k}_\lambda (b^2_{\lambda\alpha x} + b^2_{\lambda\alpha y}) \end{bmatrix}$$

or

$$\Phi_{\lambda\alpha} = m_\alpha [(\mathbf{b}_{\lambda\alpha} \cdot \mathbf{b}_{\lambda\alpha}) \mathbf{E} - \mathbf{b}_{\lambda\alpha} \mathbf{b}_{\lambda\alpha}]$$

where $\mathbf{E}$ is the identity dyadic.

Equation (36), then, is the general equation of rotational motion for each body λ in the system of n connected rigid bodies expressed in terms of barycenter vectors and a new inertia dyadic Φ_λ about the barycenter of the body λ.

IV. Constraint Torque Elimination

Equation (36), the set of differential equations describing the attitude of each rigid body in the system, can be solved for the unknown $\dot{\omega}_\lambda$ values once the system's geometric and physical properties are known, the applied torques and forces are specified, and the characteristics of each joint are described. Actually, the preceding statement is not quite correct, in that the system unknowns usually will not be limited to angular accelerations but will also include certain unknown torques of constraint at one or more joints. This occurs when a joint does not permit more than two degrees of rotational freedom between adjoining rigid bodies. As a result, although three degrees of rotational freedom are possible for a body, the nature of the joint, i.e., whether it is a single- or two-gimbal joint, will reduce the degrees of freedom to one or two and will inject constrained modes of rotation and unknown constraint torques.

Reaction torques at the joints are described in Eq. (36) by the term

$$\sum_{\beta \neq \lambda} \mathbf{L}_{\lambda\beta}$$

which may be broken down into more detail as follows (Ref. 1):

$$\sum_{\beta \neq \lambda} \mathbf{L}_{\lambda\beta} = \sum_{\beta \neq \lambda} \mathbf{L}^F_{\lambda\beta} + \sum_{i=1}^{n_c} \delta_{\lambda i} L^c_i \mathbf{u}^c_i \tag{37}$$

where

$\mathbf{L}^F_{\lambda\beta}$ = reaction torque on body λ at the joint connecting λ to β due to their relative motion about gimbal or hinge axes

n_c = number of constrained modes of rotation; $n_c = 3n - n'$, where n' is the number of degrees of freedom in the system

$\mathbf{u}^c_i$ = a unit vector directed along the axis of constrained motion

L^c_i = magnitude of the constraint torque directed along $\mathbf{u}_i$

$$\delta_{\lambda i} = \begin{cases} 0 \text{ if } \mathbf{u}^c_i \text{ is not defined for a joint on body } \lambda \\ +1 \text{ if } L^c_i \mathbf{u}^c_i \text{ is a torque on } \lambda \\ -1 \text{ if } L^c_i \mathbf{u}^c_i \text{ is a torque on } \beta \end{cases}$$

It now becomes convenient to move from consideration of the vector differential equation dealing with a single body of the system to a matrix representation of the entire set of equations describing the system's rotations. Beginning with Eq. (36), an equivalent representation in terms of matrices can be given by

$$\sum_\alpha A_{\lambda\alpha}\dot{\omega}_\alpha = E_\lambda + U_\lambda L^c \tag{38}$$

where

$A_{\lambda\lambda}$ = the 3×3 matrix of body λ components of Φ_λ

$A_{\lambda\alpha}\dot{\omega}_\alpha (\alpha \neq \lambda)$ = the 3×1 matrix of body λ components of the vector $\mathbf{D}_{\lambda\alpha} \cdot \dot{\boldsymbol{\omega}}_\alpha$ produced by the term

$$-m\mathbf{b}_{\lambda\alpha} \times (\dot{\boldsymbol{\omega}}_\alpha \times \mathbf{b}_{\alpha\lambda}) = m\mathbf{b}_{\lambda\alpha} \times (\mathbf{b}_{\alpha\lambda} \times \dot{\boldsymbol{\omega}}_\alpha)$$
$$= \mathbf{D}_{\lambda\alpha} \cdot \dot{\boldsymbol{\omega}}_\alpha$$

where

$\mathbf{D}_{\lambda\alpha} = m\,[\mathbf{b}_{\alpha\lambda}\mathbf{b}_{\lambda\alpha} - (\mathbf{b}_{\alpha\lambda} \cdot \mathbf{b}_{\lambda\alpha})\,\mathbf{E}]$

$\dot{\omega}_\alpha$ = 3×1 matrix of body α components of $\dot{\boldsymbol{\omega}}_\alpha$

U_λ = $3 \times n_c$ matrix whose ith column consists of the body λ components of $\delta_{\lambda i}\mathbf{u}^c_i$

L^c = $n_c \times 1$ matrix made up of L^c_i

E_λ = 3×1 matrix of body λ components of the remaining terms of Eq. (36), namely

$$-\boldsymbol{\omega}_\lambda \times \Phi_\lambda \cdot \boldsymbol{\omega}_\lambda + \mathbf{L}_\lambda + \sum_{\beta \neq \lambda} \mathbf{L}^F_{\lambda\beta} + \mathbf{b}_{\lambda\lambda} \times \mathbf{F}_\lambda$$
$$+ \sum_{\alpha \neq \lambda} \mathbf{b}_{\lambda\alpha} \times [\mathbf{F}_\alpha + m\boldsymbol{\omega}_\alpha \times (\boldsymbol{\omega}_\alpha \times \mathbf{b}_{\alpha\lambda})]$$

A system matrix equation can now be constructed from the n-equations given by Eq. (38):

$$A\,\dot{\omega} = E + UL^c \tag{39}$$

where

$$A = \left[\begin{array}{c|cccc} A_{11} & A_{12} & \cdots & \cdots & A_{1n} \\ \hline A_{21} & A_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & A_{\lambda\lambda} & \vdots \\ \vdots & & & & \ddots \\ A_{n1} & \cdots & \cdots & \cdots & A_{nn} \end{array}\right], \quad 3n \times 3n$$

$$\omega = \begin{bmatrix} \omega_1 \\ \hline \omega_2 \\ \hline \vdots \\ \hline \omega_n \end{bmatrix}, \quad 3n \times 1 \qquad E = \begin{bmatrix} E_1 \\ \hline E_2 \\ \hline \vdots \\ \hline E_n \end{bmatrix}, \quad 3n \times 1$$

$$U = \begin{bmatrix} U_1 \\ \hline U_2 \\ \hline \vdots \\ \hline U_n \end{bmatrix}, \quad 3n \times n_c$$

Additional equations are required to specify the constraint conditions and thereby provide sufficient information for the solution of both $\dot{\omega}$ and L^c. A pair of joined, rigid bodies is shown in Fig. 6 to illustrate the constraint condition.

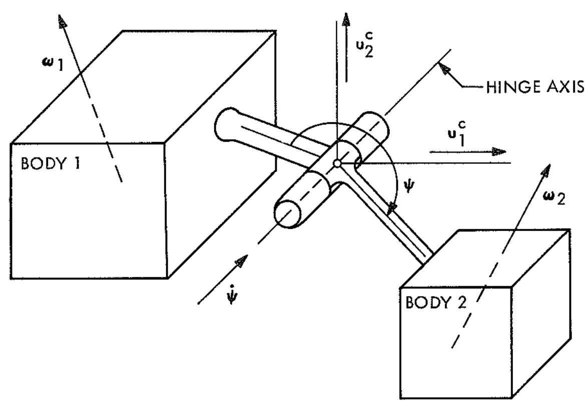

Fig. 6. A pair of joined, rigid bodies

The two bodies are "free" (subject to friction, spring constants, etc., in the hinge) to rotate relative to one another about the single hinge axis. However, no relative rotation is permitted about any axis perpendicular to the hinge axis. It is apparent that the relative vector rate of rotation $\dot{\boldsymbol{\Psi}}$ can have no components in the plane perpendicular to the hinge axis. Two vectors, $\mathbf{u}_1^c$ and $\mathbf{u}_2^c$, are needed to provide a basis in the plane perpendicular to the hinge axis. One can then describe the constraint as follows:

$$\mathbf{u}_1^c \cdot \dot{\boldsymbol{\Psi}} = \mathbf{u}_1^c \cdot (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) = 0$$

and

$$\mathbf{u}_2^c \cdot (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) = 0$$

or, in general,

$$\mathbf{u}_i^c \cdot (\boldsymbol{\omega}_\lambda - \boldsymbol{\omega}_\alpha) = 0 \tag{40}$$

The matrix U contains the components of each $\mathbf{u}_i^c$ described in the coordinate system of each body to which it applies and accounts for the sign of the constraint torque shared by that body. A matrix equivalent to Eq. (40) for the entire system can be written as

$$U^T \omega = 0 \tag{41}$$

From a systems or control analyst's point of view, the actual values of the various torques of constraint L_i^c are usually of little or no interest as long as their effects are included correctly in the dynamic model. Between Eqs. (39) and (41) these torques can be eliminated in a straightforward manner (Ref. 1).

Differentiating Eq. (41), one obtains

$$U^T \dot{\omega} + U^T \dot{\omega} = 0 \tag{42}$$

Through the use of Eq. (39),

$$U^T [A^{-1}E + A^{-1}UL^c] = -\dot{U}^T \omega$$

$$L^c = -(U^T A^{-1} U)^{-1} (U^T A^{-1} E + \dot{U}^T \omega)$$

and finally, through substitution back into Eq. (39),

$$A\dot{\omega} = E - U(U^T A^{-1} U)^{-1} (U^T A^{-1} E + \dot{U}^T \omega)$$

or

$$\dot{\omega} = A^{-1}E - A^{-1}U(U^T A^{-1} U)^{-1} (U^T A^{-1} E + \dot{U}^T \omega) \tag{43}$$

Usually, A and $U^T A^{-1} U$ will not be easily inverted except by the aid of a computer. It would then be convenient to allow the machine to solve the pair of matrix equations, Eqs. (39) and (42), numerically (by Gaussian elimination) from the combined form:

$$\left[\begin{array}{c|c} A & U \\ \hline U^T & 0 \end{array}\right] \begin{bmatrix} \dot{\omega} \\ \hline L^c \end{bmatrix} = \begin{bmatrix} E \\ \hline -\dot{U}^T \omega \end{bmatrix} \tag{44}$$

where Eq. (44) is of order $(3n + n_c)$.

Another method, described by Velman (Refs. 3 and 4), for the elimination of constraint torques from the system matrix equations makes use of the pseudoinverse of U^T (Refs. 5 and 6). The pseudoinverse of U^T is defined by

$$U^{T*} = U(U^T U)^{-1}$$

so that $U^T U^{T*} = I =$ identity matrix.

Multiplying Eq. (42) through by U^{T*}, one obtains

$$U^{T*} U^T \dot{\omega} = -U^{T*} \dot{U}^T \omega$$

or

$$G\dot{\omega} = M(t) \tag{45}$$

where

$$G = U^{T*} U^T = U(U^T U)^{-1} U^T, \qquad 3n \times 3n \text{ (symmetric)}$$

$$M = -U^{T*} \dot{U}^T \omega = -U(U^T U)^{-1} \dot{U}^T \omega, \qquad 3n \times 1$$

If the matrix F is defined by

$$G + F = I$$

Then Eq. (39) can be written as

$$A(G+F)\dot{\omega} = E + UL^c$$

Multiplying through by F, one obtains

$$\begin{aligned} FAG\dot{\omega} + FAF\dot{\omega} &= FE + FUL^c \\ &= FE + [I - U(U^T U)^{-1} U^T] UL^c \end{aligned}$$

$$FAM + FAF\dot{\omega} = FE$$

or

$$(FAF)\dot{\omega} = FE - FAM$$

However, although L^c has been removed from the matrix equation, this result is not suitable for computer solution of $\dot{\omega}$ since FAF is singular. But, by adding $G\dot{\omega}$ to each side, this obstacle is overcome:

$$(FAF + G)\dot{\omega} = FE + (I - FA)M \tag{46}$$

or

$$A'\dot{\omega} = E' \tag{47}$$

where $(FAF + G)^{-1}$ exists and Eq. (47) is of order $3n$.

Of course, the product FAF must be computed, and, as Velman points out, unless FAF can be obtained without executing all $2 \cdot (3n)^3$ multiplications, the direct numerical solution of Eq. (46) will not be even as efficient as that of Eq. (44).

Unfortunately, in the barycenter formulation, the F (or G) matrix is not diagonal (since angular velocities are not expressed as relative quantities) and this discourages speedy handling of the FAF product.

As an example of how F appears in the barycenter approach, the simple two-body system of Fig. 6 may be examined. Bodies 1 and 2 are connected by a single-degree-of-freedom hinge.

Arbitrarily one can align the body 1 fixed basis, [**a**], in such a way that the hinge axis is parallel to $\mathbf{a}_x$ and the hinge constraint unit vectors $\mathbf{u}_2^c$ and $\mathbf{u}_1^c$ are in the $\mathbf{a}_z$ and $\mathbf{a}_y$ directions respectively. Then U_1 is given by

$$U_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where column 1 contains the components of $\mathbf{u}_1^c$ modified by the sign assumed for L_1^c on body 1 (positive in this case). Column 2 contains the components of $\mathbf{u}_2^c$ in body 1 and assumes L_2^c is positive for body 1.

Assuming that the body 2 fixed basis [**b**] is identical to [**a**] when $\Psi = 0$, the constraint matrix for body 2 is

$$U_2 = \begin{bmatrix} 0 & 0 \\ -\cos\Psi & \sin\Psi \\ -\sin\Psi & -\cos\Psi \end{bmatrix}$$

where columns 1 and 2 represent the components of $\mathbf{u}_1^c$ and $\mathbf{u}_2^c$ in the body 2 basis modified by the signs of L_1^c and L_2^c for body 2 (which must be negative since they were assumed positive for body 1). Thus the matrix U is constructed from U_1 and U_2 as

$$U = \begin{bmatrix} U_1 \\ \hdashline U_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -\cos\Psi & \sin\Psi \\ -\sin\Psi & -\cos\Psi \end{bmatrix} \tag{48}$$

From the definition for U^{T*} and G one finds that

$$G = U(U^T U)^{-1} U^T = U \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} U^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & -0.5\cos\Psi & -0.5\sin\Psi \\ 0 & 0 & 0.5 & 0 & 0.5\sin\Psi & -0.5\cos\Psi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5\cos\Psi & 0.5\sin\Psi & 0 & 0.5 & 0 \\ 0 & -0.5\sin\Psi & -0.5\cos\Psi & 0 & 0 & 0.5 \end{bmatrix}$$

and, therefore, F is given by

$$F = I - G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5\cos\Psi & 0.5\sin\Psi \\ 0 & 0 & 0.5 & 0 & -0.5\sin\Psi & 0.5\cos\Psi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.5\cos\Psi & -0.5\sin\Psi & 0 & 0.5 & 0 \\ 0 & 0.5\sin\Psi & 0.5\cos\Psi & 0 & 0 & 0.5 \end{bmatrix} \tag{49}$$

(It is easy to verify that $FG = 0$, $FF = F$, $GG = G$, and $GF = 0$.)

While the system matrix equation, Eq. (47), has been freed of the unknown constraint torques, the constraint conditions are necessarily, of course, still embodied in this equation. As a result, n_c of the total $3n$ components of the system's $\dot{\omega}$ column matrix can be expressed as functions of the remaining $3n - n_c$ components. In other words, the system of n rigid bodies possesses only $3n - n_c$ degrees of freedom, and it is these components that are of primary interest to the analyst.

The first step in identifying the free components of angular acceleration (or velocity) is to designate one of the n bodies as the "base" body. The base body will then be assigned three degrees of rotational freedom, by definition. Although any of the bodies could be taken as the base body, it will usually be clear that one particular body of the system should be so designated—perhaps, as in the case of a space vehicle, because it carries certain optical or inertial sensing devices designed to provide a very stable platform for other system parts requiring precise inertial orientations.

With the base body having been chosen, a set of constrained components of angular acceleration must be chosen from the remaining bodies of the system. The number of constrained acceleration components chosen from each body must correspond to the number of constraint conditions imposed at the hinge which connects it either directly or indirectly, by an intermediate chain, to the base body. Since the system is connected in the form of a tree, it will always be possible, even though a body shares several joints with other bodies, to pick that single joint which leads from the body in question to the base body.

Equation (47) can then be rearranged and partitioned in such a way that

$$\left[\begin{array}{c|c} A'_{FF} & A'_{FC} \\ \hline A'_{CF} & A'_{CC} \end{array}\right] \left\{\begin{array}{c} \dot{\omega}_F \\ \hline \dot{\omega}_C \end{array}\right\} = \left\{\begin{array}{c} E'_F \\ \hline E'_C \end{array}\right\} \tag{50}$$

and $\dot{\omega}_F$ can be obtained by a numerical solution to the relation

$$[A'_{FF} - A'_{FC}(A'_{CC})^{-1} A'_{CF}]\, \dot{\omega}_F = E'_F - A'_{FC}(A'_{CC})^{-1} E'_C \tag{51}$$

(provided that $(A'_{CC})^{-1}$ exists) where

$\dot{\omega}_F = n_D \times 1$ column matrix of "free" or unconstrained angular acceleration components

$\dot{\omega}_C = n_c \times 1$ column matrix of constrained angular acceleration components.

$n_D =$ number of system degrees of rotational freedom $= 3n - n_c$

$A'_{FF} = n_D \times n_D$

$A'_{CC} = n_c \times n_c$

The angular velocities ω_C can then be obtained as functions of ω_F from Eq. (41). For example, in the system of Fig. 6, body 1 is chosen as the base body and ω_{2y}, ω_{2z} the components of ω_C. From Eqs. (41) and (48),

$$\omega_{1y} - \cos\Psi\, \omega_{2y} - \sin\Psi\, \omega_{2z} = 0$$

$$\omega_{1z} + \sin\Psi\, \omega_{2y} - \cos\Psi\, \omega_{2z} = 0$$

and

$$\omega_{2y} = \omega_{1y}\cos\Psi - \omega_{1z}\sin\Psi$$

$$\omega_{2z} = \omega_{1z}\cos\Psi + \omega_{1y}\sin\Psi$$

On the other hand, since the product FAF in Eq. (46) is not easily handled (F is not diagonal), much the same approach, i.e., of matrix partitioning, may be applied to Eq. (44) where the constraint vector ω_C is enlarged to include the torques of constraint L^c.

Recently, Hooker (Ref. 7) has described a technique for explicitly eliminating the constraint torques from the system equations, thereby obtaining $3n - n_c$ scalar equations, i.e., the same number as the number of system degrees of freedom. The technique is based on the selection of a base or main body and the description of the other bodies' angular velocities in terms of the main body's ω and relative rotation rates at the joints. By re-expressing Eq. (36) in these terms and summing over all bodies λ ($\lambda = 1, 2, \cdots, n$) one obtains a vector equation in which all components of constraint torque cancel. Additional scalar equations for the *relative* angular accelerations are obtained by selective summations of Eq. (36) and suitable coordinate projections.

This technique is not employed here, although it appears to promise a computational advantage over methods dealing with $3n$ system equations (as in this report). The advantage is not always clear, however, since additional numerical operations are needed to put the system equations in the form described by Hooker.

The method chosen here for the development of a general-purpose subroutine (MLTBDY) capable of solving the dynamical equations for a multi-rigid-body system is based directly on Eq. (43). Figure 7 illustrates the steps

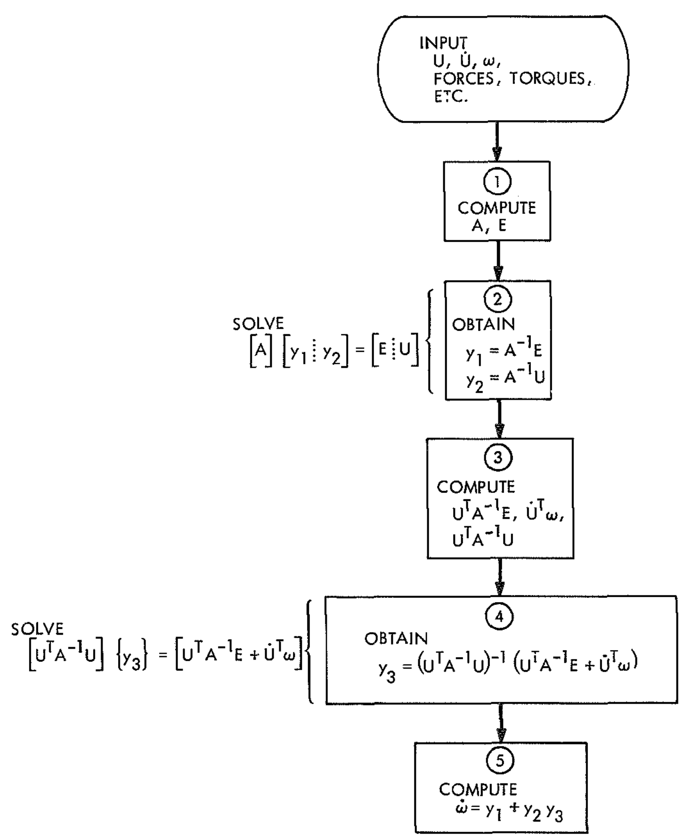

Fig. 7. Procedure for solution of system equation

involved in obtaining the solution for $\dot{\omega}$ components. Following the formation of A and E, step 2, the major time-consuming operation, requires the solution of the matrix equation

$$[A]\,[y_1 \mid y_2] = [E \mid U] \tag{52}$$

The time required to solve the equation above for a given n does not change appreciably when n_c varies (U is $3n \times n_c$). Step 4 requires the solution of a matrix equation of order n_c and, while it constitutes the next greatest time-consuming operation, is relatively fast compared with step 2. Thus the approach taken results in a computation speed consistent with solving a matrix equation of order $3n$.

V. Symmetric Rotors

It is clear from the general equation of rotation, Eq. (36), for a particular body λ, that "acceleration coupling" between λ and the remaining system bodies does not occur provided that $\mathbf{b}_{\lambda\alpha} = 0$ for all $\alpha \neq \lambda$. Under these circumstances, a computational advantage accrues since the order of the system matrix equation is reduced through the decoupling from body λ rotational acceleration terms. While certain coupling effects remain between λ and the rest of the system, they appear only as gyroscopic $(\boldsymbol{\omega} \times I \cdot \boldsymbol{\omega})$ terms or rotational reactions at the connecting joint—terms which enter only into the right-hand side of Eq. (39).

The requirement that $\mathbf{b}_{\lambda\alpha} = 0$ for $\alpha \neq \lambda$ is certainly met for any rigid symmetric rotor whose axis of rotation is fixed to another rigid body of the system. Since the symmetry assures that the rotor's center of mass is on the rotation axis, the location of the "joint" connecting the rotor to the remaining system can arbitrarily be shifted along this axis and placed at the rotor's center of mass. Thus the center of mass and the connecting joint coincide, ensuring that $\mathbf{b}_{\lambda\alpha} = 0$ for all $\alpha \neq \lambda$ (λ is the rotor). Of course the same result is obtained for the idealized case of a rotating sphere attached to the system, the point of rotation being fixed in another rigid body. However, the application to symmetric rotors will be pursued here since such rotors are rather frequently employed in space vehicle configurations.

To develop the explicit relationships between the symmetric rotor and the system it is necessary only to consider the rotor equation and the equation of the body to which the rotor is attached. Assume then that the rotor, labeled body 2, is connected to body 1 in such a way that its axis of rotation is fixed in body 1. From Eq. (36), an equivalent set of matrix equations can be written:

$$\phi_1 \dot{\omega}_1 + \tilde{\omega}_1 \phi_1 \omega_1 = L_1 + L_{12} + \sum_{\beta \neq 1,2} L_{1\beta} \tag{53}$$

$$\phi_2 \dot{\omega}_2 + \tilde{\omega}_2 \phi_2 \omega_2 = L_{21} \tag{54}$$

The vector cross-product $\boldsymbol{\omega} \times \boldsymbol{v}$ is represented by the matrix operation $\tilde{\omega} v$, where

$$\tilde{\omega} = \text{skew symmetric matrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

and

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3]$$

$$v^T = [v_1 \quad v_2 \quad v_3]$$

It is assumed in Eqs. (53) and (54) that external forces are not applied to body 1 or 2 but that body 1 may be connected to other rigid bodies. Thus the term

$$\sum_{\beta \neq 1,2} L_{1\beta}$$

is retained for generality. Also, an external torque term is supplied to body 1 while none is assumed to be applied to the rotor. The usual magnetic torquing of a rotor is included as part of the hinge reaction torque $\mathbf{L}_{12}\,(= -\mathbf{L}_{21})$ since it is inherently internal. Of course, any rotor bearing friction or damping characteristic will be a part of $\mathbf{L}_{12}$, as well as any constraint torque.

If bodies 1 and 2 were the only bodies in the system, Eqs. (53) and (54) could be combined in the form

$$\left[\begin{array}{c|c|c} \phi_1 & 0 & U_1 \\ \hline 0 & \phi_2 & U_2 \\ \hline U_1^T & U_2^T & 0 \end{array}\right] \left\{\begin{array}{c} \dot{\omega}_1 \\ \hline \dot{\omega}_2 \\ \hline L^c \end{array}\right\} = \left\{\begin{array}{c} -\tilde{\omega}_1 \phi_1 \omega_1 + L_1 + L_{12}^B \\ \hline -\tilde{\omega}_2 \phi_2 \omega_2 + L_{21}^B \\ \hline -\dot{U}_1^T \omega_1 - \dot{U}_2^T \omega_2 \end{array}\right\} \tag{55}$$

where

$L_{12}^B = -L_{21}^B =$ reaction torques about the rotor bearing axis (in general, 3×1).

Equation (55), in fact, is somewhat more complicated than it need be, since two components of $\boldsymbol{\omega}_2$ are constrained and can be written as explicit functions of the components of $\boldsymbol{\omega}_1$.

An approach to the solution of Eqs. (53) and (54) that results in a desirable simplification will now be pursued. It involves, first, the decomposition of $\boldsymbol{\omega}_2$ into a sum of $\boldsymbol{\omega}_1$ and a relative rotation rate of the rotor with respect to body 1. That is,

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \Omega\,\mathbf{r}$$

where $\mathbf{r}$ is a unit vector directed along the rotor's axis of rotation and Ω is the scalar rate of *relative* rotation.

Also,

$$\boldsymbol{\omega}_2 = [\mathbf{u}_2]^T \omega_2$$

where

$[\mathbf{u}_2]^T = [\mathbf{u}_{2x}\mathbf{u}_{2y}\mathbf{u}_{2z}]$ = matrix of unit vectors along orthogonal coordinate axes fixed in body 2 (rotor)

and

$\omega_2 = [\omega_{2x}\omega_{2y}\omega_{2z}]^T$ = matrix of $\boldsymbol{\omega}_2$ components in the body-2-fixed basis

A coordinate transformation matrix T may be defined in such a way that

$$[\mathbf{u}_2] = T^{12}[\mathbf{u}_1]$$

Therefore

$$\boldsymbol{\omega}_2 = [\mathbf{u}_1]^T T^{21}\omega_2 = [\mathbf{u}_1]^T \omega_1 + [\mathbf{u}_1]^T \Omega r$$

since

$$\boldsymbol{\omega}_1 = [\mathbf{u}_1]\,\omega_1$$
$$\mathbf{r} = [\mathbf{u}_1]\,r$$

and

$$\omega_2 = T^{12}(\omega_1 + \Omega r), \qquad (T^{21} = (T^{12})^T = (T^{12})^{-1}$$
$$\dot{\omega}_2 = T^{12}(\dot{\omega}_1 + \dot{\Omega} r) + \dot{T}^{12}(\omega_1 + \Omega r)$$

It can be shown that

$$\Omega\tilde{r} = -(T^{12})^T\dot{T}^{12}$$

Therefore,

$$\dot{\omega}_2 = T^{12}(\dot{\omega}_1 + \dot{\Omega} r - \Omega\tilde{r}\omega_1 - \Omega\tilde{r}\Omega r)$$

But $\Omega^2\tilde{r}r$ is zero since it is a cross product of $\mathbf{r}$ with itself. Therefore,

$$\dot{\omega}_2 = T^{12}(\dot{\omega}_1 + \dot{\Omega} r - \Omega\tilde{r}\omega_1)$$

Expanding Eq. (54), one obtains

$$\phi_2 T^{12}(\dot{\omega}_1 + \dot{\Omega} r - \Omega\tilde{r}\omega_1) + (\widetilde{T^{12}\omega_1} + \widetilde{T^{12}\Omega r})\,\phi_2 T^{12}(\omega_1 + \Omega r) = L_{21}$$

If the equation is multiplied through by T^{21} (transformation to the $[\mathbf{u}_1]$ basis) the following equation results:

$$T^{21}\phi_2 T^{12}(\dot{\omega}_1 + \dot{\Omega} r - \Omega\tilde{r}\omega_1) + T^{21}(\widetilde{T^{12}\omega_1} + \widetilde{T^{12}\Omega r})\,\phi_2 T^{12}(\omega_1 + \Omega r) = T^{21}L_{21} \tag{56}$$

The term $T^{21}\phi_2 T^{12}$ can be shown to be the inertia matrix of the rotor in the body 1 basis and will be designated as I. Thus, the terms in Eq. (56) can be evaluated one by one as

$$T^{21}\phi_2 T^{12} = I$$
$$T^{21}(\widetilde{T^{12}\omega_1})\,\phi_2 T^{12}\omega_1 = T^{21}(\widetilde{T^{12}\omega_1})\,T^{12}(T^{21}\phi_2 T^{12})\,\omega_1 = \tilde{\omega}_1 I\omega_1$$
$$T^{21}(\widetilde{T^{12}\Omega r})\,\phi_2 T^{12}\omega_1 = T^{21}(\widetilde{T^{12}\Omega r})\,T^{12}(T^{21}\phi_2 T^{12})\,\omega_1 = \Omega\tilde{r}I\omega_1$$
$$T^{21}(\widetilde{T^{12}\omega_1})\,\phi_2 T^{12}\Omega r = \tilde{\omega}_1 I\Omega r$$
$$T^{21}(\widetilde{T^{12}\Omega r})\,\phi_2 T^{12}\Omega r = \Omega^2\tilde{r}Ir = 0$$

Then Eq. (56) becomes

$$I(\dot{\omega}_1 + \dot{\Omega} r - \Omega\tilde{r}\omega_1) + \tilde{\omega}_1 I\omega_1 + \Omega\tilde{r}I\omega_1 + \tilde{\omega}_1 I\Omega r = T^{21}L_{21} \tag{57}$$

The statement

$$\tilde{r}Ir = 0$$

holds since ϕ_2 has been defined (in the $[\mathbf{u}_2]$ basis) as

$$\phi_2 = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ 0 & 0 & I_S \end{bmatrix}, \qquad \begin{aligned} I_S &= \text{rotor spin moment of inertia} \\ I_T &= \text{rotor transverse moment of inertia} \end{aligned}$$

and

$$\mathbf{r} = [\mathbf{u}_2]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\mathbf{u}_1]^T T^{21} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\mathbf{u}_1]^T r$$

For the same reason, it can be verified that

$$\tilde{r}I - I\tilde{r} = 0$$

and Eq. (57) is further simplified to

$$I\dot{\omega}_1 + \tilde{\omega}_1 I\omega_1 + I\dot{\Omega}r - (\widetilde{I\Omega r})\,\omega_1 = T^{21}L_{21}$$

Since

$$Ir = T^{21}\phi_2 T^{12} T^{21} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = I_S r$$

then

$$I\dot{\omega}_1 + \tilde{\omega}_1 I\omega_1 + \dot{h}r - h\tilde{r}\omega_1 = T^{21}L_{21} \tag{58}$$

where $h = I_S\Omega$ = relative angular momentum of the rotor in body 1 (scalar).

Equation (58) may now be added to Eq. (53), since both are expressed in the $[\mathbf{u}_1]$ basis. The result is

$$(\phi_1 + I)\,\dot{\omega}_1 + \tilde{\omega}_1(\phi_1 + I)\,\omega_1 + \dot{h}r - h\tilde{r}\omega_1 = L_1 + \sum_{\beta \neq 1,2} L_{1\beta} \tag{59}$$

Notice that the hinge reaction torques $\mathbf{L}_{12} = -\mathbf{L}_{21}$ cancel in Eq. (59). The additional equation required for solution of the system is the scalar differential equation obtained from Eq. (54) by multiplying both sides by e_3^T (equivalent to a dot multiplication of the original vector equation by $\mathbf{u}_{2z}$) is

$$\mathbf{u}_{2z} = [\mathbf{u}_2]^T e_3 = [\mathbf{u}_2]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e_3^T\phi_2\dot{\omega}_2 + e_3^T\tilde{\omega}_2\phi_2\omega_2 = e_3^T L_{21}$$

Therefore,

$$I_S\dot{\omega}_{2z} = L_{21z} \tag{60}$$

Since

$$\begin{aligned}
\boldsymbol{\omega}_2 &= \boldsymbol{\omega}_1 + \Omega\mathbf{r} \\
\dot{\omega}_2 &= T^{12}(\dot{\omega}_1 + \dot{\Omega}r - \Omega\tilde{r}\omega_1) \\
\dot{\omega}_{2z} &= e_3^T\dot{\omega}_2 = e_3^T T^{12}(\dot{\omega}_1 + \dot{\Omega}r - \Omega\tilde{r}\omega_1) \\
r &= T^{21}e_3 \\
\dot{\omega}_{2z} &= r^T\dot{\omega}_1 + r^T\dot{\Omega}r - r^T\Omega\tilde{r}\omega_1 \\
&= r^T\dot{\omega}_1 + \dot{\Omega}, \qquad \text{since } r^T\tilde{r} = 0 \\
I_S r^T\dot{\omega}_1 + \dot{h} &= L_{21z} \\
\dot{h} &= L_{21z} - I_S r^T\dot{\omega}_1
\end{aligned}$$

or

$$r\dot{h} = \dot{h}r = rL_{21z} - rI_S r^T\dot{\omega}_1 \tag{61}$$

Also,

$$
\begin{aligned}
\omega_2 &= T^{12}(\omega_1 + \Omega r) \\
\Omega r &= T^{21}\omega_2 - \omega_1 \\
\Omega &= r^T T^{21}\omega_2 - r^T\omega_1 \\
h\widetilde{r}\omega_1 &= I_S(r^T T^{21}\omega_2 - r^T\omega_1)\widetilde{r}\omega_1 \\
&= -\widetilde{\omega}_1 r I_S r^T T^{21}\omega_2 + \widetilde{\omega}_1 r I_S r^T\omega_1
\end{aligned}
\tag{62}
$$

Through the use of Eqs. (61) and (62), Eq. (59) may be re-expressed as

$$
(\phi_1 + I - I_{z2})\,\dot{\omega}_1 + \widetilde{\omega}_1(\phi_1 + I - I_{z2})\,\omega_1 + \widetilde{\omega}_1 I_{z2}(\omega_1 + \Omega r) = L_1 + \sum_{\beta \neq 1,2} L_{1\beta} - rL_{21z} \tag{63}
$$

where

$$
I_{z2} = rI_S r^T = T^{21} e_3 I_S e_3^T T^{12}
$$

It is clear that, since $I = T^{21}\phi_2 T^{12}$, the operation $I - I_{z2}$ simply removes from I those terms involving I_S, the rotor spin axis moment of inertia. The value of $\Omega(t)$ to be placed in Eq. (63) is obtained from Eq. (61):

$$
\Omega = \int_0^t \frac{L_{21z}}{I_S}\,dt - r^T[\omega_1 - \omega_{10}] + \Omega_0 \tag{64}
$$

$$
\omega_{10} = \omega_1(0), \qquad \Omega_0 = \Omega(0)
$$

As a result of all the manipulation, the presence of a symmetric rotor attached to body 1 does not in any way change the order of the matrix equation that must be solved to obtain the system's unconstrained rotational acceleration components. The only additions are the integration shown in Eq. (64), modification of the ϕ_1 inertia matrix to reflect the rotor's x and y moments of inertia, the gyroscopic torque term shown in Eq. (63), and the bearing reaction torque L_{21z}.

VI. Derivative Evaluation Subroutines

The practical application of the barycenter formulation will, in most cases, require the use of machine computation to solve the resulting system of differential equations. While, to some extent, each dynamical system to be studied has certain unique characteristics, it is possible to systematize, to a large degree, the form of the equations and to derive general purpose algorithms to aid in their solution. It is for this reason that FORTRAN subroutines MLTBDY, MLTBDL, and MLTBD were written. These are intended to relieve the analyst of the drudgery involved in programming the matrix manipulations, coordinate transformations, matrix inversions, etc., necessary to implement the general system of equations discussed in previous sections.

The primary output of these routines is the solution for $\dot{\omega}_F$, the matrix vector of unconstrained rotational accelerations, although the $\dot{\omega}_C$ elements may also be obtained directly if desired.

A. Subroutine MLTBDY

The subroutine MLTBDY was developed to obtain the solution to the complete system of equations as embodied in Eq. (36). Two computational approaches to the problem were considered. The first of these directly implements matrix equation (43) which, it should be remembered, represents the system with constraint torques eliminated by direct substitution (Ref. 1). An alternative approach was to deal with the combined system matrix (Eq. 44) but to partition this to obtain only the $\dot{\omega}_F$ components and thereby avoid solving a matrix equation of order $(3n + n_c)$. Partitioning requires the solution of two smaller-order matrix equations with some time saving.

Both approaches were programmed and tested for accuracy and speed with the first method showing up slightly faster than the partitioning of Eq. (44). Since the solution of Eq. (43) also produces all components of $\dot{\omega}$, including the constrained group, this approach was chosen as the best general-purpose method, allowing the analyst either to take the easy way out and obtain all $\dot{\omega}$ components from the program or, if he is so disposed, to obtain $\dot{\omega}_C$ by writing out the relations to $\dot{\omega}_F$ explicitly and solving these algebraic equations. Of course, both ways of getting $\dot{\omega}_C$ could be used just as a check on the program. Figure 7 charts the numerical steps taken to obtain $\dot{\omega}$ by using the form of Eq. (43).

1. Example: Spacecraft and scanning platform. To illustrate the use of MLTBDY, a simple example of some interest from an attitude control standpoint will be examined, namely a spacecraft of the *Mariner* type carrying a sizable instrument platform capable of mechanical articulation. This particular problem may be of interest from a number of aspects, chief among these (1) the question of spacecraft attitude-control gas consumption (if mass expulsion is used) for a particular platform scan sequence and (2) the question of platform pointing accuracy as well as spacecraft attitude error (or, indirectly, antenna pointing error) resulting from platform scanning activity.

In the 2-body system, the spacecraft and platform are assumed to share a connecting joint, which, in this case, will be taken as a single-degree-of-freedom joint or hinge. However, provision is made for orienting the hinge axis in any direction. Figure 8 illustrates the location of the system's body-fixed frames, hinge axis, and angular references. Coordinate transformations may be derived as shown below.

Let angle γ rotate the hinge axis h anywhere in the x_1–y_1 plane, and assume that y_2 is always parallel to h. Also assume that x_2 is parallel to x_1 when $\gamma = 0$ and when $\alpha = 0$. The position of body 2 relative to body 1 then may be described by two successive rotations, γ and hinge angle α.

Thus, for rotation γ,

$$\mathbf{i}_2' = \cos\gamma\,\mathbf{i}_1 + \sin\gamma\,\mathbf{j}_1$$

$$\mathbf{j}_2' = -\sin\gamma\,\mathbf{i}_1 + \cos\gamma\,\mathbf{j}_1$$

$$\mathbf{k}_2' = \mathbf{k}_1$$

where $\mathbf{i}_2'$–$\mathbf{j}_2'$–$\mathbf{k}_2'$ are unit vectors along x_2–y_2–z_2 after rotation γ.

For rotation α,

$$\mathbf{i}_2 = \cos\alpha\,\mathbf{i}_2' + \sin\alpha\,\mathbf{k}_2'$$

$$\mathbf{j}_2 = \mathbf{j}_2'$$

$$\mathbf{k}_2 = -\sin\alpha\,\mathbf{i}_2' + \cos\alpha\,\mathbf{k}_2'$$

Therefore,

$$\begin{Bmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha\cos\gamma & \cos\alpha\sin\gamma & \sin\alpha \\ -\sin\gamma & \cos\gamma & 0 \\ -\sin\alpha\cos\gamma & -\sin\alpha\sin\gamma & \cos\alpha \end{bmatrix} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} \tag{65}$$

where $\mathbf{i}_2$–$\mathbf{j}_2$–$\mathbf{k}_2$ are directed along the final position of x_2–y_2–z_2.

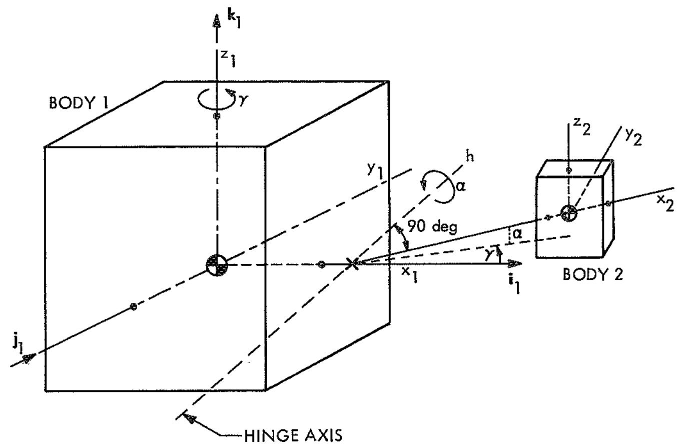

Fig. 8. Two-body system coordinates and hinge location

The assumption that y_2 is always parallel to the hinge axis means that the y_2 component of body 2's angular velocity $\boldsymbol{\omega}_2$ is a free component while the x_2 and z_2 components of $\boldsymbol{\omega}_2$ are constrained to be functions of $\boldsymbol{\omega}_1$. The constraint relations may be written in terms of angular velocity components from Eq. (65):

$$\boldsymbol{\omega}_2 \cdot \mathbf{i}_2 = \omega_{2x} = \cos\alpha\,[\cos\gamma\,\omega_{1x} + \sin\gamma\,\omega_{1y}] + \sin\alpha\,\omega_{1z} \tag{66}$$

$$\boldsymbol{\omega}_2 \cdot \mathbf{k}_2 = \omega_{2z} = -\sin\alpha\,[\cos\gamma\,\omega_{1x} + \sin\gamma\,\omega_{1y}] + \cos\alpha\,\omega_{1z} \tag{67}$$

where

$$\omega_{1x} = \boldsymbol{\omega}_1 \cdot \mathbf{i}_1$$

$$\omega_{1y} = \boldsymbol{\omega}_1 \cdot \mathbf{j}_1$$

$$\omega_{1z} = \boldsymbol{\omega}_1 \cdot \mathbf{k}_1$$

Also, the hinge angle rate of change $\dot{\alpha}$ is given by

$$\dot{\alpha} = -\omega_{2y} - (\omega_{1x}\sin\gamma - \omega_{1y}\cos\gamma) \tag{68}$$

The subroutine MLTBDY of course requires that the location of any axes of constraint associated with connecting joints of the system be specified. In this case, since only one degree of freedom is supplied by the joint, two mutually orthogonal axes of constraint must be chosen. Arbitrarily, one can assume these constraint axes are fixed in body 1. Let

$\mathbf{u}_1 = \cos\gamma\,\mathbf{i}_1 + \sin\gamma\,\mathbf{j}_1$ = first constraint axis direction

$\mathbf{u}_2 = \mathbf{k}_1$ = second constraint axis direction

Vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors perpendicular to each other and to a unit vector along the free axis of rotation, $\mathbf{j}_2$. In terms of body 2 coordinates,

$$\mathbf{u}_1 = \cos\alpha\,\mathbf{i}_2 - \sin\alpha\,\mathbf{k}_2$$

$$\mathbf{u}_2 = \sin\alpha\,\mathbf{i}_2 + \cos\alpha\,\mathbf{k}_2$$

A matrix U is used by MLTBDY to obtain the unit vector components of system constraint torques. These components are given in the coordinate frames of those bodies that experience constraint torques at joints that connect them to other system bodies. The matrix U is composed of submatrices U_i, where U_i is a $3 \times n_c$ matrix of body i components of constraint torque unit vectors for each of the n_c total system constraints. If the kth constraint, for example, does not apply to body i, since it occurs at a joint not shared by i, the kth column of U_i has zero elements. In this example,

$$U_1 = \begin{bmatrix} \cos\gamma & 0 \\ \sin\gamma & 0 \\ 0 & 1 \end{bmatrix} \qquad U_2 = \begin{bmatrix} -\cos\alpha & -\sin\alpha \\ 0 & 0 \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

Note that in the first column of U_1 are the components of $\mathbf{u}_1$, constraint axis 1, since the constraint torque is exerted about that axis. The torque is also arbitrarily assumed to be positive in the direction $\mathbf{u}_1$ for body 1. Similarly, in the second column of U_1 are the body 1 components of $\mathbf{u}_2$, with the corresponding constraint torque on body 1 assumed positive in that direction.

Columns of U_2 contain the body 2 components of $\mathbf{u}_1$ and $\mathbf{u}_2$ but with a negative sign affixed to each to account for the fact that the associated constraint torques must be in the opposite direction for body 2.

Matrix U is formed from the U_i matrices by

$$U_1 = \begin{bmatrix} U_1 \\ \hline U_2 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline U_n \end{bmatrix}, \qquad n = \text{number of system bodies}$$

In this case, $U = \begin{bmatrix} U_1 \\ \hline U_2 \end{bmatrix}, \qquad (6 \times 2)$

$$U = \begin{bmatrix} \cos\gamma & 0 \\ \sin\gamma & 0 \\ 0 & 1 \\ -\cos\alpha & -\sin\alpha \\ 0 & 0 \\ \sin\alpha & -\cos\alpha \end{bmatrix} \tag{69}$$

The subroutine must also be supplied the time derivative dU/dt,

$$\frac{dU}{dt} = \dot{U} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dot{\alpha}\sin\alpha & -\dot{\alpha}\cos\alpha \\ 0 & 0 \\ \dot{\alpha}\cos\alpha & \dot{\alpha}\sin\alpha \end{bmatrix}, \qquad (\gamma = \text{constant}) \tag{70}$$

It remains then, insofar as the characteristics of the system connecting joints are concerned, to describe the nature of any reaction torques acting about the hinge axes. For the single hinge axis present here, a simple spring-damper type connection will be assumed. Since y_2 is always parallel to the hinge axis, the total hinge reaction torque T_H on body 2 will be

$$T_H = \mathbf{T}_H \cdot \mathbf{j}_2 = -K_S(\alpha_C - \alpha) + D_S\dot{\alpha}$$

where

α_C = commanded hinge angle

K_S = hinge spring constant

D_S = hinge viscous damping coefficient

The components of $\mathbf{T}_H$ in body 1 are obtained from Eq. (65):

$$\mathbf{T}_H = T_H\mathbf{j}_2,$$

$$\mathbf{T}_H = \sin\gamma\, T_H\mathbf{i}_1 - \cos\gamma\, T_H\mathbf{j}_1$$

Finally, it is necessary only to add the spacecraft's (body 1's) control system which, as mentioned earlier, is that of the *Mariner* Mars 1969 series. Figure 9 presents

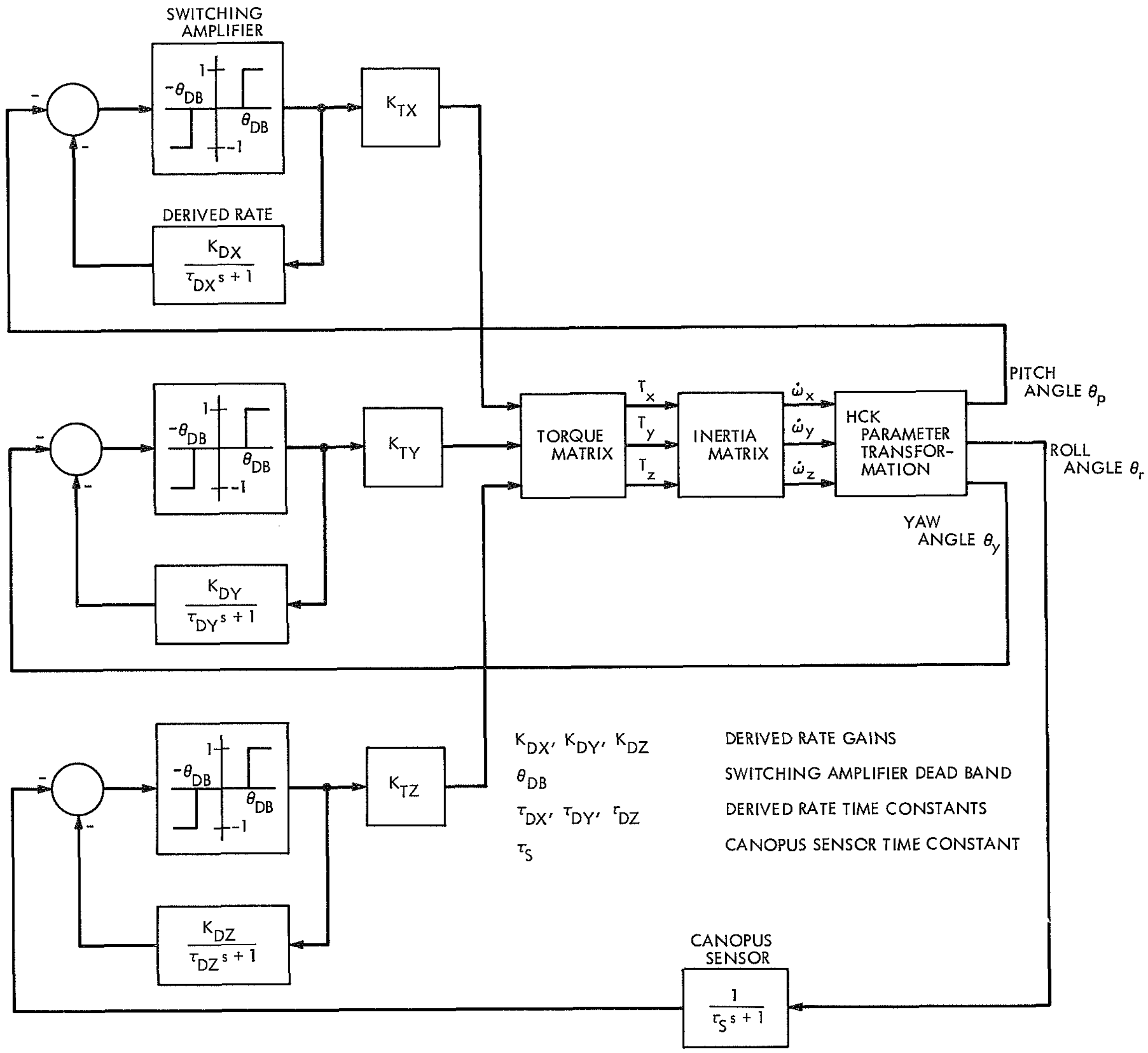

Fig. 9. Spacecraft cruise attitude control system

a block diagram of the attitude control system's cruise configuration. Pitch, yaw, and roll switching amplifier outputs are normalized to ± 1. Torques about the craft's x, y, and z axes are assumed to be the result of pure couples; i.e., no net forces are applied to body 1 by the gas jets. Derived rate feedback in each axis is characterized by both a "charge" (switching amplifier on) and a "discharge" (switching amplifier off) time constant of approximately 10 and 20 sec respectively.

2. ***Subroutine MLTBDY call statements.*** To summarize, information that must be supplied to MLTBDY includes:

(1) The number of rigid bodies in the system

(2) The mass of each body

(3) The inertia matrix of each body (with respect to a coordinate frame fixed in the body and origin at its mass center)

(4) The location of each connecting joint (in body-fixed coordinates)

(5) Body-to-body coordinate transformation matrices

(6) A description of the nature of each connecting joint (location of rotating axes and constrained axes, expressions for restoring torque, damping, friction, etc.)

(7) The angular velocity components of each body

(8) External torques and forces applied to each body (in body-fixed coordinates)

Access can be made to MLTBDY by either of two entry points. The first of these is by way of the following call statement:

CALL MLTBDY (NB, N3, NC, BMASS, IXX, IYY, IZZ, IXY, IXZ, IYZ, CIX, CIY ,CIZ)

where

NB = number of rigid bodies in the system (integer)

N3 = 3 × NB (integer)

NC = number of constraint conditions on the system (integer)

BMASS = array of rigid body mass values starting with body 1

IXX, IYY, IZZ, IXY, IXZ, IYZ = arrays of rigid body inertias starting with body 1

CIX, CIY, CIZ = arrays of vector components locating system connecting joints in each body-fixed coordinate frame

The terms NB, N3, and NC must be declared as integers in the simulation main program. The arguments of MLTBDY should be dimensioned as follows:

DIMENSION BMASS(NB), IXX(NB), IYY(NB), IZZ(NB), IXY(NB), IXZ(NB), IYZ(NB), CIX(11 × NB-1), CIY(11 × NB-1), CIZ(11 × NB-1)

To illustrate, for this two-body example,

INTEGERS: NB = 2, N3 = 6, NC = 2

DIMENSIONED VARIABLES: BMASS(2), IXX(2), IYY(2), IZZ(2), IXY(2), IXZ(2), IYZ(2), CIX(21), CIY(21), CIZ(21)

The terms CIX, CIY, and CIZ are the components of vectors $\mathbf{c}_{ij}$ discussed in Section II. If, for example, a vector $\mathbf{c}_{12}$ were drawn from the center of mass of body 1 to the joint on body 1, leading from body 1 to body 2, and its x-y-z components in the body 1 frame were (−1.7, 5.6, 0.23), then

CIX(12) = −1.7

CIY(12) = 5.6

CIZ(12) = 0.23

Also, if a vector $\mathbf{c}_{21}$, drawn from body 2's mass center to the joint on body 2 and leading from body 2 to body 1, had components (0.85, −4.1, 7.3) in the body 2 frame, then

CIX(21) = 0.85

CIY(21) = −4.1

CIZ(21) = 7.3

It is important to remember also that in a system of more than two bodies, a joint on body 2, which directly connects body 2 to body 3, for example, may also indirectly connect body 2 to bodies 6 and 7, so that, by definition,

$$\mathbf{c}_{23} = \mathbf{c}_{26} = \mathbf{c}_{27}$$

or

CIX(23) = CIX(26) = CIX(27)

CIY(23) = CIY(26) = CIY(27)

CIZ(23) = CIZ(26) = CIZ(27)

All of these values must be input to MLTBDY even though many are redundant. Note that the subscript of CIX, CIY, or CIZ is of the form CIX(ij) and not CIX(i, j). The use of the latter, doubly subscripted form would have minimized storage space, but the coded single subscript form is required if these values are to be supplied conveniently to a DSL/90 Simulation Language program via the TABLE card. To repeat, all components CIX, CIY, and CIZ *must* be input for all combinations of $i = 1, 2, \cdots$, NB and $j = 1, 2, \cdots$, NB *except* for $i = j$.

The execution of the statement CALL MLTBDY (NB, N3, NC, BMASS, . . .) initializes the subroutine with the physical constants of the system and need be done only once. The subroutine is then ready for subsequent calls with *variable* information and the computation of the system angular accelerations. This is accomplished with the following statement:

CALL MLTRAT (NB, N3, NC, TX, TY, TZ, FX, FY, FZ, U, UD, T, WX, WY, WZ, WDOT)

where

NB, N3, NC = integers as previously defined

TX, TY, TZ = arrays of torque components applied to each system body (including torque about the hinge axis)

FX, FY, FZ = arrays of force components externally applied to each body

U = array of constraint torque unit vector components as defined above

UD = time derivative of the matrix U

T = array of body-to-body coordinate transformation matrices

WX, WY, WZ = arrays of body angular velocity components

WDOT = array of body angular acceleration components

The arguments of MLTRAT should be dimensioned in the main simulation program as follows:

DIMENSION TX(NB), TY(NB), TZ(NB), FX(NB), FY(NB), FZ(NB), U(3 × NB, NC), UD(3 × NB, NC), T(NB, NB, 3, 3), WX(NB, NB), WY(NB, NB), WZ(NB, NB)

DOUBLE PRECISION WDOT(19)

For this particular example, U and UD have already been defined in terms of the angles γ and α. The terms FX, FY, and FZ are zero. Applied torques TX, TY, and TZ are given by

$$\left.\begin{aligned} TX(1) &= K_{TX}AMP_X + \sin\gamma \\ &\qquad \times [-K_S(\alpha_c - \alpha) + D_S\dot{\alpha}] \\ TY(1) &= K_{TY}AMP_Y - \cos\gamma \\ &\qquad \times [-K_S(\alpha_c - \alpha) + D_S\dot{\alpha}] \\ TZ(1) &= K_{TZ}AMP_Z \end{aligned}\right\} \begin{array}{l}\text{Body 1} \\ \text{applied torques} \\ +\text{ hinge torque}\end{array}$$

$$\left.\begin{aligned} TX(2) &= 0 \\ TY(2) &= -K_S(\alpha_c - \alpha) + D_S\dot{\alpha} \\ TZ(2) &= 0 \end{aligned}\right\} \begin{array}{l}\text{Body 2} \\ \text{applied torques} \\ +\text{ hinge torque}\end{array}$$

where AMP_X, AMP_Y, and AMP_Z are the outputs of the pitch, yaw, and roll switching amplifiers and K_{TX}, K_{TY}, and K_{TZ} are constants of proportionality.

Coordinate transformation matrices $T(m, n, i, j)$ are defined as follows. The subscript m refers to that body frame in which a particular coordinate is *presently* described, and n refers to the body frame in which the coordinate is *to be* described after the transformation. Subscripts i and j ($i = 1, 2, 3$ and $j = 1, 2, 3$) refer to the nine elements of the three-dimensional transformation matrix. Thus, a vector **v**, whose components in body 1 are (v_{x1}, v_{y1}, v_{z1}), can be described in body 2 by:

$$\begin{Bmatrix} v_{x2} \\ v_{y2} \\ v_{z2} \end{Bmatrix} = T(1, 2, i, j) \begin{Bmatrix} v_{x1} \\ v_{y1} \\ v_{z1} \end{Bmatrix}$$

where

$$T(1, 2, i, j) = [B_{ij}], \qquad \begin{aligned} i &= 1, 2, 3 \\ j &= 1, 2, 3 \end{aligned}$$

For this problem,

$$T(1, 2, i, j) = \begin{bmatrix} \cos\alpha\cos\gamma & \cos\alpha\sin\gamma & \sin\alpha \\ -\sin\gamma & \cos\gamma & 0 \\ -\sin\alpha\cos\gamma & -\sin\alpha\sin\gamma & \cos\alpha \end{bmatrix}$$

and $T(2, 1, i, j) = T(1, 2, j, i)$.

The array T must be defined in the program for every combination of m and n *except* $m \geqq n$, where m (or n) ranges from 1 to NB.

The angular velocity components, WX, WY, and WZ, are carried as doubly subscripted arrays in the program MLTBDY as well as the main calling program. By definition, WX(i, j) refers to the x_j-axis component of body i's angular velocity transformed into the body j coordinate frame. Note that in DSL/90 it will be necessary, if a subscripted variable is to be printed or plotted using the standard DSL/90 PRINT, PREPAR, and GRAPH statements, to redefine these variables in terms of nonsubscripted variables.

WDOT, the output array of MLTRAT, is a double-precision vector array (single subscript) whose elements are the body angular acceleration components given in the following order:

$$WDOT\,(1) = \dot{\omega}_{x1}$$
$$WDOT\,(2) = \dot{\omega}_{y1}$$
$$WDOT\,(3) = \dot{\omega}_{z1}$$
$$WDOT\,(4) = \dot{\omega}_{x2}$$
$$WDOT\,(5) = \dot{\omega}_{y2}$$
$$\vdots$$
$$WDOT\,(N3) = \dot{\omega}_{z\,(NB)}$$

3. Spacecraft–scan platform simulation program. A listing of the DSL/90 program used to simulate platform scanning effects on a *Mariner* Mars 1969 type of cruise attitude control system is given in Appendix B. The entire program is executed under the NOSORT option, which is necessary in DSL/90 if subscripted variables are to appear on the left side of any "equals" sign. The starting IF statement performs those operations that are only required once, i.e., calling MLTBDY, initializing the Hamilton–Cayley–Klein (HCK) parameters of body 1 (spacecraft), and finding the sine and cosine of the fixed angle γ (GA).

Sine and cosine of α (AL) are computed next, in the section of the main program which is executed at every integration step. The constrained components of platform angular velocity may now be computed, as in Eqs. (66) and (67), followed by a definition of the subscripted variables needed by MLTRAT.

Through the use of the HCK package* the inertial Z (sun line) and X axes are transformed to the spacecraft body-fixed frame by ITOB. Pitch, yaw, and roll angles of the craft may be calculated as shown. Body-to-body coordinate transformations, as developed in Eq. (65), are then calculated, as well as the hinge angle rate of change $\dot{\alpha}$ from Eq. (68).

The next section embodies the attitude control system dynamics using the SWAMP (switching amplifier with minimum-on-time) block available on the DSL/90 system tape. The attitude-control system applied torques are directly proportional to the switching amplifier outputs.

The commanded hinge angle α_c (AC) is derived using the Fortran IV AMOD function, a switch, and an integrator. The result is a sawtooth function of time with maximum and minimum values of +10 and 0 deg respectively and a period of 20 sec. This is obtained by driving the integrator with a $\pm 1°$/sec (± 0.01745 rad/sec) rate $\dot{\alpha}_c$. Hinge axis torque developed through the spring–damper system is given next and is added to gas-jet torque for transmission to MLTRAT.

Remaining are the definitions of the constraint matrix U and $\dot{U}$ (UDOT), given in Eqs. (69) and (70). The call to MLTRAT may now be executed, producing WDOT. Integration of the appropriate WDOT elements results in the needed angular velocity components of the spacecraft and platform. Finally, a call to HCK computes the spacecraft HCK parameter rates of change, which are subsequently integrated.

System parameters and initial conditions follow in the listing. Notice that the platform has a mass of 1 slug and principal moments of inertia of 7., 5., and 10. slug-ft². The hinge location is assumed to be 1 ft from the spacecraft mass center along its x axis and 0.5 ft from the platform mass center on its $-x$ axis. Angle $\gamma = 45$ deg (0.7854 rad), the spacecraft is initially at rest, and the craft is so positioned that its initial pitch, yaw, and roll angles are slightly less than the gas jet deadband value of 4.3 mr. The polarity of the pitch, yaw, and roll angles at $t = 0$ are such that the start-up of platform scanning motion will almost immediately drive all angles out of the deadband and turn all jets on.

4. Spacecraft–scan platform simulation results. Figures 10–16 picture the results of the simulation for a platform sawtooth scanning sequence of 90 sec duration. Although the responses are largely self-explanatory, note that pitch and yaw angle responses are quite similar (except for opposite polarity) since γ was deliberately chosen (45 deg) to couple scan motion equally into the two axes. Roll, of course, is only very slightly disturbed by the scan motion. Scan reversal is clearly visible in the plots of spacecraft angular velocities, along with gas jet pulsing by the derived rate feedback. For the simple system presented here, fuel consumption can easily be obtained by integrating applied gas jet torque, or equally as straightforward would be the description of the platform pointing vector in inertial space to obtain pointing error. The addition of a few arithmetic statements (including one integrator) can add a spinning rotor to body 1 for an examination of the effect of spin-stabilization, still with basically a two-body system as far as MLTBDY is concerned.

*Kopf, E. H., JPL internal document, Oct. 24, 1966.

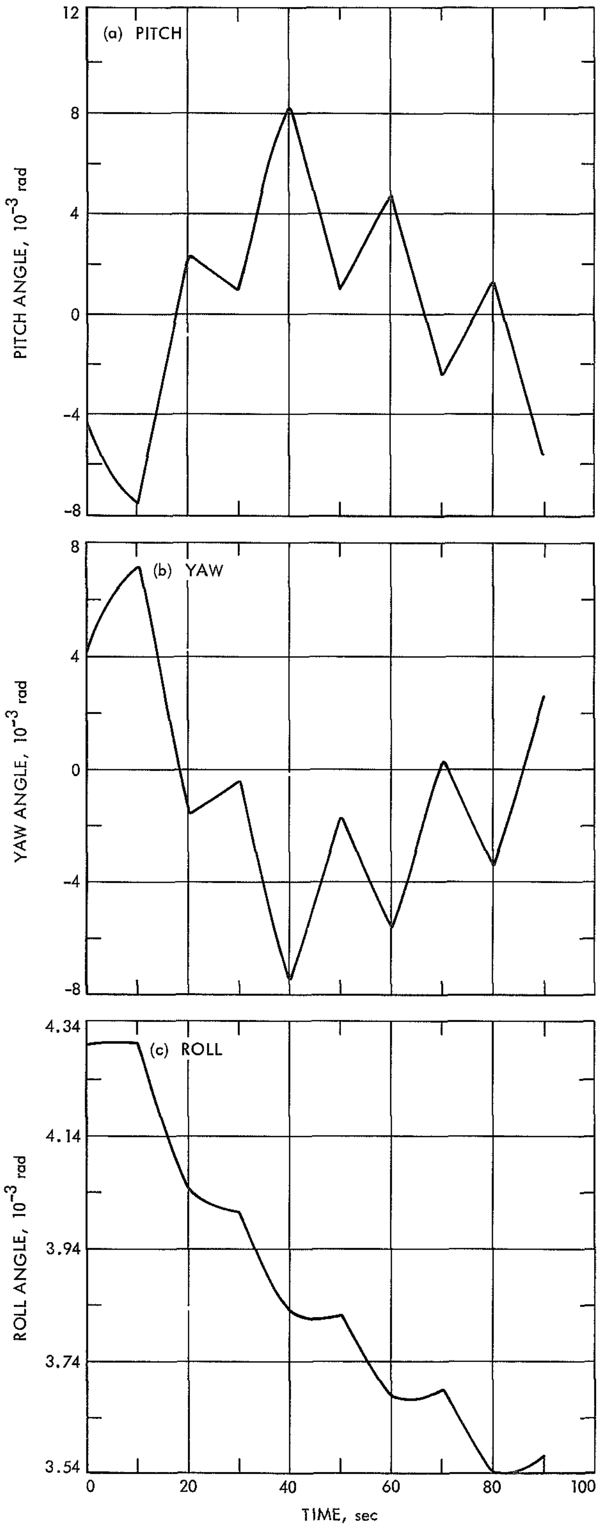

Fig. 10. Spacecraft attitude angles vs time

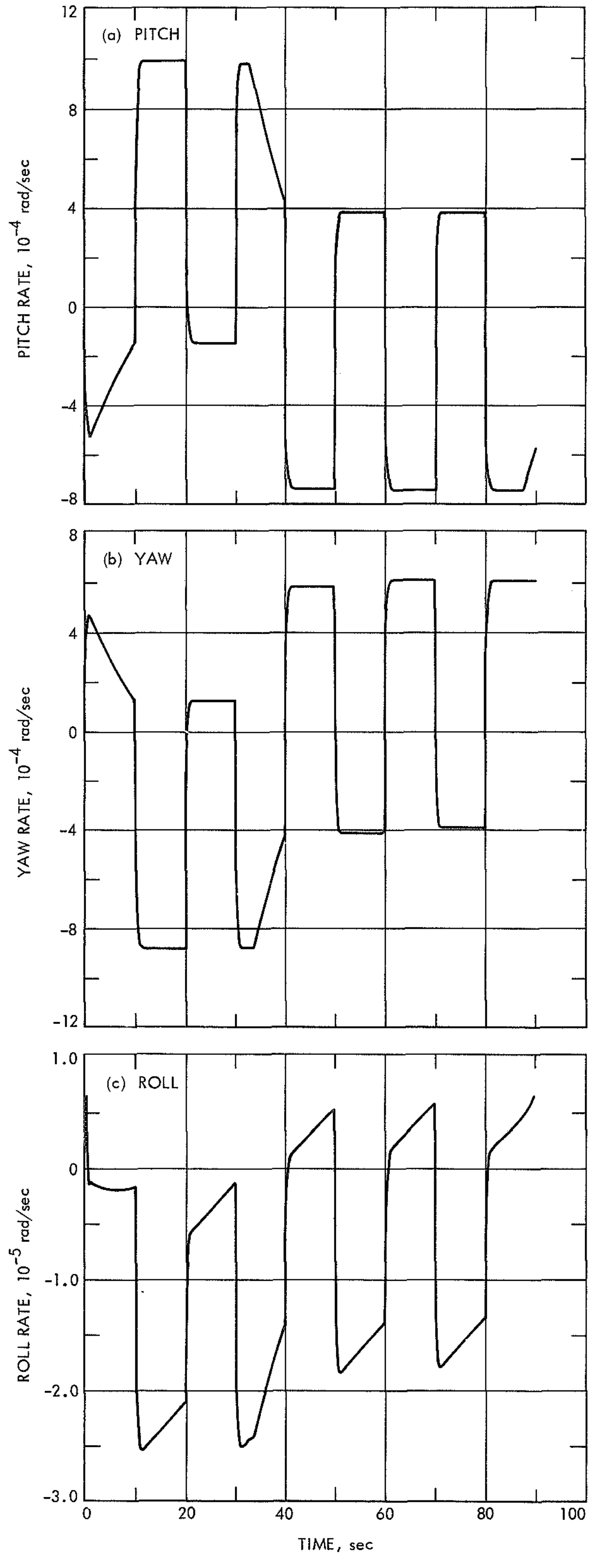

Fig. 11. Spacecraft angular rates vs time

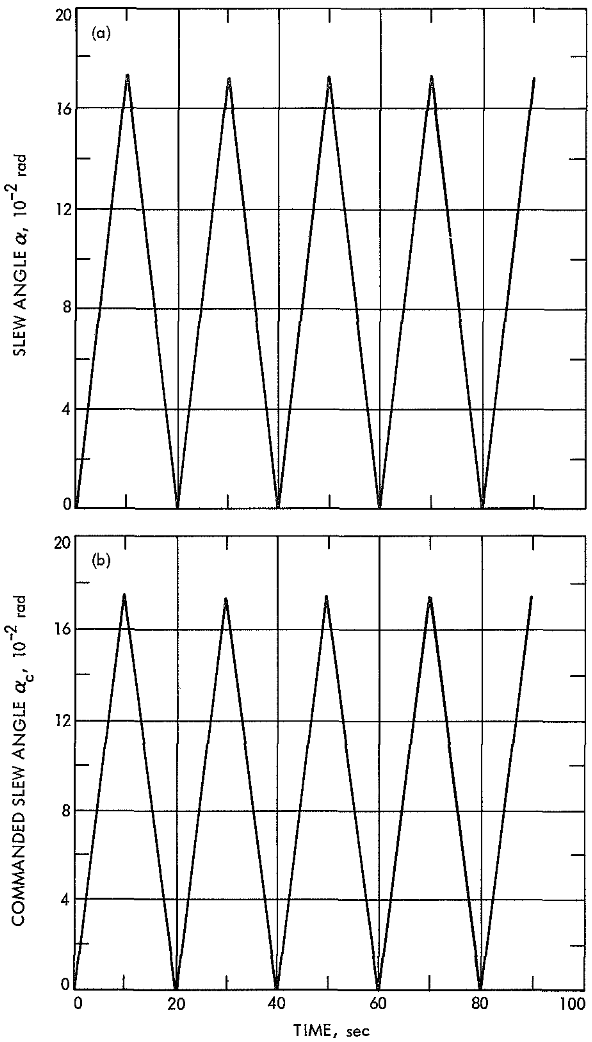

Fig. 12. Platform slew angles vs time

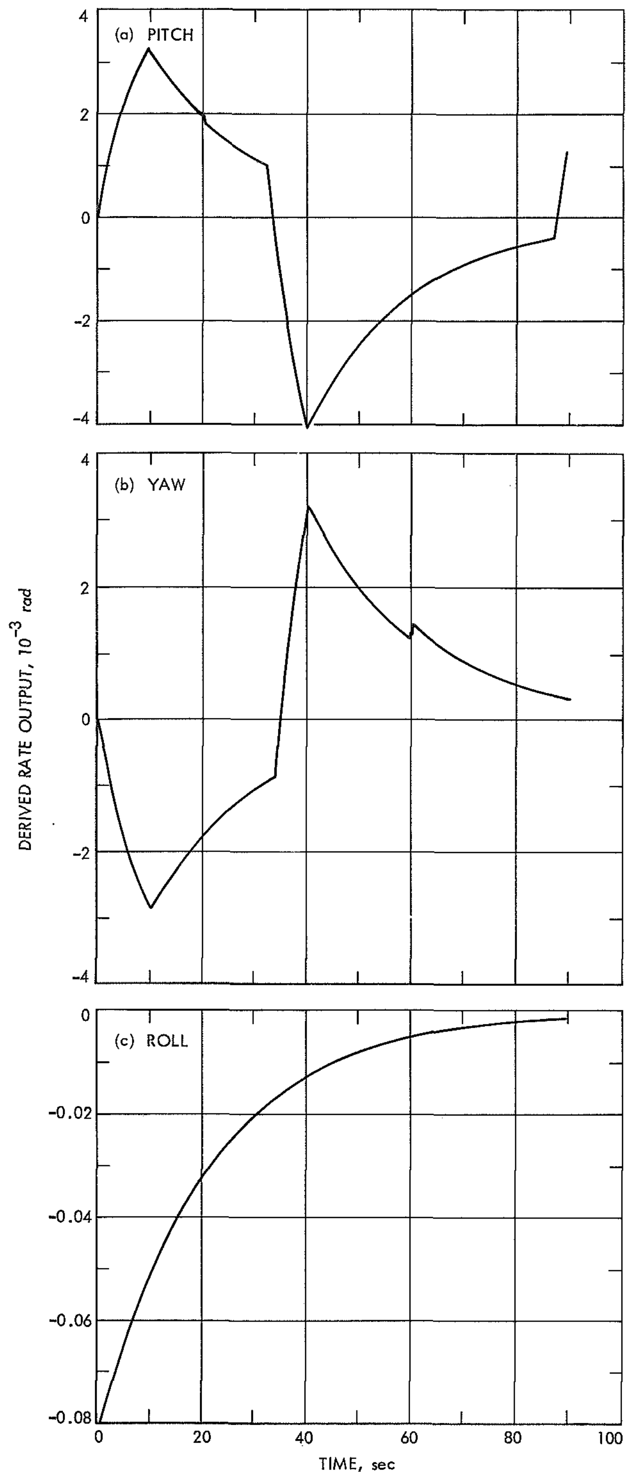

Fig. 13. Spacecraft derived rate outputs vs time

A listing of the MLTBDY subroutine is given in Appendix C. At present, MLTBDY is limited to five bodies in the DSL/90 version because of core storage capacity. This can be increased substantially for Continuous System Simulation Language (CSSL III) on the Univac 1108, but the program is probably most practically used with respect to only a few bodies.

B. Subroutine MLTBDL

While the use of MLTBDY is intended to provide an exact solution for the rotational dynamics of a system of hinged rigid bodies, no matter how large the relative angular displacement of adjacent bodies, it was also found that MLTBDY, in a modified form, could perhaps prove even more useful and efficient when applied to systems of rigid bodies when the relative rotations of the bodies may be assumed to be "small," say, less than ±5 deg. The program, called MLTBDL, has been developed in conjunction with CSSL III and is intended to facilitate the analytical task involved in such problems as spacecraft autopilot design and simulation. At present, the program

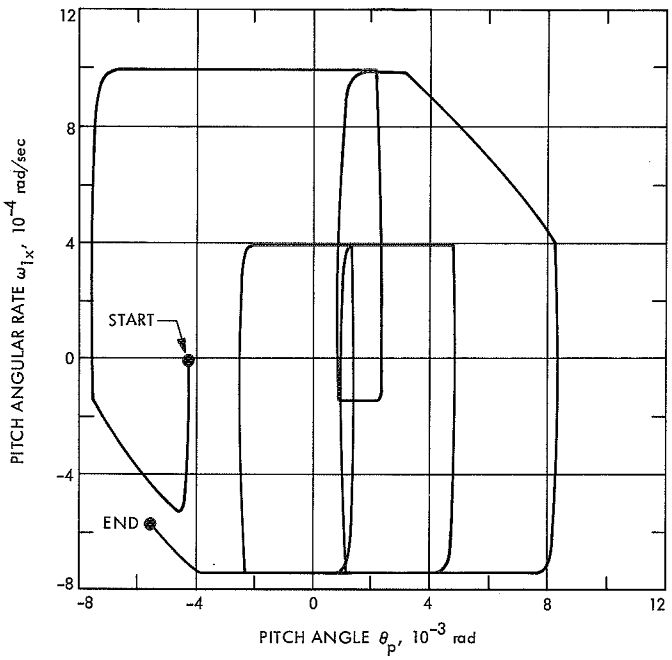

Fig. 14. Pitch axis phase plane

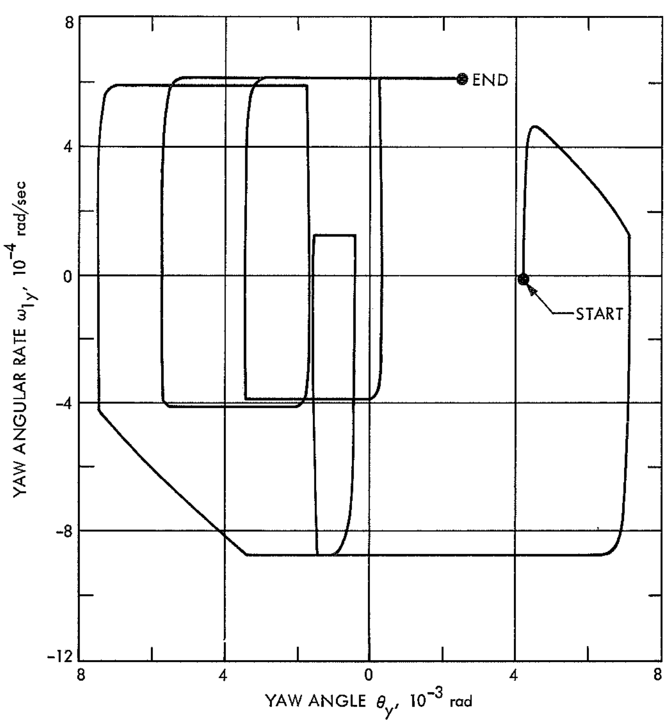

Fig. 15. Yaw axis phase plane

is capable of handling systems of up to nine interconnected rigid bodies. The discussion that follows illustrates the application of the program to the simulation of a 5-body configuration representing a spacecraft bus symmetrically hinged to four solar panels.

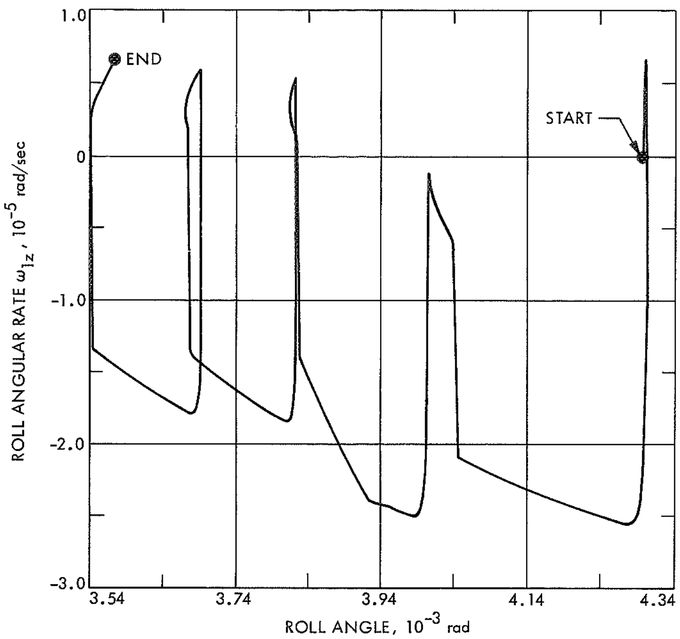

Fig. 16. Roll axis phase plane

1. Spacecraft configuration. Figure 17 shows the structural configuration under consideration. A right-handed coordinate frame is fixed in the rigid central body, or "bus," and arranged along lines of symmetry of the solar panel array. Coordinate frames are also fixed to each of the solar panels and aligned in such a way that the axes z_2, z_3, z_4, and z_5 are all parallel to z_1 of the bus. Each of the panel y axes is along the long axis of symmetry.

For this example, it is assumed that relative rotation between panel and bus is possible only about hinge axes parallel to the respective panel x axes. However, since the relative rotations are small, the body-to-body coordinate transformation matrices are assumed to be constant. Thus, the body 1 (bus)-to-body 2 (panel) transformation is

$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = t_{12} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

and, similarly,

$$t_{13} = t_{23} = t_{34} = t_{45} \cong \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

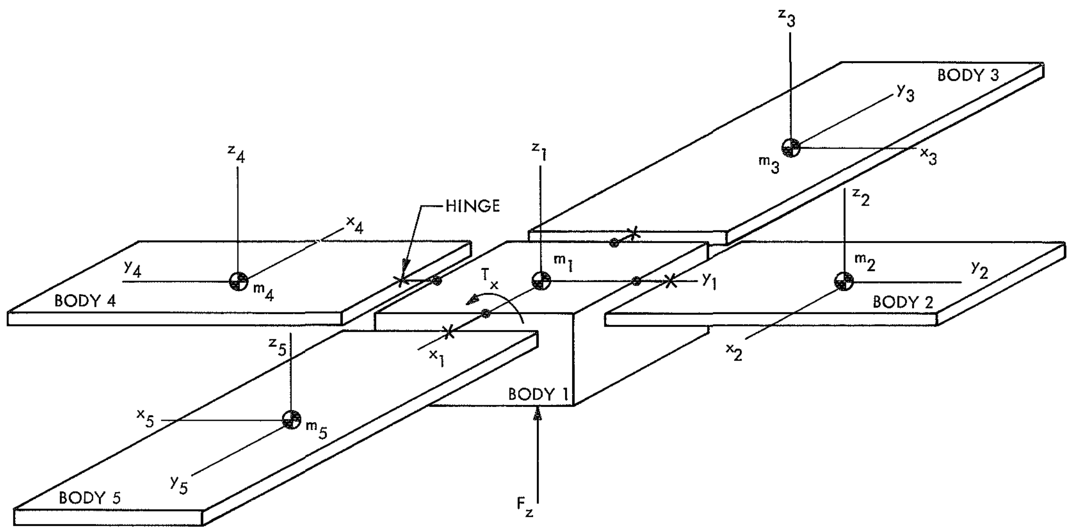

Fig. 17. Spacecraft–solar panel rigid body configuration

$$t_{14} = t_{24} = t_{35} \cong \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$t_{15} = t_{25} \cong \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The joints connecting the bus to the four panels are arbitrarily located in the x_1–y_1 plane, a distance of 3 ft from the z_1 axis. In the panel body-fixed frames, each joint is located -3.25 ft down the associated panel's y axis.

The inertia matrices of the bus and panels with respect to their coordinate frames are assumed to be as follows (Note: by definition, each coordinate frame must have its origin at the body's center of mass):

$$I_1 = \text{bus inertia matrix} = \begin{bmatrix} 550. & 0 & 0 \\ 0 & 552. & 0 \\ 0 & 0 & 800. \end{bmatrix} \text{ slug-ft}^2$$

where

$$m_1 = 54.3 \text{ slugs}$$

$$m_2 = m_3 = m_4 = m_5 = 1.55 \text{ slugs}$$

Like the subroutine MLTBDY, the modified subroutine MLTBDL requires the specification not only of the body-to-body transformation matrices and the location of the connecting joints in body-fixed coordinates, but also the direction of certain unknown torques of constraint at each joint. Since, in this example, the four panel joints are assumed to be single-degree-of-freedom hinges, there are two axes at each joint about which the panel is constrained against movement relative to the central bus. While the two axes may be anywhere in the plane perpendicular to the hinge axis (e.g., axis x_2) as long as they are orthogonal, they are most conveniently taken as the panel y and z axes, i.e., y_2, z_2, y_3, z_3, y_4, z_4, y_5, and z_5. Thus, there exist, in total, eight unknown torques of constraint whose direction must be described to MLTBDL in terms of their unit vector components *in each body directly experiencing their effect.*

A matrix U is used by MLTBDL to obtain the necessary components, where U is composed of submatrices U_i, each of which is a $3 \times n_c$ matrix of body i components of constraint torque unit vectors for each of the n_c total system constraints. If, for example, the kth constraint does not apply to body i, since it occurs at a joint not shared by i, the kth column of U_i has zero elements.

For this problem, U_1 is given by

$$U_1 \cong \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

where each column contains the body 1 components of unit vectors along y_2, z_2, y_3, z_3, y_4, z_4, y_5, and z_5. The constraint torques are arbitrarily assumed to be positive in these directions when applied to body 1. For body 2,

$$U_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since constraints 3–8 do not apply to this panel, columns 3–8 in U_2 are zero. Given the body-to-body transformations described earlier and the fact that constraint torques 1 and 2 are assumed to be in the direction y_2 and z_2 when applied to body 1, these torques must be in the opposite direction when applied to body 2, and the unit components in body 2 are therefore as shown in the first two columns of U_2. Similarly,

$$U_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$U_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Matrix U is then formed from the U_i matrices by

$$U = \begin{bmatrix} U_1 \\ \hline U_2 \\ \hline U_3 \\ \hline U_4 \\ \hline U_5 \end{bmatrix}, \qquad (15 \times 8)$$

The time derivative $\dot{U}$ would, of course, be zero if one were to take seriously the assumption that the elements of U are constant. However, while the expressions for U_2, U_3, U_4, and U_5 are exact, since the constraints were assumed to lie along y_2, z_2, y_3, z_3, y_4, z_4, y_5, and z_5, the matrix U_1 elements in reality are trigonometric functions of the relative angular displacements of the panels and bus. That is,

$$U_1 = \begin{bmatrix} 0 & 0 & -\cos\theta_3 & \sin\theta_3 & 0 & 0 & \cos\theta_5 & -\sin\theta_5 \\ \cos\theta_2 & -\sin\theta_2 & 0 & 0 & -\cos\theta_4 & \sin\theta_4 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & \sin\theta_3 & \cos\theta_3 & \sin\theta_4 & \cos\theta_4 & \sin\theta_5 & \cos\theta_5 \end{bmatrix}$$

where θ_2, θ_3, θ_4, and θ_5 are panel angular displacements relative to the bus (assumed positive in the x_2, x_3, x_4, and x_5 directions respectively). As a result, in the first approximation,

$$\dot{U}_1 = \frac{dU_1}{dt} \cong \begin{bmatrix} 0 & 0 & 0 & \omega_{3x} - \omega_{1y} & 0 & 0 & 0 & -(\omega_{5x} + \omega_{1y}) \\ 0 & -(\omega_{2x} - \omega_{1x}) & 0 & 0 & 0 & \omega_{4x} + \omega_{1x} & 0 & 0 \\ \omega_{2x} - \omega_{1x} & 0 & \omega_{3x} - \omega_{1y} & 0 & \omega_{4x} + \omega_{1x} & 0 & \omega_{5x} + \omega_{1y} & 0 \end{bmatrix}$$

and

$$\dot{U} \cong \begin{bmatrix} \dot{U}_1 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

Finally, it is necessary to describe the hinge reaction torque characteristics and the nature of external forces and/or torques which might be applied to the system bodies. In this case, a simple spring–damper type of hinge connection is assumed for each panel. Thus, the hinge restoring torque applied to body 2 will be of the form

$$T_{x2} = -K_{p2}(\alpha_2 - \theta_x) - d_{p2}(\omega_{2x} - \omega_{1x}) \qquad (71)$$

where

$$\alpha_2(t) \cong \int_0^t \omega_{2x}(t)\,dt + \alpha_2(0)$$

$$\theta_x(t) \cong \int_0^t \omega_{1x}(t)\,dt + \theta_x(0)$$

The relative rotation of the panel with respect to the bus, θ_2, is of course identical to $\alpha_2 - \theta_x$. Likewise, the hinge restoring torques applied to bodies 3, 4, and 5 can be given by

$$T_{x3} = -K_{p3}(\alpha_3 - \theta_y) - d_{p3}(\omega_{3x} - \omega_{1y}) \tag{72}$$

$$T_{x4} = -K_{p4}(\alpha_4 + \theta_x) - d_{p4}(\omega_{4x} + \omega_{1x}) \tag{73}$$

$$T_{x5} = -K_{p5}(\alpha_5 + \theta_y) - d_{p5}(\omega_{5x} + \omega_{1y}) \tag{74}$$

Of necessity, then, the reaction torques on body 1 from hinge rotations will be

$$T_{x1} = -T_{x2} + T_{x4} \tag{75}$$

$$T_{y1} = -T_{x3} + T_{x5} \tag{76}$$

As for external forces and torques, these will be limited, for purposes of illustration, to (1) the application of a constant thrust F_z to body 1 along its z axis and (2) the application of a variable torque $T_x(t)$ about the body 1 x axis.

2. ***Subroutine MLTBDL call statements.*** Access can be made to MLTBDL by either of two entry points. The first is through the following call statement:

CALL MLTBDL (NB, N3, NC, BMASS, U, T, IXX, IYY, IZZ, IXY, IXZ, IYZ, CIX, CIY, CIZ, FKC)

where

NB = number of rigid bodies in the system (integer)

N3 = 3 × NB (integer)

NC = number of constraint conditions on the system (integer)

BMASS = array of rigid body mass values starting with body 1

IXY, IXZ, IYZ, IXX, IYY, IZZ = arrays of rigid body inertias starting with body 1

CIX, CIY, CIZ = array of vector components locating system connecting joints in each body-fixed coordinate frame.

U = array of constraint torque vector components as defined above

T = array of body-to-body coordinate transformation matrices

FKC = array of floating point indicators such that if FKC $(i) \neq 1.$, then the value of WDOT (i) is not requested and will be set to zero $(i = 1, 2, \cdots, \text{N3})$

The terms NB, N3, and NC must be declared as integers in the simulation main program. The arguments of MLTBDL should be dimensioned as follows:

DIMENSION BMASS (NB), IXX (NB), IZZ (NB), FKC (N3), IXY (NB), IXZ (NB), IYZ (NB), CIX (11 × NB-1), U (N3, NC), T (NB, NB, 3, 3), CIY (11 × NB-1), CIZ (11 × NB-1)

In CSSL III, an ARRAY statement may be used instead of DIMENSION for all floating point arrays with no more than three subscripts.

To illustrate, for this 5-body example,

INTEGERS: NB = 5, N3 = 15, NC = 8
DIMENSIONED VARIABLES: BMASS (5), IXX (5), IYY (5), IZZ (5), U (15,8), T (5,5,3,3), IXY (5), IXZ (5), IYZ (5), CIX (54), CIY (54), CIZ (54), FKC (15)

The term T must be defined in the program for every combination of m and n *except* $m \geqq n$, where m (or n) ranges from 1 to NB. Also, as in MLTBDY, C (ij) must be supplied for all combinations of i and j *except* for $i = j$.

The execution of the statement CALL MLTBDL (NB, N3, NC, BMASS, . . . initializes the subroutine with the physical constants of the system and need be done only once. The subroutine is then ready for repeated computation of the system angular accelerations. This is accomplished with the following statement:

CALL MLTRAT (NB, N3, NC, TX, TY, TZ, FX, FY, FZ, UD, WX, WY, WZ, WDOT)

where

NB, N3, NC = integers as previously defined

TX, TY, TZ = arrays of torque components applied to each system body (including torque about the hinge axis)

FX, FY, FZ = arrays of force components externally applied to each body

UD = time derivative of the matrix U

WX, WY, WZ = arrays of body angular velocity components

WDOT = array of body angular acceleration components

The arguments of MLTRAT should be dimensioned in the main simulation program as follows:

DIMENSION TX(NB), TY(NB), TZ(NB), FX(NB), FY (NB), FZ (NB), UD (N3, NC), WX (NB), WY (NB), WZ (NB), WDOT (43)

The angular velocity components WX, WY, and WZ are carried as doubly subscripted arrays in the subroutine MLTBDL but are *singly* subscripted in the main calling program. By definition, WX (i) refers to the x_i-axis component of body i's angular velocity. (Note that in DSL/90 and CSSL III it will be necessary, if a subscripted variable is to be printed or plotted using the standard OUTPUT, PRINT, PREPAR, and GRAPH statements, to redefine these variables in terms of nonsubscripted variables).

WDOT, the output array of MLTRAT, is a vector array (single subscript) whose elements are the body angular acceleration components given in the following order:

$$WDOT(1) = \dot{\omega}_{x1}$$

$$WDOT(2) = \dot{\omega}_{y1}$$

$$WDOT(3) = \dot{\omega}_{z1}$$

$$WDOT(4) = \dot{\omega}_{x2}$$

$$WDOT(5) = \dot{\omega}_{y2}$$

$$\vdots$$

$$WDOT(N3) = \dot{\omega}_{z\,(NB)}$$

3. ***Attitude dynamics simulation program.*** A listing of the CSSL III program used to simulate the vibrations of the five-body system under the application of certain disturbances is shown in Appendix D. In the interest of simplicity and brevity, the simulation was not broadened to include a control system for maintaining the system attitude. However, it is certainly intended that such a controller, typically a three-axis gimbaled-engine or jet-vane autopilot, would be added to the computation as the usual condition under which MLTBDL is used.

After the system variables have been dimensioned and their types have been specified, parameter values are input using the CONSTANT or DATA statement. These include the mass and inertia values, location of the system joints, transformation matrix elements, U matrix elements, and the hinge spring and damper coefficients.

Note that, since FKC (i) = 1. for all i, all WDOT components of the system are being requested.

The program's INITIAL section, in addition to output formatting instructions, contains the call to MLTBDL which is executed only at $t = 0$ (t = TIME). Thus all the system constants are transmitted at this point to initialize the subroutine.

The system differential equations are solved in the program's DYNAMIC section under the NOSORT option. Nonzero elements of $\dot{U}$ (UD) are defined in terms of angular velocity components as derived above. Next, those panel angular velocity components that are *constrained* to body 1 are explicitly expressed in terms of body 1 angular velocity components. (This need not have been done. These same components could have been obtained by integrating the appropriate elements of WDOT but with some added computation time.)

The arrays WX, WY, and WZ are next defined in terms of their nonsubscripted variable equivalents. Remaining, prior to the call to MLTRAT, are the definitions of applied forces and torques (including the hinge torques). Equations (71–76) are embodied in the statements defining TX (1), TY (1), TX (2), TX (3), TX (4), and TX (5). Included in TX (1) is the applied torque function TORQ, which is constructed from three step functions. The applied thrust FZ (1) = 300. lb is also inserted here.

A call to MLTRAT returns the desired system angular accelerations in WDOT, which are then redefined so that they may be printed using the CSSL III OUTPUT statement. Integration of the appropriate WDOT elements results in the desired "free" components of system angular velocity, i.e., all three components of body 1 and the body x components of each panel. The liberty was taken in this example of integrating the angular velocity components *directly* to obtain the inertial angular position of the bus and panels, since any large rotations of the bus would be strictly about the x-axis. In general, however, the anticipation of any large complex motion of the bus in inertial space would require the use of the four Hamilton–Cayley–Klein parameters to represent its position. The small relative angular motion of the panels with respect to the bus would be obtained by integrating the *relative* angular velocities, e.g.,

$$\theta_2 = \int_0^t (\omega_{2x} - \omega_{1x})\, dt, \qquad \text{etc.}$$

4. Attitude dynamics simulation results. Figures 18–22 show the resulting dynamic response of the system to an applied thrust of 300 lb and an applied torque about body 1's x-axis. The torque profile is given in Fig. 18; a 100-ft-lb torque for 2 sec in one direction, 2 sec in the opposite direction, and then zero. The effect on body 1's x-component of angular velocity is clear from Fig. 19a, with coupling from panel vibrations barely visible near the 4-sec mark. Component ω_{1y} in Fig. 19b responds to panel vibrations induced by the applied thrust. A very small, second-order type of disturbance is induced into the system's z-axis, leaving a constant residual rate. Figure 20 indicates how panel bodies 2 and 4 are rotated by the bus through the hinge spring compliance, with panel bodies 3 and 5 deflecting under 300 lb of thrust.

The bus is caused to rotate approximately 33 deg (0.585 rad) about x_1 because of the torque pulse, as shown in Fig. 21a. Very slight y and z rotations are also induced, although the z rotation is constantly increasing (negatively). In Figs. 22a and 22c, panel bodies 2 and 4 rotate to almost the same inertial position in x as the bus, the difference amounting to that caused by the continuously applied force on body 1. Bodies 3 and 5 are displaced to a small negative level (about 0.5–0.7 deg) in response to the linear acceleration of the bus. Figure 23 pictures part of the program's printed output.

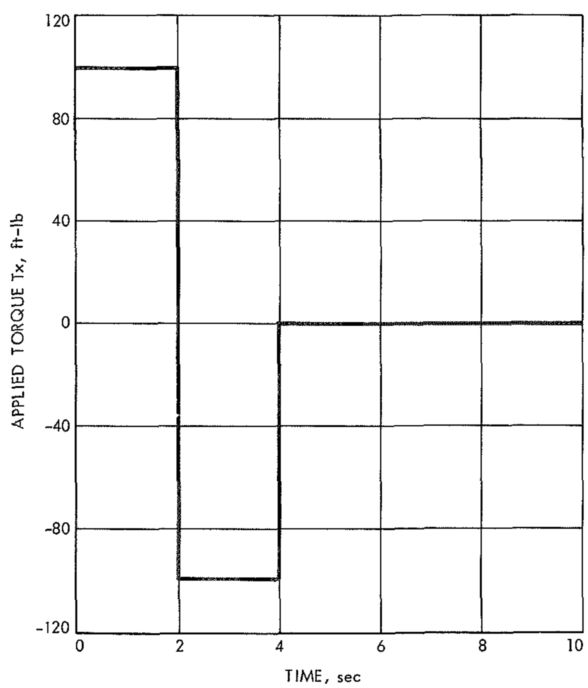

Fig. 18. Applied torque vs time

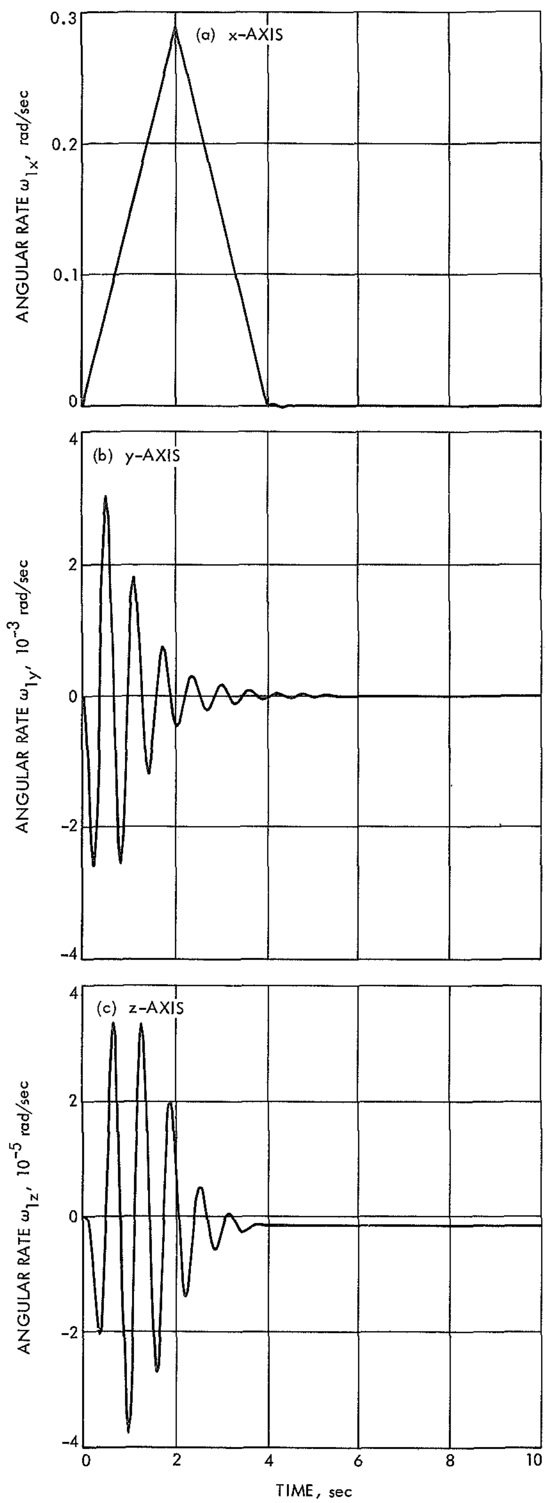

Fig. 19. Body 1 components of angular velocity vs time

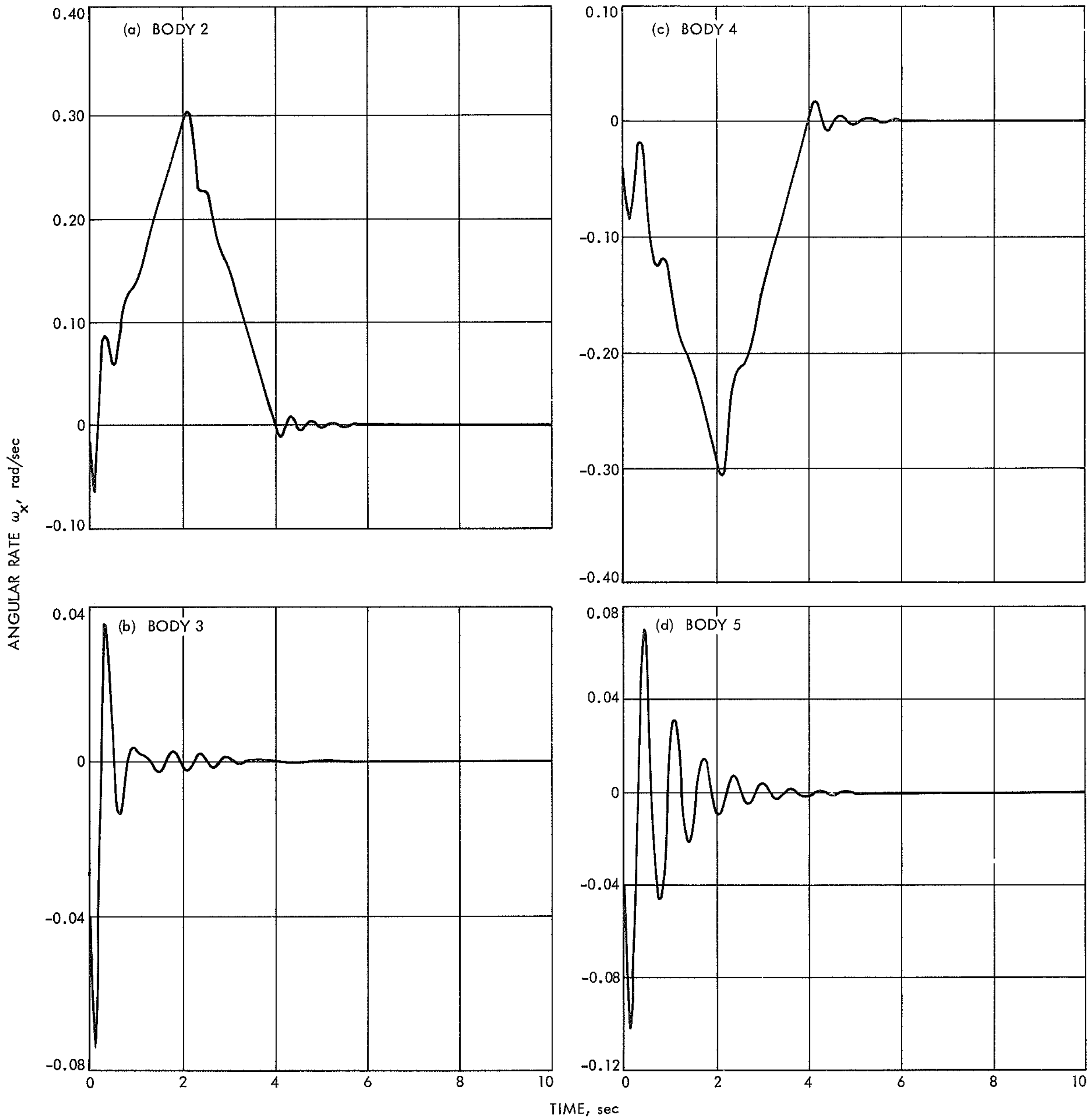

Fig. 20. Solar panel hinge rates vs time

To summarize, the system configuration is representative of a class of spacecraft structures frequently considered in conjunction with thrust vector control system design studies and simulations. The 9-body limit of MLTBDL is convenient at the moment but may be increased without great effort if necessary. The MLTBDL program listing is given in Appendix E. A further modified version of MLTBDL is also available which assumes $\dot{U} = 0$ and which totally eliminates any second-order dynamic terms in the equations of the following type: $\omega \times I \cdot \omega$. This latter, fully linearized version, MLTBD (see Appendix F), can be executed somewhat faster but will give less accurate results for the relatively undisturbed portions of the system.

VII. Discussion and Conclusions

The emergence of the barycenter formulation derived by Hooker and Margulies has understandably generated a good deal of enthusiasm among analysts faced with the problem of predicting the rotational motions of complex rigid-body systems. When the approach is applied to systems of more than two bodies, it is greatly superior to the Lagrangian formulation. Laborious and error-prone

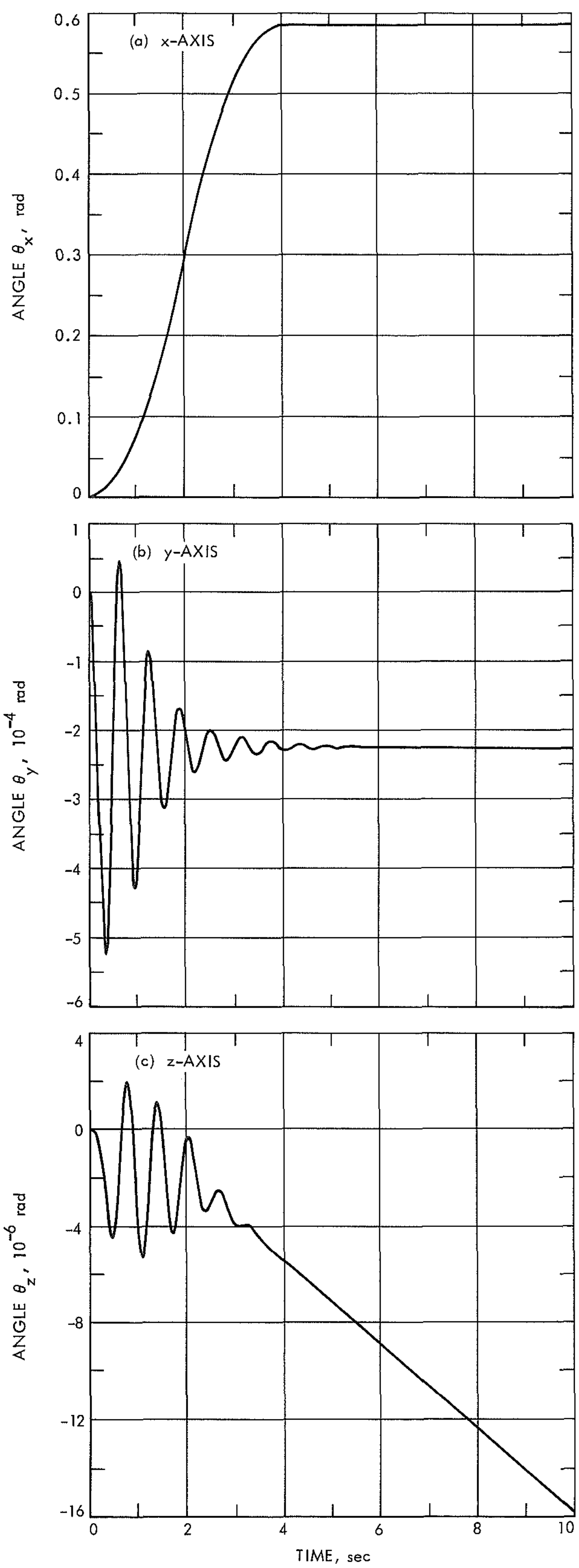

Fig. 21. Body 1 rotations vs time

differentiations are avoided, as well as the problem of eliminating unwanted translational coordinates. Also, the approach of Lagrange, which is carried out without explicit reference to body-fixed axes, often leaves unclear how to include the effects of active control laws or internal torques and forces.

On the other hand, the use of Euler's equations in the barycenter approach necessitates the appearance of constraint torques that are present at the system's connecting joints. These torques never appear in the equations of motion resulting from the Lagrangian method. However, Hooker (Ref. 7) has shown how these torques of constraint may be eliminated from the equations to reduce the system to minimal order (i.e., one scalar equation for each degree of freedom).

The objective of this report has been to describe the barycenter formulation but, more important, to describe the use of computational tools that have been developed to quickly and efficiently apply this systematic approach to practical problems of attitude dynamics and control. Specifically, the subroutine MLTBDY was devised to routinely perform the chore of solving the equations of motion for the unknown angular accelerations. While the constraint torques have been algebraically eliminated in MLTBDY, the subroutine must still deal with a $3n \times 3n$ system of equations (n = number of bodies) in reaching a solution. Thus, both the "free" and the constrained components of angular acceleration are computed for the user. While this works some computational disadvantage compared with the minimal order techniques of Hooker (Ref. 7) or the nested-body approach of Russell (Ref. 8), MLTBDY is probably simpler for the user to apply, and a number of analytical preliminaries may be saved to some advantage.

The philosophy behind the development of the subroutine MLTBDY is based on the assumption that the analyst will employ one of the commonly available, high-level simulation languages such as DSL/90, CSMP/360, CSSL III, MIMIC, etc., to compute the dynamic response of his system. These languages not only provide "integrator blocks" for solving the ordinary differential equations involved, but they generally supply a variety of special purpose blocks which simulate devices such as pulse generators, quantizers, filters, limiters, delays, noise generators, etc. It is in this spirit that MLTBDY is presented—as another general-purpose block, albeit on a higher plane of sophistication, to relieve the analyst of repetitive pro-

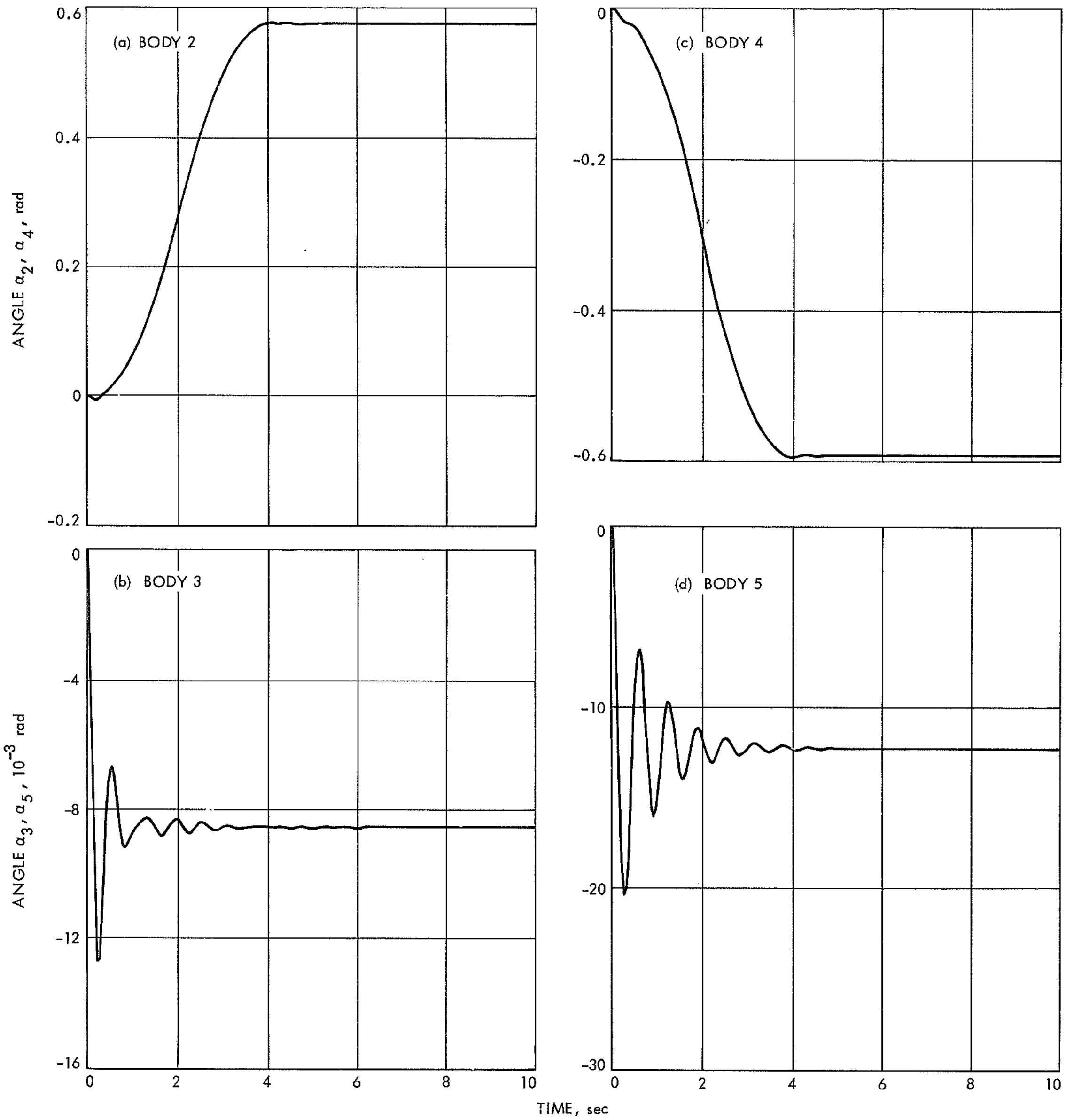

Fig. 22. Solar panel hinge axis rotations vs time

gramming and concern for detail and thereby to free him for dealing with problems on a more conceptual level.

MLTBDY is probably most practically used for systems of 2–5 bodies, since the computational problem quickly gets out of hand as the number of bodies increases. Often, however, the relative motions of the connected rigid bodies are small. To accommodate this situation, two linearized versions of MLTBDY were programmed—MLTBD and MLTBDL. In these routines, the number of bodies that can be included is quite large since repetitive matrix equation solutions are not required.

For the future, some tasks remain which would seem to offer worthwhile advantages in the solution of such dynamical systems. One of these might be the development of a multi-rigid-body dynamics subroutine capable of handling, in an efficient computational manner, the "mixed" case, i.e., one in which a few rigid bodies in a system undergo large relative rotations and the rest experience small relative rotations. Also of some interest would be the development of a general-purpose program to allow the representation of a system of connected rigid bodies and *flexible* appendages as well, combining the barycenter formulation with modal models of the flexible parts.

```
SPACECRAFT AND 4-PANEL SOLAR ARRAY VIBRATIONS TEST

BODY 1-SPACECRAFT BUS
BODY 2-SOLAR PANEL
BODY 3-SOLAR PANEL
BODY 4-SOLAR PANEL
BODY 5-SOLAR PANEL

INERTIAS

IXX          IYY          IZZ          IXY          IXZ          IYZ

BODY 1

.55000+03    .55200+03    .80000+03    .00000       .00000       .00000

BODIES 2-5

.57500+01    .21000+01    .78000+01    .00000       .00000       .00000

MASSES

M1           M2           M3           M4           M5

.54300+02    .15500+01    .15500+01    .15500+01    .15500+01
```

Fig. 23. Program printed output

PANEL HINGE SPRING AND DAMPER CONSTANTS

KP2	KP3	KP4	KP5
.40000+04	.30000+04	.25600+04	.20000+04
DP2	DP3	DP4	DP5
.11500+03	.10000+03	.90000+02	.50000+02

Fig. 23 (contd)

TIME = 0.000000	W1XD	= .178110	W1YD	= -5.421011-20	W1ZD	= -1.847514-18
	W2XD	= -1.34351	W3XD	= -1.22184	W4XD	= -1.10016
	W5XD	= -1.22184	W1X	= 0.000000	W1Y	= 0.000000
	W1Z	= 0.000000	W2X	= 0.000000	TORQ	= 100.000
	W3X	= 0.000000	W4X	= 0.000000	W5X	= 0.000000
	THX	= 0.000000	THY	= 0.000000	THZ	= 0.000000
	A2	= 0.000000	A3	= 0.000000	A4	= 0.000000
	A5	= 0.000000	W2Y	= 0.000000	W2Z	= 0.000000
	W3Y	= 0.000000	W3Z	= 0.000000	W4Y	= 0.000000
	W4Z	= 0.000000	W2YD	= 6.413056-17	W2ZD	= -9.024137-19
	W3YD	= -.178110	W3ZD	= 1.662527-17	W4YD	= -6.516055-17
	W4ZD	= 1.543158-17	W5YD	= .178110	W5ZD	= 6.397893-18
	W5Y	= 0.000000	W5Z	= 0.000000		
TIME = 1.000000-01	W1XD	= .130907	W1YD	= -1.820902-02	W1ZD	= -1.862460-05
	W2XD	= .157180	W3XD	= -.177596	W4XD	= -.375722
	W5XD	= -.478024	W1X	= 1.556080-02	W1Y	= -9.986891-04
	W1Z	= -4.737453-07	W2X	= -6.518407-02	TORQ	= 100.000
	W3X	= -7.460904-02	W4X	= -7.785336-02	W5X	= -9.108628-02
	THX	= 8.195172-04	THY	= -3.391531-05	THZ	= -9.596463-09
	A2	= -4.559427-03	A3	= -4.627586-03	A4	= -4.510791-03
	A5	= -5.187151-03	W2Y	= -9.986891-04	W2Z	= -4.737453-07
	W3Y	= -1.556080-02	W3Z	= -4.737453-07	W4Y	= 9.986891-04
	W4Z	= -4.737453-07	W2YD	= -1.820898-02	W2ZD	= -9.326362-05
	W3YD	= -.130907	W3ZD	= -1.164060-03	W4YD	= 1.820905-02
	W4ZD	= 4.358630-05	W5YD	= .130907	W5ZD	= 1.414291-03
	W5Y	= 1.556080-02	W5Z	= -4.737453-07		
TIME = .200000	W1XD	= .124083	W1YD	= -7.649667-03	W1ZD	= -9.698004-05
	W2XD	= .979136	W3XD	= .616490	W4XD	= .334011
	W5XD	= .490281	W1X	= 2.770909-02	W1Y	= -2.600580-03
	W1Z	= -6.010609-06	W2X	= 5.462377-03	TORQ	= 100.000
	W3X	= -4.526494-02	W4X	= -7.599544-02	W5X	= -8.817151-02
	THX	= 2.988962-03	THY	= -2.227284-04	THZ	= -2.665364-07
	A2	= -8.246402-03	A3	= -1.129613-02	A4	= -1.280626-02
	A5	= -1.497089-02	W2Y	= -2.600580-03	W2Z	= -6.010609-06
	W3Y	= -2.770909-02	W3Z	= -6.010609-06	W4Y	= 2.600580-03
	W4Z	= -6.010609-06	W2YD	= -7.649534-03	W2ZD	= -1.548344-04
	W3YD	= -.124083	W3ZD	= -1.279171-03	W4YD	= 7.649958-03
	W4ZD	= 2.859248-05	W5YD	= .124084	W5ZD	= 2.418232-03
	W5Y	= 2.770909-02	W5Z	= -6.010609-06		
TIME = .300000	W1XD	= .162639	W1YD	= 2.314356-02	W1ZD	= -1.012573-04
	W2XD	= .381494	W3XD	= .466450	W4XD	= .370422
	W5XD	= .848300	W1X	= 4.199173-02	W1Y	= -1.810183-03
	W1Z	= -1.739724-05	W2X	= 8.060260-02	TORQ	= 100.000
	W3X	= 1.562568-02	W4X	= -3.454402-02	W5X	= -1.423980-02

Fig. 23 (contd)

Appendix A
Vector Identity Proof

Vector Identity Proof

The following proof is given for rearranging the sequence of successive vector cross products without changing the value of the vector result. The identity proves useful in the derivations of Sections II and III.

Show that

$$\mathbf{c} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{c})] = -\boldsymbol{\omega} \times [\mathbf{c} \times (\mathbf{c} \times \boldsymbol{\omega})] \qquad \text{(A-1)}$$

If one uses the identity (Ref. 9)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{d}$$

then Eq. (A-1) becomes

$$\mathbf{c} \times [(\boldsymbol{\omega} \cdot \mathbf{c})\,\boldsymbol{\omega} - \mathbf{c}] = -\boldsymbol{\omega} \times [(\mathbf{c} \cdot \boldsymbol{\omega})\,\mathbf{c} - \boldsymbol{\omega}] \qquad \text{(A-2)}$$

Since

$$\mathbf{c} \times (-\mathbf{c}) = 0, \qquad \boldsymbol{\omega} \times \boldsymbol{\omega} = 0$$

Then Eq. (A-2) is

$$\mathbf{c} \times [(\boldsymbol{\omega} \cdot \mathbf{c})\,\boldsymbol{\omega}] = -\boldsymbol{\omega}\,[\times(\mathbf{c} \cdot \boldsymbol{\omega})\,\mathbf{c}] \qquad \text{(A-3)}$$

Also, since

$$\boldsymbol{\omega} \cdot \mathbf{c} = \mathbf{c} \cdot \boldsymbol{\omega} = k = \text{constant}$$

and

$$\mathbf{c} \times k\boldsymbol{\omega} = -\boldsymbol{\omega} \times k\mathbf{c}$$

then Eq. (A-3) becomes

$$\mathbf{c} \times k\boldsymbol{\omega} = k\mathbf{c} \times \boldsymbol{\omega}$$

and the identity is proved.

Appendix B
Program Listing for Spacecraft–Scan Platform Simulation

```
TITLE 3-AXIS GAS-JET CONTROLLED SPACECRAFT AND SCANNING PLATFORM
*
INTGER NC,NB,N3,I,J,K,L
*
STORAG IXX(2),IYY(2),IZZ(2),IXY(2),IXZ(2),IYZ(2)
STORAG BMASS(2),LIX(22),LIY(22),LIZ(22)
STORAG PHIZ(2),THETZ(2),PSIZ(2)
*
D      DOUBLE PRECISION WDOT(19)
*
D      DIMENSION WX(2,2),WY(2,2),WZ(2,2),TEX(2),TEY(2),TEZ(2),THX(2),
D     1THY(2),THZ(2),          P(6,2),PZ(2),PA(2),PB(2),PC(2),T(2,2,3,3),
D     2                PD(6,2),FX(2),FY(2),FZ(2)
*
NOSORT
*
       IF(TIME.GT.0.) GO TO 100
*
*      INITIALIZE MULTI-RIGID-BODY DYNAMICS SUBROUTINE (MLTBDY)
*
       CALLMLTBDY(NB,N3,NC,BMASS,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,LIZ)
*
*      INITIALIZE HCK PARAMETERS
*
       PZ1IC,PA1IC,PB1IC,PC1IC = INITZ(PHIZ(1),THETZ(1),PSIZ(1))
*
*      SINE AND COSINE OF GAMMA
*
       SG = SIN(GA)
       CG = COS(GA)
100    CONTINUE
*
*      SINE AND COSINE OF ALPHA
*
       SA = SIN(AL)
       CA = COS(AL)
*
*      CONSTRAINED COMPONENTS OF ANGULAR VELOCITY
*
       W2X = CA*(CG*W1X + SG*W1Y) + SA*W1Z
       W2Z = -SA*(CG*W1X + SG*W1Y) + CA*W1Z
*
*      REDEFINE SUBSCRIPTED VARIABLES
*
       WX(1,1) = W1X
       WY(1,1) = W1Y
       WZ(1,1) = W1Z
       WX(2,2) = W2X
       WY(2,2) = W2Y
       WZ(2,2) = W2Z
       PZ(1) = PZ1
       PA(1) = PA1
       PB(1) = PB1
       PC(1) = PC1
*
*      FIND BODY 1 COMPONENTS OF SUN LINE AND CANOPUS DIRECTION
*
       D1,D2 = MATRIX(PZ1,PA1,PB1,PC1)
       NX1,NY1,NZ1 = ITOB(0.,0.,1.,D1,D2)
       LX1,LY1,LZ1 = ITOB(1.,0.,0.,D1,D2)
```

```
*
*     BODY 1 PITCH, YAW, AND ROLL ANGLES
*
      THETP = ATAN2(NY1,NZ1)
      THETY = ATAN2(-NX1,NZ1)
      THETR = ATAN2(-LY1,LX1)
*
*     COORDINATE TRANSFORMATION MATRICES
*
      T(1,2,1,1) = CA*CG
      T(1,2,1,2) = CA*SG
      T(1,2,1,3) = SA
      T(1,2,2,1) = -SG
      T(1,2,2,2) = CG
      T(1,2,2,3) = 0.
      T(1,2,3,1) = -SA*CG
      T(1,2,3,2) = -SA*SG
      T(1,2,3,3) = CA
      DO 105 I=1,3
      DO 105 J=1,3
105   T(2,1,I,J) = T(1,2,J,I)
*
*     HINGE ANGLE,ALPHA,RATE OF CHANGE
*
      ALDOT = -W2Y - (W1X*SG - W1Y*CG)
*
*     ATTITUDE CONTROL SYSTEM
*
      XSNS = THETP
      YSNS = THETY
      ZSNS = INTGRL(.0042990,(THETR-ZSNS)/TRS)
      XIN = 1000.*(-XSNS-TGX*W1X) - XDER
      YIN = 1000.*(-YSNS-TGY*W1Y) - YDER
      ZIN = 1000.*(-ZSNS-TGZ*W1Z) - ZDER
      AMPX = SWAMP(1,DBX,MOT,XIN)
      AMPY = SWAMP(2,DBY,MOT,YIN)
      AMPZ = SWAMP(3,DBZ,MOT,ZIN)
      TDRX = FCNSW(AMPX,TCX,TDX,TCX)
      TDRY = FCNSW(AMPY,TCY,TDY,TCY)
      TDRZ = FCNSW(AMPZ,TCZ,TDZ,TCZ)
      XDER = INTGRL(0.,(KDX*AMPX-XDER)/TDRX)
      YDER = INTGRL(0.,(KDY*AMPY-YDER)/TDRY)
      ZDER = INTGRL(0.,(KDZ*AMPZ-ZDER)/TDRZ)
*
*     EXTERNAL TORQUES (GAS JET)
*
      EXT1 = KTX*AMPX
      EYT1 = KTY*AMPY
      EZT1 = KTZ*AMPZ
      TEX(1) = EXT1
      TEY(1) = EYT1
      TEZ(1) = EZT1
*
*     COMMANDED HINGE ANGLE,ALPHAC,DRIVE FUNCTION
*
      ZONK = AMOD(TIME,20.)
      ACR = .01745
      IF(ZONK.GE.10.) ACR=-.01745
      AC = INTGRL(0.,ACR)
*
```

```
*      HINGE + EXTERNAL TORQUE COMPONENTS
*
       THY(2) = -KS*(AC-AL) + DS*ALDOT
       THX(1) = -T(2,1,1,2)*THY(2) + TEX(1)
       THY(1) = -T(2,1,2,2)*THY(2) + TEY(1)
       THZ(1) = -T(2,1,3,2)*THY(2) + TEZ(1)
*
*      CONSTRAINT MATRIX
*
       P(1,1) = CG
       P(2,1) = SG
       P(3,2) = 1.
       P(4,1) = -CA
       P(4,2) = -SA
       P(6,1) = SA
       P(6,2) = -CA
       PD(4,1)=ALDOT*SA
       PD(4,2)=-ALDOT*CA
       PD(6,1)=-PD(4,2)
       PD(6,2)=PD(4,1)
*
*      BODY ANGULAR ACCELERATION SOLUTION
*
       CALL MLTRAT(NB,N3,NC,THX,THY,THZ,FX,FY,FZ,P,PD,T,WX,WY,WZ,WDOT)
*
*      BODY ANGULAR VELOCITY COMPUTATION
*
       W1X = INTGRL(W1XIC,WDOT(1))
       W1Y = INTGRL(W1YIC,WDOT(2))
       W1Z = INTGRL(W1ZIC,WDOT(3))
       W2Y = INTGRL(W2YIC,WDOT(5))
*
*      HINGE ANGLE CALCULATION
*
       AL = INTGRL(ALIC,ALDOT)
*
*      HCK PARAMETER RATES
*
       PZ1DOT,PA1DOT,PB1DOT,PC1DOT = HCK(PZ1,PA1,PB1,PC1,W1X,W1Y,W1Z)
*
*      HCK PARAMETER CALCULATION
*
       PZ1 = INTGRL(PZ1IC,PZ1DOT)
       PA1 = INTGRL(PA1IC,PA1DOT)
       PB1 = INTGRL(PB1IC,PB1DOT)
       PC1 = INTGRL(PC1IC,PC1DOT)
*
*      BODY CONSTANTS
PARAM NC=2,NB=2,N3=6
TABLE BMASS(1-2) = 24.0,1.0
TABLE IXX(1-2) = 110.,7.0,IYY(1-2) = 140.,5.0,IZZ(1-2) = 185.,10.
TABLE IXY(1-2) = -1.69,0., IXZ(1-2) = .31,0., IYZ(1-2) = 3.24,0.
TABLE LIX(12) = 1.,LIX(21) = -.5
TABLE LIY(12) = 0.,LIY(21) = 0.
TABLE LIZ(12) = 0.,LIZ(21) = 0.
PARAM KS=1000.,DS=200.
*      ATTITUDE CONTROL PARAMETERS
PARAM KDX = 43.6, KDY = 43.6, KDZ = 42.6
PARAM TCX = 10.0, TCY = 10.0, TCZ = 10.6
PARAM DBX = 4.30, DBY = 4.30, DBZ = 4.30
```

```
PARAM TDX = 20.0, TDY = 20.0, TDZ = 21.2
PARAM MOT = .020, TRS = .5
PARAM KTX = .0495, KTY = .063, KTZ = .08325
PARAM TGX=0.,TGY=0.,TGZ=0.
*      INITIAL ANGULAR VELOCITY COMPONENTS
INCON W1XIC = 0., W1YIC = 0., W1ZIC = 0.
INCON W2YIC = 0.
*      INITIAL EULER ANGLES OF BODY 1
TABLE PHIZ(1)=-.78071,THETZ(1)=-.0060,PSIZ(1)=.78500
PARAM GA=.7853982
INCON ALIC=0.
INTEG MILNE
RELERR AC=1.E-5,XDER=1.E-5,YDER=1.E-5,ZDER=1.E-5,AL=1.E-5,W1X=1.E-7
CONTRL DELT=.1,FINTIM=90.,CLKTIM=1800.
PRINT 1.,W1X,W1Y,W1Z,W2X,W2Y,W2Z,AL,THETP,THETY,THETR,AMPX,AMPY,AMPZ,...
      XDER,YDER,ZDER,XSNS,YSNS,ZSNS,EXT1,EYT1,EZT1,XIN,YIN,ZIN,TDRX,...
      TDRY,TDRZ,AC,DELT
PREPAR .1,THETP,THETY,THETR,W1X,W1Y,W1Z,AL,AC,XDER,YDER,ZDER,EXT1,...
      EYT1,EZT1
GRAPH ,,TIME,THETP
GRAPH ,,TIME,THETY
GRAPH ,,TIME,THETR
GRAPH ,,TIME,W1X
GRAPH ,,TIME,W1Y
GRAPH ,,TIME,W1Z
GRAPH ,,TIME,AL
GRAPH ,,TIME,AC
GRAPH ,,TIME,EXT1
GRAPH ,,TIME,EYT1
GRAPH ,,TIME,EZT1
GRAPH ,,TIME,XDER
GRAPH ,,TIME,YDER
GRAPH ,,TIME,ZDER
GRAPH ,,THETP,W1X
GRAPH ,,THETY,W1Y
GRAPH ,,THETR,W1Z
END
STOP
```

Appendix C
Subroutine MLTBDY Listing

```
      SUBROUTINE MLTBDY(NB,N3,NC,MB,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,LIZ)
C
C     ADJUSTABLE DIMENSIONS
C
      DIMENSION         MB(1),IXX(1),IYY(1),IZZ(1),IXY(1),IXZ(1),IYZ(1),
     1                  LIX(1),LIY(1),LIZ(1),TM(1)
C
C     ADDITIONAL DIMENSIONED VARIABLES
C
      DOUBLE PRECISION A(19,19),                    BMASS(5)
      DOUBLE PRECISION DET1,DET2
      DOUBLE PRECISION PTAP(8,8),BB(8)
      DOUBLE PRECISION B(19,9)
      DIMENSION PDTW(8),PTAE(8)
      DIMENSION C(3,3,5),DX(5,5,5),DY(5,5,5),          DFX(5),DFY(5)
     1,DZ(5,5,5),        LX(5,5),LY(5,5),LZ(5,5),AP(15,15),
     2                   CPX(5),CPY(5),CPZ(5),WDX(5,5),WDY(5,5),
     3WDZ(5,5),WWDX(5,5),WWDY(5,5),WWDZ(5,5),DWWDX(5,5),DWWDY(5,5),
     4DWWDZ(5,5),HX(5),HY(5),HZ(5),
     6                           DFZ(5),FEX(5,5),FEY(5,5),FEZ(5,5)
      REAL LX,LY,LZ,MB,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,LIZ
C
C     INITIAL CALCULATION OF BARYCENTER VECTORS W.R.T. BODY C.G.S
C                         AND HINGE POINTS
C
      NB3 = N3
      NC1 = NC + 1
      TM(1) = 0.
      DO 5 J=1,NB
      BMASS(J) = MB(J)
5     TM(1) = TM(1) + MB(J)
      DO 14 I=1,NB
      DO 14 J=1,NB
      IF(I.EQ.J) GO TO 16
      K = 10*I + J
      LX(I,J) = LIX(K)
      LY(I,J) = LIY(K)
      LZ(I,J) = LIZ(K)
      GO TO 14
16    LX(I,J) = 0.
      LY(I,J) = 0.
      LZ(I,J) = 0.
14    CONTINUE
      DO 13 N=1,NB
      DO 13 J=1,NB
      DX(N,J,N) = LX(N,J)
      DY(N,J,N) = LY(N,J)
      DZ(N,J,N) = LZ(N,J)
      DO 13 K=1,NB
      DX(N,J,N) = DX(N,J,N) - (BMASS(K)/TM(1))*LX(N,K)
      DY(N,J,N) = DY(N,J,N) - (BMASS(K)/TM(1))*LY(N,K)
13    DZ(N,J,N) = DZ(N,J,N) - (BMASS(K)/TM(1))*LZ(N,K)
C
C     CALCULATION OF AUGMENTED INERTIA DYADICS FOR EACH BODY
C
      DO 31 N=1,NB
      C(1,1,N) = IXX(N)
      C(1,2,N) =-IXY(N)
      C(1,3,N) =-IXZ(N)
      C(2,2,N) = IYY(N)
```

```
      C(2,3,N) =-IYZ(N)
      C(3,3,N) = IZZ(N)
      DO 30 J=1,NB
      C(1,1,N) = C(1,1,N) +BMASS(J)*(DY(N,J,N)**2 + DZ(N,J,N)**2)
      C(1,2,N) = C(1,2,N) -BMASS(J)*DX(N,J,N)*DY(N,J,N)
      C(1,3,N) = C(1,3,N) -BMASS(J)*DX(N,J,N)*DZ(N,J,N)
      C(2,2,N) = C(2,2,N) +BMASS(J)*(DX(N,J,N)**2 + DZ(N,J,N)**2)
      C(2,3,N) = C(2,3,N) -BMASS(J)*DY(N,J,N)*DZ(N,J,N)
30    C(3,3,N) = C(3,3,N) +BMASS(J)*(DX(N,J,N)**2 + DY(N,J,N)**2)
      C(2,1,N) = C(1,2,N)
      C(3,1,N) = C(1,3,N)
31    C(3,2,N) = C(2,3,N)
      RETURN
      ENTRY MLTRAT(NB,N3,NC,TX,TY,TZ,FX,FY,FZ,P,PD,T,WX,WY,WZ,E)
      DIMENSION FX(1),FY(1),FZ(1),T(NB,NB,3,3),TX(1),TY(1),TZ(1),
     1                                  WX(NB,NB),WY(NB,NB),WZ(NB,NB)
     2          ,P(N3,NC),PD(N3,NC)
      DOUBLE PRECISION E(19,1)
C
C     (A) MATRIX CONSTANT ELEMENTS
C
      DO 32 N=1,NB
      DO 32 I=1,3
      DO 32 J=1,3
      K = 3*(N-1) + I
      L = 3*(N-1) + J
32    A(K,L) = C(I,J,N)
C
C     EXTERNAL FORCES
C
      DO 33 J=1,NB
      FEX(J,J) = FX(J)
      FEY(J,J) = FY(J)
33    FEZ(J,J) = FZ(J)
C
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF ANGULAR
C                    VELOCITY VECTORS
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF
C                 EXTERNAL FORCE VECTORS
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF BARYCENTER-
C                         -TO-HINGE VECTORS
C
      DO 17 I=1,NB
      DO 17 J=1,NB
      IF(I.EQ.J) GO TO 25
      WX(I,J)=T(I,J,1,1)*WX(I,I)+T(I,J,1,2)*WY(I,I)+T(I,J,1,3)*WZ(I,I)
      WY(I,J)=T(I,J,2,1)*WX(I,I)+T(I,J,2,2)*WY(I,I)+T(I,J,2,3)*WZ(I,I)
      WZ(I,J)=T(I,J,3,1)*WX(I,I)+T(I,J,3,2)*WY(I,I)+T(I,J,3,3)*WZ(I,I)
      FEX(I,J) = T(I,J,1,1)*FEX(I,I)+T(I,J,1,2)*FEY(I,I)+T(I,J,1,3)*FEZ(
     1I,I)
      FEY(I,J) = T(I,J,2,1)*FEX(I,I)+T(I,J,2,2)*FEY(I,I)+T(I,J,2,3)*FEZ(
     1I,I)
      FEZ(I,J) = T(I,J,3,1)*FEX(I,I)+T(I,J,3,2)*FEY(I,I)+T(I,J,3,3)*FEZ(
     1I,I)
25    CONTINUE
      DO 17 K=1,NB
      IF(I.EQ.J) GO TO 17
      IF(I.EQ.K) GO TO 17
      DX(I,J,K)=T(I,K,1,1)*DX(I,J,I)+T(I,K,1,2)*DY(I,J,I)
     1          +T(I,K,1,3)*DZ(I,J,I)
```

```
      DY(I,J,K)=T(I,K,2,1)*DX(I,J,I)+T(I,K,2,2)*DY(I,J,I)
     1          +T(I,K,2,3)*DZ(I,J,I)
      DZ(I,J,K)=T(I,K,3,1)*DX(I,J,I)+T(I,K,3,2)*DY(I,J,I)
     1          +T(I,K,3,3)*DZ(I,J,I)
17    CONTINUE
C
C     VECTOR CROSS PRODUCTS DESCRIBING SYSTEM ROTATIONAL COUPLING.
C       (QUADRATIC TERMS INVOLVING THE CONNECTING BODY ANGULAR
C       VELOCITIES AND THE MUTUAL BARYCENTER-HINGE VECTORS)
C
      DO 23 N=1,NB
      CPX(N) = 0.
      CPY(N) = 0.
      CPZ(N) = 0.
      DO 23 L=1,NB
      IF(N.EQ.L) GO TO 23
      WDX(L,N) = WY(L,N)*DZ(L,N,N) - WZ(L,N)*DY(L,N,N)
      WDY(L,N) = WZ(L,N)*DX(L,N,N) - WX(L,N)*DZ(L,N,N)
      WDZ(L,N) = WX(L,N)*DY(L,N,N) - WY(L,N)*DX(L,N,N)
      WWDX(L,N) = TM(1)*(WY(L,N)*WDZ(L,N) - WZ(L,N)*WDY(L,N)) + FEX(L,N)
      WWDY(L,N) = TM(1)*(WZ(L,N)*WDX(L,N) - WX(L,N)*WDZ(L,N)) + FEY(L,N)
      WWDZ(L,N) = TM(1)*(WX(L,N)*WDY(L,N) - WY(L,N)*WDX(L,N)) + FEZ(L,N)
      DWWDX(L,N) = DY(N,L,N)*WWDZ(L,N) - DZ(N,L,N)*WWDY(L,N)
      DWWDY(L,N) = DZ(N,L,N)*WWDX(L,N) - DX(N,L,N)*WWDZ(L,N)
      DWWDZ(L,N) = DX(N,L,N)*WWDY(L,N) - DY(N,L,N)*WWDX(L,N)
      CPX(N) = CPX(N) + DWWDX(L,N)
      CPY(N) = CPY(N) + DWWDY(L,N)
      CPZ(N) = CPZ(N) + DWWDZ(L,N)
23    CONTINUE
      DO 27 N=1,NB
      DFX(N) = DY(N,N,N)*FEZ(N,N) - DZ(N,N,N)*FEY(N,N)
      DFY(N) = DZ(N,N,N)*FEX(N,N) - DX(N,N,N)*FEZ(N,N)
27    DFZ(N) = DX(N,N,N)*FEY(N,N) - DY(N,N,N)*FEX(N,N)
C
C     (A) MATRIX TIME VARYING ELEMENT COMPUTATION (TRANSFORMED TO
C         PROVIDE PROPER BODY-N COORDINATES WHEN MULTIPLIED
C           BY ANGULAR VELOCITY VECTORS OF BODIES-K
C           IN K COORDINATES--N NOT EQUAL TO K)
C
      DO 210 N=1,NB
      DO 210 L=1,NB
      IF(N.GE.L) GO TO 210
      K = 3*(N-1)
      LL = 3*(L-1)
      AP(K+1,LL+1) = -TM(1)*(DY(L,N,N)*DY(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
      AP(K+1,LL+2) =  TM(1)*DX(L,N,N)*DY(N,L,N)
      AP(K+1,LL+3) =  TM(1)*DX(L,N,N)*DZ(N,L,N)
      AP(K+2,LL+1) =  TM(1)*DY(L,N,N)*DX(N,L,N)
      AP(K+2,LL+2) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
      AP(K+2,LL+3) =  TM(1)*DY(L,N,N)*DZ(N,L,N)
      AP(K+3,LL+1) =  TM(1)*DZ(L,N,N)*DX(N,L,N)
      AP(K+3,LL+2) =  TM(1)*DZ(L,N,N)*DY(N,L,N)
      AP(K+3,LL+3) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DY(L,N,N)*DY(N,L,N))
      DO 21 I=1,3
      DO 21 J=1,3
      KK = K + I
      NN = LL + J
      A(KK,NN) = AP(KK,LL+1)*T(L,N,1,J) + AP(KK,LL+2)*T(L,N,2,J)
     1           +AP(KK,LL+3)*T(L,N,3,J)
      A(NN,KK) = A(KK,NN)
```

```
21    CONTINUE
210   CONTINUE
C
C     ANGULAR MOMENTUM VECTOR COMPONENTS FOR BODY N
C
      DO 22 N=1,NB
      K3 = 3*N
      K2 = K3-1
      K1 = K3-2
      HX(N) = A(K1,K1)*WX(N,N)+A(K1,K2)*WY(N,N)+A(K1,K3)*WZ(N,N)
      HY(N) = A(K2,K1)*WX(N,N)+A(K2,K2)*WY(N,N)+A(K2,K3)*WZ(N,N)
22    HZ(N) = A(K3,K1)*WX(N,N)+A(K3,K2)*WY(N,N)+A(K3,K3)*WZ(N,N)
C
C     (E) VECTOR ELEMENT CALCULATION
C
      DO 24 N=1,NB
      K = 3*(N-1)
      E(K+1,1) = HY(N)*WZ(N,N) - HZ(N)*WY(N,N) + TX(N)
     1         + CPX(N) + DFX(N)
      E(K+2,1) = HZ(N)*WX(N,N) - HX(N)*WZ(N,N) + TY(N)
     1         + CPY(N) + DFY(N)
24    E(K+3,1) = HX(N)*WY(N,N) - HY(N)*WX(N,N) + TZ(N)
     1         + CPZ(N) + DFZ(N)
      DO 300 N=1,NB3
300   B(N,1) = E(N,1)
      DO 302 N=1,NC
      DO 302 J=1,NB3
302   B(J,N+1) = P(J,N)
      CALL MATINV(A,19,NB3,B,NC1,DET1)
      DO 400 J=1,NC
      PTAE(J) = 0.
      DO 400 I=1,NB3
400   PTAE(J) = PTAE(J) + B(I,1)*P(I,J)
      DO 401 K=1,NC
      DO 401 J=1,NC
      PTAP(J,K) = 0.
      DO 401 I=1,NB3
401   PTAP(J,K) = PTAP(J,K) + P(I,J)*B(I,K+1)
      DO 403 J=1,NC
      PDTW(J) = 0.
      DO 403 I=1,NB
      PDTW(J) = PDTW(J) + PD(3*I-2,J)*WX(I,I)
      PDTW(J) = PDTW(J) + PD(3*I-1,J)*WY(I,I)
403   PDTW(J) = PDTW(J) + PD(3*I,J)*WZ(I,I)
      DO 402 I=1,NC
402   BB(I) = PTAE(I) + PDTW(I)
      CALL MATINV(PTAP,8,NC,BB,1,DET2)
      DO 404 J=1,NB3
      E(J,1) = B(J,1)
      DO 404 I=1,NC
404   E(J,1) = E(J,1) - B(J,I+1)*BB(I)
12    CONTINUE
      RETURN
      END
```

Appendix D
Spacecraft–Solar Panel Vibration Simulations

```
PROGRAM SPACECRAFT APPENDAGE VIBRATIONS TEST
        INTEGER NB,N3,NC
        DIMENSION T(5,5,3,3)
        ARRAY BMASS(5),IXX(5),IYY(5),IZZ(5),IXY(5),IXZ(5),IYZ(5)
        ARRAY WDOT(43),PD(15,8),TX(5),TY(5),TZ(5),WX(5),WY(5),WZ(5)
        ARRAY FX(5),FY(5),FZ(5),FKC(15),LIX(55),LIY(55),LIZ(55),P(15,8)
        CONSTANT NB=5,N3=15,NC=8
        CONSTANT TFINAL = 10.
        CONSTANT FKC=15*1.
        CONSTANT BMASS=54.3,4*1.55,IXX=550.,4*5.75,IYY=552.,4*2.1,...
             IZZ=800.,4*7.8,IXY=5*0.,IXZ=5*0.,IYZ=5*0.
        DATA LIX(13)/-3./LIX(15)/3./LIY(12)/3./LIY(14)/-3./
        DATA LIY(21)/-3.25/LIY(31)/-3.25/LIY(41)/-3.25/LIY(51)/-3.25/
        DATA LIY(23)/-3.25/LIY(24)/-3.25/LIY(25)/-3.25/LIY(32)/-3.25/
        DATA LIY(34)/-3.25/LIY(35)/-3.25/LIY(42)/-3.25/LIY(43)/-3.25/
        DATA LIY(45)/-3.25/LIY(52)/-3.25/LIY(53)/-3.25/LIY(54)/-3.25/
        DATA T(1,2,1,1)/1./T(1,2,2,2)/1./T(1,2,3,3)/1./
        DATA T(1,3,1,2)/1./T(1,3,2,1)/-1./T(1,3,3,3)/1./
        DATA T(1,4,1,1)/-1./T(1,4,2,2)/-1./T(1,4,3,3)/1./
        DATA T(1,5,1,2)/-1./T(1,5,2,1)/1./T(1,5,3,3)/1./
        DATA T(2,3,1,2)/1./T(2,3,2,1)/-1./T(2,3,3,3)/1./
        DATA T(2,4,1,1)/-1./T(2,4,2,2)/-1./T(2,4,3,3)/1./
        DATA T(2,5,1,2)/-1./T(2,5,2,1)/1./T(2,5,3,3)/1./
        DATA T(3,4,1,2)/1./T(3,4,2,1)/-1./T(3,4,3,3)/1./
        DATA T(3,5,1,1)/-1./T(3,5,2,2)/-1./T(3,5,3,3)/1./
        DATA T(4,5,1,2)/1./T(4,5,2,1)/-1./T(4,5,3,3)/1./
        DATA P(1,3)/-1./P(1,7)/1./P(2,1)/1./P(2,5)/-1./P(3,2)/1./
        DATA P(3,4)/1./P(3,6)/1./P(3,8)/1./P(5,1)/-1./P(6,2)/-1./
        DATA P(8,3)/-1./P(9,4)/-1./P(11,5)/-1./P(12,6)/-1./P(14,7)/-1./
        DATA P(15,8)/-1./
        CONSTANT DP2=115.,DP3=100.,DP4=90.,DP5=50.
        CONSTANT KP2=4000.,KP3=3000.,KP4=2560.,KP5=2000.
        CONSTANT CLKTIM=120.
INITIAL
        PAGE EJECT $ TITLE      SPACECRAFT AND 4-PANEL SOLAR ARRAY...
             VIBRATIONS TEST $ PAGE SKIP,4 $ TITLE  BODY 1 - SPACECRAFT...
             BUS $ TITLE  BODY 2 - SOLAR PANEL $ TITLE  BODY 3 - SOLAR...
             PANEL $ TITLE  BODY 4 - SOLAR PANEL $ TITLE  BODY 5 - SOLAR...
             PANEL $ PAGE SKIP,4 $ TITLE INERTIAS $ PAGE SKIP,2 $ TITLE...
             IXX,IYY,IZZ,IXY,IXZ,IYZ $ PAGE SKIP,2 $ TITLE BODY 1 $ PAGE...
             SKIP,1 $ OUT IXX(1),IYY(1),IZZ(1),IXY(1),IXZ(1),IYZ(1)
             PAGE SKIP,1 $ TITLE BODIES 2-5 $ PAGE SKIP,1 $ OUT IXX(2),...
             IYY(2),IZZ(2),IXY(2),IXZ(2),IYZ(2) $ PAGE SKIP,4 $ TITLE ...
             MASSES $ PAGE SKIP,2 $ TITLE M1,M2,M3,M4,M5 $ PAGE SKIP,2
             OUT BMASS(1),BMASS(2),BMASS(3),BMASS(4),BMASS(5)
             PAGE EJECT $ TITLE PANEL HINGE SPRING AND DAMPER CONSTANTS
             PAGE SKIP,2 $ TITLE KP2,KP3,KP4,KP5 $ PAGE SKIP,2 $ OUT ...
             KP2,KP3,KP4,KP5 $ PAGE SKIP,4 $ TITLE DP2,DP3,DP4,DP5
             PAGE SKIP,2 $ OUT DP2,DP3,DP4,DP5 $ PAGE EJECT
        CALL MLTBDL(NB,N3,NC,BMASS,P,T,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,...
                   LIZ,FKC)
END
DYNAMIC
        IF(TIME.GE.TFINAL) GO TO S1
        DERIVATIVE VIB
        VARIABLE TIME=0.0
```

```
CINTERVAL CI=.05
ALGORITHM IAL=8,JAL=8
STPCLK CLKTIM
OUTPUT  2,W1XD,W1YD,W1ZD,W2XD,W3XD,W4XD,W5XD,W1X,W1Y,W1Z,W2X,...
        TORQ,W3X,W4X,W5X,THX,THY,THZ,A2,A3,A4,A5,W2Y,W2Z,W3Y,W3Z,...
           W4Y,W4Z,W2YD,W2ZD,W3YD,W3ZD,W4YD,W4ZD,W5YD,W5ZD,W5Y,W5Z
PREPAR W1X,W1Y,W1Z,W2X,W3X,W4X,W5X,THX,THY,THZ,A2,A3,A4,A5,W2Y,...
         W2Z,W3Y,W3Z,W4Y,W4Z,W5Y,W5Z,TORQ
NOSORT
INTEGER NB,N3,NC
PD(1,4)=W3X-W1Y
PD(1,8)=-W5X-W1Y
PD(2,2)=-W2X+W1X
PD(2,6)=W4X+W1X
PD(3,1)=W2X-W1X
PD(3,3)=W3X-W1Y
PD(3,5)=W4X+W1X
PD(3,7)=W5X+W1Y
W2Y = W1Y
W2Z = W1Z
W3Y = -W1X
W3Z = W1Z
W4Y = -W1Y
W4Z = W1Z
W5Y = W1X
W5Z = W1Z
WX(1)=W1X
WY(1)=W1Y
WZ(1)=W1Z
WX(2)=W2X
WY(2) = W2Y
WZ(2) = W2Z
WX(3)=W3X
WY(3) = W3Y
WZ(3) = W3Z
WX(4)=W4X
WY(4) = W4Y
WZ(4) = W4Z
WX(5)=W5X
WY(5) = W5Y
WZ(5) = W5Z
TX(1)=KP2*(A2-THX)+DP2*(W2X-W1X) - KP4*(A4+THX)-DP4*(W4X+W1X)
TY(1)=KP3*(A3-THY)+DP3*(W3X-W1Y) - KP5*(A5+THY)-DP5*(W5X+W1Y)
TX(2)=-KP2*(A2-THX)-DP2*(W2X-W1X)
TX(3)=-KP3*(A3-THY)-DP3*(W3X-W1Y)
TX(4)=-KP4*(A4+THX)-DP4*(W4X+W1X)
TX(5)=-KP5*(A5+THY)-DP5*(W5X+W1Y)
TORQ1 = 100.*STEP(0.,TIME)
TORQ2 = -200.*STEP(2.,TIME)
TORQ3 = 100.*STEP(4.,TIME)
TORQ = TORQ1+TORQ2+TORQ3
TX(1) = TX(1) + TORQ
FZ(1)=300.
CALL MLTRAT(NB,N3,NC,TX,TY,TZ,FX,FY,FZ,PD,WX,WY,WZ,WDOT)
W1XD=WDOT(1)
W1YD=WDOT(2)
W1ZD=WDOT(3)
```

```
        W2XD=WDOT(4)
        W2YD  =  WDOT(5)
        W2ZD  =  WDOT(6)
        W3XD=WDOT(7)
        W3YD  =  WDOT(8)
        W3ZD  =  WDOT(9)
        W4XD=WDOT(10)
        W4YD  =  WDOT(11)
        W4ZD  =  WDOT(12)
        W5XD=WDOT(13)
        W5YD  =  WDOT(14)
        W5ZD  =  WDOT(15)
        W1X=INTEG(WDOT(1),0.)
        W1Y=INTEG(WDOT(2),0.)
        W1Z=INTEG(WDOT(3),0.)
        W2X=INTEG(WDOT(4),0.)
        W3X=INTEG(WDOT(7),0.)
        W4X=INTEG(WDOT(10),0.)
        W5X=INTEG(WDOT(13),0.)
        THX=INTEG(W1X,0.)
        THY=INTEG(W1Y,0.)
        THZ=INTEG(W1Z,0.)
        A2=INTEG(W2X,0.)
        A3=INTEG(W3X,0.)
        A4=INTEG(W4X,0.)
        A5=INTEG(W5X,0.)
        END
        END
END
TERMINAL
S1..    CONTINUE
END
END
```

Appendix E
Subroutine MLTBDL Listing

```
      SUBROUTINE MLTBDL(NB,N3,NC,MB,P,T,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,
     1                  LIZ,FKC)
C
C     ADJUSTABLE DIMENSIONS
C
      DIMENSION MB(1),IXX(1),IYY(1),IZZ(1),IXY(1),IXZ(1),IYZ(1),LIX(1),
     1    LIY(1),LIZ(1),TM(1),FKC(1),P(N3,NC),T(NB,NB,3,3)
C
C     ADDITIONAL DIMENSIONED VARIABLES
C
      DIMENSION Q(27,27),AP(27,27)
      DOUBLE PRECISION A(43,43),B(43,1),WRK(200)
      DIMENSION BMASS(9),PDTW(16),C(3,3,9),DFX(9),DFZ(9),DFY(9),
     1    DX(9,9,9),DY(9,9,9),DZ(9,9,9),LX(9,9),LY(9,9),
     2    LZ(9,9),CPX(9),CPY(9),CPZ(9),WDX(9,9),WDY(9,9),
     3    WDZ(9,9),WWDX(9,9),WWDY(9,9),WWDZ(9,9),DWWDX(9,9),
     4    DWWDY(9,9),DWWDZ(9,9),HX(9),HY(9),HZ(9),FEX(9,9),
     5    FEY(9,9),FEZ(9,9)
      EQUIVALENCE (AP(1),Q(1))
      REAL LX,LY,LZ,MB,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,LIZ
C
C     INITIAL CALCULATION OF BARYCENTER VECTORS W.R.T. BODY C.G.S
C                           AND HINGE POINTS
C
      NB3 = N3
      NC1 = NC + 1
      NT=N3+NC
      TM(1) = 0.
      DO 5 J=1,NB
      BMASS(J) = MB(J)
    5 TM(1) = TM(1) + MB(J)
      DO 14 I=1,NB
      DO 14 J=1,NB
      IF(I.EQ.J) GO TO 16
      K = 10*I + J
      LX(I,J) = LIX(K)
      LY(I,J) = LIY(K)
      LZ(I,J) = LIZ(K)
      GO TO 14
   16 LX(I,J) = 0.
      LY(I,J) = 0.
      LZ(I,J) = 0.
   14 CONTINUE
      DO 13 N=1,NB
      DO 13 J=1,NB
      DX(N,J,N) = LX(N,J)
      DY(N,J,N) = LY(N,J)
      DZ(N,J,N) = LZ(N,J)
      DO 13 K=1,NB
      DX(N,J,N) = DX(N,J,N) - (BMASS(K)/TM(1))*LX(N,K)
      DY(N,J,N) = DY(N,J,N) - (BMASS(K)/TM(1))*LY(N,K)
   13 DZ(N,J,N) = DZ(N,J,N) - (BMASS(K)/TM(1))*LZ(N,K)
C
C     CALCULATION OF AUGMENTED INERTIA DYADICS FOR EACH BODY
C
      DO 31 N=1,NB
      C(1,1,N) = IXX(N)
      C(1,2,N) =-IXY(N)
```

```
      C(1,3,N) =-IXZ(N)
      C(2,2,N) = IYY(N)
      C(2,3,N) =-IYZ(N)
      C(3,3,N) = IZZ(N)
      DO 30 J=1,NB
      C(1,1,N) = C(1,1,N) +BMASS(J)*(DY(N,J,N)**2 + DZ(N,J,N)**2)

      C(1,2,N) = C(1,2,N) -BMASS(J)*DX(N,J,N)*DY(N,J,N)
      C(1,3,N) = C(1,3,N) -BMASS(J)*DX(N,J,N)*DZ(N,J,N)
      C(2,2,N) = C(2,2,N) +BMASS(J)*(DX(N,J,N)**2 + DZ(N,J,N)**2)
      C(2,3,N) = C(2,3,N) -BMASS(J)*DY(N,J,N)*DZ(N,J,N)
   30 C(3,3,N) = C(3,3,N) +BMASS(J)*(DX(N,J,N)**2 + DY(N,J,N)**2)
      C(2,1,N) = C(1,2,N)
      C(3,1,N) = C(1,3,N)
   31 C(3,2,N) = C(2,3,N)
C
C     (A) MATRIX CONSTANT ELEMENTS
C
      DO 32 N=1,NB
      DO 32 I=1,3
      DO 32 J=1,3
      K = 3*(N-1) + I
      L = 3*(N-1) + J
   32 A(K,L) = C(I,J,N)
      DO 15 N=1,NB
      K3 = 3*N
      K2 = K3 - 1
      K1 = K3 - 2
      Q(K1,K1) = A(K1,K1)
      Q(K1,K2) = A(K1,K2)
      Q(K1,K3) = A(K1,K3)
      Q(K2,K1) = A(K2,K1)
      Q(K2,K2) = A(K2,K2)
      Q(K2,K3) = A(K2,K3)
      Q(K3,K1) = A(K3,K1)
      Q(K3,K2) = A(K3,K2)
   15 Q(K3,K3) = A(K3,K3)
C
C     INVERSE BODY-TO-BODY CO-ORDINATE TRANSFORMATION MATRICES
C
      DO 50 I=1,NB
      DO 50 J=1,NB
      IF(J.GE.I) GO TO 50
      DO 49 K=1,3
      DO 49 L=1,3
   49 T(I,J,K,L) = T(J,I,L,K)
   50 CONTINUE
C
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF BARYCENTER-
C                                    -TO-HINGE VECTORS
C
      DO 17 I=1,NB
      DO 17 J=1,NB
      DO 17 K=1,NB
      IF(I.EQ.J) GO TO 17
      IF(I.EQ.K) GO TO 17
      DX(I,J,K)=T(I,K,1,1)*DX(I,J,I)+T(I,K,1,2)*DY(I,J,I)
     1         +T(I,K,1,3)*DZ(I,J,I)
      DY(I,J,K)=T(I,K,2,1)*DX(I,J,I)+T(I,K,2,2)*DY(I,J,I)
     1         +T(I,K,2,3)*DZ(I,J,I)
```

```
            DZ(I,J,K)=T(I,K,3,1)*DX(I,J,I)+T(I,K,3,2)*DY(I,J,I)
     1         +T(I,K,3,3)*DZ(I,J,I)
   17 CONTINUE
C
C     (A) MATRIX TIME VARYING ELEMENT COMPUTATION (TRANSFORMED TO
C         PROVIDE PROPER BODY-N COORDINATES WHEN MULTIPLIED
C           BY ANGULAR VELOCITY VECTORS OF BODIES-K
C           IN K COORDINATES--N NOT EQUAL TO K)
C
      DO 210 N=1,NB
      DO 210 L=1,NB
      IF(N.GE.L) GO TO 210
      K = 3*(N-1)
      LL = 3*(L-1)
      AP(K+1,LL+1) = -TM(1)*(DY(L,N,N)*DY(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
      AP(K+1,LL+2) =  TM(1)*DX(L,N,N)*DY(N,L,N)
      AP(K+1,LL+3) =  TM(1)*DX(L,N,N)*DZ(N,L,N)
      AP(K+2,LL+1) =  TM(1)*DY(L,N,N)*DX(N,L,N)
      AP(K+2,LL+2) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
      AP(K+2,LL+3) =  TM(1)*DY(L,N,N)*DZ(N,L,N)
      AP(K+3,LL+1) =  TM(1)*DZ(L,N,N)*DX(N,L,N)
      AP(K+3,LL+2) =  TM(1)*DZ(L,N,N)*DY(N,L,N)
      AP(K+3,LL+3) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DY(L,N,N)*DY(N,L,N))
      DO 21 I=1,3
      DO 21 J=1,3
      KK = K + I
      NN = LL + J
      A(KK,NN) = AP(KK,LL+1)*T(L,N,1,J) + AP(KK,LL+2)*T(L,N,2,J)
     1          +AP(KK,LL+3)*T(L,N,3,J)
      A(NN,KK) = A(KK,NN)
   21 CONTINUE
  210 CONTINUE
C
C     -A- MATRIX CONTRIBUTIONS FROM THE CONSTRAINT MATRIX,P
C
      DO 101 I=1,N3
      DO 101 J=1,NC
      JC = J + N3
      A(I,JC) = P(I,J)
  101 A(JC,I) = A(I,JC)
C
C     INVERT -A- MATRIX
C
      CALL AINVD(A,43,NT,$12,WRK)
      RETURN
      ENTRY MLTRAT(NB,N3,NC,TX,TY,TZ,FX,FY,FZ,PD,W1,W2,W3,E)
      DIMENSION FX(1),FY(1),FZ(1),TX(1),TY(1),TZ(1),W1(1),W2(1),W3(1),
     1          PD(N3,NC)
      DIMENSION WX(9,9),WY(9,9),WZ(9,9),E(43,1)
C
C     EXTERNAL FORCES
C
      DO 33 J=1,NB
      WX(J,J) = W1(J)
      WY(J,J) = W2(J)
      WZ(J,J) = W3(J)
      FEX(J,J) = FX(J)
      FEY(J,J) = FY(J)
   33 FEZ(J,J) = FZ(J)
```

```
C
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF ANGULAR
C                         VELOCITY VECTORS
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF
C                   EXTERNAL FORCE VECTORS
C
      DO 25 I=1,NB
      DO 25 J=1,NB
      IF(I.EQ.J) GO TO 25
      WX(I,J)=T(I,J,1,1)*WX(I,I)+T(I,J,1,2)*WY(I,I)+T(I,J,1,3)*WZ(I,I)
      WY(I,J)=T(I,J,2,1)*WX(I,I)+T(I,J,2,2)*WY(I,I)+T(I,J,2,3)*WZ(I,I)
      WZ(I,J)=T(I,J,3,1)*WX(I,I)+T(I,J,3,2)*WY(I,I)+T(I,J,3,3)*WZ(I,I)
      FEX(I,J) = T(I,J,1,1)*FEX(I,I)+T(I,J,1,2)*FEY(I,I)+T(I,J,1,3)*FEZ(
     1I,I)
      FEY(I,J) = T(I,J,2,1)*FEX(I,I)+T(I,J,2,2)*FEY(I,I)+T(I,J,2,3)*FEZ(
     1I,I)
      FEZ(I,J) = T(I,J,3,1)*FEX(I,I)+T(I,J,3,2)*FEY(I,I)+T(I,J,3,3)*FEZ(
     1I,I)
   25 CONTINUE
C
C     VECTOR CROSS PRODUCTS DESCRIBING SYSTEM ROTATIONAL COUPLING.
C       (QUADRATIC TERMS INVOLVING THE CONNECTING BODY ANGULAR
C       VELOCITIES AND THE MUTUAL BARYCENTER-HINGE VECTORS)
C
      DO 23 N=1,NB
      CPX(N) = 0.
      CPY(N) = 0.
      CPZ(N) = 0.
      DO 23 L=1,NB
      IF(N.EQ.L) GO TO 23
      WDX(L,N) = WY(L,N)*DZ(L,N,N) - WZ(L,N)*DY(L,N,N)
      WDY(L,N) = WZ(L,N)*DX(L,N,N) - WX(L,N)*DZ(L,N,N)
      WDZ(L,N) = WX(L,N)*DY(L,N,N) - WY(L,N)*DX(L,N,N)
      WWDX(L,N) = TM(1)*(WY(L,N)*WDZ(L,N) - WZ(L,N)*WDY(L,N)) + FEX(L,N)
      WWDY(L,N) = TM(1)*(WZ(L,N)*WDX(L,N) - WX(L,N)*WDZ(L,N)) + FEY(L,N)
      WWDZ(L,N) = TM(1)*(WX(L,N)*WDY(L,N) - WY(L,N)*WDX(L,N)) + FEZ(L,N)
      DWWDX(L,N) = DY(N,L,N)*WWDZ(L,N) - DZ(N,L,N)*WWDY(L,N)
      DWWDY(L,N) = DZ(N,L,N)*WWDX(L,N) - DX(N,L,N)*WWDZ(L,N)
      DWWDZ(L,N) = DX(N,L,N)*WWDY(L,N) - DY(N,L,N)*WWDX(L,N)
      CPX(N) = CPX(N) + DWWDX(L,N)
      CPY(N) = CPY(N) + DWWDY(L,N)
      CPZ(N) = CPZ(N) + DWWDZ(L,N)
   23 CONTINUE
      DO 27 N=1,NB
      DFX(N) = DY(N,N,N)*FEZ(N,N) - DZ(N,N,N)*FEY(N,N)
      DFY(N) = DZ(N,N,N)*FEX(N,N) - DX(N,N,N)*FEZ(N,N)
   27 DFZ(N) = DX(N,N,N)*FEY(N,N) - DY(N,N,N)*FEX(N,N)
C
C     ANGULAR MOMENTUM VECTOR COMPONENTS FOR BODY N
C
      DO 22 N=1,NB
      K3 = 3*N
      K2 = K3-1
      K1 = K3-2
      HX(N) = Q(K1,K1)*WX(N,N)+Q(K1,K2)*WY(N,N)+Q(K1,K3)*WZ(N,N)
      HY(N) = Q(K2,K1)*WX(N,N)+Q(K2,K2)*WY(N,N)+Q(K2,K3)*WZ(N,N)
   22 HZ(N) = Q(K3,K1)*WX(N,N)+Q(K3,K2)*WY(N,N)+Q(K3,K3)*WZ(N,N)
C
C     (E) VECTOR ELEMENT CALCULATION
```

```
      C
            DO 24 N=1,NB
            K = 3*(N-1)
            E(K+1,1) = HY(N)*WZ(N,N) - HZ(N)*WY(N,N) + TX(N)
           1         + CPX(N) + DFX(N)
            E(K+2,1) = HZ(N)*WX(N,N) - HX(N)*WZ(N,N) + TY(N)
           1         + CPY(N) + DFY(N)
      24    E(K+3,1) = HX(N)*WY(N,N) - HY(N)*WX(N,N) + TZ(N)
           1         + CPZ(N) + DFZ(N)
            DO 403 J=1,NC
            PDTW(J) = 0.
            DO 403 I=1,NB
            PDTW(J) = PDTW(J) + PD(3*I-2,J)*WX(I,I)
            PDTW(J) = PDTW(J) + PD(3*I-1,J)*WY(I,I)
      403   PDTW(J) = PDTW(J) + PD(3*I,J)*WZ(I,I)
            DO 42 N=1,NC
            JC = N3 + N
      42    E(JC,1) = -PDTW(N)
            IC = N3 + NC
            DO 520 I=1,IC
            B(I,1) = 0.
*DIAGNOSTIC* THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.
            IF(FKC(I).NE.1.) GO TO 520
            DO 52 J=1,IC
            B(I,1) = B(I,1) + A(I,J)*E(J,1)
      52    CONTINUE
      520   CONTINUE
            DO 53 I=1,IC
      53    E(I,1) = B(I,1)
      12    CONTINUE
            RETURN
            END

END OF COMPILATION:          1 DIAGNOSTICS.
```

Appendix F
Subroutine MLTBD Listing

```
      SUBROUTINE  MLTBD(NB,N3,NC,MB,P,T,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,
     1                    LIZ,FKC)
C
C     ADJUSTABLE DIMENSIONS
C
      DIMENSION MB(1),IXX(1),IYY(1),IZZ(1),IXY(1),IXZ(1),IYZ(1),LIX(1),
     1    LIY(1),LIZ(1),TM(1),FKC(1),P(N3,NC),T(NB,NB,3,3)
C
C     ADDITIONAL DIMENSIONED VARIABLES
C
      DIMENSION AP(27,27)
      DOUBLE PRECISION A(43,43),B(43,1),WRK(200)
      DIMENSION BMASS(9),           C(3,3,9),DFX(9),DFZ(9),DFY(9),
     1    DX(9,9,9),DY(9,9,9),DZ(9,9,9),LX(9,9),LY(9,9),
     2    LZ(9,9),CPX(9),CPY(9),CPZ(9),
     3               WWDX(9,9),WWDY(9,9),WWDZ(9,9),DWWDX(9,9),
     4    DWWDY(9,9),DWWDZ(9,9),                         FEX(9,9),
     5    FEY(9,9),FEZ(9,9)
      REAL LX,LY,LZ,MB,IXX,IYY,IZZ,IXY,IXZ,IYZ,LIX,LIY,LIZ
C
C     INITIAL CALCULATION OF BARYCENTER VECTORS W.R.T. BODY C.G.S
C                            AND HINGE POINTS
C
      NB3 = N3
      NC1 = NC + 1
      NT=N3+NC
      TM(1) = 0.
      DO 5 J=1,NB
      BMASS(J) = MB(J)
    5 TM(1) = TM(1) + MB(J)
      DO 14 I=1,NB
      DO 14 J=1,NB
      IF(I.EQ.J) GO TO 16
      K = 10*I + J
      LX(I,J) = LIX(K)
      LY(I,J) = LIY(K)
      LZ(I,J) = LIZ(K)
      GO TO 14
   16 LX(I,J) = 0.
      LY(I,J) = 0.
      LZ(I,J) = 0.
   14 CONTINUE
      DO 13 N=1,NB
      DO 13 J=1,NB
      DX(N,J,N) = LX(N,J)
      DY(N,J,N) = LY(N,J)
      DZ(N,J,N) = LZ(N,J)
      DO 13 K=1,NB
      DX(N,J,N) = DX(N,J,N) - (BMASS(K)/TM(1))*LX(N,K)
      DY(N,J,N) = DY(N,J,N) - (BMASS(K)/TM(1))*LY(N,K)
   13 DZ(N,J,N) = DZ(N,J,N) - (BMASS(K)/TM(1))*LZ(N,K)
C
C     CALCULATION OF AUGMENTED INERTIA DYADICS FOR EACH BODY
C
      DO 31 N=1,NB
      C(1,1,N) = IXX(N)
      C(1,2,N) =-IXY(N)
      C(1,3,N) =-IXZ(N)
      C(2,2,N) = IYY(N)
```

```
      C(2,3,N) =-IYZ(N)
      C(3,3,N) = IZZ(N)
      DO 30 J=1,NB
      C(1,1,N) = C(1,1,N) +BMASS(J)*(DY(N,J,N)**2 + DZ(N,J,N)**2)
      C(1,2,N) = C(1,2,N) -BMASS(J)*DX(N,J,N)*DY(N,J,N)
      C(1,3,N) = C(1,3,N) -BMASS(J)*DX(N,J,N)*DZ(N,J,N)
      C(2,2,N) = C(2,2,N) +BMASS(J)*(DX(N,J,N)**2 + DZ(N,J,N)**2)
      C(2,3,N) = C(2,3,N) -BMASS(J)*DY(N,J,N)*DZ(N,J,N)
   30 C(3,3,N) = C(3,3,N) +BMASS(J)*(DX(N,J,N)**2 + DY(N,J,N)**2)
      C(2,1,N) = C(1,2,N)
      C(3,1,N) = C(1,3,N)
   31 C(3,2,N) = C(2,3,N)
C
C     (A) MATRIX CONSTANT ELEMENTS
C
      DO 32 N=1,NB
      DO 32 I=1,3
      DO 32 J=1,3
      K = 3*(N-1) + I
      L = 3*(N-1) + J
   32 A(K,L) = C(I,J,N)
C
C     INVERSE BODY-TO-BODY CO-ORDINATE TRANSFORMATION MATRICES
C
      DO 50 I=1,NB
      DO 50 J=1,NB
      IF(J.GE.I) GO TO 50
      DO 49 K=1,3
      DO 49 L=1,3
   49 T(I,J,K,L) = T(J,I,L,K)
   50 CONTINUE
C
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF BARYCENTER-
C                               -TO-HINGE VECTORS
C
      DO 17 I=1,NB
      DO 17 J=1,NB
      DO 17 K=1,NB
      IF(I.EQ.J) GO TO 17
      IF(I.EQ.K) GO TO 17
      DX(I,J,K)=T(I,K,1,1)*DX(I,J,I)+T(I,K,1,2)*DY(I,J,I)
     1          +T(I,K,1,3)*DZ(I,J,I)
      DY(I,J,K)=T(I,K,2,1)*DX(I,J,I)+T(I,K,2,2)*DY(I,J,I)
     1          +T(I,K,2,3)*DZ(I,J,I)
      DZ(I,J,K)=T(I,K,3,1)*DX(I,J,I)+T(I,K,3,2)*DY(I,J,I)
     1          +T(I,K,3,3)*DZ(I,J,I)
   17 CONTINUE
C
C     (A) MATRIX TIME VARYING ELEMENT COMPUTATION (TRANSFORMED TO
C         PROVIDE PROPER BODY-N COORDINATES WHEN MULTIPLIED
C           BY ANGULAR VELOCITY VECTORS OF BODIES-K
C           IN K COORDINATES--N NOT EQUAL TO K)
C
      DO 210 N=1,NB
      DO 210 L=1,NB
      IF(N.GE.L) GO TO 210
      K = 3*(N-1)
      LL = 3*(L-1)
      AP(K+1,LL+1) = -TM(1)*(DY(L,N,N)*DY(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
```

```
      AP(K+1,LL+2) =  TM(1)*DX(L,N,N)*DY(N,L,N)
      AP(K+1,LL+3) =  TM(1)*DX(L,N,N)*DZ(N,L,N)
      AP(K+2,LL+1) =  TM(1)*DY(L,N,N)*DX(N,L,N)
      AP(K+2,LL+2) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DZ(L,N,N)*DZ(N,L,N))
      AP(K+2,LL+3) =  TM(1)*DY(L,N,N)*DZ(N,L,N)
      AP(K+3,LL+1) =  TM(1)*DZ(L,N,N)*DX(N,L,N)
      AP(K+3,LL+2) =  TM(1)*DZ(L,N,N)*DY(N,L,N)
      AP(K+3,LL+3) = -TM(1)*(DX(L,N,N)*DX(N,L,N) + DY(L,N,N)*DY(N,L,N))
      DO 21 I=1,3
      DO 21 J=1,3
      KK = K + I
      NN = LL + J
      A(KK,NN) = AP(KK,LL+1)*T(L,N,1,J) + AP(KK,LL+2)*T(L,N,2,J)
     1          +AP(KK,LL+3)*T(L,N,3,J)
      A(NN,KK) = A(KK,NN)
   21 CONTINUE
  210 CONTINUE
C
C     -A- MATRIX CONTRIBUTIONS FROM THE CONSTRAINT MATRIX,P
C
      DO 101 I=1,N3
      DO 101 J=1,NC
      JC = J + N3
      A(I,JC) = P(I,J)
  101 A(JC,I) = A(I,JC)
C
C     INVERT -A- MATRIX
C
      CALL AINVD(A,43,NT,$12,WRK)
      RETURN
      ENTRY MLTRAT(NB,N3,NC,TX,TY,TZ,FX,FY,FZ,W1,W2,W3,E)
      DIMENSION FX(1),FY(1),FZ(1),TX(1),TY(1),TZ(1),W1(1),W2(1),W3(1)
      DIMENSION WX(9,9),WY(9,9),WZ(9,9),E(43,1)
C
C     EXTERNAL FORCES
C
      DO 33 J=1,NB
      WX(J,J) = W1(J)
      WY(J,J) = W2(J)
      WZ(J,J) = W3(J)
      FEX(J,J) = FX(J)
      FEY(J,J) = FY(J)
   33 FEZ(J,J) = FZ(J)
C
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF ANGULAR
C                      VELOCITY VECTORS
C     BODY-TO-BODY COORDINATE TRANSFORMATIONS OF
C                  EXTERNAL FORCE VECTORS
C
      DO 25 I=1,NB
      DO 25 J=1,NB
      IF(I.EQ.J) GO TO 25
      WX(I,J)=T(I,J,1,1)*WX(I,I)+T(I,J,1,2)*WY(I,I)+T(I,J,1,3)*WZ(I,I)
      WY(I,J)=T(I,J,2,1)*WX(I,I)+T(I,J,2,2)*WY(I,I)+T(I,J,2,3)*WZ(I,I)
      WZ(I,J)=T(I,J,3,1)*WX(I,I)+T(I,J,3,2)*WY(I,I)+T(I,J,3,3)*WZ(I,I)
      FEX(I,J) = T(I,J,1,1)*FEX(I,I)+T(I,J,1,2)*FEY(I,I)+T(I,J,1,3)*FEZ(
     1I,I)
      FEY(I,J) = T(I,J,2,1)*FEX(I,I)+T(I,J,2,2)*FEY(I,I)+T(I,J,2,3)*FEZ(
     1I,I)
```

```
      FEZ(I,J) = T(I,J,3,1)*FEX(I,I)+T(I,J,3,2)*FEY(I,I)+T(I,J,3,3)*FEZ(
     1I,I)
   25 CONTINUE
C
C     VECTOR CROSS PRODUCTS DESCRIBING SYSTEM ROTATIONAL COUPLING.
C       (QUADRATIC TERMS INVOLVING THE CONNECTING BODY ANGULAR
C       VELOCITIES AND THE MUTUAL BARYCENTER-HINGE VECTORS)
C
      DO 23 N=1,NB
      CPX(N) = 0.
      CPY(N) = 0.
      CPZ(N) = 0.
      DO 23 L=1,NB
      IF(N.EQ.L) GO TO 23
      WWDX(L,N) = FEX(L,N)
      WWDY(L,N) = FEY(L,N)
      WWDZ(L,N) = FEZ(L,N)
      DWWDX(L,N) = DY(N,L,N)*WWDZ(L,N) - DZ(N,L,N)*WWDY(L,N)
      DWWDY(L,N) = DZ(N,L,N)*WWDX(L,N) - DX(N,L,N)*WWDZ(L,N)
      DWWDZ(L,N) = DX(N,L,N)*WWDY(L,N) - DY(N,L,N)*WWDX(L,N)
      CPX(N) = CPX(N) + DWWDX(L,N)
      CPY(N) = CPY(N) + DWWDY(L,N)
      CPZ(N) = CPZ(N) + DWWDZ(L,N)
   23 CONTINUE
      DO 27 N=1,NB
      DFX(N) = DY(N,N,N)*FEZ(N,N) - DZ(N,N,N)*FEY(N,N)
      DFY(N) = DZ(N,N,N)*FEX(N,N) - DX(N,N,N)*FEZ(N,N)
   27 DFZ(N) = DX(N,N,N)*FEY(N,N) - DY(N,N,N)*FEX(N,N)
C
C     (E) VECTOR ELEMENT CALCULATION
C
      DO 24 N=1,NB
      K = 3*(N-1)
      E(K+1,1) = TX(N) + CPX(N) + DFX(N)
      E(K+2,1) = TY(N) + CPY(N) + DFY(N)
   24 E(K+3,1) = TZ(N) + CPZ(N) + DFZ(N)
      IC = N3 + NC
      DO 520 I=1,IC
      B(I,1) = 0.
*DIAGNOSTIC*  THE TEST FOR EQUALITY BETWEEN NON-INTEGERS MAY NOT BE MEANINGFUL.
      IF(FKC(I).NE.1.) GO TO 520
      DO 52 J=1,IC
      B(I,1) = B(I,1) + A(I,J)*E(J,1)
   52 CONTINUE
  520 CONTINUE
      DO 53 I=1,IC
   53 E(I,1) = B(I,1)
   12 CONTINUE
      RETURN
      END

END OF COMPILATION:          1 DIAGNOSTICS.
```

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16. Abstract

The results of attempts to put into practice the apparent advantages of the "barycenter formulation" of rigid-body rotational dynamics is described. The end product is a FORTRAN subroutine capable of computing the angular accelerations of each body in a system composed of several point-connected rigid bodies.

A 3-body system is used to illustrate the concept of the connection barycenter. Extension of the barycenter formulation of the dynamical equations to the general case of n bodies is then derived. Some discussion is devoted to the computational problem of handling interbody torques of constraint. An efficient procedure for accommodating the presence of symmetric rotors in the system is also developed.

Two space vehicle attitude dynamics and control simulations of some interest are used to illustrate the application of the computer subroutine MLTBDY: one example is a spacecraft, under three-axis control, subject to the perturbations of a mechanically scanning platform, while the other is a rigid space vehicle hinged to four large solar-cell panels and under the influence of a trajectory-correcting rocket engine.

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